April 2007

DeMoivre's Theorem

Carl W. David
University of Connecticut, Carl.David@uconn.edu

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C. W. David
Department of Chemistry
University of Connecticut
Storrs, Connecticut 06269-3060
(Dated: April 16, 2007)

I. SYNOPSIS

DeMoivre's theorem is of great utility in some parts of physical chemistry, and is re-introduced here.

II. INTRODUCTION

DeMoivre's Theorem (sometimes also called Euler's) is a lynchpin for carrying out integration over sines and cosines, and worth the effort of mastering.

Once learned, the need for remembering formula for the double angles, half angles, sums (or differences) of angles, etc., will vanish.

Take the time, learn the Theorem, and you will be a better person.

III. DERIVATION

We start with complex numbers, of the form

$$z = x + \imath y$$

which has an Argand diagram which looks like Figure 1

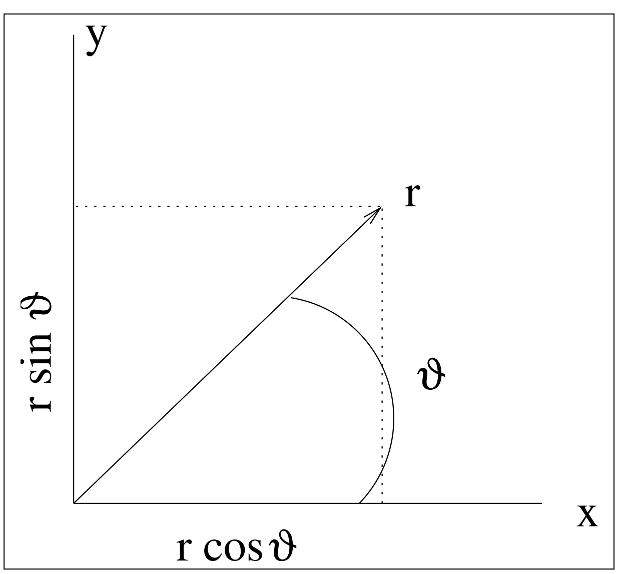

FIG. 1: The Argand diagram. $x = r\cos\vartheta$ and $y = r\sin\vartheta$ come from elementary trigonometry.

We know that $x = r\cos\vartheta$ and $y\ r\sin\vartheta$ by elementary trigonomentry, and that the inverse relations obtain

$$r = \sqrt{x^2+y^2} \qquad \vartheta = \arctan\frac{y}{x}$$

If we expand $e^{\imath\vartheta}$ in a Taylor Series one has

$$e^{\imath\vartheta} = 1 + \imath\vartheta + \frac{1}{2!}(\imath\vartheta)^2 + \frac{1}{3!}(\imath\vartheta)^3 + \frac{1}{4!}(\imath\vartheta)^4 + \cdots$$

which results in

$$e^{\imath\vartheta} = 1 + \imath\vartheta - \frac{1}{2!}(\vartheta)^2 - \frac{1}{3!}\imath(\vartheta)^3 + \frac{1}{4!}(\vartheta)^4 + \cdots$$

where we form two alternating series, one imaginary, the other real. The $\sqrt{-1}$ can be factored from the imaginary series, so that we finally obtain (recognizing the Taylor expansions of sine and cosine):

$$e^{\imath\vartheta} = \cos\vartheta + \imath\sin\vartheta$$

which is DeMoivre's (Euler's) Theorem.

IV. USAGE

$$e^{\imath\vartheta} = \cos\vartheta + \imath\sin\vartheta \tag{4.1}$$

becomes, upon substituting $-\varphi$ for ϑ

$$e^{-\imath\varphi} = \cos\varphi + \imath\sin -\varphi \tag{4.2}$$

or,

$$e^{-\imath\vartheta} = \cos\vartheta - \imath\sin\vartheta \tag{4.3}$$

since it doesn't matter what letter we have on each side of the equation, and the sine is an odd function (cosine is even).

A. Adding, the Cosine

Adding the two equations we have

$$e^{\imath\vartheta} + e^{-\imath\vartheta} = \cos\vartheta - \imath\sin\vartheta + \cos\vartheta + \imath\sin\vartheta = 2\cos\vartheta$$

so

$$\cos\vartheta = \frac{e^{\imath\vartheta} + e^{-\imath\vartheta}}{2} \tag{4.4}$$

Typeset by REVTEX

B. Subtracting, the Sine

$$\sin\vartheta = \frac{e^{\imath\vartheta} - e^{-\imath\vartheta}}{2} \tag{4.5}$$

C. Double Angle Formula

Given

$$e^{+\imath\vartheta} = \cos\vartheta + \imath\sin\vartheta \qquad e^{+\imath\vartheta} = \cos\vartheta + \imath\sin\vartheta$$

the product of these two is

$$e^{+\imath\vartheta}e^{+\imath\vartheta} = (\cos\vartheta + \imath\sin\vartheta)(\cos\vartheta + \imath\sin\vartheta) = \cos^2\vartheta - \sin^2\vartheta + \imath(2\sin\vartheta\cos\vartheta)$$

But

$$e^{+\imath\vartheta}e^{+\imath\vartheta} = e^{\imath(\vartheta+\vartheta)} = e^{\imath(2\vartheta)} = (\cos 2\vartheta + \imath\sin 2\vartheta) = \cos^2\vartheta - \sin^2\vartheta + \imath(2\sin\vartheta\cos\vartheta)$$

which implies two equalities, one for the real part, and one for the imaginary part of these two equalities:

$$\cos 2\vartheta = \cos^2\vartheta - \sin^2\vartheta$$

and

$$\sin 2\vartheta = 2\sin\vartheta\cos\vartheta$$

.

V. THE SUM OF TWO ANGLES FORMULA

Given

$$e^{+\imath\vartheta} = \cos\vartheta + \imath\sin\vartheta \qquad e^{+\imath\varphi} = \cos\varphi + \imath\sin\varphi$$

the product of these two is

$$e^{+\imath\vartheta}e^{+\imath\varphi} = (\cos\vartheta + \imath\sin\vartheta)(\cos\varphi + \imath\sin\varphi) = \cos\vartheta\cos\varphi - \sin\vartheta\sin\varphi + \imath\left(\cos\vartheta\sin\varphi + \sin\vartheta\cos\varphi\right)$$

but

$$e^{\imath(\vartheta+\varphi)} = \cos(\vartheta+\varphi) + \imath\sin(\vartheta+\varphi)$$

implying (using the fact the the Real part of the l.h.s. equals the Real part of the r.h.s.,

$$\cos(\vartheta+\varphi) = \cos\vartheta\cos\varphi - \sin\vartheta\sin\varphi$$

and likewise for the Imaginary part, i.e.,

$$\sin(\vartheta+\varphi) = \cos\vartheta\sin\varphi + \sin\vartheta\cos\varphi$$

VI. OTHER FORMULA

It is clear that one can play many games using DeMoivre's (Euler's) Theorem, when it comes to needing a specialized formula for a special reason. As an example, we are often required to integrate something like

$$\int \sin^2\vartheta d\vartheta$$

or

$$\int \sin\vartheta\cos\vartheta d\vartheta$$

(although the latter integral is trivial [1]) over a domain. These integrals are often handled by replacing the sines and cosines by exponetials using you know what, and expanding the polynomial, regrouping, and then either integrating directly, or recovering simple sines and cosines and then integrating, whichever seems simpler and/or more straightforward.

VII. AN EXAMPLE

Consider the integral

$$\int_{-\pi}^{\pi} \sin^2 nx dx = ? \tag{7.1}$$

$$= \int_{-\pi}^{\pi} \left(\frac{e^{\imath nx} - e^{-\imath nx}}{2\imath} \right)^2 dx$$

$$= -\frac{1}{4} \int_{-\pi}^{\pi} \left(e^{2\imath nx} + e^{-2\imath nx} - 2 \right) dx$$

$$= -\frac{1}{4} \int_{-\pi}^{\pi} \left(2 \cos 2nx - 2 \right) dx$$

$$= -\frac{1}{4} \left(2 \frac{\sin 2nx}{2n} - 2x \right) \Bigg|_{-\pi}^{\pi} dx$$

which is

$$-\frac{2}{4} (-\pi - (-(-\pi))) = \pi$$

[1] $\int udu$