September 2005

Stationary Markovian Equilibrium in Overlapping Generation Models with Stochastic Nonclassical Production

Olivier F. Morand
University of Connecticut

Kevin L. Reffett
Arizona State University

University of Connecticut

Department of Economics Working Paper Series

Stationary Markovian Equilibrium in Overlapping Generation Models with Stochastic Nonclassical Production

Olivier F. Morand
University of Connecticut

Kevin L. Reffett
Arizona State University

Working Paper 2005-52

September 2005

---

341 Mansfield Road, Unit 1063
Storrs, CT 06269–1063
Phone: (860) 486–3022
Fax: (860) 486–4463
http://www.econ.uconn.edu/

Abstract

This paper provides new sufficient conditions for the existence, computation via successive approximations, and stability of Markovian equilibrium decision processes for a large class of OLG models with stochastic nonclassical production. Our notion of stability is existence of stationary Markovian equilibrium. With a nonclassical production, our economies encompass a large class of OLG models with public policy, valued fiat money, production externalities, and Markov shocks to production. Our approach combines aspects of both topological and order theoretic fixed point theory, and provides the basis of globally stable numerical iteration procedures for computing extremal Markovian equilibrium objects. In addition to new theoretical results on existence and computation, we provide some monotone comparative statics results on the space of economies.

Journal of Economic Literature Classification: C62, E13, O41

The authors would like to thank John Coleman, Manjira Datta, Jaime Erikson, Len Mirman, Seppo Heikkilä, Manuela Santos, John Stachurski and Yiannis Vailakis for very helpful discussions on topics related to this paper. All mistakes are our own.

1 Introduction

The problem of global stability of Markovian equilibrium decision policies (MEDP) and the existence of stationary Markov equilibrium (SME) for stochastic overlapping generations (OLG) models with production is an important question that has not been thoroughly examined. Existing studies have been exclusively focused on models with simple classical production technologies, and it is not clear whether their results and methods can be extended to models with production non-convexities and public policy. Further, these studies almost always follow topological approaches only capable of generating existential results (i.e., existence of Markov equilibria, of dynamic indeterminacies, of sunspot equilibria). It is then difficult to see how the current theoretical results can be applied to construct successive approximation algorithms converging to actual equilibrium objects. Given the importance of numerical solutions in many applied OLG models in public finance, economic demography and geography, macroeconomics, and growth theory, this is a serious shortcoming. The lack of constructive methods is also a serious impediment to the study of stability in the sense of comparative statics results (with respect to ordered changes in some of the parameters of the economy) for the set of Markovian equilibrium.

In this paper we develop a new monotone iterative approach for studying the questions of existence, stability, and computation of Markovian equilibrium for a large class of OLG models with nonclassical stochastic production and Markov shocks. We prove the existence of SME and provide successive approximation methods for obtaining extremal "pure" SME corresponding to each MEDP, thus directly addressing the question of stability for the class of economies under consideration through the use of constructive methods. By "monotone" (or "isotone") approach we mean a collection of arguments and techniques relying primarily on the preservation of order of particular mappings as opposed to topological properties of such mappings. It is very important to note that we work with stochastic OLG models with very general "nonclassical" one-sector production technologies, a class of models known for their potential to exhibit endogenous (expectational) fluctuations and/or local indeterminacies.[1] In particular, we argue that although local indeterminacies in the perfect foresight Markovian equilibrium might be possible for our environments (e.g., as in the infinite horizon economies studied in Benhabib and Farmer [4], Farmer and Guo [26], and Benhabib and Wen [5]), stationary Markov equilibrium do exist from initial conditions. This arguments makes the important point that although local analysis of MEDP around steady states might indicate local indeterminacies, the underlying equilibrium dynamics in the stochastic model still converges to a (non-degenerate) invariant distribution.

The paper makes four important new contributions to the existing literature on the existence, stability, characterization, and computation of equilibrium objects in stochastic OLG models. First, we prove a new result concerning the

[1] Our production specification is consistent with works in infinite horizon economies with public policy of Coleman [11], Boldrin and Rustichini [8], Benhabib and Farmer [4], and Farmer and Guo [26], for instance.

existence of SME in economies where the exogenous shocks follow a quasi compact first-order Markov process and the MEDP is only measurable. Since the quasi-compactness of the Markov process for shocks is equivalent to Doeblin's condition, and since we made no assumptions on the MEDP beyond its measurability, this existence result is quite general. Second, with an additional isotonicity assumption for the Markov process for shocks and under conditions sufficient to obtain monotone MEDP, we present explicit computational algorithms converging (in order and topology) to extremal SME. We thus remove the need for the standard regularity conditions (namely, the Feller property) when computing extremal SME for isotone Markov processes. Third, we address the question of existence and construction of measurable isotone MEDP in a large class of OLG models with potentially nonclassical production through the application of the "Euler equation method" of Coleman [11]. This application of the Euler equation method to OLG models is entirely new, and is shown to yield powerful results. Fourth, we combine all these results and techniques and derive sufficient conditions under which there exists a "pure" SME in wide class of OLG models, which includes that of Wang[56] [57], and develops algorithms to construct extremal "pure" SME. We view our approach as an attempt to provide a detailed computational method for constructing *particular* MEDP and their associated SME, to complement the earlier work of Wang [56][57] as well as to generalize some of the results established in Erikson, Morand, and Reffett [23] for models with stochastic production and independently and identically distributed (iid) shocks.

To discuss these contributions in greater details, we first compare several methods of proof of existence of SME for economies where the Markov process of the state vector is represented by a transition function P_h (the associated operator on probability measures is denoted T_h^*). Specifically, we consider economies where P_h results from the combination of a measurable selection h from equilibrium correspondence Ψ with a given transition function Q characterizing the Markov process for the exogenous variables. All these approaches share the common insight of equating SME to fixed points of a particular mapping; they differ, however, in their choice of fixed point argument.[2] In all our subsequent discussion, unless specified otherwise, the state space (or support for limiting distributions) is assumed to be compact.[3]

A first approach relies on continuity, and is developed for instance in Stokey & al. [51]and in Grandmont and Hildenbrand [25]. In Stokey & al. [51] it is the Feller property of P_h (i.e., the "weak continuity" of T_h^*), that implies that all sequences of N-period averages contain a subsequence weakly converging to an invariant measure (Theorem 12.10). In Grandmont and Hildenbrand [25], the continuity of h combined with the Feller property of Q imply the existence of

[2] This include the literature defining SME as distributions on ergodic sets, as for instance in Duffie & al. [15].

[3] Recent work by Stachurski [50] indicates the possibility to generalize our results to cases where the shock space is unbounded, by applying Lyapunov methods to characterize conditions under which SME are both (i) nondegenerate, and (ii) globally stable (e.g., unique). We will pursue this approach is subsequent work.

an invariant measure by Schauder's Theorem. These existence results are very useful, but in the absence of additional properties of P_h (such as monotonicity) there is no systematic way to compute any of the invariant measures, nor is there any possibility to rule out trivial or degenerate fixed points. Furthermore, the Feller property of P_h generally rests on the continuity of the selection h, and often turns out to often be difficult to obtain: This is a serious drawback of the continuity approach that has been applied in previous work.

A second approach emphasizes convexity, as in the work of Blume[7], Nachman[41], and Duffie & al. [15]. Blume [7] shows that if the multivalued stochastic kernel $\{P_h,\, h \in \Psi\}$ is convex valued and uhc it has an invariant measure by the Fan-Glicksberg/Himmelberg fixed point theorem. Duffie & al. [15] extends Blume's result to non-compact state spaces by introducing self justified sets, while Nachman[41] emphasizes the possibility of convexifying the equilibrium correspondence by considering mixed strategy, i.e. randomizations over the set of selections from the equilibrium correspondence (the "pure" strategies, or "non-sunspots" equilibria). The advantage of the convexity approach is its generality, and it is basically the strategy for the proof of existence of SME in OLG models with Markov shocks of Wang [57]. The drawbacks are twofold: First, the convex-valued requirement implies that it is generally not possible to show that the invariant measure is associated with a non-sunspot equilibrium. Second, as in the continuity approach, there is no information concerning the computation of any *particular* MEDP/SME (e.g., extremal MEDP/SME equilibria). As in the first approach, stability of equilibrium from the perspective of perturbations on the space of economies seems difficult to assess.

The third approach exploits monotonicities in the underlying equilibrium decision processes, and is best presented in the work of Hopenhayn and Prescott[34] as well as in the more general results of Heikkilä and Salonen [33]. In this approach, the existence of a SME follows from an application of a version of Tarski's fixed point theorem for an increasing self map (T_h^* when h is isotone) on a chain complete lattice (the space of probability distributions on a compact support), as shown in Hopenhayn and Prescott ([34], Theorem 1). With the addition of the Feller property it can be shown that a monotone Markov process has extremal invariant measures which can be computed via successive approximation.[4] Further, Hopenhayn and Prescott ([34], Theorem 2) show that a monotone Markov process satisfying a Monotone Mixing Condition (MMC) has a unique invariant measure. Thus, absent the MMC, this monotone approach has nothing to say about the computation of extremal invariant measures unless P_h is assumed to have the Feller property, and this again puts great restrictions on the computational power of the monotone approach and on its potential to address stability issues.

Our existence result is distinct from the existing ones because it neither requires the Feller property nor the isotonicity of P_h, and it does not rely on the convexity/uhc property of the multivalued stochastic kernel $\{P_h,\, h \in \Psi\}$. Our proof, however, relies on the critical property of quasi-compactness of Q, which

[4] This result is well-known (see, for instance, Stokey & al. [51] Exercise 12.12)

implies the quasi-compactness of the operator P_h associated to any measurable h. Our contribution to the computation of extremal SME, although in the line of the monotone approach of Hopenhayn and Prescott[34], does not rely on the Feller property of P_h, but, rather, on the much weaker[5] assumption of order continuity of the operator T_h^* along monotone recursive T_h^*-sequences.[6] This is a very important result because we demonstrate that the isotonicity of h (together with the isotonicity of Q) *regardless of its continuity properties*, is sufficient for order continuity of T^* along monotone T^*-sequences, and thus for computing extremal SME. With our result, the monotone approach can truly be called a monotone method since existence, characterization, and computational and stability results can then be derived on the basis of algebraic and order properties only, and without appeal to topological properties (except for the compactness of the state space).

A contribution of this paper concerns the problem of finding sufficient conditions for the existence of measurable and isotone MEDP in a large class of stochastic *non-optimal* OLG models. Some of these issues have been addressed in Hopenhayn and Prescott [34], Nishimura and Stachurski [43], and Nishimura et al [44] in the context of infinite horizon economies with *iid shocks,* but none of these papers directly study MEDP in *non-optimal* economies (either infinite horizon or OLG). In addition, even for Pareto optimal economies with Markov shock, Hopenhayn and Prescott's conditions are *extremely* strong.[7] Mirman, Morand, and Reffett [37] have obtained new results for a much more general class of infinite horizon non-optimal economies with Markov shocks (which includes the models studied in Hopenhayn and Prescott [34] but also many other environments), but only in the context of infinitely lived agent models.

In this paper we propose to use the "Euler equation method" pioneered by Coleman[11], and applied to infinitely-lived agents models for instance in Greenwood and Huffman [29], Datta & al.[18], Datta & al.[19] [20], and Morand and Reffett[39]. The Euler equation method we use though is quite different than in these papers, and uses the isotonicity and order continuity of an operator defined from the Euler equation and operating on a countable chain complete poset of increasing measurable functions to show existence of MEDP in that poset, and to present algorithms converging to extremal MEDP through successive approximations.[8] The method extends the results for OLG models with iid shocks recently obtained in Erikson & al. [23]. Finally, combining our

[5]Much weaker in the sense that an increasing stochastic kernel with the Feller property is order continuous along monotone T^*-sequences.

[6]See Section 2 of the paper for the definition of order continuity.

[7]In particular, their assumption of strict complementarity of $\Gamma(x, z) = [0, f(x, z)]$ in Proposition 2 is generally not satisfied unless f is Leontieff, and such assumption is critical to apply Topkis results on preservation of supermodularity under maximization (See Lemma 1 in Hopenhayn and Prescott[34]).

[8]In Miao and Santos [36] a MEDP exists as a particular selection from a Markovian correspondence, but the authors provide no information converning the computation of such selection. We improve upon Mia and Santos [36] by providing sharper continuity characterizations of MEDP and results concerning the set of Markovian equilibrium, and by using an approcah that ties numerical solutions directly to theoretical methods.

new findings concerning the construction of invariant measures with our results on existence and construction of monotone measurable selections, we address the problem of existence and construction of SME. This problem has been the subject of fruitful research, beginning with the work of Galor and Ryder [28] on deterministic OLG models, extended to stochastic models with iid shocks in Wang[56], and later with Markov shocks in Wang [57]. In contrast to all the existing work, we give sufficient conditions for the existence of a "pure" stationary Markov equilibrium, (i.e., a SME associated with a non-sunspot THME in the terminology of Wang [57] and Duffie & al.[15]), and for each MEDP we produce algorithms to compute extremal SME within a class of OLG models allowing for non-convexities in production. All our proofs rely heavily on a constructive version of Tarski's fixed point theorem for countable chain complete lattice which we establish early in the paper.

The paper is organized as follows. In the next section of the paper, we presents the set of economies and the lattice theoretic concepts used in this paper, including an order-based fixed point theorem for countable chain complete lattices, and a discussion of the various partially ordered sets relevant to our analysis. In section 3, we apply the Euler equation method to a large class of overlapping generation models and obtain existence and computation results for the set of MEDP. Section 4 addresses the issue of existence of SME. In this section, we adapt some new results concerning the construction of SME first presented in Morand [38] to our OLG setting.

2 Prerequisite Tools and Results

Our emphasis throughout the paper is on ordered spaces and order preserving functions and mappings, so lattice theory is the proper set of tools for our analysis, but we keep its presentation to a bare minimum.[9] We discuss specific ordered sets (of functions, and of probability measures) in which we will search for equilibrium objects as fixed points of some order preserving mappings. Our results, although in the spirit of Tarski's famous fixed point theorem, are more than just existential since we propose constructive algorithm converging to obtain the extremal fixed points. These algorithms are based on our own order-based fixed point theorem which we develop below. Also, measurability will turn out to be a critical issue, and we obtain it from the measurable maximum theorem which we state without proof at the end of this section.

2.1 Results from the Theory of Ordered Sets

We begin with a brief summary of some concepts in the theory of (partially) ordered sets and lattice theory the we shall use in the sequel. Recall a *partially ordered set* (or *poset*) is a pair $(P, \leq)$ where the set P endowed with a partial order $\leq$ (i.e., a reflexive, antisymmetric, and transitive binary relation). If

[9] For a thorough venture into lattice theory for economics, consult Topkis [53] and Veinott [55].

$(P, \leq)$ is a poset, then an *upper bound* (resp., *lower bound*) of $A \subset P$ is any element u (resp. v) such that $\forall p \in A$, $u \geq p$ (resp. $v \leq p$). A *chain* is a linearly ordered subset of P. A sequence is a subset of P of the form $\{p_n\}_{n \in \mathbb{N}}$. If $p_i \leq p_{i+1}$ (resp. $p_i \geq p_{i+1}$) for all $i \in N$ the sequence is called *increasing* (resp., *decreasing*). A *countable chain* is a linearly ordered subset of P that can be written in the form of a "double sequence" $\{p_n\}_{n=-\infty}^{+\infty}$ with $p_i \leq p_{i+1}$ for all $i \in \mathbb{Z}$.

A *lattice* is a poset $(P, \leq)$ such that any two elements p and p' in P, $\inf\{p, p'\}$ and $\sup\{p, p'\}$ (i.e., the greatest lower bound and lowest upper bound of the set $\{p, p'\}$) exist in P. In this case, which case we denote $\inf\{p, p'\}$ and $\sup\{p, p'\}$ respectively by $p \wedge p'$ and $p \vee p'$; a poset $(P, \leq)$ that is a lattice is a *complete lattice* if the greatest lower bound (glb) and lowest upper bound (lub) of any subset $P' \subset P$ exists in P, in which case they are respectively denoted $\wedge_P P'$ and $\vee_P P'$. A poset $(P, \leq)$ is *countably chain complete* if the $\vee_P C$ and $\wedge_P C$ of any countable chain C of P exist in P. Finally, if $(P, \leq_P)$ and $(L, \leq_L)$ are posets, a function $F : (P, \leq_P) \to (L, \leq_L)$ is said to be increasing if it is *order-preserving*, i.e.,

$$\forall (p, p') \in P \times P,\ p \geq_P p' \text{ implies } F(p) \geq_L F(p').$$

2.1.1 Spaces for Candidate MEDP

Next, we discuss the properties of function spaces in which we will search for Markovian equilibrium decision policies (MEDP). In the remainder of this paper,we will denote by the (minimal) state space for candidate MEDP as follows: let $X = [0, k_{\max}] \subset \mathbb{R}$ where $k_{\max} > 0$, $Z = [z_{\min}, z_{\max}] \subset \mathbb{R}$ where $0 < z_{\min} \leq z_{\max}$, $S = X \times Z$, and $S^* =]0, k_{\max}\} \times Z$. All compact subsets of $\mathbb{R}^n$ that we shall use will be endowed with the standard pointwise partial order $\leq$ and the usual topology on $\mathbb{R}^n$. We will use $\mathcal{B}(S)$ to denote the Borel algebra corresponding to the set S.

We now develop two function spaces of interest in our work. Given a function $w : S \longrightarrow X$ that is bounded, increasing in s,[10] continuous (and thus $\mathcal{B}(S)$-measurable), denote by W the set of all functions $h : S \to X$ such that $\forall s \in S$ $0 \leq h(s) \leq w(s)$. Endow W with the standard pointwise order $\geq$ on a function space defined as follows:

$$w_1 \geq w_2 \text{ iif } \forall s \in S,\ w_1(s) \geq w_2(s).$$

Finally, we say that a function $h \in W$ is *non-trivial* if it is strictly positive on S^* that is:

$$\forall s \in S^*,\ h(s) > 0.$$

[10] We say that a function $h \in (W, \leq)$ is *increasing in s* (equivalently *isotone on* $(S, \leq)$) if:

$$\forall (s, s') \in S \times S,\ s' \geq s \text{ implies that } h(s') \geq h(s).$$

By making successive restrictions on the functions h, we can consider several subsets of W, each also endowed with the pointwise order $\geq$. First we add some monotonicity restrictions to the elements of W to define the following subsets G and H:

(i) $G = \{h \in W$ and h increasing on $(S, \leq)\}$,

(ii) $H = \{h \in W$ and h increasing in x for all $z \in Z\}$.

Lemma 1 *$(G, \leq)$ and $(H, \leq)$ are complete lattices with minimal and maximal elements 0 and w respectively.*

Proof. It is easy to verify that glb and lub are constructed by pointwise inf and sup, that is:

$$\forall D \subset G \text{ and } \forall s \in S, \;\; \wedge_G D(s) = \inf\{h(s), h \in D\} \text{ and } \; \vee_G D(s) = \sup\{h(s), h \in D\},$$

and similarly in H. It is important to note that for any increasing sequence $\{h_n\}_{n \in \mathbb{N}}$ in H. $\vee\{h_n\}_{n\in\mathbb{N}}(s) = \sup_{n\in\mathbb{N}} h_n(s) = \lim_{n\to\infty} h_n(s)$ for all $s \in S$.■

In many cases, we will seek MEDP with topological properties, i.e., semi-continuity properties. Consider then the following two subsets U and L in $H \subset W$, each endowed with the pointwise order $\geq$:

(iii) $U = \{h \in H$ and upper semicontinuous ("usc") in x for all $z \in Z\}$,

(iv) $L = \{h \in H$ and lower semicontinuous ("lsc") in x for all $z \in Z\}$.■

Lemma 2 *$(U, \leq)$ and $(L, \leq)$ are complete lattices with minimal and maximal elements 0 and w, respectively.*

Proof. The lower envelope of a family of usc functions is usc (see, for instance Aliprantis and Border[1]), hence:

$$\forall D \subset U \text{ and } \forall s \in S, \;\; \wedge_U D(s) = \inf\{h(s), h \in D\},$$

and since U has the top element w, by Davey and Priestley[11] [21] $(U, \leq)$ is a complete lattice. It is important to note that, although the upper envelope is increasing, it is not necessarily usc. However:

$$\forall D \subset U \text{ and } \forall s \in S, \;\; \vee_U D(s) = \inf_{s<t}\{\sup\{h(t), h \in D\}\}.$$

Symmetric results hold for $(L, \leq)$.■

Finally, we add the algebraic requirement of $\mathcal{B}(S)$-measurability to the sets G, H, U, and L, and correspondingly define the sets G_m, H_m, U_m and L_m.

[11] To show that a partially ordered set is a complete lattice sometimes requires much less work that the definition of completeness would have us believe: Davey and Priestly demonstrate that a non-empty poset $(P, \leq)$ is a complete lattice if and only if P has a top (resp. bottom) element and for any $P' \subset P$, $\wedge_P P'$ (resp. $\vee_P P'$) exists (in P).

Lemma 3 *$(G_m, \leq)$, $(H_m, \leq)$, $(U_m, \leq)$ and $(L_m, \leq)$ are countably chain complete Posets with minimal and maximal elements 0 and w, respectively.*

Proof. The pointwise limit of a sequence of $\mathcal{B}(S)$-measurable function is $\mathcal{B}(S)$-measurable. (See for example, Halmos [30], Theorem 20.A). Thus, $(G_m, \leq)$, $(H_m, \leq)$, $(U_m, \leq)$ and $(L_m, \leq)$ are each $\sigma-$complete lattices.

(ii) In particular, for any increasing sequence of functions $\{h_n\}_{n\in\mathbb{N}}$ in H_m or in G_m:

$$\vee_{H_m,G_m}\{h_n\}_{n\in\mathbb{N}}(s) = \sup_{n\in\mathbb{N}} h_n(s) = \lim_{n\to+\infty} h_n(s) \text{ for all } s \in S,$$

while a symmetric result holds for decreasing sequences of functions. Therefore $(G_m, \leq)$ and $(H_m, \leq)$ are countably chain complete Posets. For an increasing sequence $\{h_n\}_{n\in\mathbb{N}}$ in $(U_m, \leq)$, the functions $\vee_U\{h_n\}_{n\in\mathbb{N}}(.)$ and $\lim_{n\to+\infty} h_n(.)$ coincide almost everywhere, which implies that $\vee_U\{h_n\}_{n\in\mathbb{N}}$ is $\mathcal{B}(S)$-measurable and therefore precisely : $\vee_{U_m}\{h_n\}_{n\in\mathbb{N}} = \vee_U\{h_n\}_{n\in\mathbb{N}}$. Also, for any decreasing sequence $\{h_n\}_{n\in\mathbb{N}}$ in $(U_m, \leq)$, $\wedge_{U_m}\{h_n\}_{n\in\mathbb{N}}(s) = \inf_{n\in\mathbb{N}}\{h_n(s)\} = \lim_{n\to\infty} h_n(s)$ for all $s \in S$. Symmetric results hold for monotone sequences in $(L_m, \leq)$, which proves the desired result in (ii).■

2.1.2 Spaces for Candidate SME

In this paper, tationary Markov equilibria will be defined as probability measures satisfying certain properties. We prove now that the space of probability measures on a compact subset $S \subset R^n$ when endowed with the first-order stochastic dominance partial order is a chain complete poset.[12] We denote the space of bounded and $\mathcal{B}(S)$-measurable real valued functions by $\mathbf{B}(S, \mathcal{B}(S))$, and by $\mathbf{C}(S, \mathcal{B}(S))$ the space of bounded continuous and $\mathcal{B}(S)$-measurable[13] real valued functions, and we use the standard inner product notation:

$$\langle f, \mu \rangle = \int_S f(s)\mu(ds),\ f \in \mathbf{B}(S, \mathcal{B}(S)) \text{ and } \mu \in \Lambda(S, \mathcal{B}(S)).$$

We denote by $\Lambda(S, \mathcal{B}(S))$ the space of probability measures defined on the measurable space $(S, \mathcal{B}(S))$, which we first endow with the stochastic order $\geq_s$ defined as follows: $\mu \geq_s \mu'$ if and only if:

$$\langle f, \mu \rangle \geq \langle f, \mu' \rangle \text{ for all increasing } f \in \mathbf{B}(S, \mathcal{B}(S)).$$

Note that since S is compact, $\mu \geq_s \mu'$ if and only if:

$$\langle f, \mu \rangle \geq \langle f, \mu' \rangle \text{ for all increasing } f \in \mathbf{C}(S, \mathcal{B}(S)),$$

[12] In Erikson & al. [23], the authors work with the space of probability measures on a compact subset S of the real line, which is shown to be a complete latice when endowed with the partial order of first-order stochastic dominance. When $S \subset \mathbb{R}^n$, this is no longer the case (See Morand [**?**] for further discussion of these issues).

[13] Given the definition of S, every bounded continuous function $f : S \to \mathcal{R}$ is $\mathcal{B}(S)$-measurable.

as shown for instance in Torres[54].

Next, we also endow $\Lambda(S,\mathcal{B}(S))$ with the weak topology for which a sequence of probability measures $\{\mu_n\}_{n\in\mathbb{N}}$ in $\Lambda(S,\mathcal{B}(S))$ is said to *weakly converge* to $\mu\in\Lambda(S,\mathcal{B}(S))$ if for all $f\in\mathbf{C}(S,\mathcal{B}(S))$:

$$\lim_{n\to\infty}\langle f,\mu_n\rangle=\langle f,\mu\rangle\,, \tag{CV}$$

in which case we write $\mu_n\Longrightarrow\mu$ and call μ the *weak limit* of the sequence $\{\mu_n\}_{n\in\mathbb{N}}$. Finally, we say that a sequence $\{\mu_n\}_{n\in\mathbb{N}}$ in $\Lambda(S,\mathcal{B}(S))$ is said to *converge in the total variation norm* to $\mu\in\Lambda(S,\mathcal{B}(S))$ if condition (CV) above holds for all $f\in\mathbf{B}(S,\mathcal{B}(S))$, and the convergence is uniform for all functions with $\|f\|\leq 1$. By definition, convergence in the total variation norm clearly implies weak convergence (see, for instance, Stokey & al.[51] for the various definitions of convergence).

We now state and demonstrate an important result concerning the order structure[14] of $(\Lambda(S,\mathcal{B}(S)),\geq_s)$ as well as the properties of monotone sequences of probability measures in $(\Lambda(S,\mathcal{B}(S)),\geq_s)$.

Lemma 4 *$(\Lambda(S,\mathcal{B}(S)),\geq_s)$ is a countable chain complete poset with minimal and maximal elements. In addition, for any increasing (decreasing) sequence $\{\mu_n\}_{n\in\mathbb{N}}$*

$$\mu_n\Longrightarrow\vee\{\mu_n\}_{n\in\mathbb{N}}\ \textit{(resp. } \mu_n\Longrightarrow\wedge\{\mu_n\}_{n\in\mathbb{N}}\textit{)}.$$

Proof. Clearly $(\Lambda(S,\mathcal{B}(S)),\geq_s)$ is a poset with minimal and maximal elements (the singular measures $\delta_{(0,z_{\min})}$ and $\delta_{(x_{\max},z_{\max})}$, respectively). Since all monotone sequences of $([0,x_{\max}]\times[z_{\min},z_{\max}],\geq)$ (and therefore order bounded) converge, by Lemma 1 in Heikkila and Salonen[33] all monotone sequences $\{\mu_n\}_{n\in\mathbb{N}}$ in $(\Lambda(S,\mathcal{B}(S)),\geq_s)$ (and thus order bounded) weakly converge.[15] Consequently, if $\{\mu_n\}_{n\in\mathbb{N}}$ is an increasing sequence and μ its weak limit, for all increasing $f\in\mathbf{C}(S,\mathcal{B}(S))$ and for all $n\in\mathbb{N}$:

$$\langle f,\mu_n\rangle\leq\lim_{i\to+\infty}\langle f,\mu_i\rangle=\langle f,\mu\rangle\,,$$

and μ is thus an upper bound for $\{\mu_n\}_{n\in\mathbb{N}}$. Given any upper bound q of $\{\mu_n\}_{n\in\mathbb{N}}$, for all increasing $f\in\mathbf{C}(S,\mathcal{B}(S))$:

$$\text{For all } i\in\mathbb{N},\ \langle f,\mu_i\rangle\leq\langle f,q\rangle\,,$$

implying that:

$$\lim_{n\to\infty}\langle f,\mu_n\rangle=\langle f,\mu\rangle\leq\langle f,q\rangle\,,$$

[14] Following a different argument, Hopenhayn and Prescott [34] prove that $(\Lambda(S,\mathcal{B}(S)),\geq_s)$ is a chain complete lattice with minimal and maximal elements.

[15] This result is also a consequence of Helly's Theorem (See, for instance, Corollary 1 to Theorem 12.9 in SLP [51]).

and therefore that $\mu \leq_s q$. Thus μ is the lowest upper bound, and we have:

$$\mu_n \Longrightarrow \vee\{\mu_n\}_{n\in\mathbb{N}}.$$

A symmetric argument easily shows that every decreasing sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ weakly converges to its lower bound, i.e:

$$\lambda_n \Longrightarrow \wedge\{\lambda_n\}_{n\in\mathbb{N}}.$$

■

2.2 An order-based fixed point theorem

Within the framework of the seminal fixed point theorem of Tarski ([52], Theorem 1) stating that an increasing map F that transforms a complete lattice $(P, \leq)$ has a non-empty set of fixed points, we prove a new result concerning the construction of extremal fixed points of such a map under some additional assumptions. First we called a *monotone recursive F-sequence* any increasing (decreasing) sequence C of the form $x \leq F(x) \leq ... \leq F^n(x) \leq ...$ (resp. $x \geq F(x) \geq ... \geq F^n(x) \geq ...$). Given this definition, we say that a function F that transforms a poset $(P, \geq)$ into itself is *order continuous along monotone recursive F-sequences* (in short "OCF") if for any increasing (decreasing) recursive F-sequence C in P such that $\vee C$ (resp. $\wedge C$) exists,

$$\vee\{F(C)\} = F(\vee C) \text{ (resp. } \wedge \{F(C)\} = F(\wedge C)).$$

This definition calls for two important remarks relating to the literature:

(a). Unlike the property of order continuity, the OCF of F does not imply that F is necessarily increasing. Consider for instance $X = [0, 1] \subset \mathbb{R}$ and the function $F : X \to X$ defined as follows:

$$F(x) = \left\{ \begin{array}{c} 0 \text{ for } x = 0 \\ (1-x) \text{ for all } x \text{ in }]0,1[\\ 1 \text{ for } x = 1 \end{array} \right\}$$

The only monotone recursive F-sequences are $\{F^n(1/2)\}_{n\in\mathbb{N}}$, $\{F^n(1)\}_{n\in\mathbb{N}}$ and $\{F^n(0)\}_{n\in\mathbb{N}}$. Since $\vee\{F^n(1/2)\}_{n\in\mathbb{N}} = 1/2 = F(\vee\{F^n(1/2)\}_{n\in\mathbb{N}})$, $\vee\{F^n(1)\}_{n\in\mathbb{N}} = 1 = F(\vee\{F^n(1)\}_{n\in\mathbb{N}})$ and $\vee\{F^n(0)\}_{n\in\mathbb{N}} = 0 = F(\vee\{F^n(0)\}_{n\in\mathbb{N}})$, clearly F is OCF[16]. Obviously F is not increasing.

(b). A function can be OCF without being continuous in the usual topological sense (F in the example above is not continuous in the usual topology on $\mathbb{R}$).

The new constructive fixed point theorem we prove next establishes that if a function F from a countable chain complete lattice $(P, \geq)$ into itself is increasing and OCF, then extremal fixed points can be obtained as glb or lub of particular

[16] The same results hold for the glb of these sequences.

monotone sequences. Although this result is based on a theorem in Dugundgi and Granas ([14], Theorem 4.2) and is also related very closely to Theorem 2.1 reported in Heikkilä [32]. The version of the theorem we now prove was first developed in Morand [38] for the study of SME in various economies. For completeness, we give a detailed proof here also:

Theorem 5 *Let $(P, \geq)$ be a countable chain complete poset with maximal and minimal element $p_{\max}$ and $p_{\min}$, respectively, and $F : (P, \geq) \to (P, \geq)$ increasing and OCF.*

(a). If there exists $a \in P$ such that $F(a) \geq a$, then $\vee\{F^n(a)\}_{n\in\mathbb{N}}$ is the minimal fixed point of F in the order interval $[a, p_{\max}]$ of $(P, \geq)$.

(b). If there exists $b \in P$ such that $b \geq F(b)$, then $\wedge\{F^n(b)\}_{n\in\mathbb{N}}$ is the maximal fixed point in order interval $[p_{\min}, b]$ of $(P, \geq)$.

Proof. (a). Since $F(a) \geq a$ and F isotone, recursively we prove that $\forall n \in N$, $F^{n+1}(a) \geq F^n(a)$. Thus $\{F^n(a)\}_{n\in\mathbb{N}}$ is an increasing sequence, and since $(P, \geq)$ is a countable chain complete poset $\vee\{F^n(a)\}_{n\in\mathbb{N}}$ exists. As F is FOC, $F(\vee\{F^n(a)\}_{n\in\mathbb{N}}) = \vee\{F^n(a)\}_{n\in\mathbb{N}}$, that is, $\vee\{F^n(a)\}_{n\in\mathbb{N}}$ is a fixed point of F. Consider any $d \geq a$ such that $F(d) = d$. Since F is increasing, $\forall n \in N$, $d \geq F^n(a)$ which implies that d is an upper bound of $\{F^n(a)\}_{n\in\mathbb{N}}$, which implies that $d \geq \vee\{F^n(a)\}_{n\in\mathbb{N}}$. As a result, $\vee\{F^n(a)\}_{n\in\mathbb{N}}$ is necessarily the minimal fixed point of F in $[a, p_{\max}]$.

(b). The proof follows a similar argument to that in (b).■

The following useful corollary to this theorem that we shall use in the sequel reads as follows:

Corollary 6 *$\vee\{F^n(p_{\min})\}_{n\in\mathbb{N}}$ and $\wedge\{F^n(p_{\max})\}_{n\in\mathbb{N}}$ are the minimal and maximal fixed points of F in $(P, \geq)$.*

2.3 The measurable maximum theorem

We require in this paper MEDP to be measurable functions, because that property is needed to construct a Markov operator whose fixed points will be stationary Markov equilibria. Since the MEDP are obtained as solutions to a maximization problem, the measurable maximum theorem is a natural way to establish their desired measurability. We state without proof this theorem (for a proof, see for instance Aliprantis and Border[1]).

Theorem 7 *Let*[17] *$\varphi : S \twoheadrightarrow X$ be a weakly measurable correspondence with nonempty compact values, and suppose $f : S \times X \to R$ is a Caratheodory*

[17] We write $S \twoheadrightarrow X$ rather than $S \to P(X)$ where $P(X)$ is the set of all subsets of X (standard notation in AB99).

function ($\mathcal{B}(S)$-measurable for all $x \in X$, and continuous in x for all $s \in S$). If we define the value function $v : S \to R$ by:

$$v(s) = \max_{x \in \varphi(s)} f(s, x),$$

and the correspondence $G : S \twoheadrightarrow X$ of maximizers by:

$$G(s) = \{x \in \varphi(s) : f(s, x) = v(s)\},$$

then (i) the value function v is $\mathcal{B}(S)$-measurable, (ii) the correspondence G is measurable, has non empty compact values, and admit a measurable selector, and (iii) if $G(s)$ is singled valued for all $s \in S$, then it is $\mathcal{B}(S)$-measurable.

3 Setup, existence and construction of extremal MEDP

We consider the class of OLG models described in Wang [57] which we modify along several dimensions. First, the lifetime utility function $U(c_1, c_2)$ representing the agents' preferences is required to be supermodular, in addition to the standard restrictions specified in Assumption 1. Second, we generalize the production function of Wang by allowing for nonconvexities in production and various forms of public policy distortions, although the constant returns to scale in private inputs, as in Wang, imply zero profits (see Assumption 2 and 2'). This setting is typical of the literature on infinite horizon nonoptimal economies (see for instance, Coleman [11]), and may be taken as the reduced form for a number of economies with frictions, as discussed for example in Greenwood and Huffman [29]. Third, we put restrictions on the Markov shock process, as described in Assumption 3 and $3'$, that are different than those in Wang [56][57].

3.1 The primitives of the economy

We now discuss the primitive notions of preferences, technologies, and stochastic structure. We begin with preference. Our assumptions on lifetime utility functions are standard (e.g., see Wang [56]) except for the additional restriction in IV where we assume c_1 and c_2 be weak complements, which is trivially satisfied when utility is separable in its two arguments.

Assumption 1. The utility function $U : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$:
I. is continuously differentiable;
II. is strictly increasing and strictly concave in each of its arguments;
III. satisfies $\forall c_2 > 0$, $\lim_{c_1 \to 0^+} U_1(c_1, c_2) = +\infty$ and $\forall c_1 > 0$, $\lim_{c_2 \to 0^+} U_2(c_1, c_2) = +\infty$;
IV. has increasing differences in (c_1, c_2).

With these assumptions on preferences, we are now ready to consider the characteristics of the nonclassical production of the economies. By nonclassical production, we mean that production is allowed to possess particular types of equilibrium "spillovers". Specifically, as in the literature on infinite horizon economies (e.g., Coleman [11], Boldrin and Rustichini [8], Benhabib and Farmer[4], Benhabib and Wen [5]) we assume that the nonclassical production will have social inputs (K, N) and constant returns to scale in private inputs (k, n), and denote by $F(k, n, K, N, z)$ the production technology. Following arguments well-known in the literature (e.g. Greenwood and Huffman [29] and Datta & al. [18]), this reduced-form production specification F can shown to represent an economy with public policy, with production nonconvexities, with monopolistic competition, or even a monetary economies. Our assumptions on F, including the monotonic properties of the equilibrium wage rate and rental rate of capital, are adapted from the literature on nonoptimal stochastic growth (see Coleman [11] and Greenwood and Huffman [29] for the corresponding assumptions in infinitely lived agent models).

Assumption 2. The production function $F(k, n, K, N, z) : \mathbb{R}_+ \times [0, 1] \times \mathbb{R}_+ \times [0, 1] \times Z \to \mathbb{R}_+$ is:

I. twice continuously differentiable in its first two arguments;

II. constant returns to scale in the private inputs (k, n);

III. such that $w(k, z) = F_2(k, 1, k, 1, z)$ is increasing in k and $\lim_{k\to 0} w(k, z) = 0$;

IV. such that $r(k, z) = F_1(k, 1, k, 1, z)$ is strictly decreasing in k and that $\lim_{k\to 0} r(k, z) = +\infty$;

V. such that there exists $k_{\max}$ with $\forall k \geq k_{\max}$ and $\exists z \in Z$ $F(k, 1, k, 1, z) \geq k_{\max}$, and $\forall k \leq k_{\max}$, $\forall z \in Z$, $F(k, 1, k, 1, z) \leq k_{\max}$.

Note that in Assumption 2, we are anticipating $n = 1 = N$ in equilibrium (as households do not value leisure), so that 2.III, 2.IV, and 2.V concern properties of the function $F(k, 1, K, 1, z)$. In particular, assumption 2.V implies that the set of feasible capital stock can be restricted to be in the compact interval $X = [0, k_{\max}]$ (as long as we place the initial date zero capital stocks $k_0 = K_0$ in X).To sharpen the characterization of equilibrium objects we will make the following additional assumption (trivially satisfied, for instance, in setup with multiplicative shock).

Assumption 2′. $w(k, z)$ and $r(k, z)$ are increasing in z.

Finally, uncertainty enters the production process in the form of a Markov process with transition function Q; as in Wang[57] shocks take their values on the compact subset $Z = [z_{\min}, z_{\max}]$ of $\mathbb{R}$. Our emphasis on monotone properties induces us to assume that the transition function for shocks is increasing (Assumption 3), an assumption also made in Hopenhayn and Prescott [34]. Unlike Wang[57], we relax the assumption of the Feller property for Q and only require it to satisfy Doeblin's condition (D), which is equivalent to the operator Q being quasi-compact (see Neveu [42], V.3.2). This is a significant step as our MEDP will not necessarily be continuous, so the Feller property is not particularly useful in this context. Finally, it is important to note that both of these assumptions are trivially satisfied in the special case of *iid* shocks (in particular, they are satisfied for the economies in Erikson & al.[23]).

Assumption 3. The random variables z_t follow a first order Markov process with stationary transition function Q such that Q is increasing, that is, for every increasing $f \in \mathbf{B}(Z, \mathcal{B}(Z))$, the function:

$$Tf(z) = \int f(z')Q(z, dz') \text{ is increasing in } z.$$

Assumption 3′. The random variables z_t follow a first order Markov process with stationary transition function Q such that there exists $\gamma \in \Lambda(Z, \mathcal{B}(Z))$ and $\varepsilon > 0$ with:

$$\forall B \in \mathcal{B}(Z),\ \gamma(B) \leq \varepsilon \text{ implies that } \forall z \in Z,\, Q(z, B) \leq 1 - \varepsilon.$$

3.2 An Euler equation method for MEDP

Having specified the class of economies under consideration, we now develop an "Euler equation" method adapted to the case of the stochastic OLG models we are considering. Specifically from the Euler equation evaluated along an equilibrium trajectory, we define a nonlinear operator A whose *non-trivial* fixed points are precisely the MEDP. This nonlinear operator has is increasing and maps a countable chain complete lattice of candidate equilibrium policies into itself; the existence of fixed points follows from a direct application of Tarski's fixed point theorem. The construction of extremal non-trivial fixed points through successive approximations then relies on the OCF properties of the operator A, as well as some additional restrictions sufficient to prove existence of a non-trivial minimal fixed point.

In many important ways, the Euler equation method generates *sharper* characterizations of particular MEDP than the purely topological methods used in Wang[57] and based upon Duffie & al. [15]. In particular, we develop conditions under which MEDP are semi continuous (either upper or lower) versus Lipschitz continuous, the latter property being important for developing error bounds (e.g., see Santos [49]).

Consider the maximization problem of a typical agent earning the competitive wage w in the first period of his life, and who must decide what part of his earning to consume immediately, and what part to set aside for future consumption. While making these decisions, the agent takes as given a law of motion h for capital stock, which he uses to compute the competitive expected return on his capital investment. Returns on labor and capital are precisely the competitive prices $w(k, z) = F_2(k, 1, k, 1, z)$ and $r(k, z) = F_1(k, 1, k, 1, z)$ issued from the firm's maximization. Thus given a candidate equilibrium law of motion for the capital stock $k' = h(k, z)$, an agent seeks to solve:

$$\max_{y \in [0, w(k,z)]} \int_Z U(w(k, z) - y, r(k', z')y)Q(z, dz'),$$

Under assumptions 1 and 2, the necessary and sufficient Euler equation associated with this household maximization problem is:

$$\begin{aligned}&\int_Z U_1(w(k,z)-y, r(h(k,z),z')y)Q(z,dz')\\ =\ &\int_Z U_2(w(k,z)-y, r(h(k,z),z')y)r(h(k,z),z')Q(z,dz').\end{aligned}$$

Recalling that we introduced in Part 1 the set W of functions $h : S \to X$ with $0 \leq h(s) \leq w(s)$ for all $s \in S$, we define a Markovian equilibrium decision policy (MEDP) as follows:

Definition 8 *A MEDP is a non-trivial function $h \in W$ such that, for all $(k,z) \in S^*$:*

$$\begin{aligned}&\int_Z U_1(w(k,z)-h(k,z), r(h(k,z),z')h(k,z))Q(z,dz') \qquad \text{(E)}\\ =\ &\int_Z U_2(w(k,z)-h(k,z), r(h(k,z),z')h(k,z))r(h(k,z),z')Q(z,dz').\end{aligned}$$

and $h(0,z) = 0$ for all $z \in Z$.

We use the Euler equation to define the non-linear operator A as follows. For a given $h \in H_m$ and for any $s = (k,z) \in S^*$, consider the equation (E') in y below[18]:

$$\begin{aligned}&\int_Z U_1(w(k,z)-y, r(h(k,z),z')y)Q(z,dz') \qquad \text{(E')}\\ =\ &\int_Z U_2(w(k,z)-y, r(h(k,z),z')y)r(y,z')Q(z,dz'), \qquad (1)\end{aligned}$$

and set:

$$Ah(s) = \left\{ \begin{array}{c} 0 \text{ if } h(s) = 0 \\ y^* \text{ if } h(s) > 0 \end{array} \right\},$$

where y^* is the unique solution to equation (E$'$). Note that Assumptions 1-3 are sufficient to establish the uniqueness of y^* since these conditions imply that the LHS of equation (E$'$) is increasing in y and satisfies $\lim_{y \to w(k,z)^-} LHS = +\infty$, and that the RHS of (E$'$) is decreasing in y and satisfies $\lim_{y \to 0^+} RHS = +\infty$. Since an element h in H_m is a MEDP if and only if it is a non-trivial fixed point of A, the search for MEDP in H_m is equivalent to finding the nontrivial fixed points of the operator A.

[18] Notice the operator here differs from the infinite horizon case significantly; in particular, the operator is not defined to be this periods "response" to a guess of next period's law of motion. In essence for this economy, we simply parameterize future prices via the marginal product of capital, and then develop implied equilibrium law of motion. We then shall iterative on this procedure from upper and lower solutions to find extremal fixed points, which are then MEDP.

3.3 Measurable MEDP

We first consider the question of existence of MEDP, which we establish is a consequence of the particular order properties of the operator A.

Lemma 9 *A is a self map on $(H_m, \leq)$.*

Proof. For any $h \in H_m$, Assumptions 1-2 imply that the RHS of (E$'$) is decreasing in k while the LHS is increasing in k. It then follows easily that the function $Ah : S \to X$ is increasing in k for each $z \in Z$. The $\mathcal{B}(S)$-measurability of Ah relies on the maximum measurable theorem. Indeed, by definition:

$$Ah(s) = \arg \max_{y \in [0, w(s)]} p(s, y) \text{ for all } s = (k, z) \in S^*,$$

where:

$$p(s,y) = p((k,z),y) = -\left| \int_Z [U_1(w(k,z) - y, r(h(k,z),z')y) - U_2(w(k,z) - y, r(h(k,z),z')y)r(y,z')]Q(z,dz') \right|.$$

Since $h \in H_m$ for all $y \in [0, w(k,z)]$ the function

$$U_1(w(k,z) - y, r(h(k,z),z')y) - U_2(w(k,z) - y, r(h(k,z),z')y)r(y,z')$$

is continuous in z' and jointly measurable in (k,z) and therefore jointly measurable in (k,z,z'), which implies that $p(s,y)$ is $\mathcal{B}(S)$-measurable (See Appendix A). Since $p(s,y)$ is clearly continuous in y, the maximum measurable theorem applies and Ah is $\mathcal{B}(S)$-measurable. Thus A maps $(H_m, \geq)$ into itself.■

Lemma 10 *A is increasing on $(H_m, \leq)$.*

Proof. The RHS of (E') is decreasing in h while the LHS is increasing in h. As a result, $h \leq h'$ implies that $Ah \leq Ah'$, i.e., A is increasing in h.■

Lemma 11 *A is order continuous along any monotone sequence, and therefore OCF.*

Proof. Consider for instance an increasing sequence $\{a_n\}_{n\in\mathbb{N}}$ in $(H_m, \leq)$. For all $s \in S$, the sequence of real numbers $\{a_n(s)\}_{n\in\mathbb{N}}$ is increasing and bounded above by $w(k, z_{\max})$, and therefore $\lim_{n\to\infty} a_n(s) = \sup\{a_n(s)\}_{n\in\mathbb{N}}$. Since A is increasing, the sequence $\{Aa_n(s)\}_{n\in\mathbb{N}}$ also satisfies $\lim_{n\to\infty} Aa_n(k,z) = \sup\{Aa_n(k,z)\}_{n\in\mathbb{N}}$. The pointwise limit of a sequence of $\mathcal{B}(S)$-measurable functions is a measurable function, and, for all $s \in S$:

$$\vee_{H_m}\{a_n\}_{n\in\mathbb{N}}(s) = \sup\{a_n(s)\}_{n\in\mathbb{N}} \text{ and } \vee_{H_m}\{Aa_n\}_{n\in\mathbb{N}}(s) = \sup\{Aa_n(s)\}_{n\in\mathbb{N}},$$

By definition of Aa_n, $\forall n \in \mathbf{N}$, and $\forall (k,z) \in S^*$: The functions U_1, U_2, and r being continuous in the relevant arguments, taking limits when n goes to infinity implies that:

$$\begin{aligned} & \int_Z U_1(w(k,z) - \\ & \sup\{Aa_n(k,z)\}_{n\in\mathbb{N}}, r(\sup\{a_n(k,z)\}_{n\in\mathbb{N}}, z') \sup\{Aa_n(k,z)\}_{n\in\mathbb{N}}) Q(z,dz') \\ = & \\ & \int_Z U_2(w(k,z) - \\ & \sup\{Aa_n(k,z)\}_{n\in\mathbb{N}}, r(\sup\{a_n(k,z)\}_{n\in\mathbb{N}}, z') \sup\{Aa_n(k,z)\}_{n\in\mathbb{N}}) r(\sup\{Aa_n(k,z)\}_{n\in\mathbb{N}}, z') Q(z,dz'), \end{aligned}$$

which implies that $A(\sup\{a_n(s)\}_{n\in\mathbb{N}}) = \sup\{Aa_n(s)\}_{n\in\mathbb{N}}$. Thus $\vee_{H_m}\{Aa_n\}_{n\in\mathbb{N}} = A\vee_{H_m}\{a_n\}_{n\in\mathbb{N}}$, and a similar argument can be made for any decreasing sequence $\{a_n\}_{n\in\mathbb{N}}$ in $(H_m, \geq)$ to establish that $\wedge_{H_m}\{Aa_n\}_{n\in\mathbb{N}} = A \wedge_{H_m} \{a_n\}_{n\in\mathbb{N}}$.■

With these lemmata in place, we are now ready to prove our first central proposition of the paper concerning existence and computation of MEDP. Let H_A be the set of fixed points of the operator A in $(H_m, \leq)$.

Proposition 12 *Under Assumptions* $1, 2$ *and* 3*, the set of fixed points* $H_A \subset (H_m, \leq)$ *is non empty. Further, this fixed point correspondence admits minimal and maximal fixed point that can be computed as* $\vee_{H_m}\{A^n 0\}_{n\in\mathbb{N}} = \vee H_A$ *and* $\wedge_{H_m}\{A^n w\}_{n\in\mathbb{N}} = \wedge H_A$*, respectively.*

Proof. Since A is increasing, OCF, and a self map on the countable chain complete poset $(H_m, \leq)$, the proposition follows from our fixed point theorem in Part 2; that is, there exists a non empty set of fixed points, the maximal fixed point in $(H_m, \leq)$ is $\wedge_{H_m}\{A^n w\}_{n\in\mathbb{N}}$ which is obtained as $\wedge_{H_m}\{A^n w\}_{n\in\mathbb{N}}(s) = \lim_{n\to\infty} A^n w(s)$, and the minimal fixed point is $\vee_{H_m}\{A^n 0\}_{n\in\mathbb{N}} = 0$.■

3.4 Increasing measurable MEDP

We now establish the existence of non-trivial fixed points of A in the space $(G_m, \leq)$ of increasing measurable MEDP. First, it is to be expected that to prove that an MEDP is increasing in the exogenous shock (we already know that it is increasing in the endogenous state k) will require additional restrictions on the primitives. Intuitively, an increase in z leads to an immediate increase in wealth as well as an expected increase in the rate of return on savings under the assumption of an increasing transition function Q.[19] As a result, young agents do not necessarily respond to an increase in z by increasing their savings. Assumption 4 below presents a set of sufficient conditions for A to be a self map on $(G_m, \leq)$. Second, recall that by construction the minimal fixed point of A

[19] Naturally, if shocks are independently distributed, an increase in z generates only a contemporaneous positive wealth effect to which an agent respond by increasing savings (see [23]). No additional assumptions on shocks are then needed.

is zero, a trivial object that we exclude from the set of MEDP. Consequently, we give in Assumption 5 a simple sufficient condition for the existence of a *non-trivial minimal fixed point of* A *in* $(G_m, \leq)$, which by definition will be the minimal MEDP in $(G_m, \leq)$. Finally, under the combined Assumption 1 through 5 we state our result concerning the construction of extremal increasing and measurable MEDP.

We begin with sufficient conditions for Ah to be increasing in z whenever h is, so that A maps the countable chain complete lattice $(G_m, \leq)$ into itself. Denote by G_A the set of fixed points of G in $(G_m, \leq)$.

Assumption 4.

(a). Utility is separable, that is $U(c_1, c_2) = u(c_1)+v(c_2)$, and$v"(c_2)c_2/v'(c_2) \geq 0$ for all $c_2 > 0$.

(b). Shocks are multiplicative, that is the production function is such that $f(k, K, z) = zf(k, K)$.

Proposition 13 *Under Assumptions* $1, 2, 2', 3$ *and* 4*,* $G_A \subset (G_m, \leq)$ *is non-empty and there exist minimal and maximal fixed points.*

Proof. We prove that A is a self map on $(G_m, \leq)$. Under Assumption 4 equation (E′) becomes:

$$u'(w(k, z) - y) = \int b(z, y, z')Q(z, dz'), \tag{E"}$$

in which we write $b(z, y, z') = v'(r(h(k, z), z')y)r(y, z')$ for simplicity. The function b has several properties.

(a). $b(z, y, z')$ is increasing in z' since

$$\begin{aligned}
\partial b/\partial z' &= v"(c_2)yr(y, z')\partial r(h(k, z), z')/\partial z' + v'(c_2)\partial r(y, z')/\partial z' \\
&= \\
&\left[v"(c_2)c_2 \left(\frac{r(y, z')}{r(h(k, z), z')}\right)\left(\frac{\partial r(h(k, z), z')/\partial z'}{\partial r(y, z')/\partial z'}\right) + v'(c_2)\right]\partial r(y, z')/\partial z' \\
&= \\
&[v"(c_2)c_2 + v'(c_2)]\,\partial r(y, z')/\partial z' \\
&\geq 0,
\end{aligned}$$

by Assumption 4.

(b). $b(z, y, z')$ is increasing in z whenever $h \in G_m$.

Consider then any $y \in [0, w(k, z_2)]$, and any $z_1 \geq z_2$. We have

$$\int b(z_2, y, z')Q(z_2, dz') \leq \int b(z_1, y, z')Q(z_2, dz') \leq \int b(z_1, y, z')Q(z_1, dz'),$$

where the first inequality results from (a) and the second from (b) and the property that Q is an increasing transition function. This establishes that the right hand side of (E$''$) is increasing in z, and since the left hand side is clearly decreasing in z, for all $h \in G_m$, $z_1 \geq z_2$ implies that $Ah(k, z_1) \geq Ah(k, z_2)$; thus A maps G_m into itself. The proof of OCF of A in $(G_m, \leq)$ is the same as that in the previous theorem in the last subsection of the paper since for any increasing sequence $\{a_n\}_{n\in\mathbb{N}}$ of $G_m \subset H_m$,

$$\vee_{G_m}\{a_n\}_{n\in\mathbb{N}} = \vee_{H_m}\{a_n\}_{n\in\mathbb{N}},$$

(a symmetric result holds for decreasing sequences) and the existence of a non-empty set of fixed points in $(G_m, \leq)$ follows from our fixed point theorem in Section 2.■

Our next assumption concerns the limit behavior of the rental rate of capital when the capital stock goes to 0. We show that it implies that A maps strictly up the order interval $]0, h_0[$ of $(G_m, \leq)$, and it is thus sufficient to establish the existence of a non-trivial minimal fixed point of A (which is then by definition the minimal MEDP in $(G_m, \leq)$).

Assumption 5. $\lim_{x\to 0^+} r(x, z_{\max}) = 0$.

Lemma 14 *Under Assumption* 5*, there exists* $h_0 \in (L_m, \leq) \cap (G_m, \leq)$ *such that:*

$$\forall s \in S^*, Ah_0(s) \geq h_0(s) > 0,$$

and:

$$\forall s \in S^*, 0 < x < h_0(s) \Longrightarrow Ax > x.$$

Proof. See Appendix C.■

We can now state an important result concerning the construction of the maximal fixed point of A in the set of functions that are isotone in (k, z), and $\mathcal{B}(S)$-measurable.

Proposition 15 *Under Assumptions* $1, 2, 2', 3, 4,$ *and* 5*, the set of MEDP in* $(G_m, \leq)$ *is nonempty, and has a maximal MEDP is* $\wedge_{G_m}\{A^n w\}_{n\in\mathbb{N}}$ *(in practice constructed as* $\wedge_{G_m}\{A^n w\}_{n\in\mathbb{N}}(s) = \lim_{n\to\infty} A^n w(s)$ *for all* $s \in S$*), while the minimal MEDP is* $\vee_{G_m}\{A^n h_0\}_{n\in\mathbb{N}}$ *(and is constructed as* $\vee_{G_m}\{A^n h_0(s)\}_{n\in\mathbb{N}} = \lim_{n\to\infty} A^n h_0(s)$ *for all* $s \in S$*).*

Proof. $\wedge_{G_m}\{a_n\}_{n\in\mathbb{N}} = \wedge_{H_m}\{a_n\}_{n\in\mathbb{N}}$ for all decreasing sequence $\{a_n\}_{n\in\mathbb{N}}$ in $(G_m, \leq)$ so $\wedge_{G_m}\{A^n w\}_{n\in\mathbb{N}}(s) = \lim_{n\to\infty} A^n w(s)$ for all $s \in S$, and by Theorem 2 $\wedge_{G_m}\{A^n w\}_{n\in\mathbb{N}}$ is the maximal fixed point of A (since as A is OCF and increasing on the countable chain complete $(G_m, \leq)$). By our fixed point theorem of Section 2, Theorem 5 and its corollary above implies that $\vee_{G_m}\{A^n h_0\}_{n\in\mathbb{N}}$ is the minimal fixed point of A in the order interval $([h_0, w], \leq) \subset (L_m, \leq) \subset (G_m, \leq)$. It is thus strictly positive. By lemma 14 above $Ax > x$ for all $0 < x < h_0(s)$, and A thus cannot have a fixed point in $[0, h_0]$ other than 0. As a result, $\vee_{G_m}\{A^n h_0\}_{n\in\mathbb{N}}$ is the minimal strictly positive fixed point of A in $(G_m, \leq)$.■

3.5 Semicontinuous MEPD

We now show next that our operator A is transformation (e.g., self-map) on the countably chain complete poset of semicontinuous (in k) functions $(U_m, \leq)$ (resp. $(L_m, \leq)$). It is then not too difficult to also prove that A is OCF on both $(U_m, \leq)$ and $(L_m, \leq)$[20] thus establishing that the existence of a nonempty set of MEDP in $(U_m, \leq)$ (resp. $(L_m, \leq)$, in each case of the domain, the operator A admitting maximal and minimal elements of the fixed point set which can be obtained by successive approximations from upper and lower solutions (i.e., "end" points that map up and down with the fixed point contained within the resulting order interval).

Lemma 16 *A is a self map on $(U_m, \leq)$ and on $(L_m, \leq)$.*

Proof. We prove that A is a self map on $(U_m, \leq)$ (the case of $(L_m, \leq)$ is symmetric). Consider $h \in (U_m, \leq) \subset (H_m, \leq)$ and any $z \in Z$. Since Ah is increasing in k, if it is right continuous at every $k \in [0, x_{\max}[$ it is then necessarily upper semicontinuous in k. Suppose on the contrary that there exists $(\widetilde{k}, z) \in [0, x_{\max}[\times Z$ where Ah is not right continuous at $\widetilde{k}$, i.e., that there exists $\Delta > 0$ such that:

$$\lim_{k_n \to \widetilde{k}^+} Ah(k_n, z) = Ah(\widetilde{k}, z) + \Delta,$$

where $k_n \to \widetilde{k}^+$ denote convergence[21] of the sequence $\{k_n\}_{n \in N}$ in X from the right (i.e., from above). By definition of Ah, we have, for all k_n, $n \in N$, and all $z \in K$:

$$\begin{array}{ll} & \int_Z u_1(w(k_n,z) - Ah(k_n,z), r(h(k_n,z),z')Ah(k_n,z))Q(z,dz') \\ = & \\ & \int_Z u_2(w(k_n,z) - Ah(k_n,z), r(h(k_n,z),z')Ah(k_n,z))r(Ah(k_n,z),z')Q(z,dz') \end{array}$$

By hypothesis, h is increasing and usc and therefore continuous to the right at $\widetilde{k}$, so

$$\lim_{k_n \to \widetilde{k}^+} h(k_n, z) = h(\widetilde{k}, z).$$

[20] It should be noted that order continuity of A in $(L_m, \leq)$ (as well as in $(U_m, \leq)$) does not follow immediately from the order continuity of A in $(G_m, \leq)$ since, for instance, $\wedge_{L_m}\{A^n w\}$ and $\wedge_{G_m}\{A^n w\}$ are generally distinct (however they a.e. coincide).

[21] Since $\{k_n\}_{n \in N}$ is a decreasing sequence and Ah is increasing in k, $\{Ah(k_n, z)\}_{n \in N}$ is a decreasing (and bounded) sequence, and therefore convergent, so the expression $\lim_{k_n \to \widetilde{k}^+} Ah(k_n, z)$ is legitimate.

By continuity of u_1, u_2,and r, letting k_n converges to $\widetilde{k}$ from the right, we have:

$$\int_Z u_1(w(\widetilde{k},z) - Ah(\widetilde{k},z) - \Delta, r(h(\widetilde{k},z),z')(Ah(\widetilde{k},\bar{z}) + \Delta))Q(z,dz')$$

$$=$$

$$\int_Z u_2(w(\widetilde{k},z) - Ah(\widetilde{k},z) - \Delta, r(h(\widetilde{k},z),z')(Ah(\widetilde{k},z) + \Delta))r(Ah(k,z) + \Delta, z')Q(z,dz')$$

But $\Delta \neq 0$ contradicts the uniqueness of the solution to (E′) given $(\widetilde{k},z)$. It must therefore be that $\Delta = 0$, which proves that Ah is right continuous at any $\widetilde{k} \in [0, k_{\max}[$, and thus upper semicontinuous.■

Proposition 17 *Under Assumptions* $1, 2, 2', 3', 4$ *and* 5, $\wedge_{G_m}\{A^n w\}_{n\in\mathbb{N}}$ *is the maximum MEDP in* $(U_m, \leq)$ *and* $\vee_{G_m}\{A^n h_0\}_{n\in\mathbb{N}}$ *is the minimum MEDP in* $(L_m, \leq)$. *As a result, if the MEDP is unique, it is necessarily continuous in* k.

Proof. By the previous lemma A is a self map on $(U_m, \leq)$. By lemma 10, A is increasing on $(H_m, \leq)$, and therefore increasing on $(U_m, \leq)$ (resp, $(L_m, \leq)$. Further since $w \in (U_m, \leq)$ and A is a self map on $(U_m, \leq)$, $A^n w \in (U_m, \leq)$ for all $n \in \mathbb{N}$. Thus $\wedge_{G_m}\{A^n w\}_{n\in\mathbb{N}}$ is the lower envelope of a family of usc functions, and is therefore usc in k on $[0, x_{\max}[$. Since $h_0 \in (L_m, \leq)$ and A is a self map on $(L_m, \leq)$, $A^n h_0 \in (L_m, \leq)$ for all $n \in \mathbb{N}$. Thus $\vee_{G_m}\{A^n h_0\}_{n\in\mathbb{N}}$ is the upper envelope of a family of lsc functions, and is therefore lsc in k on $]0, x_{\max}]$.■

Finally, we note that the maximum MEDP in $(L_m, \leq)$ (minimum MEDP in $(U_m, \leq)$) can easily be constructed by altering the maximum MEDP in $(U_m, \leq)$ (resp. minimum MEDP in $(L_m, \leq)$) at its discontinuity points.

4 Stationary Markov equilibria

In this section, we follow the work of Grandmont and Hildenbrand [25], Futia [27] and Hopenhayn and Prescott [34], and define a stationary Markov equilibrium (SME) in the form of an invariant distribution. We initially postulate the existence of a MEDP h in $(W, \leq)$ which is only assumed to be $\mathcal{B}(S)$-measurable[22], and we only require the transition function Q to be quasi-compact: Under these conditions, we demonstrate that there always exists a SME associated with h. We then discuss the existing literature on existence of SME obtained as fixed points of particular mappings, and show that our result is new. This existence result, however, offers no information on how to compute this SME. Most importantly, it offers no guarantees that the candidate SME is not a *trivial* one (i.e., all the probability mass concentrated at stationary capital stock equal to 0).

[22] We proved in the previous section that finding a MEDP in $(H_m, \leq)$ $(\subset (W, \leq))$ requires no additional assumptions beyond the standard ones.

In light of this possibility, we prove two additional results. First, under the additional combined assumptions of isotonicity of h (i.e., $h \in (G_m, \leq)$) and of Q (i.e., Q is now assumed to be quasi compact and increasing) we show that minimal and maximal SME associated with h can be constructed through successive (monotone) approximations. This is a new finding, since existing computational results rely on a different -although generally more restrictive- set of assumptions on h or Q (or on both). Second, we present sufficient conditions within the class of OLG models that we study under which there exists a non trivial SME associated with the minimal MEDP. We also give sufficient conditions under which the minimal nontrivial SME associated with the minimal MEDP can be constructed through successive approximations. All conditions are expressed as restrictions on the set of primitives of the problem.

4.1 Definition of SME and existence results

Given a $\mathcal{B}(S)$-measurable function h in $(W, \leq)$ and with no restrictions on Q, it is well known (see, for instance, Stokey & al. [51]) that the state vector $s = (x, z) \in S$ follows a first-order Markov process with transition function P_h defined by:

$$\forall A \times B \in \mathcal{B}(S),\ P_h(x, z; A, B) = \{ \begin{array}{c} Q(z, B) \text{ if } h(x, z) \in A \\ 0 \text{ otherwise.} \end{array}$$

Equivalently:

$$P_h(x, z; A, B) = \int_B I_A(h(x, z)) Q(z, dz'),$$

where $I_A(h(x, z))$ is the indicator function of A (i.e., $I_A(h(x, z)) = 1$ if and only if $h(x, z) \in A$, and 0 otherwise). In the notations of Duffie & al. [15] (S, P_h) is a Time-Homogenous Markov Equilibrium (THME), more precisely a nonsunspot THME (see Wang [57] Definition 3.4). Associated with the transition function P_h are the operators $T_h : \mathbf{M}(S, \mathcal{B}(S)) \to \mathbf{M}(S, \mathcal{B}(S))$ defined by:

$$\forall f \in \mathbf{M}(S, \mathcal{B}(S)),\ Tf(s) = \int_S f(s') P_h(s, ds'),$$

and $T_h^* : \Lambda(S, \mathcal{B}(S)) \to \Lambda(S, \mathcal{B}(S))$ defined by:

$$\forall D \in \mathcal{B}(S),\ \mu_{t+1}(D) = T_h^* \mu_{t+1}(D) = \int_S P_h(s; D) \mu_t(ds).$$

The quantity $T_h^* \mu_{t+1}(D)$ is thus the probability that the next period value of the state vector lies in the set D if the current period state vector is drawn according to the probability measure μ_t, and if all agents follow the optimal decision rule h. This leads us to define a (pure) stationary Markov equilibrium[23] (in short, SME) for the MEDP h as follows:

[23]Since this SME is associated with a known law of motion (the MEDP h), we may call it a "pure" SME, in contrast to Duffie & al. [15] and Wang [57] where there is no such information.

Definition 18 *A (pure) SME associated with the MEDP h is a probability measure* $\mu \in \Lambda(S, \mathcal{B}(S))$ *such that:*

$$\text{For all } D \in \mathcal{B}(S),\ \mu(D) = T_h^* \mu(D) = \int_S P_h(s; D)\mu(ds).$$

In light of this definition the study (i.e., the issues of existence, uniqueness, and computation) of SME is identical to that of the fixed points of T_h^* in $\Lambda(S, \mathcal{B}(S))$.[24] We first remind the reader of the two classes of results concerning the existence of SME obtained as fixed points of the operator T_h^* in the theorem immediately below. Recall that the transition function P_h has the Feller property if, $T(\mathbf{C}(S, \mathcal{B}(S))) \subset \mathbf{C}(S, B(S))$. This is equivalent to T_h^* being "weakly continuous", that is, for all sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ in $\Lambda(S, \mathcal{B}(S))$:

$$\lambda_n \Longrightarrow \lambda \text{ implies } T_h^* \lambda_n \Longrightarrow T_h^* \lambda.$$

Theorem 19 *(i). If* $h \in (G_m, \leq)$ *and Q satisfies Assumption* 3*, then there exists a SME associated with h.*

(ii). If P_h *has the Feller property, then there exists a SME associated with h.*

Proof. (i) Both h and Q increasing imply that T_h^* is an increasing self map on $(\Lambda(S, \mathcal{B}(S)), \leq)$ (and S has the minimal element $(0, z_{\min})$), and the result follows from Corollary 2 in HP[34]. (ii) See Stokey & al. ([51]Theorem 12.10). Consider any $f \in \mathbf{C}(S, \mathcal{B}(S))$, any sequence $\{s_n = (x_n, z_n)\}_{n\in\mathbb{N}}$ in S converging to $s = (x, z) \in S$, and $h \in \mathbf{C}(S, \mathcal{B}(S))$. Since $S = [0, x_{\max}] \times [z_{\min}, z_{\max}] \subset \mathbb{R}^2$, $f(h(k, z), z')$ is uniformly continuous on $S \times Z$. As a result, $\forall \varepsilon > 0$, $\exists\ N \in \mathbb{N}$ such that $\forall z' \in Z$ and $\forall n \geq N$:

$$f(h(x, z), z') - \varepsilon \leq f(h(x_n, z_n), z') \leq f(h(x, z), z') + \varepsilon,$$

and by integration:

$$\begin{aligned}\int_Z f(h(x, z), z')Q(z_n, dz') - \varepsilon \quad &\leq \quad \int_Z f(h(x_n, z_n), z')Q(z_n, dz') \\ &\leq \quad \int_Z f(h(x, z), z')Q(z_n, dz') + \varepsilon \text{ for all } n \geq N.\end{aligned}$$

If Q has the Feller property, then for any $g \in \mathbf{C}(S, \mathcal{B}(S))$, $\int_S g(s')Q(s, ds')$ is continuous. In particular,

$$\lim_{n\to\infty} \int_Z f(h(x, z), z')Q(z_n, dz') = \int_Z f(h(x, z), z')Q(z, dz'),$$

[24] Note that in Wang [57] terminology, (S, P_h, μ) is a SME.

and the previous inequalities imply that:

$$\lim_{n\to\infty}\int_Z f(h(x_n,z_n),z')Q(z_n,dz') = \int_Z f(h(x,z),z')Q(z,dz'),$$

which proves that P_h has the Feller property since:

$$\int_Z f(x',z')P_h(x,z';dx*dz) = \int_Z f(h(x,z),z')Q(z,dz').$$

Thus, if h is continuous and Q has the Feller property, then P_h has the Feller property.■

As shown in the theorem above, current arguments establishing the existence of a SME through a fixed-point approach require either the isotonicity of both Q and h, or the Feller property of P_h, the latter generally relying on the continuity of h. Our contribution is to demonstrate the new result that the $\mathcal{B}(S)$-measurability of the MEDP h is sufficient for the existence of a SME without isotonicity or continuity properties of h, but as long as Q is quasi-compact.

Theorem 20 *If Q satisfies Assumption $3'$, then for any $\mathcal{B}(S)$-measurable function h in W there exists a SME associated with h.*

Proof. The first critical step in this proof is to note that Assumption $3'$ on Q implies that P_h satisfies Doeblin's condition (identified as condition (D) in Doob[**?**] or Stokey & al. [51], for instance). Indeed, given any arbitrary $\rho \in \Lambda(X,\mathcal{B}(X))$, consider the probability measure $\gamma' = \rho \otimes \gamma \in \Lambda(S,\mathcal{B}(S))$, where γ is the probability measure in Assumption $3'$. By definition $P_h(x,z;A\times B) \leq Q(z,B)$, and Assumption $3'$ then implies that there exists $\varepsilon > 0$ such that:

$$\begin{aligned}\forall A\times B &\in \mathcal{B}(S),\ \gamma'(A\times B) \leq \gamma(B) \leq \varepsilon \\ &\Rightarrow \quad \forall z \in Z,\ P_h(x,z;A\times B) \leq Q(z,B) \leq 1-\varepsilon,\end{aligned}$$

which is precisely condition (D) for P_h. The second step rests on a theorem in Stokey & al. ([51], Theorem 11.9), which says that if P_h satisfies Doeblin's condition, then for any $\mu_0 \in \Lambda(S,\mathcal{B}(S))$, the sequence $\{\lambda_n = (1/n)\sum_{i=0}^{n-1} T_h^{*i}\mu_0\}_{n\in\mathbb{N}}$ converges in the total variation norm and its limit is an invariant of T_h^*. It is therefore a SME associated with h.■

4.2 OCF property and computation of extremal SME

We turn now to the problem of computing the extremal SME through successive approximations, for which we first state the existing results in a theorem below. This theorem shows that when P_h cannot be shown to have the Feller property, not much can be said about the construction of extremal SME unless P_h satisfies a Monotone Mixing Condition. This is a problem, since we already noticed that the Feller property of P_h often relies on the continuity of the MEDP h, something that is generally difficult to obtain in many problems including the OLG setup of this paper.[25]

We address this problem in the remaining of this subsection of the paper by showing that under the assumption of isotonicity of h and Q, the quasi-compactness of Q (a weaker requirement than that of the Feller property) is sufficient to permit the construction of extremal SME by successive (monotone) approximations. Before establishing this result, we first present the existing computational results in the theorem below.

Theorem 21 *(i). If $h \in (G_m, \leq)$ and Q satisfies Assumption 3, and P_h has the Feller property, then the sequences $\{T_h^{*n}\delta_{(x_{\max},z_{\max})}\}_{n\in\mathbb{N}}$ and $\{T_h^{*n}\delta_{(0,z_{\min})}\}_{n\in\mathbb{N}}$ converge weakly respectively to the maximal and minimal SME associated with h.*

(ii). If $h \in (G_m, \leq)$ and P_h satisfies the Monotone Mixing Condition, then for all μ in $\Lambda(S, \mathcal{B}(S))$, the sequence $\{T_h^{*n}\mu\}_{n\in\mathbb{N}}$ converges weakly to the unique SME associated with h.

Proof: (i). Since $\delta_{(x_{\max},z_{\max})}$ (the probability measure associating mass one to the maximal point of S) is the maximal element of $\Lambda(S, \mathcal{B}(S))$, then, necessarily:

$$T_h^*\delta_{(x_{\max},z_{\max})} \leq_s \delta_{(x_{\max},z_{\max})}.$$

If T_h^* is increasing, then the sequence $\{T_h^{*n}\delta_{(x_{\max},z_{\max})}\}_{n\in\mathbb{N}}$ is decreasing, and therefore weakly convergent (to is glb, which we denote μ). The weak continuity (i.e., the Feller property) of T_h^* then implies that $\mu = T_h^*\mu$. Next, consider any fixed point λ of T_h^*. Since T_h^* is increasing:

$$\lambda \leq \delta_{(x_{\max},z_{\max})} \Rightarrow \lambda = T_h^{*n}\lambda \leq_s T_h^{*n}\delta_{b(x_{\max},z_{\max})} \text{ for all } n \in \mathbb{N},$$

and λ is therefore a lower bound for $\{T_h^{*n}\delta_{(x_{\max},z_{\max})}\}_{n\in\mathbb{N}}$, which implies that $\lambda \leq_s \mu$. Symmetric arguments can be made for the increasing sequence $\{T_h^{*n}\delta_{(0,z_{\min})}\}_{n\in\mathbb{N}}$. (ii) See Hopenhayn and Prescott,[34] Theorem 2.■

It is easy to see in the proof of (i) above that the topological property of weak continuity of the isotone operator T_h^* is not necessary, and that the OCF property of T_h^* is sufficient for the proof. Indeed, recall that T_h^* is OCF (short for "order continuous along monotone recursive T_h^*-sequences"), if for any increasing (decreasing) sequence of the form $C = \{T_h^{*n}\gamma\}_{n\in\mathbb{N}}$ in $\Lambda(S, \mathcal{B}(S))$:

$$\vee_{\Lambda(S,\mathcal{B}(S))}\{T_h^*(C)\} = T_h^*(\vee_{\Lambda(S,\mathcal{B}(S))}C) \text{ (resp. } \wedge_{\Lambda(S,\mathcal{B}(S))}\{T_h^*(C)\} = T_h^*(\wedge_{\Lambda(S,\mathcal{B}(S))}C)),$$

which turns out to be equivalent to:

$$\lim_{n\to\infty} T_h^{*n}\gamma = T_h^*(\lim_{n\to\infty} T_h^{*n}\gamma),$$

[25] There are results giving sufficient conditions for the existence of a continuous selection out of a usc correspondence, but these are of very little use for computational purposes.

for any monotone sequence $\{T_h^{*n}\gamma\}_{n\in\mathbb{N}}$, i.e., the weak limit of the sequence $\{T_h^{*n}\gamma\}_{n\in\mathbb{N}}$ is a fixed point of T_h^*.[26]

We now prove the important result that the OCF property of T_h^* is satisfied without any topological assumption on h**.** This is an important result, and a significant addition to the constructive theorem above, since it implies that the continuity of h is irrelevant for the construction of extremal fixed points of T_h^*.

Proposition 22 *For any function h in $(H_m, \leq)$ and Q satisfying Assumption 3*′*, T_h^* is OCF.*

Proof. Consider an increasing sequence of the form $\{T_h^{*n}\mu_0\}_{n\in\mathbb{N}}$ in $\Lambda(S, \mathcal{B}(S))$. Denoting $\mu = \vee_{\Lambda(S,\mathcal{B}(S))}\{T_h^{*n}\mu_0\}_{n\in\mathbb{N}}$ its weak limit, by definition, for all isotone $f \in \mathbf{C}(S, \mathcal{B}(S))$ and all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that:

$$\forall n \geq N, \ |\langle f, T_h^{*n}\mu_0\rangle - \langle f, \mu\rangle| < \varepsilon$$

Given the sequence $\{\lambda_n = (1/n)\sum_{i=0}^{n-1} T_h^{*i}\mu_0\}_{n\in\mathbb{N}}$, any isotone $f \in \mathbf{C}(S, \mathcal{B}(S))$, and $n > N$:

$$\begin{aligned}
|\langle f, \lambda_n\rangle - \langle f, \mu\rangle| &= \left|(1/n)\left\langle f, \sum_{i=0}^{N-1}(T_h^{*i}\mu_0 - \mu)\right\rangle + (1/n)\left\langle f, \sum_{i=N}^{n-1}(T_h^{*i}\mu_0 - \mu)\right\rangle\right| \\
&\leq (1/n)\left|\left\langle f, \sum_{i=0}^{N-1}(T_h^{*i}\mu_0 - \mu)\right\rangle\right| + (1/n)\sum_{i=N}^{n-1}\left|\langle f, T_h^{*i}\mu_0\rangle - \langle f, \mu\rangle\right| \\
&\leq (1/n)\left|\left\langle f, \sum_{i=0}^{N-1}(T_h^{*i}\mu_0 - \mu)\right\rangle\right| + \varepsilon.
\end{aligned}$$

This clearly implies that:

$$\lim_{n\to\infty} |\langle f, \lambda_n\rangle - \langle f, \mu\rangle| = 0,$$

proving that the sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ weakly converges to μ. On the other hand, since P_h satisfies condition (D) by Theorem 11.9 in Stokey & al [51] the sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ converges in the total variation norm (and therefore weakly converges), and its limit is an invariant measure of T_h^*, which by uniqueness of the limit is necessarily μ. This establishes the order continuity along monotone recursive T_h^*-sequences (a symmetric argument is easily made for decreasing recursive T_h^*-sequences).■

Finally, when h belongs to $(G_m, \leq) \subset (H_m, \leq)$ and Q increasing and quasi-compact, the operator T_h^* is then increasing and also OCF (by the previous proposition since $(G_m, \leq) \subset (H_m, \leq)$). Given that $\Lambda(S, \mathcal{B}(S))$ is a chain complete lattice with minimal and maximal elements, the following result is a direct consequence of our constructive fixed point of Section 2.

[26] Note that, for an isotone operator T_h^* on a countable chain complete Poset, weak continuity implies order continuity along monotone T_h^*-sequences, hence our claim that OCF is less restrictive than weak continuity.

Theorem 23 *If $h \in (G_m, \leq)$ and Q satisfies Assumptions 3 and 3′, the sequences $\{T_h^{*n}\delta_{(x_{\max},z_{\max})}\}_{n\in\mathbb{N}}$ and $\{T_h^{*n}\delta_{(0,z_{\min})}\}_{n\in\mathbb{N}}$ converge weakly respectively to the maximal and minimal SME associated with the MEDP h.*

4.3 Non-trivial SME in stochastic OLG models

As noted before, one problem concerning the existence and computation of extremal SME is the possibility that the set of SME reduces to a single "trivial" probability measure, where by "trivial" we mean a probability measure for which all mass is concentrated at $k = 0$. Indeed for a MEDP h satisfying $h(x, z) < x$ for all $(x, z) \in S^*$, any SME is necessarily trivial, and an obvious case when this happens is if when $\forall(x, z) \in S^*$, $w(x, z) < x$.

We want to find sufficient conditions preventing the existence of a unique but trivial SME, and we want to state these conditions in terms of the primitives of the problem, unlike Galor and Ryder [28] or Wang[56][57]. We show that under Assumption 6 below, there exists a minimal MEDP $h_{\min}$ and there also exists a non-trivial SME associated with this $h_{\min}$. Furthermore, under the assumption of isotonocity of Q, we show that there exists a non-trivial minimal SME associated with the minimal MEDP $h_{\min}$, and we give an algorithm converging monotonically it.

We first make the following technical assumption we shall use in the sequel:

Assumption 6.

1. There exists a right neigborhood $\Delta \subset X$ of 0 such that, for all $k \in \Delta$, $w(k, z_{\min}) \geq k$.
2. The following inequality holds:

$$\begin{aligned} &\lim_{k\to 0^+} u_1(w(k, z_{\min}) - k, r(k, z_{\max})k) \\ < \\ &\lim_{k\to 0^+} u_2(w(k, z_{\min}) - k, r(k, z_{\max})k)r(k, z_{\min}). \end{aligned}$$

Note that with separable utility, assumption 6 becomes:

$$\lim_{k\to 0^+} u'(w(x, z_{\min}) - x) < \lim_{k\to 0^+} v'(r(x, z_{\max})x)$$

We have the following important result.

Proposition 24 *Under Assumptions* $1, 2, 3, 3\prime$ *, and* 6*, there exists a non-trivial pure SME, which we construct as associated with the minimal MEDP in* $(H_m, \leq$ *).*

Proof summary. A complete proof can be found in Appendix B, and we here just present the successive steps of the proof to show the constructive nature of the argument.

Step 1. Under Assumption 6 we show that there exists a function $g_0 \in (G_m, \leq) \subset (H_m, \leq)$ and a right neighborhood $\Theta =]0, k_0[$ of 0 such that:

(i) $g_0(x, z) \geq x$ on Θ,

(ii) $Ag_0(s) > g_0(s)$ for all $s \in S^*$,

(iii) if $0 < x < g_0(s)$, then $Ax > x$.

Step 2. We will now exploit the results established in Step 1 to construct the minimal (non-trivial) MEDP. Since A is isotone on $(H_m, \leq)$ and g_0 belongs to $(H_m, \leq)$ (ii) above implies that A maps the subset $[g_0, w]_m$ of H_m into itself. Given that $([g_0, w]_m, \leq)$ is an order interval of $(H_m, \leq)$, it is also a countable chain complete lattice. Recalling that A is OCF, our fixed point theorem of Section 2 implies the existence non-empty set of MEDP in $([g_0, w], \leq)$, with minimal and maximal elements.

Next, (iii) implies that A cannot have a fixed point in $(H_m, \leq)$ smaller than g_0 other than 0. As a result, $h_{\min} = \vee_{H_m}\{A^n g_0\}_{n\in\mathbb{N}}$ is the minimal strictly positive fixed point of A in $(H_m, \leq)$ and $h_{\min} \geq g_0$, and by (i) and (ii), $h_{\min}(x, z) > x$ at least on the open interval $\Theta =]0, k_0[$.

It is important to note that the function g_0 is in fact an element of $(G_m, \leq)$, but is also constructed to be continuous in z for all x and lower semi continuous in x for all z (see the complete proof in Appendix B). This implies that $h_{\min}$, as the upper envelope of the family $\{A^n g_0\}_{n\in\mathbb{N}}$ of lsc functions in x and in z, is lsc in x and in z. This also implies that, under Assumption 6, $h_{\min}$ is the minimal strictly positive fixed point of A in $(G_m, \leq)$ and $h_{\min} \geq g_0$, and thus the minimal MEDP in $(G_m, \leq)$.

Step 3. We show that the singular measure $\delta = \delta_{(k_0/2, z_{\min})}$ is such that $T^{*n}_{h_{\min}}\delta \geq_s \delta$ for all $n \in \mathbb{N}$. There are now two cases.

1. If Q is not isotone and only quasi compact, then we know the sequence $\{\lambda_n = (1/n)\sum_{i=0}^{n-1} T^{*n}_{h_{\min}}\delta\}_{n\in\mathbb{N}}$ converges to an invariant measure μ, which is non-trivial since $\lambda_n \geq_s \delta$ for all $n \in \mathbb{N}$, and thus $\mu \geq_s \delta$.

2. If we assume that Q is isotone (in addition to being quasi-compact), then $T^*_{h_{\min}}$ is an isotone operator and the sequence $\{T^{*n}_{h_{\min}}\delta\}_{n\in\mathbb{N}}$ is increasing. By the quasi-compactness of Q, $\mu = \vee\{T^{*n}_{h_{\min}}\delta\}_{n\in\mathbb{N}}$ is then a fixed point of T^*_h and a non-trivial SME since $\mu \geq_s \delta$.

It is important to notice that our result concerning the existence of a non-trivial SME thus does not depend on isotonic properties of Q, $h_{\min}$, or $P_{h_{\min}}$, although it obviously holds when Q is an increasing transition function. What we gain with the monotonicity of Q is the possibility of constructing the minimal SME by successive approximations. This is because the sequence $\{T^*_{h_{\min}}\delta\}_{n\in\mathbb{N}}$ is then an increasing sequence, so the quasi compactness of Q guarantees that its lub is a fixed point of $T^*_{h_{\min}}$ and therefore a non-trivial SME. We announce that result in the following proposition.

Proposition 25 *Under Assumption* $1, 2, 3, 3'$, *and* 6, *there exists a non trivial minimal SME associated with the minimal MEDP* $h_{\min}$. *This SME has support in the set* $E = [k_0, x_{\max}] \times Z$ *and can be obtained as* $\vee\{T^{*n}_{h_{\min}}\delta_{(k, z_{\min})}\}_{n\in\mathbb{N}}$ *for any* $k \in \Theta =]0, k_0[$.

Proof. The state space $S = [0, x_{\min}] \times [z_{\min}, z_{\max}]$ can be partitioned in three disjoint sets with different ergodic properties. The three sets are $\{0\} \times Z$, $\Theta \times Z$ (recall that $\Theta =]0, k_0[$ is introduced in Assumption 4 above), and $E = [k_0, x_{\max}] \times Z$.

(i) The set $\{0\} \times Z$ is obviously ergodic since $h(0, z) = 0$ for any MEPD h.

(ii). The set $\Theta \times Z$ is transcient since given any MEDP h and associated transition P_h there is a positive probability of leaving that set and no probability of returning in it. Indeed, consider any $x_0 \in \Theta$. The sequence $\{x_n\}_{n\in\mathbb{N}}$ recursively defined by $x_{n+1} = h_{\min}(x_n, z_{\min})$ for $n \in \mathbb{N}$, is strictly increasing, bounded above by $x_{\max}$, and therefore convergent. If x is its limit, then the lower semicontinuity of $h_{\min}(., z_{\min})$ implies that $h_{\min}(x, z_{\min}) = x$, so that $x > k_0$ necessarily since $h_{\min}(k, z_{\min}) > g_0(k, z_{\min}) > k$ for $k \in]0, k_0]$. This (together with the isotonicity of $h_{\min}$ in z) implies that for any $s \in \Theta \times Z$ there exists $n \in \mathbb{N}$ such that $P^n_{h_{\min}}(s; \Theta \times Z) = 0$, which implies that $P^n_h(s; \Theta \times \times Z) = 0$ for any MEDP h since $h \geq h_{\min}$. Also, for all $(x, z) \in E$ $h_{\min}(x, z) > g_0(x, z) = k_0$, which implies that for all MEDP h:

$$P_h(s; \Theta \times Z) = 0 \text{ for all } s \in E,$$

(and $P_h(0, z; [k_0, x_{\max}] \times Z) = 0$ as well).

(iii). The set $E = [k_0, x_{\max}] \times Z$ is invariant since for all $s \in E$, $h(s) \in [k_0, x_{\max}]$ and therefore:

$$P_h(s; E) = 1.$$

Next, for any given $k \in]0, k_0[$, $\vee\{T^{*n}_{h_{\min}} \delta_{(k, z_{\min})}\}_{n\in\mathbb{N}}$ is a fixed point of $T^*_{h_{\min}}$ whose support belongs to E since $\Theta \times Z$ is transient and $\{0\} \times Z$ is ergodic. Consider any $k' \in \Theta$. Clearly, $\delta_{(k', z_{\min})} \leq_s \vee\{T^{*n}_{h_{\min}} \delta_{(k, z_{\min})}\}_{n\in\mathbb{N}}$ and by isotonicity of $T^*_{h_{\min}}$:

$$\vee\{T^{*n}_{h_{\min}} \delta_{(k', z_{\min})}\}_{n\in\mathbb{N}} \leq_s \vee\{T^{*n}_{h_{\min}} \delta_{(k, z_{\min})}\}_{n\in\mathbb{N}}.$$

Also, $\delta_{(k, z_{\min})} \leq_s \vee\{T^{*n}_{h_{\min}} \delta_{(k', z_{\min})}\}_{n\in\mathbb{N}}$ and therefore:

$$\vee\{T^{*n}_{h_{\min}} \delta_{(k, z_{\min})}\}_{n\in\mathbb{N}} \leq_s \vee\{T^{*n}_{h_{\min}} \delta_{(k', z_{\min})}\}_{n\in\mathbb{N}},$$

which implies that $\vee\{T^{*n}_{h_{\min}} \delta_{(k, z_{\min})}\}_{n\in\mathbb{N}} = \vee\{T^{*n}_{h_{\min}} \delta_{(k', z_{\min})}\}_{n\in\mathbb{N}}$. Finally, consider any other SME λ with support included in E. Since $\delta_{(k, z_{\min})} \leq_s \lambda$ for all $k \in \Theta$, by isotonicity of $T^*_{h_{\min}}$:

$$\vee\{T^{*n}_{h_{\min}} \delta_{(k, z_{\min})}\}_{n\in\mathbb{N}} \leq_s \lambda,$$

which prove that $\vee\{T^{*n}_{h_{\min}} \delta_{(k, z_{\min})}\}_{n\in\mathbb{N}}$ is the smallest SME with support included in E.

4.4 Equilibrium Comparative statics

We now discuss the possibility of generating comparative statics results for the set of SME with respect to the space of economies by way of an example.

Consider two productions functions f and f' such that the associated wage rates are $w \geq w'$. Noting that Ah is increasing in w in equation (E') in Section 3, necessarily for any h, $A(h; F_1) \geq A(h; F_2)$. By a recursive argument, it is easy to see that the extremal fixed points of A are then increasing in w (see also Proposition 2.3 in Heikkilä [32] on fixed point comparative statics results for increasing operators on chain complete posets). This implies that the set of MEDP are increasing in w in the weak induced set order, and that the extremal MEDP are increasing in w in the pointwise order sense (See Topkis, [53], p.38 for a discussion of the weak induced set ordering). We now show that the pointwise ordering of two MEDP implies the ordering of their corresponding Markov operators in the following sense.

Proposition 26 *For any Q satisfying Assumption 3 and $\mathcal{B}(S)-$measurable functions $h' \geq h$ in $(H_m, \leq)$:*

$$\forall \lambda \in \Lambda(S, B(S)),\ T_{h'}^{*n}\lambda \geq_s T_h^{*n}\lambda \text{ for all } n \in \mathcal{N}$$

Proof. We prove the result by a recursive argument. First, for any increasing $f \in \mathbf{C}(S, \mathcal{B}(S))$:

$$h' \geq h \Longrightarrow \forall (k,z) \in S,\ \int f(h'(k,z), z')Q(z, dz') \geq \int f(h(k,z), z')Q(z, dz'),$$

and thus, $\forall \lambda \in \Lambda(S, \mathcal{B}(S))$:

$$\int f(s)T_{h'}^{*}\lambda(ds) = \int\int f(h'(k,z), z')Q(z, dz')\lambda(dk * dz) \geq \int\int f(h(k,z), z')Q(z, dz')\lambda(dk * dz) = \int f(s)T_h^{*}\lambda(ds)$$

so $T_{h'}^{*}\lambda \geq_s T_h^{*}\lambda$. Next, suppose now that $T_{h'}^{*n-1}\lambda \geq_s T_h^{*n-1}\lambda$ for $n \geq 2$.

$$\begin{aligned} \int f(s)T_{h'}^{*n}\lambda(ds) &= \int\int f(h'(k,z), z')Q(z, dz')T_{h'}^{*n-1}\lambda(ds) \\ &\geq \\ & \int\int f(h'(k,z), z')Q(z, dz')T_{h}^{*n-1}\lambda(ds) \\ &\geq \\ \int\int f(h(k,z), z')Q(z, dz')T_{h}^{*n-1}\lambda(ds) &= \int f(s)T_h^{*n}\lambda(ds), \end{aligned}$$

which implies that $T_{h'}^{*n}\lambda \geq_s T_h^{*n}\lambda$.■

A direct application of this previous proposition then implies that the pointwise ordered change in w, which implies a pointwise ordered change in the extremal MEDP, induces a ordered change (in the stochastic order sense) in the extremal SME.

Corollary 27 *The extremal SME are increasing (in the stochastic order sense) in w.*

5 Appendix

5.1 Appendix A

Lemma. Consider any function $g : X \times Z \times Z \to \mathcal{R}$ bounded and $\mathcal{B}(X \times Z \times Z)$-measurable. Then

$$j(k, z) = \int g(k, z, z')Q(z, dz'),$$

is $\mathcal{B}(X \times Z)$-measurable.

Proof. The proof is in three steps.

Step 1. for any $A \times B \times C \in \mathcal{B}(X \times Z \times Z)$, consider the indicator function $\chi_{A\times B\times C}$. We have:

$$j(k, z) = \int \chi_{A\times B\times C}(k, z, z')Q(z, dz') = \left\{ \begin{array}{c} Q(z, C) \text{ if } k \in A \text{ and } z \in B \\ 0 \text{ otherwise} \end{array} \right\},$$

so:

$$\{(k, z),\ j(k, z) \leq a\} = \left\{ \begin{array}{c} \emptyset \text{ is } s \in (X \times Z) \setminus (A \times B) \\ A \times (B \cap \{z, Q(z, C) \leq a\}) \end{array} \right\}.$$

Since $Q(., C)$ is a measurable function for all $C \in \mathcal{B}(Z)$, $A \times (B \cap \{z, Q(z, C) \leq a\}) \in \mathcal{B}(X \times Z)$. Thus, j is $\mathcal{B}(X \times Z)$-measurable.

Step 2. Consider any $B(X \times Z \times Z)$-measurable bounded simple function $\Phi : X \times Z \times Z \to \mathcal{R}$. The standard representation of Φ is:

$$\Phi = \sum_{i=1}^{n} a_i \chi_{A_i\times B_i\times C_i},$$

and:

$$j(k, z) = \int \sum_{i=1}^{n} a_i \chi_{A_i\times B_i\times C_i}(k, z, z')Q(z, dz') = \sum_{i=1}^{n} a_i \int \chi_{A_i\times B_i\times C_i}(k, z, z')Q(z, dz'),$$

which is $\mathcal{B}(X \times Z)$-measurable since the sum of n $\mathcal{B}(X * Z)$-measurable functions.

Step 3. For any $B(X \times Z \times Z)$-measurable bounded function $g : X \times Z \times Z \to \mathcal{R}$, there exists an increasing sequence of $B(X \times Z \times Z)$-measurable bounded simple functions $\{\Phi_n\}$ converging pointwise to g. By the Monotone Convergence Theorem:

$$j(k, z) = \int g(k, z, z')Q(z, dz') = \lim_{n\to\infty} \int \Phi_n(k, z, z')Q(z, dz').$$

This shows that the sequence of $\mathcal{B}(X \times Z)$-measurable functions $\int \Phi_n(k, z, z')Q(z, dz')$ converges pointwise to $j(k, z)$, which therefore is $\mathcal{B}(X \times Z)$-measurable.

5.2 Appendix B

The proof is in two parts. Part 1 establishes the existence of $g_0 : X \times Z \to X$, isotone and $\mathcal{B}(S)$-measurable that is mapped up by the operator A and part 2 shows the existence of a probability measure μ_0 that is mapped up T_h^*, where h is any fixed point of A in the interval $[g_0, w]$.

Part 1. By continuity of all functions in k, the inequality in Assumption 6 implies that there exists of $\Theta =]0, k_0] \subset \Delta \subset X$ such that, $\forall k \in \Theta$:

$$\begin{aligned} & u_1(w(k, z_{\min}) - k, r(k, z_{\max})k) \\ < \; & \\ & u_2(w(k, z_{\min}) - k, r(k, z_{\max})k) r(k, z_{\min}). \end{aligned}$$

Consequently, $\forall k \in \Theta =]0, k_0]$:

$$\begin{aligned} & \int_Z u_1(w(k,z) - k, r(k,z')k) Q(z, dz') \\ \leq \; & \\ & u_1(w(k, z_{\min}) - k, r(k, z_{\max})k) \\ < \; & \\ & u_2(w(k, z_{\min}) - k, r(k, z_{\max})k) r(k, z_{\min}) \\ \leq \; & \\ & \int_Z u_2(w(k,z) - k, r(k,z')k) r(k,z') Q(z, dz'). \end{aligned}$$

Next, consider $g_0 \in (G_m, \leq)$ defined as:

$$g_0(k,z) = \begin{array}{c} 0 \text{ if } k = 0,\ z \in Z \\ k \text{ if } 0 < k \leq k_0,\ z \in Z \\ k_0 \text{ if } k \geq k_0,\ z \in Z \end{array} .$$

Clearly g_0 is isotone in (k, z), continuous in z for all k (since constant in z), and continuous in k for all z and therefore $\mathcal{B}(S)$-measurable (since it is a Caratheodory function). We show now that $Ag_0(k, z) > g_0(k, z)$ for all $(k, z) \in X^* \times Z$. Suppose first that there exists $0 < k \leq k_0$ and $z \in Z$ such that $Ag_0(k, z) \leq g_0(k, z) = k$. Then:

$$\begin{aligned} & \int_Z u_1(w(k,z) - k, r(k,z')k) Q(z, dz') \\ < \; & \\ & \int_Z u_2(w(k,z) - k, r(k,z')k) r(k,z') Q(z, dz') \\ \leq \; & \\ & \int_Z u_2(w(k,z) - Ag_0(k,z), r(k,z') Ag_0(k,z)) r(Ag_0(k,z), z') Q(z, dz') \\ = \; & \\ & \int_Z u_1(w(k,z) - Ag_0(k,z), r(k,z') Ag_0(k,z)) Q(z, dz'). \end{aligned}$$

where the first inequality stems from a result just above, the second inequality from $u_{22} \leq 0$, $u_{12} \geq 0$ and r decreasing in its first argument, and the equality follows from the definition of Ag_0. Thus, we have:

$$\begin{aligned} & \int_Z u_1(w(k,z) - k, r(k,z')k)Q(z,dz') \\ < & \\ & \int_Z u_1(w(k,z) - Ag_0(k,z), r(k,z')Ag_0(k,z))Q(z,dz'). \end{aligned}$$

which is contradicted by the hypothesis that $u_{11} \leq 0$ and $u_{12} \geq 0$. It must therefore be the case that, for all $(k,z) \in]0,k_0] * Z$, $Ag_0(k,z) > g_0(k,z) = k$. Also, for any $k > k_0$ and $z \in Z$:

$$Ag_0(k,z) \geq Ag_0(k_0,z) > g_0(k_0,z) = k_0 = g_0(k,z),$$

since Ag_0 is isotone. Thus, $Ag_0 > g_0$ on $X^* \times Z$, and by Theorem 2 there exists a non-empty set of fixed points of A in the order interval $([g_0,w],\leq) \subset (G_m,\leq)$, and the minimal fixed point of A in $([g_0,w],\leq) \subset (G_m,\leq)$ is $\vee_{G_m}\{A^n g_0\} = h_{\min}$. Recall from Example 3 in Section 2 that the increasing sequence $\{A^n g_0\}_{n\in\mathbb{N}}$ of functions in $(G_m,\leq) \subset (H_m,\leq)$:

$$\vee_{H_m,G_m}\{A^n g_0\}_{n\in\mathbb{N}}(s) = \lim_{n\to\infty} A^n g_0(s).$$

Since g_0 is continuous in k for all z, $A^n g_0$ has the same property for all $n \in \mathbb{N}$, and $h_{\min}$ is therefore lsc in k for all z as the upper envelope of a family of continuous functions.

We next prove that there cannot be a fixed point of A in $(G_m,\leq)$ or in $(H_m,\leq)$ that is smaller than $h_{\min}$ other than 0. For any $(k,z) \in]0,k_0] \times Z$, consider any y such that $0 < y < g_0(k,z) = k$, and suppose that $Ay \leq y$. Since $y < k \leq k_0$ by assumption $w(y,z) \geq y$ and we have:

$$\begin{aligned} & \int_Z u_1(w(y,z) - y, r(y,z')y)Q(z,dz') \\ < & \\ & \int_Z u_2(w(y,z) - y, r(y,z')y)r(y,z')Q(z,dz') \\ \leq & \\ & \int_Z u_2(w(y,z) - Ay, r(y,z')Ay)r(Ay,z')Q(z,dz') \\ = & \\ & \int_Z u_1(w(y,z) - Ay, r(y,z')Ay)Q(z,dz'), \end{aligned}$$

where the first inequality follows from Assumption 6, the second from $Ay \leq y$, and the equality from the definition of Ay. Summarizing, we have:

$$\begin{aligned} &\int_Z u_1(w(y,z)-y, r(y,z')y)Q(z,dz') \\ < \\ &\int_Z u_1(w(y,z)-Ay, r(y,z')Ay)Q(z,dz'), \end{aligned}$$

which contradicts $Ay \leq y$ under the assumption that u_{11} and u_{12}. Thus it must be that $Ay > y$ for all $y \in]0, k_0]$. Thus:

$$\forall s \in S^*,\ 0 < y < g_0(s)(\leq k_0) \Longrightarrow Ay > y,$$

and there cannot be a strictly positive fixed point of A in $(G_m, \leq)$ or in $(H_m, \leq$)that does not belong to the order interval $([g_0, w], \leq)$: $h_{\min}$ is therefore the minimum strictly positive fixed point of A in $(G_m, \leq)$ or in $(H_m, \leq)$.

Part 2. Consider $\mu_0 = \delta_{(k_0/2, z_{\min})}$ which concentrates all the mass at $(k_0/2, z_{\min})$. Since $h_{\min}(k_0/2, z_{\min}) > k_0/2$ the support of $T^*_{h_{\min}}\mu_0$ is included in $]k_0/2, x_{\max}] \times Z$, which implies that $T^*_{h_{\min}}\mu_0 \geq_s \mu_0$ (in fact $T^*_{h_{\min}}\mu_0 >_s \mu_0$). If T^*_h is isotone, then we know that $\vee\{T^{*n}_{h_{\min}}\mu_0\}_{n\in\mathbf{N}}$ is a SME. If T^*_h is not isotone, it is easy to prove recursively that:

$$\text{support of } T^{*n}_{h_{\min}}\mu_0 \text{ is included in }]k_0/2, x_{\max}] \times Z,$$

using the property that $h(x,z) > k_0/2$ for all $(x,z) \in]k_0/2, x_{\max}] * Z$. As a result, $T^{*n}_{h_{\min}}\mu_0 \geq_s \mu_0$ for all $n \in \mathbb{N}$, and $\lambda_n = (1/n)\sum_{i=0}^{n-1} T^{*i}_h \mu_0 \geq_s \mu_0$ Recall that the weak limit μ of the sequence $\{\lambda_n = (1/n)\sum_{i=0}^{n-1} T^{*i}_h \mu_0\}_{n\in\mathbb{N}}$ is a SME so that $\mu \geq \mu_0$ since $\lambda_n \geq_s \mu_0$ for all n.

5.3 Appendix C

First, fix $k \in X^*$. For any $z \in Z$:

$$\begin{aligned} &\lim_{x\to 0^+}\int_Z u_1(w(k,z)-x, r(x,z')x)Q(z,dz') \\ = \\ &\int u_1(w(k,z),0)Q(z,dz') \\ \leq \\ &u_1(w(k,z_{\min}),0). \end{aligned}$$

Thus there exists $\Psi =]0, \overline{x}]$, such that, for all $x \in \Psi$ and all $z \in Z$:

$$\begin{aligned} &\int_Z u_1(w(k,z)-x, r(x,z')x)Q(z,dz') \\ < \\ &2u_1(w(k,z_{\min}),0). \end{aligned} \tag{E0}$$

Secondly, if $\lim_{x\to 0^+} r(x, z_{\max})x = 0$ then for all $k \in X^*$:

$$\lim_{x\to 0^+, x\in]0, w(k,z_{\min})[} u_2(w(k, z_{\min}) - x, r(x, z_{\max})x) = \lim_{x\to 0^+} u_2(w(k, z_{\min}), r(x, z_{\max})x) = +\infty.$$

The expression $u_2(w(k, z_{\min}) - x, r(x, z_{\max})x)$ can therefore be made arbitrarily large in a right neighborhood of 0. Thus there exists $\Omega =]0, \overline{k}]$ with $0 < \overline{k} \leq w(k, z_{\min})$ and $M > 0$ such that, for all $x \in \Omega$,

$$u_2(w(k, z_{\min}) - x, r(x, z_{\max})x) > M.$$

For any $x \in \Omega$:

$$\begin{aligned}
& \int_Z u_2(w(k, z) - x, r(x, z')x)r(x, z')Q(z, dz') \\
\geq \quad & \\
& \int_Z u_2(w(k, z_{\min}) - x, r(x, z_{\max})x)r(x, z')Q(z, dz') \\
\geq \quad & \\
& \int_Z Mr(x, z')Q(z, dz') \\
\geq \quad & \\
& Mr(x, z_{\min})
\end{aligned} \tag{E1}$$

where the first inequality stems from $u_{12} \geq 0$ and u_2 decreasing, and the second from above. This last expression can be made arbitrarily large, independently of z, by choosing x in Ω sufficiently close to 0. That is, it is always possible to choose x, independently of z, sufficiently small in $\Omega \cap \Psi$ so that

$$Mr(x, z_{\min}) \geq 2u_1(w(k, z_{\min}), 0) \tag{E2}$$

Pick such an x and set $\delta_0(k, z) = x$ for all $z \in Z$. Combining (E0), (E1), (E2), $x = \delta_0(k, z)$ necessarily satisfies, for all $z \in Z$, the following inequality:

$$\begin{aligned}
& \int_Z u_2(w(k, z) - x, r(x, z')x)r(x, z')Q(z, dz') \\
> \quad & \\
& \int_Z u_1(w(k, z) - x, r(x, z')x)Q(z, dz')
\end{aligned} \tag{E3}$$

That is, by construction, we have:

$$\begin{aligned}
& \int_Z u_2(w(k, z) - \delta_0(k, z), r(\delta_0(k, z), z')\delta_0(k, z))r(\delta_0(k, z), z')Q(z, dz') \\
> \quad & \\
& \int_Z u_1(w(k, z) - \delta_0(k, z), r(\delta_0(k, z), z')\delta_0(k, z))Q(z, dz')
\end{aligned}$$

By repeating the same operation for each k in X^*, and setting $\delta_0(0,z) = 0$, we thus construct a function $\delta_0 : X \times Z \to X$ constant in z, and therefore increasing in z.

Note that if $k' \geq k$ then $\delta_0(k,z)$ necessarily satisfies:

$$\begin{aligned} & \int_Z u_2(w(k',z) - \delta_0(k,z), r(\delta_0(k,z),z')\delta_0(k,z))r(\delta_0(k,z),z')Q(z,dz') \\ > & \\ & \int_Z u_1(w(k',z) - \delta_0(k,z), r(\delta_0(k,z),z')\delta_0(k,z))Q(z,dz'). \end{aligned}$$

Consequently, $\delta_0(k',z)$ can always be chosen to be at least as great as $\delta_0(k,z)$. In other words, the function $\delta_0 : X \times Z \to X$ can be constructed to be increasing in k.

Note that, by construction, for any $(k,z) \in X^* \times Z$, any $0 < x' < \delta_0(k,z)$ also satisfies (E0), (E1), (E2) and therefore (E3).

In particular, the function $p_0 : X \times Z \to X$ defined as:

$$p_0(k,z) = \min_{k' \geq k} \{\delta_0(k',z)\}.$$

satisfies (E3) for any $(k,z) \in X^* * Z$, is increasing in k for all z, and constant in z for all k (and thus continuous in z for all k). Finally, the function h_0 defined as follows:

$$h_0(k,z) = \{ \begin{array}{c} \sup_{0<k'<k} p_0(k',z) \text{ for } k \in X^*, \; z \in Z \\ 0 \text{ for } k = 0, \, z \in Z \end{array}$$

is smaller than p_0 (and therefore than δ_0, hence it satisfies (E3)), increasing in k for all z, constant in z for all k, and lower semicontinuous in k for any given $z \in Z$.

We now prove that $\forall (k,z) \in X^* \times Z, \; Ah_0(k,z) > h_0(k,z) > 0$. Since $h_0(k,z) > 0$ by construction, suppose then that there exists $k \in X^*$ and $z \in Z$ such that $Ah_0(k,z) \leq h_0(k,z)$. As a result:

$$\begin{aligned} & \int_Z u_1(w(k,z) - h_0(k,z), r(h_0(k,z),z')h_0(k,z))Q(z,dz') \\ < & \\ & \int_Z u_2(w(k,z) - h_0(k,z), r(h_0(k,z),z')h_0(k,z))r(h_0(k,z),z')Q(z,dz') \\ \leq & \\ & \int_Z u_2(w(k,z) - Ah_0(k,z), r(h_0(k,z),z')Ah_0(k,z))r(Ah_0(k,z),z')Q(z,dz') \\ = & \\ & \int_Z u_1(w(k,z) - Ah_0(k,z), r(h_0(k,z),z')Ah_0(k,z))Q(z,dz'). \end{aligned}$$

where the first inequality stems from (E3), the second from the assumptions on the primitives, and the equality follows from the definition of $Ah_0(k,z)$. Summarizing, we have:

$$\int_Z u_1(w(k,z) - h_0(k,z), r(h_0(k,z),z')h_0(k,z))Q(z,dz')$$
$$<$$
$$\int_Z u_1(w(k,z) - Ah_0(k,z), r(h_0(k,z),z')Ah_0(k,z))Q(z,dz').$$

which is contradicted by the hypothesis that $u_{11} \leq 0$ and $u_{12} \geq 0$. Thus, necessarily, $Ah_0(k,z) > h_0(k,z)$ and A maps h_0 strictly up. Finally, from the Remark above, recall that for a given $(k,z) \in X^* * Z$, any x' such that $0 < x' < h_0(k,z) <$ necessarily satisfies:

$$\int_Z u_1(w(k,z) - x, r(x,z')x)Q(z,dz')$$
$$<$$
$$\int_Z u_2(w(k,z) - x, r(x,z')x)r(x,z')Q(z,dz'),$$

and it must therefore be the case that $Ax' > x'$.

6 References

References

[1] Aliprantis, C. and K. Border. 1999. *Infinite Dimensional Analysis: A Hitchhiker's Guide*. Springer Verlag Press.

[2] Barbie, M., M Hagedorn, and A. Kaul. 2000. Dynamic efficiency and Pareto optimality in a stochastic OLG model with production and social security. MS. U. of Bonn.

[3] Barro, Robert J., 1990. Government Spending in a Simple Model of Endogenous Growth, *Journal of Political Economy* 98, 103-125.

[4] Benhabib, J. and R. Farmer. 1994. Indeterminancy and increasing returns. *Journal of Economic Theory,* 63, 19-41.

[5] Benhabib, J. and Y. Wen. 2004. Indeterminacy, aggregate demand, and the real business cycle. *Journal of Monetary Economics,* 51, 503-530.

[6] Birkhoff, G. 1967. *Lattice Theory.* AMS Press.

[7] Blume, L. 1982. New techniques for the study of stochastic equilibrium processes. *Journal of Mathematical Economics, 9,* 61-70.

[8] Boldrin, M. and A. Rustichini. 1994. Growth and indeterminacy in dynamic models with externalities, *Econometrica,* 62, 323-342.

[9] Brock, W. and L. Mirman. 1972. Optimal growth and uncertainty: The discounted case, *Journal of Economic Theory*, 4, 479-513.

[10] Chattopadhyay, S. and P. Gottardi. 1999. Stochastic OLG models, market structure, and optimality. *Journal of Economic Theory,* 89, 21-67.

[11] Coleman, W.J. 1991. Equilibrium in a production economy with an income tax, *Econometrica*, 59, 1091-1104.

[12] Coleman, W. J., II. 2000. Uniqueness of an equilibrium in infinite-horizon economies subject to taxes and externalities, *Journal of Economic Theory* 95, 71-78.

[13] Diamond, P. 1965. National debt in a neoclassical growth model, *American Economic Review*, 55, 1126-1150.

[14] Dugundji, J. and A. Granas. 2003. *Fixed Point Theory*. Springer-Verlag, New York Inc.

[15] Duffie, D., Geanakoplos, J.D., Mas-Colell, A. McLennann, A. 1994. Stationary markov equilibria, *Econometrica* 62, 745-782.

[16] Demange, G. and G. Laroque. 1999. Social security and demographic shocks. *Econometrica,* 67, 527-542.

[17] Demange, G. and G. Laroque. 2000. Social security, optimality, and equilibria in a stochastic OLG economy. *Journal of Public Economic Theory,* 2, 1-23.

[18] Datta, M., Mirman, L., and K. Reffett. 2002. Existence and uniqueness of equilibrium in distorted dynamic economies with capital and labor, *Journal of Economic Theory*, 103, 377-410.

[19] Datta, M., Mirman, L., Morand, O., and K. Reffett. 2002. Monotone methods for Markovian equilibrium in dynamic economies, *Annals of Operations Research*, 114, 117-144.

[20] Datta, M., Mirman, L., Morand, O., and K. Reffett. 2005. Markovian equilibrium in infinite horizon economies with incomplete markets and public policy, *Journal of Mathematical Economics*, forthcoming.

[21] Davey, B. and H. Priestley. 1990. *Introduction to Lattices and Order*. Cambridge Press.

[22] Dechert, W. and K. Yamamoto. Asset valuation and production efficiency in an OLG models with production shocks. *Review of Economic Studies,* 59, 389-405.

[23] Erikson, J., O. Morand, and K. Reffett (2005) Isotone Recursive Methods for Overlapping Generations Models with Stochastic Nonclassical Production. MS.

[24] Geanokoplos, J. and H. Polemarchakis. 1991. Overlapping generations models. in *Handbook of Mathematical Economics, vol 4.* ed: Hildenbrand, Werner, and Sonnenschein. 1899-1960.

[25] Grandmont, J. and W. Hildenbrand. 1974. Stochastic processes of temporary equilibrium. *Journal of Mathematical Economics,* 1, 244-277.

[26] Farmer, R. and J. Guo. 1994. Real business cycles and the animal spirits hypothesis. *Journal of Economic Theory,* 63, 42-72.

[27] Futia, Carl A. 1982. Invariant distributions and the limiting behavior of Markovian economic models, *Econometrica* 50 (2), 377-408.

[28] Galor, O. and Ryder, H. 1992. Existence, uniqueness and stability of equilibrium in an overlapping-generation model with productive capital, *Journal of Economic Theory*, 49, 360-375.

[29] Greenwood, J., and G. Huffman.1995. On the existence of nonoptimal equilibria in dynamic stochastic economies, *Journal of Economic Theory*, 65, 611-623.

[30] Halmos, P. 1950. *Measure Theory,* Van Nostrand Press.

[31] Hausenchild, N. 2002. Capital accumulation in a stochastic overlapping generation model with social security, *Journal of Economic Theory*, 106, 201-216.

[32] Heikkilä, S. 2004. Existence results for operator equations in abstract spaces and an application. *Journal of Mathematical Analysis and Applications,* 262-273.

[33] Heikkilä, S. and H. Salonen 1996. On the existence of extremal stationary distributions of Markov processes. Research Report, Department of Economics, University Turku.

[34] Hopenhayn, H. and E. Prescott. 1992. Stochastic monotonicity and stationary distributions for dynamic economies, *Econometrica*, 60(6), 1387-1406.

[35] Kubler, F. and H. Polemarchakis. 2004. Stationary Markov Equilibria for overlapping generations, *Economic Theory*, 1-22.

[36] Miao, J. and M. Santos. 2005. Existence and computation of Markov equilibria for dynamic non-optimal economies. MS. Arizona State University.

[37] Mirman, L., O. Morand, and K. Reffett. 2004. A Qualitative approach to Markovian equilibrium in infinite horizon economies with capital. MS.

[38] Morand, O. 2005. Computing Stationary Markov Equilibria. MS. University of Connecticut.

[39] Morand, O. and K. Reffett. 2003. Existence and uniqueness of equilibrium in nonoptimal unbounded infinite horizon economies, *Journal of Monetary Economics*, 50, 1351-1373.

[40] Morand, O. and K. Reffett. 2005. Stationary Markov Equilibria in Overlapping generations models with Stochastic Production. MS. Arizona State University.

[41] Nachban, D. 1988. Stochastic equilibria. *Journal of Mathematical Economics,* 17, 69-75.

[42] Neveu, J. 1965. *Mathematical Foundations of the Calculus of Probability.* Holden-Day Press.

[43] Nishimura, K. and J. Stachurski. 2005. Stability of stochastic optimal growth models: a new approach. *Journal of Economic Theory,* 122, 110-118.

[44] Nishimura, K., J. Stachurski, and Rudnicki. 2005. Stochastic growth with nonconvexities. *Journal of Mathematical Economics,* forthcoming.

[45] Okuno, M. and I. Zilcha. 1980. On the efficiency of competitive equilibrium in infinite horizon monetary economies. *Review of Economic Studies,* 47, 797-807.

[46] Reffett, K. 2005. Mixed Monotone Recursive Methods. MS. Arizona State University.

[47] Romer, P. 1986. Increasing Returns and long-run growth, *Journal of Political Economy*, 94, 1002-1037.

[48] Samuelson, P.A. 1958. An exact consumption-loans model of interest with or without the social contrivance of money, *Journal of Political Economy*, 66. 467-482

[49] Santos, Manuel. 2000. The numerical accuracy of numerical solutions using Euler residuals. *Econometrica, 68, 1377-140*.

[50] Stachurski, J. 2003. Stochastic growth with increasing returns: stability and path dependence, *Studies in Nonlinear Dynamics and Econometrics,* 7 (2).

[51] Stokey, N., R.E. Lucas and E. Prescott. 1989. *Recursive methods in economics dynamics*. Harvard Press.

[52] Tarski, A. 1955. A lattice-theoretical fixpoint theorem and its applications, *Pacific Journal of Mathematics*, 5, 285-309.

[53] Topkis, D. 1998. *Supermodularity and Complementarity.* Princeton Press.

[54] Torres 1990.

[55] Veinott, A. 1992. Lattice programming: qualitative optimization and equilibria. MS. Stanford.

[56] Wang, Y. 1993. Stationary equilibria in an overlapping generations economy with stochastic production, *Journal of Economic Theory*, 61, 423-435.

[57] Wang, Y. 1994. Stationary Markov equilibrium in an OLG model with correlated shocks, *International Economic Review*, 35(3), 731-744.

[58] Zilcha, I. 1990. Dynamic efficiency in OLG models stochastic production. *Journal of Economic Theory,* 52, 364-379.