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Kwamie Dunbar
University of Connecticut and Sacred Heart University

University of Connecticut

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Kwamie Dunbar
University of Connecticut and Sacred Heart University

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341 Mansfield Road, Unit 1063
Storrs, CT 06269–1063
Phone: (860) 486–3022
Fax: (860) 486–4463
http://www.econ.uconn.edu/

Abstract

In this paper we propose a methodology that we believe improves the effectiveness of several common assumptions underlying Modern Portfolio Theory's dynamic optimization framework. The paper derives a general outline of a stochastic nonlinear-quadratic control for analyzing and solving a non-linear mean-variance optimization problem. The study first develops and then investigates the role of unsystematic (credit) risk in this continuous time stochastic asset allocation model where the wealth generating process has a non-negative constraint. The paper finds that given unsystematic risk, wealth constraints and higher order moments the market price of risk is non-constant and the investor's optimal terminal return may be lower than previously indicated by a number of classical models. This result provides a convenient solution to practitioners seeking to evaluate competing investment strategies.

Journal of Economic Literature Classification: G0, G10, C02, C15

Keywords: Dynamic Optimization; Credit Risk; Mean-Variance Analysis; Linear Quadratic Control; Credit Default Swaps; Capital Market Line; Gram-Charlier expansion; unsystematic risks

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1.0 Introduction

Recent credit events in global financial markets have resulted in significant losses to a number of retirement and investment portfolios because a number of asset allocation models have systematically underestimated the probability of the improbable. This rare and costly experience demonstrates that a number of pervasive asset allocation assumptions in mean-variance theory may be implausible. As a result of these observations we propose a nonlinear dynamic optimization model that addresses perceived weaknesses in the following assumptions underlying the generally accepted classical linear mean-variance approach; *(a) Investment returns are only a function of market-risk (Markowitz (1952, 1959)), (b) The capital market line (CML) connecting the risk-free asset to the efficient frontier's optimal market portfolio is linear and characterized by a normal distribution (Zhou et al (2000), (Lintner (1965), Mossin (1966), Stapleton et al (1983)), (c) The market price of risk is constant (Li (2007)) and (d) Investors have relatively constant risk aversion (Samuelson (1963).* The proposed model is consistent with salient financial facts, such as modern portfolio theory and extreme value theory.

The purpose of this study as laid out in subsequent sections of this paper demonstrates that an investor's risk adjusted terminal return $\left(\xi_n^*\right)$ may be lower than indicated by the standard dynamic mean-variance framework (Dunbar (2008)). As a result we show that the non-linear super efficient asset-return state space (*definition 1*) bridging the risk-free rate and the efficient market portfolio (efficiency frontier) will also be lower. The analysis demonstrates that a lower capital market line is a result of the portfolio's higher order moments (skew (χ) and kurtosis (φ)) and total investment (systematic and unsystematic) risk[2].

DEFINITIONS 1:

(*i*) *The super efficient portfolio space is referred to as the CML in Modern Portfolio Theory and is the imaginary line bridging the gap between the risk-free asset and the most efficient market portfolio. The super efficient state space is assumed to be the path of all super optimal portfolios for a specific portfolio mix.*

We believe that these results will have an impact in three areas of modern portfolio theory (MPT). First, institutional investors need to predict the market price of risk (η) in order to evaluate the riskiness of trading strategies. In order to do this, it is necessary to accurately predict the hypothetical slope of the CML state space. Our approach allows us to do a better job of making these predictions than standard MPT. Second, the approach provides a mechanism for reasonably determining the total risk adjusted terminal returns (ξ_n^*) of an investor in a market experiencing macroeconomic disruptions. Unsystematic (credit) risk has emerged as a major concern to the preservation of investor's wealth (ξ_0) and as such should be treated as one of the risk components determining the agent's terminal return. Finally, from an academic perspective the results shed light on the marginal utility of risk adjusted returns given unsystematic risk (Γ) and arbitrary stochastic processes.

The paper develops a general framework of a stochastic nonlinear-quadratic (NLQ) control for studying the mean-variance optimization problem which is then used to derive and later evaluate the nonlinear asset risk-return space (the non-linear capital market line - NCML).

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[2] In this paper systematic risk is considered market risk $r(t)$ and unsystematic risk is considered to be credit risk $c(t)$, see definitions 2(i) and 2(ii).

In this specification of the model we include a pooled risk measure that includes unsystematic risk and a probability density function that best fits the characteristically non-normal portfolio returns; this risk-neutral computational approach decomposes the nonlinear optimization problem into two sub problems, where the first uses convex optimization theory to find the random variables representing the optimal terminal wealth (ξ_n) process. In the second step we use a Jondeau-type (2001) constrained Gram-Charlier expansion to derive a stochastic *nonlinear quadratic* NLQ control model, predicated on trading strategies that are specified in terms of monetary amounts invested in individual assets; this is laid out in proposition 4.0.

Proposition 1.0: *The market price of credit risk (η) is not constant given unsystematic risks, the cumulants of skewness and kurtosis. Hence the CML can be written as* $CML: E(r_C) = \left(r_F + \sigma_P \frac{E(r_M) - r_F}{(\sigma_\Phi + \sigma_\Gamma)_M} \right) (g(\xi))$ *where* r_F *is the risk free rate,* r_M *is the market return,* $(\sigma_\Phi + \sigma_\Gamma)$ *is the pooled systematic and unsystematic risk of the market, and* $g(\xi)$ *is the Gram-Charlier expansion with the skew and kurtosis parameters.*

For the study's extended model we let $\mathbb{R}$ be a real line and $\mathbb{R}_+$ be the set of nonnegative real numbers governing the dynamic process. Consider the nonlinear quadratic functional equation of type;

$$x(t) = \left[f(t,x) \right] \left(g(t,s,x) \right) \tag{1.1}$$

for all $t \in \mathbb{R}_+$, where $x: \mathbb{R}_+ \to \mathbb{R}$, $f: R_+ x \mathbb{R} \to \mathbb{R}$, $g: R_+ x \mathbb{R} \to \mathbb{R}$. By a *solution* of the NLQ function in expression 1.1 we mean $x \in C(\mathbb{R}_+, \mathbb{R})$ that satisfies equation 1.1, where $x(t)$ is the nonlinear capital market line (NCML) and where $C(\mathbb{R}_+, \mathbb{R})$ is the space of continuous real-valued functions (super efficient portfolios) on $\mathbb{R}_+$. The presence of the Gram-Charlier expansion

$\left(g\left(t,s,x\right)\right)$ in equation 1.1 results in a nonlinear specification of the traditional LQ framework. Now, when $\Gamma=\chi_k=\varphi_u=0$ we say the NLQ model converges to the linear quadratic class of models. Where Γ =credit risk, χ_k = skewness and φ_u = kurtosis.

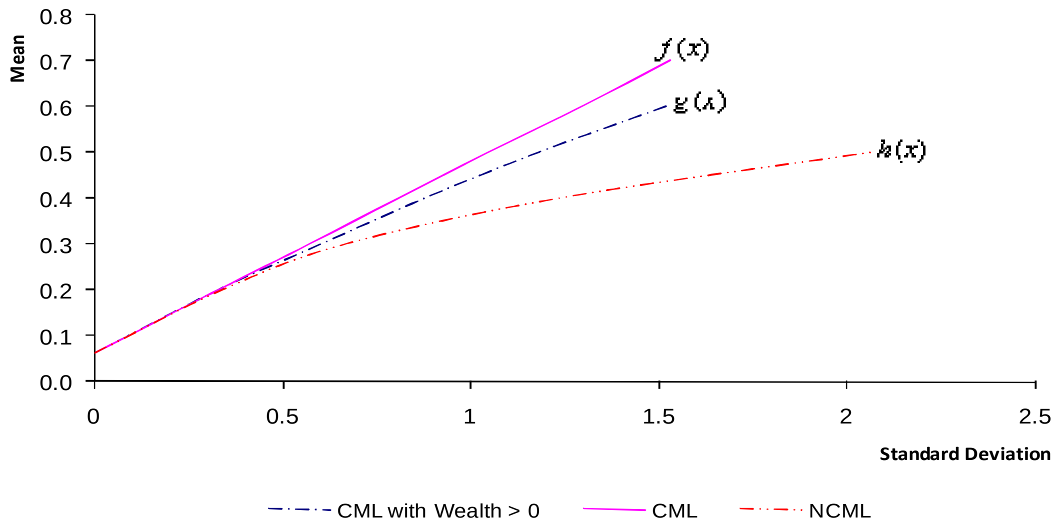
To build our dynamic model, we use the models of Bielecki et al (2005) and Zhou and Li (2000) as our benchmark case. We then develop the extended nonlinear quadratic framework in expression 1.1 to test the impacts of unsystematic (Γ) risks and nonlinearity on the state space connecting the super efficient portfolio to the optimal market portfolio of the efficiency frontier. Our closed form solution enables us to study the impacts of these risk and their associated distributional effects[3] on asset allocation strategies.

We follow Bielecki *et al* (2005) in modeling an optimization problem in which wealth is constrained to be positive $(\xi_0>0)$. However we allow unsystematic risk and an embedded Gram-Charlier expansion[4] to enter the model, so that we can examine the effects of the higher order moments of skew (χ) and kurtosis (φ) on the dynamic portfolio allocation (NCML) path. Interestingly, the study results demonstrate that the NCML obtained indirectly via the corresponding efficiency frontier is not linear but curvilinear, see figure 1. This illustrates that the distribution of super efficient portfolios bridging the risk-free and efficient market portfolio resides in a nonlinear state space.

[3] There is a vast amount of empirical evidence suggesting that asset return distributions are negatively skewed and fat tailed (Fama (1965), Longstaff *et al* (2005), Duffie (1996)).

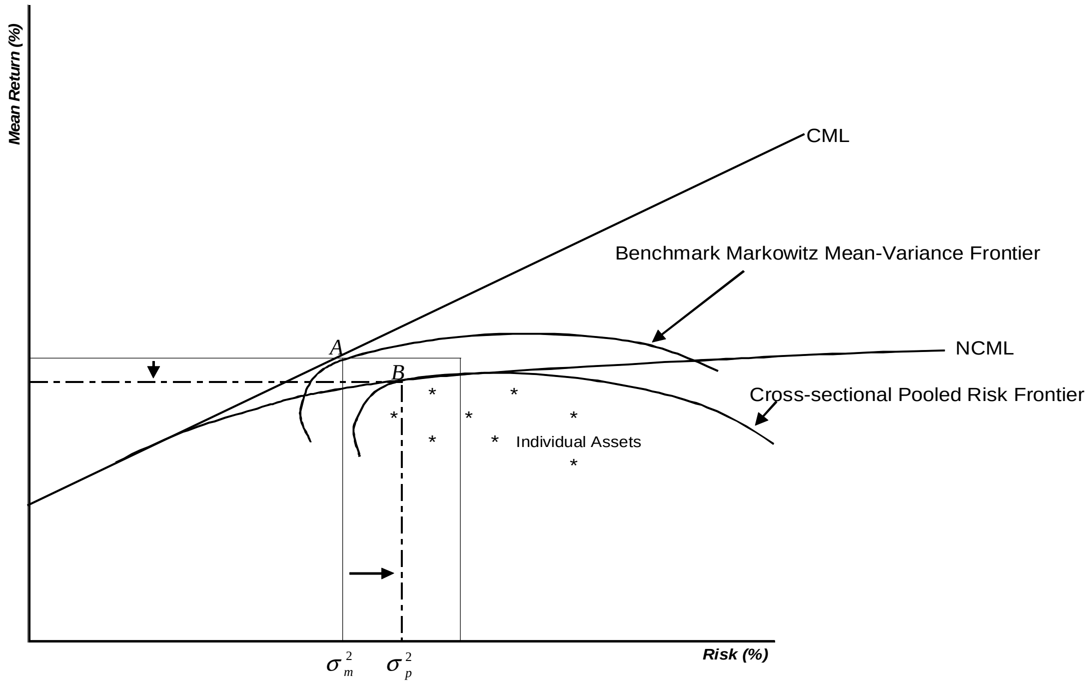
[4] The Gram-Charlier expansion has become popular in finance as a generalization of the normal density. The Gram-Charlier expansion is a polynomial expansion of the normal density function that provides a parsimonious representation of a distribution with skewness and kurtosis.

Fig. 1: Chart Depicting the Curvilinear NCML and the Traditional CML



Additionally, the portfolio mix illustrated in figure 1 indicates that the market price of risk (η) varies across the investment risk spectrum. Modern Portfolio Theory (MPT) suggests that the slope of the CML is positively sloped, linear and represents the market price of risk (η) for the efficient portfolio in the market. However, contrary to this longstanding view of traditional Markowitz-type models, the NCML illustrates that the market price of risk is not constant given the curvilinear shape of the NCML; proposition 2. In fact figure 2 demonstrates that, given a portfolio's non-constant market price of risk, an investor can quite clearly evaluate the riskiness of alternate trading strategies such as those at points A and B. Moreover, figure 2 also indicates that there are circumstances when the NCML will converge to the traditional model. We find that when unsystematic risk approaches zero the NCML nests the CML model.

Figure 2: Risk Return Characteristics of an Investment under Cumulative Market and Credit Risk: A Dynamic Horizon



Note: Point A represents the efficient allocative market-risk adjusted returns from a risk-free investment and an investment in A. Point B represents the efficient allocative pooled-risk investment returns as credit risk increases and investors move to a lower expected terminal return because of a flight to quality. The NCML represents the nonlinear expansion of the traditional linear capital market line.

Following Hull (2007), total investment risk can be subdivided into systematic and unsystematic risks. The NLQ control model developed here uses a measure of total investment risk[5] where the unsystematic (credit) risk component is simulated by credit default swaps *(definition (2(ii))*, a widely accepted proxy of credit risk (*definition 2(i)*) in a number of empirical models (Longstaff *et al* (2005) and Das *et al* (2006)). The study assumes that investment risk is a pooled measure (σ_p) of both systematic (Φ) and unsystematic risk (Γ). We define systematic risk as that risk which is common to all securities within specific markets (such as the effect of interest rates on the equity market), while unsystematic (credit) risk or idiosyncratic risk is

[5] Total investment risk is defined as a fulsome measure of systematic and unsystematic risk $(\sigma_\Phi + \sigma_\Gamma)$ as defined in Dunbar (2008).

defined as the risk associated with individual securities. Whereas there is an abundance of work on the systematic risk component of total portfolio risk, the first paper to have emerged that addresses the unsystematic component of portfolio risk in dynamic portfolio optimization was that by Dunbar (2008), who used a pooled risk measure $\left(\tilde{\text{var}}(\sigma_p^2)dZ_t\right)$ to capture the effects of both systematic and unsystematic risk.

The study uses a fulsome measure of risk that captures the effects of both systematic and unsystematic risks on the investor's terminal portfolio. As discussed in section 3, it is assumed that the cumulative (Systematic and unsystematic) pooled risk measure $\left(\tilde{\text{var}}(\sigma_p^2)dZ_t\right)$ is a probability weighted average of market risk $\left(\sigma_\Phi^2\right)$ and credit risk $\left(\sigma_\Gamma^2\right)$. In addition, we assume that this cross-sectional pooled risk model nests the familiar mean-variance dynamic asset allocation model. Under conditions of diminishing credit risk this cumulative risk model converges to the study's benchmark Markowitz dynamic mean-variance framework.

Finally, given the empirical evidence of Fama (1965) and others on the non-normal nature of asset returns, we are interested in the effects of skewness (χ) and kurtosis (φ) on the optimal market portfolio choice by an investor. Obviously an investor's demand for risky assets in his market portfolio is affected by his preferences and his utility function. Traditional methodologies involving the CML state space assume constant marginal utility over all levels of risk and returns. The NCML on the other-hand demonstrates that as risk increases the utility associated with a given expected terminal return decreases (figure 2). Here we say this investor is risk averse as he derives less utility from the same level of return as risk increases. Hence, within the nonlinear dynamic optimization framework, while the amount of risk an agent will take on is positively correlated with expected returns, we find that satiation eventually sets in because of diminishing marginal utility of terminal wealth, as depicted in the curvilinear shape of the NLQ curve.

The remainder of the paper is organized into four sections. In section 2 we discuss developments in the modern portfolio framework and examine recent extensions in portfolio optimization. Section 3 lays out the basic setup of the model investigated in this paper. This section introduces the general dynamic framework of the model and discusses the technical background for optimal dynamic asset allocation, giving some overview of current dynamic asset allocation methodologies and the analytical procedure for including the credit risk proxy to the optimization process. Section 4 derives the extended nonlinear dynamic optimization model under credit risk and a constrained Gram-Charlier distribution. The model is later calibrated to U.S. interest rate, stock return, credit and market risk data. The later area of this section presents some representative calculations and discussions on the main empirical findings regarding the role of credit risk and higher order cumulants in the dynamic optimization discussion. Section 6 summarizes the finding and proposes areas of possible future research.

2.0 Review of Approaches in Dynamic Asset Allocation

In recent years portfolio management theory has evolved from the single period mean-variance model of Markowitz[6] (1952, 1959) to the multi-period continuous time portfolio selection model of Merton (1969, 1971, 1973). Markowitz's earlier work was followed by that of a number of researchers in particularly that of Merton's celebrated approach which has become one of the corner stones of modern finance and which has inspired literally hundreds of extensions and applications (Hakansson (1971), Samuelson (1986) and Pliska (1997), etc). Quite recently, research on multi-period portfolio selections have been dominated by expected utility maximization of the terminal wealth ($E\left[U\left(\xi_n\left(\tau\right)\right)\right]$) where U may be a power, log, exponential or quadratic utility function. For the problem of maximizing the expected utility of the investor's wealth at a fixed planning horizon, Merton used dynamic programming and partial differential equation theory to derive and analyze the relevant Hamilton-Jacobi-Bellman (HJB) equation.

Building on the work of Zhou and Li (2000) this paper develops a general framework of a stochastic nonlinear-quadratic (NLQ) control for studying the mean-variance optimization problem and evaluating the nonlinear asset risk-return space (NCML). Zhou and Li (2000) used linear-quadratic (LQ) optimal control theory to solve a continuous-time, mean-variance problem with assets having deterministic coefficients. In their LQ formulation, the dollar amount, rather than proportional wealth, in individual assets was used to define the trading strategy. This lead to a dynamic system that is linear in both the state (the level of wealth) and

[6] Not only have this model and its single variance seen widespread use in the financial industry, but also the basic concepts underlying this model have become the corner stone of classical finance theory.

the control (trading strategy) variables. Together with the quadratic form of the objective function, this formulation falls naturally into the realm of stochastic LQ control. Exploiting the stochastic LQ control theory, Zhou et al (2003) and others later extended the initial continuous-time, mean-variance research into areas such as equity with random drift and diffusion coefficients, regime switching markets, constraints in short selling (Lim (2002)) and mean-variance hedging of a given contingent claim.

Recently Bielecki *et al* (2005) moved the debate further by examining the wealth process in the optimal trading strategy. They examined a methodological weakness in the traditional framework where a bankrupt investor can keep trading, borrowing money even though his wealth is negative. The ability of trading even though the value of an investor's portfolio is strictly negative is unrealistic. Following Bielecki *et al* (2005) the trading strategies of this study were expressed as a proportion of wealth in the individual assets, where the monetary value of the portfolios were automatically constrained to be strictly positive. Given the recent spate of asymmetric disruptions in global financial markets, the implied assumptions of no credit risk and a normally distributed linear CML is largely impractical in a real world setting. This brings us to the subject of this paper; Non-linear dynamic asset allocation: the effects of arbitrary stochastic processes and unsystematic risk (Γ) on the super efficient portfolio space.

Dunbar (2008) using stochastic LQ control theory illustrated that an agent's optimal terminal return is lower than that indicated by the traditional Markowitz approach because of the presence of credit risk. In this dynamic optimization approach Dunbar (2008) used a pooled risk measure to capture the effects of both market and credit risk or total investment risk. The results of the study's empirical analyses illustrated that the exclusion of credit risk from any portfolio optimization analysis overstates the familiar risk frontier by overstating the investor's optimal terminal investment returns $\xi_0(n)$ during periods of elevated credit risks, because existing models implicitly assume the non-existence of credit events.[7]

To overcome the restrictions imposed by the usual normal assumption a number of recent empirical studies have used a Gram-Charlier type expansion. For example Knight and Satchell (1997) develop an option pricing model using a Gram-Charlier expansion for the underlying asset. In a similar framework, Abken et al (1996) used a Gram-Charlier expansion to approximate risk-neutral densities (RND). Gallant and Tauchen (1989) used Gram-Charlier expansions to describe deviations from normality of the innovations in a GARCH framework. The Gram-Charlier expansion allows for additional flexibility over a normal density because they naturally introduce the skewness (χ) and kurtosis (φ) of the distribution as parameters. However because these functions are polynomial approximations, they have the drawback of yielding negative values for certain parameters. To overcome this drawback we follow the approach developed by Jondeau (2001) and apply positivity constraints to the Gram-Charlier expansion to guarantee positive values over the range of the parameters of the study. The appealing feature of the Gram-Charlier expansion is that we do not need to make distributional assumptions which may be hard to justify.

[7] A credit event is defined as a sudden progressive change in an asset's credit standing, brought on by events such as a default or bankruptcy that raises doubts about the asset's ability to repay its future obligations or payoff.

To illustrate the analytical flexibility and potential of the extended dynamic methodology, a simple empirical specification was tested under scenarios involving higher order moments, credit and market risk experiences to see how closely the results reflect actual market conditions. The study adopts changes in historical CDS bid-ask spreads as a proxy of credit risk. This is in keeping with the approach by a number of studies in the literature that have used CDS spreads as determinants of default risk, such as Longstaff *et al* (2005) and Das *et al* (2006).

DEFINITIONS 2:

(*i) Credit risk is measured by changes in the credit default swap (CDS) of each firm.*

(ii) Credit-default swaps (CDS) are financial instruments used to speculate on the ability of borrowers to repay debt. In the event of a default the CDS contract pays the buyer face value in exchange for the underlying securities or the cash equivalent should a country or company fail to adhere to its debt agreements. A rise in the price of a CDS contract indicates deterioration in the perception of credit quality; a decline, the opposite.

3.0 The General Model Structure: Dynamic Framework and Technical Background

This section contains the general framework of the dynamic optimization model investigated in this paper. The study develops two alternative models that were used as the primary tools for investigating the instantaneous space (*CML*) describing the set of super efficient portfolios that combines the risk-free asset[8] to the investor's optimal dynamic terminal solution given both market and credit risk. We first develop a benchmark model that allows us to determine the investor's attitude to systematic (market) risk; next we creatively exposed the agent to unsystematic (credit) risk through a more fulsome risk measure so as to determine

[8] The concept of a risk-free asset is used to describe an asset whose returns are certain and fully expected, and which is based on the expected growth rate of the overall economy in the long-run, adjusted for any short-term liquidity risks.

changes in the instantaneous capital market space (CML), the efficient frontier and their responses to changes in skew and kurtosis. In the benchmark case we follow Bielecki *et al* (2005) in modeling portfolio optimization under wealth constraints and using their analytically tractable approach for deriving the CML from the instantaneous efficiency frontier. In addition we follow the usual conditions for a dynamic portfolio optimization strategy[9] where the risky security is allowed to follow a geometric Brownian motion and a constant risk-free rate.

For the reasons discussed in section 1 we relaxed the dynamic optimization linearity assumptions using instead some measure of nonlinearity for a better fit. As such we formulate an optimization problem by assuming that there are *n* investment opportunities, with random return rates, $R_1,\ldots,R_n$ through the next year. In addition given that the true probability distribution function (PDF) of the random asset return rates are unknown, yet believed to be similar to a normal one, it is quite natural to approximate it with a PDF of the form

$$g\left(\xi_0\right)=p_n\left(\xi_0,\theta\right)\phi\left(\xi_0\right) \tag{3.1}$$

where $\phi\left(\xi_0\right)$ is the standard zero mean and unit variance density, $\theta\in\mathbb{R}^m$ is a vector of distribution parameters and where $p_n\left(\xi_0\right)$ is chosen so that $g\left(\xi_0\right)$ has the same first moments as the PDF of ξ_0. This procedure which is referred to as the Gram-Charlier expansion allows for the use of a semi-nonparametric device to overcome the restrictions of the usual normality assumption (Jarrow and Rudd (1982), Longstaff (1995), Backus *et al* (1997)). This leads to proposition 3.0 which suggests;

[9] In a dynamic context we construct mean-variance efficient portfolios by optimally allocating wealth across securities as the expected returns and variance-covariance changes over time. As discussed in footnote 2 we may hedge the change in the investment opportunity set, however the hedged payoffs will be lower because of credit risk.

Proposition 3.0: *If an agent's demand for risky assets in a mean-variance optimization problem exhibits a non-constant market price of risk then the optimal portfolio choice is influenced not only by the distribution's mean and covariance but also by skewness(χ) and kurtosis(φ). When the market price of risk is constant this is no longer the case.*

In the proposed model there is a finite amount of wealth ξ_o and an investment objective where the expected terminal return $[(1+r_i)\xi_0]$ is maximized under the conditions that the chance of losing more than some fixed amount $b>0$ is smaller than α, where $\alpha \in (0,1)$, the value at risk constraint. Now let $\frac{\xi_0}{n_{1i}},...,\frac{\xi_0}{n_{nm}}$ be the amounts invested in the *n* opportunities and let the net increase in the value of our terminal investment after a year be random and equals $G(\xi_n, r) = \sum_{i=1}^{n} r_i \xi_i \quad i = 1,2,...,n$, where $r_i = \frac{\xi_n}{\xi_0}$ is a quarterly random return rate of asset *i*.

We assume that the expected value is nonlinear in both ξ_0 and ξ_n;

$$\mathbb{E}\left[G(\xi, r)\right] = \sum_{i=1}^{n} \overline{r}_i \gamma \xi_i \quad where\ \gamma = g(\xi) = p_n(\xi, \theta)\phi(\xi) \tag{3.2}$$

when $\gamma = \phi(\xi)$ equation 3.2 is linear, this is a special case of the nonlinear specification.

Now with $\overline{r_i} = \mathbb{E}[r_i]$. The optimization problem takes the form;

$$\text{Minimize} \ \sum_{i=1}^{n} b_n (1-\overline{r})_i \gamma \xi_i$$

$$\text{Subject to} \quad \mathbb{P}\left\{\sum_{i=1}^{n} r_i \gamma \xi_i \geq -b\right\} \geq 1-\alpha \qquad \xi \geq 0 \tag{3.3}$$

Wealth (ξ_0) enters the model as the capital invested in the portfolio's n assets. If we invest an amount ξ_i in an asset i, after a year the value of this investment is $(1+r_i)\xi_n$. Following Bielecki *et al* (2005), we impose constraints on $\xi_0 \geq 0$ so that wealth does not turn negative and we also assume that $\overline{r} \neq 0$, otherwise no improvement over $\xi_0 = 0$ would be possible, see proposition 2.1 in Bielecki *et al* (2005) .

The study considers a pure exchange, frictionless economy with a finite horizon $[0, \tau]$ for a fixed $[\tau > 0]$. Following the usual conditions of portfolio optimization, trading can be discrete or continuous and traded are equity products, defaultable and default-free zero coupon bonds as are bank time deposits of all maturities. The portfolio of U.S. Treasury bonds serves as the numeraire. The underlying uncertainty in the economy is represented by a fixed filtered complete probability space $\left(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t^{\mathcal{B}}\}_{\tau \geq 0}\right)$ on which is defined a standard $\{\mathcal{F}_t^{\mathcal{B}}\}_{t \geq 0}$ adapted *B*-dimensional Brownian motion $W(t) \equiv \left(W^1(t), \ldots, W^m(t)\right)'$ (Duffie (1992)). The probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the filtration $\{\mathcal{F}_t^{\mathcal{B}} \mid t \leq a \leq \tau\}(-\infty \leq t < \tau \leq +\infty)$, Hilbert space $\mathbb{R}$ equipped with the inner product $\langle . , . \rangle$ and a Euclidean norm $\|\bullet\|_{\mathbb{R}}$, defines the Banach space. Now given the Lagrangian specification;

$$\mathcal{L}_{\mathcal{F}}^2(0, \tau, \mathbb{R}) = \left\{ \varphi(\bullet) \left| \begin{array}{l} \varphi(\bullet)\, is\, an\, \mathcal{F}_t - adapted, \mathbb{R} - valued\ measurable\ process\, on [\tau, t] \\ and\ E\int_t^\tau \|\varphi(\tau, w)\|_R^2\, dt < +\infty \end{array} \right. \right\} \tag{3.4}$$

With Euclidean norm;

$$\|\varphi(\bullet)\|_{\mathcal{F},2} = \left(E\int_t^\tau \|\varphi(\tau, \xi)\|_{\mathbb{R}}^2\, dt \right)^{\frac{1}{2}} < +\infty \tag{3.5}$$

The following assumptions are made:

1. *The portfolio considered in this paper is assumed to be self financing and continuously rebalanced.*
2. *Financial Markets are dynamically complete.*
3. *For the rational investor it is assumed that the value of the expected terminal wealth $E\,\xi_n(\tau)$ satisfies $E\,\xi_n(\tau) \geq \xi_0^{e^{\int_0^{\tau} r(s)ds}}$.*
4. *It is assumed that volatility in the credit default swaps of firms is a proxy of credit risk in financial markets.*
5. *It is assumed that in the familiar Markowitz mean-variance framework σ_p^2 is a probability weighted average of market risk $\left(\sigma_\Phi^2\right)$ and credit risk $\left(\sigma_\Gamma^2\right)$*
6. *$\xi_0(t)$ is predictable with respects to $\mathcal{F}_{(0)}$ and meets the usual integrating conditions*[10]

Now, as stated in equation 3.4 we denote $\mathcal{L}_{\mathcal{F}}^2\left(0,\tau;\mathbb{R}^m\right)$ the set of all R^m- valued, measurable stochastic processes $\mathcal{F}_t$ adapted to $\{\mathcal{F}_t\}_{t\geq 0}$, such that $E\int_0^T |f(t)|^2\,dt < +\infty$. a.s.[11]

where τ is a fixed terminal time and $\left(\Omega, \mathcal{F}, P\{\mathcal{F}_t\}_{t>0}\right)$ is a filtered complete probability space on which is defined a standard $\mathcal{F}_t$ adapted m-dimensional Brownian motion $\xi_0(t)$. And where;

(a) $\mathcal{L}_{\mathcal{F}}^2\left(0,\tau;R^m\right)$ is the Hilbert space of a $\xi-valued$ integrable function on $\left[0,\tau\right]$ endowed with the norm $\left(\int_0^T |f(t)|_x^2\,dt\right)^{\frac{1}{2}}$ for a given Hilbert space ξ_0.

(b) We have a certain space of super efficient portfolios (CML)[12] *S defined on a domain $D\subset\mathbb{R}^m$ and a function $\psi\in S$. The space S is represented by $C\left(\left[0,\tau\right]\right)$ a continuous*

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[10] Harrison-Pliska (1981) and Duffie (1996)

[11] Throughout this paper a.s. signifies that the corresponding statement holds true with probability 1.

[12] With this definition, the capital market line is: $CML = \mu_{p*} = r_f + \sigma_p \dfrac{E(r_m) - r_f}{\sigma_\Phi}$, where, given the various values of sigma (portfolio standard deviation), a straight line can be traced indicating the super-optimal (risk-return) portfolio space originating at the risk-free rate R_f. A proof is found in Ingersoll, *Theory of Financial Decision Making* (1987, p.89).

function on compact set D. $C\left(\left[0,\tau\right];\xi\right)$ *is defined as the Banach space of a* $\xi-$*valued continuous function on* $\left[0,\tau\right]$ *endowed with the maximum norm* $\|\bullet\|$ *for a given Hilbert space* ξ.

Now as proposed earlier suppose there is a market in which $m+1$ assets are traded continuously that includes a time deposit whose price process $P_0(t)$ is subject to the following stochastic differential equation;

$$\begin{cases} d\mathbb{P}_0(t) = r(t)\mathbb{P}_0(t)dt + \left(\sigma_\Phi^2 + \sigma_\Gamma^2\right)dW_t & t \in \left[0,\tau\right] \\ \mathbb{P}_0(0) = \mathbb{P}_0 > 0 \end{cases} \tag{3.6}$$

where the interest rate of the time deposit ($r_t > 0$) is a uniformly bounded $\mathcal{F}_t$ adapted, scalar valued stochastic process, and where total investment risk is represented by expression 3.16; $\sigma_p = \left(\sigma_\Phi + \sigma_\Gamma\right)$. Also present is an equity investment whose price is stochastic, risky in both nominal and real terms in the economy and the real price follows an $It\hat{o}'s$ process that is represented as;

$$\begin{cases} d\mathbb{P}_i(t) = \mathbb{P}_i(t)\left\{u_i(t)dt + \sum_{j=1}^{m}\sigma_{ij}(t)W^j(t)\right\}, & t \in \left[0,\tau\right] \\ \mathbb{P}_i(0) = \mathbb{P}_i > 0 \end{cases} \tag{3.7}$$

here $r_i > 0$ is the expected real return on the investor's equity investment per unit of time, $(\sigma_{ij} > 0 = (\sigma_p)_{ij} > 0)$ is the volatility or dispersion vector[13] of the real return on equity per unit of time and where $u_i(t) > 0$ can be considered as the appreciation rate. The volatility or dispersion rate of the equity investment can be represented as;

$$\sigma_i(t) = \left(\sigma_{ij}(t),...,\sigma_{im}(t)\right) : \left[0,\tau\right] \to \mathbb{R}^m$$

[13] We assume that the volatility matrix $\left[\sigma_\Phi,\ \sigma_\Gamma\right]$ has full rank. This assumption ensures that neither the bond nor the stock is a redundant asset in the economy.

$$\Rightarrow \sigma_p(t) = \left((\sigma_\Phi(t)+\sigma_\Gamma(t))_{ij}, \ldots, (\sigma_\Phi(t)+\sigma_\Gamma(t))_{im}\right) : [0,\tau] \to \mathbb{R}^m$$

We assume the usual conditions of non-degeneracy that defines the pooled covariance matrix as;

$$\sigma(t)\sigma(t)' \geq \delta I \quad \forall t \in [0,\tau]$$

$$\Rightarrow \sigma_p(t) = (\sigma_\Phi(t)+\sigma_\Gamma(t))(\sigma_\Phi(t)+\sigma_\Gamma(t))' \geq \delta I \qquad \forall t \in [0,\tau]$$

Where:

$$\pi_i(t) \equiv N_i(t)P_i(t) \qquad i = 0,1,2,3,\ldots,m$$

denotes the total value of the investor's wealth in the i^{th} time deposit or stock. We call $\pi_i(t) = (\pi_1(t),\ldots,\pi_m(t))'$ a portfolio of the investor. The objective of the investor is to maximize the mean terminal wealth $E\xi_n(\tau)$, and at the same time to minimize the variance of the terminal wealth.

$$\mathbb{V}\,\xi_n(\tau) \equiv E\left[\xi_n(\tau) - E\xi_n(\tau)\right]^2 = E\xi_n(\tau)^2 - \left[E\xi_n(\tau)\right]^2 \tag{3.8}$$

3.1 The Optimal Wealth Process in the Benchmark Model

Now following Bielecki *et al* (2005) consider an agent whose total wealth at time $\tau > 0$ is denoted by $\xi(t)$. Further assume that the trading of shares takes place continuously in a self financing fashion (no income or consumption) and transaction costs. Then $\xi(\bullet)$ satisfies the benchmark time t optimal wealth;

$$\begin{cases} d\xi(t) = \left\{ r_1(t)\xi(t) + \sum_{i=1}^{m}\left[b_1(t) - r(t)\right]\pi_i(t) \right\} dt + \sum_{j=1}^{m}\sum_{i=1}^{m}\sigma_{ij}(t)dW^i(t) \\ \xi(0) = \xi_0 \geq 0 \end{cases} \tag{3.9}$$

Where portfolio risk is a function of only systematic risk[14] (σ_Φ) and $\pi_i(t),\ i=0,1,2,3,..,n$ denotes the total market value of the agent's wealth in the i^{th} asset. Hence $N_i(t)\equiv \pi_i(t)/S_i(t)$ is the number of shares of the i^{th} asset held by the agent at time *t*. This implies that $\pi_0(t)+\pi_1(t)+...+\pi_m(t)=\xi(t).\ \pi_0(t)$ is the time t-vale of the bank time deposit and $\pi(\bullet)=(\pi_1(\bullet),...,\pi_m(\bullet))'$ the portfolio of the agent.

Now given that the excess return stream is represented as

$$B(t)=\left(b_1(t)-r(t),...,b_m(t)-r(t)\right) \tag{3.10}$$

and the risk premium process;

$$\theta(t)\equiv\left(\theta_1(t),...\theta_m(t)\right)=B(t)(\sigma(t))^{-1} \tag{3.11}$$

from notation in 3.10 and 3.11, the time *t* optimal wealth given systematic risk now becomes;

$$\begin{cases} d\xi(t)=\left[r(t)\xi(t)+B(t)\pi(t)\right]dt+\pi(t)'\sigma(t)dW(t) \\ \xi(0)=\xi_0 \end{cases} \tag{3.12}$$

where ξ_0 is the initial state, $W(t)\equiv\left(W^1(t),...,W^m(t)\right)$ is a given m-dimensional Brownian motion over $[0,\tau]$ on a given filtered space $\left(\Omega,\mathcal{F},P\{\mathcal{F}_t\}_{t>0}\right)$, and $u(\bullet)\in\mathcal{L}^2_{\mathcal{F}}\left(0,\tau;\mathbb{R}^m\right)$ is a control.

3.2 The Optimal Wealth Process under Systematic and Unsystematic risk

Now in order to introduce unsystematic risk (Γ) to the model developed in section 3.0 we need to modify the traditional optimization problem in equation 3.3 to introduce total

[14] The extended model uses a cross-sectional pooled risk variable in expression 3.16, which allows the study to capture both systematic (Φ) and unsystematic risk (Γ).

portfolio risk[15] (systematic (Φ) and unsystematic (Γ)) in the objective function. We define total portfolio risk as $Va\tilde{r}\left(\sigma_{TP}^2\right)dZ_t = \sigma_\Phi^2 + \sigma_\Gamma^2$, where the agent's risk exposure is not only a function of market risk but also credit risk. Interestingly, the study's cross-sectional pooled risk framework is assumed to nest the traditional mean-variance market risk model. With the embedded unsystematic (credit) risk parameter of equation 3.16 the general constrained controlled linear stochastic differential notation in 3.12 can be simplified for mathematical ease but without loss of generality to the following linear Ito's stochastic differential equation (SDE) in 3.13 below;

$$\begin{cases} d\xi(t) = \{A(t)\xi(t) + B(t)u(t) + f(t)\}dt + \sum_{j=1}^{m} D_j(t)u(t)dW^j(t) \\ \xi(0) = \xi_0 - (d-u), \;\; Otherwise \end{cases} \tag{3.13}$$

where we introduce the following assumptions for the coefficients of the above problem;

$$\begin{aligned}
&A(t) \text{ and } f(t) \text{ are scalars;} \\
&u(\bullet) \in L_F^2(0,\tau;\mathbb{R}_+^m); \\
&\xi(t) = \xi(t) - (d-u); \\
&A(t) = r(t); \\
&f(t) = (d-u)r(t); \\
&B(t) = (b_1(t) - r(t), ..., b_m(t) - r(t); \\
&D_j(t) = (\sigma_{\Phi ij}(t) + \sigma_{\Gamma ij}(t)), ..., (\sigma_{\Phi mj}(t) + \sigma_{\Gamma mj}(t)); \\
&B(t)' \in \mathbb{R}_+^m \text{ and } D_j(t)' \in \mathbb{R}^m \; (j = 1, ..., m) \text{ are column vectors.} \\
&\text{The matrix } \sum_{j=1}^{m} D_j(t)' D_j(t) \text{ is non Singular.}
\end{aligned}$$

The solution $\xi(\bullet)$ of the SDE representation in 3.13 is called the response of the control $u(\bullet)$, and $(\xi(\bullet), u(\bullet))$ is called the admissible pair, where the objective of the optimal

[15] Here we assume that an asset's total risk consists of both systematic and unsystematic risk.

control problem is to minimize some specified cost function over $\mathcal{L}_{\mathcal{F}}^{2}\left(0,\tau;\mathbb{R}^{m}\right)$. In the absence of credit risk, the agent's optimal wealth depicted in expression 3.13 converges to the benchmark dynamic optimization model in equation 3.14.

$$\begin{cases} d\xi(t)=\left\{r(t)\xi(t)+\sum_{i=1}^{m}(b_i(t)-r(t))u_j(t)\right\}dt+\sum_{j=1}^{m}\sum_{i=1}^{m}(\sigma_{kij}(t))u_i(t)dW^{j}(t) \\ \xi(0)=\xi_0, \ Otherwise \end{cases} \tag{3.14}$$

where the risk premium in 3.11 is now represented as

$$\theta(t)\equiv\left(\theta_1(t),...\theta_m(t)\right)=B(t)(\sigma_{\Phi}(t)+\sigma_{\Gamma}(t))^{-1}$$

Following Vasicek (1977), it is assumed that credit risk c_t (like market risk r_t) follows an Ornstein-Uhlenbeck diffusion process;[16]

$$dc_t=\kappa(\overline{c}_t-c_t)dt+\sigma_p^2 dz_c \tag{3.15}$$

where $\overline{c}$ is the long-run mean, σ_p^2 is the volatility, and κ is the mean reversion.

From assumption *5* the investor's cross-sectional pooled risk frontier is represented as,

$$Va\tilde{r}\left(\sigma_p^2\right)dz_t=Va\tilde{r}\left(\sigma_{\Phi}^2+\sigma_{\Gamma}^2\right)dz_t \tag{3.16}$$

$$\Rightarrow Va\tilde{r}\left(\sigma_p^2\right)dz_t\approx\frac{(n-1)\sigma_{\Phi}^2+(m-1)\sigma_{\Gamma}^2}{n+m-2} \tag{3.17}$$

where *n* and *m* are the number of observations in both sets of risk data.

[16] Where $\overline{c}$, σ_p^2 and κ are positive constants and dz_c is standard Brownian motion.

4.0 Solution to the Non-linear Asset Allocation Problem

In this section, we derive the extended non-linear dynamic efficiency frontier (involving a cumulative Pooled Risk frontier), in the variance-expected return space $\left(\left(\sigma_{ij}^2(\xi_t)\right), E\left(\xi_\tau\right)\right)$. Now consider the following optimization problem parameterized by the terminal $\xi_n \in \mathbb{R}$;

Minimize Var $\xi_0\left(\tau\right) \equiv E\xi_0\left(\tau\right)^2 - \xi_n{}^2$,

$$\text{Subject to } \begin{cases} E\xi\left(\tau\right) = \xi_n \\ \xi_0\left(\tau\right) \geq 0 \\ \pi\left(\bullet\right) \in L_F^2\left(0, \tau; \mathbb{R}^m\right) \\ \left(\xi\left(\bullet\right), \pi\left(\bullet\right)\right) \\ \sigma_p(\tau) = \sigma_\Phi + \sigma_\Gamma \end{cases} \tag{4.2}$$

where $\xi(\tau) \geq 0$ is the positivity constraint on wealth ensuring the investor doesn't invest when bankrupt and $\sigma_p(\tau)$ is the cumulative investment risk facing an investor. As stated in section 1, the NLQ control model is predicated on a set of trading strategies that are based on monetary amounts (wealth ξ_0) invested in the portfolios individual assets. As a result of this monetary approach we state proposition 3.1 from Bielecki *et al* (2005).

Proposition 4.0: *Assume that* $\xi\left(\bullet\right)$ *is a wealth process under an admissible portfolio. If* $\xi\left(t\right) \geq 0,\ \forall t \in \left[0, \tau\right]$

$$\xi\left(t\right) = \rho\left(t\right)^{-1} E\left(\rho(\tau)\xi(\tau) \,|\, \mathcal{F}_t\right),\ \forall t \in \left[0, \tau\right]$$

where it follows that $\xi(t) \geq 0,\ \forall t \in \left[0, \tau\right]$, *and* $\rho\left(\bullet\right)$ *which is the discount factor satisfies*

$$\begin{cases} d\rho(t) = \rho(t)\left[-r(t)dt - \theta(t)dW(t)\right] \\ \rho(0) = 1 \end{cases}$$

To solve the optimization problem in equation 4.2 we follow the risk-neutral computational approach used by Pliska (1986), and Bielecki *et al* (2005) where the optimization problem is decomposed into two sub-problems. The first involves finding the random variable representing the optimal terminal wealth $\left(\xi_n^*\right)$ while the second sub-problem, identifies the trading strategy $\pi(\bullet)$ that replicates the optimal terminal wealth $\left(\xi_n^*\right)$. Assuming the solution to $\left(\xi^*\right)$ exist, then the efficient terminal valued portfolio and the associated wealth process are given respectively as;

$$\begin{cases} d\xi(\tau) = \left[r(t)\xi(t) + B(t)\pi(t)\right]dt + \pi(t)'(\sigma_\Phi(t) + \sigma_\Gamma(t))dW(t) \\ \xi(\tau) = \xi^* \end{cases} \tag{4.3}$$

where $\sigma_\Phi(t)$ and $\sigma_\Gamma(t)$ represents market and credit risk respectively. The unique variance minimizing portfolio solution to expression 4.2 corresponding to $\xi_n > 0$ and $\left(\xi^*(\bullet), \xi_n(\bullet)\right)$ provides a unique solution to expression 4.3.

The optimal solution to 4.2 is the variance minimizing portfolio $\operatorname{var}\xi^*(\tau)$; where $\xi_0 \geq 0$ *and* $\xi_n \geq 0$ and where the variance minimizing portfolio corresponding to $\xi_n = \dfrac{\xi_0}{E\left[\rho(\tau)\right]}$ is a risk free portfolio. If the optimal portfolio choice and strategy $\left(\xi^*(\bullet), \pi(\bullet)\right)$ satisfies problem 4.2, then $\xi^*(\tau)$ is optimal for expression 4.4 and $\left(\xi^*(\bullet), \pi(\bullet)\right)$ satisfies 4.3. Operationally we solve the optimization problem in 4.2 by first transforming the sub-problem in 4.4 to an equivalent set of Lagrangian Multipliers in constraints on wealth; equation 4.5.

$$
\begin{array}{l}
Minimize\ W\xi^2 - \xi_n \\
Subject\ to \begin{cases} E\xi_n = \xi_n \\ E\left[\rho(\tau)\xi\right] = \xi_0 \\ \xi \in \mathcal{L}^2_{\mathbb{F}_T}\left(\Omega;\mathbb{R}^m\right) \\ \sigma_p(\tau) = \sigma_\Phi + \sigma_\Gamma \end{cases}
\end{array}
\tag{4.4}
$$

Lemma 4.1: *Let $Var\,\xi(T)$ be a local minimum of the portfolio problem 4.2 and let $\hat{\Lambda}(\hat{\xi})$ be the set of Lagrangian multipliers $\lambda \in \mathbb{R}^m_+$ and $\hat{\mu} \in \mathbb{R}^p$ satisfying equation 4.5. Where (i) The set $\hat{\Lambda}(\hat{\xi})$ is convex and closed, and (ii) If problem 4.2 satisfies Robinson's condition at $\hat{\xi}$, then the set $\hat{\Lambda}(\hat{\xi})$ is also bounded[17].*

As stated above, the techniques we follow allow us to develop optimality conditions for problem 4.2. Now following Bielecki *et al* (2005) and from lemma 4.1 there exist $\lambda \in \mathbb{R}^m_+$ *and* $\hat{\mu} \in \mathbb{R}^p$ *such that*

$$
\begin{cases} E\left[(\lambda - \mu\rho(\tau))^+\right] = \xi_n \\ E\left[\rho(\tau)(\lambda - \mu\rho(\tau))^+\right] = \xi_0 \end{cases}
\tag{4.5}
$$

Where (λ, μ) solves the equivalent set of linear Lagrangian multipliers[18]. Now if (λ, μ) satisfies 4.5 then $\xi^* = \left(\lambda - \mu\rho(\tau)\right)^+$ must be an optimal solution of 4.2. A proof is found in Bielecki *et al* (2005, pp 225-226).

Furthermore from lemma 4.1 we say the set $\hat{\Lambda}(\hat{\xi})$ of multipliers (λ, μ) satisfying the above conditions are convex and compact. A proof is in Ruszczynski ((2006), pp. 115-117). We solve for the Lagrangian multipliers of the simultaneous expression in equation 4.5 (This is done by considering the constraints as a set of two equations and solving for the unknown

[17] The proof is in Ruszczynski (2006) Nonlinear Optimization, pp 117-118.

[18] Since these equations are linear, the solution is straight forward: $\lambda = \frac{\xi_n E\left[\rho(\tau)^2\right] - \xi_0 E\left[\rho(\tau)\right]}{\operatorname{var}\rho(\tau)}$ and $\mu = \frac{\xi_n E\left[\rho(\tau)\right] - \xi_0}{Var\,\rho(\tau)}$

multipliers). From theorem 5.1 of Bielecki *et al* (2005) $\lambda = \xi_n$ and $\mu = 0$ when $\frac{\xi_0}{\xi_n} = E\left[\rho(\tau)\right]$ leading the terminal wealth under the corresponding variance minimizing portfolio $(\pi(\bullet))$, to produce $\xi_0(\tau) = \left(\lambda - \mu\rho(\tau)\right)^+ = \lambda = \xi_n$.

Now given the $\lambda(\bullet)$ and $\mu(\bullet)$ that satisfies equation 4.5 and some ξ^*, then the indirect efficient frontier satisfying the Hamilton-Jacobi-Bellman equating is derived from the following parameterized equations;

$$\begin{cases} E\left[\xi^*(\tau)\right] = \xi_n \\ Var\,\xi^*(\tau) = \lambda(\xi_n)\xi_n - \mu(\xi_n)x_0 - \xi_n^2, \qquad \dfrac{\xi_0}{E\left[\rho(\tau)\right]} \le \xi_n \le \dfrac{\xi_0}{\alpha} \end{cases} \tag{4.6}$$

Where $\left(\lambda(\xi_n), \mu(\xi_n)\right)$ is the unique solution to 4.5 parameterized by ξ_n. Moreover all the efficient portfolios are those variance minimizing portfolios corresponding to

$$\xi_n \in \left[\frac{\xi_0}{E\left[\rho(\tau)\right]}\right], \frac{\xi_0}{\alpha}$$

However since the exact probability distribution of the random variables representing the optimal terminal wealth function is unknown we will not assume a normal distribution as Bielecki *et al* (2005) but will use a Gram-Charlier expansion to approximate the true distribution. Where the Jondeau (2001) Gram-Charlier risk-neutral density is represented as;

$$g(\xi_0;\phi,\delta) = \left[1 + \frac{b_3}{\sqrt{6}}(\xi_0{}^3 - 3\xi_0) + \frac{b_4}{\sqrt{24}}(\xi_0{}^4 - 6\xi_0{}^2 + 3)\right]\phi(\xi_0) \tag{4.7}$$

$$\Rightarrow g(\xi_0;\phi,\delta) = \left[1 + \phi(\xi_0{}^3 - 3\xi_0) + \delta(\xi_0{}^4 - 6\xi_0{}^2 + 3)\right]\phi(\xi_0)$$

where $\xi_n = \left(\log(\xi_0) - \mu\right)/\sigma_p$, $\phi = \frac{\chi}{6}$, $\delta = \frac{\varphi}{24}$.

Given the solution to sub-problem 1 in expression 4.4 we then seek to determine the trading strategies that replicate the optimal terminal wealth. The unique variance minimizing portfolio for expression 4.2 corresponding to $\xi_n > 0$ where $\alpha < r_i < 1-\alpha$ is given by

$$\pi^*(t) = \left(\sigma_\Phi(t) + \sigma_\Gamma(t)\right)^{-1} \xi_n^*(t) = \left(\sigma_p\right)^{-1} \xi_n^*(t) \tag{4.8}$$

where σ_p represents the pooled systematic and unsystematic investment risk discussed in section 3.2 (expression 3.16), and also where $\left(\xi^*(\bullet), \xi_n^*(\bullet)\right)$ is the unique solution to a Backward Stochastic Differential Equation (BSDE) represented as;

$$\begin{cases} d\xi_0(t) = \left[r(t)\xi_0(t) - \theta(t)\xi_n(t)\right]dt + \xi_n(t)'dW(t) \\ \xi_0(\tau) = \left(\lambda - \mu\rho(\tau)\right)^+ \end{cases} \tag{4.9}$$

Following Bielecki *et al* (2005), the unique variance minimizing portfolio strategy for expression 4.2 corresponding to ξ_n with $\dfrac{\xi_0}{E\left[\rho(\tau)\right]} \le \xi_n \le \dfrac{\xi_0}{\alpha}$, is a replicating portfolio for a European put option written on the fictitious equity asset $\mu\rho(\bullet)$ with a strike price $\lambda > 0$ and maturity τ.

$$P_e\left(\mu\rho(\bullet), \lambda, \tau\right) = \begin{cases} dy(t) = y(t)\left[r(t) - \left|\theta(t)\right|^2 dW(t)\right], \\ y(0) = \mu e^{\left\{-\int_0^T \left[2r(s) - \left|\theta(s)\right|^2\right]ds\right\}}, \qquad y(\tau) = \mu\rho(\tau) \end{cases} \tag{4.10}$$

Now given an investor who needs to evaluate the riskiness of a set of trading strategies derived from the solution of the variance minimizing problem in expression 4.2 and the replicating trading strategies of a European put option in 4.10. We can evaluate the functions $g\left(\xi_n\right)$ and $f\left(\xi_n\right)$ over some interval $C\left[x_1, x_2\right]$, where the slope of the function over the interval gives the market price of risk of the investment strategy.

$$\Rightarrow Slope\,\xi_n = \lim_{\Delta\xi \to 0} \frac{f(\xi_n + h) - f(\xi_n)}{\Delta \xi_n} \tag{4.11}$$

$$= \lim f'(\xi_n) = a\xi_n + b$$

with $f'(\xi_n) = a\xi_n + b$, where a and b are some constant, the slope and instantaneous rate of change of the function $f(\xi_n)$ can be evaluated at any point $C[x_1, x_2]$ by substituting the derived value of (ξ_n) into $f'(\xi_n) = a\xi_n + b$ and taking the limits.

4.1 An Empirical Non-Linear Optimization Solution to the Terminal Wealth Problem

In determining the variance minimizing frontier in equation 4.2, we first assume that the efficient portfolios are the variance minimizing portfolios corresponding to $E\xi_n(T) > \xi_0 e^{\int_0^T r(s)ds}$, as such the empirical model corresponding to the efficient portfolio and associated wealth process is given as

$$\pi^*(t) = g(-d_+(t, y(t))\left(\left(\sigma_\Phi(t) + \sigma_\Gamma(t)\right)\left(\sigma_\Phi(t) + \sigma_\Gamma(t)\right)'\right)^{-1} B(t)' y(t) \tag{4.1.1}$$

$$\Rightarrow \pi^*(t) = -\left(\left(\sigma_\Phi(t) + \sigma_\Gamma(t)\right)\left(\sigma_\Phi(t) + \sigma_\Gamma(t)\right)'\right)^{-1} \left[\xi_n^*(t) - \lambda g(-d_-(t, y))e^{-\int_t^T r(s)ds}\right]$$

and where;

$$\xi_n^* = \lambda g(-d_-(t, y(t)))e^{-\int_t^T r(s)ds} - g(-d_+(t, y(t)))y(t)$$

where[19] $g(\bullet)$ with $g(\xi)=\left[1+\frac{b_3}{\sqrt{6}}(a^3-3a)+\frac{b_4}{\sqrt{24}}(a^4-6a^2+3)\right]\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{v^2}{2}dv}\right)$ (4.1.2)

here expression 4.1.2 is the cumulative distribution function (developed in 4.7) for the unknown terminal wealth process coefficients, and $\left(\sigma_{\Phi}(t)'+\sigma_{\Gamma}(t)'\right)$ is the pooled credit and market-risk matrix. Where $y(t)$ is defined as

$$y(t)=\mu e^{\left\{-\int_0^T\left[2r(s)-|\theta(s)|^2\right]ds\right\}}e^{\left\{\int_0^t\left[r(s)-\frac{3}{2}|\theta(s)|^2\right]ds-\int_0^t\theta(s)dW(s)\right\}} \tag{4.1.3}$$

$$d_+(t,y)=\frac{\ln(y/\lambda)+\int_t^T\left[r(s)+\frac{1}{2}|\theta(s)|^2\right]ds}{\sqrt{\int_t^T|\theta(s)|^2\,ds}} \tag{4.1.4}$$

$$d\_(t,y)=d_+(t,y)-\sqrt{\int_t^T|\theta(s)|^2\,ds} \tag{4.1.5}$$

As discussed in section 3, (λ,μ) is the unique solution that satisfies equation 4.6, whilst $\theta(t)=\left(\theta_1(t),...,\theta_m(t)\right)=B(t)\left(\sigma_m(t)'+\sigma_c(t)'\right)^{-1}$ includes the cumulative pooled risk measure of investment risk. Now if $\xi_n(\tau)>\xi_0 e\int_0^T r(s)ds$ then (λ,μ) satisfies expression 4.5 which may be represented as

$$\begin{cases}\lambda g\left(\frac{\ln(\lambda/\mu)+\int_0^T\left[r(s)-\frac{1}{2}|\theta(s)|^2\right]ds}{\sqrt{\int_0^T|\theta(s)|^2\,ds}}\right)-\mu e^{-\int_0^T\left[r(s)-|\theta(s)|^2\right]ds}g\left(\frac{\ln(\lambda/\mu)+\int_0^T\left[r(s)-\frac{3}{2}|\theta(s)|^2\right]ds}{\sqrt{\int_0^T|\theta(s)|^2\,ds}}\right)=x_0e^{\int_0^T r(s)ds}\\ \lambda g\left(\frac{\ln(\lambda/\mu)+\int_0^T\left[r(s)+\frac{1}{2}|\theta(s)|^2\right]ds}{\sqrt{\int_0^T|\theta(s)|^2\,ds}}\right)-\mu e^{-\int_0^T r(s)ds}g\left(\frac{\ln(\lambda/\mu)+\int_0^T\left[r(s)-\frac{1}{2}|\theta(s)|^2\right]ds}{\sqrt{\int_0^T|\theta(s)|^2\,ds}}\right)=z\end{cases} \tag{4.1.6}$$

[19] We derive $g(\bullet)$ from expression 1.1 in section 1

And the analytical representation of the efficient frontier in expression 4.6

$$\begin{cases} E\left[\xi_n^*(\tau)\right] = \dfrac{\Psi e^{\int_0^T r(t)dt} g_1(\Psi) - g_2(\Psi)}{\Psi g_2(\Psi) - e^{-\int_0^T \left[r(s) - |\theta(s)|^2\right]ds} g_3(\Psi)} \xi_0 \\ Var\, \xi_n^*(\tau) = \left[\dfrac{\Psi}{\Psi g_1(\Psi) - e^{-\int_0^T \left[r(s) - |\theta(s)|^2\right]ds} g_2(\Psi)} - 1\right]\left[E\xi_n^*(\tau)\right]^2 - \dfrac{\xi_0}{\Psi g_1(\Psi) - e^{-\int_0^T \left[r(s) - |\theta(s)|^2\right]ds} g_2(\Psi)} E\xi_n^*(\tau), \qquad \Psi \in (0, \infty] \end{cases} \tag{4.1.7}$$

Where $\Psi = \dfrac{\lambda}{\mu}$

In the benchmark model we develop the traditional mean-variance frontier and its corresponding state space of efficient portfolios; where the CML which is depicted as a straight positively sloped line. An example of the benchmark case is that of Zhou and Li (2000) given as;

$$Er^*(T) = r_f(\tau) + \sqrt{\left(e^{\int_0^T |\theta(t)|^2} - 1\right)} \sigma_{r^*(\tau)} \tag{4.1.8}$$

where $r_f(\tau) \equiv \left(e^{\int_0^T r(t)dt} - 1\right)$ is the risk free rate over $\left[0, \tau\right]$ and where $\sigma_{r^*(T)}$ denotes the standard deviation of $r^*(\tau)$. Now consider the empirical example of Zhou and Li (2000) where an agent has a time deposit with an interest rate of $r(t) = 0.06$, an equity investment $B(t) = 0.12$ and weighted risk measure of $(\sigma_m + \sigma_c) = 0.35$. Assume also that the investor has an endowment ξ_0=\$1million and expects a terminal payoff ξ_n=\$1.2million at $\tau = 1$. In addition, following work by Corrado and Su (1997) we select values for skew between -1.10 to 0.10 and values for kurtosis between 2.39 to 3.80. When both χ and φ=0 we assume a normal distribution. For the S&P500 index historical rates of return have a skewness in the range of (-1.12) to 0.15 and a kurtosis in the range of 0.22 to 8.92.

Estimating the CML via the efficient frontier as in Zhou and Li (2000) leads to

$$Er^*(\tau) = 0.0618 + 0.416\sigma_{r^*(T)} \tag{4.1.9}$$

While in the case where there is a pooled-risk measure $\left(\sigma_p\right)$ and a Gram-Charlier expansion $g(\bullet)$, the NCML is estimated as following from expression 4.1.7,

where;

expression 3.11 yields $\theta(t) = 0.2$;

expression 4.1.6 yields $$\begin{cases} \lambda g\left(\dfrac{\ln(\lambda/\mu)-0.02}{0.2}\right) - \mu e^{0.1} g\left(\dfrac{\ln(\lambda/\mu)-0.18}{0.2}\right) = e^{0.06} \\ \lambda g\left(\dfrac{\ln(\lambda/\mu)+0.14}{0.2}\right) - \mu e^{0.06} g\left(\dfrac{\ln(\lambda/\mu)-0.02}{0.2}\right) = 1.2 \end{cases} \tag{4.1.10}$$

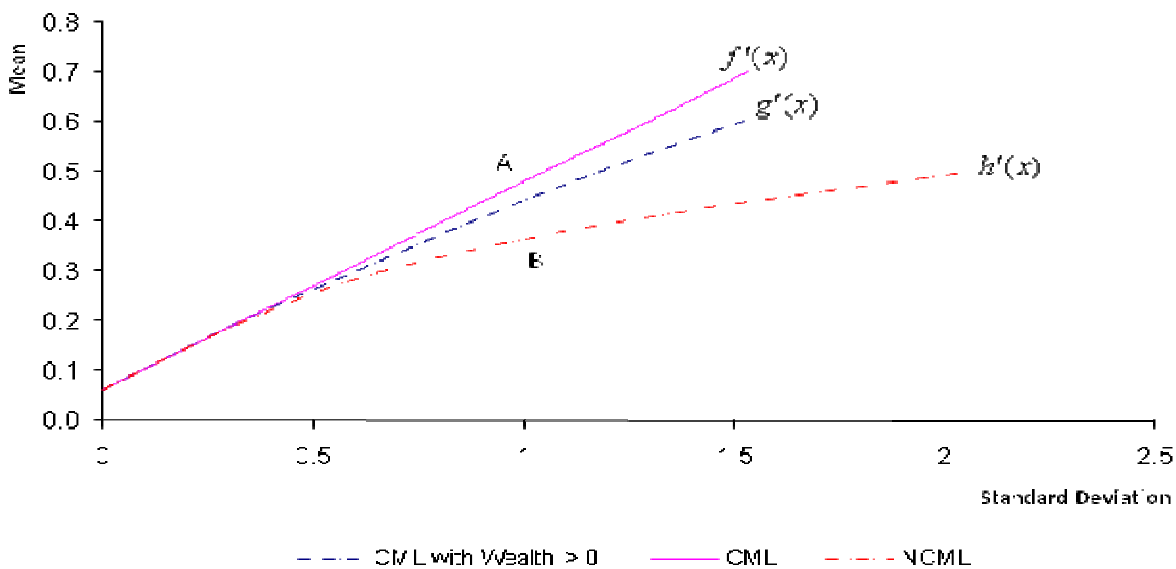
$\lambda = 1.595 \quad \mu = 0.91$

Hence the state space of the NCML can be shown as;

$$\begin{aligned} Er^*(1) &= \frac{e^{0.06}\Psi g\left(\dfrac{\ln\Psi+0.14}{0.2}\right) - g\left(\dfrac{\ln\Psi-0.02}{0.2}\right)}{\Psi g\left(\dfrac{\ln\Psi-0.02}{0.2}\right) - e^{0.1} g\left(\dfrac{\ln\Psi-0.18}{0.2}\right)} - 1 \\ \sigma^2_{p^*(1)} &= \left[\frac{\Psi}{\Psi g\left(\dfrac{\ln\Psi+0.14}{0.2}\right) - e^{-0.06} g\left(\dfrac{\ln\Psi-0.02}{0.2}\right)} - 1\right]\left[Er^*(1)\right]^2 - \frac{1}{\Psi g\left(\dfrac{\ln\Psi+0.14}{0.2}\right) - e^{-0.06} g\left(\dfrac{\ln\Psi-0.02}{0.2}\right)}\left[Er^*(1)+1\right] \end{aligned} \tag{4.1.11}$$

Figure 3 which depict both the benchmark and the NCML super efficient portfolio space (derived from equations 4.1.9 and 4.1.11) illustrates that the market price of risk is lower under the NCML's case $\left(CML\,\eta \geq NCML\,\eta\right)$. Expression 4.11 is used to derive the market price of risk for expressions 4.1.9 and 4.1.11. The lower NCML market price of risk for the equilibrium market portfolio $\left(f'(x) \geq g'(x)\right)$ indicates reluctance by the agent to reach for higher yields during periods of rising credit risk. Conversely, in a market characterized by rising σ_p, a lower demand for an increasingly risky set of assets results in a lower market price of risk for those assets, because of the lower expected terminal return depicted in figure 3.

Fig 3: Illustrates the Market price of Risk in the NCML and the Traditional CML



Moreover, we illustrate that the agent's expected utility is not constant and is variant with risk levels and the asymmetric parameters of skew and kurtosis. From the approach of Kritzman and Rich (1998) in determining the influence of time on expected utility, we follow their approach to determine if an investor will maintain the same percentage exposure to risky assets at higher levels of unsystematic risks. From figure 3 we know that investors do not have constant risk aversion given both systematic and unsystematic risk. Now following Kritzman and Rich (1998) and deriving expected utility when returns are random we consider the Zhou et al (2000) example above where we assume the risky investment has a 50% chance of the gain indicated by the CML at point (A) and a 50% chance of the return at point (B) on the NCML curve. We assume that this investor will be indifferent to a risky investment and the time deposit returning 0.06%. Now from figure 3 the CML's guaranteed constant expected utility return is $\ln(CML\,\xi_n\,wealth) = 0.47\%$ and the NCML's probability weighted chance of systematic and unsystematic risk adjusted returns $\left[0.5x\ln(CML\,\xi_n\,Wealth) + 0.5(\ln(Unsystematic\,risk\,adjusted\,\xi_n)\right] = 0.41\%$ suggests that at a given level of risk this investor has a lower expected utility, because of a possible aversion to risking credit or unsystematic risks. Hence this investor would not allocate a constant proportion of wealth to risky assets; this partly explains the tendency for investors to demonstrate a flight to the safety of government bonds or other high quality fixed income products during a credit risk crisis.

5.0 Conclusion

Optimization problems arise in many disciplines beyond economics, statistics, business and mathematics, with new optimization problems appearing all the time leading scientists to be constantly analyzing their properties and solutions. Quite frequently we find that these models have to be adjusted or modified to reflect real life experiences. As such this study proposes improvements to the generalized stochastic *linear Quadratic* (LQ) control methodology for dynamic portfolio optimization to include unsystematic (credit) risk which has recently emerged as a major concern to the preservation of investor's wealth. In fact the framework discussed in section 4 of this study addresses a number of other perceived weaknesses in the traditional framework that have led to a systematical underestimation of the probability of the improbable.

The traditional model's exclusion of an implied unsystematic risk measure understates the investor's fulsome risk exposure. Traditional dynamic optimization models indicate that the state space governing the set of super-optimal portfolio returns depends only on the rates of return of the efficient portfolio and the covariance of the assets. However the extended model (NCML) developed in this study indicates otherwise. We find that the NCML is curvilinear indicating a variable market price of risk across the risk spectrum. In fact the NCML addresses the issue of what happens when the investor doesn't have constant relative risk aversion, by linking the expected utility of the investor to the flight to quality puzzle.

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