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1 Introduction

Many interesting questions concerning public policy in economics requires developing a class of models that allow for the possibility of violations of the second welfare theorem of Arrow and Debreu. For example in the endogenous growth literature, the role of human capital is often stressed as being central to explaining the differences in long run growth rates among developed countries. Central to such analysis is a failure of markets to incorporate external effects into the private decisions of agents concerning the accumulation of human capital. Unfortunately, this interesting characteristic of human capital (as in the seminal work in Romer [38] for example) implies that the second welfare theorem becomes useless. As a consequence, questions concerning the existence and the characterization of competitive equilibrium becomes complicated to address. Similarly, a significant part of the research in public finance and monetary theory is developing models where fiscal and monetary agents can play an active role in determining the equilibrium dynamics governing capital accumulation, investment, consumption and output. However, in models with public taxation and distortionary monetary policy (e.g., the many references to models in Lucas and Stokey [31], Coleman [14] [15] [16] and Greenwood and Huffman [25]), equilibrium is again nonoptimal, another example of the failure of the “Negishi” based methods based upon the second welfare theorem.¹

In this survey, we discuss an emerging class of methods, which we identify as the “monotone map methods”, first pioneered in the work of Coleman [14], but which relevance to more general environments has been demonstrated in the more recent work of Greenwood and Huffman [25], Coleman [15],[16], Datta, Mirman, and Reffett [18], and Datta, Mirman, Morand, and Reffett [19] [33], that integrate Negishi type approaches with the existence and characterization of competitive equilibrium. These methods are powerful, and are based upon some important results developed in the literature in mathematics and operations studying fixed point of operators on partially ordered spaces. Results in this literature vary from primarily topological as in the work of Krasnoselskii [29] [30], and Amann [5], to lattice based as in the work of Tarski [47] , Topkis [51] [52], Vives [55] and Zhou [56]. Although some of the methods are topological, and often require the under-

¹See Kehoe [28] for an excellent survey of the so called Negishi approach to these problems.
lying domain of the continuous compact operators to be subsets of partially ordered Banach spaces with particularly desirable properties, other methods are primarily lattice based and require very little (if anything) concerning the continuity of operators, but necessitates very strong completeness properties for the underlying domains of the operators. What is very interesting about these monotone methods (whether topological or lattice based), and what is unlike methods based upon fixed point theorems by Brouwer, Schauder, or Fan-Glicksberg, is that they are constructive and can therefore be used as the basis for a systematic study of the theoretical properties of numerical methods that are generally used in the applied literature to compute numerical solutions to such models. This paper focuses primarily on such order-based constructions, and shows how to related these approaches to recent studies concerning the numerical accuracy of simulation methods for nonoptimal models to similar versions of such arguments developed for the Pareto optimal models considered in Santos [44]. In this paper, the possibility of extending the work of Santos [44] appears to hinge critically on the smoothness of equilibrium in strongly concave environments.

We begin the paper by considering a class of nonoptimal environments that are basically versions of the models studied in the work of Brock and Mirman [13] (e.g., single sector production, identical agents, uncertainty, rational expectations, inelastic labor supply) amended to allow for possible violations of the second welfare theorem (e.g., situations where there are taxes, distortionary monetary policy, and/or production externalities associated with capital accumulation). These are the prototype models considered in Coleman [14] and Greenwood and Huffman [25]. We first present the monotone map methods that are typically used in the existing literature that are built around equilibrium versions of the household’s Euler equations (as in Coleman [14]). We then develop a new monotone map approach which is not based upon equilibrium Euler equations, but where the operator is defined from the Bellman’s equation and generate a sequence that has fixed points on a complete lattice of functions. This operator is related to those in the existing literature but also use information about the value function to obtain additional characterizations of the equilibrium that are not available using a pure Euler equation approach. In our discussion, we also show how to develop an alternative version of the monotone map procedure that does not involve any smoothness considerations (e.g., we do not use Euler equations). This new method is developed in a recent paper of Mirman, Morand, and Reffett [33]. We conclude by discussing how to extend the
methods discussed in the paper to models with more general specifications of the models primitive economic data. In particular, we consider models with endogenous labor supply and models that allow for unbounded equilibrium growth.

In economies where the second welfare theorem applies, the important works of Amir, Mirman, and Perkins [3], Hopenhayn and Prescott [26], and Amir [4] provides some results on the application of lattice methods to the problems of existence, uniqueness, and characterization of the decentralized equilibrium, and represent an alternative approach to the research program based on differential topological methods pioneered in the work of Balasko [7] [8], Araujo and Scheinkman [6], and Santos [41]. Since much of work bin comparative analysis in macroeconomics requires economic environments where the second welfare theorem does not hold, this paper identifies how lattice methods can be used to study existence, uniqueness and comparative analysis in many distorted environments.

The remainder of the paper is as follows. In the second section of the paper we discuss the methods of Coleman [14] and Greenwood and Huffman [25]. In section three we present an alternative monotone map method based upon the properties of the best response map of each agent, to construct an operator directly from Bellman’s equation that exploits the supermodularity properties of the environment. We then compare the two methods. We show that, as opposed to the topological constructions that underlie the approaches in Coleman [14][16], the analysis of operators constructed from the best response mapping relies only upon the order structure and properties of a set of functions, and that we can directly apply Tarski’s fixpoint theorem to a complete lattice of functions. In this sense, our work generalizes some of the results in Hopenhaym and Prescott [26]. Section four shows how the methods of Coleman [14] can be extended to models with unbounded growth and endogenous labor supply, and Section five concludes.

2 An Euler Equation Method for a Smooth Strongly Concave Environment

2.1 Taste, Technology and Distortion

For each period and state, the preferences are represented by a monotone, continuous period utility index $u(c_i)$, where $c_i \in \mathbb{K} \subset \mathbb{R}_+$ is period $i$ consump-
tion. Letting $\theta^i = (\theta_1, \ldots, \theta_i)$ denote the history of the shocks until period $i$, a household’s lifetime preferences are defined over infinite sequences indexed by dates and histories $c = (c_\theta^i)$ and are given by:

$$U(c) = E_o \left\{ \sum_{i=0}^{\infty} \beta^i u(c_i) \right\}$$

where the summation is with respect to the probability structure of future histories of the shocks $\theta^i$ given the history of shocks, the transition matrix $\chi$ representing the stochastic process of shocks, and the optimal plans up to a given date $i$. We make the following assumption:

**Assumption 1.** The function $u : K \mapsto R$ is bounded, twice continuously differentiable, strictly increasing, strictly concave, $u'(c) < 0$, and bounded away from zero. In addition, $u'(c)$ satisfies the standard Inada conditions:

$$\lim_{c \to 0} u'(c) = \infty \quad \text{and} \quad \lim_{c \to \infty} u'(c) = 0.$$ 

Each period, households are endowed with a unit of time which they supply inelastically to competitive firms. With the capital-labor ratio denoted by $k$, and the per capita counterpart of this measurement by $K$, we assume that the production possibilities facing the household are summarized by a function $f(k, K, z; t)$ where $t \in \mathbb{T}$, $\mathbb{T}$ the set of continuous functions non-decreasing in their arguments, and is thus a closed sublattice in the Banach lattice of continuous functions with the standard topology of uniform convergence induced by the Sup norm (denoted the $C^0$ topology). Here we restrict the shock process so that the operation of integration preserve increasing differences, as in Hopenhaym and Prescott: [26]

**Assumption 1':** The transition function $\chi(z, dz')$ in an increasing transition function.

**Assumption 2.** The function $f$ satisfies the following conditions:

1. $f(0, K, z; t) = 0$ for all $K \in \mathbb{K}$, $z \in \mathbb{Z}$ and $t \in \mathbb{T}$.
2. $f$ is uniformly continuous, twice continuously differentiable, strictly increasing and strictly concave in its first argument, and the partial derivative with respect to the first variable $f_1$ bounded away from zero.
**Assumption 3.** There exist \( \hat{k}(z) > 0 \) such that \( f(\hat{k}(z), K, z, t) + (1 - \delta)\hat{k}(z) = \hat{k}(z) \) and \( f(k, K, z, t) < k \) for all \( k > \hat{k}(z) \) and for all \( z \in Z \).\(^2\)

Assumptions 1 and 2 are standard in the literature. These assumptions jointly imply that the utility function \( v(k, k', z) = u[f(k, K, z, t) - k'] \) is strongly concave in \( (k, k') \) in the sense of Montrucchio \(^3\) for each \( (K, z, \tau) \). Assumption 3 is a standard feature in stochastic growth literature (See Brock and Mirman \([13]\)), and implies that \( \hat{k} = \sup_z \hat{k}(z) \) exists. As a consequence, the state space for the endogeneous variable \( k \) (and for output) can be defined on the compact set \( \mathbb{K} = [0, \hat{k}] \). Note that since the domain of \( f \) is compact, uniform continuity is implied by continuity.

The parameter vector \( t \) may represent the actions of a government and can be interpreted in many ways. For instance, in an economy with only a state contingent capital income tax (as in Coleman \([14]\)), one can rewrite the modified technology as follows:

\[
f(k, K, z; t) = (1 - t_1(K, z))g(k, z) + t_2(K, z)
\]

where \( g \) is the undistorted production function, \( t_1(K, z) : \mathbb{S} \rightarrow [0, 1] \), and \( t_2(K, z) \) is interpreted as a lump sum transfer. If we define the standard lexicographic partial order on \( \mathbb{T} \) as \( t'(K, z) \succeq t(K, z) \) if and only if either \( t_1'(K, z) < t_1(K, z) \) for all \( (K, z) \in \mathbb{S} \), or \( t_1'(K, z) = t_1(K, z) \) and \( t_2'(K, z) \geq t_2(K, z) \), then \( f(k, K, z; t) \) is increasing in \( t \).\(^3\) Many other cases of distorted economies can be handled within this language under a similar interpretation. In all cases, the government budget at equilibrium is such that the revenues exactly match the expenditures.

The dynamic decision problem for the household is simple. The aggregate states for the economy, denoted \( \mathbb{S} \), belong to the product space \( \mathbb{S} = \mathbb{K} \times \mathbb{Z} \), and since each representative household enters a period with an individual stock of capital, \( k \), the state variables for the household are represented by

\(^2\)These restrictions on preferences and technology in Assumptions 1 and 2 are made to operationalize the Euler equation approach developed in the next section, and they conform with the standard assumptions made in the stochastic growth literature (e.g., Brock and Mirman \([13]\)). It is well-known that weaker Inada type conditions are possible on both \( u \) and \( f \) which are sufficient to guarantee that policies in equilibrium are interior. For more discussion, see later part of our paper.

\(^3\)Finally, because of the dimensionality of the parameter space, we assume that \( T \) is a closed pointwise compact subset of a Banach lattice of continuous functions \( C(K, z) \) which is endowed with the standard pointwise partial order and \( C^0 \) uniform topology. It is therefore a complete lattice (i.e., compact in the interval topology of Frink \([24]\)).
the vector $s \in \mathbb{K} \times \mathcal{S}$. For a given $t \in \mathbb{T}$ we define the household’s feasible correspondence $\Gamma(k, K, z, t)$ for the distorted economy as the set of actions $(c, k')$ satisfying the following constraints:

$$c + k' = f(k, K, z; t); \text{ and } c, k' \geq 0$$

Under Assumption 1, $\Gamma(k, K, z; t)$ is a well-behaved correspondence for each $(k, K, z), t \in \mathbb{T}$. In particular, since $f$ and $t$ are assumed to be continuous, $\Gamma$ is a non-empty, compact and convex-valued, continuous correspondence for each state $s = (k, K, z)$. Also, since the distortions are such that $t$ is increasing in $K$, for each $z$, the correspondence is expanding in $(k, K)$. The correspondence is also expanding in $t \in \mathbb{T}$, i.e., for $t' \geq t$ in the partial order structure on $\mathbb{T}$, $\Gamma(k, K, z; t) \subseteq \Gamma(k, K, z; t')$ for all $(k, K, z)$. In addition, we assume that households consider that the per capita capital stock evolves according to:

$$K' = \kappa(K, z; t)$$

where for any given $t$, $\kappa(., . , t) : \mathbb{K} \times \mathbb{Z} \to \mathbb{K}$ is continuous in its arguments.

The household solves the dynamic decision problem that is summarized in terms of the Bellman equation:

$$J(s) = \sup_{(c, k') \in \Gamma(s, t)} \{u(c) + \beta \int_{\mathbb{Z}} J(s') \chi(z, dz')\} \quad (1)$$

Standard arguments show the existence a $J \in \mathcal{V}$ that satisfies this functional equation for each $\sigma$, where $\mathcal{V}$ is the space of bounded, continuous real valued functions with the sup norm (see, for instance, Stokey, Lucas and Prescott [46]). In addition, standard arguments also establish that $J$ is strictly concave in $k$. Following arguments in Mirman and Zilcha [37], the concavity of $J$ also implies that $J$ is once differentiable in $k$.

We are now prepared to define equilibrium.

**Definition.** A (recursive) competitive equilibrium for this economy consists of a function $t$, a value function for the household $J(s)$, and the associated individual decisions $c$ and $k'$ such that: (i) $J(s)$ satisfies the household’s Bellman equation (1), and $c, k'$ solve the optimization problem in the Bellman’s equation given $t$; (ii) all markets clear: i.e., $k' = \kappa(S) = K'$ and (iii) the government budget equilibrates.\footnote{This last condition $t_1 F = t_2$ in our example of distortion following Coleman ([14]).}
2.2 Existence of Equilibrium

Recalling that the parameter \( t \in \mathbb{T} \) might represent a distortionary equilibrium, we cannot construct an equilibrium solution that is based upon the second welfare theorem. We adopt an alternative strategy, which we name an “Euler equation approach”,\(^5\) and construct an equilibrium by iterating on an operator based on the household’s decision problems. Specifically, we look for a fixed point of an monotone operator defined implicitly in the Euler equation on the space of policy functions. Recalling that an operator \( A \) on a partially ordered set \((\mathbb{X}, \succeq)\) is said to be a monotone operator if \( h' \succeq h \) implies \( Ah' \succeq Ah \) for all \((h, h') \in \mathbb{X}\), we remind the reader of Tarski’s fixed point theorem:

**Theorem 1 Veinott [54].** Let \((\mathbb{X}, \succeq)\) be a complete lattice, and \( A : X \to X \) a monotone increasing mapping. Then the set of fixed point of \( A \) is a non-empty complete lattice.

More precise characterizations of the set of fixed points can be obtained, but they require establishing more properties of the mapping \( A \). In particular, we will be using the following version of Tarski’s fixpoint theorem:

**Theorem 2 Knaster-Tarski fixed point theorem ([17]).** Let \((\mathbb{X}, \succeq)\) be a partially ordered set with the property that every countable chain in \( X \) has a supremum and an infimum. Let \( A : X \to X \) be monotone and continuous, and assume that there exists some \((a, b)\) in \( \mathbb{X}^2 \) with \( A(a) \succeq a \) and \( A(b) \preceq b \). Then \( f \) has a fixed point and \( A^n(a) \) converges to a minimal fixed point in the set \([a, b]\) and \( A^n(b) \) converges to a maximal fixed point in the set \([a, b]\).

The first operator we use is borrowed from Coleman [14] and is used to prove existence of equilibrium. Unfortunately, as shown in Coleman, this operator can be shown to have desirable concave properties only for specific environment (i.e., CES utility) and therefore cannot be used to establish uniqueness in the most general setup.\footnote{This is in contrast to the “value function” or the “Bellman equation” approach, in which one looks for a fixed point of the Bellman’s operator in the space of value functions. In a non-smooth environment, the Bellman equation approach is useful while the Euler equation approach need not be.} We are not able to use the same operator for our second result, the uniqueness of equilibrium. However, we develop a second operator, similar to the operator studied in...
Coleman [16], which fixed points coincide with the fixed points of the first operator, and we show that the second operator has at most one fixed point. This same line of argument is developed for a model with labor-leisure choice in Datta, Mirman and Reffet [18].

The Euler equation associated with the optimal policy function from the right side of the Bellman equation in (1) (after appealing to the envelope condition) is:

\[
u'(c) = \beta \int \phi(z) \left[ u'(f - c, z') r(f - c, z') \right] d\Phi(z; dz'). \tag{2}
\]
in which we omit the arguments in each function for simplicity. Consider the space, denoted \(H^0_+\), of consumption functions \(h\) such that:

(i). \(h: \mathcal{S} \to \mathbb{K}\) and \(h\) continuous;
(ii). 0 \(\leq h(k, z) \leq f(K, K, z, \tau)\) for all \((k, z) \in \mathcal{S}\);
(iii). 0 \(\leq h(k', z) - h(k, z) \leq f(k', k', z; \tau) - f(k, k, z, \tau)\) for all \(k' \geq k\).

Equip \(H^0_+\) with the standard uniform \(C^0\) topology and the partial order \(\geq\) defined as \(h' \geq h\) if and only if \(h'(S) \geq h(S)\) for all \(S \in \mathcal{S}\). Notice that condition (iii) imposes that both \(h\) and \(f - h\) be increasing, and implies some very important properties for \(H^0_+\).

**Lemma 3** Under Assumption 1 and 2:
1. The set \(H^0_+\) is an equicontinuous set of function;
2. Every monotone sequence of elements of \(H^0_+\) converges pointwise to an element of \(H^0_+\);
3. The convergence is uniform on any compact subset of the state space;
4. With the additional Assumption 3, the set \(H^0_+\) is compact in the uniform topology.

**Proof:** (1). Equicontinuity is here induced by the double monotonicity in (iii) of the elements of \(H^0_+\) in conjunction with the uniform continuity of \(F(k, z) = f(k, k, z)\). The assumption of uniform continuity of \(F\) on its domain implies that:

\[\forall \varepsilon > 0, \exists \delta > 0 \mid |k - k'| < \delta \implies |F(k', z) - F(k, z)| < \varepsilon\]

For all \(h\) in \(H^0_+\), properties (iii) writes: For all \(k' \geq k\),

\[0 \leq h(k', z) - h(k, z) \leq F(k', z) - F(k, z)\]
Combining this last inequality with the uniform continuity of $F$ leads to:

$$\forall \varepsilon > 0, \exists \delta > 0 \ |k - k'| < \delta \implies \forall c \in \mathbb{H}_+^0, \ |h(k', z) - h(k, z)| < \varepsilon$$

which demonstrates that $\mathbb{H}_+^0$ is an equicontinuous set of functions. Notice that this result does not require compactness of the state space $S$.

(2) and (3). As a consequence, the Azerla-Ascoli theorem (See Royden [39]) applies: Each sequence $\{g_n\}_{n=0}^\infty$ of elements of $\mathbb{H}_+^0$, because the closure of the set $\{g_n(k, z) : 0 \leq n < \infty\}$ is necessarily compact since by definition $0 \leq g_n(k, z) \leq f(k, k, z; \tau)$, therefore has a subsequence that converges toward a continuous function $g$, and the convergence is uniform on compact subset of $S$. It is easy to see that if the sequence $\{g_n\}_{n=0}^\infty$ is monotone, then the whole sequence converges pointwise to the limit of the convergent subsequence, denoted $g$. Moreover, because all $g_n$ and $f - g_n$ are increasing, $g$ and $f - g$ are also increasing, in addition to being continuous and in the set $[0, f]$. Consequently, $g$ belongs to $\mathbb{H}_+^0$.

(4). Assumption 3 implies that the state space is compact, and the Azerla-Ascoli therefore implies that $\mathbb{H}_+^0$ is compact in the topology induced by the Sup norm (i.e., the uniform topology).

The nonlinear operator $A$ on $\mathbb{H}_+^0$ (it will be shown later that $A$ maps $\mathbb{H}_+^0$ into itself) is defined implicitly as the zero of the following operator $Z$:

$$Z(h, \tilde{h}, k, z; \tau) = \Psi_1(\tilde{h}) - \Psi_2(h, \tilde{h}, k, z; \tau),$$

where,

$$\Psi_1 = u'(\tilde{h})$$

and:

$$\Psi_2 = \beta \int_Z u'(h(f - \tilde{h}, z'), z')r(f - \tilde{h}, z'; \tau)\chi(z, dz').$$

That is, the operator $A$ is defined as:

$$A(h; \tau) = \{\tilde{h} : Z(h, \tilde{h}, k, z; \tau) = 0 \text{ for } h > 0 \text{ and } \tilde{h} = 0 \text{ for } h = 0\}.$$
Lemma 4 Under Assumptions 1, 2 and 3:

1. For any \( h \in \mathbb{H}_0^+ \), there exists a unique \( Ah \) for all \( (k, z) \) such that \( Z(h, Ah, k, z; \tau) = 0 \).
2. \( A \) is a self map or, \( A : \mathbb{H}_0^+ \to \mathbb{H}_0^+ \).
3. \( A \) is a continuous operator on \( \mathbb{H}_0^+ \);
4. \( A \) is a monotone operator on \( \mathbb{H}_0^+ \).
5. There exists a maximal fixed point \( Ah^* \in \mathbb{H}_0^+ \) such that \( \lim_{n \to \infty} A^n f \to Ah^* \) uniformly.
6. The maximal fixed point is strictly positive.

Proof: The proofs of (1), (2), (3), (4) are in Coleman [14]. It is important to note that neither (1), (2), nor (4) rely on compactness of the state space, and are therefore true under Assumption 1 and 2 only. Part (5) follows from the version of Tarski theorem stated at the beginning of the section, noticing that compact \( \mathbb{H}_0^+ \) and from the previous proposition. Notice that Tarski theorem does not rule out the possibility for the zero consumption to be the only fixed point of \( A \). However, it is easy to show that the zero consumption plan is not optimal, a feature of the model that relies crucially on the assumption of unbounded marginal utility at zero. Coleman [14] proves Part (6) under some restriction of the shocks while Greenwood and Huffman [25] provides a more general proof of the strict positivity of the maximal fixed point.\(^6\)

We can now state our existence result.

Proposition 5 Under Assumption 1-3, there exists an equilibrium.

2.3 Uniqueness of Equilibrium

Coleman [14] establishes the uniqueness of the fixed point of the mapping \( A \) by restricting the utility function (see, assumption 7).\(^7\) In this paper, we demonstrate uniqueness for the general class of utility function satisfying Assumption 1 by introducing another operator, denoted \( \hat{A} \), and show that \( \hat{A} \) is a pseudo concave and \( x_0 \)-monotone operator.

\(^6\)Some of these properties are true under Assumption 1-2 only, which will be addressed in Subsection 2.3.

\(^7\)Actually Coleman’s [14] construction can be generalized to the case of constant absolute risk aversion in addition to constant relative risk aversion by using a very similar construction.
Definition. An operator \( \tilde{A} \) on \( H^0_+ \) is pseudo concave if for any strictly positive function \( c \) in \( H^0_+ \), any \( 0 < t < 1 \), and for all \( (k, z) \in S \), \( (\tilde{A}tc)(k, z) > t(\tilde{A}c)(k, z) \).

Definition. An operator \( \tilde{A} \) on \( H^0_+ \) is \( x_0 \)-monotone if it is monotone and if for any strictly fixed point \( c_1 \) of \( \tilde{A} \) there exists some \( k_0 > 0 \) such that the following is true: For any \( 0 \leq k_1 \leq k_0 \) and any \( c_2 \in \mathbb{H}_+^0 \) such that \( c_1(k, z) \geq c_2(k, z) \), \( c_1(k, z) \geq (\tilde{A}c_2)(k, z) \) for all \( k \geq k_1 \) and all \( z \).

Theorem 6 [14] An operator \( \tilde{A} \) that is pseudo concave and \( x_0 \)-monotone has at most one strictly fixed point.

We construct the operator \( \tilde{A} \) as follows. First define the set of functions, denoted \( M \) and endowed with the standard partial pointwise order, \( m : \mathbb{R}_+ \times Z \to \mathbb{R} \) such that:

(i). \( m \) is continuous,

(ii). For all \( (K, z) \in \mathbb{R}_+ \times Z \), \( 0 \leq m(K, z) \leq F(K, z) \)

(iii). For any \( K = 0 \), \( m(K, z) = 0 \).

For any \( m \in M \), consider the function \( \Psi(m(K, z)) \) implicitly defined by:

\[
u'(\Psi(m(K, z))) = 1/m(K, z) \quad \text{for} \quad m > 0, \text{0 elsewhere}\]

Clearly, \( \Psi \) is continuous, increasing, \( \lim_{m \to 0} \Psi(m) = 0 \), and \( \lim_{m \to F(K, z)} \Psi(m) = F(K, z) \). Using the function \( \Psi \), we denote:

\[
\widehat{Z}(m, \tilde{m}, K, z) = 1/\tilde{m}
\]

and consider the operator \( \hat{A} \):

\[
\hat{A}(m) = \{ \tilde{m} / \widehat{Z}(m, \tilde{m}, K, z) = 0 \text{ for } m > 0, \text{0 elsewhere}\}
\]

Since \( \widehat{Z} \) is strictly increasing in \( m \) and strictly decreasing in \( \tilde{m} \), and since \( \lim_{\tilde{m} \to 0} \widehat{Z} = +\infty \) and \( \lim_{m \to F(K, z)} \widehat{Z} = -\infty \), for each \( m(K, z) > 0, \) with \( K > 0 \), and \( z \in Z \) there exists a unique \( \hat{A}m(K, z) \).
It is easy to show that to each fixed point of the operator $A$ corresponds a fixed point of the operator $\hat{A}$. Indeed, consider $x$ such that $Ax = x$ and define $y = 1/u'(x)$ (or, equivalently $\Psi(y) = x$). By definition, $x$ satisfies:

$$u'(x(K, z)) = \beta E_z\{H(F(K, z) - x(K, z), z') * u'(x(F(K, z) - x(k, z), z'))\}$$

for all $(K, z)$. Substituting the definition of $y$ into this expression, this implies that:

$$1/y = \beta E_z\{[H(F(K, z) - \Psi(y(K, z))), z']/[y(F(K, z) - \Psi(y(k, z), z'))]\}$$

which shows that $y$ is a fixed point of $\hat{A}$.

**Lemma 7** The operator $\hat{A}$ is pseudo concave and $x_0$-monotone, and therefore has at most one strictly positive fixed point.

Proof: Recall that $\hat{A}$ is pseudo concave if, for any strictly positive $m$ and any $0 < t < 1$, $\hat{A}tm(K, z) > t\hat{A}m(K, z)$ for all $K > 0$ and for all $z \in Z$. Since $\hat{Z}$ is strictly decreasing in its second argument, a sufficient condition for this to be true is that:

$$\hat{Z}(tm, t\hat{A}m, K, z) > \hat{Z}(tm, \hat{A}tm, K, z) = 0 \quad (3)$$

By definition:

$$\hat{Z}(tm, t\hat{A}m, K, z) = 1/t\hat{A}m$$

and

$$\beta E_z\{[H(F(K, z) - \Psi(t\hat{A}m(K, z)), z')]/[tm(F(K, z) - \Psi(t\hat{A}m(K, z)), z')]\}$$

so that:

$$t\hat{Z}(tm, t\hat{A}m, K, z) = 1/\hat{A}m$$

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\[
\beta E_z\{[H(F(K, z) - \Psi(t\hat{A}m(K, z)), z')] / [m(F(K, z) - \Psi(t\hat{A}m(K, z)), z')]\}
\]

Since \( \Psi \) is increasing and \( H(K', z')/m(K', z') \) is decreasing in \( K' \):

\[
1/\hat{A}m - \beta E_z\{[H(F(K, z) - \Psi(t\hat{A}m(K, z)), z')] / [m(F(K, z) - \Psi(t\hat{A}m(K, z)), z')]\} > 0
\]

and \( \hat{Z}(tm, t\hat{A}m, K, z) > 0 \) so that condition (3) obtains.

The condition that \( \lim_{k\to0} f_1(k, K, z) = \infty \) for all \( K > 0 \), all \( z \) in Assumption 1 (ii) implies that \( H(0, z') = \infty \) for all \( z' \). Given that \( \hat{A} \) is monotone, this latter condition is sufficient for the operator \( \hat{A} \) to be \( x_0 \)-monotone (Lemma 9 and 10 in Coleman [14]).

All fixed points of \( A \) - and at least one of them is strictly positive - are also fixed points of \( \hat{A} \), which has at most one fixed point. Thus, necessarily, the strictly fixed point of \( A \) is unique, and we can state our existence result.

**Proposition 8** Under Assumption 1-3, there exists a unique strictly positive equilibrium.

### 3 A Value Function Iteration Method

In this section we relax some of the smoothness restrictions on the primitives assumed in the previous sections, and appeal to supermodularity and lattice theoretic methods to prove existence of and to characterize the equilibrium. The new set of assumptions encompasses a larger set of environments that the one considered before, although our analysis also applies to the setup of the previous section. However, abandoning the assumption of differentiability and smoothness implies that we cannot work with the Euler equation, and
that we have to build a method straight from the Bellman’s equation. We show how the seminal work of Hopenhayn and Prescott [26] can be applied to distorted economies and can be build upon to show existence of equilibrium in distorted setups and to begin characterize the dependence of equilibrium on the distortion.

Mirman, Morand and Reffett (2001) pushes the analysis further and show that these methods can be generalized to distorted setups in which the strict concavity assumptions on the primitives is also relaxed. However, in such setups the analysis is complicated by the fact that the optimal policy is no longer a one-to-one mapping, but a correspondence. Nevertheless, Mirman, Morand and Reffett (2001) generate existence and characterization results by constructing an argument based on a generalization of Tarski’s fixpoint theorem to correspondences.

For the rest of this section, we make the following assumptions on the primitives of taste and technology:

**Assumption 4.** The primitive economic data satisfy the following:

(i) period utility index $u: K \rightarrow R$ is bounded, continuous, strictly concave, and strictly increasing in $c$;

(ii) For any $t \in T$, the production function $f(\cdot; t): K \times K \times Z$ is continuous and non-decreasing in all of its arguments, and satisfies $f(0, K, z; t) = 0$ for all $(K, z, t) \in K \times Z \times T$;

(iii) there exist $\hat{k}(z, t) > 0$ such that $f(\hat{k}(z, t), \hat{k}(z, t), z, t) + (1-\delta)\hat{k}(z, t) = \hat{k}(z, t)$ and $f(k, k, z, t) < k$ for all $k > k(z, t)$ for all $z \in Z$ and $t \in T$;

(iv) the period utility index $u$, the production function $f$, and the parameter $t \in T$ are such that $\Upsilon(y, k, K, z, t(K, z)) = u(f(k, K, z; t) - y)$ is supermodular in $(x, y)$ where $x = (k, t)$ for each $(K, z)$.

First, notice that supermodularity and the property of having increasing differences are equivalent properties for functions defined on compact domains that are products of closed intervals of $R$, and therefore lattices. This is the case in the rest of the paper, and we therefore use the terms “increasing differences” and “supermodularity” equivalently. Second, the following lemma is a consequence of Assumption 4(i) and (ii).

**Lemma 9** The function $\Upsilon(y, k, K, z, t(K, z))$ is strictly supermodular in $(x', y)$ where $x' = (k, K)$ given any $(z, t)$.

Proof. By Assumption 4(ii) $f$ is non-decreasing in all its arguments so that $f(k', K', z; t) \geq f(k, K, z; t)$ whenever $(k', K') \geq (k, K)$, this last inequality being in the pointwise partial order on $R^2$. Consequently:
\[ u(f(k', K', z; t) - y) - u(f(k, K, z; t) - y) \]

is strictly increasing in \( y \) since \( u \) is strictly concave by Assumption 4(i). This establishes that \( \Upsilon(y, k, K, z, t(K, z)) \) has increasing differences in \((x, y)\) where \( x = (k, K) \), which is equivalent to supermodularity.

Finally, we also to restrict the shock process so that the operation of integration (or more literally summation in our case) perserves supermodularity, and therefore follow Hopenhayn and Prescott [26] by imposing the following restriction:

**Assumption 5:** The transition function \( \chi(z, dz) \) in an increasing transition function.

Although Assumption 4 (i) requires strict concavity of the period utility index and, together with Assumption 4 (ii), implies strict supermodularity of \( \Upsilon \), our existence result only requires that \( \Upsilon \) be supermodular and therefore holds under the weaker assumption of concavity of \( u \).

### 3.1 The Household Decision Problem

For a given \( t \in T \) the household’s feasible correspondence \( \Gamma(k, K, z, t) \) is the set of actions \((c, k')\) satisfying the following constraints:

\[ c + k' = f(k, K, z; t); \text{ and } c, k' \geq 0 \]

and is a well-behaved correspondence for each \((k, K, z)\), \( t \in T \) under Assumption 1. In particular, since \( f \) and \( t \) are assumed to be continuous, \( \Upsilon \) is a non-empty, compact and convex-valued, continuous correspondence for each state \( s = (k, K, z) \). Also, since \( t \) is increasing in \( K \), the correspondence is expanding in \((k, K)\). The correspondence is also expanding in \( t \in T \), i.e., for \( t' \geq t \) in the pointwise order structure on \( T \), \( \Gamma(k, K, z; t) \subseteq \Gamma(k, K, z; t') \) for all \((k, K, z)\).

To complete the description of the aggregate economy, we assume that households take as given the following recursion on the per-capita aggregate capital stock \( K' \):

\[ K' = h(K, z) \]
where $h \in C_1 \subset C$ and $C$ is the set of bounded functions defined on a compact $S$, and $C_1$ is a subset of $C$ for which we also require $h$ to be socially feasible, i.e., for each $t \in T$, $0 \leq h(K, z) \leq f(K, K, z, t)$ for all $(K, z)$, and non-decreasing in its first argument for each $z$, i.e., $h(K', z) - h(K, z) \geq 0$ when $K' \geq K$. Equipped with the Sup norm and the standard pointwise order structure, $C_1$ is a subset of the Banach lattice $C$ of bounded functions defined on $S$. Notice that we do not require $h$ to be continuous: Indeed, we placed no restrictions other than continuity and monotonicity on $f$ and therefore cannot expect that policy function to be continuous.

The following proposition established that sufficient structure for applying Tarski-related fixed point theorems exists.

**Proposition 10** $C_1$ is a complete convex sublattice of the Banach lattice of bounded functions $C$.

We are now ready to represent the typical household’s decision problem for a decentralized competitive equilibrium. Consider a household entering the period in state $(k, K, z)$ for a given $h$ and $t$. For any given $h \in C_1$ and $t \in T$, the value function $v(k, K, z; t, h)$ necessarily satisfies:

$$v(k, K, z; t, h) = \sup_{c, k' \in \Gamma(k, K, z; t)} \{u(c) + \beta \int v(k', h(K, z), z'; t, h) \chi(z, dz')\} \quad (4)$$

Defining the operator $T^C$ as:

$$T^C v(k, K, z; t, h) = \sup_{0 \leq y \leq f(k, K, z, t)} \{u(f(k, K, z, t) - y) + \beta \int v(y, h(K, z), z'; t, h) \chi(z, dz')\}$$

and applying the standard version of the theorem of the maximum (see Berge [10]) and the contraction mapping theorem, it is easy to show that $T^C$ delivers a unique value function $v^*(k, K, z; t, h)$ for each pair $(t, h)$, as stated in the following lemma:

**Lemma 11** For $h \in C_1$ and $t \in T$, there exists a unique function $v^*$ bounded, weakly increasing, concave and continuous in its first argument satisfying Bellman’s functional equation (4).
Denote:

\[ \gamma(k, K, z; t, h) = \{ y \mid y = \arg \max_{0 \leq y \leq f(k, K, z; t)} \left\{ u(f(k, K, z; t) - y) \right\} \]

\[ + \beta \int_z v^*(y, h(K, z), z'; t, h) \chi(z, dz') \]

the optimal policy associated with the value function \( v^*(k, K, z; t, h) \). Since the right hand side in the definition of \( \gamma \) is strictly concave in \( y \), the optimal policy exists for each \((k, K, h, t)\). In addition, the right hand side is also supermodular in \((x, y)\) where \( x = (k, K) \), so that \( \gamma \) is increasing in \((k, K)\) for each \( z \in Z \) by application of Hopenhayn and Prescott [26], which relies on an important result in Topkis (see [52], Theorem 2.8.1).

**Lemma 12** Given any \( t \in T \) and \( h \in C_1 \), under Assumptions 4-5, the value function \( v(k, K, z) \) has an optimal policy \( \gamma(k, K, z) \) which is increasing in \((k, K)\) for each \( z \in Z \).

### 3.2 Existence of Equilibrium

Some of the assumptions required to prove of existence in the previous section of this paper have been relaxed in this section, so existence has to be established through a different path. Our strategy is to construct a non-linear correspondence \( A \) which maps a complete lattice of functions \( C_1 \) into itself \( C_1 \) and to show that this correspondence is monotone increasing on \( C_1 \) in the pointwise partial order. Since \( C_1 \) is a lattice, it follows from Tarski’s theorem that the set of fixed points of \( A \) is not empty.\(^8\)

Specifically, for a given \( h \) and a given \( t \), we define \( Ah(t) \) as the optimal policy along a candidate equilibrium trajectory, that is, we impose the equilibrium condition \( k = K \) in the optimal policy so that:

\[ Ah(K, z) = \gamma(K, K, z; h; t) \]

This construction defines an operator \( A \) with the following properties:

\(^8\)Note that under a more general setup relaxing the assumption of strict concavity, Zhou (CITE) develops a version of Tarski theorem for correspondence that can be applied to demonstrate existence of equilibrium (See Mirman, Morand and Reffett 2001).
Lemma 13 Given any \( t \in T \), under Assumptions 4-5, for any \( h \), \( Ah(K, z) \in C_1 \) and \( A \) is increasing in \( h \) in the partial pointwise order.

The following proposition is then a direct consequence of Tarski's theorem previously cited.

Proposition 14 Under Assumption 4-5, the set of equilibrium is a non-empty complete lattice.

Because a complete lattice is, a fortiori, chain complete, and the operator \( A \) has an excessive point (the zero consumption, which satisfies \( A0 \geq 0 \)) and a deficient point (the production function, which satisfies \( Af \leq f \)), the mapping \( A \) has a minimal and a maximal fixed point, from Tarski theorem. Further developments of these order-based methods also suggest algorithms to compute the minimal and maximal fixed points, as demonstrated in Mirman, Morand and Reffett 2001.

Proposition 15 For each \( t \in T \), the maximal fixed point of the operator \( Ah(K, z) \) can be computed as \( h_u(t) = \lim_{n \to \infty} A_u^n f(K, K, z; t) \).

3.3 Monotone Comparative Analysis

Comparative analysis for this economy discusses how the set of fixed points \( E(t) \in P(C_1) \) of the operator \( A \) changes with respect to changes in the parameter \( t \in T \). Recall that \( T \) is endowed by pointwise partial order. When comparing the elements of \( P(C_1) \), we will use the two order relationships that are defined below (See Veinott [54] for a discussion of the ordinal structure of the various partial orders on \( P(C_1) \)). Let \( X \) be a partially ordered set and \( P(X) \) the power set of \( X \):

(i). Weak induced set order (see Shannon [45] or Topkis [52]): The weak induced set order on \( P(X) \setminus \emptyset \), denoted by \( \geq_w \), is such that \( B \geq_w B' \) if for each \( x \in B \), there exists \( x' \in B' \) such that \( x \geq x' \), and for each \( x' \in B' \) there exists \( x \in B \) such that \( x \geq x' \).

(ii). Induced set ordering (discussed in Topkis [52]): The induced set ordering on \( P(X) \setminus \emptyset \), denoted by \( \geq_s \), in section 2, is such that \( B \geq_s B' \) if for each \( x \in X \) and each \( x' \in X' \), \( x' \wedge x \in B' \) and \( x' \vee x \in B \).

Clearly, when \( X \) is a lattice, the strong order \( \geq_s \) is a stronger ordering than the weak order \( \geq_w \) in the sense that \( B \geq_s B' \) implies that \( B \geq_w B' \).
While it is clear that $\geq_{ss}$ implies $\geq_w$ and $\geq_s$, we were unable to obtain sufficient conditions to generate comparative analysis results in $\geq_{ss}$.

Our strategy for comparative analysis is simple: We show that $A$ varies monotonically in the parameter $t$ on $T$ in the weak set order $\geq_w$, as well as in the strong set order $\geq_s$. This result depends critically on the supermodularity property of the return function postulated in Assumption 4 (iv). Consider the subset of $V_t$ of $V$ of functions $v(k,K,z,t)$ that are supermodular in $(k,t)$, and define:

$$T^C v(k,K,z,t) = \sup_y \{ u(f(k,K,z,t(K,z)) - y) + \beta \int v(y, h(K,z), z', t) \chi(z, dz') \}$$

The second term on the right side of this equality is supermodular in $(y,t)$ and therefore in $((k,y),t)$, while the first term is by Assumption 4 (iv) supermodular in $((k,y),t)$ for each $(K,z)$. Consequently:

$$T^C v(k,K,z,t) = \sup_y \{ H(k,K,y,z,t) \}$$

is supermodular in $(k,t)$ by Topkis ([52], theorem 2.7.6), which proves that $T^C V_t \subset V_t$. Since $V_t$ is a closed subspace of $V$, then by Corollary 1 of Theorem 3.2 in Stokey, Lucas and Prescott [46], the value function satisfying the Bellman’s equation belongs to the subset $V_t$ of $V$. Consequently, the optimal policy function $\gamma(k,K,z,t)$ is increasing in $(k,t)$ by a similar argument to the one developed in Lemma 5. This property provides the basis for the following result:

**Theorem 16** Under Assumptions 4-5, for all $t' \geq_T t$, the set of fixed points $E(t)$ of the nonlinear operator $A$ satisfies: (i) $E(t') \geq_w E(t)$, and (ii) $E(t') \geq_s E(t)$ on $C_1$.

### 4 Extensions

We show that the methods developed in the first section of the paper can be tailored to cover more complex environments. In particular, it is relatively straightforward to show how to relax the assumption of compactness of the state space (Assumption 3) and still establish existence and uniqueness of equilibrium for a set of unbounded growth models (Morand and Reffett...
2001). Such results fill an important gap in the existing literature. The methods also apply to models in which labor is an input to the production of the consumption good, in addition to capital. However, such a modification of the production function increases the dimension of the choice set of households and requires defined a more complicated set of functions on which a monotone increasing operator can be constructed from the Euler equations.

4.1 Unbounded Growth

Existence and uniqueness can also be established in setups in which Assumption 3 does not hold, so that growth is unbounded, by using the same operator $A$. This type of models encompasses endogenous growth models with externalities, even when there are no stationary representation as in Greenwood and Huffman [25], provided that utility is bounded. The difficulty arises from the fact that the set of functions on which $A$ operates is much larger than in the compact state space case, since it includes some unbounded elements. Fortunately, it can be shown that this set still has sufficient properties for a fixpoint argument to apply. Specifically, the argument is based on the following fixpoint theorem (for proof of the theorem, see Morand and Reffett [36]):

**Theorem 17** Let $[\underline{y}, \overline{y}]$ be an order interval of a lattice $(E, \leq)$, and $A : [\underline{y}, \overline{y}] \to [\underline{y}, \overline{y}]$ a continuous and increasing function. If every sequence in $[\underline{y}, \overline{y}]$ has a convergent subsequence that converges to an element in $[\underline{y}, \overline{y}]$ then $A$ has a minimal fixed point $\underline{x}$ and a maximal fixed point $\overline{x}$. Moreover, $\overline{x} = \lim_{k \to \infty} A^k(\underline{y})$ and $\underline{x} = \lim_{k \to \infty} A^k(\overline{y})$, and the sequences $\{A^k(\underline{y})\}_{k=0}^\infty$ and $\{A^k(\overline{y})\}_{k=0}^\infty$ are increasing and decreasing, respectively.

Consider the order interval $[\underline{y}, \overline{y}] = [0, F] = \mathbb{H}_0^+$, which is an equicontinuous set of functions under the assumption of uniform continuity of $F$ on its domain. The operator $A$ is a monotone self map (Lemma 4), with $AF \leq F$ and $A0 \geq 0$, and every sequence of elements of $\mathbb{H}_0^+$ has a convergent subsequence that converges in $\mathbb{H}_0^+$, so that every monotone sequence is convergent in $\mathbb{H}_0^+$, and the convergence is uniform on compact subsets of the state space (Lemma 3). In particular, denote by $\bar{c}$ the limit of the sequence $\{A^k(\overline{y})\}_{k=0}^\infty$.

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9When utility is not assumed to be bounded, Alvarez and Stokey (CITE) demonstrate existence and uniqueness for the class of return functions that are homogeneous of degree $\theta \leq 1$. 
Then, if $A$ is continuous, by definition of continuity at $\tilde{c}$, the equality $A\tilde{c} = \tilde{c}$ is true, and $\tilde{c}$ (resp. $\tilde{c}$) are the maximal and minimal fixed points by the previous theorem.

Thus, the only requirement of Theorem 17 left to be demonstrated is that the operator $A$ is continuous. If that is the case, continuity at $\tilde{c}$ implies that for any sequence $\{c_n\}$ converging to $\tilde{c}$, the sequence $\{Ac_n\}$ converges to $A\tilde{c}$. As a consequence, the sequence $\{A^n(\tilde{y})\}$ which converges to $\tilde{c}$ will also converge to $A\tilde{c}$, and by uniqueness of the limit, $A\tilde{c} = \tilde{c}$. Continuity (at $\tilde{c}$) is therefore a sufficient condition for establishing that $A\tilde{c} = \tilde{c}$, but a far from necessary one, since we are only interested in one particular sequence converging to $\tilde{c}$. Indeed, it is possible to directly demonstrate that $A\tilde{c}(K, z) = \tilde{c}(K, z)$ without having to first prove that $A$ is continuous in $\tilde{c}$, as stated in the following proposition.

**Lemma 18** Proposition 19 The limit $\tilde{c}$ of the sequence $\{A^k(\tilde{y})\}_{k=0}^{\infty}$ is a fixed point of the operator $A$, that is, $A\tilde{c} = \tilde{c}$.

**Proof.** Rearranging the notations, pick any $K = (x, y)$ in $\mathbb{R}_+ \times \mathbb{R}_+$ and consider $s = (K, z)$. Assume, without loss of generality, that $x \geq y$. The sequence $\{c_{n+1}\}_{n=0}^{\infty} = \{Ac_n\}_{n=0}^{\infty}$ converges to $\tilde{c}$ pointwise, so that:

for all $z$ in $Y$, $F(s) - Ac_n(s)$ converges to $F(s) - \tilde{c}(s)$

and, since $H$ is continuous:

for all $z$ in $Y$, $H(F(s) - Ac_n(s))$ converges to $H(F(s) - \tilde{c}(s))$

We also know that the convergence of sequence $\{c_n\}_{n=0}^{\infty}$ toward $\tilde{c}$ is uniform on the compact space $Y = [0, F(x, x, z^{\text{max}})] \times [0, F(x, x, z^{\text{max}})] \times Z$. Consequently:

for all $z$ in $Z$, $c_n(F(s) - Ac_n(s))$ converges to $\tilde{c}(F(s) - \tilde{c}(s))$

Note that the uniform convergence toward $\tilde{c}$ is essential in establishing this result. Indeed, for all $z$:

$$|c_n(F(s) - Ac_n(s)) - \tilde{c}(F(s) - \tilde{c}(s))|$$

\[\leq\]
\[ |c_n(F(s) - Ac_n(s)) - \bar{c}(F(s) - Ac_n(s))| \]

\[ + \]

\[ |\bar{c}(F(s) - Ac_n(s)) - \bar{c}(F(s) - \bar{c}(s))| \]

The first absolute value on the right side of the inequality above is bounded above by \( \sup |c_n - \bar{c}| \) on the compact \( Y \), which can be made arbitrarily small because of the uniform convergence on the compact \( Y \). The second absolute value can be made arbitrarily small by equicontinuity of \( \bar{c} \).

Then, by continuity of \( u' \):

for all \( z \) in \( Z \), \( u'[c_n(F(s) - Ac_n(s))] \) converges to \( u'\bar{c}(F(s) - \bar{c}(s)) \)

Thus:

\[ \beta E_z\{u'[c_n(F(s) - Ac_n(s))]H(F(s) - Ac_n(s))\} \text{ converges to } \beta E_z\{u'\bar{c}(F(s) - \bar{c}(s))\}H(F(s) - \bar{c}(s)) \]

The term on the left is exactly \( u'(Ac_n(s)) \) which we know converges to \( u'(\bar{c}(s)) \). By uniqueness of the limit:

\[ u'(\bar{c}(s)) = \beta u'\bar{c}(F(s) - \bar{c}(s))\bar{c}(F(s) - \bar{c}(s)) \]

which demonstrates that, for all \( s \), \( Ac(s) = \bar{c}(s) \).

Finally, noting that a completely symmetric reasoning applies for the sequence defined by \( c_0 = \bar{y} \) and \( c_{n+1} = Ac_n \) in which \( \bar{y} \) is a deficient point, i.e. \( \bar{y} \leq \bar{y} \), we can now state the following proposition.

**Proposition 20** Under Assumption 1-2, there exists a maximal fixed point of \( A \) in \( [0, F] \), which can be obtained as the limit of the sequence \( A^n(F) \).

Uniqueness follows from a similar argument to the one presented before, exploiting the pseudo concavity and \( x_0 \)-monotonicity of an operator \( \hat{A} \).\(^{10}\)

\(^{10}\)A benign modification of the proof theorem 6 in Section 2 is required.
4.2 Models with Capital and Elastic Labor Supply

We alter the setting of Section 2 to incorporate elastic labor supply, in addition to a state contingent wage and capital tax, while also allowing for production externalities. Many economies fit this notation. Production still takes place in perfectly competitive markets for both output goods and the factors of production, but households nowl have preferences defined over both consumption and leisure, so their unit of time will no longer be supplied inelastically. We therefore alter the assumptions in section two of the paper so that preferences are now represented by a period utility index $u(z_t)$ where $z_t = (c_t, l_t) \in \mathbb{R}_+ \times [0, 1]$. Again letting $\theta^t = (\theta_1, ..., \theta_t)$ denote the history of the shocks until period t, the households lifetime preferences are additively separable and defined over infinite sequences indexed by dates and histories $z = (z_{\theta^t})$ and are now given as follows

$$U(z) = E_o \left\{ \sum_{t=0}^{\infty} \beta^t u(z_t) \right\}.$$ 

We now change assumption one given the additional considerations introduce by defining preferences over leisure: the period utility function $u: \mathbb{R} \times [0, 1] \mapsto \mathbb{R}$ satisfies the following assumption:

**Assumption 6:**

(i) The period utility function $u$ is continuously differentiable, strictly increasing, and strictly concave in $(c, l)$.

(ii) The partial derivatives $u_c(c, l)$ and $u_l(c, l)$ satisfy the Inada conditions:

$$\lim_{c \to 0} u_c(c, l) = \infty, \lim_{c \to \infty} u_c(c, l) = 0, \lim_{l \to 0} u_l(c, l) = \infty.$$

(iii) $u_{cd}(c, l) \geq 0$

With the exception of (iii), which is simply normality, these assumptions on period utility then are standard. (iii) is particularly important in our case, as we require that the marginal rate of substitution $\frac{w}{u_c}$ to be non-decreasing in $c$ and $\frac{w}{u_l}$ to be non-increasing in $l$ to construct a monotone operator in our case. However, one should note that Assumption 6 (iii) is not sufficient to have the period utility function supermodular in $(c, l, k, K)$.

Each household is endowed with a unit of time, and enters into a period with an individual stock of capital $k$. We assume a decentralization where firms do not face dynamic decision problems. Households own firms as well
as both the factors of production and rent these factors of production in competitive markets. In addition, to allow for externalities in the production process, we assume that the production technologies of the firms depend on per capita aggregates. Each period, firms rent capital $k$ and labor $n$ from households, sell output goods in competitive markets, and then return all profits to households at the end of the period. Let $f : \mathbb{K} \times [0, 1] \times \mathbb{K} \times [0, 1] \times \Theta$ summarize the production possibilities for the firm in any given period, $\mathbb{K}$ is a compact set to be described in more detail later. Technology satisfies the following assumption:

**Assumption 7**: The production function satisfies,

(i) $f(0, 0, K, N, \theta) = 0$ for all $(K, N, \theta) \in \mathbb{K} \times [0, 1] \times \Theta$,

(ii) $f(k, n, K, N, \theta)$ is continuous, increasing, differentiable; in addition, it is concave and homogeneous of degree one in $(k, n)$.

(iii) $f(k, n, K, N, \theta)$ also satisfies the standard Inada conditions in $(k, n)$ for all $(K, N, \theta) \in \mathbb{K} \times [0, 1] \times \Theta$; i.e. $\lim_{k \to 0} f(k, n, K, N, \theta) = \infty, \lim_{n \to 0} f(k, n, K, N, \theta) = \infty, \lim_{k \to \infty} f(k, n, K, N, \theta) = 0$.

(iv) There exists a $\hat{k}(\theta) > 0$, such that $f(\hat{k}(\theta), 1, \hat{k}(\theta), 1, \theta) + (1 - \delta)\hat{k}(\theta) = \hat{k}(\theta)$ and $f(k, 1, k, 1, \theta) < k$ for all $k > \hat{k}(\theta)$, for all $\theta \in \Theta$.

Assumption 7 is completely standard in the stochastic growth literature (c.f., Brock and Mirman [13]). In particular Assumption 7 (iv) implies that the state space for capital stock and output can be defined as the compact set $\mathbb{K} = [0, \bar{k}]$.

Assume that firms maximize profits under perfect competition, and denote $\bar{r}(K, \theta)$ the rental rate on capital and $\bar{w}(K, \theta)$ the wage rate. The factor prices are continuous functions of the aggregate state variable. The representative firm’s maximum profit is,

$$\Pi(\bar{r}, \bar{w}, K, N, \theta) = \sup_{k, n} f(k, n, K, N, \theta) - \bar{r}k - \bar{w}n$$

Anticipating the standard definition of competitive equilibrium with $k = K$ and $n = N(S)$, for $S \in \mathbb{S}$ prices in the factor markets are $\bar{r} = f_k$ and $\bar{w} = f_n$. Also, given the assumed structure on the firm’s decision problem, the Theorem of the Maximum ([10]) implies that $\Pi$ is a continuous function, and that solutions to the firm’s problem exist.

The household solves a standard dynamic capital accumulation problem, which we describe by parametrizing the aggregate economy faced by a typical decision maker. Again define $\mathbb{C}$ to be the space of bounded, continuous
functions with domain $S$ and range $\mathbb{R}_+$. If the aggregate per capita capital stock is $K$, then households assume that per capita consumption decisions $C$, and per capita labor supply $N$, and the recursion of the capital stock $K'$ are given by,

\[ K' = \kappa(S); \quad C = C(S); \quad N = N(S); \quad C, \; \kappa, N \in \mathbb{C} \]

The aggregate economy thus consists of functions $\Omega = (w, r, \kappa, C, N)$ from a space of functions with suitable restrictions needed to parameterize the household’s decision problem in the second-stage. Assume that the policy-induced equilibrium distortions have the following standard form,

\[ r = [1 - \pi_n(K, \theta)] \bar{r}, \quad w = [1 - \pi_n(S)] \bar{w}, \]

where $\pi = [\pi_k, \pi_n]$ is a continuous mapping $S \to [0, 1] \times [0, 1]$. We assume regularity conditions on the distorted prices $w$ and $r$.

**Assumption 8**: The vector of distortions $\pi = [\pi_k, \pi_n]$ is such that the distorted wage $w = (1 - \pi_n(K, \theta)) \bar{w}(K, N(K, \theta), \theta)$ and the distorted rental rate $r = (1 - \pi_n(K, \theta)) \bar{r}(K, N(K, \theta), \theta)$ satisfy the following:

(i) $w : \mathbb{K} \times \Theta \to \mathbb{R}_+$ is continuous, at least once-differentiable and (weakly) increasing in $K$,

(ii) $r : \mathbb{K}_+ \times \Theta \to \mathbb{R}_+$ is continuous and decreasing in $K$ such that,

\[ \lim_{K \to 0} r(K, \theta) \to \infty. \]

In other words, we assume that the distorted wage and rental rates behave geometrically as the non-distorted rates $\bar{w}$, $\bar{r}$ (which are the marginal products of labor and capital, respectively).

Let the lump-sum transfers to each agent be given by a function $d(S) = \pi_k K + \pi_n N(K, \theta)$, and the household’s total income (taking into account the elastic nature of labor supply) is therefore given as $y(s) = rk + wN + (1 - \delta)k + \Pi$, the sum of distorted rental and wage incomes, undepreciated capital and profits where $s$ is the individual household’s state, $s = (k, S) = (k, K, \theta)$. Note that $y(s)$ is a continuous function. The household’s feasible correspondence $\Psi(s)$ for the distorted economy then consists of the set of $(c, k') \in \mathbb{R}_+^2$ and $l \in [0, 1]$ that satisfy the following constraint,

\[ c + w(1 - l) + k' = y, \]

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given \((k, K, \theta) \gg 0\). Notice that \(\Psi(s)\) is well behaved: In particular, since \(\Pi\) is continuous, \(\Psi\) is a non-empty, compact and convex-valued, continuous correspondence.

Next, we state the decision problem for the household. At the beginning of any period the aggregate state for the economy is given by \(S \in S\). Each household enters the period with their individual capital stock \(k \in K\), so their individual state is \(s \in K \times S\). Then the households dynamic decision problem is summarized by the Bellman equation,

\[
J(s) = \sup_{(c, l, k') \in \Psi(s)} u(c, l) + \beta \int_\Theta J(s') \chi(\theta, d\theta')
\]

Standard arguments show the existence a \(J \in \mathbb{J}\) that satisfies this functional equation, where \(\mathbb{J}\) is the space of bounded, continuous functions with the uniform norm. In addition, since \(u\) is strictly concave in \(c\), standard arguments also establish that \(J\) is strictly concave in its first argument, \(k\). Following arguments in Mirman and Zilcha [37], the concavity of \(J\) also implies that the envelope theorem applies and \(J\) is once differentiable in \(k\). A standard argument shows that the first order conditions for this problem can be written as

\[
\frac{u_i(c, l)}{u_c(c, l)} = (1 - \pi_n(S)) f_n(K, N, \theta) \quad (5)
\]

\[
u_c(c, l) = \beta \int_\Theta u_c(c(K', \theta'), l(K', \theta')) r(K', \theta') \chi(\theta, d\theta'). \quad (6)
\]

where the ‘\(^\prime\)’ notation refers to next period value of the particular variable.

In equilibrium, \(c(s) = C(S), k = K, n = N(S)\). The next period capital stock, in equilibrium, is then written as \(K' = y - C\). Also, for later reference, define \(\hat{l}(S)\) as the solution to,

\[
\frac{u_i(f(K, 1 - \hat{l}(S), \theta), \hat{l}(S))}{u_c(f(K, 1 - \hat{l}(S), \theta), \hat{l}(S))} = (1 - \pi_n(S)) f_n(K, 1 - \hat{l}(S), \theta).
\]

Notice that \(\hat{l}\) is the amount of leisure that is compatible with utility maximization if everything today is currently invested. In general, the amount
of consumption is less than \( f \) and leisure, which is positively related to consumption, is therefore less than \( \hat{l}(S) \). This establishes a useful upper bound on output in any state which in the sequel we will use to define a candidate set of consumption functions.

Now consider defining the following function \( l^*(c, K, \theta) \) implicitly in the first order condition (5)

\[
\frac{u_c(c, l^*(c, K, \theta))}{u_c(c, l^*(c, K, \theta))} = (1 - \pi_n(S))f_n(K, 1 - l^*(c, K, \theta), K, \theta).
\]

Under Assumptions 6 and 7, one can easily establish that \( l^*(c, K, \theta) \) is decreasing in \( c \) and increasing in \( K \). Given a candidate equilibrium function \( c(S) \), we rewrite the Euler equation (6) in equilibrium using \( l^* \) as follows,

\[
u_c(c, l^*(c, K, \theta)) = \beta \int \frac{u_c(c(F_c - c, \theta'), l^*(c(F_c - c, \theta'), K', \theta'))r(F_c - c, \theta')}{(\theta, d\theta')}
\]

where \( F_c = f(K, 1 - l^*(c(K, \theta), K, \theta), \theta) + (1 - \delta)K \). We then use this last equation to define a nonlinear operator that yields a strictly positive fixed point in the space of consumption functions. This fixed point is an equilibrium for the economy.

Define \( F^u(S) = F^u(K, \theta) = f(K, 1 - \hat{l}(K, \theta), \theta) + (1 - \delta)K \) and the space \( \mathbf{H} \) of functions \( h : \mathbb{S} \to \mathbb{R} \) such that:

(i) \( h \) is continuous,
(ii) \( h(S) \in [0, F^u(S)] \);
(iii) \( u_c(h(S), l^*(h(S), S)) \) is decreasing in \( h \);
(iv) \( u_c(h(S), l^*(h(S), S)) \) is decreasing in \( K \).

Equip \( \mathbf{H} \) with the sup norm. Note that the assumption the marginal utility of consumption is decreasing in \( h \) means that the space \( \mathbf{H} \) differs from the space of consumption functions studied in Coleman [15]. It is easily verified that for the preferences considered in this model, the restriction \( u_c \) decreasing in \( h \) is implied. However, since the class of preferences studied in this paper is larger than that studied in Coleman, additional restriction is necessary on the space of consumption functions.

Define the mapping \( Z : \mathbf{H} \times \mathbf{Y} \times \mathbf{K} \times \mathbf{Z} \to \mathbb{R} \) where \( \mathbf{Y} \subset \mathbb{R}_+ \), as

\[
Z(h, \zeta, K, \theta) = \Psi_1(\zeta, K, \theta) - \Psi_2(h, \zeta, K, \theta)
\]
where
\[ \Psi_1 = u_c(\zeta, l^*(\zeta, K, \theta)), \]
and
\[ \Psi_2 = \beta \int_\Theta u_c(h(F_\zeta - \zeta, \theta'), l^*(h(F_\zeta - \zeta, \theta'), F_\zeta - \zeta, \theta')) r(F_\zeta - \zeta, \theta') \chi(\theta, d\theta'). \]

Here \( F_\zeta = f(K, 1 - l^*(\zeta, K, \theta) + (1 - \delta)K \). Then define the nonlinear operator \( A : \mathcal{H} \to \mathcal{H}' \) as follows:
\[ Ah(K, \theta) = \{ \zeta \text{ such that } Z(h, \zeta, K, \theta) = 0, h > 0; Ah(K, \theta) = 0 \text{ elsewhere} \} \]

where \( \mathcal{H}' \) will be shown below to be exactly \( \mathcal{H} \).

Proposition 21 Under Assumptions 6-8, for any \( h \in \mathcal{H} \), there exists a unique \( Ah = \hat{h} \) such that \( Z(h, \hat{h}, K, \theta) = 0 \), for any \( (K, \theta) \).

Proof: Consider a given \( h \in \mathcal{H} \) and \( (K, \theta) \in S \). Notice first that \( F_\hat{h} - \hat{h} \) is decreasing in \( \hat{h} \) from the fact that \( N = 1 - l^*(\hat{h}, K, \theta) \) is decreasing in \( \hat{h} \). Given Assumption 6 and Assumption 8 (ii), the second term in \( Z \) in (7) is strictly increasing in \( \hat{h} \) since \( h \) is such that \( u_c(h(K, \theta), l^*(h(K, \theta), K, \theta)) \) is decreasing in \( K \). Also, under Assumption 6 (iii), the first term in \( Z \) in (7) is strictly decreasing in \( \hat{h} \), so \( Z \) is strictly decreasing in \( \hat{h} \). Assumption 6 (ii) implies that as \( \hat{h} \to f \), the second term in \( Z \) approaches infinity, while the first term remains finite. As a consequence, \( Z \) tends toward \( -\infty \). Likewise, under Assumption 6 (ii), as \( \hat{h} \to 0 \), the first term of \( Z \) approaches \( \infty \) while the second term remains finite, therefore \( Z \to \infty \). Therefore given continuity assumptions on preferences and distorted prices, \( \hat{h} \) exists and is unique. Continuity implies that \( Ah_n \to Ah \) if \( h_n \to h \).

Our setup is more general than Greenwood and Huffman [25], which only consider the case where \( u_{cl} = 0 \), and Coleman [15], which allows for \( u_{cl} \geq 0 \) and also some cases where \( u_{cl} < 0 \) but considers a restricted homothetic class of preferences and imposes more restrictions (jointly on utility, production functions and distortions) to study the case of negative cross partials of \( u \). The same case of negative cross-partial of \( u \) can be handled in our setting also. At this stage, we are unable to capture more general cases of negative cross partials of \( u \) than Coleman [15], therefore, we focus only on the \( u_{cl} \geq 0 \) case. We have the following important result:
Theorem 22  Under Assumptions 6-8, \( Ah \in H \).

**Proof:** Under the continuity assumptions on preferences, technologies, and distorted prices, continuity of \( Ah \) is obvious. Also it is straightforward that \( Ah \) belongs to the interval \([0, F^u(K, \theta)]\), otherwise the equality, \( Z(h, Ah, K, \theta) = 0 \), cannot be met since the second term in \( Z \) is not defined. To prove that \( Ah \in H \), we need to check two more properties of \( Ah \). First, consider \( h_2 \geq h_1 \), with \((h_1, h_2) \in H^2\). We show \( u_c(Ah, l^*(Ah, K, \theta)) \) is decreasing in \( Ah \), i.e. that \( u_c(Ah_2, l^*(Ah_2, K, \theta)) \leq u_c(Ah_1, l^*(Ah_1, K, \theta)) \).

Define \( \tilde{h}_1 = Ah_1 \) such that \( Z(\tilde{h}_1, Ah_1, K, \theta) = 0 \) and \( \tilde{h}_2 = Ah_2 \) such that \( Z(h_2, Ah_2, K, \theta) = 0 \). Recall that,

\[
\Psi_2(h, \tilde{h}, K, \theta) = \beta \int_{\Theta} u_c(h(F_\tilde{h} - \tilde{h}, \theta'), l^*(h(F_\tilde{h} - \tilde{h}, \theta'), F_\tilde{h} - \tilde{h}, \theta'))r(F_\tilde{h} - \tilde{h}, \theta')\chi(\theta, d\theta'),
\]

(8)

where \( F_\tilde{h} = f(K, 1-l^*(\tilde{h}, K, \theta)) + (1-\delta)K \). Since \( h_2 \) is in \( H \), \( \Psi_2(h_2, \tilde{h}_2, K, \theta) \leq \Psi_2(h_1, \tilde{h}_1, K, \theta) \). That is, \( \Psi_2 \) is decreasing \( h \). Since \( \Psi_1 \) is independent of \( h \), the solution \( \tilde{h}_2 \) must be such that \( \Psi_1(\tilde{h}_2, K, \theta) \leq \Psi_1(\tilde{h}_1, K, \theta) \). Or, \( u_c(Ah_2, l^*(Ah_2, K, \theta)) \leq u_c(Ah_1, l^*(Ah_1, K, \theta)) \), for all \((K, \theta)\). This verifies that \( u_c(Ah, l^*(Ah, K, \theta)) \) is decreasing in \( Ah \).

Finally, to complete the proof we need to show that \( Ah \) is such that \( u_c(Ah(K, \theta), l^*(Ah(K, \theta), K, \theta)) \) is decreasing in \( K \). We show that for any \( K_2 \geq K_1, u_c(\tilde{h}_2, l^*(\tilde{h}_2, K_2, \theta)) \leq u_c(\tilde{h}_1, l^*(\tilde{h}_1, K_1, \theta)) \) where \( \tilde{h}_1 = Ah_1 \) for \((K_1, \theta)\) and \( \tilde{h}_2 = Ah_2 \) for \((K_2, \theta)\), that is, \( Z(h_1, \tilde{h}_1, K_1, \theta) = 0 \) and \( Z(h_2, \tilde{h}_2, K_2, \theta) = 0 \). Note that, \( F_\tilde{h} - \tilde{h} \) is increasing in \( K \), since \( l^*(\tilde{h}, K, \theta) \) is decreasing in \( K \), the marginal products of capital and labor are positive and also by Assumption 8 (ii) \( r \) is decreasing in \( K \). From equation (8), \( \Psi_2(h_1, \tilde{h}_1, K_2, \theta) \leq \Psi_2(h_1, \tilde{h}_1, K_1, \theta) \) and from the definition of \( Z \), \( \Psi_2(h_1, \tilde{h}_1, K_1, \theta) = \Psi_1(\tilde{h}_1, K_1, \theta) \). Also, \( l^*(\tilde{h}, K, \theta) \) decreasing in \( K \) and \( u_{cd} \geq 0 \) imply that \( \Psi_1(\tilde{h}_1, K_2, \theta) \leq \Psi_1(\tilde{h}_1, K_1, \theta) \). Therefore, as \( K \) increases both \( \Psi_1 \) and \( \Psi_2 \) decrease. Using the fact that \( \Psi_1(\tilde{h}_2, K, \theta) \leq \Psi_1(\tilde{h}_1, K, \theta) \), for all \((K, \theta)\), demonstrated above we get,

\[
\Psi_1(\tilde{h}_2, K_2, \theta) \leq \max\{\Psi_1(\tilde{h}_1, K_2, \theta), \Psi_2(h_1, \tilde{h}_1, K_2, \theta)\} \leq \Psi_1(\tilde{h}_1, K_1, \theta),
\]

which verifies \( u_c(\tilde{h}_2, l(\tilde{h}_2, K_2, \theta)) \leq u_c(\tilde{h}_1, l(\tilde{h}_1, K_1, \theta)) \).

Notice that \( H \) is a non-empty, convex subset of a space of continuous, bounded real-valued functions but is not compact. We define the following
subset of $H$ denoted by $\bar{H}$ to be the set of functions $h \in H$ such that

$$0 \leq |h(K_2, \theta) - h(K_1, \theta)| \leq |F(K_2, l^*(h(K_2, \theta), K_2, \theta) - F(K_1, l^*(h(K_1, \theta), K_1, \theta))|$$

for all $K_2 \geq K_1$. A standard argument shows that the space of consumption functions $H \subset \bar{H}$ is a closed, pointwise compact, and equicontinuous set of functions. Then by a standard application of Arzela-Ascoli, $\bar{H}$ is a compact order convex, order interval in $H$. Notice that the restriction on consumption in the space $\bar{H}$ that distinguishes it from $H$ implies that the investment function $K' = F_h - h$ is an increasing functions of the current capital stock $K$ which follows because $F_h$ is increasing in $K$ (since $l^*$ is decreasing in $K$, the marginal products of capital and labor are positive).

We now define the standard pointwise partial order on $H$ as follows: $h' \geq h$ if $h'(S) \geq h(S)$ for all $S \in \mathbb{R} \times \Theta$, and we adopt the same order on the subspace $\bar{H}$. We have the following results:

**Lemma 23** $\bar{H}$ is a complete lattice.

**Proof:** The interval topology on a partially order set $X$ is that topology that has the property that each closed set is either $X$, the empty set, or can be represented as the intersection of sets that are finite unions of closed intervals in $X$. A lattice $X$ is complete if every subset set of $X$ has a sup. Frink [24] shows that a lattice is compact in its interval topology iff it is a complete lattice. The converse is shown in Birkhoff [11]. Notice that $\bar{H}$ is compact in its interval topology. Therefore it is a complete lattice.

**Proposition 24** $Ah \in \bar{H}$. Furthermore, under assumption 6-8, $A$ is monotone on $H$.

**Proof:** We previously established that $Ah \in H$, and it simply remains to show that for all $K_2 \geq K_1$:

$$0 \leq |Ah(K_2, \theta) - Ah(K_1, \theta)| \leq |F(K_2, l^*(Ah(K_2, \theta), K_2, \theta) - F(K_1, l^*(Ah(K_1, \theta), K_1, \theta))|$$
Consider any \( K_2 \geq K_1 \). Since \( Ah \in \mathcal{H}, \Psi_1(Ah(K_2, \theta), K_2, \theta) \leq \Psi_1(Ah(K_1, \theta), K_2, \theta) \).

But then:

\[
\Psi_2(h, Ah(K_2, \theta), K_2, \theta) \leq \Psi_2(h, Ah(K_1, \theta), K_2, \theta) \leq \Psi_2(h, Ah(K_1, \theta), K_1, \theta)
\]

for a solution at \( Z = 0 \). This requires that:

\[
0 \leq| Ah(K_2, \theta) - Ah(K_1, \theta)| \leq | F(K_2, \mu'(Ah(K_2, \theta), K_2, \theta)) - F(K_1, \mu'(Ah(K_1, \theta), K_1, \theta)) |
\]

To demonstrate monotonicity, consider \( h' \geq h \), \((h, h') \in \mathcal{H}^2\). Note that \( u_c \) is strictly decreasing in \( h \), so that the second term in \( Z, \Psi_2 \), is decreasing in \( h \), while the first term, \( \Psi_1 \), is independent of \( h \). Also, \( Z(h, Ah, K, \theta) = 0 \) by definition of the operator \( A \) therefore \( Z(h', Ah, K, \theta) \geq 0 \). \( Z \) is decreasing in its second argument by Assumptions 6 (i), 7 (ii) and 8 (ii), hence the solution \( Ah' \) must be such that \( Ah' \geq Ah \).

**Proposition 25** Under Assumptions 6-8, among the set of fixed points of \( A : \mathcal{H} \to \mathcal{H} \) there exists a maximal fixed point \( h^* \in \mathcal{H} \) such that \( \lim_{n \to \infty} A^n F \to Ah^* = h^* \), uniformly. Further, the maximal fixed point is strictly positive (\( h^* > 0 \)).

**Proof:** The first result (existence and convergence) follows from the fact that \( \mathcal{H} \) is complete and \( A \) is a monotone self map of \( \mathcal{H} \). Note \( AF \leq F \) and \( A0 \geq 0 \), therefore by application of Tarski’s theorem the operator \( A \) has a fixed point. In addition, because \( A \) is continuous, \( A^n F \) converges to a maximal fixed point \( Ah^* \) in the set \( \mathcal{H}_I = \{ h \mid h \in \mathcal{H}, h \leq F \} \), and since \( S \) is compact, the convergence in uniform. The second property (positivity) follows from an obvious modification of the main theorem in Greenwood and Huffman [25].

Uniqueness of equilibrium can be established through an argument related to that in section 2 of the paper. Simplifying the notation by writing \( f^u(K, \theta) = f(K, 1 - \bar{L}(K, \theta), \theta) \), we define the set \( \mathcal{M} \) of functions \( m : K \times \Theta \to K \) such that:

(i) \( m \) is continuous;
(ii) \( 0 \leq m(K, \theta) \leq \frac{1}{\mu_c(f^u(K, \theta), \bar{L}(K, \theta))} \) for \( K > 0 \); \( m(K, \theta) = 0 \) for \( K = 0 \),
(iii) \( \frac{r(K', \theta)}{m(K', \theta)} < \frac{r(K, \theta)}{m(K, \theta)} \) for \( K' > K \).
The space \( M \) can be interpreted as the space of the reciprocal of the marginal utility functions. Our argument consist in defining a suitable operator on the space \( M \), showing that it has a unique strictly positive fixed point, and relating the fixed points of the new operator to the old operator \( Ah \). Define the function \( H(m, K, \theta) \) for each \( m \in M \) implicitly as follows (the following lemma makes sure that this definition is meaningful),

\[
u_c(H(m(K, \theta), K, \theta), l(H(m(K, \theta), K, \theta), K, \theta)) = \frac{1}{m(K, \theta)}, m > 0; \text{ and } H(m, K, \theta) = 0, m = 0.
\]

Note that then \( H(m(K, \theta), K, \theta) = h(K, \theta) \) pointwise.

**Lemma 26** The mapping \( H(m, K, \theta) \) is well-defined for each \((m, K, \theta)\).

**Proof:** As \( m \to 0 \), \( H(m, K, \theta) \to 0 \), for all \((K, \theta)\) and as \( m \to \frac{1}{u_c(f^*(K, \theta), l(K, \theta))} \), \( H(m, K, \theta) \to f(K, 1 - \hat{\iota}, \theta) \). Also, note that \( H(\frac{1}{u_c(c, l(c, K, \theta))}, K, \theta) = c \), for all \((K, \theta)\), and \( H \) is continuous. Therefore, \( H \) is well-defined.

To characterize \( H(m, K, \theta) \), consider \( m' \geq m \) in the pointwise partial order on \( M \), and define \( h_2 = H(m', K, \theta) \) and \( h_1 = H(m, K, \theta) \). Notice that \( m' \geq m \) implies \( h_2 \geq h_1 \). We can now show that \( f(k, 1 - l(H(m, K, \theta) , K, \theta)) - H(m, K, \theta) \) is decreasing in \( m \) by the definition of \( H(m, K, \theta) \). Consider

\[
\Delta(h, f_h - h, \theta) = \beta \int u_c(h(f_h - h, \theta'), l(h(f_h - h, \theta'), f_h - h, \theta')) r(f_h - h, \theta') \chi(\theta, d\theta')
\]

For \( m' \geq m \), we have the following inequality

\[
u_c(Ah_1, l(Ah_1, K, \theta)) = \Delta(h_1, f_{Ah_1} - Ah_1, \theta) \geq \Delta(h_2, f_{Ah_1} - Ah_1, \theta)
\]

Therefore, for such a perturbation of \( h \), the mapping \( Z \) used in the definition of \( Ah \) is now nonnegative, and the first term in the definition of \( Z \) must decrease and the second term must increase in a solution \( Ah_2 \). The latter implies \( f_{Ah_2} - Ah_2 \leq f_{Ah_1} - Ah_1 \). By the definition of \( H(m, K, \theta) \), the quantity \( f(K, 1 - l(H(m, K, \theta), K, \theta)) - H(m, K, \theta) = f_{H(m)} - H(m) \) must be decreasing in \( m \). Since \( m' \) and \( m \) where arbitrary, that completes the proof of the claim.

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Following section 2, define the mapping

\[ \hat{Z}(m, \bar{m}, K, \theta) = \frac{1}{\bar{m}} - \beta \int_0^1 r \left( f(m - H(m, K, \theta), \theta') \right) \chi(\theta, d\theta'), \]  

(9)

where \( f(m - H(m, K, \theta)) = f(K, 1 - l(H(m, K, \theta), K, \theta)) - H(m, K, \theta) \) and the associated operator \( \hat{A} \) as:

\[ \hat{A}(m) = \{ \bar{m} \in M \mid \hat{Z}(m, \bar{m}, K, \theta) = 0, \text{ for } m > 0; 0 \text{ elsewhere} \} \]

**Lemma 27** The operator \( \hat{A} \) is a well-defined self map.

*Proof:* Recall \( f(m - H(m, K, \theta)) \) is decreasing in \( m \). Also that \( \hat{Z} \) is strictly increasing in \( m \), and strictly decreasing in \( \bar{m} \). Also for fixed \( m > 0, K > 0, \bar{m} \to 0 \) implies that \( \hat{Z} \to -\infty \). Similarly, as \( \bar{m} \to \frac{1}{u_c(f_l(l(K, \theta))} \), \( \hat{Z} \to \infty \). Consequently there is a unique \( \hat{A}m \) for each \( m > 0, K > 0, \) all \( \theta \). Note also that if \( K' > K \), \( \hat{Z}(m, \bar{m}, K', \theta) > \hat{Z}(m, \bar{m}, K, \theta) \). Therefore, \( \hat{A}m(K', \theta) > \hat{A}m(K, \theta) \), when \( K' > K \). Since \( r \) decreasing in \( K \), \( \frac{\bar{m}}{m} \) is decreasing in \( K \). Thus \( \hat{A}m \in M \).

We can relate the orbits of the operator \( \hat{A}^n \) to the operator \( A^n \) by the following construction. Consider some \( h_0 \in \hat{H} \). For such an \( h_0 \), there exists an \( m_0 = \frac{1}{u_c(h_0, l(h_0, K, \theta))} \in M \) such that \( H(h_0, l(h_0, K, \theta)) = h_0 \). By definition, \( \hat{Z}(m_0, \hat{A}m_0, K, \theta) = \hat{Z}(H(\frac{1}{u_c(h_0, l(h_0, K, \theta))}, K, \theta), \hat{A}H(h_0, l(h_0, K, \theta)), K, \theta) = Z(h_0, Ah_0, K, \theta) \). Therefore, \( h_1 = Ah_0 = H(\frac{1}{u_c(Ah_0, l(Ah_0, K, \theta))}) = H(\hat{A}m_0) \). A similar argument establishes \( A^n h_0 = H(\hat{A}^n m_0) \), for all \( n \).

**Lemma 28** \( \hat{A} \) has a strictly positive fixed point.

*Proof:* We know that \( m = 0 \) is a fixed point of \( \hat{A} \). To verify that \( \hat{A} \) has strictly positive fixed points in \( M \), consider the trajectory of \( \hat{A} \) from \( m_0 = \frac{1}{u_c(f^u_l(K, \theta))} \). It is easily verified that \( 0 < \hat{A}m_0 \leq m_0 \). Compute \( h_0(K, \theta) = H(m_0(K, \theta)) = H(\frac{1}{u_c(h_0, l(h_0, K, \theta))}) \leq H(\frac{1}{u_c(f^u_l(K, \theta))}) \). Notice that \( h_0(K, \theta) \) is equal to an orbit of the operator \( Ah \in \hat{H} \), namely, it is the same as \( Af^u(K, \theta) \). Therefore, \( h_0 \) is the optimal plan associated with a one-period distorted dynamic economy. Similar calculations show \( h_n(K, \theta) = H(A^{n-1} m_0(K, \theta)) \) is the optimal plan associated with a \( n \)-period economy.
Since $A$ and $\hat{A}$ are continuous, and they both map compact sets to compact sets, $h^* = \lim_{n \to \infty} h_n(K, \theta) = \lim_{n \to \infty} H(A^m(0, K)) \in \mathcal{H}$. Therefore following an argument in Coleman [15], we can associate a value function with each orbit $A^n f^u$ which is strictly concave. Therefore $h^* > 0$, and $m^* = \lim_{n \to \infty} H(A^n 0) > 0$. So $\hat{A}$ has a strictly positive fixed point.

Finally, uniqueness of this strictly positive fixed point rests upon the pseudo concavity and $x_0$-monotonicity of the operator $\hat{A}$.

**Theorem 29** Under Assumptions 6-8, $h^* > 0$ is the unique equilibrium.

**Proof:** Since $\hat{Z}$ is increasing in $m$, and decreasing in $\tilde{m} = \hat{A}m$, $\hat{A}m_1 \geq \hat{A}m_2$ for $m_1 \geq m_2$. Also the Inada condition in Assumption 6 is sufficient for $\hat{A}$ to be $K_0$-monotone (see Coleman [14], Lemma 9, 10), and a sufficient condition for pseudo-concavity is,

$$\hat{Z}(tm, t\hat{A}m, K, \theta) > \hat{Z}(tm, \hat{A}m, K, \theta).$$

which is always true since $m \in M$, and $r$ decreasing in $K$. Hence, $\hat{Z}(tm, t\hat{A}m, K, \theta) = \frac{1}{m} - \beta \int_{\theta}^{r(f_m - H(t\tilde{m}), \theta')} \chi(\theta, d\theta') > 0$, and $\hat{Z}(tm, \hat{A}m, K, \theta) = 0$. Therefore by Theorem 8 in Coleman ([14]), $\hat{A}$ has at most one strictly positive fixed point, and therefore $A$ has exactly one strictly positive fixed point (following the same argument as in Section 2).

5 Suggestions for Future Research

In this paper we have discussed how to construct competitive equilibrium for a broad class of smooth, strongly concave infinite horizon economies. The method first restricts attention to continuous Markovian equilibrium. In each case, one first constructs an closed and equicontinuous subset of bounded continuous functions in the uniform topology. This is a compact set in this topology, and in addition also a complete lattice. We then develop various methods for establishing equilibrium for each parameter describing the aggregate policy environment. Since our operators are all single valued, we are able to use various versions of order based fixed arguments (either based upon topological constructions or lattice based constructions) to establish equilibrium. Since equilibrium in each of our environments is unique,
we can do strong comparative analysis in the sense of Milgram and Shannon [35] in strong set orders.

What is critical in this environment is the role of strong concavity. When the value function can be shown to be appropriately concave, the best responses of each agent is a single valued continuous mapping, and this greatly simplifies the analysis. If one relaxes the strict concavity property in competitive environments, the methods have to be changed. The Euler equation methods in Section two seem doomed, for once we lose strong concavity of the value function, continuous selections no longer in general exist in the best response mappings, and so Euler equations methods have little appeal. Mirman, Morand, and Reffett [33] we study this case and show existence by extending the value iteration methods of section three of this paper and apply Zhou’s generalization of the theorem of Tarski (see Zhou [56]). Further when using the so-called induced set order of Topkis [52], we are able to show that monotone comparative analysis like that obtained in section three is once again available. Having lost uniqueness, we are unable to provide strong comparative analysis results like those discussed in Milgram and Shannon [35].

Additionally, the methods could also prove very useful in studying dynamic games (for instance policy games between two countries). It is well known that in such environments, even if return functions are assumed to be strictly concave, the value functions for such dynamic games are generally not even concave. While topological methods based upon single valued operators seem difficult to apply, the value function iteration methods of Section 3, however, appears very promising because the latter methods are based upon supermodularity properties, and not concavity properties. For at least the symmetric equilibrium case discussed in Sundaram [?], it would appear that the methods in Mirman, Morand, and Reffett [33] might work. This problem, and many others are now the subject for future research.

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