Functionally graded materials (FGMs) are non-homogenous and tailored to have a spatial variation of properties. The gradual modification of material properties is quite effective in reducing stresses. Finite element analysis of nonhomogeneous materials can be performed using an assemblage of either graded or homogeneous elements. A graded finite element samples the material property at more than one integration points, while a homogeneous element has constant property at all integration points based on property of the element at the centroid. In this dissertation, a six-node incompatible graded finite element is developed.

This research aims to show significance of six-node incompatible (QM6) element over four-node compatible (Q4) graded elements in terms of accuracy of the results and computation time. The numerical solution is obtained using UMAT capability of the ABAQUS software. The results are compared with the exact solution (e.g. stress due to far field tension loads for graded infinite plates). Incompatible graded element is shown to give better performance in terms of accuracy over Q4 element and computationally efficient than an eight-node compatible (Q8) element in two-dimensional plane elasticity. Thus six-node incompatible (QM6) is recommended for modelling FGMs.

Furthermore, dynamic loading characteristics of the shock tube onto sandwich steel beams as an efficient and accurate alternative to time consuming and complicated fluid structure interaction using finite element modelling is introduced. Improved accuracy of 3D dynamic analysis using
eight node incompatible brick elements (C3D8I) is demonstrated through this dynamic analysis example and results are compared to lower-order compatible brick elements (C3D8).

Keywords: Functionally graded material, Grade finite elements, Quadrilateral elements, Incompatible elements, Isotropically graded, Orthotropically graded, Dynamic analysis, Corrugated core, Sandwich beams, Russell error
Development and Application of Incompatible Graded Finite Elements for Analysis of Nonhomogeneous Materials

Asmita Rokaya

B.E., Civil Engineering, Tribhuvan University, Nepal, 2014

A Dissertation Submitted in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy At the University of Connecticut 2020
APPROVAL PAGE

Doctor of Philosophy Dissertation

Development and Application of Incompatible Graded Finite Elements for Analysis of Nonhomogeneous Materials

Presented by

Asmita Rokaya, B.E.

Major Advisor

Dr. Jeongho Kim

Associate Advisor

Dr. Ramesh Malla

Associate Advisor

Dr. Wei Zhang

University of Connecticut

2020
ACKNOWLEDGEMENTS

First and most of all, I am grateful to my advisor Dr. Jeongho Kim, from whom I received invaluable support and assistance. His expertise in formulating the research topic and methodology gave my thesis direction and the form it is in. He has provided immense insight into this research. I am also thankful for the support he provided during my PhD program.

I would like to extend my appreciation to my associate advisor Dr. Ramesh Malla, who gave me detailed suggestions on my dissertation. He gave me valuable comments and feedbacks for the improvement of this dissertation. I am also thankful to my associate advisor Dr. Wei Zhang for his valuable support and suggestion in regards to this dissertation. In addition, I would like to acknowledge reviewers Dr. Dianyun Zhang, Dr. Shinae Jang and, Dr. Arash Zaghi for their relevant and invaluable feedback. I would also like to thank all the faculty members of the Civil and Environmental Engineering department for their encouragement and support.

I would like to acknowledge Pratt and Whitney for financial support while working on my research. The continued support from them for four years, through the funded project, helped me while I worked on my research. I am grateful to my mother, Laxmi Sharma, my father Angalal Rokaya, and my brother Pranjal Rokaya, for encouraging me in numerous ways to finish this program. I would like to extend my appreciation to my friends in the civil and environmental engineering department, especially Dr. Mostafa, Dr. Manish, Bijaya, Sukirti, David, Joseph, Lilia, Toby, Brendon and others who directly or indirectly helped me during my study.
Development and Application of Incompatible Graded Finite Elements for Analysis of Nonhomogeneous Materials

Asmita Rokaya, PhD.
University of Connecticut, 2020

Functionally graded materials (FGMs) are non-homogenous and tailored to have a spatial variation of properties. The gradual modification of material properties is quite effective in reducing stresses. Finite element analysis of nonhomogeneous materials can be performed using an assemblage of either graded or homogeneous elements. A graded finite element samples the material property at more than one integration points, while a homogeneous element has constant property at all integration points based on property of the element at the centroid. In this dissertation, a six-node incompatible graded finite element is developed.

This research aims to show significance of six-node incompatible (QM6) element over four-node compatible (Q4) graded elements in terms of accuracy of the results and computation time. The numerical solution is obtained using UMAT capability of the ABAQUS software. The results are compared with the exact solution (e.g. stress due to far field tension loads for graded infinite plates). Incompatible graded element is shown to give better performance in terms of accuracy over Q4 element and computationally efficient than an eight-node compatible (Q8) element in two-dimensional plane elasticity. Thus six-node incompatible (QM6) is recommended for modelling FGMs.

Furthermore, dynamic loading characteristics of the shock tube onto sandwich steel beams as an efficient and accurate alternative to time consuming and complicated fluid structure interaction using finite element modelling is introduced. Improved accuracy of 3D dynamic analysis using
eight node incompatible brick elements (C3D8I) is demonstrated through this dynamic analysis example and results are compared to lower-order compatible brick elements (C3D8).
# Table of Contents

Title Page ................................. i
Approval Page .......................... ii
Acknowledgement ...................... iii
Abstract ................................. iv
Table of Contents ...................... vi
List of Tables ........................... ix
List of Figures .......................... x

Chapter 1  Introduction to Functionally graded materials and incompatible graded elements.  1
    1.1 Introduction ...................... 1
    1.2 Review of literature .......... 3
    1.3 Shear Locking ................... 7
    1.4 Incompatible graded element formulation for isotropic elements  9
    1.5 Incompatible graded finite element for orthotropic materials 15
    1.6 Stability check for incompatible graded elements. 20
    1.7 Dynamic analysis in 3D using incompatible elements. 25
    1.8 Motivation for proposed research 26
        1.8.1 Objectives of proposed research 27
        1.8.2 Organization of the dissertation 29

Chapter 2  Incompatible Graded Finite Elements for Analysis of Isotropic Grade Elements. 29
    2.1 Introduction ..................... 29
    2.2 Elasticity solutions of non-homogenous materials. 30
    2.3 Numerical examples. .......... 32
    2.4 Results and discussions. ..... 34
    2.5 Mesh refinement Study ....... 45
    2.6 Conclusion ..................... 47

Chapter 3  Incompatible Graded Finite Elements for Orthotropic Functionally Graded Materials 48
    3.1 Introduction ..................... 48
    3.2 Russell error .................... 49
    3.3 Elasticity solutions for orthotropic functionally graded materials 51
<table>
<thead>
<tr>
<th>Chapter 3</th>
<th>Numerical Examples</th>
<th>52</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.4.1</td>
<td>Circular Orthotropic Radially Graded Disc</td>
<td>52</td>
</tr>
<tr>
<td>Chapter 3</td>
<td>Orthotropic Functionally Graded Plate</td>
<td>57</td>
</tr>
<tr>
<td>Chapter 3</td>
<td>Graded Fiberglass Carbon Composites</td>
<td>65</td>
</tr>
<tr>
<td>Chapter 3</td>
<td>Radially Graded Curved Beam</td>
<td>68</td>
</tr>
<tr>
<td>Chapter 3</td>
<td>Conclusions</td>
<td>74</td>
</tr>
<tr>
<td>Chapter 4</td>
<td>An Accurate Analysis for Corrugated Sandwich Steel Beams under Dynamic Impulse</td>
<td>76</td>
</tr>
<tr>
<td>4.1</td>
<td>Introduction</td>
<td>76</td>
</tr>
<tr>
<td>4.2</td>
<td>Material Description and Test Setup</td>
<td>78</td>
</tr>
<tr>
<td>4.2.1</td>
<td>Sandwich Steel Beams with Graded Corrugated Core</td>
<td>78</td>
</tr>
<tr>
<td>4.2.2</td>
<td>Shock tube Test</td>
<td>79</td>
</tr>
<tr>
<td>4.2.3</td>
<td>Strain Rate-dependent Constitutive Law</td>
<td>81</td>
</tr>
<tr>
<td>4.3</td>
<td>Finite Element Analysis of Corrugated Sandwich Beams</td>
<td>82</td>
</tr>
<tr>
<td>4.4</td>
<td>Improved Loading Assumptions</td>
<td>83</td>
</tr>
<tr>
<td>4.5</td>
<td>Deformed shapes and mid-span deflection</td>
<td>85</td>
</tr>
<tr>
<td>4.6</td>
<td>Error Estimation of Front Face Deflections</td>
<td>93</td>
</tr>
<tr>
<td>4.7</td>
<td>Additional FEA results and discussion</td>
<td>95</td>
</tr>
<tr>
<td>4.7.1</td>
<td>Energy Quantities</td>
<td>96</td>
</tr>
<tr>
<td>4.7.2</td>
<td>Von Mises stress histories</td>
<td>97</td>
</tr>
<tr>
<td>4.7.3</td>
<td>Plastic strain histories</td>
<td>98</td>
</tr>
<tr>
<td>4.7.4</td>
<td>Contact reaction force at supports</td>
<td>100</td>
</tr>
<tr>
<td>4.8</td>
<td>Parametric study: Reverse core arrangement</td>
<td>101</td>
</tr>
<tr>
<td>4.9</td>
<td>Results and Discussion</td>
<td>102</td>
</tr>
<tr>
<td>4.9.1</td>
<td>Deformed shapes and mid-span deflection: reverse core arrangement</td>
<td>102</td>
</tr>
<tr>
<td>4.9.2</td>
<td>Energy Quantities: reverse core arrangement</td>
<td>107</td>
</tr>
<tr>
<td>4.9.3</td>
<td>Contact reaction force at supports</td>
<td>110</td>
</tr>
<tr>
<td>4.9.4</td>
<td>Comparative Discussion on Graded Core Arrangement</td>
<td>111</td>
</tr>
<tr>
<td>4.10</td>
<td>Comparison of lower order C3D8 elements with C3D8I elements</td>
<td>114</td>
</tr>
<tr>
<td>4.11</td>
<td>Equivalent properties modelling for simplified model.</td>
<td>118</td>
</tr>
<tr>
<td>4.11.1</td>
<td>Homogenization of multilayered cores.</td>
<td>122</td>
</tr>
<tr>
<td>4.12</td>
<td>Concluding remarks</td>
<td>123</td>
</tr>
<tr>
<td>4.13</td>
<td>Acknowledgement</td>
<td>125</td>
</tr>
</tbody>
</table>
LIST OF TABLES

Table 1 Core density for various arrangements 79

Table 2 Summary of maximum deflection for back panel and front panel 91

Table 3 Comparison of Russell errors 94

Table 4 Plastic energy absorption percentage 96

Table 5 Plastic energy absorption percentage 108

Table 6 Comparison of Support reaction between Normal core and Reversed core arrangements 111

Table 7 Comparison of Mid span deflections between Normal core and Reversed core arrangements 112

Table 8 Comparison of Plastic energy absorption between Normal core and Reversed core arrangements 113

Table 9 Comparison of mid-span deflection of corrugated beam with the homogenous beam 120
# LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1-1</td>
<td>Functionally Graded material.</td>
<td>1</td>
</tr>
<tr>
<td>1.3-1</td>
<td>Deformation of a Q4 element subjected to pure bending.</td>
<td>8</td>
</tr>
<tr>
<td>1.3-2</td>
<td>Deformation of rectangular block in pure bending.</td>
<td>9</td>
</tr>
<tr>
<td>1.4-1</td>
<td>(a) Q4 element in physical space. (b) QM6 element in physical space with</td>
<td>11</td>
</tr>
<tr>
<td></td>
<td>curved lines in the boundary showing the added displacement functions.</td>
<td></td>
</tr>
<tr>
<td>1.4-2</td>
<td>(a) and (b) Homogenous Q4 and QM6 elements, respectively.</td>
<td>14</td>
</tr>
<tr>
<td>1.4-3</td>
<td>(a) and (b) Graded Q4 and QM6 elements, respectively.</td>
<td>14</td>
</tr>
<tr>
<td>1.5-1</td>
<td>QM6 element in physical space with curved lines in boundary showing added</td>
<td>20</td>
</tr>
<tr>
<td></td>
<td>displacement functions.</td>
<td></td>
</tr>
<tr>
<td>1.6-1</td>
<td>Eigen-analysis for Q4 homogenous materials.</td>
<td>23</td>
</tr>
<tr>
<td>1.6-2</td>
<td>Eigen-analysis for Q4 Non-homogenous materials ($\beta=1$).</td>
<td>23</td>
</tr>
<tr>
<td>1.6-3</td>
<td>Eigen-analysis for QM6 homogenous materials ($\beta=0$).</td>
<td>24</td>
</tr>
<tr>
<td>1.6-4</td>
<td>Eigen-analysis for QM6 Non-homogenous materials ($\beta=1$).</td>
<td>25</td>
</tr>
<tr>
<td>1.7-1</td>
<td>Steel sandwich beams with four corrugated layers.</td>
<td>26</td>
</tr>
<tr>
<td>2.3-1</td>
<td>(a) Geometry of plate loaded in tension load perpendicular to the gradation.</td>
<td>33</td>
</tr>
<tr>
<td>2.4-1</td>
<td>Non-averaged nodal stress results for tension load applied perpendicular to</td>
<td>35</td>
</tr>
<tr>
<td></td>
<td>exponential material gradation.</td>
<td></td>
</tr>
<tr>
<td>2.4-2</td>
<td>(a) Stress distribution for tension load applied parallel to exponential</td>
<td>36</td>
</tr>
<tr>
<td></td>
<td>material gradation. (b) Stress distribution for tension load applied parallel</td>
<td></td>
</tr>
<tr>
<td></td>
<td>to linear gradation.</td>
<td></td>
</tr>
<tr>
<td>2.4-3</td>
<td>(a) Strain distribution for loading applied parallel to exponential gradation</td>
<td>37</td>
</tr>
<tr>
<td></td>
<td>(b) Strain distribution for loading applied parallel to linear gradation.</td>
<td></td>
</tr>
<tr>
<td>2.4-4</td>
<td>Stress distribution for Q4 and QM6 elements with bending load applied</td>
<td>38</td>
</tr>
<tr>
<td></td>
<td>perpendicular to the exponential material gradation.</td>
<td></td>
</tr>
<tr>
<td>2.4-5</td>
<td>(a) Stress distribution for tension load applied perpendicular to exponential</td>
<td>39</td>
</tr>
<tr>
<td></td>
<td>material gradation for T3 and QM6 elements. (b) Stress distribution for</td>
<td></td>
</tr>
<tr>
<td></td>
<td>tension load applied perpendicular to exponential material gradation for</td>
<td></td>
</tr>
<tr>
<td></td>
<td>T6 and QM6 elements.</td>
<td></td>
</tr>
<tr>
<td>2.4-6</td>
<td>(a) and (b) Triangular elements (T3 and T6) (a) regular set up; (b)</td>
<td>40</td>
</tr>
<tr>
<td></td>
<td>diagonals swapped.</td>
<td></td>
</tr>
<tr>
<td>2.4-7</td>
<td>Stress distribution for tension load applied perpendicular to exponential</td>
<td>40</td>
</tr>
<tr>
<td></td>
<td>material gradation for (a) T3 and QM6 elements (regular mesh in Fig. 8(a)).</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(b) T6 and QM6 elements. (Mesh swapped in Fig. 8(b)).</td>
<td></td>
</tr>
<tr>
<td>2.4-8</td>
<td>(a) Stress distribution for tension load applied parallel to exponential</td>
<td>41</td>
</tr>
<tr>
<td></td>
<td>material gradation for T3 and QM6 elements. (b) Stress distribution for</td>
<td></td>
</tr>
<tr>
<td></td>
<td>tension load applied parallel to exponential material gradation for T6 and</td>
<td></td>
</tr>
<tr>
<td></td>
<td>QM6 elements.</td>
<td></td>
</tr>
<tr>
<td>2.4-9</td>
<td>(a) Stress distribution for tension load applied parallel to exponential</td>
<td>42</td>
</tr>
<tr>
<td></td>
<td>material gradation for T3 and QM6 elements. (b) Stress distribution for</td>
<td></td>
</tr>
<tr>
<td></td>
<td>tension load applied parallel to exponential material gradation for T6 and</td>
<td></td>
</tr>
<tr>
<td></td>
<td>QM6 elements. (Diagonal of mesh swapped).</td>
<td></td>
</tr>
</tbody>
</table>
Figure 2.4-10 (a) Stress distribution comparison for tension load applied parallel to linear material gradation for T3 and QM6 elements. (b) Stress distribution for tension load applied parallel to exponential material gradation for T6 and QM6 elements.

Figure 2.4-11 (a) Stress distribution for tension load applied parallel to linear material gradation for T3 and QM6 element. (b) Stress distribution for tension load applied parallel to exponential material gradation for T6 and QM6 element (Diagonal of triangular mesh swapped).

Figure 2.4-12 (a) Stress distribution for the bending load applied perpendicular to exponential material gradation for T3 and QM6 elements. (b) Stress distribution for the bending load applied perpendicular to exponential material gradation for T3 and QM6 elements.

Figure 2.4-13 (a) Stress distribution for the bending load applied perpendicular to exponential material gradation for T3 and QM6 elements. (b) Stress distribution for the bending load applied perpendicular to exponential material gradation for T3 and QM6 elements. (Diagonal of mesh swapped).

Figure 2.5-1 Stress distribution for tension load applied parallel to exponential gradation.

Figure 2.5-2 Stress distribution for tension load applied parallel to linear gradation.

Figure 3.4-1 Geometry of circular disc with radially varying orthotropic properties.

Figure 3.4-2 Non-averaged nodal stress results using Q4, QM6 and Q8 graded elements: (a) $\sigma_{xx}$, (b) $\sigma_{yy}$ (c) $\sigma_{xy}$.

Figure 3.4-3 Non-averaged nodal stress results: (a) $\sigma_{RR}$, (b) $\sigma_{t\theta}$

Figure 3.4-4 Russell error comprehensive factor for non-averaged nodal shear stress results for circular plate disc under internal pressure.

Figure 3.5-1 Geometry of orthotropic plate loaded in far-field tension.

Figure 3.5-2 Non-averaged nodal stress for tension load applied perpendicular to exponential material gradation.

Figure 3.5-3 Non-averaged nodal stress for tension load applied perpendicular to exponential material gradation.

Figure 3.5-4 Russell error comprehensive factor for non-averaged nodal normal stress results for the orthotropic plate under tension.

Figure 3.5-5 Non-averaged nodal stress results for tension load applied parallel to material gradation.

Figure 3.5-6 Non-averaged nodal stress results for tension load applied parallel to material gradation.

Figure 3.5-7 Russell error comprehensive factor for non-averaged nodal normal stress for the orthotropic plate under tension.

Figure 3.5-8 (a) and (b) Non-averaged shear stress results for tension load applied perpendicular to material gradation.

Figure 3.5-9 Shear stress in orthotropic graded plate discretized with (a) Q4 graded; (b) QM6 graded, and Q8 graded elements, respectively.

Figure 3.5-10 Non-averaged nodal shear stress for bending stress applied perpendicular to material gradation.
Figure 3.6-1 (a) and (b) Non-averaged nodal stress results for the bending stress applied perpendicular to material gradation.
Figure 3.6-2 (a) and (b) Non-averaged nodal stress results for the compressive load applied perpendicular to material gradation.
Figure 3.6-3 Russell error comprehensive factor for non-averaged stresses for the orthotropic composite plate under compression.
Figure 3.7-1 The geometry of the curved quarter circle beam.
Figure 3.7-2 Radial stress $\sigma_r$ versus r in the curved beam.
Figure 3.7-3 Russell error comprehensive factor for non-averaged radial stress results for the curved beam under moment load.
Figure 3.7-4 Circumferential stress $\sigma_{\theta\theta}$ versus r in the curved beam.
Figure 3.7-5 Russell error comprehensive factor for non-averaged circumferential stress results for the curved under moment load.
Figure 3.7-6 Circumferential and radial stress plots for the curved beam under moment loading.

Figure 4.2-1 Steel beams with four corrugated layers.
Figure 4.2-2 Shock tube test setup [16]; (b) schematic of test arrangement; (c) Reflected pressure measured by the sensor during shock tube testing for various cores.
Figure 4.2-3 Stress-strain curves of (a) Steel 1018 and (b) Steel 1008 at different strain rates
Figure 4.3-1 Three-dimensional finite element mesh of the corrugated beam
Figure 4.4-1 Deformed shapes of corrugated graded core beams captured by high speed camera.
Figure 4.4-2 Pressure distributions over the loaded area adopted in the current finite element analysis.
Figure 4.5-1 ABBC: (a) Deformed shapes (b) Mid-span deflection of front and back faces. Only one test out of the two was successful for this core arrangement.
Figure 4.5-2 AABC: (a) Deformed shapes (b) Mid-span deflection of front and back faces
Figure 4.5-3 ABCC: (a) Deformed shapes and von Mises stress; (b) Mid-span deflection of front and back faces
Figure 4.5-4 Deformed shapes for AACC core arrangement at critical times (b) Mid-span deflection histories of front and back face for AACC core arrangement.
Figure 4.7-1 Plastic energy absorbtion by substrates and core for (a) AABC (b) ABBC (c) AACC and (d) ABCC
Figure 4.7-2 Von Mises stress histories at the center of front and back face for (a) ABBC (b)AABC (c)ABCC and (d) AACC.
Figure 4.7-3 Plastic strain histories at the center of front and back face for (a) ABBC (b)AABC (c)ABCC and (d) AACC
Figure 4.8-1 Reflected pressure assumed for various cores arranged in the reverse order.
Figure 4.9-1 (a) Deformed shapes of CBBA (b) Comparison of mid-span deflections: CBBA vs ABBC
Figure 4.9-2 (a) Deformed shapes of CBAA (b) Comparison of mid-span deflections: CBAA vs AABC
Figure 4.9-3 (a) Deformed shapes of CCBA (b) Comparison of mid-span deflections: CCBA vs ABCC
Figure 4.9-4 (a) Deformed shapes of CCAA (b) Comparison of mid-span deflections: CCAA vs AACC
Figure 4.9-5 Plastic energy absorption by substrates and core for (a) CBBA (b) CBAA (c) CCBA and (d) CCAA
Figure 4.9-6 Contact reaction force between the specimen and support.
Figure 4.10-1 C3D8 element
Figure 4.10-2 Front face and back face deflection comparison between C3D8 and C3D8I elements for AABC, AACC, ABBC and ABCC core arrangements respectively.
Figure 4.11-1 Corrugated beam represented by homogenous beam.
Figure 4.11-2 Cross section of the lengthwise corrugated beam
Figure 4.11-3 Deflection of the beam with corrugated core. (tc= 0.3 and tf=1)
Figure 4.11-4 Deflection of the beam with equivalent homogenous section.
Figure 4.11-5 Deflection of the beam: (a) Corrugated core (b) Homogenous core.
1 Introduction to Functionally graded materials and incompatible graded elements.

1.1 Introduction

Functionally graded materials exhibit continuous variation of material properties, which result from the non-homogenous microstructure [1]. Their material properties, such as Poisson's ratio, Young's modulus of elasticity, shear modulus, vary with location. Spatial gradation of material properties may arise due to thermal gradients in harsh thermal environments and the physical arrangement of constituent materials at constant temperatures. Due to the smooth transition from one material to another in graded materials, properties such as thermal stresses, residual stresses, stress concentration can be reduced. Functionally graded materials also eliminate the sharp interfaces existing in the composite material, which is where the failure is initiated [2]. In Figure 1.1-1, a composite material is composed of two materials, and the mixture is functionally graded. On the right side, the properties of the material can be seen to be varying according to the composition of the material. It is different from traditional composites because there is no distinct interface between the two materials.

![Functionally Graded material](image)

Figure 1.1-1 Functionally Graded material.
Finite element analysis is a computerized method for predicting how a structure reacts to the application of forces such as heat, tension, bending, and other physical effects. It is based on discretizing the structure into a large number of elements. The behavior of these elements is described by partial differential equations. Finite element analysis of the response of nonhomogeneous materials can be performed using an assemblage of either graded or homogeneous elements. A graded finite element samples the material property gradient at more than one integration point within an element, while a homogeneous element takes constant properties at the centroid of the element. Finite elements which model functionally graded elements have nodes and gauss points where the spatial variation of field quantities are calculated. In reality, actual variation in the region spanned by an element is infinite. Thus using finite elements analysis gives us an approximation of the solution. The solution largely depends on the choice of elements and mesh size. Lower order elements that have linear displacement functions are computationally efficient, but they can give inaccurate results in some cases. Higher-order elements that have quadratic displacement functions generally provide more accurate solutions but are computationally expensive. In general, the formulation of elements in structural mechanics relies on long-established tools of stress-strain relations, strain-displacement relations, and energy consideration [3].

In this study, the graded incompatible six-node quadrilateral element is developed, and its characteristics are assessed by comparing it to lower-order compatible elements (Q4 & T3 elements) as well as higher-order compatible elements (Q8) for functionally graded materials in Abaqus.
1.2 Review of literature

Different researches have been done in the field of development and implementation of the functionally graded element using FEA. A considerable amount of research work has been done in the field of development and implementation of the functionally graded element using FEA. Santare et al [4] compared linearly and exponentially graded materials to conventional homogenous elements. His study has shown that graded element surpasses the performance of homogenous elements in some loading cases. The Generalized isoparametric formulation has been employed by Kim and Paulino [5] to investigate homogenous and graded elements for non-homogenous material with loadings such as bending, traction and fixed grip loading applied perpendicular or parallel to the gradation. Higher-order element (Q8) was shown to provide a better solution than the elements with linear shape functions (Q4). Significant studies related to the investigation of mechanical properties of FGM have been conducted. Graded elements were used by Kim and Paulino [6-10] to investigate fracture mechanics of FGMs, and to model non-homogenous isotropic and orthotropic materials using generalized isoparametric formulation. Thermal and residual stresses of FGMs were investigated using graded elements [11-12]. Functionally graded materials possess numerous advantages such as improved thermal attributes [13] and have great potential for application where operation condition is severe [14]. Application of graded finite elements has been investigated for a wide range of fields such as asphalt pavements [15]; cohesive zone material [16]; and functionally graded piezoelectric actuators [17].

Materials can be either isotopically graded, which means that the properties vary in one direction only, or they can be orthotopically graded. Due to the processing techniques such as plasma spray [18], electron beam vapor deposition [19], and functionally graded materials tend to orthotropic
So far, considerable studies related to orthotropic functionally graded materials have been done. Investigation of fracture mechanics of orthotropic graded elements is done in [20-22]. In these studies, several formulations for evaluation of fracture parameters of orthotropic plates are developed. Additive manufacturing is one of the areas of application of orthotropic FGMs. Functionally graded additive manufacturing allows a change in material properties with the position which can produce efficient structures [23]. Additive manufacturing of carbon fibres is widely investigated these days. The product is generally orthotropic in nature because carbon composites itself has orthotropic properties. Additively manufactured ceramics with orthotopically graded structure is developed and characterized [25]. Additive manufacturing of carbon fibre reinforced composites is investigated [26]. Generally, FEM packages have software limitations for functionally graded additive manufacturing because they allow discrete material definitions [27]. It should be noted that, additive manufacturing allows production of structures with varying density and porosity. Several authors have studied the relations between such physical and mechanical properties [28]. Knowledge of mechanical properties allows robust characterization and implementation in FEM.

Compatible finite elements such as Q4 and Q8 have been developed and used for linear and nonlinear static and dynamic analysis [29-30]. The Compatibility of elements indicates that there should be no gaps or overlaps developed in the structure after deformation [30]. One of the main causes of inaccuracies in a lower-order finite element such as Q4 and T3 element is their inability to represent stress gradients within an element. The Q4 element is also subjected to shear locking when it is subjected to bending.

Incompatible displacements were first introduced to the rectangular isoparametric finite elements [31]. The bilinear displacement field of Q4 element was enhanced by adding two quadratic terms
in each of the displacement fields. Later, a patch test restriction was introduced, which eliminated the displacement compatibility requirements [32]. Patch test is the necessary and sufficient condition for a finite element analysis convergence. Various forms of Irons patch test were performed by Taylor et al. [33]. It was discovered that Q6 element does not pass patch test unless it is parallelogram. The modification was done to the incompatible element stiffness matrix to satisfy convergence [34]. Wilson [35] showed that due to shear locking, classical four-node quadrilateral and eight nodes cannot be used to simulate the behavior of real structures. It was shown that incompatible displacement modes corrected to pass patch test significantly enhances the performance of quadrilateral and hexahedral isoparametric elements. Several other authors have also modified incompatible graded elements to have a different forms of non-conforming finite elements.

Incompatible elements have quadratic expressions in their displacement field which allows them to represent pure bending. Modification to the stiffness matrix is done to satisfy convergence requirements. Modified incompatible elements (QM6) elements showed significantly improved performance of quadrilateral and hexahedral isoparametric elements because of reduced shear locking [35]. The behavior of orthotropic FGMs under various loading is studied and compared with analytical solutions [5]. It is shown that higher order (Q8) graded elements are better than conventional homogenous elements. Zhang et al. [36] modified classical QM6 elements to form non-confirming axisymmetric elements. It was also shown to pass the patch tests and the numerical test results showed a good element performance. Similarly, Wachspress [37] modified the two linear combinations of the four basis functions associated with the side nodes of Q8 elements to form the QP6 element. It was shown that the QP6 element and the QM6 element both give very similar results hence have identical performance.
The application of gradation can be extended to the sandwich structures. Sandwich structures have been largely used in the naval and aerospace industry to protect main structures from explosives and blast loading. Theoretical, Numerical, and experimental studies of sandwich beams under dynamic loading has been reported in various literature [38-41]. Preliminary assessment of the sandwich beam structure done by Xue and Hutchinson [38] shows a significant capability of the sandwich beam to sustain higher impulse than the monolithic counterpart. Fleck and Deshpande [39] categorized failure of the beam under blast loading into three stages: Fluid-structure interaction, Core compression, and beam stretching and bending. Their study on clamped beam subject to shock loading implies decreased impulse transmitted to the structure as a result of fluid-structure interaction. The study of solid beams and sandwich beam with honeycomb cores under different levels of impulse indicated significantly lower back face deflection [39]. Dynamic loading can be imposed onto the sandwich using various methods such as explosives [42], projectile impact [43] and shock tube loading. [44].

Numerical modeling of shock tube load requires a two-step approach, first pressure profile in the model should be matched with the experimental pressure profile. Several iterations of the model without the beam has to be run with varying pressure profile in the high-pressure region of the shock tube. An alternative to this approach is to apply pressure profile generated from shock tube as a time dependent, non-uniformly distributed pressure [46]. The approach used in [47] overestimates deflection of beams compared to the experiment. In our study, we have made an improved loading assumption based on the deformation history of the top plate. The time period at which the dynamic air pressure interacts with the top plate of the sandwich panel is estimated from experimental images captured by high speed camera, and is used in our loading history. Based
on this, loading area is varied with time of deformation of the top plate. Numerical results for sandwich beams with four different graded cores are studied herein and verified with experimental data.

1.3 Shear Locking

Shear locking is exhibited by the four-node plane element and eight-node solid element [3]. When these element formulations are specifically used to simulate beam bending behavior, they display over-stiffness due to spurious shear strain. For a Q4 element as shown in Figure 1.3-1, when it is subjected to bending, element is overly stiff and cannot produce desired displacement modes associated with the pure bending. It is observed that the top and the bottom sides remain unchanged whereas side edges have horizontal displacement. Thus, shear locking caused by the inability of the element’s displacement field to model the kinematics associated with bending. This results in spurious shear stress development in addition to the bending stress. This phenomena can be described by series of equations in terms of strain energy of the element.

The strain energy for an element is given by:

\[ U = \frac{1}{2} \int \{\varepsilon\}^T [E] \{\varepsilon\} dV \]  \hspace{1cm} (1.3-1)

where, for 2D case, \( \{\varepsilon\} = [\varepsilon_x \quad \varepsilon_y \quad \gamma_{xy}]^T \)

\( E \) is the modulus. \( \varepsilon \) Denotes strain and \( V \) is the volume of the element.

When Q4 element is subjected to the pure bending, the horizontal displacement of the side edges is equal to \( \theta_1 b/2 \). So the element strains become,

\( \varepsilon_x = -\frac{\theta_1 y}{a} \quad \varepsilon_y = 0 \quad \gamma_{xy} = -\frac{\theta_1 x}{a} \)  \hspace{1cm} (1.3-2)
The horizontal displacement of the sides is as shown in Figure 1.3-1.

Note that Shear strain is Non-Zero, which should have been zero in this case. There is also shear energy associated with this shear strain. If this shear energy is very high, the element becomes very stiff to bending.

Figure 1.3-1 Deformation of a Q4 element subjected to pure bending.

Hence the strain energy of the element due to bending moment is equal to:

$$\frac{M_1 \theta_1}{2} = U_1$$

(1.3-3)

Consider an exact solution using Euler-Bernoulli beam theory. If we solve the equation analytically then we obtain following strains:

$$\varepsilon_x = -\frac{\theta_1 y}{a} \quad \varepsilon_y = -v \frac{\theta_1 y}{a} \quad \gamma_{xy} = 0$$

(1.3-4)

In equation (1.3-5), we can see that shear stain is Zero. Shear strain in Y-direction is an approximation. Shear strain ($\varepsilon_y$) becomes zero, if the Poisson’s ratio is equal to zero. The strain energy due to bending in the element is given by equation (1.3-6), we can see that for the case
when $\theta_1 = \theta_2$. Since, Strain energy of the element, is greater than element in Figure 1.3-2, and the moment $M_1$ is greater than $M_2$. This prompts that due to shear locking, accuracy of the solution is compromised.

$$\frac{M_2 \theta_2}{2} = U_2$$

(1.3-6)

Figure 1.3-2 Deformation of rectangular block in pure bending.

1.4 Incompatible graded element formulation for isotropic elements

When element formulations that are subject to shear locking are specifically used to simulate beam bending behavior, they display over-stiffness due to spurious shear strain. The remedial measure for this phenomenon is to add bending modes or two internal degrees of freedom per element displacement modes [31]. This allows the elements to curve between the nodes and model
bending. The added internal degrees of freedom are not connected to other elements; hence modes associated with these internal degrees are incompatible. QM6 element has additional displacement terms as below:

\[
\begin{align*}
    u &= \sum_i N_i u_i + (1-\xi^2) a_1 + (1-\eta^2) a_2 \\
    v &= \sum_i N_i v_i + (1-\xi^2) a_3 + (1-\eta^2) a_4
\end{align*}
\]  

(1.4-1)  
\( (1.4-2) \)

For plane element, index \( i \) runs from 1 to 4 and \( N_i \) are shape functions of a quadrilateral element given by:

\[
N_i(\xi, \eta) = \frac{1}{4} (1 + \xi \xi_i)(1 + \eta \eta_i) \tag{1.4-3}
\]

Where \((\xi, \eta)\) denote intrinsic coordinates in the interval \([-1,1]\) and \((\xi_i, \eta_i)\) denote the local coordinates of node \( i \). Figures 1(a) and 1(b) show the plane Q4 and QM6 elements, respectively, in a physical space. The added quadratic displacement in the QM6 element is shown by the dashed curved lines at the boundary of the quadrilateral.
Thus, QM6 element has 8 nodal degree of freedom and 4 generalized degree of freedom given by $a_i$. Element stiffness matrices for QM6 element can be generated by numerical integration, with $[B]$ given by equation (4).

$$ [B] = [B_d \ B_a] \quad (1.4-4) $$

Where $[B]$ is the strain-displacement matrix of shape function derivatives. $[B_d]$ operates on nodal degree of freedom and $[B_a]$ operates on node-less degree of freedom. Hence, $[B_d]$ is identical to the $[B]$ of a Q4 element. To obtain $[B_a]$, $[B_d]$ or $[B]$ for the four-node element is appended and is constructed from equation (1.4-5). Equation (1.4-5) maps strains of an element in natural coordinate system to the displacements of an element. The Strains are mapped back to the x and y coordinate of the element using jacobian matrix. In the next step, strains in the x and y coordinate can be expressed in terms of Strain-displacement matrix, $[B]$. Strain-displacement relation is given by equation (1.4-6). Strain-displacement matrix of a Q4 element is 8x3 matrix whereas for a QM6 element it is 12x3 matrix.
Where, \( N_{1,\eta} \) is derived from the shape function of quadrilateral Q4 and is equal to \( \frac{-1 - \xi}{4} \).

Other terms in equation (1.4-5) are derived similarly. The last 4 columns in matrix given by are equation (1.4-5) multiplied by \( a_i \) to get strains.

\[
\varepsilon = [B]\{d\} \quad (1.4-6)
\]

However, Q6 elements formulated in this way fails to represent constant stress or constant strain states unless they are rectangular. The strain energy of elements is given as:

\[
U = \frac{1}{2} (\int \sigma^T \varepsilon \, dV) = \frac{1}{2} (\int \sigma^T B \, dV) \hat{u} + \frac{1}{2} (\int \sigma^T B_a \, dV) a
\]

\[ (1.4-7) \]

Q4 element fulfills both compatibility and completeness requirement irrespective of the shape of the element. The incompatible (QM6) element will also fulfill completeness if the strain energy associated with the incompatible modes vanish for all constant strain/states. Let a vector of constants \([\sigma_0]\) represent any state of uniform stress. We desire that degree of freedom remain zero when a typical element displays an arbitrary constant stress state \([\sigma_0]\). This requires that load terms associated with \( a_i \) be zero.

\[
\frac{1}{2} (\int \sigma_0^T B_a dV) a = \frac{1}{2} \sigma_0^T (\int B_a \, dV) a = 0
\]

\[ (1.4-8) \]

Thus, QM6 element would satisfy the requirement:

\[
\int B_a \, dV = 0
\]

\[ (1.4-9) \]
Strain displacement matrix can be modified to satisfy the completeness requirement and this modification is given by:

\[ B^m_a = B_a - \frac{1}{V} \int_V B_a dV \]  

(1.4-10)

Q6 element with modified strain displacement is called QM6 element. The remedy that converts a Q6 element to a QM6 element is a kind of ‘selective integration’ which means use of different integration rules to treat different parts of stiffness matrix integrand is implemented. Drawback associated with incompatible elements is that there is lack of a bound-on displacement which is a less important factor than the accuracy of parent elements.

Principle of virtual work yields following relation between nodal forces and nodal displacements:

\[ f^e = k^e d \]  

(1.4-11)

Where, \( k^e \) is element stiffness matrix \( f^e \) is the element force vector.

Numerical integration is performed based on evaluation of stiffness matrix at the Gaussian integration points. Elemental stiffness is given by the following equation:

\[ k^e = \int_{\Omega^e} B^{eT} D(x) B^e d\Omega \]  

(1.4-12)

Where, \( k^e \) is element stiffness matrix, \( D(x) \) is constitutive matrix which is a function of spatial position of the element. \( \Omega_e \) is the domain of element. The integrand in equation (1.4-12) is evaluated at each integration point over the element. For general solids, strain displacement relation is given by:

\[ \varepsilon = B d \]  

(1.4-13)

Where, \( d \) is nodal displacement vector. The stresses are not constant within the quadrilateral element. Stress relation is established using constitutive relation:
\[ \sigma = D(x)\varepsilon \]  

(1.4-14)

Figure 1.4-2 shows the homogenous Q4 and Q6 elements with constant young’s modulus (E) within the element. This means that same values are assigned at the four gauss points which are indicated by the crosses. Similarly, Figure 1.4-3 shows elements with a gradient which indicates varying Young’s modulus within the element. The value of Young’s modulus at each of these Gauss points are different.

Figure 1.4-2 (a) and (b) Homogenous Q4 and QM6 elements, respectively.

Figure 1.4-3 (a) and (b) Graded Q4 and QM6 elements, respectively.
1.5 Incompatible graded finite element for orthotropic materials

Orthotropic elements have anisotropic properties in two coordinates which are x and y. For orthotropic elements, stresses relation can be described using constitutive relation:

$$\varepsilon_i = \sum_{j=1}^{n} a_{ij} \sigma_j$$

(1.5-1)

Where $a_{ij}$ contracted notation for compliance tensor $S_{ijkl}$. $i$ and $j$ typically represent orthotropic directions (x and y).

The stresses and strains for 3-D formulation are as follows:

$$\begin{align*}
\sigma_1 &= \sigma_{11} \\
\sigma_2 &= \sigma_{22} \\
\sigma_3 &= \sigma_{33} \\
\sigma_4 &= \sigma_{23} \\
\sigma_5 &= \sigma_{31} \\
\sigma_6 &= \sigma_{12}
\end{align*}$$

$$\begin{align*}
\varepsilon_1 &= \varepsilon_{11} \\
\varepsilon_2 &= \varepsilon_{22} \\
\varepsilon_3 &= \varepsilon_{33} \\
\varepsilon_4 &= 2\varepsilon_{23} \\
\varepsilon_5 &= 2\varepsilon_{31} \\
\varepsilon_6 &= 2\varepsilon_{12}
\end{align*}$$

(1.5-2)

For plane stress, the relation between total strains and stresses can be expressed in the form given below:

$$\begin{align*}
\left[ \begin{array}{l}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33} \\
\varepsilon_{23} \\
\varepsilon_{31} \\
\varepsilon_{12}
\end{array} \right] &=
\frac{1}{E_{11}}
\begin{bmatrix}
1 & -v_{21} & -v_{31} & 0 & 0 & 0 \\
-v_{12} & E_{22} & E_{33} & 0 & 0 & 0 \\
-v_{13} & E_{22} & E_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2G_{23}} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2G_{31}} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2G_{12}}
\end{bmatrix}
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{23} \\
\sigma_{31} \\
\sigma_{12}
\end{bmatrix}
\end{align*}$$

(1.5-3)
Not all the terms associated in the equation above are independent. Some of the properties may be correlated using Maxwell’s theorem:

\[
\frac{v_{23}}{E_{22}} = \frac{v_{32}}{E_{33}}, \quad \frac{v_{31}}{E_{33}} = \frac{v_{13}}{E_{11}}, \quad \frac{v_{12}}{E_{11}} = \frac{v_{21}}{E_{22}} \tag{1.5-4}
\]

For 2-D elements, the stiffness matrix of an element \( (k^e) \) with thickness \( t \), can be represented by:

\[
[k^e]_{8\times8} = \int \int [B]^T[E][B]t \, dx \, dy \tag{1.5-5}
\]

Here, \([B]\) is a 3x12 strain-displacement matrix of shape function derivatives. For an incompatible QM6 element, \([B]\) matrix consists of \([B_d]\), which operates on the nodal degree of freedom and \([B_a]\), which operates on the node-less degree of freedom. Incompatible graded element has bending modes or two internal degrees of freedom per element displacement modes \([32]\). These additional terms allow the elements to curve between the nodes and model bending. The added internal degrees of freedom are not connected to other elements. Hence modes associated with these internal degrees are node less. QM6 element has additional displacement terms given by additional quadratic terms in equation(1.5-6) and(1.5-7).

\[
u = \sum_i N_i u_i + (1-\xi^2)a_1 + (1-\eta^2)a_2 \tag{1.5-6}
\]

\[
u = \sum_i N_i v_i + (1-\xi^2)a_3 + (1-\eta^2)a_4 \tag{1.5-7}
\]

For plane element, index \( i \) runs from 1 to 4 and \( N_i \) are shape functions of a quadrilateral element given by:

\[
[B] = [B_d \quad B_a] \tag{1.5-8}
\]
Where \((\xi, \eta)\) denote intrinsic coordinates in the interval \([-1,1]\) and \((\xi_i, \eta_i)\) denote the local coordinates of node \(i\).

\([E]\) is a 3X3 constitutive matrix which relates stresses and strain. For orthotropic plane stress elements with constant Poisson’s ratio, it is given in equation (1.5-9).

\[
\begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{12}
\end{bmatrix} = \frac{1}{1 - \nu^2} \begin{bmatrix}
E_{11} & -\nu E_{22} & 0 \\
-\nu E_{22} & E_{22} & 0 \\
0 & 0 & G_{12}
\end{bmatrix} \begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{12}
\end{bmatrix}
\] (1.5-9)

Here, 1 and 2 direction represent Cartesian coordinates \(x\) and \(y\) respectively.

Since orthotropic elements have varying modulus, the terms \(E_{11}, E_{22}\) and \(G_{12}\) are independent. Strains for an orthotropic element are affected by these properties which can be seen from equation (1.5-9). For a physical element, \(x, y\)-axes can be transformed to natural coordinates axes \((\xi, \eta)\) by using the Jacobian matrix. Implementing Jacobian to a function \((\phi)\) in Cartesian coordinates yields:

\[
\begin{bmatrix}
\phi_x \\
\phi_y
\end{bmatrix} = [J]^{-1} \begin{bmatrix}
\phi_{\xi} \\
\phi_{\eta}
\end{bmatrix}
\] (1.5-10)

Function \(\phi_x\) represents displacements derivatives which can be either \(u\) or \(v\).

In numerical analysis, strains are obtained by multiplying derivatives of shape functions with the displacements. For QM6 element, displacement is 12x1 matrix \([d]\) and, shape function derivatives are a 4x12 matrix \([N_{\xi,\eta}]\). This can be represented as:

\[
[u_{\xi,\eta}] = [N_{\xi,\eta}][d]
\] (1.5-11)
Here, \( u_{\xi,\eta} \) is displacement derivatives in \( \xi - \eta \) coordinates, \([N_{\xi,\eta}]\) is shape function derivative matrix \([d]\) is displacement matrix for QM6 element.

Finally, strains in x-y coordinates are given by:

\[
\begin{pmatrix}
  u_x \\
  u_y \\
  v_x \\
  v_y 
\end{pmatrix} =
\begin{bmatrix}
  I_{11} & I_{12} & 0 & 0 \\
  I_{21} & I_{22} & 0 & 0 \\
  0 & 0 & I_{11} & I_{12} \\
  0 & 0 & I_{21} & I_{22}
\end{bmatrix}
\begin{pmatrix}
  u_{\xi} \\
  u_{\eta} \\
  v_{\xi} \\
  v_{\eta}
\end{pmatrix}
\tag{1.5-12}
\]

Here, \([I_{11} I_{12} I_{21} I_{22}]\) is inverse of Jacobean matrix \((J^{-1})\).

For an element, using strain displacement relations \((\varepsilon = [B] [d])\), strain-displacement matrix \([B]\) can be obtained from equation \((1.5-9)-(1.5-11)\). By expanding \((1.5-10)\) and replacing in \((1.5-5)\) the equation for stiffness matrix is obtained for an element. Analytical solution of this equation is tedious, and thus numerical integration steps are suggested in [3]. Equation \((1.5-13)\) can be expanded by replacing equation \((1.5-12)\). It is given by:

\[
{[u_{\xi,\eta}]} = 
\begin{bmatrix}
  u(1,1) & 0 & u(1,3) & 0 & u(1,5) & 0 & u(1,7) & 0 & u(1,9) & 0 & 0 & 0 \\
  u(2,1) & 0 & u(2,3) & 0 & u(2,5) & 0 & u(2,7) & 0 & 0 & u(2,9) & 0 & 0 \\
  0 & u(1,1) & 0 & u(1,3) & 0 & u(1,5) & 0 & u(1,7) & 0 & 0 & u(3,12) & 0 \\
  0 & u(2,1) & 0 & u(2,3) & 0 & u(2,5) & 0 & u(2,7) & 0 & 0 & 0 & u(4,12)
\end{bmatrix}
\begin{bmatrix}
  d
\end{bmatrix}
\tag{1.5-13}
\]

Where, \( u(1,1) = I_{11} N_{1,\xi} + I_{12} N_{1,\eta} \)

\( u(2,1) = I_{21} N_{1,\xi} + I_{22} N_{1,\eta} \)

\( u(1,5) = I_{11} N_{3,\xi} + I_{12} N_{3,\eta} \)

\( u(2,5) = I_{21} N_{3,\xi} + I_{22} N_{3,\eta} \)

\( u(1,9) = -2 \Gamma_{11} \xi \)

\( u(1,3) = I_{11} N_{2,\xi} + I_{12} N_{2,\eta} \)

\( u(2,3) = I_{21} N_{2,\xi} + I_{22} N_{2,\eta} \)

\( u(1,7) = I_{11} N_{4,\xi} + I_{12} N_{4,\eta} \)

\( u(2,7) = I_{21} N_{4,\xi} + I_{22} N_{4,\eta} \)

\( u(3,12) = -2 \Gamma_{11} \eta \)
Also, $N_{1,\eta}$ is derived from the shape function of quadrilateral Q4 and is equal to $\frac{-(1-\xi)}{4}$. Other terms in equation (1.5-12) are derived similarly. Plane stress relation gives the following equation:

\[
\begin{pmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{pmatrix} = 
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 1 & 0
\end{bmatrix} 
\begin{bmatrix}
u_{\xi,\eta}
\end{bmatrix}
\tag{1.5-14}
\]

Thus $[B]$ can be obtained from the above equation. Initial evaluation of the term under integral in equation an orthotropic QM6 element gives additional terms consisting of $E_{11}r_{11}4\xi^2, E_{22}r_{11}4\eta^2, 2E_{11}r_{11}\xi\eta$ and so on. Equation (1.5-5 can be written in terms of natural coordinate, and the limits of integration can be set. This gives equation ((1.5-14)).

\[
[k^e]_{8x8} = \int_{-1}^{1} \int_{-1}^{1} [B]^T [E] [B] t J d\xi d\eta
\tag{1.5-15}
\]

Above integral will give 8X8 stiffness matrix which includes orthotropic components ($E_{11}, E_{22},$ and $G_{12}$). Principle of virtual work yields following relation between nodal forces($f^e$) and nodal displacements (d):

\[
f^e = k^e d
\tag{1.5-16}
\]

Where $k^e$ is element stiffness matrix, $f^e$ is the element force vector.

Hence for orthotropic QM6 elements, the forces are a function of modulus in different directions and nodal displacements. The additional bending modes will help to reduce the shear strains in orthotropic directions with different properties.
Figure 1.5-1 shows the plane QM6 elements in a physical space for an orthotropic element. Added quadratic displacements in the QM6 element are indicated by the dotted curved lines at the boundary of the quadrilateral.

![Figure 1.5-1 QM6 element in physical space with curved lines in boundary showing added displacement functions.](image)

1.6 Stability check for incompatible graded elements.

Principle of minimum potential energy which is the basis for finite element approximation is given by:

\[
\pi(u) = \frac{1}{2} \int_V \varepsilon^T D \varepsilon \, dV - \int_V \rho f^T u \, dV - \int_{\partial V} \mathbf{t}^T \mathbf{u} \, dS
\]  

(1.6-1)

The formulation of incompatible graded elements is based on the above equation. Approximation of the strain terms in equation(1.6-2 is given by (1.4-1) and (1.4-2). For the calculation of the second term (body forces) and third term (traction forces), displacements of four node elements is used. Thus, the enrichment functions in the QM6 element only expand the solution space \( \{u, v\} \).
\[ u = \sum_i N_i u_i \] (1.6-2)

\[ v = \sum_i N_i v_i \] (1.6-3)

In the previous section, it has been explained how the stiffness matrix is modified for the incompatible graded element so that additional displacements do not contribute to the overall work done by the element. This allows the QM6 element to pass the patch test. In this section, we examine the stiffness matrix of the incompatible graded element. Paulino et al. [27] performed a weak patch test for non-homogenous materials modeled with graded finite elements. The stiffness matrix for any element consists of the matrix product of eigenvectors and eigenvalue. With the finite element method, convergence can be proved if the element shape functions and nodal variables represent complete polynomials up to the order that depends on the governing differential equation [48]. Additionally, to assess the convergence of finite element, consistency, and stability tests are traditionally performed. In his study, the eigenvalue test for the single element stability check for Q4 and Q8 homogenous and graded elements were conducted. Deformation-equivalent loads for homogenous and FGM cases were shown to be different by imposing displacement vectors for tension, shear, and bending cases.

In this section, the Eigen value test is performed for the single element stability check. Eigenvalues for Q4 homogenous and graded as well as QM6 homogenous and graded elements are calculated. The element taken is a square with unit length. Poisson’s ratio for the element is 0.3 and gradation is defined by \( E_1 = 1, E_2 = 2.718(\beta = 1) \).

For the QM6 element, there are four additional degrees of freedom, therefore the resulting stiffness matrix is 12X12. However, the strain energy within the element is minimized with respect to additional degrees of freedom so that additional displacements can be eliminated. This is done through a standard static condensation procedure. Thus the resulting stiffness matrix becomes 8X8.
matrix. Thus the geometry of the element stays same as Q4 element. The Jacobian for the QM6 is same as the Q4 element. For additional shape functions, Jacobian is calculated at the center of the elements.

\[
\frac{\partial}{\partial \xi} (N_{5-6})|_{\xi, \eta=0} \cdot \frac{\partial}{\partial \eta} (N_{5-6})|_{\xi, \eta=0}
\]

(1.6-4)

Figure 1.6-1 and Figure 1.6-2 present Eigen-value results for Q4 homogenous and non-homogenous elements, respectively. Q4 element with full integration has 3 rigid body modes (2 translation and 1 rotation), 2 bending modes and 3 shear modes (Constant strain modes).

From the results, it can be seen that there are 8 modes of deformation and there are 3 rigid body motions with eigenvalues equal to zero. This shows that element is stable for use in finite element simulation.

However, there are significant differences between the eigenvalues of the homogenous and non-homogenous elements. The total energy \((U_i = (\lambda_i/2 , i = 1,2, ..., \text{NDofs})\) is seen to increase for FGM case.
Figure 1.6-1 Eigen-analysis for Q4 homogenous materials.

Figure 1.6-2 Eigen-analysis for Q4 Non-homogenous materials ($\beta=1$).
Figure 1.6-3 and Figure 1.6-4 present Eigen-value results for QM6 homogenous and non-homogenous elements respectively. The modes associated with locking (bending modes) for QM6 homogenous elements have lower eigenvalues compared to the Q4 homogenous elements. This is because of the kinematic relaxation that the extra modes provide in the case of QM6 homogenous element. In the case of the non-homogenous elements, the eigenvalues of QM6 elements are lower for all shear and bending modes compared to the non-homogenous Q4 element, which also indicates that the strain energy associated with these modes are less for the QM6 non-homogenous element.

Figure 1.6-3 Eigen-analysis for QM6 homogenous materials ($\beta=0$).
Thus it can be seen since strain energy for QM6 non-homogenous elements are significantly different from Q4 non-homogenous elements. This implies that poor results may be obtained if Q4 non-homogenous elements are used instead of QM6 element for the analysis of graded materials.

**1.7 Dynamic analysis in 3D using incompatible elements.**

The application of incompatible elements in 3D is widely used for dynamic analysis. Dynamic analysis is time-dependent analysis, which is required when the loading occurs in a short duration such as shock loadings, impulse loadings, etc. The critical time is calculated based on the element size. In dynamic analysis, computation time is significantly saved by using incompatible (C3D8I) elements. C3D8I elements have 8X8 gauss quadrature and nine incompatible modes. In this dissertation, the dynamic analysis of a graded sandwich beam under shock loading is done using incompatible graded elements. Sandwich beams are used as cladding on the main structure so that

Figure 1.6-4 Eigen-analysis for QM6 Non-homogenous materials (β=1).
in the event of a blast, they can absorb blast energy and minimize damage to the main structure [49]. Sandwich beams have gained attention due to their high stiffness/strength and stiffness/weight ratio. Dynamic loading, particularly generated by shock tube, can be used to evaluate blast resistance properties of sandwich beams with corrugated cores. In this study, we have focused on improved loading assumptions based on the deformation history of the top plate for shock tube loading. The beam is made up of two substrates at the front and the back and the corrugated cores, as shown in Figure 1.7-1. Core arrangement is made in a graded manner i.e., the thickness of the four cores decreases or increases along with the thickness of the beam.

![Figure 1.7-1 Steel sandwich beams with four corrugated layers.](image)

A shock tube test was performed at the University of Rhode Island [5048]. Along with improved loading assumptions for the shock tube load, numerical core optimization of the cores is done to identify the cores with maximum blast resistance properties. This research aims at the demonstration of the superiority of incompatible elements over compatible elements for dynamic analysis.

### 1.8 Motivation for proposed research

Functionally graded elements are quite effective in reducing thermal and residual stresses because of spatial variation in properties, which is why they have a wide range of applications in aerospace applications. Gradation in materials can occur even without a change in properties. For
example, when there is thermal loading, variation in properties across the element is induced. Plane four-node element is not enough to capture the stress-strain behaviour of the element. Shear locking is exhibited by the four-node plane element. When these element formulations are specifically used to simulate beam bending behaviour, they display over-stiffness due to spurious shear strain. The remedial measure for this phenomenon is to add bending modes or two internal degrees of freedom per element displacement modes or two internal degrees of freedom per element displacement modes.

Incompatible elements have quadratic expressions in their displacement field, as explained in chapter 1, which allows them to represent pure bending. The motivation of this research is to be able to efficiently analyse the graded materials numerically. This can be done using incompatible elements.

1.8.1 Objectives of proposed research

Key objectives of this research can be summarized in 3 points.

Objective 1: Develop incompatible graded elements for isotropic graded materials and demonstrate the accuracy over lower-order compatible elements.

A six-node incompatible graded finite element is to be developed and studied. Such an element is recommended for use since it is more accurate than a four-node compatible element and more efficient than the eight-node compatible element in two-dimensional plane elasticity. The objective of this research is to show the superiority of the QM6 element by comparison of six-node incompatible (QM6) with four-node compatible (Q4) graded elements as well as other triangular elements T3 (3-node triangular element) and T6 (six-node triangular element). Several plane elasticity problems may be taken whose analytical solution is available from the literature.
Gradation of material can be either linear or exponential. Accuracy and computation time are considered in determining the functionality of the QM6 element.

**Objective 2: Develop incompatible graded elements for orthotropic graded materials and demonstrate the accuracy over lower-order compatible elements.**

With the increasing use of composite materials, the need for analysis of orthotropic graded plates is necessary. Orthotropic elements have varying material properties such as Young’s moduli (E11, E22), in-plane shear modulus (G12), and Poisson’s ratio (v12). Incompatible graded element is to be developed for orthotropic functionally graded plates and radially curved beams. Elasticity solutions for the stresses are used to compare QM6 elements with Q4, Q8, and triangular elements. Stress Results for circular discs, plates with properties of composite material, and radially curved beams are compared.

**Objective 3: Improve accuracy of 3D dynamic analysis using incompatible elements and demonstrate the accuracy over lower-order compatible elements.**

The third objective is to study the sandwich beam under dynamic loading, particularly generated by the shock tube. Numerical modelling of shock tube load requires a two-step approach; at first, the pressure profile in the model should be matched with the experimental pressure profile. Several iterations of the model without the beam has to be run with varying pressure profile in the high-pressure region of the shock tube. An alternative to this approach is to apply the pressure profile generated from the shock tube as a time-dependent, non-uniformly distributed pressure [47,51]. The approach used in [47] overestimates the deflection of beams compared to the experiment. In our research, we take the time-varying loaded area into account with the aid of captured deformation images. We assume that the loaded area is expanded as the beam deflects. This approach enables accurate prediction of beam deflection. Incompatible element formulation
In addition, core optimization is to be done with numerical simulation by changing the graded core layups.

### 1.8.2 Organization of the dissertation

Chapter 1 contains the basics of incompatible graded elements. The theory of incompatible elements is explained in detail in this chapter. Introduction to the shear locking, as well as the formulation of incompatible graded elements, are presented in this chapter. Chapter 2 covers the incompatible graded finite elements for the analysis of isotropic graded elements. Chapter 2 investigates the incompatible graded finite elements for the analysis of isotropic graded elements. (Objective 1). Chapter 3 presents the incompatible graded elements for orthotropic functionally graded materials. (Objective 2). Improvement of accuracy of 3D dynamic analysis using incompatible elements and its accuracy over lower-order compatible elements is discussed in chapter 4 (Objective 3). The dissertation concludes with Chapter 5, which summarizes the main results.

### 2 Incompatible Graded Finite Elements for Analysis of Isotropic Graded Elements.

#### 2.1 Introduction

In this chapter, comparison between six-node incompatible (QM6) and four-node compatible (Q4) graded elements is shown for isotropic graded elements. Numerical solution is obtained from ABAQUS using UMAT capability of the software and exact solution is provided as reference for comparison. A graded plate with exponential and linear gradation subjected to traction and bending load is considered. Additionally, three-node triangular (T3) and six-node triangular (T6) graded
elements are compared to QM6 element. Incompatible graded element is shown to give better performance in terms of accuracy and computation time over other element formulations for functionally graded materials (FGMs).

This chapter is organized into five sections. Section 2 presents elasticity solutions of non-homogenous materials. Section 3 presents the numerical examples for isotopically graded plates. Section 4 presents the results and discussion. Section 5 presents mesh refinement study. Finally, Section 6 concludes this chapter.

### 2.2 Elasticity solutions of non-homogenous materials.

This section reviews some closed-form solutions for nonhomogeneous elasticity problems. We consider an infinitely long plate, graded along its finite width, under tension and bending. These closed-form solutions will be used as reference solutions for the present study. Erdogan and Wu [52] and Kim and Paulino [5] provided exact solutions for functionally graded plate of infinite length and finite width under symmetric loading conditions such as tension and bending. Consider the graded plate illustrated by Figure 2.3-1 with the Poisson's ratio assumed as constant for a plate with graded modulus perpendicular to the loading and as zero for plate with loading parallel to the gradation.

For exponential variation,

\[ E(x) = E_1 e^{\beta x} \]  

(2.2-1)

Where, \( \beta \) is the length scale factor characterized by,

\[ \beta = \frac{1}{L} \ln\left(E_1/E_2\right), \text{ where, } E_1 = 1 \text{ and } E_2 = 8. \]

For linear variation of the modulus,
\[ E(x) = E_1 + \beta x \]  

(2.2-2)

Where, \( \beta \) is the length scale factor characterized by,

\[ \beta = \frac{1}{L} (E_2 - E_1) \]  

(2.2-3)

Where, \( L \) is the length of the FGM plate, \( E_1 = E(x=0) \) and \( E_2 = E(x=W) \).

Along y direction, displacement component is given by \( v \). Strain component in this direction is given by:

\[ \varepsilon_y(x) = \frac{\partial v}{\partial y} \]  

(2.2-4)

\[ \sigma_{yy}(x) = \varepsilon_y(x) \frac{E(x)}{1-\nu^2} \]  

(2.2-5)

Where \( \nu = \) Poisson’s ratio

For infinitely long plate, stresses become unidirectional. The stresses for tension loading and bending load can be given by the following expression:

\[ \sigma_{yy}(x) = \frac{E(x)}{1-\nu^2} (Cx+D) \]  

(2.2-6)

Where, \( C \) and \( D \) can be determined from the boundary conditions for tension load and bending loading as given below:

\[ \int_0^w \sigma_{yy}(x) \, dx = N, \quad \int_0^w x \, \sigma_{yy}(x) \, dx = M \]  

(2.2-7)

Where, \( N \) is the tension load resultant and \( M \) is bending load.
For tension loading,

\[ C = \frac{B}{A} (e^{\beta w} - 1)(\beta W - 2) \]  
\[ D = \frac{B}{A} \beta W * e^{\beta w} (3 - \beta W) + \beta W - 2e^{\beta w} + 4) \]

Similarly, for bending, C and D are defined as below:

\[ C = \frac{-E}{A} \beta (e^{\beta w} - 1) \]  
\[ D = \frac{E}{A} (\beta e^{\beta w} - e^{\beta w} + 1) \]

Where, A, B and E are as follows:

\[ A = e^{\beta w} ((w\beta)^2 - e^{\beta w}) - 1 \]  
\[ B = \frac{\beta + N}{2*E(x)} (1 - v^2) \]  
\[ E = \frac{M}{E(x)} \beta (1 - v^2) \]

With \( E = E(x) \) where \( E_1 = E(x=0) \) and \( E_2 = E(x=L) \).

### 2.3 Numerical examples.

A square plate is modelled in Abaqus [53Error! Reference source not found.]. The plate consists of 81 elements. The plate is subjected to loading (either bending or tensile) at the upper edge. The stress distribution was obtained by applying forces at the nodes. The magnitude of the tensile force is obtained by using MATLAB to get traction ((2.2-7) to force relation. The values of exact forces obtained at nodes is parabolic in nature as shown in Figure 2. These forces values are 0.39, 0.86, 0.96, 1.05, 1.12, 1.17, 1.16, 1.06, 0.86 and 0.29 from nodes 1 to 10 respectively at the upper edge.
of the plate. Similarly, in case of bending load, load is applied using analytical field equation as shown in Figure 2.3-1 (b). Boundary condition is simply supported at the bottom edge as shown in Figure 2.3-1.

Figure 2.3-1 (a) Geometry of plate loaded in tension load perpendicular to the gradation. (b) Geometry of plate loaded in bending load perpendicular to the gradation. Linear and exponential variation of modulus along the width $E=E(x)$ where $E_1 = E(x=0)$ and $E_2 = E(x=L)$.

Young’s modulus of elasticity is varied using user subroutines in Abaqus. Figure 2(c) shows the linear and exponential profiles of $E$ along the width of the plate. For homogenous element, layered transition of Young’s modulus is applied. The value of $E$ at the centroid location of an element is considered. Which means that $E$ is discrete and changes from element to element along the width. For graded element, continuous variation of $E$ is defined in the subroutine. Exponential and linear Young’s modulus variation are given by equation(2.2-1) and(2.2-2) respectively.
The plate is discretized using Q4 and QM6 as well as triangular T3 and T6 elements. For loading applied perpendicular to material gradation, Poisson’s ratio is assumed to be constant; while for loading parallel to material gradation, Poisson’s ratio is zero. Nodal stress values without averaging is taken at y=0 for comparison. The 2 x 2 Gauss quadrature is taken for Q4 and QM6 quadrilateral elements. 1 Gauss point for T3 (3-node triangular element), and 3 for T6 (six-node triangular element) are used.

2.4 Results and discussions.

Figure 2.4-1 compares a normal stress $\sigma_{yy}$ versus $x$ in the plate with exponentially graded modulus subjected to tension load perpendicular to the gradation with the exact solution. The plate (L=W=9) is discretized with 9x9 mesh with Q4 and QM6 isoparametric elements. Nodal stresses results at y=0 are compared to the exact solution. Graded elements have a significant improvement in correlation to the exact results over the homogenous plate. The QM6 graded element gives a better result than Q4 graded element at each node. The homogenous Q4 and QM6 elements both provide a piecewise zigzag linear solution. Note that average nodal stresses between two internal nodes (eight internal nodes) are similar regardless of using either graded or homogenous element. However, the stresses at the both edges (at x=0 and 9) are more critical than any internal nodes in most of engineering applications (e.g. edge stresses in a medium as crack initiation trigger and nodal stresses (non-averaging) at the interface between two dissimilar media). In this example, the stress at x=0 using homogeneous elements deviates from desired solution. But graded elements (QM6 and Q4) captured edge stresses very accurately.
Figure 2.4-1 Non-averaged nodal stress results for tension load applied perpendicular to exponential material gradation.

Figure 2.4-2 compares a nodal stress $\sigma_{xx}$ versus $x$ for the FGM plate with exponentially graded modulus subjected to tension load parallel to the gradation with the exact solution. The mesh for the plate is 9x9 discretized with Q4 and QM6 isoparametric elements. The homogeneous Q4 element provided exact solution in the whole region; however, piecewise linear results are seen in the Q4 graded case. The Q4 graded element is not recommended for use in this case. Average nodal stresses are exact in internal nodes, but edge stresses deviate from exact solutions. It is promising that QM6 element eliminates this issue. Note that QM6 yields exact solution in case of graded and homogenous elements. This newly developed incompatible QM6 element is capable of representing accurate stress (both edge and internal stresses) solutions for graded materials with general material gradation.
To further investigate the effect of material gradation type, Figure 2.4-2 (b) compares $\sigma_{xx}$ vs $x$ for the FGM plate with linearly graded modulus subjected to tension load parallel to the gradation with the exact solution. The mesh for the plate is 9x9 discretized with Q4 and QM6 isoparametric elements. In the case of linearly gradation, we can see that worse response from graded Q4 is observed. Stress variation decreases over the width. The other elements, Q4 homogenous as well as both QM6 homogenous and graded, provide exact result. The accurate, thus promising, response of QM6 graded elements is not affected by gradation type.

![Figure 2.4-2](image)

**Figure 2.4-2** (a) Stress distribution for tension load applied parallel to exponential material gradation. (b) Stress distribution for tension load applied parallel to linear gradation.

To provide in-depth assessment, Figure 2.4-3(a) and Figure 2.4-3 (b) show strain variation in plate along the width for exponential and linear variation of properties when loading is applied parallel to material gradation. Piecewise constant strain variation is seen in each Q4 graded element leading to piecewise constant stress. Conversely, QM6 captures accurate strain distributions due to quadratic incompatible displacement modes.
Figure 2.4-3 (a) Strain distribution for loading applied parallel to exponential gradation (b) Strain distribution for loading applied parallel to linear gradation.

To study the effect of far-field loading type, Figure 2.4-4 compares $\sigma_{yy}$ vs $x$ for the FGM plate with exponentially graded modulus subjected to bending load perpendicular to the gradation with the exact solution. QM6 graded elements provide the closest solution to the exact results. Q4 Homogenous and QM6 Homogenous results are piecewise linear and similar in values. Although intermediate values of the nodal stresses can be averaged to get stress close to the exact solution, stresses at the edge deviate from the exact solution.
So far, the response of quadrilateral elements is studied and compared. The usage of triangular graded element is increasing and worth investigating. Figure 2.4-5 (a) and Figure 2.4-5 (b) compare $\sigma_{yy}$ vs $x$ for the FGM plate with exponentially graded modulus subjected to tension load perpendicular to the gradation with the exact solution for T3 and T6 elements, respectively, with QM6 element. The stresses are taken at $y=0$. T3 has a constant strain formulation with one gauss quadrature. Due to one gauss quadrature per element, graded and homogenous T3 elements give same stress results. QM6 graded gives a closer solution to exact solution whereas T3 graded element has a large deviation from the exact solution and provides stepwise variation in stress. T3 graded element is not recommended for use unless mesh is highly refined. On the other hand, the results obtained from T6 element formulation are comparable to Q4 and QM6 elements. T6 and QM6 homogenous element gives stepwise stress variation. These stresses are obtained by averaging element nodal stresses from two triangular elements (i.e. elements 1-4-3 and 1-2-3) as shown in Figure 2.4-6.
Figure 2.4-5  (a) Stress distribution for tension load applied perpendicular to exponential material gradation for T3 and QM6 elements. (b) Stress distribution for tension load applied perpendicular to exponential material gradation for T6 and QM6 elements.

Figure 2.4-6 (a) and (b) Triangular elements (T3 and T6) (a) regular set up; (b) diagonals swapped.

Figure 2.4-7 compares the stresses when the diagonal of triangular element T3 and T6 is swapped (see Figure 2.4-6(b)) T3 element still gives a larger deviation in edge stress from exact
solution when the diagonal are reversed. The results are satisfactory without apparent difference in T6 element.

Figure 2.4-7 Stress distribution for tension load applied perpendicular to exponential material gradation for (a) T3 and QM6 elements (regular mesh in Fig. 8(a)), (b) T6 and QM6 elements (Mesh swapped in Fig. 8(b)).

Figure 2.4-8 (a) compares $\sigma_{xx}$ vs $x$ for the FGM plate with exponentially graded modulus subjected to tension load parallel to the gradation with the exact solution at $y=0$ for T3 element and QM6 element. The exact solution is $\sigma_{xx}=1$. QM6 provides exact solution in case of graded as well as homogenous case. Large variation in stress is seen in case of T3 graded and homogenous element. Steps in stress variation is constant across the width of the plate. Figure 2.4-8 (b) shows and compares $\sigma_{xx}$ vs $x$ for the FGM plate with exponentially graded modulus subjected to tension load parallel to the gradation with the exact solution at $y=0$ for T6 and QM6 element. T6 graded element gives a very close approximation of the exact solution. Quite different from previous results, homogenous elements provided a better approximation than the graded case. T6 graded gives a close approximation of the exact solution though we can see stepwise variation. T6
homogenous elements give exact solution. Both QM6 graded and homogenous elements provide exact solution.

Figure 2.4-8 (a) Stress distribution for tension load applied parallel to exponential material gradation for T3 and QM6 elements. (b) Stress distribution for tension load applied parallel to exponential material gradation for T6 and QM6 elements.

Figure 2.4-9 compares the stresses when the diagonal of triangular element T3 and T6 is swapped. Some variation in stress is seen in case of T3 graded and homogenous element case when the diagonals are swapped. The result shows that there is no apparent difference in T6 element when the diagonal is reversed. T3 element performs worst as expected.
Figure 2.4-9 (a) Stress distribution for tension load applied parallel to exponential material gradation for T3 and QM6 elements. (b) Stress distribution for tension load applied parallel to exponential material gradation for T6 and QM6 elements. (Diagonal of mesh swapped).

Figure 2.4-10 (a) compares $\sigma_{xx}$ vs $x$ for the FGM plate with linearly graded modulus subjected to tension load parallel to the gradation with the exact solution at $y=0$ for T3 element and QM6 element. The exact solution is $\sigma_{xx} = 1$. QM6 provides exact solution in case of graded as well as homogenous case. Large variation in edge as well as intermittent stress is seen in case of T3 element. Figure 2.4-10 (b) shows and compares $\sigma_{xx}$ vs $x$ for the FGM plate with linearly graded modulus subjected to tension load parallel to the gradation with the exact solution at $y=0$ for T6 element and QM6 element. There is a very good agreement between the exact solution and T6 homogenous elements. Small discrepancy is still visible for T6 graded element. As expected, the accuracy of T3 element is worse than Q4, QM6 and T6 element. T6 elements being quadratic do not perform as well as QM6 in this case.
Figure 2.4-10 (a) Stress distribution comparison for tension load applied parallel to linear material gradation for T3 and QM6 elements. (b) Stress distribution for tension load applied parallel to exponential material gradation for T6 and QM6 elements.

Figure 2.4-11 compares the stresses when the diagonal of triangular elements, T3 and T6 is swapped. T3 element gives a large deviation in edge stress from exact solution when the diagonal is reversed. The results show that there is no apparent difference in T6 element.

Figure 2.4-11 (a) Stress distribution for tension load applied parallel to linear material gradation for T3 and QM6 element. (b) Stress distribution for tension load applied parallel to exponential material gradation for T6 and QM6 element (Diagonal of triangular mesh swapped).

Figure 2.4-13 (a) shows and compares $\sigma_{yy}$ vs $x$ for the FGM plate with exponentially graded modulus subjected to bending load perpendicular to the gradation with exact solution. The mesh for the plate is 9x9 discretized with T3 and QM6 isoparametric elements. T3 element is compared to QM6 element. QM6 graded gives exponentially decreasing solution close to exact solution. Homogenous QM6 and T3 element both give exponentially decreasing piecewise linear solution. Larger variation in stress along the width is seen in case of homogenous/graded T3 element. Figure
2.4-13 (b) compares $\sigma_{yy}$ vs $x$ for the FGM with exponentially graded modulus subjected to bending load perpendicular to the gradation with the exact solution for T6 and QM6 element. Graded T6 and QM6 element provide a very similar response close to the exact solution. T6 Homogenous case provides similar approximation as QM6 homogenous element. Although T6 element provides identical results to QM6 element, T6 element is computationally more expensive. Hence, it can be deduced that QM6 is more efficient. Figure 2.4-13 (a) and Figure 2.4-13 (b) give comparison between T3 and T6 with QM6 element respectively when the diagonal of triangular element is swapped. Results for the diagonal swapped case for T3 element correspond to similar stress variation as the original element arrangement case. In this case, stresses are overestimated across the width of the element. The results show that there is no apparent difference in T6 element.

![Stress Distribution](image)

Figure 2.4-12 (a) Stress distribution for the bending load applied perpendicular to exponential material gradation for T3 and QM6 elements. (b) Stress distribution for the bending load applied perpendicular to exponential material gradation for T3 and QM6 elements.
Figure 2.4-13 (a) Stress distribution for the bending load applied perpendicular to exponential material gradation for T3 and QM6 elements. (b) Stress distribution for the bending load applied perpendicular to exponential material gradation for T3 and QM6 elements. (Diagonal of mesh swapped).

2.5 Mesh refinement Study

In section 2.4, stress distribution for tension load applied parallel to exponential and linear gradation is compared for the Q4 elements. To demonstrate the effect of mesh on the results, Mesh refinement study is done. The geometry of the plate loaded in tension in Figure 2.3-1 is further divided into 18 elements. Figure 2.5-1 and Figure 2.5-2 show the results for exponentially graded elements and Linear graded elements respectively. It can be observed that there is slight reduction of error, when mesh of the element is refined. This is due to sampling of the gradation properties at more gauss points. However Q4 graded elements still cannot give the exact solution. This implies that even with computationally expensive refined Q4 graded elements, QM6 elements still give better results.
Figure 2.5-1 Stress distribution for tension load applied parallel to exponential gradation.

Figure 2.5-2 Stress distribution for tension load applied parallel to linear gradation.
2.6 Conclusion

This chapter evaluates the accuracy of the incompatible graded element (QM6) for tension and bending loading cases for isotropic functionally graded materials. Various element formulation, Q4 as well as triangular T3 and T6 elements were compared to the QM6 element. Numerical stress results were compared with the exact solution. Following observation can be made from this study:

- Incompatible graded element QM6 gives a very accurate solution for isotropic functionally graded materials when the tension load is applied parallel to the gradation.

- For tension load applied perpendicular to the gradation, QM6 also performs better than Q4, T3, and T6 elements. For bending, the loading performance of the T6 element is comparable to the QM6 element. However, two T6 elements are computationally more expensive than one QM6 element. Thus QM6 is preferred.

- In the bending, T3 and Q4 elements are stiff due to parasitic shear. Shear locking is apparent in Q4 and T3 elements. The Higher-order T6 element provides improved approximation over the T3 element due to higher-order strain. In our examples, we found the comparable performance of both T6 and quadrilateral elements. QM6 element still gives the best apparent result over other elements.

- Mesh refinement study done for Q4 elements show that QM6 elements give a better results even when the more integration points are used for Q4 elements in certain loading cases.

Mixed elements using quadratic and triangular elements is a practical solution in most numerical simulation. Certain boundary conditions require the use of triangular elements. Thus, including the triangular element in our study is quite relevant. Due to the addition of quadratic terms in $\xi$ and $\eta$ in the case of the QM6 element, it gives improved performance. QM6 element is accurate and
efficient over Q4 and T6 elements and is recommended for the analysis of functionally graded materials.

3 Incompatible Graded Finite Elements for Orthotropic Functionally Graded Materials

3.1 Introduction

An incompatible graded element (modified Q6 element, QM6) is developed and studied for isotropic functionally graded materials [54]. Modified Q6 graded element is shown to give much better performance in terms of accuracy over Q4 graded elements, and other triangular elements. With the increasing use of composite materials and additive manufacturing technologies, the need for the analysis of orthotropic graded materials is necessary. In this study, we have developed an incompatible graded element using UMAT in Abaqus for modelling orthotropic graded plates and radially graded curved beams with orthotropic properties.

Curved beams have application in numerous engineering structures such as bridges as well in aerospace structures. Significant amounts of investigation are done for isotropic homogenous curved beams in terms of analytical solution. Vibration and buckling of functionally graded orthotropic cylindrical shells are investigated [55]. A generalized FEM based modeling approach for anisotropic functions for FGM cantilever beams. Using FGMs allow the design of specific stress field in a beams is studied [56]. Stress distribution across non-homogeneous circular beam subjected to pure bending is derived using curved beam approximation [57]. Similarly, Wang and Liu [58] presented elasticity solutions beam. Analytical expressions for displacements and stress resultants of curved FGM beams are obtained by Tufekci et al. [59]. Initial value method was used
to solve differential equations in their study. We have used the same “beam” problem in [59] and compared analytical solutions with numerical solutions for radial and circumferential stresses. Additionally, an example of a circular disc with orthotropic properties varying in the radial direction is included. Numerical solutions for normal stresses and shear stresses generated along the radius is presented. To represent and compare data more efficiently Russell error [60-61] is used. Russell error measurement is a suitable technique as it is not biased towards either of the transient response and hence is a proper technique for statistical evaluation of multiple point systems.

This chapter is organized into five sections. Section 2 presents Russell error formulation. Section 3 presents the elasticity solution for functionally graded orthotropic plates. Section 4 presents the results and discussion along with error estimates. Finally, section 5 concludes this chapter.

3.2 Russell error

Russell error is a mathematical error used for quantifying transient data for magnitude and phase error [60]. The basics behind Russell error is to quantify the transient data $f$ of length $N$ as a vector with magnitude and direction:

$$F = S \hat{\phi}$$  \hspace{1cm} (3.2-1)

Where $\hat{\phi}$ a unit vector for phase error and $S$ is scalar magnitude for magnitude error. Relative magnitude error between two vectors $\vec{f_1}$ and $\vec{f_2}$ can be expressed by:

$$M = \frac{s_1^2 - s_2^2}{s_1 s_2}$$  \hspace{1cm} (3.2-2)
With $S_{1,2} = \sqrt{\sum_{i=1}^{N} f(i)^2}$.

The obtained relative error is unbounded, but we need to combine phase error, which is bounded, to magnitude error in order to obtain comprehensive error. It is desirable to have the measure of magnitude error on the same relative scale as phase error. Maintaining the unbiased nature of sign, the magnitude error factor is defined as:

$$\varepsilon_m = \text{sign}(M) \log_{10}(1 + |M|) \quad (3.2-3)$$

The phase error is determined based on phase correlation, which is the normal correlation computed on set of data that fluctuates according to time. We obtain relative phase correlation

$$A = \frac{\sum_{i=1}^{N} f1(i)f2(i)}{\sum_{i=1}^{N} f1(i)^2 + \sum_{i=1}^{N} f2(i)^2} \quad (3.2-4)$$

between two vectors $\vec{f1}$ and $\vec{f2}$ can be expressed as:

Further, it is given that phase correlation between $\vec{f1}$ and $\vec{f2}$ are equivalent to the phase shift between two trigonometric functions ranging from 1 to -1 is given by

$$\varepsilon_p = \cos^{-1}(A)/\Pi \quad (3.2-5)$$

The magnitude and phase error can be combined into a single comprehensive error $C_R$:

$$\varepsilon_c = \frac{\pi}{4} (\varepsilon_p^2 + \varepsilon_m^2) \quad (3.2-6)$$

This error measure is not biased towards any of the response and hence is suitable for statistical evaluation of multiple point systems. To better quantify the difference between the results for QM6 and other elements, this measure is used.
3.3 Elasticity solutions for orthotropic functionally graded materials

Exact solutions for functionally graded plates under various loadings are derived in [4]. For orthotropic plates with exponential variation, elastic moduli can be defined as below:

\[ \begin{align*}
E_{11}(x) &= E_{11}^0 e^\beta_{11} x \\
E_{22}(x) &= E_{22}^0 e^\beta_{22} x \\
G_{12}(x) &= G_{12}^0 e^\beta_{12} x
\end{align*} \] (3.3-1)

Where \( \beta \) is the length scale factor characterized by the following equation:

\[ \beta = \beta_{11} = \beta_{22} = \beta_{12} = \frac{1}{W} \log \left( \frac{E_{11}^w}{E_{11}^0} \right) \] (3.3-2)

Here, \( W \) is the width of the FGM plate. For linear variation, elastic moduli can be defined as:

\[ \begin{align*}
E_{11}(x) &= E_{11}^0 + \beta_{11} x \\
E_{22}(x) &= E_{22}^0 + \beta_{22} x \\
G_{12}(x) &= G_{12}^0 + \beta_{12} x
\end{align*} \] (3.3-3)

With,

\[ \begin{align*}
\beta_{11} &= \frac{1}{W} (E_{11}^w - E_{11}^0), \\
\beta_{22} &= \frac{1}{W} (E_{22}^w - E_{22}^0) \text{ and } \\
\beta_{12} &= \frac{1}{W} (G_{12}^w - G_{12}^0)
\end{align*} \] (3.3-4)

For fixed grip conditions, normal stress under plane stress condition is given by:

\[ \sigma_{yy}(x) = \varepsilon_{yy}(x) \frac{E_{22}(x)}{1 - v_{t2}^2} \] (3.3-5)

With \( \varepsilon_{yy}(x, \pm \infty) = \varepsilon_0 \), the stress distribution becomes:

\[ \sigma_{yy}(x) = \varepsilon_0 \frac{E_{22}(x)}{1 - v_{t2}^2} \] (3.3-6)

For the tension and the bending loads, the membrane resultant (\( N \)) and the bending moment (\( M \)) are defined by
\[ N = \sigma_t w \quad \text{and} \quad M = \frac{\sigma_b w^2}{6} \quad (3.3-7) \]

Using the compatibility conditions and expressing \( \varepsilon_{yy} = Cx + D \), normal stress can be expressed as:

\[ \sigma_{yy}(x) = \frac{E_{22}(x)}{1 - \nu_{12}^2} (Cx + D) \quad (3.3-8) \]

Where, \( C \) and \( D \) can be determined from the boundary conditions as given below:

\[
\int_{0}^{w} \sigma_{yy}(x) \, dx = N, \quad \int_{0}^{w} x \cdot \sigma_{yy}(x) \, dx = M \quad (3.3-9)
\]

The constants \( C \) and \( D \) are different for tension and the bending. For tension loading,

\[
C = -\frac{\beta_{22}N}{A}, \quad D = \frac{(E_{22}^0 + \beta_{22}W) \cdot N}{A} \quad (3.3-10)
\]

Similarly, for bending, \( C, D, \) and \( A \) are defined as below:

\[
C = \frac{-36 + M \cdot (E_{22}^0 + \beta_{22}W)}{6 \cdot A \cdot w^2}, \quad D = \frac{6 \cdot M \cdot (E_{22}^0 + \beta_{22}W)}{6 \cdot A \cdot w^2} \quad (3.3-11)
\]

\[
A = \frac{1}{6} \beta_{22}^2 w^3 + \beta_{2} E_{22}^0 w^2 + (E_{22}^0)^2 w \quad (3.3-12)
\]

### 3.4 Numerical Examples

#### 3.4.1 Circular Orthotropic Radially Graded Disc

Recent advancement in additive manufacturing has allowed manufacturing of structures with well-defined varying porosities in the radial direction. There is an intrinsic relation between porosity/density of the structure to material moduli. Various relationships between porosity and
moduli were investigated by Choren et al. [62]. A practical example of a circular orthotropic disc is taken in our study as shown in Figure 3.4-1 with orthotropic properties linearly varying in the radial direction as follows:

\[
E_{11}(R) = E_{11}^0 + \beta_{11} R \\
E_{22}(R) = E_{22}^0 + \beta_{22} R \\
G(R) = G_{12}^0 + \beta_{12} R
\]  

\[E_{11}^0 = 1 \quad E_{22}^0 = 0.1 \quad v_{12} = 0.3 \quad G_{12}^0 = 0.5\]

and, \[E_{11}^{R_{out}} = 7 \quad E_{22}^{R_{out}} = 0.7 \quad v_{12} = 0.3 \quad G_{12}^{R_{out}} = 0.5\]

and, \[\beta_{11} = 7/3 \quad \beta_{22} = 0.7/3 \quad \beta_{12} = 3.5/3\]

Where, R is the radial coordinate. Note that the ratio of elastic moduli in two orthogonal 1 and 2 directions are 10. The radius of the inner circle (R\text{in}) = 1 and the outer radius (R\text{out}) = 4. The circular disc is meshed with ten equally spaced elements in the radial direction. In-plane stresses along the mid-section of the disc shown by dotted lines in Figure 2 are compared using Q4, QM6, and Q8 graded elements.

![Figure 3.4-1 Geometry of circular disc with radially varying orthotropic properties.](image-url)
Figure 3.4-2 (a), 3(b) and 3(c) compare normal stresses $\sigma_{xx}$, $\sigma_{yy}$ and shear stress $\sigma_{xy}$, respectively, obtained using Q4, QM6 and Q8 graded elements. Stresses are extracted from nodes along the middle section of the disc.

It is observed that the higher-order element Q8 provides better accuracy than other lower-order elements. As shown in Figure 3.4-2 (a), the overall deviation of $\sigma_{xx}$ for Q4 graded element is larger than QM6 and Q8 graded elements. Averaging nodal stresses is a typical practice in finite element technology, but stresses at inner and outer edge nodes (e.g., nodes at $x=1$ and $x=4$) where no averaging can be done may not be accurate unless a sufficient number of elements are used on these boundary regions. Some difference in the $\sigma_{xx}$ stress is seen in Figure 3.4-2 (a) on the inner edge with 20-30% error compared to Q8 solutions. On the other hand, the normal stress $\sigma_{yy}$ is not much affected by the choice of element types.

A significant difference in the shear stress, however, is seen in Figure 3.4-2 (c) even with nodal averaging schemes used. Due to shear locking in the Q4 graded element, this element shows highest shear stresses among the three. The Q4 graded element shows gradual increase in shear stress as we move toward the outer radius of the disc. This shear locking can be resolved by using QM6 graded element. Although Q8 graded element produces most accurate solutions, QM6 graded element gives computationally efficient solutions with good accuracy obtained.
Figure 3.4-2 Non-averaged nodal stress results using Q4, QM6 and Q8 graded elements: (a) $\sigma_{xx}$, (b) $\sigma_{yy}$ (c) $\sigma_{xy}$.

Figure 3.4-3 (a) and (b) show radial and circumferential stresses along the radial direction. The Q4 element predicts significantly lower radial stresses and higher circumferential stresses on most
of the disc. The performance is poor even with nodal averaging. Yet, the QM6 and Q8 elements are in good match with each other.

![Graph](image)

(a) (b)

Figure 3.4-3 Non-averaged nodal stress results: (a) $\sigma_{RR}$, (b) $\sigma_{\theta\theta}$

Figure 3.4-4 shows Russell comprehensive error factors for Q4, QM6, and Q8 graded elements compared to the exact solution for shear stresses. It shows a significantly higher error for Q4 graded element than QM6 and Q8 graded elements. QM6 graded element is more efficient in computational than Q8 element because of fewer number of degree of freedoms. The total run time for the circular disc with Q8 elements is 10 seconds and for QM6 elements 7 seconds. For a small 2D model such as this, computational time difference does not seem significant. However, for bigger models and 3D analysis incompatible elements become very efficient.
Figure 3.4-4 Russell error comprehensive factor for non-averaged nodal shear stress results for circular plate disc under internal pressure.

3.5 Orthotropic Functionally Graded Plate

An orthotropic functionally graded plate is modelled using Abaqus [53] as shown in Figure 6. The plate is simply supported at one end. Orthotropic properties of the plate such as Young’s moduli and shear modulus are graded given by Equations (3.3-1)-(3.3-3). The constitutive law is implemented using UMAT in Abaqus. The plate is loaded parallel and perpendicular to the gradation as shown in Figure 3.5-1. The length of the plate is 18, and the width is 9. The plate is discretized with quadrilateral Q4, QM6, and Q8 elements as well as triangular T3 and T6 elements.
Figure 3.5-1 Geometry of orthotropic plate loaded in far-field tension.

Figure 3.5-2 (a) and (b) compare the normal stress $\sigma_{yy}$ versus $x$ in an orthotropic plate using graded and homogenous Q4, QM6 and Q8 elements with the exact solution. It is seen that QM6 and Q8 graded elements provide a very close solution to the analytical solution. Q4 graded elements have the most significant deviation to the analytical solution. Both QM6 and Q8 homogeneous elements give an almost identical solution but with piecewise zig-zag patterns. Q4 homogeneous element also gives a piecewise zigzag solution with a larger deviation from the analytical solution. Stresses on the two edges obtained using homogeneous elements deviate from the exact solution. QM6 graded elements provide accurate stress results over the entire domain.
Figure 3.5-2 Non-averaged nodal stress for tension load applied perpendicular to exponential material gradation.

Figure 3.5-3 (a) and (b) compare the normal stress $\sigma_{yy}$ versus $x$ in an orthotropic plate using triangular elements (T3 and T6) with QM6 element. Element nodal stresses for triangular elements are obtained by averaging stresses at the corner nodes. QM6 elements give a very close solution to the analytical solution and its accuracy is equivalent to T6 elements. T3 element has a constant strain formulation, and too stiff to represent the bending and so deviates from the analytical solution.
Figure 3.5-3 Non-averaged nodal stress for tension load applied perpendicular to exponential material gradation.

Figure 3.5-4 (a) shows bar graph comparison of Russell comprehensive error factors for Q4, QM6 and Q8 graded element subjected to tension. The figure clearly shows that QM6 and Q8 graded elements give an insignificant error. T3 graded element gives the highest error (not shown). Figure 3.5-4 (b) shows a comparison for error using homogenous elements. It can be observed that all homogenous elements give a similar deviation from the exact solution. Q8 homogenous provides a better solution than other homogenous elements.
Figure 3.5-4 Russell error comprehensive factor for non-averaged nodal normal stress results for the orthotropic plate under tension.

To further investigate the effect of material gradation direction, Figure 3.5-5 compares $\sigma_{xx}$ vs. $x$ in an orthotropic plate for Q4, QM6 and Q8 graded elements with the exact solution. The plate is loaded parallel to the gradation. The mesh for the plate is 18x9 discretized with Q4, QM6 and Q8 isoparametric elements. Q4 graded element performs very poorly but QM6 and Q8 graded elements capture the exact solution.

![Figure 3.5-5 Non-averaged nodal stress results for tension load applied parallel to material gradation.](image)

The usage of triangular graded elements is increasing and worth investigating. Figure 3.5-6 compares $\sigma_{xx}$ vs. $x$ for the FGM plate subjected to tension load parallel to the gradation for T3 and
T6 elements with the exact solution. T3 graded element gives a piecewise linear solution; however, T6 graded element gives a very close approximation to the exact solution.

Figure 3.5-6 Non-averaged nodal stress results for tension load applied parallel to material gradation.

Figure 3.5-7 shows a Russell comprehensive error factor for Q4, QM6, Q8, T3 and T6 graded elements subjected to tension load parallel to the gradation. QM6, Q8 and T6 elements give zero error. Q4 graded element gives the highest error among the graded elements.
Figure 3.5-7 Russell error comprehensive factor for non-averaged nodal normal stress for the orthotropic plate under tension.

Figure 3.5-8 (a) and (b) compare the shear stresses $\sigma_{xy}$ in the orthotropic plate under tension for Q4, QM6, and Q8 graded and homogenous elements, respectively. QM6 and Q8 elements behave almost identical for both graded and homogenous elements. However, it is seen that Q4 overestimates shear in the orthotropic plate. It can be concluded that Q4 element, whether or not graded and homogeneous, is the worst choice to use for graded orthotropic solids such as fibre-reinforced or woven-fabric composites because of spurious shear.
Figure 3.5-8 (a) and (b) Non-averaged shear stress results for tension load applied perpendicular to material gradation.

Figure 3.5-9 compares the contour plots for shear stress $\sigma_{xy}$ in orthotropic plate obtained from FEA. For Q4 graded elements, a zigzag pattern is seen with alternating higher and lower values of shear stresses. QM6 and Q8 graded elements show a uniform value close to zero.
Figure 3.5-9 Shear stress in orthotropic graded plate discretized with (a) Q4 graded; (b) QM6 graded, and Q8 graded elements, respectively.

To study the effect on different loading types, the bending stress of unit magnitude is applied on the plate perpendicular to the gradation. Figure 3.5-10 (a) and (b) compare the shear stress $\sigma_{xy}$ in the plate under the bending stresses. Q4 elements exhibit large shear stresses. QM6 graded and Q8 graded elements perform almost identically and give minimal error.

![Graph](image)

(a) ![Graph](image)

(b) Figure 3.5-10 Non-averaged nodal shear stress for bending stress applied perpendicular to material gradation.

### 3.6 Graded Fiberglass Carbon Composites

Fiberglass carbon composites have a wide range of applications and are inherently orthotropic in nature. Dissimilar orthotropic stiffness is one of the major characteristics of such composites. Properties of additively manufactured of carbon fibres reinforced thermoplastic composite has been studied by Ning et al. [27]. Additive manufacturing technique for composites has been applied in [66] in which the feasibility of short carbon fibres as reinforcement in fused decomposition
modelling is shown. Such additively manufactured parts have varying moduli. The present study addresses a theoretical approach to analysing such an orthotropic plate with carbon fiberglass properties. Properties of T300/913 composite is used in this example. The Young’s moduli and shear modulus are assumed to vary exponentially. The composite plate is loaded with either the bending stresses or compression. Bending stress of magnitude equal to one is applied perpendicular to the gradation using analytical equation at the edge of the plate. For compression, 1-unit compression load is applied at the edge similar to the tension load application in previous examples. Compression and the bending load are both applied perpendicular to the gradation.

Variations of moduli in composite plate is assumed by following relation:

\[
E_{11}(x) = E_{11}^0 e^{\beta x} \quad E_{22}(x) = E_{22}^0 e^{\beta x} \quad G(x) = G_{12}^0 e^{\beta x}
\]  

(3.6-1)

Where, \(E_{11}^0 = 16.5\text{GPA}\) \(E_{22}^0 = 1.1\text{GPA}\) \(v_{12}=0.31\) \(G_{12}^0=0.575\text{GPA}\) and \(E_{11}^W = 132\text{GPA}\) \(E_{22}^W = 8.8\text{GPA}\) \(v_{12}=0.31\) \(G_{12}^W=4.5\text{GPA}\)

Figure 3.6-1 (a) and (b) compare normal stresses for Q4 and Q8 elements with QM6 element for the bending stress loading. Homogenous elements give piecewise solution, and graded elements give a very smooth solution that matches with the analytical solution. It is seen that Q4, QM6, and Q8 give very similar response for both graded and homogenous cases.
Figure 3.6-1 (a) and (b) Non-averaged nodal stress results for the bending stress applied perpendicular to material gradation.

Figure 3.6-2 (a) and (b) show the comparison for normal stresses for compressive loading. It can be observed that QM6 and Q8 graded elements give almost exact solution, but Q4 graded element gives some variation especially visible at the intermediate nodes. Hence considering different load scenarios such as compression, QM6 graded element performs better over Q4 graded elements.
Figure 3.6-2 (a) and (b) Non-averaged nodal stress results for the compressive load applied perpendicular to material gradation.

Figure 3.6-3 shows Russell comprehensive error factors for Q4, QM6 and Q8 graded elements subjected to compression perpendicular to the gradation. FEA results are compared to the exact solution in order to calculate the error. Graded elements give smaller error than homogenous elements (not seen). Q8 graded element gives the least error, and Q4 element the largest error.

![Graph showing error factors for different graded elements](image)

Figure 3.6-3 Russell error comprehensive factor for non-averaged stresses for the orthotropic composite plate under compression.

### 3.7 Radially Graded Curved Beam

A curved beam with isotropic material is modelled as a quarter circle as shown in Figure 19 using Abaqus [53]. The internal and external radii of the circle are 0.5m and 0.6m, respectively. A bending moment of 10 KN-m is applied to the free end. The beam is modelled as a cantilever beam with one end fixed as shown in Figure 3.7-1. The section is meshed with a mesh size of 0.02 m. Loading is applied as bending stress with the equation: $120*(0.55-X)$ at the free end. Young’s
modulus is varied radially and is given by equation (3.7-1). Variation of modulus through radial direction is obtained by using UMAT in Abaqus. Stresses (radial and circumferential) extracted at the fixed end are compared.

\[
E(r) = E_1 \left( \frac{R}{R_c - W/2} \right)^2 \frac{\lambda \left( R/(R_c - W/2) - 1 \right)}{o \left( R_c + W/2 \right) \left( R_c - W/2 \right) - 1}
\]  

where, \( \lambda = \ln \left( \frac{E_1}{E_2} \right) - 2 \ln \frac{R_c + W/2}{R_c - W/2} \)

\[
R = r - R_I
\]

Where, \( R_I \) - Internal radius of the curved beam, \( W \) - Width of the beam.
First order shear deformation theory is used to analyse stresses and deformation of a curved beam under in-plane loadings in [56]. The effect of shear deformation is considered by using the well-known kinematic relations in polar coordinates. These relations are as follow:

Using beam theory assumptions, stress-strain relations become.

\[ \varepsilon_{\theta\theta} = \frac{1}{r} \frac{\partial u_{\theta}}{\partial \theta} + \frac{u_r}{r} \]

(3.7-3)

\[ \varepsilon_{rr} = \frac{\partial U_r}{\partial r} \]

\[ \sigma_{\theta\theta} = E(r) \varepsilon_{\theta\theta} \]  

(3.7-4)

Detailed derivation can be found in [56]

Figure 3.7-2 compares radial stress \( \sigma_{rr} \) versus \( r \) in the curved beam obtained using Q4, QM6, and Q8 graded elements. The results are compared to the exact solution derived and tabulated in [59]. Radial stresses are obtained by transforming the results into cylindrical coordinates. Compared to the circumferential stresses, radial stresses are very low. It is worth noting that the curved beam is graded in the radial direction. Due to stress concentration, corner nodes have higher value of stresses for all the element cases. Stresses are compared at \( \Theta = 5.4^\circ \), at this location stresses are free from stress concentration influence. It can be observed that Q8 elements give an excellent correlation. QM6 elements gives a close approximation. Q4 elements have a large deviation from the exact solution.
Figure 3.7-2 Radial stress $\sigma_{rr}$ versus $r$ in the curved beam.

Figure 3.7-3 shows bar graph comparison of Russell comprehensive error factors for Q4, QM6 and Q8 graded elements for radial stresses. Error is based on the difference with the analytical solution. A very high value of error can be observed for Q4 elements. QM6 and Q8 elements give a relatively low error.
Figure 3.7-3 Russell error comprehensive factor for non-averaged radial stress results for the curved beam under moment load.

Figure 3.7-4 compares circumferential stress $\sigma_{\theta\theta}$ versus $r$ in the curved beam at $\Theta=0$. The results are compared to the exact solution derived in [58]. Similar values are also given in [59]. Circumferential stresses are obtained by transforming the results into cylindrical coordinates. Stresses for Q4 element is seen to have piecewise variation. This variation decreases as we move along the nodes from left to right. QM6 and Q8 element predict the analytical results very well without any piecewise variation. Since the gradation of material is in the radial direction, circumferential stress prediction is not much affected by choice of element. It is safe to say that all Q4, QM6, and Q8 graded elements predict circumferential stress with minimal error.
Figure 3.7-4 Circumferential stress $\sigma_{\theta\theta}$ versus $r$ in the curved beam.

Figure 3.7-5 shows bar graph comparison of Russell comprehensive error factors for Q4, QM6 and Q8 graded elements for circumferential stresses. Error is calculated from the difference of the stress results from FEA with the analytical solution. All elements give a very low error. The magnitude of error is slightly less for QM6 and Q8 elements than Q4.

Figure 3.7-5 Russell error comprehensive factor for non-averaged circumferential stress results for the curved under moment load.
Figure 3.7-6 shows stress contour plots for circumferential stresses and radial stresses for the curved beam. It can be observed that there is stress concentration at the ends of the beam which results in higher stresses at the corners. Circumferential stresses vary uniformly along the radius. Higher circumferential is observed at the inner side of the curved beam.

Figure 3.7-6 Circumferential and radial stress plots for the curved beam under moment loading.

### 3.8 Conclusions

This chapter addresses the features of a new incompatible graded element for modeling orthotropic functionally graded materials. From the study, the following observations can be made:

- Due to the addition of quadratic terms in $\xi$ and $\eta$ in QM6 graded element, it gives an improved performance for orthotropic graded materials. QM6 graded element is accurate over Q4 and T6 graded element and more efficient in computational times than Q8 graded element.

- From the observation of radial and circumferential stresses in the curved beam, it can be also concluded that QM6 graded element has a comparable performance with Q8 element. QM6 graded element is recommended over Q4 graded element because of its accuracy.
• Very high shear stresses are seen in orthotropic materials when using Q4 graded elements. This is mainly due to shear locking and material orthotropy. It is evident that QM6 element avoids this phenomenon and gives accurate shear stress results.

• Error quantification is done using Russell error and presented in bar graphs. The numeric value of error supported the advantage of using QM6 graded element in comparison to Q4 elements.

Based on the present 2D numerical study, QM6 graded element is recommended for analysis of orthotropic graded materials. Moreover, the computational efficiency of incompatible eight-node brick (B8) graded elements will be huge compared to 20-node brick (B20) element in 3D analysis.
4 An Accurate Analysis for Corrugated Sandwich Steel Beams under Dynamic Impulse

4.1 Introduction

Dynamic loading particularly generated by shock tube is used in this study to evaluate blast resistance properties of sandwich beams with corrugated cores. Numerical modeling of the shock tube load requires a two-step approach, first pressure profile in the model should be matched with the experimental pressure profile. Several iterations of the model without the beam has to be run with varying pressure profile in the high-pressure region of the shock tube. An alternative to this approach is to apply pressure profile generated from shock tube as a time dependent, non-uniformly distributed pressure [47,51]. The approach used in [47] overestimates deflection of the beams compared to the experiment. In our study, we have made an improved loading assumption based on the deformation history of the top plate. The time period at which the dynamic air pressure interacts with the top plate of the sandwich panel is estimated from experimental images captured by high speed camera, and is used in our loading history. Based on this, loading area is varied with time of deformation of the top plate. Numerical results for sandwich beams with four different graded cores are studied herein and verified with experimental data. Errors between front face deflections predicted by the finite element analysis (FEA) and experimental data are quantified by using the Russell error [60]. Difference in the magnitude and the phase of the two transient data is measured. Comprehensive error which combines magnitude and phase errors between experimental data and FEA results gives a very good correlation for the current study.

In recent years, study of core arrangement has gained attention due to it’s a significant contribution to blast resistance. Core arrangement is crucial for optimization of blast performance...
using sandwich beams. Analytical and numerical investigation done by Tilbrook et al. [72] shows that core strength and blast impulse magnitude are important in minimizing back face deflection and support reaction. Liang et al. [73] showed that angle of corrugation and leg length are important parameters for optimum design. Apetre et al. [74] theoretically demonstrated that functionally graded core reduces shear stress when there is no abrupt change between stiffness of the face sheet and the core. Wang et al. [73] observed that gradually graded foam with least density foam towards shock loading outperforms in terms of damage resistance. As a result of gradual core compression due to least mismatch in wave impedance, least energy was transmitted to the back face when lowest density core is at the shock loading. Experimental studies conducted by Gardner et al. [76] with similar core configuration with added polyuria layer showed a better energy absorption and less damage when stepwise compression of the core is allowed. Better blast resistance performance of the cores with soft cores facing the shock loading is shown in [77,78]. Finite element simulation study on dynamic response of metallic sandwich spherical shell with graded aluminum foam cores in [79] concluded that core layer arrangement with lower to higher density from loading end to outward gives optimal resistance to blast loading. A comparative study done for a quasi-static loading case by Vaidya et al. [80] shows smaller deflection of back face when soft cores are towards the loading. In our study, we study both the graded core layers with least density core placed in front and the reversely graded core layers with highest density core in front to elucidate the gradation effects onto the behavior of the sandwich beam.

This chapter is organized as follows: Section 2 contains information about the graded corrugated sandwich beam under study, shock tube experiment and also constitutive law for the materials of the beam. Section 3 describes finite element model set up. Section 4 presents loading assumptions. Section 5 discusses deformation and mid span deflections from FEA and compares it with the
experimental results. Section 6 presents error quantification by Russell error method. Section 7 follows with more discussion on FEA results such as plastic energy absorption, reaction force, von misses stress and plastic strain. Section 8 illustrates consideration of reverse core arrangement study in this chapter. Section 9 discusses some results such as deformation, mid span deflection and plastic energy absorption properties of reverse core arrangement as well as comparative discussion on normal core arrangement with reverse core arrangement. Section 9 shows comparison between incompatible elements and compatible elements for the analysis of improved loading model. Section 10 discusses homogenization scheme for corrugated cores. Section 11 concludes our work.

4.2 Material Description and Test Setup

In this section, material properties of the corrugated beam specimen are explained. In addition, shock tube test and constitutive relation for the materials is discussed.

4.2.1 Sandwich Steel Beams with Graded Corrugated Core

Four steel beams with graded corrugated cores are studied using enhanced loading assumptions in this chapter. These corrugated beams are supposed to be attached to the structural components such as beams and columns. The beam is made up of two substrates at the front and the back and the corrugated cores as shown in Figure 4.2-1. The face plates have dimension of 50.8(width) x 203.2(length) x 3(thickness) in mm and is 250 g a piece. The beams are made of hypoeutectic steels, where substrates are steel 1018 as received and corrugated cores are steel 1008 after heated to 900°C and furnace cooled, which makes it soft and ductile. The substrates and the corrugated layers are spot-welded together through both ends of the surfaces in contact.
Figure 4.2-1 Steel beams with four corrugated layers.

The cores are arranged with non-uniform thicknesses. The height of each corrugated layer is around 6mm. Non-uniform thicknesses for the four corrugated layers are considered in this chapter to investigate the effect of the core arrangement onto the dynamic behavior. In the following sections, A refers to 0.762mm, B refers to 0.508mm and C refers to 0.254mm having an average mass of 60 g, 37 g and 18 g, respectively. Table 1 lists the core density and relative density are graded cores for ABBC, AABC, ABCC and AACC. The core arrangements vary the wave impedance in core compression thus giving different capability to each arrangement.

Table 1 Core density for various arrangements

<table>
<thead>
<tr>
<th>Parameter</th>
<th>ABBC</th>
<th>AABC</th>
<th>ABCC</th>
<th>AACC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Core density (kg/m³)</td>
<td>488.19</td>
<td>549.22</td>
<td>427.17</td>
<td>501.04</td>
</tr>
</tbody>
</table>

4.2.2 Shock tube Test

Shock tube test was performed as shown in Figure 4.2-2(a). This shock tube of length of 8 m consists of high pressure driver section and low pressure driven section separated by a diaphragm. The driver section is pressurized with helium gas and when pressure reaches a critical value, diaphragm is ruptured. The gas travels in the driven section. In this way, this planar shock wave is imparted onto the specimen [73]. The beam specimen span is 152.40 mm between two simply supported boundary condition and is kept normal to the muzzle of the shock tube as shown in Figure 4.2-2 (b). Two pressure transducers at distance of 180 mm and 20 mm distance from face
of the muzzle as shown in Figure 4.2-2 (b) measure the initial and reflected shock pressures. Two tests of each specimen are performed under shock loading is performed. Figure 4.2-2 (c) shows a pressure profile measured from this shock tube for four corrugated graded cores.

Figure 4.2-2 Shock tube test setup [16]; (b) schematic of test arrangement; (c) Reflected pressure measured by the sensor during shock tube testing for various cores.

The deflection and velocity of the beams was captured using Photon SA1 high-speed digital camera. The camera has a frame capture speed of 20000 fps with image resolution of $512 \times 512$ pixels for 2s time duration [47].
4.2.3 Strain Rate-dependent Constitutive Law

Strain rate-dependent constitutive relations of Steel 1018 (see Figure 4.2-3(a)) is obtained using the Split-Hopkinson Pressure Bar (SHPB) technique for various high strain rates at room temperature. The SHPB experiments were compressive in nature and the specimens used were cylindrical, 6.35mm in diameter and 2.54 mm thick. The stress-strain curves of steel 1018 under strain rates (2000/s, 2500/s and 3100/s) obtained from SHPB tests as well as the quasi-static stress-strain curve obtained by quasi-static testing are shown in Figure 4.2-3(a). Bilinear hardening curve with linear strain-rate dependence was used for material model of Steel 1018 in the finite element simulation. For Steel 1018, the Young’s modulus is 190GPa and the stress (MPa)-strain model was defined as:

\[
\sigma = (500 + 200 \varepsilon^p)(1 + 0.0003 \varepsilon) MPa
\]  
(4.2-1)

The quasi-static constitutive curve for Steel 1008 was experimentally obtained as shown in Figure 4.2-3(b), and the bilinear hardening model is also used. Since steel 1008 and steel 1018 are low-carbon steels, Steel 1008 is also assumed to follow the strain-hardening behavior and strain-rate dependence. For Steel 1008, the Young’s modulus is 190GPa and the strain model was defined as:

\[
\sigma = (200 + 400 \varepsilon^p)(1 + 0.0003 \varepsilon) MPa
\]  
(4.2-2)
Figure 4.2-3 Stress-strain curves of (a) Steel 1018 and (b) Steel 1008 at different strain rates

4.3 Finite Element Analysis of Corrugated Sandwich Beams

Numerical analysis is carried out by using 3-D model in ABAQUS/explicit. Symmetric quarter model of the beam is used to reduce the computation time as shown in Figure 4.3-1. 3 layers of elements in the front and bottom substrate and 1 layer of elements in the corrugated layers is generated by meshing. Total number of elements in the mesh is around 50,000. Meshing is done using hexagonal elements. The element formulation used is 8-node linear brick elements enhanced with incompatible modes or C3D8I element formulation. C3D8I element formulation is better than C3D8 element because it removes shear locking as well as ensures reduction in volumetric locking. To improve beam bending behavior, incompatible mode element is used. C3D8I elements also gives reduced computational time over C3D8 element and quadratic element formulation.
Figure 4.3-1 Three-dimensional finite element mesh of the corrugated beam

The support is modeled as a rigid body and meshed with about 20000 elements. The contact between the specimen and the rigid support is defined as frictionless in the tangential direction and hard contact in the normal direction. The substrates and the corrugated layers are assumed to have perfect bonding. Additional Damping is not introduced in the model as most of the energy is dissipated as plastic energy [47].

4.4 Improved Loading Assumptions

Fluid structure interaction for shock tube loading requires a two-step approach. First pressure profile in the model should be matched with the experimental pressure profile. Several iterations of the model without the beam must be run with various magnitudes of pressure profile in the high-pressure region of the shock tube. This approach becomes tedious and challenging [81]. Another alternative approach to this is given by Zhang et al. [47] and Yazici et al. [51] in which pressure profile generated from shock tube is applied on to the beam as a time dependent, non-uniformly distributed pressure (see Figure 4.4-1).
The transient pressure undergone by the top plate while impacted is quite dependent upon the deformation of the steel beam, primarily of the top plate, which is in contact with the dynamic impulse. Taking this alternative approach in this chapter requires the time-varying deformation history of the top plate captured by high-speed camera, which will be interacting with the incoming pressure, so-called fluid-interaction, Zhang et al. [47] extended the loaded area on the beam from 38.1 (radius of the muzzle) to 76.2 mm at zero time that is not consistent with the captured image of the deformed beam as seen in Figure 4.4-1(a). As a result, the front face deflections are overestimated in this approach. In this chapter, we take the time-varying loaded area into account with the aid of captured deformation images. We assume that the loaded area is expanded as the beam deflects. The loaded area is expanded after 0.5 ms. As such, we consider two different shock tube loadings onto the beam at times 0 s and 50 ms, respectively. At t=0 s, reflected pressure profile is imparted onto the beam on a uniformly and is equal to the diameter of the shock tube muzzle which is 38.1 mm. The loading area is extended further at t= 0.5 ms, with distribution field of load as shown in Figure 4.4-1(b).

(a) T= 0 sec  
(b) T=0.5 ms

Figure 4.4-1 Deformed shapes of corrugated graded core beams captured by high speed camera.
Figure 4.4-2 Pressure distributions over the loaded area adopted in the current finite element analysis.

4.5 Deformed shapes and mid-span deflection

(b)

Figure 4.5-1 depicts deformed shapes and compares mid-span deflection of ABBC calculated by the present approach in comparison with the experiment and FEA results by Zhang et al. [47]. The latter approach overestimates the deformation of the front plate and underestimates that of the back plate. The reason is that the larger deformation of the front substrate is due to the
expanded loaded area at time t=0 sec. The real-time deformed shapes of each arrangement were obtained based on sequential images taken by a high-speed camera [82]. From the deformed shapes, it is seen that Core crushing of first C layer takes place almost immediately as the load is applied. At around 1ms, the front substrate slaps against the second B layer. During this time, beam bending also occurs. Beam bending and core crushing is seen to be coupled when we see the beam failure progression around 2ms. Maximum deflection is seen around 2 ms in the front substrate. Where as in the back substrate maximum deflection occurs around 2.5 ms in our FEA and around 3 ms in the test. This is due to debonding of the cores that occurs after 2.5 ms in the test. We cannot see debonding in our FEA results because of the perfect bonding that we have assumed. Stress in the substrates and cores can also be seen in the beam.

(b)

Figure 4.5-1(a). Gradual progression of stress transfer from front substrate to the cores and then to the back substrate is visible. At around 3 ms, when the loading has subdued, it can be deduced that there is equal distribution of stress in the beam.
Figure 4.5-1 (b) we have compared our FEA mid span deflection of ABBC as well as deflection given in [47] with the test 2. Only test 2 was successful among two tests. Significant improvement in correlation can be seen in the plot.
Figure 4.5-1 ABBC: (a) Deformed shapes (b) Mid-span deflection of front and back faces. Only one test out of the two was successful for this core arrangement.

Figure 4.5-2 depicts deformed shapes and compares mid-span deflection of AABC calculated by the present approach in comparison with the experiment and FEA results by Zhang et al. [47]. Similar to the ABBC case, the latter approach overestimates the deformation of the front plate [47]. From the deformed shapes, it is seen that Core crushing of first C layer takes place as soon as the load is applied. Local buckling of the second B layer is seen to take place around 1ms. Beam bending and core crushing is seen to be coupled from 2ms to 3ms. Core compression of second B layer is seen although it does not collapse completely. Higher Stress in the back substrate can be seen at t=2ms. The stresses transfer to the back face sooner than ABBC because of resistance to collapse of two stronger layers AA. Similar to above case, only one test is successful in this case. From the comparison plot of mid span deflection in Figure 4.5-2(b), significant improvement in mid span deflection may be seen from our current approach.
Figure 4.5-2 AABC: (a) Deformed shapes (b) Mid-span deflection of front and back faces
Figure 4.5-3 depicts deformed shapes and compares mid-span deflection of ABCC calculated by the present approach in comparison with the experiment and FEA results by Zhang et al. [47Error! Reference source not found.]. Similar to the previous cases, the latter approach overestimates the deformation of the front plate, almost in the entire time period. In the deformed shapes obtained from the experiment and FEA, it is seen is core crushing of the two C layers takes place around 1ms. Core compression of subsequent B layer starts after that it is coupled with bending mechanism. Gradual progression of stress transfer can be seen in this core arrangement as well. Stress transfer to the back substrate is visible from 1ms onwards. Some stress is left in the front substrate at 3ms.
Figure 4.5-3 ABCC: (a) Deformed shapes and von Mises stress; (b) Mid-span deflection of front and back faces

Figure 4.5-4 depicts deformed shapes and compares mid-span deflection of AACC calculated by the present approach in comparison with the experiment and FEA results by Zhang et al. [47]. Similar to the previous cases, the latter approach overestimates the deformation of the front plate [47] almost in the entire time period. Core crushing of starts around 5ms, after which core crushing is combined with the bending of the beam. It is seen that two core cores (C-C) cores crushes completely around 3ms, while A-A core do not compress due to sudden change in the density and A-A being higher density layer. Stress transfer seems to the back substrate is visible around 2ms. Similar value of stress is seen between 2ms and 3ms in the core arrangement.
Figure 4.5-4 Deformed shapes for AACC core arrangement at critical times (b) Mid-span deflection histories of front and back face for AACC core arrangement.
Table 2 summarizes the test and FEA results for the four core arrangements considered. The back face deflection shows the least deflection for AABC and ABBC, in which case, the wave impedance is minimum. Thus these cores perform better than AACC in which there is sudden change in the core layer thickness.

Table 2 Summary of maximum deflection for back panel and front panel.

<table>
<thead>
<tr>
<th>Graded core arrangements</th>
<th>ABBC</th>
<th>AABC</th>
<th>ABCC</th>
<th>AACC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Back Panel deflection (mm) FEM</td>
<td>7.93</td>
<td>7.08</td>
<td>9.16</td>
<td>9.00</td>
</tr>
<tr>
<td>Back Panel deflection (mm) Experiment</td>
<td>5.10</td>
<td>6.60</td>
<td>13.19</td>
<td>10.80</td>
</tr>
<tr>
<td>Front Panel deflection (mm) FEM</td>
<td>17.56</td>
<td>14.77</td>
<td>23.2</td>
<td>22.68</td>
</tr>
<tr>
<td>Front Panel deflection (mm) Experiment</td>
<td>18.08</td>
<td>13.47</td>
<td>27.4</td>
<td>26.91</td>
</tr>
</tbody>
</table>

4.6 Error Estimation of Front Face Deflections

To better evaluate the error estimation for the front face deflections, we used a Russell error estimation. Russell error is a mathematical error used for quantifying transient data for magnitude and phase error [60], and has been used to correlate and validate finite element results. The basics behind Russell error is to quantify the transient data \( f \) of length \( N \) as a vector with magnitude and direction:

\[
\vec{f} = S\hat{\phi}
\]  

(4.6-1)

Where, \( \hat{\phi} \) is a unit vector for phase error and \( S \) is scalar magnitude for magnitude error. Relative magnitude error between two vectors \( \vec{f}_1 \) and \( \vec{f}_2 \) can be expressed by:

\[
M = \frac{(S_1^2 - S_2^2)}{S_1S_2}
\]  

(4.6-2)
With \( S = \sqrt{\sum_{i=1}^{N} f(i)^2} \).

Phase error is measure of phase correlation between two vectors. Relative phase correlation between two vectors \( \overrightarrow{f_1} \) and \( \overrightarrow{f_2} \) can be expressed as:

\[
A = \frac{\sum_{i=1}^{N} f_1(i)f_2(i)}{\sum_{i=1}^{N} f_1(i)^2 \ast \sum_{i=1}^{N} f_2(i)^2}
\]  

(4.6-3)

Further, it is given that phase correlation between \( \overrightarrow{f_1} \) and \( \overrightarrow{f_2} \) is equivalent to phase shift between two trigonometric function ranging from 1 to -1 is given by

\[
\varepsilon_p = \cos^{-1}(A)/\Pi
\]  

(4.6-4)

Phase error and magnitude are combined together as a comprehensive error. Since the magnitude error is unbounded, it will dominate the comprehensive error (RC). Hence to bring the magnitude error and phase error to the same scale, the magnitude error is expressed as:

\[
\varepsilon_m = \text{sign}(M) \log(1+|M|)
\]  

(4.6-5)

Use of a comprehensive error factor substantially reduces the comparison effort. Combined comprehensive error is combination between magnitude and phase error with an arbitrary constant of \( \Pi/4 \)

\[
\varepsilon_m = \text{sign}(M) \log(1+|M|)
\]  

(4.6-6)
In this chapter, \( \vec{f_1} \) and \( \vec{f_2} \) are transient and measured response for front face deflections respectively.

Table 3 provides the Russell error of the front face deflections for the four core arrangements. Comprehensive correlation are given by correlation defined by Russell [60-61] as: Excellent - RC<0.15, Acceptable-0.15<RC≤0.28, and Poor RC>0.28. There is an excellent correlation between the experiment and the present approach, while acceptable correlation between the experiment and the previous approach by Zhang et al. [47]

Table 3: Comparison of Russell errors between the present approach and Zhang et al. [47]

<table>
<thead>
<tr>
<th>Core arrangement</th>
<th>Present FEA</th>
<th>Zhang at al. [47]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Magnitude error</td>
<td>Phase error</td>
</tr>
<tr>
<td>ABBC</td>
<td>0.0099</td>
<td>0.0105</td>
</tr>
<tr>
<td>AABC</td>
<td>0.0954</td>
<td>0.0368</td>
</tr>
<tr>
<td>ABCC</td>
<td>0.0954</td>
<td>0.0368</td>
</tr>
<tr>
<td>AACC</td>
<td>0.0376</td>
<td>0.0257</td>
</tr>
</tbody>
</table>

4.7 Additional FEA results and discussion

In the following section, additional FEA results which signifies blast performance of corrugated beam such as Plastic energy absorption, Contact force, Stresses and Strains are discussed.
4.7.1 Energy Quantities

Figure 4.7-1 shows the energy absorbed by the sandwich beams with different graded corrugated layers. The plastic energy is dissipated mostly by the core as shown in Table 4. Plastic energy absorption in the front substrate begins around 0.5 ms. Plastic energy absorption in the back substrate starts later around 1.5 ms. Higher total amount of plastic energy is absorbed by ABCC and AACC than ABBC and AABC. In terms of percentage, the corrugated cores AABC and ABBC absorb higher percentage of plastic energy in the core than AACC and ABCC. Least percentage of plastic energy is absorbed by AABC in the back substrate.
Figure 4.7-1 Plastic energy absorption by substrates and core for (a) AABC (b) ABBC (c) AACC and (d) ABCC

Table 4: Plastic energy absorption percentage

<table>
<thead>
<tr>
<th>Core</th>
<th>Total plastic energy absorption</th>
<th>Percentage (%) absorbed by Core</th>
<th>Percentage (%) absorbed by Front substrate</th>
<th>Percentage (%) absorbed by Back substrate</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABBC</td>
<td>57.3</td>
<td>75.3</td>
<td>16.5</td>
<td>8.9</td>
</tr>
<tr>
<td>AABC</td>
<td>52.4</td>
<td>81.4</td>
<td>14.5</td>
<td>4.1</td>
</tr>
<tr>
<td>ABCC</td>
<td>70.6</td>
<td>71.0</td>
<td>23.6</td>
<td>6.3</td>
</tr>
<tr>
<td>AACC</td>
<td>76.9</td>
<td>70.0</td>
<td>22.1</td>
<td>7.8</td>
</tr>
</tbody>
</table>

4.7.2 Von Mises stress histories

Figure 4.7-2 below shows the von Mises stress for the graded core arrangements. Von Mises stress at the center of front face starts to increase almost immediately when the load is applied. For the sandwich plates with corrugated layers ABBC and AABC, with relatively smoothly graded layers, stress in the back face starts to increase after a time lag of about 0.25ms. In case of ABCC and AACC cores, the stress begins to increase at around 1ms. This difference in time is due to delay in stress wave reaching the back face in case of ABCC and AACC cores. Stress value reaches yield stress values in all the cases. After the core densification, stress in the back face reaches maximum value. The maximum stress value in both front and back face have a similar value. Stress in both the faces starts to decrease as the magnitude of the pressure load applied decreases.
Maximum stress in the back face is sustained for a longer period in case of AACC and ABCC core arrangement. This implies that back face endures higher stress for longer period in AACC and ABCC than in case of ABBC and AABC.

Figure 4.7-2 Von Mises stress histories at the center of front and back face for (a) ABBC (b) AABC (c) ABCC and (d) AACC.

4.7.3 Plastic strain histories

Figure 4.7-3 shows plastic strain histories of Center of the back substrate and the front substrate. Plastic strain begins to increase at around 0.6ms that is when the load is fully applied. After 0.6
ms, core compression starts to take place. As the stress waves are transferred to the back substrate, the back substrate begins to experience plastic strain at around 1.5ms. Plastic strain reaches maximum value at the center of front substrate around 1.25ms for ABBC and AABC cores. For ABCC and AACC cores plastic strain value reaches maximum value around 1.50 ms.

In case of ABBC and AABC stepwise plastic strain increment is seen in the back face where as in AACC and ABCC, strain increment is relatively abrupt. In addition, Higher value of strains are sustained by back substrate with cores composed of two soft cores at the loading i.e. ABCC and AACC. This is because two soft cores go through extensive compression and transfer more stress towards the back. Whereas, in case of ABBC and AABC cores, single soft C core at the front goes through compression while comparatively stronger second layer B resists the compression.

![Graphs showing plastic strain over time for ABBC and AABC cores.](image-url)
Figure 4.7-3 Plastic strain histories at the center of front and back face for (a) ABBC (b)AABC (c)ABCC and (d) AACC

4.7.4 Contact reaction force at supports

Figure 0-4 shows the contact reaction force between the support and the specimen for the arrangements: ABBC, AABC, ABCC and AACC. Since the separation in the normal direction between the support and the corrugated sandwich plate is not allowed, the contact force may have a negative value at some time. Core AACC starts to develop higher contact forces much later than the rest of the core arrangements. ABCC core develops higher contact forces between support and specimen at a later time than ABBC and AABC core arrangements. This is because transfer of stress wave to the support occurs later due to initial core crushing of C-C layers in the front. It is seen that AABC core has the smallest value of maximum contact force while AACC core arrangement has the highest contact force.
4.8 Parametric study: Reverse core arrangement

In this section, numerical study is done for graded corrugated cores with same density in the reversed order with thickest core layer facing the shock loading. Studies have shown a significant contribution of core arrangement on blast performance of the beam [77]. Since test data for reflected impulse loading for the reversed core systems is unavailable, we make a simple assumption for the impact loading by averaging impulse obtained from constituent cores. For instance with the CBBA core, the 1st A core layer on the impact side takes 25 percentage of loading from AAAA and the 4th C core layer takes 25 percentage of loading from CCCC. In addition, the BB core layers in the middle are assumed to take 50 percent of loading from BBBB core. Figure 4.8-1 shows the postulated pressure profiles used in our analysis. This postulated pressure assumption gives similar pressure profiles given in Figure 4.2-2 (c).
4.9 Results and Discussion

4.9.1 Deformed shapes and mid-span deflection: reverse core arrangement

Figure 4.9-1 shows deformed shapes and compares mid-span deflection of the reversed core arrangement, i.e. CBBA. For the reverse core arrangements, we see that core crushing of the layer with least density placed in the back starts around 0.5 ms. At around 1 ms, the front beam starts to bend. Higher crushing of the 4th core layer closest to the back face is observed afterward. Larger deformations and stresses are transmitted from the first two core layers toward the back face as seen in times 2 ms-3ms. The front plate in the reverse core system has a larger deflection over the time. This may be due to higher impulse onto the back face and stepwise crushing starting from the back face.
Figure 4.9-1 (a) Deformed shapes of CBBA (b) Comparison of mid-span deflections: CBBA vs ABBC

Figure 4.9-2 shows deformed shapes and compares mid-span deflection of the reversed core arrangement, i.e. CBAA. For this case, we see similar response in terms of core crushing of the 4th core layer with least density placed in the back and the front beam bending. The first two front core layers are stiffer than the other two layers, so the crushing primarily occurs in those two back
core layers. Larger deformations and stresses are transmitted from the first two core layers toward the back face, but some stresses remaining in the front face as seen at times 2 ms-3ms. The front and back faces tend to deform together. Compared to the original core specimen, the front plate in the reverse core system has a larger deflection over the time from 1.2 ms.

Figure 4.9-2  (a) Deformed shapes of CBAA  (b) Comparison of mid-span deflections: CBAA vs AABC
Figure 4.9-3 shows deformed shapes and compares mid-span deflection of the reversed core arrangement, i.e. CCBA. In this case, we observe core crushing of the two core layer (CC) with least density placed in the back. The two core layers in the back are more compliant than the first two layers, so the crushing severely occurs in those back core layers. Core crushing of the second C layer occurs along with bending of the back substrate around 1.5 ms. Similar to the CBAA case, larger deformations and stresses are transmitted from the first two core layers toward the back face, but some stresses remaining in the front face as seen at times 2 ms-3ms. The front and back faces tend to deform together. Mid-span displacements are similar in between the original core specimen, i.e. ABCC and the reverse core system, i.e. CCBA.
Figure 4.9-4 shows deformed shapes and compares mid-span deflection of the reversed core arrangement, i.e. CCAA. In this case, we observe severe core crushing of the two core layer (CC) with least density placed in the back. The two core layers in the back are much more compliant than the first two layers, so the crushing severely occurs in those back core layers. Different from the previous cases, the back face is confined with the contact between the crushed back core layers and the stiffer front core layers, thus making the front core take higher stresses. Mid-span displacements are higher in the original core specimen until time 1.6 ms, but the reversed core system takes higher deformations afterward.
4.9.2 Energy Quantities: reverse core arrangement

Figure 4.9-5 shows the plastic energy absorbed by the sandwich beams with reversed corrugated layers. Plastic energy absorption percentage by substrate and core is summarized in Table 5. Higher percentage of plastic energy is dissipated by the core. Bottom substrate starts energy
dissipation around 1.7 ms, which is when stress reaches bottom substrate. Top substrate and bottom substrate dissipate similar amount of plastic energy. When we compare plastic energy absorbed by the whole structure, it can be observed that CBBA core arrangement absorbs highest amount of plastic energy while CBAA absorbs least amount of plastic energy. In terms of percentage, it is seen that CBBA core dissipates more plastic energy. In case of the core arrangement CCBA, bottom substrate starts dissipating plastic energy faster than the top substrate. This is due to instantaneous core crushing of the C layer at the back and stress wave affecting the back substrate early.

(a)  

(b)
Figure 4.9-5 Plastic energy absorption by substrates and core for (a) CBBA (b) CBAA (c) CCBA and (d) CCAA

Table 5 Plastic energy absorption percentage

<table>
<thead>
<tr>
<th>Core</th>
<th>Total PE absorption</th>
<th>Percentage (%) absorbed by Core</th>
<th>Percentage (%) absorbed by Front substrate</th>
<th>Percentage (%) absorbed by Back substrate</th>
</tr>
</thead>
<tbody>
<tr>
<td>CBBA</td>
<td>97.4</td>
<td>76.3</td>
<td>14.5</td>
<td>9.2</td>
</tr>
<tr>
<td>CBAA</td>
<td>67.0</td>
<td>87.7</td>
<td>7.5</td>
<td>4.8</td>
</tr>
<tr>
<td>CCBA</td>
<td>86.7</td>
<td>71.3</td>
<td>11.8</td>
<td>15.7</td>
</tr>
<tr>
<td>CCAA</td>
<td>80.0</td>
<td>82.9</td>
<td>10.4</td>
<td>6.7</td>
</tr>
</tbody>
</table>
4.9.3 Contact reaction force at supports

Figure 4.9-6 shows the contact reaction force between the support and the specimen for the arrangements: CBBA, CBAA, CCAA and CCBA. Since the separation in the normal direction between the support and the corrugated sandwich plate is not allowed, the contact force may have a negative value at some time. It can be deduced from the contact force at the support in the reverse core arrangement that until 1.25 ms, there is less amount of stress in the contact between support and specimen. During this time core compression of the least density layer at the back takes place. As the coupling of bending and core compression takes place, after 1.25 ms, the contact forces in the support starts to increase significantly. The reverse loading cases show a significantly higher reaction force at the support than the original core arrangement. CCAA gives the highest reaction force, while CBBA gives the least.

![Graph showing contact reaction force](image)

Figure 4.9-6 Contact reaction force between the specimen and support.
4.9.4 Comparative Discussion on Graded Core Arrangement

Graded corrugated cores with same density were reversed in order with highest density layer facing the shock loading. Numerical study of these cores allowed us to see the failure behavior of the beam. Experimental study of two core arrangements done by Wang et al. [44] showed arrangement with least density foam towards blast loading had higher core compression and smaller back face deflection than arrangement where higher density core was facing shock tube loading. Studies have shown a significant contribution of core arrangement on the blast performance of the beam. Zhang et al. [47] studied three core arrangements and showed monotonically increasing core gradation gives a superior performance.

When core arrangement is reversed, It is observed that mode of failure is still core compression and combined core compression and bending. Core compression starts from lower density layer which is near the support. Higher core compression near the support leads to higher damage in the back face. Table shows comparison of maximum support reaction. It is seen that reversed core arrangement have higher support reaction. Least maximum support reaction is seen in case of CBBA core and highest maximum support reaction in case of CCAA core arrangement. The strong layers in the front resists compression and the load is transferred to the subsequent soft layers. Due to this, higher reaction force is seen in the support.
Table 6 Comparison of Support reaction between Normal core and Reversed core arrangements.

<table>
<thead>
<tr>
<th>Core</th>
<th>Maximum Reaction Force at the support</th>
<th>Reverse core arrangement</th>
<th>Maximum Reaction Force at the support</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABBC</td>
<td>2483</td>
<td>CBBA</td>
<td>3702</td>
</tr>
<tr>
<td>AABC</td>
<td>2268</td>
<td>CBAA</td>
<td>7096</td>
</tr>
<tr>
<td>ABCC</td>
<td>2663</td>
<td>CCBA</td>
<td>5552</td>
</tr>
<tr>
<td>AACC</td>
<td>3079</td>
<td>CCAA</td>
<td>9551</td>
</tr>
</tbody>
</table>

Table presents maximum mid span deflection in the reversed core arrangements. Maximum mid span deflection is an important criterion for the design than the final deflection so we have taken maximum mid span deflection comparison into account it is seen that when the cores are reversed both front face as well as back face maximum deflections are higher. Similar to the reasoning for higher support reaction, higher back face deflection is due to transfer of load to the soft cores near the back face.
Table 7 Comparison of Mid span deflections between Normal core and Reversed core arrangements.

<table>
<thead>
<tr>
<th>Graded core arrangements</th>
<th>Back Panel deflection FEM</th>
<th>Front panel deflection FEM</th>
<th>Reverse graded core arrangement</th>
<th>Back Panel deflection FEM</th>
<th>Front panel deflection FEM</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABBC</td>
<td>7.93</td>
<td>18.08</td>
<td>CBBA</td>
<td>8.35</td>
<td>20.36</td>
</tr>
<tr>
<td>AABC</td>
<td>7.08</td>
<td>14.77</td>
<td>CBAA</td>
<td>8.33</td>
<td>18.27</td>
</tr>
<tr>
<td>ABCC</td>
<td>9.16</td>
<td>23.20</td>
<td>CCBA</td>
<td>10.08</td>
<td>25.28</td>
</tr>
<tr>
<td>AACC</td>
<td>9.00</td>
<td>22.68</td>
<td>CCAA</td>
<td>14.73</td>
<td>24.95</td>
</tr>
</tbody>
</table>

Table gives the plastic energy absorption by the core in terms of percentage absorption. It is seen that higher energy is absorbed by the back face in general when the cores are reversed. It can be observed that Core absorbs higher percentage of plastic energy in case of reversed arrangement. Also, front substrate absorbs less energy in reversed core arrangement than in the normal core arrangement. Because of comparatively higher magnitude of loading in CBBA reversed loading case, higher plastic energy is absorbed by this core arrangement.
Table 8 Comparison of Plastic energy absorption between Normal core and Reversed core arrangements.

<table>
<thead>
<tr>
<th>Core</th>
<th>Total PE absorption</th>
<th>Percentage (%) absorbed by Core</th>
<th>Percentage (%) absorbed by Front substrate</th>
<th>Percentage (%) absorbed by Back substrate</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABBC</td>
<td>57.3</td>
<td>75.3</td>
<td>16.5</td>
<td>8.2</td>
</tr>
<tr>
<td>CBBA</td>
<td>97.4</td>
<td>76.3</td>
<td>14.5</td>
<td>9.1</td>
</tr>
<tr>
<td>AABC</td>
<td>53.1</td>
<td>81.4</td>
<td>14.5</td>
<td>4.1</td>
</tr>
<tr>
<td>CBAA</td>
<td>67.0</td>
<td>87.7</td>
<td>7.5</td>
<td>4.8</td>
</tr>
<tr>
<td>AACC</td>
<td>76.9</td>
<td>70.0</td>
<td>22.1</td>
<td>7.8</td>
</tr>
<tr>
<td>CCAA</td>
<td>80.0</td>
<td>82.9</td>
<td>10.4</td>
<td>6.7</td>
</tr>
<tr>
<td>ABCC</td>
<td>70.6</td>
<td>70.3</td>
<td>23.7</td>
<td>6.5</td>
</tr>
<tr>
<td>CCBA</td>
<td>86.7</td>
<td>71.3</td>
<td>15.7</td>
<td>11.8</td>
</tr>
</tbody>
</table>

4.10 Comparison of lower order C3D8 elements with C3D8I elements

For dynamic problems in abaqus C3D8 elements and C3D8I elements are available to discretize the model. The C3D8 element is a general purpose linear brick element, fully integrated (2x2x2 integration points). Due to the full integration, the element will behave badly for material with high values of Poisson’s ratio or elements which have plastic behavior. The element also behaves badly under bending load. C3D8I element also uses full integration. It is general purpose brick element with 9 incompatible modes. The advantage of this element is that, it has additional bending modes which helps to give a better solution. Equation (4.10-3)- (4.10-1) give the displacement
function of an 8-node solid element with 9 additional degrees of freedom. To demonstrate the superior behavior of C3D8I elements over C3D8 elements, we have used improved-loading models.

Figure 4.10-1 C3D8 element

\[ [B] = [B_d \ B_a] \] (4.10-2)

\[
u = \sum_i N_i u_i + (1-\xi^2)a_1 + (1-\eta^2)a_2 + (1-\zeta^2)a_7 \] (4.10-3)

\[
v = \sum_i N_i v_i + (1-\xi^2)a_3 + (1-\eta^2)a_4 + (1-\zeta^2)a_8 \] (4.10-4)

\[
v = \sum_i N_i v_i + (1-\xi^2)a_5 + (1-\eta^2)a_6 + (1-\zeta^2)a_9 \] (4.10-5)

Figure (4.10-2) compares mid-span deflection different corrugated cores discretised with C3D8 and C3D8I elements. Front face and back face deflections are compared with the test results. From the observation of the deflection plot, it can be clearly seen that C3D8I elements perform far better than the C3D8 element. It can be concluded that C3D8 elements give highly inaccurate solution due to over stiffening. Although the computation time for C3D8 is lower than C3D8I elements, incompatible mode elements (C3D8I) is recommended.
Figure 4.10-2 Front face and back face deflection comparison between C3D8 and C3D8I elements for AABC, AACC, ABBC and ABCC core arrangements respectively.
4.11 Equivalent properties modelling for simplified model.

Modelling corrugated beams is tedious process compared to modeling a monolithic beams in finite element software. FEM was used to get equivalent properties of sandwich structures with various cores [88]. Although commercial codes allow one to analyze corrugated structures by meshing all corrugations, it requires significant computational time [90]. The structural parameters of corrugated beam affects its mechanical performance. To model a large structure with corrugation, the complex structure can be regarded as homogenous section. This will make the structure simplified thereby making it easier to model as well reduce the computation process. Several authors have investigated homogenization approach using equivalent properties [89, 90,92].

![Diagram of corrugated beam represented by homogenous beam.](image)

Figure 4.11-1 Corrugated beam represented by homogenous beam.

For a sinusoidal corrugated case as shown in Figure 4.11-1, we can evaluate young’s modulus using formula given in [91]. Modulus for these structure depend on the shapes and corrugation. For a length wise corrugation, modulus and bending stiffness is calculated.
For a corrugated core as shown in Figure 4.11-2, the dimensionless middle surface length is given by:

\[ s_0 = \int_0^1 \sqrt{1 + c_0^2 \cos^2(2\pi \xi)} \, d\xi \]  

(4.11-1)

Where,

\[ C_0 = \pi t_c (1 - x_0) / a_0 \]

\[ x_0 = \frac{t_0}{t_c} \]

\[ \xi = \frac{x}{a_0} \]

![Figure 4.11-2 Cross section of the lengthwise corrugated beam](image)

The bending modulus of the corrugated beam can be calculated using equation (4.11-2).

\[ I_z = \frac{1}{12} t_c^3 \left[ (2x_1 (4x_1^2 + 6x_1 + 3) + (x_0^3) / s_0) \right] \]  

(4.11-2)

Where,

\[ x_1 = \frac{t_f}{t_c} \]
Modulus of the section with different core thickness \( (t_0) \) and flange thickness \( (t_f) \) is calculated. Then the equivalent modulus of a homogenous section with the same overall thickness is calculated. Basic assumption made for these equivalent section is that bonding quality between core and the face panels is perfect and its possible effects on local stiffness and behavior are not considered [93]. The deflection of both corrugated and homogenous section is calculated using Abaqus. Length of the simply supported beam is 750 mm. Two loads (700 N) are applied at the distance of 250 mm from each end. The mid-span deflection of corrugated beam and the homogenous beams are tabulated in Table 9. It can be seen that homogenous beams give almost same results as the corrugated beams.

Table 9 Comparison of mid-span deflection of corrugated beam with the homogenous beam.

<table>
<thead>
<tr>
<th>c(core thickness)</th>
<th>Tf(flange thickness)</th>
<th>FEA deflection of corrugated beam</th>
<th>FEA Deflection of homogenous beam</th>
<th>Percentage error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>1</td>
<td>31.21</td>
<td>31.26</td>
<td>0.16</td>
</tr>
<tr>
<td>0.5</td>
<td>1</td>
<td>29.45</td>
<td>29.43</td>
<td>0.20</td>
</tr>
<tr>
<td>0.3</td>
<td>1.5</td>
<td>18.35</td>
<td>18.84</td>
<td>0.26</td>
</tr>
<tr>
<td>0.3</td>
<td>2</td>
<td>12.6</td>
<td>12.97</td>
<td>0.29</td>
</tr>
</tbody>
</table>

Figure 4.11-2 and Figure 4.11-3 show deflection of the corrugated core and equivalent section respectively. It can be observed that both beams exhibit similar bending behavior with same magnitude of deflection.
Figure 4.11-3 Deflection of the beam with corrugated core. (tc = 0.3 and tf=1)

Figure 4.11-4 Deflection of the beam with equivalent homogenous section.
4.11.1 Homogenization of multilayered cores.

For a multilayered core, equivalent modulus is evaluated for each cores. Magnucka-Blandzi et al. [90] analyzed shearing effect for sandwich beams with corrugated cores. The study was carried out for corrugated cores in crosswise and lengthwise direction. Elastic constants for the corrugated beam was analyzed in their study. Based on the expressions in their study, we calculated the bending modulus of sinusoidal cores. Equation 4.11-3 gives the modulus of a single layer of corrugated core. For dynamic analysis, the density of the corrugated core can be calculated based on the actual mass of the core and dividing it by the volume.

\[
Ex = \left( \frac{x_0^3}{s_0} \right) E \quad (4.11-4)
\]

For homogenization, we have added another layer of core to previous example. Figure 4.11-2 and Figure 4.11-3 show deflection of the multilayered corrugated core and equivalent section respectively. Each homogenous layer is assigned equivalent modulus corresponding to the core section. It can be observed that both beams exhibit similar bending behavior with same magnitude of deflection.

(a)
4.12 Concluding remarks

Four corrugated cores under shock loading is studied numerically in this study. The back face and front face deflections, plastic energy absorption, Von misses stress, Plastic strain and the contact force between support and specimen for all the corrugated sandwich beams are discussed. Simplified loading assumption as an alternative to fluid structure interaction is proposed. The previous assumption of the pressure load as a trapezoidal load has been modified. Variation in loading area is added and is dependent on time. The time is chosen based on the deflection of the front substrate.

A better correlation is seen through improved loading assumption. Error quantification is done using Russell error. Russell error for Front face deflection comparison between current results and test results indicates an excellent correlation while previous approach by Zhang et al. [47] gives just acceptable correlation.

AABC and ABBC showed similar performance. Least deflection as well as highest plastic energy absorption is seen in these two cores. Cores with 2 soft layers near supports (C-C) show higher deflection, less plastic energy absorption, higher support reaction as well as higher strain. Thus,
graded corrugated beams having AABC and ABBC give better blast performance than AACC and ABCC.

Core compression starts with the least density layers i.e. C layers in all the cases. Which is accompanied by coupling of core compression and beam bending. It is seen to be the common method for beam failure. Core layer arrangement is seen to be important factor in blast performance rather than the overall relative density of the cores.

In addition to the study above, Study of Graded corrugated cores with same density, reversed in order with highest density layer facing the shock loading is done in this study. Using the postulated loading and applying the loading on to the beams using the current approach, we could deduce the following:

Analysis of reversed core arrangements show beam failure mode to be core compression as well as combined core compression and beam bending. However, Core densification initiates from the soft cores which are near the back face or the main structure. Stress waves are thus transferred to the furthest layer before the layer near the shock loading starts to compress. The deflection of beam in each case, show an initial lower deflection than when soft cores are facing the load, this suggests resistance of stronger cores at the front. The deflection in the beam increases after core densification of the soft core at the back is complete. The maximum deflection of the reversed core is higher than the deflection of beam with normal core arrangement. Also, in each case increased support reaction and higher plastic energy absorption in the back face when cores are reversed.

- Among the Reversed arrangements, CCAA core gave a higher deflection in front face and back face as well as very high reaction force. Energy was dissipated as plastic energy. It is seen that higher amount of plastic energy was absorbed by the reversed core arrangements. Generally, higher plastic energy was absorbed in the back face. It can be concluded that,
while designing sandwich beam for blast performance, core arrangement plays a vital role than density of the sandwich core.

From comparative study between the incompatible elements and lower order compatible element for dynamic analysis, it can be seen that lower order compatible element produces highly inaccurate results. Hence, Incompatible elements are recommended.

### 4.13 Acknowledgement

We gratefully acknowledge the financial support from the U.S. Department of Homeland Security (Award 2008-ST-061-TS0002-02) to the University of Connecticut through Center for Resilient Transportation Infrastructure (Director: Prof. Michael Accorsi). We also acknowledge material processing and characterization by Prof. Rainer Hebert’s group at University of Connecticut and strain rate dependent properties and shock tube test provided by Prof. Arun Shukla’s group at University of Rhode Island.

### 5 Conclusion

The three objectives of the research presented in this dissertation are explained in chapter 1. Chapter 1 also includes theory of incompatible graded elements. This includes both isotropic and orthotropic cases. In chapter 1, stability of incompatible graded and homogenous elements are studied. It is shown that incompatible graded element is stable and has different eigenvalues compared to Q4 graded elements. In chapter 2 incompatible graded elements are developed for isotropic graded materials. In this chapter, stresses are compared for several isotropic graded cases. The superiority of QM6 graded elements is clearly shown over lower-order compatible elements. In chapter 3, orthotropic gradation is studied with the help of various examples. Russel error is
used to quantify the error magnitude between QM6 element and Q4 element. Chapter 3 further reinforces the objective of this dissertation. Incompatible graded elements for orthotropic graded materials demonstrate accuracy over lower-order compatible elements. Thus incompatible graded elements are recommended for analysis of graded materials.

Chapter 4 introduces accurate loading scheme for dynamic load modelling of the shock tube. In this chapter, accurate loading scheme is demonstrated for the shock tube loading. Core optimization is done through parametric study. Comparison between incompatible element and lower order compatible elements is done for dynamic models for the improved loading model. The front and back face deflections are compared. Incompatible elements clearly give highly accurate solution compared to lower order compatible element. This chapter also introduces homogenization scheme for corrugated cores, which will help to simply modeling of corrugated sandwich beams.
6 References


7 Appendix

Appendix 1

1. First computational model is created.

2. Next step is to define property by calling UMAT in Abaqus.
   i. Following is exponential modulus variation that for isotropic and orthotropic materials respectively.

   \[ E(x) = E_1 e^{\beta x} \quad \text{(isotropic)} \]

   \[ E_{11}(x) = E_{11}^0 e^{\beta_{11} x} \quad E_{22}(x) = E_{22}^0 e^{\beta_{22} x} \quad G_{12}(x) = G_{12}^0 e^{\beta_{12} x} \quad \text{(orthotropic)} \]

   Where, \( \beta \) is length scale characterized by \( \beta = \frac{1}{L} \ln(E_1 / E_2) \). \( E_1 \) and \( E_2 \) are modulus at two different edges respectively.

   ii. Tensor coefficients related to normal stresses are updated. For plane stress orthotropic elements these components are given by DDSDE matrix:

   \[
   [\text{DDSDE}] = \frac{E}{(1-\nu^2)} \begin{bmatrix}
   E_{11} & -\nu E_{22} & 0 \\
   -\nu E_{22} & E_{22} & 0 \\
   0 & 0 & G_{12}
   \end{bmatrix}
   \]

   For an isotropic elements, \( E_{11} = E_{22} = E \) and \( G_{12} = \frac{E}{2(1+\nu)} \)

   iii. Stresses are updated at the gauss points. For loop is used to update stresses. For a two-dimensional plane elements two in-plane components exists therefore NTENS=2. For three dimensional elements NTENS=3.

   For I=1:NTENS
   For J=1:NTENS
     \[
     \text{STRESS}(I) = \text{STRESS}(J) + \text{DDSDDE}(IJ) * \text{DSTRAN}(J)
     \]
   End
   End

   Here, DSTRAN denotes array of strain increments, STRESS is passed as array of stress tensor.