

August 2006

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Carl W. David

University of Connecticut, Carl.David@uconn.edu

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Recommended Citation

David, Carl W., "Alternative Formulations for Angular Momentum Operators, Cartesian and Spherical Polar Forms" (2006). *Chemistry Education Materials*. 18.

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Alternative Formulations for Angular Momentum Operators, Cartesian and Spherical Polar Forms

C. W. David
Department of Chemistry
University of Connecticut
Storrs, Connecticut 06269-3060
 (Dated: August 1, 2006)

I. CARTESIAN AND SPHERICAL POLAR FORMS

$$\begin{aligned} y &= r \sin \vartheta \sin \varphi \\ z &= r \cos \vartheta \end{aligned}$$

It is of value to inspect the angular momentum operator in terms of angles rather than Cartesian coördinates. Remember that

$$x = r \sin \vartheta \cos \varphi \quad \text{and}$$

$$\vec{L} = \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{pmatrix} = \hat{i}(yp_z - zp_y) + \hat{j}(zp_x - xp_z) + \hat{k}(xp_y - yp_x) \quad (1.1)$$

We start with a feast of partial derivatives:

$$\left(\frac{\partial r}{\partial x}\right)_{y,z} = \sin \vartheta \cos \varphi \quad (1.2)$$

$$\left(\frac{\partial r}{\partial y}\right)_{x,z} = \sin \vartheta \sin \varphi \quad (1.3)$$

$$\left(\frac{\partial r}{\partial z}\right)_{x,y} = \cos \vartheta \quad (1.4)$$

$$\left(\frac{\partial \vartheta}{\partial x}\right)_{y,z} = \frac{\cos \vartheta \cos \varphi}{r} \quad (1.5)$$

$$\left(\frac{\partial \vartheta}{\partial y}\right)_{x,z} = \frac{\cos \vartheta \sin \varphi}{r} \quad (1.6)$$

$$\left(\frac{\partial \vartheta}{\partial z}\right)_{x,y} = -\frac{\sin \vartheta}{r} \quad (1.7)$$

and finally

$$\left(\frac{\partial \varphi}{\partial x}\right)_{y,z} = -\frac{\sin \varphi}{r \sin \vartheta} \quad (1.8)$$

$$\left(\frac{\partial \varphi}{\partial y}\right)_{x,z} = \frac{\cos \varphi}{r \sin \vartheta} \quad (1.9)$$

$$\left(\frac{\partial \varphi}{\partial z}\right)_{x,y} = 0 \quad (1.10)$$

which we employ on the defined x-component of the angular momentum, thus

$$L_x \equiv yp_z - zp_y = -i\hbar \sin \vartheta \sin \varphi \frac{\partial}{\partial z} - \left(-i\hbar \cos \vartheta \frac{\partial}{\partial y}\right)$$

where

$$\frac{\partial}{\partial z} = \left(\frac{\partial r}{\partial z}\right)_{x,y} \frac{\partial}{\partial r} + \left(\frac{\partial \vartheta}{\partial z}\right)_{x,y} \frac{\partial}{\partial \vartheta} + \left(\frac{\partial \varphi}{\partial z}\right)_{x,y} \frac{\partial}{\partial \varphi}$$

$$\frac{\partial}{\partial z} = (\cos \vartheta) \frac{\partial}{\partial r} + \left(\frac{\sin \vartheta}{r}\right) \frac{\partial}{\partial \vartheta} + (0) \frac{\partial}{\partial \varphi} \quad (1.11)$$

or

$$\frac{\partial}{\partial z} = \cos \vartheta \frac{\partial}{\partial r} + \left(\frac{\sin \vartheta}{r}\right) \frac{\partial}{\partial \vartheta} \quad (1.12)$$

and

$$\frac{\partial}{\partial y} = \left(\frac{\partial r}{\partial y}\right)_{x,z} \frac{\partial}{\partial r} + \left(\frac{\partial \vartheta}{\partial y}\right)_{x,z} \frac{\partial}{\partial \vartheta} + \left(\frac{\partial \varphi}{\partial y}\right)_{x,z} \frac{\partial}{\partial \varphi}$$

$$\frac{\partial}{\partial y} = (\sin \vartheta \sin \varphi) \frac{\partial}{\partial r} + \left(\frac{\cos \vartheta \sin \varphi}{r}\right) \frac{\partial}{\partial \vartheta} + \left(\frac{\cos \varphi}{r \sin \vartheta}\right) \frac{\partial}{\partial \varphi} \quad (1.13)$$

and

$$\frac{\partial}{\partial x} = \left(\frac{\partial r}{\partial x} \right)_{y,z} \frac{\partial}{\partial r} + \left(\frac{\partial \vartheta}{\partial x} \right)_{y,z} \frac{\partial}{\partial \vartheta} + \left(\frac{\partial \varphi}{\partial x} \right)_{y,z} \frac{\partial}{\partial \varphi}$$

$$\frac{\partial}{\partial x} = (\sin \vartheta \cos \varphi) \frac{\partial}{\partial r} + \left(\frac{\cos \vartheta \cos \varphi}{r} \right) \frac{\partial}{\partial \vartheta} + \left(-\frac{\sin \varphi}{r \sin \vartheta} \right) \frac{\partial}{\partial \varphi} \quad (1.14)$$

so,

$$\begin{aligned} L_x \equiv yp_z - zp_y = & -i\hbar r \sin \vartheta \sin \varphi \left((\cos \vartheta) \frac{\partial}{\partial r} + \left(-\frac{\sin \vartheta}{r} \right) \frac{\partial}{\partial \vartheta} \right) \\ & - \left(-i\hbar r \cos \vartheta \left((\sin \vartheta \sin \varphi) \frac{\partial}{\partial r} + \left(\frac{\cos \vartheta \sin \varphi}{r} \right) \frac{\partial}{\partial \vartheta} + \left(\frac{\cos \varphi}{r \sin \vartheta} \right) \frac{\partial}{\partial \varphi} \right) \right) \end{aligned} \quad (1.15)$$

At constant r, the partial with respect to r loses meaning, and one has

$$\begin{aligned} \frac{L_x}{-i\hbar} = & r \sin \vartheta \sin \varphi \left(+ \left(-\frac{\sin \vartheta}{r} \right) \frac{\partial}{\partial \vartheta} \right) \\ & - r \cos \vartheta \left(\left(\frac{\cos \vartheta \sin \varphi}{r} \right) \frac{\partial}{\partial \vartheta} + \left(\frac{\cos \varphi}{r \sin \vartheta} \right) \frac{\partial}{\partial \varphi} \right) \end{aligned} \quad (1.16)$$

which leads to (combining the ϑ partial derivative terms and cancelling the r terms)

$$L_x = -i\hbar \left(\sin \varphi \frac{\partial}{\partial \vartheta} + \frac{\cos \vartheta \cos \varphi}{\sin \vartheta} \frac{\partial}{\partial \varphi} \right) \quad (1.17)$$

and

$$\begin{aligned} L_y \equiv zp_x - xp_z = & -i\hbar r \cos \vartheta \left(\sin \vartheta \cos \varphi \frac{\partial}{\partial r} + \left(\frac{\cos \vartheta \cos \varphi}{r} \right) \frac{\partial}{\partial \vartheta} + \left(-\frac{\sin \varphi}{r \sin \vartheta} \right) \frac{\partial}{\partial \varphi} \right) \\ & - i\hbar r \sin \vartheta \cos \varphi \left(\cos \vartheta \frac{\partial}{\partial r} + \left(\frac{\sin \vartheta}{r} \right) \frac{\partial}{\partial \vartheta} + (0) \frac{\partial}{\partial \varphi} \right) \end{aligned} \quad (1.18)$$

which becomes

$$\begin{aligned} L_y = & -i\hbar \left(\left(\frac{r \cos^2 \vartheta \cos \varphi}{r} \right) \frac{\partial}{\partial \vartheta} + \left(-\frac{r \cos \vartheta \sin \varphi}{r \sin \vartheta} \right) \frac{\partial}{\partial \varphi} \right) \\ & - i\hbar \left(\left(\frac{r \sin^2 \vartheta \cos \varphi}{r} \right) \frac{\partial}{\partial \vartheta} \right) \end{aligned} \quad (1.19)$$

so that

$$L_y = -i\hbar \left(\cos \varphi \frac{\partial}{\partial \vartheta} - \frac{\cos \vartheta \sin \varphi}{\sin \vartheta} \frac{\partial}{\partial \varphi} \right) \quad (1.20)$$

which is

$$-i\hbar \left(\sin \vartheta \cos \varphi \frac{\partial}{\partial y} - \sin \vartheta \sin \varphi \frac{\partial}{\partial x} \right) \quad (1.21)$$

Finally, for L_z we have

$$l_z = xp_y - yp_x$$

which becomes

$$\begin{aligned} & -i\hbar r \left(\sin \vartheta \cos \varphi (\sin \vartheta \sin \varphi) \frac{\partial}{\partial r} + \left(\frac{\cos \vartheta \sin \varphi}{r} \right) \frac{\partial}{\partial \vartheta} + \left(\frac{\cos \varphi}{r \sin \vartheta} \right) \frac{\partial}{\partial \varphi} \right) \\ & - \sin \vartheta \sin \varphi (\sin \vartheta \cos \varphi) \frac{\partial}{\partial r} + \left(\frac{\cos \vartheta \cos \varphi}{r} \right) \frac{\partial}{\partial \vartheta} + \left(-\frac{\sin \varphi}{r \sin \vartheta} \right) \frac{\partial}{\partial \varphi} \end{aligned} \quad (1.22)$$

$$L_z = -i\hbar \frac{\partial}{\partial \varphi} \quad (1.23)$$

II.

We need to form L^2 , i.e.,

$$L^2 = L_x \cdot L_x + L_y \cdot L_y + L_z \cdot L_z \quad (2.1)$$

$$L_x^2 = -\hbar^2 \left(\sin \varphi \frac{\partial}{\partial \vartheta} + \frac{\cos \vartheta \cos \varphi}{\sin \vartheta} \frac{\partial}{\partial \varphi} \right) \left(\sin \varphi \frac{\partial}{\partial \vartheta} + \frac{\cos \vartheta \cos \varphi}{\sin \vartheta} \frac{\partial}{\partial \varphi} \right) \quad (2.2)$$

$$L_y^2 = -\hbar^2 \left(\cos \varphi \frac{\partial}{\partial \vartheta} - \frac{\cos \vartheta \sin \varphi}{\sin \vartheta} \frac{\partial}{\partial \varphi} \right) \left(\cos \varphi \frac{\partial}{\partial \vartheta} - \frac{\cos \vartheta \sin \varphi}{\sin \vartheta} \frac{\partial}{\partial \varphi} \right) \quad (2.3)$$

$$L_z^2 = -\hbar^2 \frac{\partial^2}{\partial \varphi^2} \quad (2.4)$$

For L_x^2 we have, upon expansion

$$\begin{aligned} \frac{L_x^2}{-\hbar^2} &= \sin \varphi \frac{\partial \sin \varphi}{\partial \vartheta} \frac{\partial}{\partial \vartheta} \Rightarrow \sin^2 \varphi \frac{\partial^2}{\partial \vartheta^2} \\ &+ \frac{\cos \vartheta \cos \varphi}{\sin \vartheta} \frac{\partial \sin \varphi}{\partial \vartheta} \frac{\partial}{\partial \varphi} \Rightarrow + \cos^2 \varphi \cot \vartheta \frac{\partial}{\partial \vartheta} + \cot \vartheta \cos \varphi \sin \varphi \frac{\partial^2}{\partial \vartheta \partial \varphi} \\ \sin \varphi \frac{\partial \frac{\cos \vartheta \cos \varphi}{\sin \vartheta} \frac{\partial}{\partial \varphi}}{\partial \vartheta} &\Rightarrow - \sin \varphi \cos \varphi \frac{\partial}{\partial \varphi} - \frac{\cos^2 \vartheta \cos \varphi \sin \varphi}{\sin^2 \vartheta} \frac{\partial}{\partial \varphi} + \frac{\cos \vartheta \cos \varphi \sin \varphi}{\sin \vartheta} \frac{\partial^2}{\partial \varphi \partial \vartheta} \\ &+ \frac{\cos \vartheta \cos \varphi}{\sin \vartheta} \frac{\partial \frac{\cos \vartheta \cos \varphi}{\sin \vartheta} \frac{\partial}{\partial \varphi}}{\partial \varphi} \Rightarrow - \frac{\cos^2 \vartheta \cos \varphi \sin \varphi}{\sin^2 \vartheta} \frac{\partial}{\partial \varphi} + \frac{\cos^2 \vartheta \cos^2 \varphi}{\sin^2 \vartheta} \frac{\partial^2}{\partial \varphi^2} \end{aligned} \quad (2.5)$$

For L_y we have

$$\begin{aligned} \frac{L_y^2}{-\hbar^2} &= \cos \varphi \frac{\partial \cos \varphi}{\partial \vartheta} \frac{\partial}{\partial \vartheta} \Rightarrow \cos^2 \varphi \frac{\partial^2}{\partial \vartheta^2} \\ &- \frac{\cos \vartheta \sin \varphi}{\sin \vartheta} \frac{\partial \cos \varphi}{\partial \vartheta} \frac{\partial}{\partial \varphi} \Rightarrow + \cot \vartheta \sin^2 \varphi \frac{\partial}{\partial \vartheta} - \frac{\cos \vartheta \sin \varphi}{\sin \vartheta} \cos \varphi \frac{\partial^2}{\partial \vartheta \partial \varphi} \\ - \cos \varphi \frac{\partial \frac{\cos \vartheta \sin \varphi}{\sin \vartheta} \frac{\partial}{\partial \varphi}}{\partial \vartheta} &\Rightarrow + \cos \varphi \sin \varphi \frac{\partial}{\partial \varphi} + \cos \varphi \frac{\cos^2 \vartheta \sin \varphi}{\sin^2 \vartheta} \frac{\partial}{\partial \varphi} - \cos \varphi \frac{\cos \vartheta \sin \varphi}{\sin \vartheta} \frac{\partial^2}{\partial \varphi \partial \vartheta} \\ &+ \frac{\cos \vartheta \sin \varphi}{\sin \vartheta} \frac{\partial \frac{\cos \vartheta \sin \varphi}{\sin \vartheta} \frac{\partial}{\partial \varphi}}{\partial \varphi} \Rightarrow + \frac{\cos^2 \vartheta \cos \varphi \sin \varphi}{\sin^2 \vartheta} \frac{\partial}{\partial \varphi} + \frac{\cos^2 \vartheta \sin^2 \varphi}{\sin^2 \vartheta} \frac{\partial^2}{\partial \varphi^2} \end{aligned} \quad (2.6)$$

and, of course

$$\frac{L_y^2}{-\hbar^2} = \frac{\partial^2}{\partial \varphi^2} \quad (2.7)$$

It follows, adding Equations 2.5, 2.6 and 2.7, that

$$\sin^2 \varphi \frac{\partial^2}{\partial \vartheta^2}$$

$$\begin{aligned}
& + \cos^2 \varphi \cot \vartheta \frac{\partial}{\partial \vartheta} + \cot \vartheta \cos \varphi \sin \varphi \frac{\partial^2}{\partial \vartheta \partial \varphi} \\
& - \sin \varphi \cos \varphi \frac{\partial}{\partial \varphi} - \frac{\cos^2 \vartheta \cos \varphi \sin \varphi}{\sin^2 \vartheta} \frac{\partial}{\partial \varphi} + \frac{\cos \vartheta \cos \varphi \sin \varphi}{\sin \vartheta} \frac{\partial^2}{\partial \varphi \partial \vartheta} \\
& \quad - \frac{\cos^2 \vartheta \cos \varphi \sin \varphi}{\sin^2 \vartheta} \frac{\partial}{\partial \varphi} \\
& \quad + \frac{\cos^2 \vartheta \cos^2 \varphi}{\sin^2 \vartheta} \frac{\partial^2}{\partial \varphi^2} \\
& \quad + \cos^2 \varphi \frac{\partial}{\partial \vartheta} \\
& + \cot \vartheta \sin^2 \varphi \frac{\partial}{\partial \vartheta} - \frac{\cos \vartheta \sin \varphi}{\sin \vartheta} \cos \varphi \frac{\partial^2}{\partial \vartheta \partial \varphi} \\
& + \cos \varphi \sin \varphi \frac{\partial}{\partial \varphi} + \cos \varphi \cot^2 \vartheta \sin \varphi \frac{\partial}{\partial \varphi} - \cos \varphi \cot \vartheta \sin \varphi \frac{\partial^2}{\partial \varphi \partial \vartheta} \\
& + \frac{\cos^2 \vartheta \cos \varphi \sin \varphi}{\sin^2 \vartheta} \frac{\partial}{\partial \varphi} + \frac{\cos^2 \vartheta \sin^2 \varphi}{\sin^2 \vartheta} \frac{\partial^2}{\partial \varphi^2} \\
& \quad + \frac{\partial^2}{\partial \varphi^2}
\end{aligned} \tag{2.8}$$

which, gathering terms for clarity becomes

$$\begin{aligned}
& \left\{ \sin^2 \varphi + \cos^2 \varphi \right\} \frac{\partial^2}{\partial \vartheta^2} \\
& + \left\{ \cos^2 \varphi \cot \vartheta + \cot \vartheta \sin^2 \varphi \right\} \frac{\partial}{\partial \vartheta} \\
& + \left\{ -2 \frac{\cos^2 \vartheta \cos \varphi \sin \varphi}{\sin^2 \vartheta} + \cos \varphi \sin \varphi + 2 \frac{\cos^2 \vartheta \cos \varphi \sin \varphi}{\sin^2 \vartheta} - \sin \varphi \cos \varphi \right\} \frac{\partial}{\partial \varphi} \\
& + \left\{ \cot \vartheta \cos \varphi \sin \varphi + \frac{\cos \vartheta \cos \varphi \sin \varphi}{\sin \vartheta} - \frac{\cos \vartheta \sin \varphi}{\sin \vartheta} \cos \varphi - \cos \varphi \cot \vartheta \sin \varphi \right\} \frac{\partial^2}{\partial \varphi \partial \vartheta} \\
& + \left\{ \frac{\cos^2 \vartheta \cos^2 \varphi}{\sin^2 \vartheta} + \frac{\cos^2 \vartheta \sin^2 \varphi}{\sin^2 \vartheta} + 1 \right\} \frac{\partial^2}{\partial \varphi^2}
\end{aligned} \tag{2.9}$$

and finally

$$\begin{aligned}
& \frac{\partial^2}{\partial \vartheta^2} \\
& \cot \vartheta \frac{\partial}{\partial \vartheta} \\
& \text{zero} \frac{\partial}{\partial \varphi} \\
& \text{zero} \frac{\partial^2}{\partial \varphi \partial \vartheta} \\
& + \left\{ \frac{\cos^2 \vartheta + \sin^2 \vartheta}{\sin^2 \vartheta} + 1 \right\} \frac{\partial^2}{\partial \varphi^2}
\end{aligned} \tag{2.10}$$

which is, ultimately

$$L^2 = -\hbar^2 \left(\frac{1}{\sin^2 \vartheta} \left[\sin \vartheta \frac{\partial \sin \vartheta}{\partial \vartheta} \frac{\partial}{\partial \vartheta} + \frac{\partial^2}{\partial \varphi^2} \right] \right) \tag{2.11}$$

We see that the operator associated with Legendre's Equation (multiplied by a constant) has emerged, meaning that Legendre Polynomials and angular momentum are intimately associated together.