8-10-2020

Homomesy for Foatic Actions on the Symmetric Group

Elizabeth Sheridan-Rossi
University of Connecticut - Storrs, elizabeth.sheridan-rossi@uconn.edu

Homomesy for Foatic Actions on the Symmetric Group

Elizabeth Sheridan Rossi, Ph.D.

University of Connecticut, 2020

ABSTRACT

In this thesis, we consider two different families of maps on the symmetric group $\mathfrak{S}_n$, each created by intertwining a bijection of Foata with dihedral involutions on permutation matrices. Iterating each map creates a cyclic action on $\mathfrak{S}_n$, partitioning it into orbits. This allows us to look at statistics that have the same average value over each orbit, called *homomesic*. The homomesy phenomenon was first proposed by Propp and Roby in 2011, and many instances have been found across a wide range of combinatorial objects and maps.

The first family of maps involves the so-called "fundamental bijection" of Rényi and Foata, which "drops parentheses" from a permutation in canonical disjoint cycle decomposition. The second, due to Foata and Schützenberger, was originally used to provide a bijective proof showing the equidistribution across $\mathfrak{S}_n$ of the inversion number and the major index. Computations in SageMath led to a number of conjectural homomesies on well-known permutation statistics. We prove many of them here, and state the remainder as open problems.

Homomesy for Foatic Actions on the Symmetric Group

Elizabeth Sheridan Rossi

M.S. New York University, 2012

B.S. Bates College, 2006

A Dissertation

Submitted in Partial Fulfillment of the

Requirements for the Degree of

Doctor of Philosophy

at the

University of Connecticut

2020

APPROVAL PAGE

Doctor of Philosophy Dissertation

Homomesy for Foatic Actions on the Symmetric Group

Presented by

Elizabeth Sheridan Rossi, B.S., M.S.

Major Advisor ______________________
Tom Roby

Associate Advisor ______________________
David Reed Solomon

Associate Advisor ______________________
Ralf Schiffler

University of Connecticut

2020

ACKNOWLEDGMENTS

There were many wonderful people who made my long journey through UConn what it was. I owe a huge debt of gratitude to them all but I cannot go without mentioning a few by name.

First and foremost I have to thank my advisor Tom Roby. We share a love of many things, including math, and it was a pleasure to teach, learn and work with him for the past 7 years. He showed extraordinary patience through the many twists and turns my life took in the process and always encouraged me to just keep moving forward one step at a time.

Appreciation goes to Ralf Schiffler for serving on my committee and to Reed Solomon for doing the same, and also for being a fantastic graduate director and advisor to me along the way. I also thank Michael La Croix and Jim Propp for contributing the main questions that became this thesis.

Sincerest of thanks goes to Amit Savkar, without whom I would probably not have been able to stay in graduate school after becoming a mother. He provided invaluable opportunities for me and was such an excellent mentor and friend.

Monique Roy was always happy to see me and talk about something other than math. Keith Conrad and Fabiana Cardetti were inspiring teachers and supports throughout. To my wonderful classmates and friends Michelle Rabideau, Dan Martin, Phanuel Mariano, Bobby McDonald, Melissa Haire, Rebecca Moran, Mike Joseph, Sangjoon Lee, and David Nichols— I may be pulling up the rear but I made it to the end!

Many other people have been a part of my lifelong mathematical journey. I have to thank my oldest math friend, Nate Stambaugh, for a million laps around the puddle, encouraging me to major in math in the first place, and never balking at my attempts to follow in his mathematical footsteps. My undergraduate advisor, Bonnie Shulman, pushed me to attend my first ever math conference and was always an excellent example of what it means to be a woman in math. My wonderful colleagues at St. Ann's have fueled my love of math for many years as well as my excitement for teaching it.

I thank my siblings, Patrick and Jane, for falsely telling people I was a genius, and though I hate to admit it, giving me confidence just for believing it. Without my parents none of this would be possible. Thank you, Mom for truly believing that I could do this and encouraging me at every step. And thank you to my father, for teaching me to love math in the first place.

They were not here when I started this process, and I can't exactly say they made it easier, but my wonderful children Harris and Lucy always helped to remind me what really matters. Lastly, I have to thank the only person who truly knows what I went through to get to the end. The person who was there through every tear, every minor celebration, every doubt, and every success. To my husband Dave, thank you, thank you, thank you. For everything.

Contents

Chapter 1

Homomesy Introduction and Background

1.1 Introduction

The field of Dynamical Algebraic Combinatorics is the study of actions on sets of discrete combinatorial objects. One can see several recurring themes within the field, namely looking at whether an action is periodic, what its order is, and subsequent study of the orbit structure. From here a search for *homomesic* statistics (those with the same average value over all orbits) is a natural next step. The use of equivarient bijections is often a useful tool in this exploration.

The *cyclic sieving phenomenon* provides many instances where homomesy may also exist. Other fruitful areas of study so far have included actions that can be built up using smaller local changes. Examples of this include toggling, as seen on independent sets of path graphs [JR18], in the rowmotion action on antichains of a poset [PR15], and in promotion on semistandard Young tableaux (which we will see in Section 1.4.3). Tom Roby's exposition on "Dynamical Algebraic Combinatorics and the Homomesy Phenomenon" [Rob16] provides many more examples.

The explorations in this dissertation were largely influenced by Jim Propp's idea to look for instances of homomesy among the fundamental combinatorial objects counted by the "Twelvefold Way" of Rota [Stan11, Section 1.9]. One such object, permutations, will be the focus of study in this thesis.

In this chapter we will discuss the homomesy phenomenon and a few introductory examples. As much of the work in this thesis relates to permutations and their properties and behavior under certain actions, we begin in Section 1.2 giving an overview of these. The examples in Section 1.4 give a taste of the range of instances of homomesy in the existing literature. In Chapter 2 we look at maps created by intertwining the Rényi–Foata map and five dihedral involutions, and the resulting homomesic permutation statistics. In Chapter 3, we look at similar intertwinings involving the same involutions and a second map, constructed by Foata and Schützenberger. We again consider and prove many homomesic statistics for these actions. We note that Chapters 2 and 3 could be read independently. A list of all homomesy results and conjectures can be seen in Appendices A and B.

1.2 Permutations

Permutations are one of the most fundamental objects in all of combinatorics and appear in many branches of mathematics.

Definition 1.2.1. A **permutation** w **of** $[n] := \{1, 2, \ldots, n\}$ is a linear ordering $w_1, w_2, \ldots, w_n$ of the elements of $[n]$. If we think of w as a word $w_1 w_2 \cdots w_n$ in the alphabet $[n]$, then such a word corresponds to the bijection $w : [n] \to [n]$ given by $w(i) = w_i$.

Definition 1.2.2. The notation $w_1 w_2 \cdots w_n$ is called **one-line notation**. We can also write a permutation in **two-line notation** as $\left(\begin{smallmatrix} 1 & 2 & \cdots & n \\ w_1 & w_2 & \cdots & w_n \end{smallmatrix}\right)$. Alternatively we can write a permutation in **cycle notation**. A **cycle** of a permutation w is a sequence $(x, w(x), w^2(x), \ldots w^{l-1}(x))$ where $w^l(x) = x$. This is essentially the orbit of x under the action of w.

It is elementary to see that every permutation can be written as a product of disjoint cycles. To write a permutation in cycle notation, we represent it as a disjoint union of its distinct cycles.

Example 1.2.3. Consider the permutation $w = 326541$ in one-line notation. Here $w(1) = 3$, $w(2) = 2$, $w(3) = 6$, $w(4) = 5$, $w(5) = 4$ and $w(6) = 1$. In cycle notation this could be written $(136)(45)(2)$. The representation of w in disjoint cycles notation is not unique, though we will later see a way to define a canonical representation.

Definition 1.2.4. A pair (w_i, w_j) is called an **inversion** of the permutation $w = w_1 w_2 \cdots w_n$ if $i < j$ and $w_i > w_j$. We denote $\mathrm{Inv}(w)$ to be the set of inversion pairs in w and $\mathrm{inv}(w)$ to be the number of inversions.

For example, the permutation 25431 has the inversions $(2,1),(5,4),(5,3),(5,1),(4,3),(4,1),(3,1)$, with $\mathrm{inv}(w) = 7$.

We write $\mathfrak{S}_n$ for $\mathfrak{S}_{[n]}$, the symmetric group of permutations on $[n]$.

Definition 1.2.5. The **permutation matrix** W associated to $w = w_1 \cdots w_n$ is the $n \times n$ matrix given by $W = [a_{ij}]$ where

$$a_{ij} = \begin{cases} 1 & \text{if } w_i = j \\ 0 & \text{else} \end{cases}$$

For example, the permutation $w = 425316$ is represented by the following matrix.

$$W = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The inversion pairs of this permutation are $(4,2)$, $(4,3)$, $(4,1)$, $(2,1)$, $(5,3)$, $(5,1)$ and $(3,1)$. In a permutation matrix, two numbers w_i and w_j make up an inversion pair (w_i, w_j) if the 1 for w_i appears above and to the right of w_j.

For example, in the matrix below, we can see that circled 1 in the first row represents the 4 in our permutation and the circled 1 in the second row represents the 2 in our permutation. The "4" appears above and to the right of the "2", illustrating the inversion pair $(4,2)$.

$$\begin{pmatrix} 0 & 0 & 0 & \textcircled{1} & 0 & 0 \\ 0 & \textcircled{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The total number of inversions of a permutation matrix W is

$$\operatorname{inv} W = \sum_{\substack{1 \le i < i' < n \\ 1 \le j' < j \le n}} a_{ij} a_{i'j'}$$

The eight elements of the dihedral group act naturally on these matrices and give corresponding actions on permutations. As an example, if we consider the permutation matrix when flipped across the vertical axis we can see that this corresponds to the map $\mathcal{C} : \mathfrak{S}_n \to \mathfrak{S}_n$, which takes a permutation $w = w_1 \cdots w_n$ to its **complement**, whose value in position i is $n + 1 - w_i$.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$425316 \mapsto 352461$$

Similarly, a horizontal flip corresponds to the reversal map, $\mathcal{R} : \mathfrak{S}_n \to \mathfrak{S}_n$, which takes a permutation $w = w_1 \cdots w_n$ to its **reversal**, whose value in position i is w_{n+1-i}.

$$\begin{pmatrix} 0&0&0&1&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&1&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&0&1 \end{pmatrix} \mapsto \begin{pmatrix} 0&0&0&0&0&1\\ 1&0&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&1&0\\ 0&1&0&0&0&0\\ 0&0&0&1&0&0 \end{pmatrix}$$

$$425316 \mapsto 613524$$

A flip over the main diagonal corresponds to the inverse map, $\mathcal{I} : \mathfrak{S}_n \to \mathfrak{S}_n$, which takes a permutation w to its **inverse** w^{-1}.

$$\begin{pmatrix} 0&0&0&1&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&1&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&0&1 \end{pmatrix} \mapsto \begin{pmatrix} 0&0&0&0&1&0\\ 0&1&0&0&0&0\\ 0&0&0&1&0&0\\ 1&0&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&1 \end{pmatrix}$$

$$425316 \mapsto 524136$$

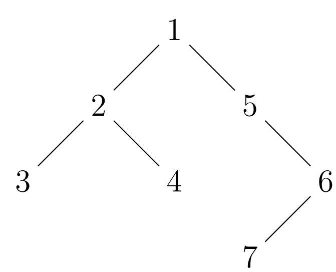
The inverse can also be computed by taking a permutation written in cycle notation, freezing the first number in each cycle, and reversing the order of the remaining numbers. So for example the permutation (7)(4512)(396) has inverse (7)(4215)(369).

In Chapter 2 and 3 we will discuss compositions of maps involving some of the dihedral actions which are also **involutions**, meaning that they are their own inverse. There we will also give more examples of these maps.

The last permutation representation that we will discuss is the visualization of the maps as trees. There are many ways to define bijections between permutations and trees, a few of which we will see in future sections. More on this topic can be seen in [Stan11, Section 1.5].

Definition 1.2.6. A **binary tree** is a tree structure in which each node has at most two children, which are referred to as the left child and the right child. An **increasing binary trees** has vertices labeled $1, 2, \ldots, n$, such that the labels along any path from the root are increasing.

Here we will give an example of a map between permutations and increasing binary trees. Define the map, T as follows. First, consider the permutation $w = w_1 w_2 \cdots w_n$ and identify w_i to be the least element of w (so for $w \in \mathfrak{S}_n$, $w_i = 1$). Now consider w to be factored as $w = u w_i v$. Let w_i be the label of the root of the binary tree, and think of $T(u)$ and $T(v)$ as left and right (respectively) subtrees. Applying these procedure recursively to $T(u)$ and $T(v)$ yields an increasing binary tree. Below we see the tree $T(3241576)$.



To reverse the map, we take a labeled increasing binary tree, and read the labels in "symmetric order", meaning we read them recursively in the order of the left subtree, the root of the tree, and the right subtree. This particular representation can shed light on certain properties of permutations, as well as properties of increasing binary trees. One can deduce from this bijection that the number of increasing binary trees with n vertices is $n!$, the number of permutations in $\mathfrak{S}_n$. [Stan11, Proposition 1.5.3]. In Chapter 3 we will see a similar tree representation for permutations and discuss related permutation statistics.

1.3 Introduction to Homomesy

Homomesy is a phenomenon that was identified by Tom Roby and James Propp in 2011 [PR15]. A group action on a set of combinatorial objects partitions the set into orbits. We call statistics **homomesic** if they have the same average value over each orbit. Examples of homomesy span the field of dynamical algebraic combinatorics and frequently also occur in examples of Reiner, Stanton and White's cyclic sieving phenomenon [RSW04]. We begin with a formal definition and an illustrative example.

Definition 1.3.1. Given a set $\mathcal{S}$, an invertible map $\tau : \mathcal{S} \to \mathcal{S}$ such that each τ-orbit is finite, and a function (or "statistic") $f : \mathcal{S} \to \mathbb{K}$ taking values in some field $\mathbb{K}$ of characteristic zero, we say that the triple $(\mathcal{S}, \tau, f)$ exhibits **homomesy** if there exists a constant $c \in \mathbb{K}$ such that for every τ-orbit $\mathcal{O} \subset \mathcal{S}$ $\frac{1}{\#\mathcal{O}} \sum_{x \in \mathcal{O}} f(x) = c$. In this situation we say that the function $f : \mathcal{S} \to \mathbb{K}$ is **homomesic** under the action of τ on $\mathcal{S}$ or more specifically **c-mesic** [PR15, Definition 1].

We begin with a basic example of homomesy. Let S_k^n be the set of binary strings of length n with exactly k 1s. Note that this set has $\binom{n}{k}$ elements. Previously we defined the term **inversion** for permutations. We can extend this definition to any **word** (an expression of the form $s_1 \ldots s_n$ where the s_i's are elements of some set S) in $[l]^n$.

Definition 1.3.2. A pair (w_i, w_j) is called an **inversion** of the word $w = w_1 w_2 \cdots w_n \in [l]^n$ if $i < j$ and $w_i > w_j$. We denote the number of inversions in the word w by $\mathbf{inv}(w)$.

In the case of a binary string, an inversion is an instance of a 1 appearing before a 0. For example in the string 10110 the first 1 appears before two 0s, the second 1 appears before one 0 and the third 1 appears before one 0; thus, there are a total of four inversions. For our example we consider elements of the set S_k^n under the action, τ, of rightward cyclic shifting. In the following examples the symbol ↰ indicates a return to the start of the orbit.

Example 1.3.3. If $n = 6$ and $k = 4$, the string 101011 under τ generates the following orbit:

$$101011 \to 110101 \to 111010 \to 011101 \to 101110 \to 010111 \,↰$$

After at most n steps we will return to our original string. The number of inversions for the elements of this orbit are $(3, 5, 7, 3, 5, 1)$.

There are a total of three orbits when $n = 6, k = 4$. The other two are as follows:

$$111100 \to 011110 \to 001111 \to 100111 \to 111011 \to 11101 \,↰$$

$$011011 \to 101101 \to 110110 \,↰$$

We could think of the later orbit as a **super orbit** of length 6 where each element is repeated. The table below illustrates the number of inversions for the elements of each cycle along with their orbit size and average number of inversions.

Orbit	Inversions	Orbit Size	Average Number of Inversions
1	$(8,4,0,2,4,6)$	6	$(8+4+0+2+4+6)/6 = 4$
2	$(3,5,7,3,5,1)$	6	$(3+5+7+3+5+1)/6 = 4$
3	$(2,4,6)$	3	$(2+4+6)/3 = 4$

Theorem 1.3.4. *The average number of inversions in an orbit under cyclic rotation for* S_k^n *is always equal to* $\frac{k(n-k)}{2}$*, which is the global average. So if* τ *is the cyclic rotation map and* f *is the number of inversions, we say that* f *is* $\frac{k(n-k)}{2}$*-mesic.*

Proof. To see why this claim is true, note first that any binary string of length n with k 1's can be converted into any other string by a sequence of transpositions switching adjacent numbers. For example we can convert the string 111100 to 101011 by the following moves where the bits being swapped are underlined in each step

$$111\underline{10}0$$

$$1110\underline{10}$$

$$11\underline{10}01$$

$$1\underline{10}101$$

$$101\underline{10}1$$

$$101011$$

This is similar to how any permutation can be generated by adjacent transpositions. We can think of 111100 and 101011 as being generators of two of the orbits shown above. We claim that replacing 10 with 01 does not change the total number of inversions in an orbit, thus showing that the average number of inversions in the orbits generated by these two strings are the same.

When the swap occurs in positions $i, i+1$ such that $1 < i < k-1$ we lose one inversion. There are $n-1$ such paired strings in an orbit of length n, essentially every string except the one where the 10 bit "rounds the corner". When the swap occurs in positions $1, k$ we gain a total of $(n-k)+(k-1) = n-1$ inversions. So the total inversion change is $(-1)\cdot(n-1)+(n-1)\cdot 1 = 0$. See the example in Figure 1.1.

Figure 1.1: Inversion changes in orbits under cyclic rotation

Orbit 1	Orbit 2	Δ inv(w)
111100	111010	−1
011110	011101	−1
001111	101110	+5
100111	010111	−1
110011	101011	−1
111001	110101	−1

It follows that the total number of inversions is the same for any super orbit, so the average number of inversions is the same for every orbit. Note that this average is equal to the global average, $\frac{k(n-k)}{2}$. The easiest way to see this is to consider the cycle which contains the binary string where all k 1s occur consecutively at the beginning, which clearly has $k(n-k)$ inversions. On the other hand, this cycle also contains the string with all k 1s appearing consecutively at the end, which has 0 inversions. So the average number of inversions for this particular string is $\frac{k(n-k)}{2}$, and thus this is the average number of inversions for all binary strings of length n with k 1s. $\square$

1.3.1 Cyclic Sieving Phenomenon

The cyclic sieving phenomenon was identified by Victor Reiner, Dennis Stanton, and Dennis White in 2004 [RSW04]. It exists in several of the known examples of homomesy. Like homomesy, CSP involves orbit structure and group actions on sets. We begin with the following definition, which we will illustrate with an example.

Definition 1.3.5. Let X be a finite set and $C = \langle c \rangle$ be a cyclic group of order n acting on a finite set X. Given a polynomial $X(q)$ with integer coefficients in a variable q, we say the triple $(X, C, X(q))$ exhibits the **cyclic sieving phenomenon** if for all integers d, the number of elements fixed by c^d equals the evaluation $X(\xi^d)$ where $\xi = e^{\frac{2\pi i}{n}}$. In particular $X(1)$ is the cardinality of X, so $X(q)$ can be regarded as a generating function for X [RSW04].

The cyclic sieving phenomenon generalizes the $q = -1$ **phenomenon** of John Stembridge which identified the case where C has order 2 and included examples involving plane partitions and Young tableaux. In Stembridge's work $X(1)$ gives the cardinality of X while $X(-1)$ gives an enumeration for symmetry classes within X [Stem94]. For our example we return to the scenario in the previous section where we let X be the set S_k^n of binary strings of length n with exactly k 1s.

Example 1.3.6. Let $n = 6$ and $k = 4$ as in Example 1.3.3 and consider the group

$C = \langle c_R \rangle$. We saw that the 15 elements of S_4^6 partition into three orbits. All 15 strings, are fixed under the action of C_R^6, and 3 are fixed under C_R^3, namely 110110, 101101 and 011011. No other strings are fixed by any other power of C_R.

Our polynomial is the well-known Gaussian binomial coefficient, defined as follows.

Definition 1.3.7. The following polynomials in $\mathbf{N}[q]$ are ***q*-analogues** of common combinatorial numbers. Notice when $q = 1$ we get n, $n!$ and $\binom{n}{k}$.

- $[n]_q := \frac{1-q^n}{1-q} = 1 + q + \cdots + q^{n-1}$
- $[n]_q! := [1]_q[2]_q \cdots [n]_q$
- $\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$

Define the polynomial $X(q) = \binom{n}{k}_q$, so in our example we have

$$X(q) = \binom{6}{4}_q = \frac{[6]_q!}{[4]_q![2]_q!} = \frac{(1+q+q^2+q^3+q^4+q^5)(1+q+q^2+q^3+q^4)}{(1+q)} =$$

$$1 + q + 2q^2 + 2q^3 + 3q^4 + 2q^5 + 2q^6 + q^7 + q^8$$

Now let $\xi = e^{2\pi i/6}$. Plugging in the sixth roots of unity into our polynomial gives the following:

$$X(\xi) = 0$$
$$X(\xi^2) = 0$$
$$X(\xi^3) = 3$$
$$X(\xi^4) = 0$$

$$X(\xi^5) = 0$$
$$X(\xi^6) = 15$$

So, for all $m \in \mathbb{Z}$, the number of elements fixed by C_R^m is given by $X(i^m)$. In other words, the triple $\left(\binom{[6]}{4}, X(q), \langle C_R \rangle\right)$ exhibits the cyclic sieving phenomenon.

1.4 Homomesy Examples

In this section we review examples of homomesy that highlight interesting subtleties about the phenomenon and show the breadth of its occurrences.

1.4.1 Dihedral Group Actions on Permutation Matrices

Our first example has to do with dihedral group actions on permutation matrices. As we saw in Section 1.2, dihedral group actions act naturally on permutation matrices. Applying the 90 degree clockwise rotation map R_{90} to a permutation matrix will form a homomesy triple. The permutations in $\mathfrak{S}_3$ break up into the following two orbits under R_{90}.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \hookleftarrow$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \hookleftarrow$$

The total number of inversions for each matrix are $(0, 3)$ for the first orbit and $(1, 2, 1, 2)$ for the second, so an average of $\frac{3}{2}$ for each.

Theorem 1.4.1. *Let S be the set of $n \times n$ permutation matrices., let R_{90} be the 90 degree rotation map. The statistic* $\operatorname{inv} W$*, the total number of inversions, is homomesic with average value* $\frac{n(n-1)}{4}$*.*

Proof. Let A be a permutation matrix where $A(i, j)$ is the entry in position (i, j), and consider an inversion pair (w_m, w_k). So $A(m, w_m) = 1$ lies in position (m, w_m) which is located above and to the right of $A(k, w_k) = 1$ in position (k, w_k). In other words, $m < k$ and $w_m > w_k$. But then after rotating by 90 degrees to positions (m', w'_m) and (k', w'_k) we will have $m' > k'$ and $w'_m < w'_k$, so there is no inversion. In other words, each inversion pair in W corresponds to a non-inversion pair after applying R_{90}. Thus, the average number of inversions in an orbit is equal to half of the maximum total inversions possible for any n, which is $\frac{1}{2} \cdot \frac{n(n-1)}{2} = \frac{n(n-1)}{4}$.

□

We can prove that inv W is also homomesic for the other dihedral group actions in similar fashion. To prove homomesy of inv W for the entire dihedral action, what we have shown, along with the following Lemma suffices.

Lemma 1.4.2 ([Rob16])**.** *Let G be a group acting on the set S, and let H be a subgroup of G. If the triple (S, H, f) exhibits homomesy, then so does the triple (S, G, f).*

The result discussed in this section has also been generalized by Behrend and Roby [BR] to alternating sign matrices. The most interesting homomesies for a dihedral action are those that are not implied by a cyclic subgroup. See [STWW15] for one example.

1.4.2 Bulgarian Solitaire

In this section we will see an example of homomesy suitably generalized on a non-invertible map. In the game *Bulgarian Solitaire*, a pack of n identical cards is divided arbitrarily into several piles. A move consists of removing one card from each pile and collecting the removed cards to form a new pile. The original version of this puzzle proposed a pack of $n = 15$ cards and the task of determining why the final position was the same regardless of the initial setup.

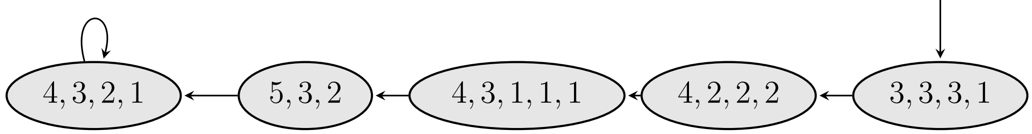
Bulgarian Solitaire gained popularity in the early 1980s. As the story goes, Konstantin Oskolkov of the Steklov Mathematical Institute of Moscow was shown the problem on a train on his way to give a talk in Leningrad. After bringing the problem back to his colleagues, it circled around mathematical communities until it ultimately caught the attention of Martin Gardner [Hop12]. Gardner included the problem in his Scientific American article *Mathematical Games: Tasks you cannot help finishing no matter how hard you try to block finishing them* [Gar83]. The name was given to it by Henrik Ericsson of the KTH Royal Institute of Technology although he remarked it was "silly because it is neither Bulgarian nor a solitaire."

More formally, Bulgarian Solitaire is a map on integer partitions.

Definition 1.4.3. A **partition** of $n \in \mathbb{N}$ is a sequence $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ of integers λ_i satisfying $\sum \lambda_i = n$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$. The number of parts of λ is called the **length** of λ and is denoted $\ell(\lambda)$.

We can think of our starting position for Bulgarian Solitaire as a partition λ of n of length $\ell = \ell(\lambda)$ which is separated into unordered piles containing $\lambda_1, \lambda_2, \ldots, \lambda_\ell$ cards. After one move of Bulgarian Solitaire our result is the partition containing the numbers $\ell, \lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_\ell - 1$. Figure 1.2 has an example where $n = 10$.

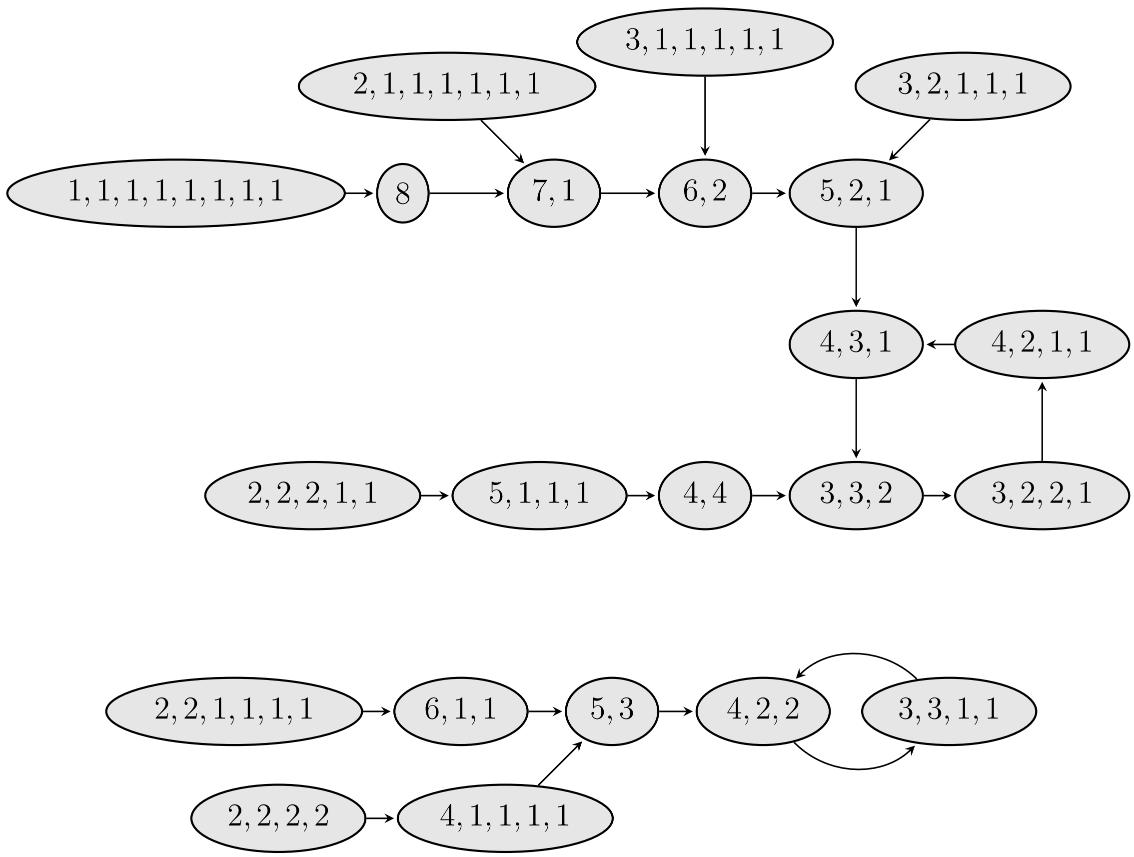
Figure 1.2: An example of Bulgarian Solitaire for $n = 10$



Notice that once the map yields $(4, 3, 2, 1)$ we enter a loop where the Bulgarian Solitaire map is the identity. In fact all partitions of the form $(j, j-1, j-2, \ldots, 1)$, called **staircase partitions**, are stable under the Bulgarian Solitaire map. If $n = 1 + 2 + 3 + \cdots + j$ is any triangular number, then the staircase partition exists, and it turns out that any sequence of moves eventually leads to the staircase partition.

For non-triangular numbers we do not have one single fixed point, as can be seen in Figure 1.3 where $n = 8$.

Figure 1.3: The action of Bulgarian Solitaire on all partitions of $n = 8$



Our original definition for homomesy stipulated that τ be an invertible map. Below we have a slightly modified version which extends to non-invertible maps like this one.

Definition 1.4.4. Let $\mathcal{S}$ be a finite set with a (not necessarily invertible) map $\tau : \mathcal{S} \to \mathcal{S}$ (called a **self-map**). Given the self-map τ and starting from some possible $a \in \mathcal{S}$, one constructs the sequence of iterates $\tau(a), \tau^2(a), \ldots$. Since $\mathcal{S}$ is finite, there exists some i, j such that $\tau^i(a) = \tau^j(a)$. The sequence $\tau^i(a), \ldots, \tau^{j-1}(a)$ is called a

recurrent cycle and a **recurrent set** is the union of these cycles. We call a statistic $f : \mathcal{S} \to \mathbb{K}$ **homomesic** if the average of f is the same over every recurrent cycle.

Of course if τ is an invertible action on a finite set $\mathcal{S}$, then this definition of homomesy specializes to Definition 1.3.1.

Theorem 1.4.5 ([Rob16, Proposition 3])**.** *Let $n = j(j-1)/2 + k$ with $0 \le k < j$, and consider the action of Bulgarian Solitaire on the set of partitions of n. Then the length statistic ℓ, which computes the number of parts of a partition λ, is homomesic with average $(j-1) + k/j$.*

In our example above where $n = 8$, so $j = 4$ and $k = 2$, we expect the average length statistic to equal $3 + \frac{2}{4} = \frac{7}{2}$. We saw that the two recurrent cycles for $n = 8$ are

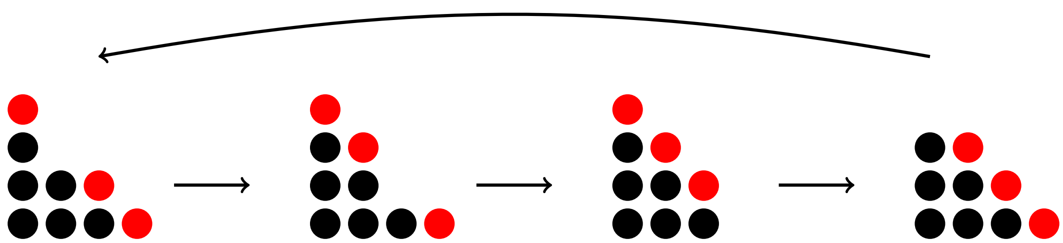
$$(4211, 431, 332, 3221) \text{ and } (422, 3311)$$

These have average length statistic

$$\frac{4+3+3+4}{4} = \frac{7}{2} \text{ and } \frac{3+4}{2} = \frac{7}{2}$$

Proof. We represent our Bulgarian Solitaire piles as columns of dots viewed as a shape made up of two parts. Black dots form the largest possible triangular number less than n, of size $j(j-1)/2$ and the k red dots are distributed down the diagonal. As long as there is at most one "distributed dot" in every column, in each step the dots will march down the diagonal of the triangle and the cycle will repeat. Any shape of this form is a **recurrent shape**. This is true because at each step we are identifying the bottom row, with its $j-1$ black dots and 0 or 1 red dot, and rotating it to become the first column in the next step. The action of removing the bottom row reduces each column by 1 dot, and rotates the position of the red dot in the lowest position (if there is one). Thus the dots will effectively "march down" the diagonal in a recurrent cycle of Bulgarian Solitaire.

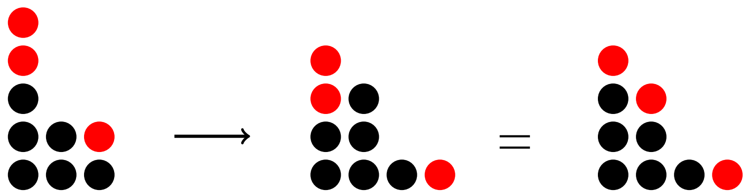
Figure 1.4: Example of a recurrent cycle where $n = 9$ is represented as the triangular number $6 = (4 \cdot 3)/2$ plus 3 distributed dots



To see that all recurrent cycles contain shapes of this form, assume this is not the case. In any partition of n, we can write the partition in order of decreasing size of the parts, and highlight the largest possible triangular number (in black) within that shape. If we did not have a recurrent shape, then we must have at least two dots in one of the columns. By applying the Bulgarian Solitaire map, this will eventually lead to a scenario where the first (largest) column has two red dots above the triangle. The next step of the action will then distribute the two dots between the largest and second largest columns giving us the distribution of one dot in each column, or a recurrent shape. See Figure 1.5 for an example.

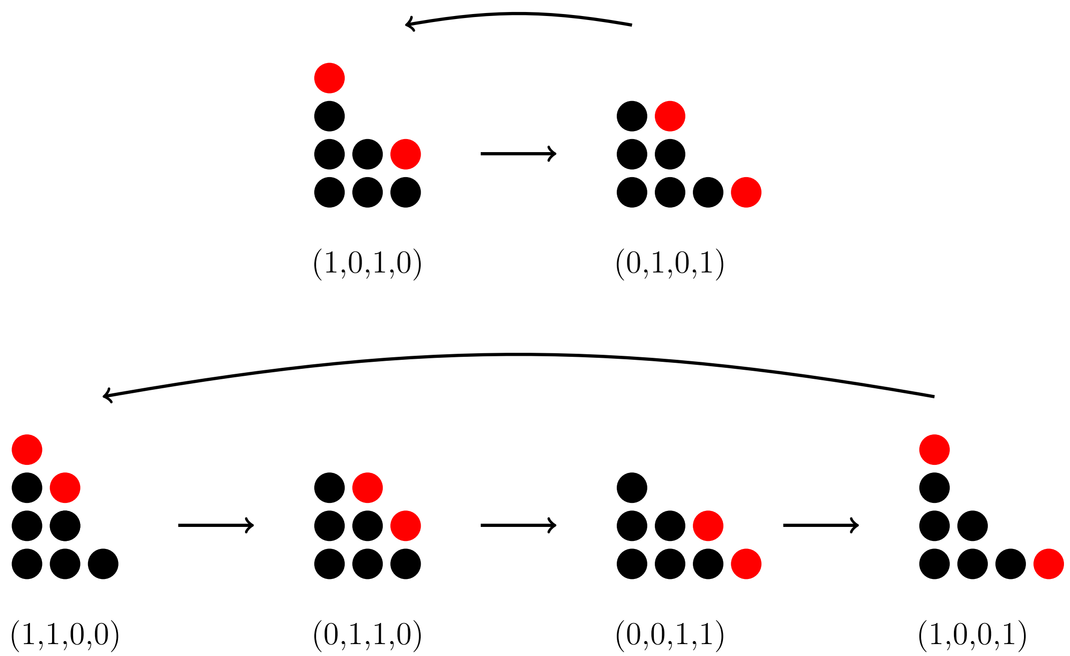
It follows that we can think of a recurrent shape of $n = j(j-1)/2 + k$ to be the j^{th} triangular number with k distributed dots. There is an equivarient bijection between shapes with k distributed dots along the j^{th} triangular number and binary strings of length j with k 1s, where the Bulgarian Solitaire map is equivalent to right cyclic

Figure 1.5: Example of the distribution of dots.



shifting of the binary strings. In Figure 1.6 we can see this for all cycles when $j = 4$ with $k = 2$.

Figure 1.6: Dot representations corresponding to bit strings where $j = 4$, $k = 2$



In each orbit (or superorbit) of length j, the number of shapes with j parts is equal to the number of strings that end in 1, which is equal to k. Similarly, the number of shapes with $j - 1$ parts is equal to the number of strings that end in 0, which is equal to $j-k$. So the total number of parts in a super orbit is

$$jk+(j-1)(j-k) = jk+j^2-j-jk+k$$
$$= j^2-j+k = \boxed{j(j-1)+k}$$

Since all super orbits are length j, it follows that the average number of parts in a super orbit (and thus in an orbit) is

$$\frac{j(j-1)+k}{j} = \boxed{(j-1)+\frac{k}{j}}$$

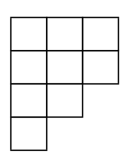
□

1.4.3 Promotion on Semistandard Young Tableaux

In this section we will see an example of the promotion map on semistandard Young tableaux. To begin, recall Definition 1.4.3 of a partition from the previous section.

Definition 1.4.6. A **Young diagram** is a visual representation of a partition λ where $\lambda = (\lambda_1,\ldots,\lambda_k) \vdash n$. It is obtained by drawing a left-justified series of boxes with λ_i boxes in the ith row. Figure 1.7 shows the Young diagram for the partition $(3,3,2,1) \vdash 9$.

Figure 1.7: Young diagram for the partition $(3,3,2,1) \vdash 9$



Definition 1.4.7. A **skew shape**, denoted λ/μ is a pair of partitions $\lambda = (\lambda_1, \lambda_2, \dots)$ and $\mu = (\mu_1, \mu_2, \dots)$ where $\mu_i \leq \lambda_i$ for each i. The **skew diagram** is the difference of the Young diagrams of λ and μ, namely the set of boxes that belong to λ but not to μ. We call a box in μ an **inner corner** of λ/μ if the boxes immediately below and to the right of it are not in μ.

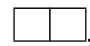
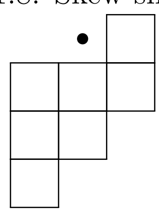
Example 1.4.8. Letting λ = the shape in Figure 1.7 and μ = , we have the skew shape λ/μ given in Figure 1.8 with • indicating the place of the inner corner.

Figure 1.8: Skew shape λ/μ



Definition 1.4.9. A **semistandard Young tableau** $\mathrm{SSYT}_k(\lambda)$ of a partition $\lambda \vdash n$ is a filling of the boxes of the Young diagram of λ with elements of the set $\{1, 2, \dots, k\}$ for some k, such that rows are weakly increasing and columns are strictly increasing. Similarly, a **skew semistandard tableau** $\mathrm{SSYT}_k(\lambda/\mu)$ is obtained by filling the boxes of the skew diagram λ/μ such that entries increase weakly along each row, and increase strictly down each column. We have an example $T \in \mathrm{SSYT}_4((3,3,2,1)/(2))$ in Figure 1.9.

In this section we look at an example of homomesy involving sums of centrally symmetric entries under promotion on semistandard Young tableaux. The action that we are looking at is called promotion and has two useful definitions. The first

Figure 1.9: Example of $T \in \mathrm{SSYT}_4((3,3,2,1)/(2))$

		3
1	1	4
2	2	
3		

involves **jeu-de-taquin** slides, and the second involves composition of toggles using Bender–Knuth involutions. We begin with the first definition.

Definition 1.4.10. Let $T \in \mathrm{SSYT}_k(\lambda)$. The **promotion** $\mathcal{P}(T)$ of T is given by the following construction.

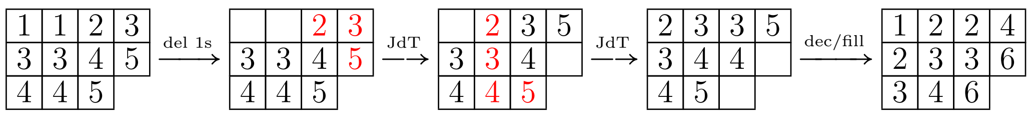
- If T has no 1's, then let $\mathcal{P}(T)$ be the result of decrementing all the values of T by 1. Otherwise, do the following.
- First, delete all the entries in the boxes of T that contain a 1.
- We then apply **jeu-de-taquin** slides as follows.

  Letting μ be the empty boxes, we begin with any inner corner of λ/μ, denote it b_0, and we identify the sequence of boxes $b_0, b_1, b_2, \ldots, b_m$ where each b_{i+1} is whichever of the boxes immediately below or to the right of b_i contains the smaller value. If the boxes have equal values, we choose the box below. If either of the boxes lies outside of λ, we choose the one in λ. The sequence ends when b_m is an inner corner of λ. Now, we use this sequence to to create a new tableau by sliding the value in b_{i+1} into b_i. If we perform this process iteratively until our shape is no longer skew, we have completed jeu-de-taquin slides and we call this shape the **rectification** of T.

- Once the shape has been rectified, we decrement all the values by 1 and then place k in all empty boxes resulting from sliding.

Example 1.4.11. We let $k = 6$, and at each step we highlight the cells participating in the jeu-de-taquin slide in red.

Figure 1.10: One iteration of promotion using jeu-de-taquin slides

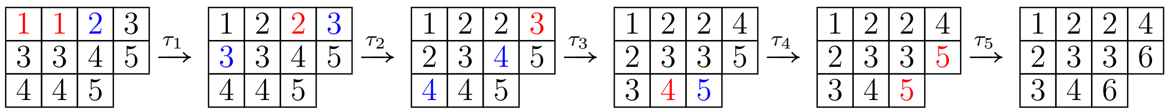


Our second definition for promotion is based on Bender–Knuth operations, which we call **toggles**. While the equivalence of these definitions is not obvious, a proof can be found in both [Gans80] and [BPS13]. There are several other instances of homomesy that come from actions involving toggles.

Definition 1.4.12. Given $T \in \mathrm{SSYT}_k(\lambda)$ and $i \in [k]$, consider each i that is paired with an $i+1$ (directly below) in the same column (and vice-versa) to be "frozen" and the remainder free. Then in each row with r free i's and s free $(i+1)$'s, τ_i replaces these with s free i's and r free $(i+1)$'s. We define the **promotion** operator to a composition of these toggles, $P(T) = \tau_{k-1} \circ \tau_{k-2} \circ \cdots \circ \tau_1(T)$.

Example 1.4.13. Let $k = 6$ and consider the semistandard Young tableau T from Example 1.4.11 . The relevant cells are highlighted with i in red and $i+1$ in blue for each iteration τ_i. Note "frozen" pairs remain in black.

Figure 1.11: One iteration of promotion using toggles

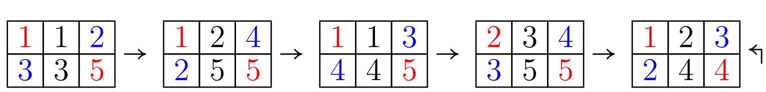


Note that the end result of the quite different processes in Example 1.4.11 and Example 1.4.13 agree. In order to discuss homomesy occurrences, we will restrict ourselves to the specific shape where $\lambda = (n^m) = (n, n, \ldots, n)$. In this particular situation it is a nontrivial theorem that $P^k(T)$ is equal to the identity [Rho10, Corollary 5.6]. (For most generic shapes the order of promotion is quite large relative to k). If we consider certain "opposite" cells in our rectangular shape, the sum of these cells has the same average value over the orbit, which is equal to the number of cells multiplied by $(k+1)/2$. We formalize this below.

Theorem 1.4.14 ([BPS13, Theorem 1.1])**.** *Let $T \in SSYT_k(n^m)$ and fix a subset of cells $R \subseteq (n^m)$ where R is symmetric with respect to $180°$ rotation around the center of (n^m). Then the statistic $\sigma^R(T) :=$ sum of entries of T whose cells are in R, is c-mesic where $c = |R|(\frac{k+1}{2})$.*

Example 1.4.15. Consider the following orbit of promotion on $T \in \mathrm{SSYT}_5(3^2)$.

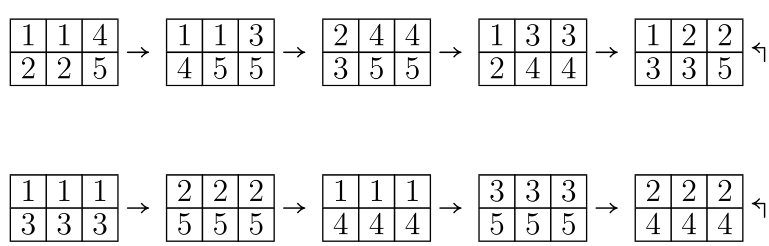
Figure 1.12: An orbit of promotion



To begin, we consider all options where $|R| = 2$. Our pairs are indicated in red, black and blue. The entries in the upper left and lower right corners (shown in red) have sums $(6,6,6,7,5)$ across the orbit, with an average value of 6. The upper middle and lower middle (in black) have sums $(4,7,5,8,6)$, again with an average of 6. Lastly, the upper right and lower left (in blue) have sums $(5,6,7,7,5)$, also average 6. So we see homomesy where $c = |2|(\frac{5+1}{2}) = 6$.

We can also consider subsets $R = 4$ where we expect $c = |4|(\frac{5+1}{2}) = 12$. First we look at the entries of the lower left, lower middle, upper middle, and upper right (blue and black) and compute the sums to be $(9,13,12,15,11)$ with average 12. Alternatively if we look at the subset consisting of the upper left, upper middle, lower middle, and lower right (red and black), we get sums $(10,13,11,15,11)$, with average also 12 as expected. Of course these also follow from adding the homomesies mentioned above. Below we show 2 additional orbits in $\mathrm{SSYT}_5(3^2)$ which both have average value $c = 3|R|$ for $\sigma^R(T)$.

Figure 1.13: Two additional orbits of promotion for $\mathrm{SSYT}_5(3^2)$



This homomesy result was presented as a conjecture by Propp and Roby, and proved by Bloom, Pechenik and Saracino [BPS13]. Promotion on rectangular tableaux was shown to exhibit cyclic sieving by Brendan Rhoades using Kazhdan–Lusztig theory [Rho10].

1.4.4 Rowmotion on Order Ideals of Posets

Our next example is another map which can be described as a composition of involutions. In later chapters we will be discussing a number of other maps which are also of this form. We begin with some relevant definitions from poset theory, but a much more comprehensive exposition of the topic can be found in Stanley's graduate text [Stan11, Chapter 3]. The following definitions all come from this text.

Definition 1.4.16. A **partially ordered set** P, or **poset**, is a set, together with a binary relation denoted $\leq$, satisfying the following three axioms:

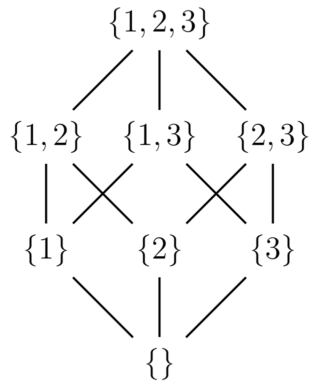
- For all $t \in P, t \leq t$ (*reflexivity*)
- If $s \leq t$ and $t \leq s$, then $s = t$ (*antisymmetry*)
- if $s \leq t$ and $t \leq u$, then $s \leq u$ (*transitivity*)

Definition 1.4.17. If $s, t \in P$, then we say that t **covers** s, denoted $s \lessdot t$ if $s < t$ and no element $u \in P$ satisfies $s < u < t$.

Definition 1.4.18. We say that two elements s and t of P are **comparable** if $s \leq t$ or $t \leq s$; otherwise s and t are **incomparable**, denoted $s \parallel t$.

Definition 1.4.19. The **Hasse diagram** of a finite poset P is the graph whose vertices are the elements of P, whose edges are the cover relations, and such that if $s < t$ then t is drawn above s.

Figure 1.14: The Hasse diagram for the poset B_3 of subsets of $\{1,2,3\}$ ordered by inclusion.



Definition 1.4.20. An **order ideal** of P is a subset I of P such that if $t \in I$ and $s \le t$, then $s \in I$. Let $J(P)$ denote the set of order ideals of the poset P, which is also a poset, ordered by inclusion.

Definition 1.4.21. An **order filter** of P is a subset F of P such that if $t \in F$ and $s \ge t$, then $s \in F$. Denote $\mathcal{F}(P)$ the set of order filters of the poset P.

Definition 1.4.22. An **antichain** of P is a subset A of P such that any two distinct elements of A are incomparable. Denote $\mathcal{A}(P)$ the set of antichains of P.

For finite posets, there is a one-to-one correspondence between antichains and order ideals. Namely, the set of maximal elements of I is an antichain A of P, conversely, $I = \{s \in P : s \le t \text{ for some } t \in A\}$. In this case we write $I = \mathbf{I}(A)$ is the order ideal generated by A. Viewing our poset upside down gives us the same bijection between antichains and filters, we will call the map from filters to antichains $\mathbf{F}^{-1}$. It should also be clear that the complement map **comp** is a bijection between order ideal and filters.

Definition 1.4.23. Let P be a finite poset, and $I \in J(P)$. **rowmotion** is the map acting either on antichains (denoted ρ_A) or on order ideals (denoted ρ_J) as follows.

$$\rho_J : J(P) \xrightarrow{\text{comp}} F(P) \xrightarrow{\mathbf{F}^{-1}} A(P) \xrightarrow{\mathbf{I}} J(P)$$

$$\rho_A : A(P) \xrightarrow{\mathbf{I}} J(P) \xrightarrow{\text{comp}} F(P) \xrightarrow{\mathbf{F}^{-1}} A(P)$$

We illustrate this definition with the example below, first for ρ_J.

Figure 1.15: Action of ρ_J on $J(P)$

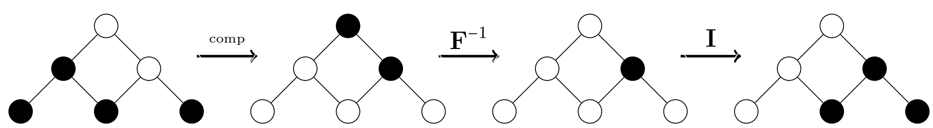
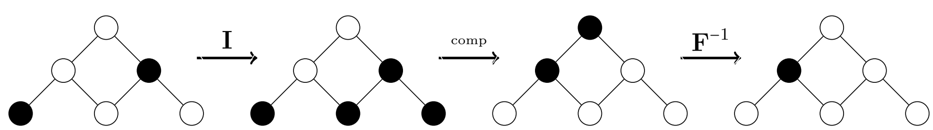


Figure 1.16: Action of ρ_A on $A(P)$



The following theorem illustrating homomesy began as a conjecture of Dmitri Panyushev [Pan09] and was later proven by Drew Armstrong, Christian Stump, and Hugh Thomas [AST13].

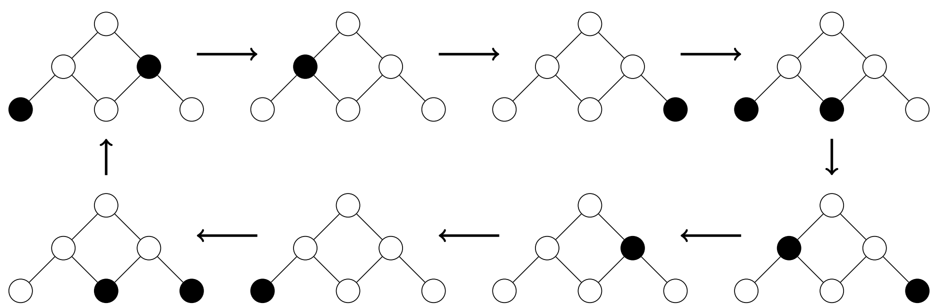
Theorem 1.4.24 ([AST13, Theorem 1.2])**.** *Let W be a finite Weyl group of rank r, with corresponding positive root poset $\Phi^+(W)$. Then for any orbit $\mathcal{O}$ under the action of ρ_A on $\mathcal{A}(\Phi^+(W))$, we have*

$$\frac{1}{|\mathcal{O}|}\sum_{A\in\mathcal{O}}|A| = r/2$$

In other words, the cardinality statistic is homomesic with respect to rowmotion acting on antichains of the positive root poset, with average half the rank.

In practice this theorem applies to a few families of posets and some special cases. For more information on Weyl groups and root posets, we refer the reader to Björner and Brenti's text on Coxeter groups [BB05] . The poset shown above is a root poset of type A_3 and leads to the following orbits under ρ_A.

Figure 1.17: One orbit for ρ_A acting on the type A_3 root posets



The cardinality of the antichains in this orbit are $(2,1,1,2,2,1,1,2)$ with average value $(2+1+1+2+2+1+1+2)/8 = 12/8 = 3/2$.

The other two orbits for A_3 are of size 4 and 2 and can be seen Figures 1.18 and .1.19.

Figure 1.18: Size 4 orbit for type A_3 root posets under ρ_A

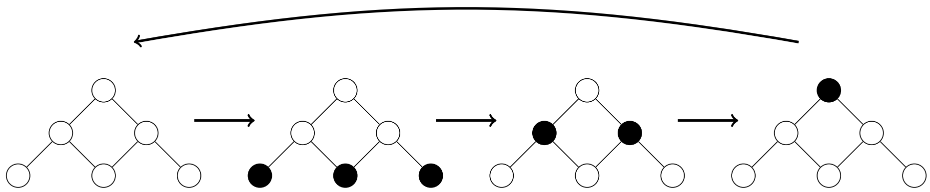
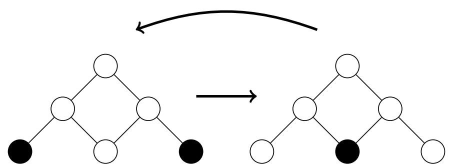


Figure 1.19: Size 2 orbit for type A_3 root posets under ρ_A



Again the average values of the antichains are $(0+3+2+1)/4 = 3/2$ for the size 4 orbit and $(2+1)/2 = 3/2$ for the size 2 orbits as predicted by the theorem.

In the following two chapters we will discuss homomesies found in maps formed by composing dihedral involutions with two different permutation maps, the Rényi–Foata map and the Foata–Schützenberger map.

Chapter 2

The Rényi–Foata Map

2.1 Introduction to the Rényi–Foata Map

In this chapter we look at homomesies for actions of the symmetric group, particularly so called *Foatic maps.* These are created by intertwining the "fundamental bijection" of Rényi and Foata, denoted $\mathcal{F} : \mathfrak{S}_n \to \mathfrak{S}_n$ with standard dihedral symmetries on $\mathfrak{S}_n$ (see Section 1.2). Similar maps involving intertwinings can be seen in the literature [BW91]. To define $\mathcal{F}$, we need first to specify a type of disjoint cycle decomposition.

Definition 2.1.1. Let $w \in \mathfrak{S}_n$. The **canonical (disjoint) cycle decomposition (CCD)** of w is the decomposition of the permutation w into disjoint cycles, where **(a)** each cycle is written with its largest element first and **(b)** the cycles are written in increasing order of first (largest) elements.

The permutation $417682953 \in \mathfrak{S}_9$ (in one-line notation) can be converted to CCD by first rewriting it in terms of cycles, and then reordering each cycle to begin with the largest number. In the second step below, we underline the largest element of each cycle.

$$417682953 \to (14\underline{6}2)(37\underline{9})(5\underline{8}) \to (6214)(85)(937)$$

Definition 2.1.2. The **Rényi–Foata Map ($\mathcal{F}$)** takes a permutation in CCD, drops the parentheses, and reinterprets the result as a permutation in one-line notation.

The following example shows how $\mathcal{F}$ acts on the permutation $w = 417682953 = (6214)(85)(937)$. We have also written $\mathcal{F}(w)$ in CCD.

$$(6214)(85)(937) \xrightarrow{\mathcal{F}} 621485937 = (2)(4)(83165)(97)$$

To compute the inverse $\mathcal{F}^{-1}$ of the Rényi-Foata map, take a permutation in one-line notation, place an open parenthesis in front of every **record** (a left-to-right maximum of a permutation written in one-line notation), then fill in the corresponding closed parentheses. The following example shows the action of the inverse map:

$$\underline{6}214\underline{8}5\underline{9}37 \xrightarrow{\mathcal{F}^{-1}} (6214)(85)(937)$$

This map converts certain natural permutation statistics to others. In particular, it shows that the number of permutations in $\mathfrak{S}_n$ with exactly k left-to-right maxima (or records) is the same as the number with exactly k cycles. This follows directly from the definition since under the inverse map $\mathcal{F}^{-1}$, records determine the beginning of cycles. This number is the *signless Stirling numbers of the first kind* and is defined by the recurrence $c(n,k) = (n-1)c(n-1,k) + c(n-1,k-1)$, $n,k \geq 1$ with initial conditions $c(n,k) = 0$ if $n < k$ or $k = 0$, except $c(0,0) = 1$ [Stan11, Lemma 1.3.6].

An additional result that follows directly from the definition has to do with **ascents** and **weak exceedances**.

Definition 2.1.3. Let $w \in \mathfrak{S}_n$. An index j for which $w_j < w_{j+1}$ is called an **ascent** of w. For $w = 62\underline{1}\underline{4}8\underline{5}9\underline{3}7$ the ascent set is $\{3,4,6,8\}$.

In later work we will look at the permutation statistic **Rasc**, which counts the number of records of a permutation which are also ascents.

Definition 2.1.4. Let $w \in \mathfrak{S}_n$. An index i for which $w_i \geq i$ is called a **weak exceedance** of w. In CCD, a weak exceedance will appear as a number in its own cycle (since $i = w_i$) or as two adjacent numbers within a cycle in the order i, w_i where $i < w_i$. The weak exceedances for $w = 417682953$ are the elements of the set $\{1,3,4,5,7\}$.

By looking at $\mathcal{F}^{-1}$ we can see that that the number of permutations with k ascents is the same as the number with $k+1$ weak exceedances. We consider again the example $w = 621485937$ with ascent set $\{3,4,6,8\}$ and look specifically at $i = 3$ and $j = 4$ where we have $w_3 = 1 < 4 = w_4$. Now under the map $\mathcal{F}^{-1}$, we observe

$$621485937 \xrightarrow{\mathcal{F}^{-1}} (6214)(85)(937)$$

This becomes a weak exceedance in cycle notation since this is now implying $w_i = 4$ is greater than $i = 1$. Note that there is also an ascent between every record and the number before it in one-line notation. We can see this for index $i = 5$ and $i - 1 = 4$ where $w_5 = 8$ is a record.

$$621485937 \xrightarrow{\mathcal{F}^{-1}} (6214)(85)(937)$$

These map to weak exceedances in CCD since the last number in a cycle is smaller than the first (so since $5 < 8$). We also get one additional weak exceedance from the last number in the last cycle that was not originally an ascent. In our example that is the $7 \mapsto 9$.

This results in $k+1$ weak exceedances as desired. This number is counted by the Eulerian number $A(d, k+1)$. These have an explicit formula as a summation but are more conveniently defined here by the recurrence $A(d,k) = (d-k)A(d-1,k-1) + (k+1)A(d-1,k)$ with $n \geq 1$, $k \geq 2$ and $A(n,1) = 1$ for $n \geq 0$ and $A(0,k) = 0$ if $k \geq 2$ [Stan11, Proposition 1.4.3].

As mentioned in Section 1.2, we will be looking at compositions of the Rényi–Foata map and its inverse, (sometimes called *intertwinings*) with five of the dihedral group actions on permutation matrices which are involutions. We eliminate the identity map for lack of interesting properties.

Definition 2.1.5. The following are the five **dihedral involutions** of $\mathfrak{S}_n$ that we will consider:

(a) $\mathcal{C} : \mathfrak{S}_n \to \mathfrak{S}_n$, which takes a permutation $w = w_1 \cdots w_n$ to its **complement** whose value in position i is $n + 1 - w_i$;

(b) $\mathcal{R} : \mathfrak{S}_n \to \mathfrak{S}_n$, which takes a permutation $w = w_1 \cdots w_n$ to its **reversal** whose value in position i is w_{n+1-i};

(c) $\mathcal{Q}^2 : \mathfrak{S}_n \to \mathfrak{S}_n$, which takes a permutation $w = w_1 \cdots w_n$ to its **rotation of the permutation matrix by 180-degrees**;

(d) $\mathcal{I} : \mathfrak{S}_n \to \mathfrak{S}_n$, which takes a permutation w to its **inverse** w^{-1};

(e) $\mathcal{D} : \mathfrak{S}_n \to \mathfrak{S}_n$, which takes a permutation w to its **rotated inverse** $\mathcal{Q}^2(\mathcal{I}(w))$.

Observe that $\mathcal{Q}^2 = \mathcal{R} \circ \mathcal{C} = \mathcal{C} \circ \mathcal{R}$. This gives a quick way to compute $\mathcal{Q}^2(w)$ by reading off $(n+1-j)$ for $j = w_n, w_{n-1}, \ldots, w_1$ (right to left).

Example 2.1.6. For our running example $w = 417682953 = (6214)(85)(937)$.

(a) $\mathcal{C}(417682953) = 693428157$

(b) $\mathcal{R}(417682953) = 359286714$

(c) $\mathcal{Q}^2(417682953) = 751824396$

(d) $\mathcal{I}((6214)(85)(937)) = (6412)(85)(973)$

(e) $\mathcal{D}((6214)(85)(937)) = \mathcal{Q}^2(\mathcal{I}((6214)(85)(937))) = (137)(25)(4698)$

The maps discussed in this chapter take on the following form, where $\mathcal{A}$ and $\mathcal{B}$ represent dihedral involution maps:

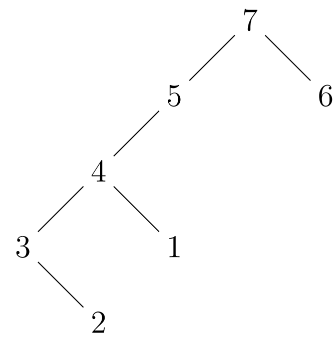
$$\mathfrak{S}_n \xrightarrow{\mathcal{F}} \mathfrak{S}_n \xrightarrow{\mathcal{A}} \mathfrak{S}_n \xrightarrow{\mathcal{F}^{-1}} \mathfrak{S}_n \xrightarrow{\mathcal{B}} \mathfrak{S}_n$$

This exploration follows from the work of Michael La Croix and Tom Roby [LR20]. They call these compositions of the Rényi–Foata map and dihedral symmetries **Foatic maps**. As a shorthand Foatic maps are labeled by the dihedral actions ($\mathcal{A}$ and $\mathcal{B}$) in the order applied. So for example the map $\mathcal{R} \circ \mathcal{F}^{-1} \circ \mathcal{C} \circ \mathcal{F}$ would be called the complement-reversal map. This gives a total of 25 possible maps to explore.

La Croix and Roby noticed homomesies for three particular Foatic maps: reversal-inversion, complement-inversion, and complement-rotation. They represented permutations using a type of decreasing binary tree, called a **heap**, to describe the structure of one particular map, reversal-inversion. This allowed them to prove that the fixed point statistic on $\mathfrak{S}_n$ (see Definition 2.2.1) is homomesic for the reveral-inversion map (denoted $\overline{\varphi}$ in Theorem 2.1.9).

Definition 2.1.7. Let S be a finite totally ordered set, and $w \in \mathfrak{S}_S$ be a permutation of S written in one-line notation. If $S \subseteq [n] \coloneqq \{1, 2, \ldots, n\}$, we call w a **partial permutation of n**. We recursively define the **heap** of w, $H(w)$ as follows. Set $H(\varnothing) = \varnothing$ (the empty word). If $w \neq \varnothing$, let m be the largest element of w, so w can be written uniquely as umv, where u and v are partial permutations (possibly empty). Set m to be the root of $H(w)$, with $H(u)$ its left subtree and $H(v)$ its right subtree.

Example 2.1.8. Consider the permutation $w = 3241576 \in \mathfrak{S}_7$. Note this is the same permutation we saw as an example in Section 1.2 after Definition 1.2.6, also visualized as a tree, albeit a different one from the one seen here. As $m = 7$ is the largest element of w, we have $u = 32415$ and $v = 6$. Similarly, as we apply this idea recursively to u, we get the sub factorization as $u'm'v'$ where $m' = 5$, $u' = 3241$ and $v' = \varnothing$. The subtree for $H(3241576)$ can be seen below.



Theorem 2.1.9 ([LR20, Theorem 14]). *1. Let φ be the map $\mathcal{F} \circ \mathcal{I} \circ \mathcal{F}^{-1} \circ \mathcal{R}$. The size of each φ-orbit (equivalently $\overline{\varphi}$-orbit) is a power of 2. Specifically if w lies in the orbit, define the* ***height*** *of the heap $H(w)$ to be the number of edges h in a maximal path from the root (to a leaf); then the size of the orbit is 2^h.*

2. Let Fix(w) *denote the number of fixed points, i.e., 1-cycles of w. Then the statistic* Fix *is* 1*-mesic with respect to the action of $\overline{\varphi}$. Equivalently,* Rasc *is* 1*-mesic with respect to the action of φ.*

3. For fixed values $i \neq j$ in $[n]$, let $\mathbf{1}_{i<j}(u)$ denote the indicator statistic of whether i occurs to the left of j in the one-line notation of u. Then $\mathbf{1}_{i<j}(u)$ is $\frac{1}{2}$-mesic with respect to the action of φ.

4. Similarly for fixed $i \in [n]$, let $\mathbf{1}_{(i,n)}$ denote the indicator statistic of whether i and n lie in the same cycle of w. Then $\mathbf{1}_{(i,n)}$ is $\frac{1}{2}$-mesic with respect to the action of $\overline{\varphi}$.

The proof of Theorem 2.1.9 relies heavily on the following lemma.

Lemma 2.1.10. *Let $w \in \mathfrak{S}_n$ have the form AnB (in one-line notation), where A and B are (possibly empty) partial permutations of n. Then the action of φ satis-*

fies $\varphi(AnB) = \varphi(B)nA$. *Thus* $H(\varphi(AnB))$ *is the heap interchanging the left and right subtrees at* n*, leaving the former unchanged and applying* φ *recursively to the latter. In particular, the action of* φ *preserves the underlying unlabeled graph of the corresponding heaps.*

The homomesies conjectured and proved in this chapter were discovered using Sage code adeptly written by Mike Joseph, as well as via numerical experiments with data from Michael La Croix. We found homomesies by checking linear combinations involving numbers and locations of the following statistics: **fixed points**, **inversions**, **weak exceedances**, **exceedances**, **ascents**, **descents**, **left-to-right max/min** and **right-to-left max/min**. Some of these we saw in Chapter 1, the rest we will define as we describe the homomesies. A list of all conjectured and proved homomesies can be seen in Appendix A.

2.2 Elementary Homomesies

There are several examples of homomesy among the 25 Foatic maps we consider. In this section we will explore some of the more elementary ones and see their proofs. In particular we will look at homomesies involving fixed points, exceedances, weak exceedances, and combinations of left-to-right and right-to-left maxima and minima.

2.2.1 Extreme Fixed Points ($\mathbf{Fix}_1 - \mathbf{Fix}_n$)

Definition 2.2.1. Let $w = w_1w_2\cdots w_n \in \mathfrak{S}_n$. A **fixed point** is $j \in [n]$ such that $w_j = j$. Equivalently, j is alone in its cycle when written in cycle notation. Let

$\mathrm{Fix}_i(w) = 1$ if i is a fixed point of w and zero otherwise; denote the total number of fixed points as $\mathrm{Fix}(w) = \sum_{i=1}^{n} \mathrm{Fix}_i(w)$.

Example 2.2.2. For the permutation $w = 416352 = (14362)(5)$, the number 5 is a fixed point, so $\mathrm{Fix}_5(w) = 1$, $\mathrm{Fix}_j(w) = 0$ for $j \neq 5$ and $\mathrm{Fix}(w) = 1$.

Theorem 2.2.3. *The statistic $Fix_1 - Fix_n$ is 0-mesic with respect to the following nine Foatic maps: reversal-reversal, rotation-reversal, complement-complement, rotation-complement, reversal-rotation, complement-rotation, inversion-rotation, rotated inversion-inversion, and inversion-rotated inversion.*

Homomesy for $\mathrm{Fix}_1 - \mathrm{Fix}_n$ is somewhat common among Foatic maps and the proofs are relatively straightforward. As an example we will show the proof for rotation-reversal. The others are similar.

Proof. We claim that if n is a fixed point in a permutation, then 1 is a fixed point exactly two iterations later within an orbit of rotation-reversal.

Consider the following example:

$$(312)(4)(5) \xrightarrow{\mathcal{F}} 31245 \xrightarrow{\mathcal{Q}^2} 12453 \xrightarrow{\mathcal{F}^{-1}} (1)(2)(4)(53) = 12543 \xrightarrow{\mathcal{R}} 34521$$

$$34521 = (42)(513) \xrightarrow{\mathcal{F}} 42513 \xrightarrow{\mathcal{Q}^2} 35142 \xrightarrow{\mathcal{F}^{-1}} (3)(5142) = 45321 \xrightarrow{\mathcal{R}} 12354 = (1)(2)(3)(54)$$

If n is a fixed point, then it appears last in its own cycle in CCD; thus, $\mathcal{F}(w)_n = n$. Since the rotation map $\mathcal{Q}^2$ complements and reverses, we get $\mathcal{Q}^2(\mathcal{F}(w))_1 = 1$. It follows that $\mathcal{F}^{-1}(\mathcal{Q}(\mathcal{F}(w)))$ has the cycle (1). In other words, $\mathcal{F}^{-1}(\mathcal{Q}(\mathcal{F}(w)))_1 = 1$. Lastly, applying the reverse map gives $\mathcal{R}(\mathcal{F}^{-1}(\mathcal{Q}(\mathcal{F}(w))))_n = 1$.

Beginning the next iteration of rotation-reversal, let $v = \mathcal{R}(\mathcal{F}^{-1}(\mathcal{Q}(\mathcal{F}(w))))$. Since $v_n = 1$, then the last cycle begins $(n1\cdots)$. So after applying $\mathcal{F}(v)$ we have n followed immediately by 1 in one-line notation. This also holds for $\mathcal{Q}^2(\mathcal{F}(v))$. It follows that $\mathcal{F}^{-1}(\mathcal{Q}^2(\mathcal{F}(v)))$ has a final cycle of the form $(n1\cdots)$, meaning $\mathcal{F}^{-1}(\mathcal{Q}^2(\mathcal{F}(v)))_n = 1$. To finish off the map, we get $\mathcal{R}(\mathcal{F}^{-1}(\mathcal{Q}^2(\mathcal{F}(v))))_1 = 1$ so 1 is a fixed point.

Thus the total number of times 1 is a fixed point will equal the total number of times n is a fixed point, and so $\mathrm{Fix}_1 - \mathrm{Fix}_n$ is 0-mesic.

□

The proofs of the other eight maps listed are very similar. We begin with a fixed point for either 1 or n, and track its behavior to arrive at a fixed point for either n or 1 (respectively) in either one or two iterations. Below we give a table illustrating the behavior of these two fixed points in each of the maps exhibiting this homomesy.

Map	**Behavior of the one-cycles** (1) **and** (n)
Reversal-Reversal	(n) appears one iteration after (1)
Complement-Complement	(1) appears one iteration after (n)
Rotation-Complement	(n) appears two iterations after (1)
Reversal-Rotation	(1) appears two iterations after (n)
Complement-Rotation	(n) appears two iterations after (1)
Inversion-Rotation	(n) appears one iteration after (1)
Rotated inversion-Inversion	(1) appears one iteration after (n)
Inversion-rotated inversion	(n) appears one iteration after (1)

2.2.2 Exceedances (wexc+exc)

We saw definitions for **ascents** and **weak exceedances** in Definitions 2.1.3 and 2.1.4. We include those again here, along with a few other necessary definitions.

Definition 2.2.4. Let $w \in \mathfrak{S}_n$. An index i for which $w_i \geq i$ is called a **weak exceedance** of w, while an index i for which $w_i > i$ is called an **exceedance** of w. Let $\operatorname{exc}(w)$ denote the number of exceedances of w and $\operatorname{wexc}(w)$ denote the number of weak exceedances of w. Note that $\operatorname{exc}(w) = \operatorname{wexc}(w) - \operatorname{Fix}(w)$.

Example 2.2.5. In the example where $w = \underline{2}\underline{7}1\underline{4}3\underline{6}\underline{9}58$, the exceedances (in red) are at positions $1, 2$, and 7 and the weak exceedances (underlined) are the exceedances along with those at positions 4 and 6.

Definition 2.2.6. Let $w \in \mathfrak{S}_n$. An index j for which $w_j \leq w_{j+1}$ is called an **ascent** of w. Let $\operatorname{asc}(w)$ denote the number of ascents in w. An index i for which $w_i \geq w_{i+1}$ is called a **descent** of w. We let

$$D_i = \begin{cases} 1 & \text{if } i \text{ is a descent} \\ 0 & \text{otherwise} \end{cases}$$

Then we can define the **major index**, $\operatorname{maj}(w) = \sum_{i=1}^{n-1} i D_i(w)$.

Example 2.2.7. In the example where $w = \underline{3}8\underline{1}975\underline{2}\underline{4}6$, the ascents are underlined.

Theorem 2.2.8. *The rotation-rotation map, $\mathcal{Q}^2 \circ \mathcal{F}^{-1} \circ \mathcal{Q}^2 \circ \mathcal{F}$, exhibits homomesy for the statistic* $\operatorname{wexc} + \operatorname{exc}$. *More specifically,* $\operatorname{wexc} + \operatorname{exc}$ *is n-mesic.*

Example 2.2.9. Below we have an orbit of rotation-rotation.

$(1)(4)(6235) = 135462$	$\xrightarrow{\mathcal{F}}$	146235	$\xrightarrow{\mathcal{Q}^2}$	245136	$\xrightarrow{\mathcal{F}^{-1}}$	$(2)(4)(513)(6) = 325416$	$\xrightarrow{\mathcal{Q}^2}$	163254
$(1)(3)(5)(642) = 163254$	$\xrightarrow{\mathcal{F}}$	135642	$\xrightarrow{\mathcal{Q}^2}$	531246	$\xrightarrow{\mathcal{F}^{-1}}$	$(53124)(6) = 241536$	$\xrightarrow{\mathcal{Q}^2}$	142635
$(1)(65324) = 142635$	$\xrightarrow{\mathcal{F}}$	165324	$\xrightarrow{\mathcal{Q}^2}$	354216	$\xrightarrow{\mathcal{F}^{-1}}$	$(3)(5421)(6) = 513246$	$\xrightarrow{\mathcal{Q}^2}$	135462

$$\vdots$$

Proof. By tracing the map carefully, we can see that certain statistics are equated at each iteration. Below we indicate these equalities, color-coded to match the example above. A description follows each assertion.

- $\mathrm{wexc}(w) = \mathrm{asc}(\mathcal{F}(w)) - 1$

  A fixed point in CCD will be a weak exceedance since $w_i = i$. By the definition of CCD, the lead element of each cycle is the largest in that cycle. So under the Foatic map, this weak exceedance became an ascent. A weak exceedance within a cycle must be some i where $i < w_i$, but within the cycle they appear in the order $i, w(i)$. Again, when the parenthesis are dropped during the application of $\mathcal{F}$ this will look like an ascent. The only exception is the last number in the last cycle. This will be a weak exceedance after applying $\mathcal{F}$ since the first number in the cycle is larger than it, but it is not an ascent.

- $\mathrm{asc}(\mathcal{F}(w)) = \mathrm{asc}(\mathcal{Q}^2(\mathcal{F}(w)))$.

  Since the rotation map $\mathcal{Q}^2$ is the same as reversal and complementation, each of which swap ascents and descents, the resulting permutation will have the same number of ascents in the same positions.

- $\mathrm{asc}(\mathcal{Q}^2(\mathcal{F}(w))) + 1 = \mathrm{wexc}(\mathcal{F}^{-1}(\mathcal{Q}^2(\mathcal{F}(w)))$.

  This assertion follows by a reversal of the logic from the first bullet.

- $\mathrm{wexc}(\mathcal{F}^{-1}(\mathcal{Q}^2(\mathcal{F}(w))) = n - \mathrm{exc}(\mathcal{Q}^2(\mathcal{F}^{-1}(\mathcal{Q}^2(\mathcal{F}(w)))$

  As mentioned, the action of rotation complements and reverses a permutation. If we have, for example, a number k such that $w_k = j$, under complementation we get $w_k = n - j$ and under reversal we get $w_{n-k} = n - j$. Thus, if $k \leq j$, then $n - k \geq n - j$. So exceedances map to non-exceedances and vise versa.

It follows that $\text{wexc}(w) = n - \text{exc}(\mathcal{Q}^2(\mathcal{F}^{-1}(\mathcal{Q}^2(\mathcal{F}(w)))$. So if we sum wexc+exc for each element of an orbit $\mathcal{O}$, we will get $n \cdot \#\mathcal{O}$, and therefore the average of wexc+exc is n.

□

Theorem 2.2.10. *The statistic* wexc + exc *is n-mesic for the rotation-inversion map,* $\mathcal{I} \circ \mathcal{F}^{-1} \circ \mathcal{Q}^2 \circ \mathcal{F}$.

Proof. We can see this as a variation of the previous theorem. The assertions for the maps $\mathcal{F}, \mathcal{Q}^2$ and $\mathcal{F}^{-1}$ certainly remain the same. The only additional fact we need is that

$$\text{wexc}(\mathcal{F}^{-1}(\mathcal{Q}^2(\mathcal{F}(w))) = n - \text{exc}(\mathcal{I}(\mathcal{F}^{-1}(\mathcal{Q}^2(\mathcal{F}(w)))$$

For a fixed point $w_i = i$ it is also true that $\mathcal{I}(w)_i = i$. Originally this number was a weak exceedance, but since it maps to itself, it is not an exceedance. Cycles of longer length freeze the first number and reverse the remaining numbers under the inversion map. Thus, if a number was a weak exceedance, meaning it was in the order i, w_i, it would then be in the order w_i, i, making it no longer a weak exceedance and vice-versa.

From here we can similarly conclude that $\text{wexc}(w) = n - \text{exc}(\mathcal{I}(\mathcal{F}^{-1}(\mathcal{Q}^2(\mathcal{F}(w)))$ and that the average of wexc+exc is n for every orbit.

□

The following theorem can also be proved in a similar way, tracing the activity of each statistic through the map.

Theorem 2.2.11. *The statistic* wexc + exc *is n-mesic for the inversion-inversion map.*

Also by similar reasoning, we can see that if we have k weak exceedances for a permutation w, under the two maps in the following theorem, we have $n+1-k$ weak exceedances in the next iteration of the orbit. Thus we get the following:

Theorem 2.2.12. *The statistic* wexc *is $\frac{n+1}{2}$-mesic for orbits of reversal-rotated inversion and complement-rotated inversion*

2.2.3 Maxima and Minima

Definition 2.2.13. Let $w = w_1w_2 \cdots w_n \in \mathfrak{S}_n$. When reading the w_i from left to right, a **left-to-right maximum**, or **record**, is the largest number that has been read thus far. The terms **right-to-left maximum**, **left-to-right minimum** and **right-to-left minimum** are defined similarly.

If w_i is a left-to-right maximum, we write $w_i \in \underrightarrow{\text{Max}}(w)$. Denote $\underrightarrow{\text{max}}(w)$ as the number of left-to-right maxima. Similarly, if $w_j \in \underleftarrow{\text{Min}}(w)$, then w_j is a right-to-left minimum and $\underleftarrow{\text{min}}(w) := \#\underleftarrow{\text{Min}}(w)$. We define $\underleftarrow{\text{Max}}(w), \underleftarrow{\text{max}}(w), \underrightarrow{\text{Min}}(w)$, and $\underrightarrow{\text{min}}(w)$ in the same fashion.

Example 2.2.14. In the permutation 251863947, the left-to-right maxima are $2, 5, 8, 9$.

Several Foatic maps yield homomesy for linear combinations of left-to-right and right-to-left maxima and minima.The following table indicates which statistics are homomesic for which Foatic maps. In each case the map is 0-mesic. The homomesy which is still a conjecture is so marked.

Map	**Homomesic Statistic(s)**
inversion-reversal	$\underrightarrow{\mathrm{max}} - \underleftarrow{\mathrm{max}}$ $\underrightarrow{\mathrm{min}} - \underleftarrow{\mathrm{min}}$
rotation-complement	$\underleftarrow{\mathrm{max}} - \underleftarrow{\mathrm{min}}$ [conj]
inversion-complement	$\underrightarrow{\mathrm{max}} - \underrightarrow{\mathrm{min}}$ $\underleftarrow{\mathrm{max}} - \underleftarrow{\mathrm{min}}$
rotation-rotation	$\underrightarrow{\mathrm{max}} - \underleftarrow{\mathrm{min}}$
rotation-inversion	$\underrightarrow{\mathrm{max}} - \underleftarrow{\mathrm{min}}$

We will prove the homomesy for the rotation-rotation map, the others (excluding the conjecture) are similar.

Theorem 2.2.15. *The statistic* $\underrightarrow{\mathrm{max}} - \underleftarrow{\mathrm{min}}$ *is* 0*-mesic with respect to orbits of rotation-rotation.*

Lemma 2.2.16. *Define* $\Psi := \mathcal{Q}^2 \circ \mathcal{F}^{-1} \circ \mathcal{Q}^2 \circ \mathcal{F}$*, the rotation-rotation map.* Ψ *takes elements of* $\underleftarrow{\mathrm{Min}}(w)$ *to their inverses under* w *and* Ψ^{-1} *takes elements of* $\underrightarrow{\mathrm{Max}}(w)$ *to their inverses under* w*. More formally, if* $w_i \in \underleftarrow{\mathrm{Min}}(w)$*, then* $\Psi(w)_{w_i} = i$*. Similarly, if* $w_j \in \underrightarrow{\mathrm{Max}}(w)$*, then* $\Psi^{-1}(w)_{w_j} = j$*.*

Example 2.2.17. We illustrate this with an example. We will view our permutations in 2-line notation to more clearly see this result. Consider the permutation

$$\Psi\left(\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \textcolor{red}{1} & 5 & 6 & 4 & \textcolor{red}{2} & \textcolor{red}{3} \end{pmatrix}\right) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \textcolor{blue}{1} & \textcolor{blue}{5} & \textcolor{blue}{6} & 2 & 4 & 3 \end{pmatrix}$$

Here the elements of $\underleftarrow{\mathrm{Min}}(w)$ are in red. The lemma asserts that $\underleftarrow{\mathrm{Min}}(w)$ are mapped to their inverses under Ψ. So $w_1 = 1, w_5 = 2, w_6 = 3 \in \overleftarrow{\text{Min}}(w)$, and $\Psi(w)_1 = 1, \Psi(w)_2 = 5$ and $\Psi(w)_3 = 6$, shown in blue.

Proof. **(1)** ($\overleftarrow{\text{Min}}$)

Let $w_i \in \overleftarrow{\text{Min}}(w)$; this forces $w_i \leq i$. For if $w_i > i$, then there would be $n - i$ spots to the right of w_i that would need to be filled with numbers greater than w_i. But there are fewer than $n - i$ numbers greater than w_i since $n - w_i < n - i$. For example, consider the following permutation from $\mathfrak{S}_6$ where we have $w_3 = 4$.

$$- - 4 - - -$$

Since 4 appears in the third position, there are three spaces to the right of it. But if $4 \in \overleftarrow{\text{Min}}(w)$, then it is in the set of smallest elements when reading from right to left. So all the numbers to the right of it need to be greater than 4, which is not possible in $\mathfrak{S}_6$. Hence we can conclude that $w_3 \leq 3$ in this example and $w_i \leq i$ in general.

As a first case, suppose that $w_i < i$. We begin with an example.

Example 2.2.18. Consider the permutation $(2)(413)(65) = 324165$ under rotation-rotation. Note that $w_6 = 5$ and $w_4 = 1 \in \overleftarrow{\text{Min}}(w)$. We will focus on $w_4 = 1$ and note $1 < 4$.

$$(2)(413)(65) \xrightarrow{\mathcal{F}} 241365 \xrightarrow{\mathcal{Q}^2} 214635 \xrightarrow{\mathcal{F}^{-1}} (21)(4)(635) \xrightarrow{\mathcal{Q}^2} (56)(3)(142)$$

We generalize our example to the following sequence of maps, where $c(j)$ represents the complement of j, namely $c(j) = n+1-j$.

$$\sim (*iw_i*) \sim \xrightarrow{\mathcal{F}} *iw_i* \xrightarrow{\mathcal{Q}^2} *c(w_i)c(i)* \xrightarrow{\mathcal{F}^{-1}} \sim (*c(w_i)c(i)*) \sim \xrightarrow{\mathcal{Q}^2} \sim (*w_i i*) \sim$$

When we write w in CCD, w_i cannot be the largest element in the cycle (since i and w_i are in the same cycle), and so we have a permutation where i appears immediately to the left of w_i and has the form $\sim (*iw_i*) \sim$ where $*$ represents any (possibly empty) string and, $\sim$ represents any allowable cycles (or possibly nothing).

Applying $\mathcal{F}$ to w drops the parentheses, leaving us with i and w_i in adjacent positions in one-line notation (shown in blue). The map $\mathcal{Q}^2$ reverse and complements the permutation, leaving us with $c(w_i)$ and $c(i)$ in adjacent positions in one-line notation (shown in purple). Now $w_i < i \implies c(w_i) > c(i)$; thus, when we apply $\mathcal{F}^{-1}$ there cannot be parenthesis between $c(w_i)$ and $c(i)$ and they will be in the same cycle (shown in orange). Lastly, $\mathcal{Q}^2$ complements within cycles, so we arrive at w_i and i in adjacent positions in the same cycle (shown in green), implying that $w_i \mapsto i$ as desired.

As a second case, suppose $i = w_i$ for $w_i \in \underleftarrow{\mathrm{Min}}(w)$, so we have w_i in its own cycle. So w has the form $\sim (w_i) \sim$ and Ψ acts as follows:

$$\sim (w_i) \sim \xrightarrow{\mathcal{F}} *w_i* \xrightarrow{\mathcal{Q}^2} *c(w_i)* \xrightarrow{\mathcal{F}^{-1}} \sim (c(w_i)) \sim * \xrightarrow{\mathcal{Q}^2} \sim (w_i) \sim$$

The important observation here is that based on the definition of CCD, the last element of the cycle before (w_i) must be smaller than w_i, and the first element of the cycle after (w_i) must be greater than w_i. This means that after applying $\mathcal{F}$, the one line notation of $\mathcal{F}(w)$ has w_i preceded by a smaller number and followed by a larger number (shown in blue).

After applying $\mathcal{Q}^2$ we will have $c(w_i)$ again preceded by a smaller number and followed by a larger number (shown in violet). It follows that in the next iteration, when applying $\mathcal{F}^{-1}$, $c(w_i)$ will be a fixed point, alone in its cycle (shown in orange). Lastly, $\mathcal{Q}^2$ complements $c(w_i)$, leaving us with the fixed point w_i, so $w_i \mapsto i$ as desired (shown in green). Thus for all $w_i \in \underrightarrow{\text{Min}}(w)$, $\Psi(w)_{w_i} = i$.

(2) ($\underrightarrow{\text{Max}}$)

Next we show the reverse claim: if $w_j \in \underrightarrow{\text{Max}}(w)$ then $\Psi^{-1}(w)_{w_j} = j$. Let $w_j \in \underrightarrow{\text{Max}}(w)$ for $w \in \mathfrak{S}_n$. As above, we have $w_j \geq j$. We will trace our permutation w via $\Psi^{-1} = \mathcal{F}^{-1} \circ \mathcal{Q}^2 \circ \mathcal{F} \circ \mathcal{Q}^2$. Consider first the case where $w_j > j$. So in CCD w takes one of two forms, detailed below.

- $\sim (* j w_j *) \sim$

  This case follows from a reversing of the logic of part (1).

- $\sim (w_j * j) \sim$

$$\sim (w_j * j) \sim \xrightarrow{\mathcal{Q}^2} \sim (c(w_j) * c(j)) \sim$$

The application of the first Q^2, we would need to translate our permutation into CCD. Since $c(w_j) < c(j)$, this means $c(w_j)$ would no longer be the first element of the cycle, instead it would be of the form $\sim (*c(j)c(w_j)*) \sim$ and the rest of the map would look like the first case. In both scenarios it is clear that $c(j)$ now maps to $c(w_j)$.

- Lastly, if $w_j = j$ and the application of Ψ^{-1} takes on a form identical to the second part of **(1)**.

Finally, we can conclude that $\Psi^{-1}(w)_{w_j} = j$, which completes the proof of the lemma.

$\square$

Proof (of Theorem 2.2.15). We first claim that if $w_i \in \underleftarrow{\text{Min}}(w)$ for $w \in \mathfrak{S}_n$, then $i \in \underrightarrow{\text{Max}}(w)$ for $\Psi^{-1}(w)$. From the lemma we infer that since $\underleftarrow{\text{Min}}(w)$ are increasing when viewed from left-to-right, the same is true of $\Psi(\underleftarrow{\text{Min}}(w))$. Similarly, since $\underrightarrow{\text{Max}}(w)$ are increasing when views from left-to-right, the same is true of $\Psi^{-1}(\underrightarrow{\text{Max}}(w))$.

Using an example we will illustrate this fact, and show why we can take it an iteration further to show that $\Psi(\underleftarrow{\text{Min}}(w)) = \underrightarrow{\text{Max}}(w)$ and $\Psi^{-1}(\underrightarrow{\text{Max}}(w)) = \underleftarrow{\text{Min}}(w)$. In the first permutation, the $\underleftarrow{\text{Min}}(w)$ are in red. In the second line the $\underrightarrow{\text{Max}}(w)$ are in blue.

$$572163948 \xrightarrow{\mathcal{Q}^2 \circ \mathcal{F}^{-1} \circ \mathcal{Q}^2 \circ \mathcal{F}} 4 * 68 * 9*$$

$$253798164 \xrightarrow{\mathcal{F} \circ \mathcal{Q}^2 \circ \mathcal{F}^{-1} \circ \mathcal{Q}^2} *1 * 2 * 4 * 5$$

Consider the first example, where the $\underleftarrow{\text{Min}}(w)$ are translated to increasing left-to right numbers. To see that these are in fact the $\underrightarrow{\text{Max}}(w)$, suppose by way of contradiction that we had some other $\underrightarrow{\text{Max}}(w)$ in between these, for example, of the form

$$4568 * 9*$$

Then, since $5 \in \underrightarrow{\text{Max}}(w)$, when we apply Ψ^{-1}, we would have

$$4568 * 9* \xrightarrow{\mathcal{F} \circ \mathcal{Q}^2 \circ \mathcal{F}^{-1} \circ \mathcal{Q}^2} *1 * 3248$$

But then 2 would have been a $\underleftarrow{\text{Min}}(w)$ in our original permutation. It follows that $\underleftarrow{\text{Min}}(w)$ are translated to $\underrightarrow{\text{Max}}(w)$ in the rotation-rotation map and the reverse is also true by similar reasoning. We conclude that $\underrightarrow{\text{max}} - \underleftarrow{\text{min}}$ is 0-mesic for rotation-rotation.

□

2.3 Fixed Point Homomesy in the Complement-Inversion Map

In [LR20], La Croix and Roby conjectured fixed point homomesy for the complement-inversion map. In this section we seek to prove this homomesy for certain specific orbits.

Definition 2.3.1. A permutation is a **k-inside-out permutation (k-IOP)** if it can be written in one-line notation via the following algorithm.

Let w be a permutation in $\mathfrak{S}_n$. Take the numbers 1 through k and arrange them according to the following pattern.

$$\begin{array}{ll} \frac{k+2}{2}, \frac{k}{2}, \cdots, k-2, 3, k-1, 2, k, 1 & \text{for } k \text{ even} \\ \frac{k+1}{2}, \frac{k+3}{2}, \frac{k-1}{2}, \cdots, k-2, 3, k-1, 2, k, 1 & \text{for } k \text{ odd} \end{array}$$

The remaining numbers $k+1, \ldots, n$ can be arranged after this prefix in any order.

Example 2.3.2. An example of permutations in this form would be 435261897 where the k-inside-out part is 435261 and $k = 6$, and the remaining part (in green) is 897. Another example would be 231647859 where $k = 3$, the k-IO part is 231, $k = 3$ and the remaining part (in green) is 647859.

We begin with an example orbit of complement-inversion.

Example 2.3.3. Consider the following complement-inversion orbit in $\mathfrak{S}_6$:

$$\begin{array}{lllllllll}
(4)(532)(61) & \xrightarrow{\mathcal{F}} & \boxed{453261} & \xrightarrow{\mathcal{C}} & 324516 & \xrightarrow{\mathcal{F}^{-1}} & (32)(4)(51)(6) & \xrightarrow{\mathcal{I}} \\
(32)(4)(51)(6) & \xrightarrow{\mathcal{F}} & 324516 & \xrightarrow{\mathcal{C}} & \boxed{453261} & \xrightarrow{\mathcal{F}^{-1}} & (4)(532)(61) & \xrightarrow{\mathcal{I}} \\
(4)(523)(61) & \xrightarrow{\mathcal{F}} & \boxed{452361} & \xrightarrow{\mathcal{C}} & 325416 & \xrightarrow{\mathcal{F}^{-1}} & (32)(541)(6) & \xrightarrow{\mathcal{I}} \\
(32)(514)(6) & \xrightarrow{\mathcal{F}} & 325146 & \xrightarrow{\mathcal{C}} & \boxed{452631} & \xrightarrow{\mathcal{F}^{-1}} & (4)(52)(631) & \xrightarrow{\mathcal{I}} \\
(4)(52)(613) & \xrightarrow{\mathcal{F}} & \boxed{452613} & \xrightarrow{\mathcal{C}} & 325164 & \xrightarrow{\mathcal{F}^{-1}} & (32)(51)(64) & \xrightarrow{\mathcal{I}} \\
(32)(51)(64) & \xrightarrow{\mathcal{F}} & 325164 & \xrightarrow{\mathcal{C}} & \boxed{452613} & \xrightarrow{\mathcal{F}^{-1}} & (4)(52)(613) & \xrightarrow{\mathcal{I}} \\
(4)(52)(631) & \xrightarrow{\mathcal{F}} & \boxed{452631} & \xrightarrow{\mathcal{C}} & 325146 & \xrightarrow{\mathcal{F}^{-1}} & (32)(514)(6) & \xrightarrow{\mathcal{I}} \\
(32)(541)(6) & \xrightarrow{\mathcal{F}} & 325416 & \xrightarrow{\mathcal{C}} & \boxed{452361} & \xrightarrow{\mathcal{F}^{-1}} & (4)(523)(61) & \xrightarrow{\mathcal{I}}
\end{array}$$

When viewing the elements of an orbit of complement-inversion in cycle notation, we note that the first number of the first cycle will alternate between k and $n+1-k$ as we move through the orbit. In our example we observe an alternation of 4 and 3. If a permutation's first cycle begins with k, under the Foatic map $\mathcal{F}$, the first number in one-line notation will be k. When we apply the complement map $\mathcal{C}$, the first number will be $n+1-k$. Since the first number begins the first cycle, applying $\mathcal{F}^{-1}$ means out first cycle will begin with $n+1-k$. Since the inverse map $\mathcal{I}$ freezes the first number in each cycle and reverses the other numbers, $n+1-k$ will remain the first number in the cycle.

In order to better understand these orbits, we will look at a type of suborbit of them. In Figure 2.1, we will zoom in on the boxed elements of this orbit. Since we are alternating between elements of the second column and the third column, which are separated by the complement map, each of the boxed permutations will begin with the same number, in this case 4. Next, we will look at the boxed elements in isolation as their own special type of orbit. We can think of these elements as forming their own 'suborbit' under the following actions.

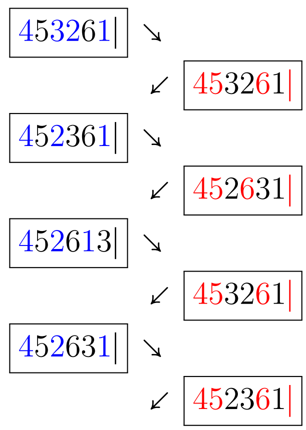
When following the map in a southwest direction ($\swarrow$) we do the following:

- Identify in red the ascending records, consider these as 'posts'
- Add an imaginary red post in position $n+1$
- Reverse the string of number in between two posts

  When following the map in a southeast direction ($\searrow$) we do the following:

- Identify in blue the descending records, consider these as 'posts'

Figure 2.1: A ‘suborbit’ of a complement-inversion orbit



- Add an imaginary blue post in position $n+1$

- Reverse the string of number in between two posts

Note that traveling in the southwest direction is the equivalent of applying the map $\mathcal{F} \circ \mathcal{I} \circ \mathcal{F}^{-1}$. This is consistent with the directions since identifying ascending records can be thought of as adding the parenthesis of $\mathcal{F}^{-1}$ by marking the first element of each cycle. Reversing between posts is precisely the action of $\mathcal{I}$ where we freeze the first element of a cycle and reverse the order of the remaining elements. To apply $\mathcal{F}$ we would then drop the parentheses, leaving us with a permutation in one line notation.

Similarly, traveling in the southeast direction is the equivalent of applying the map $\mathcal{C} \circ \mathcal{F} \circ \mathcal{I} \circ \mathcal{F}^{-1} \circ \mathcal{C}$. Here we note that the descending records of a permutation are in the same positions as the ascending records of the complement. Thus we mark descending records first, then apply the same map as above.

Viewing complement-inversion orbits in this way will allow us to leverage the former results from La Croix and Roby which prove fixed point homomesy for the reversal-inversion map. First, we restate part 2 of Theorem 2.1.9 as a lemma to our result.

Lemma 2.3.4. *The statistic* Fix *is* 1*-mesic with respect to the action of* $\overline{\varphi} = \mathcal{I} \circ \mathcal{F}^{-1} \circ \mathcal{R} \circ \mathcal{F}$.

La Croix and Roby conjectured that the statistic Fix is also 1-mesic for complement-inversion and complement-rotation, and we add here that this is also likely the case for reversal-rotation. What follows is a proof that Fix takes on the same average value for k-IOP orbits of complement-inversion.

Theorem 2.3.5. *The statistic Fix has average value* 1 *for orbits of complement-inversion that contain at least one* k*-IOP.*

Proof. k-IOPs have the property that the k-IO part of the permutation is frozen as you travel through a boxed suborbit. In the following proof we will refer to this as the 'frozen prefix' and the remaining part of the permutation as the 'free part'.

To understand this type of permutation more clearly, consider the following example:

$$
\begin{array}{lclclclc}
(43)(52)(61)(8)(97) & \xrightarrow{\mathcal{F}} & \boxed{435261897} & \xrightarrow{\mathcal{C}} & 675849213 & \xrightarrow{\mathcal{F}^{-1}} & (6)(75)(84)(9213) & \xrightarrow{\mathcal{I}} \\
(6)(75)(84)(9312) & \xrightarrow{\mathcal{F}} & 675849312 & \xrightarrow{\mathcal{C}} & \boxed{435261798} & \xrightarrow{\mathcal{F}^{-1}} & (43)(52)(61)(7)(98) & \xrightarrow{\mathcal{I}} \\
(43)(52)(61)(7)(98) & \xrightarrow{\mathcal{F}} & \boxed{435261798} & \xrightarrow{\mathcal{C}} & 675849312 & \xrightarrow{\mathcal{F}^{-1}} & (6)(75)(84)(9312) & \xrightarrow{\mathcal{I}} \\
(6)(75)(84)(9213) & \xrightarrow{\mathcal{F}} & 675849213 & \xrightarrow{\mathcal{C}} & \boxed{435261897} & \xrightarrow{\mathcal{F}^{-1}} & (43)(52)(61)(8)(97) & \xrightarrow{\mathcal{I}}
\end{array}
$$

In this particular example, our frozen prefix, 435261, will be the same in every boxed element, due to its alternating nature. The decreasing numbers will alternate with the increasing numbers, freezing our entire prefix at each step.

Because of this phenomenon, we can focus on the remaining three numbers of the permutation. These three numbers are all greater than the frozen prefix by definition, so they will never be in red in the left column. So when traveling in the southeast direction, these numbers are acted on by reversal. As we know, traveling in the southwest direction acts on these numbers by the map $\mathcal{F}\mathcal{I}\mathcal{F}^{-1}$. So then we can think of this free part as a smaller orbit of reversal-inversion. More specifically, if our entire orbit is size $2m$, this is an orbit of reversal inversion of size m. In this example, it corresponds to the orbit from $\mathfrak{S}_3$, containing the elements (2)(31) and (1)(32).

In general, any orbit containing a k-IOP in the second column will follow a similar orbit structure. Since La Croix and Roby have already shown that reversal-inversion orbits have fixed point homomesy, it follows that the number of fixed points in the free part will equal the size of the free part orbit. In our example that number is 2. Note that the size of the overall orbit containing this type of permutations will be twice the size of the free part orbit. So the total number of fixed points contributed by the free part is $\frac{1}{2} \cdot \#\mathcal{O}$.

The remaining fixed points in the orbit will come from the frozen prefix part. For k odd, $\mathcal{F}^{-1}(w)$ yields 1 fixed point (namely $\frac{k+1}{2}$). In the forward direction, $\mathcal{F}^{-1}(\mathcal{C}(w))$ will then yield 2-cycles followed by a larger cycle beginning with n, but no fixed points. This structure will remain after applying $\mathcal{I}$. It follows that the number of fixed points contributed by the frozen prefix is $\frac{1}{2} \cdot \#\mathcal{O}$. For k even, we can argue analogously. In this example that fixed point will be (6).

It follows that the total number of fixed points in orbits of this type is equal

$\frac{1}{2} \cdot \#\mathcal{O} + \frac{1}{2} \cdot \#\mathcal{O} = \#\mathcal{O}$ so Fix has average value 1.

□

2.4 Near Homomesies

Both the inversion-reversal and inversion-complement maps have a slew of statistics that are nearly homomesic, i.e. statistics that have the same average value for most orbits, but fail for a few specific larger orbits, which happen to have odd length. The statistics that come up for most orbits under inversion-reversal are $\underrightarrow{\max} - \underleftarrow{\max}$, $\underrightarrow{\min} - \underleftarrow{\min}$, inv, D_n, $D_i + D_{n-i}$ (for $i \in [n-1]$), des $= \sum_{i=1}^{n-1} D_i$ and cdes $= \sum_{i=1}^{n} D_i$. For inversion-complement the statistics of interest are inv, D_i for any $i \in [n]$ (and thus des, cdes, and maj by linear combination), $\underrightarrow{\max} - \underrightarrow{\min}$ and $\underleftarrow{\max} - \underleftarrow{\min}$. The phenomenon for these two maps actually comes from an interesting orbit structure that explains not only why several orbits have the same average value for certain statistics, but also the reason that we get exceptions.

2.4.1 The Inversion-Reversal Map

The inversion-reversal map has a nice map structure that lends itself to several near homomesies. Unfortunately many of these break at $n = 8$ because of a few anomalous orbits. In this section we seek to explain the structure of this map.

Example 2.4.1. Below is an orbit of inversion-reversal. Notice that when written in one-line notation, we can pair up permutations that are reversals of each other. We have done this by color coding the permutations. By arranging the orbit structure in a certain way, we can create an orbit where the middle two permutations are reverses, and then the one above and below that pair are reverses, and then the pair two above and two below are reverses and so on.

$$\begin{aligned}
(32)(4)(51)(6) &= 532416\\
(2)(4)(6351) &= 625413\\
(1)(32)(5)(64) &= 132654\\
(53)(6142) &= 465231\\
(21)(5)(634) &= 214653\\
(4)(62513) &= 356412\\
(1)(32)(645) &= 132564\\
(614253) &= 456231\\
(52134)(6) &= 314526\\
(653421) &= 614235
\end{aligned}$$

To explain this structure, we will begin with the middle two permutations. For these permutations to be reverses is equivalent to saying that the map $\mathcal{R} \circ \mathcal{F}^{-1} \circ \mathcal{I} \circ \mathcal{F}$ is equivalent to reversal. This will be true for any permutation w that has $\mathcal{F}^{-1} \circ \mathcal{I} \circ \mathcal{F}$ equal to the identity map, in other words any permutation w where $\mathcal{F}(w)$ is an involution. In Example 2.4.1 we have two permutations with this property, the first of the middle two permutations, and the last permutation.

$$\mathcal{F}((21)(5)(634)) = \mathcal{F}(214653) = 215634 = (21)(53)(64)$$

$$\mathcal{F}((653421)) = \mathcal{F}(614235) = 653421 = (43)(52)(61)$$

In order to further understand this structure, we begin with one of the permutations that is paired with its reversal. In other words, a permutation w where $\mathcal{F}(w)$ is an involution and $\mathcal{F}(w) = \mathcal{I}(\mathcal{F}(w)))$. Certainly if a permutation is its own inverse, then $\mathcal{I}$ acts like the identity.

$$215634 = (21)(53)(64) \xrightarrow{\mathcal{I}} 215634 = (21)(53)(64)$$

As we build out from this part of the map in both directions to recreate the orbit, we are applying $\mathcal{F}^{-1}$ in both cases, so we get the same permutations at those steps.

$$(21)(5)(634) \xrightarrow{\mathcal{F}} 215634 \xrightarrow{\mathcal{I}} 215634 \xrightarrow{\mathcal{F}^{-1}} (21)(5)(634)$$

Similarly, since $\mathcal{R}$ is an involution, we also get the same permutations in each direction at this step.

$$\begin{array}{ccccccccc} & & & & & & & (4)(62513) = 356412 & \xrightarrow{\mathcal{R}} \\ (21)(5)(634) = 214653 & \xrightarrow{\mathcal{F}} & 215634 & \xrightarrow{\mathcal{I}} & 215634 & \xrightarrow{\mathcal{F}^{-1}} & & (21)(5)(634) = 214653 & \xrightarrow{\mathcal{R}} \\ (4)(62513) = 356412 & & & & & & & & \end{array}$$

This continues at every step, building out from the center. Again we have color coded to show that we have the same permutation as we work our way outward.

$$\begin{array}{ccccccccc} & & & & & & & (1)(32)(645) = 132564 & \xrightarrow{\mathcal{R}} \\ (53)(6142) = 465231 & \xrightarrow{\mathcal{F}} & 536142 & \xrightarrow{\mathcal{I}} & 462513 & \xrightarrow{\mathcal{F}^{-1}} & & (4)(62513) = 356412 & \xrightarrow{\mathcal{R}} \\ (21)(5)(634) = 214653 & \xrightarrow{\mathcal{F}} & 215634 & \xrightarrow{\mathcal{I}} & 215634 & \xrightarrow{\mathcal{F}^{-1}} & & (21)(5)(634) = 214653 & \xrightarrow{\mathcal{R}} \\ (4)(62513) = 356412 & \xrightarrow{\mathcal{F}} & 462513 & \xrightarrow{\mathcal{I}} & 536142 & \xrightarrow{\mathcal{F}^{-1}} & & (53)(6142) = 465231 & \xrightarrow{\mathcal{R}} \\ (1)(32)(645) = 132564 & & & & & & & & \end{array}$$

The key is that as we build out, we come to a reversal pair in the left hand column of each iteration.

If we do not have a permutation w within our orbit such that $\mathcal{F}(w)$ is an involution, then we will not get the same matching as we otherwise would. However, there exists some other permutation which is the reverse of w in another orbit. Since we could apply the same reasoning to a pair of elements of the two orbits, if we consider these statistics on these orbits together, we would still get the same average values. This orbit structure leads to "near-homomesies"— statistics that have the same average value when restricted to just orbits that contain involutions and lead us to the following theorem.

Theorem 2.4.2. *The orbits of inversion-reversal which contain an involution have the property that the statistics* $\underrightarrow{\max} - \underleftarrow{\max}$ *and* $\underrightarrow{\min} - \underleftarrow{\min}$ *have average value* 0*,* inv *has average value* $\frac{n(n-1)}{4}$*,* D_n *has average value* $\frac{1}{2}$*,* $D_i + D_{n-i}$ *has average value* 1*,* des *has average value* $\frac{n}{2}$*, and* cdes *has average value* $\frac{n+1}{2}$*.*

Proof. **Maxima and Minima**: If a number is a left-to-right maxima, when the reverse is applied it becomes a right-to-left maxima. The same is true for left-to-right minima becoming right-to-left minima. Consider the following two permutations which are reverses of each other. In the first example the $\underrightarrow{\max}$ and $\underleftarrow{\max}$ are in red and in the second the $\underrightarrow{\min}$ and $\underleftarrow{\min}$ are in blue.

$$748231965 \xrightarrow{\mathcal{R}} 569132847$$
$$748231965 \xrightarrow{\mathcal{R}} 569132847$$

Thus, since every permutation and its reverse appear once within the cycle, both $\underrightarrow{\max} - \underleftarrow{\max}$ and $\underrightarrow{\min} - \underleftarrow{\min}$ have average value 0.

Inversions: $(a,b) \in \operatorname{Inv}(w) \iff (a,b) \notin \operatorname{Inv}(\mathcal{R}(w))$. Consider the following permutations which are reversals of each other.

$$415236 \xrightarrow{\mathcal{R}} 632514$$

The inversion pairs for the permutation on the left are $(4,1),(4,2),(4,3),(5,2)$ and $(5,3)$. For the permutation on the right hand side we have inversion pairs $(6,3),(6,2),(6,5),(6,1),\ (6,4),(3,2),(3,1),(2,1),(5,1)$ and $(5,4)$. So $\mathrm{inv}(w) = 5$ and $\mathrm{inv}(\mathcal{R}(w)) = 10$. Note that the total possible inversions for $\mathcal{S}_n$ is $\frac{n(n-1)}{2}$, in this case 15. Therefore, if an orbit has the property that every elements reverse is also in the orbit, then the total number of inversions in the orbit is $\frac{\#\mathcal{O}}{2} \cdot \frac{n(n-1)}{2}$ meaning that the average number of inversions is $\frac{n(n-1)}{4}$.

Descents: If we can show that both $D_i + D_{n-i}$ and D_n show the same average values over orbits, then des and cdes follow since these are just linear combinations of the previous two statistics.

We begin with D_n. If n is a descent in a permutation w, then $w_n > w_1$. When we reverse w, these numbers will switch positions and $\mathcal{R}(w)_n < \mathcal{R}(w)_1$. Thus, if D_n is a descent in a permutation, then it will not be a descent in the reverse of a permutation. So n will be a descent in half of the permutations in an orbit, and D_n has average value $\frac{1}{2}$.

By a similar reasoning, if i is a descent of a permutation w, so $w_i < w_{i+1}$, then it follows that $\mathcal{R}(w)_{n+1-i} > \mathcal{R}(w)_{n-i}$. For example, consider the following two permutations which are reverses of each other.

$$748231954 \xrightarrow{\mathcal{R}} 569132847$$

Take for example, the index 3, where $w_3 = 8$. Since $w_4 = 2$, we have that 3 is a descent since $2 < 8$. When we reverse this permutation, the numbers 8 and 2 will appear in the reverse order. So the location of 2, which is $n - 3 = 9 - 3 = 6$ is not a descent.

We conclude that the total $D_i + D_{n-i} = \#\mathcal{O}$ for each i. So the statistic $D_i + D_{n-i}$ has average value 1. It follows that des$= \sum_{j=1}^{n-1} D_j = \#\mathcal{O} \cdot \frac{n}{2}$, so des has average value $\frac{n}{2}$. Similarly, cdes$= \sum_{k=1}^{n} D_k = \#\mathcal{O} \cdot \frac{n}{2} + \#\mathcal{O} \cdot \frac{1}{2}$, so cdes has average value $\frac{n+1}{2}$.

□

2.4.2 The Inversion-Complement Map

We have a very similar phenomenon with the inversion-complement map.

Example 2.4.3. The following is an example of an orbit in $\mathfrak{S}_6$.

$$\begin{array}{rl}
(3)(4)(6251) & = 653412 \\
(2)(63154) & = 521643 \\
(3)(5)(6214) & = 413652 \\
(653124) & = 241635 \\
(1)(65432) & = 162345 \\
(4)(53)(621) & = 615432 \\
(541)(623) & = 536142 \\
(413)(652) & = 364125 \\
(641253) & = 256134 \\
(1)(2)(43)(65) & = 124365
\end{array}$$

As in Example 2.4.1 , we have a similar pairing system, only here our pairs are complements of each other. The reason for this is similar to that for the inversion-reversal example. Beginning in the middle, we can identify a permutation w where

$\mathcal{F}(w)$ is an involution. From here we can build out in both directions, matching permutations for each map. The only difference from our explanation in the previous section is that we replace the reversal map with the complement map.

As long as we start with a permutation where $\mathcal{F}(w)$ is an involution, we will get the pairing seen above. If no such permutation is in the orbit, we will not have this property. However, if we take any w in one of these exception orbits, we can find its complement in another orbit, and these two orbits together will have the same average values for these statistics as in the nicer orbits.

Theorem 2.4.4. *The orbits of inversion-complement which contain an involution have the property that the statistics* $\underset{\longrightarrow}{\max} - \underset{\longrightarrow}{\min}$ *and* $\underset{\longleftarrow}{\max} - \underset{\longleftarrow}{\min}$ *have average value* 0*,* inv *has average value* $\frac{n(n-1)}{4}$*,* D_i *has average value* $\frac{1}{2}$ *for all* $i \in [n]$*,* des *has average value* $\frac{n-1}{2}$*,* cdes *has average value* $\frac{n}{2}$*, and* maj *has average value* $\frac{n(n-1)}{4}$*.*

Proof. **Maxima and Minima**: As in the inversion-reversal example, these homomesies are the most straightforward. If a number is a left-to-right maxima, under the complement map this number will be a left-to-right minima, and likewise for right-to-left maxima and minima. It follows that both $\underset{\longrightarrow}{\max} - \underset{\longrightarrow}{\min}$ and $\underset{\longleftarrow}{\max} - \underset{\longleftarrow}{\min}$ have average value 0.

Inversions: Again, this homomesy is similar to the inversion-reversal case. If a pair (i, j) is an inversion pair, then under the complement map, (i, j) will not be in an inversion. It follows that the statistic inv has average value $\frac{n(n-1)}{4}$.

Descents: Note that if i is a descent, then $w_i > w_{i+1}$. Under the complement map, $\mathcal{C}(w)_i < \mathcal{C}(w)_{i+1}$. So if i is a descent for a permutation, it is not a descent for the complement of that permutation. It follows that the total number of descents for any pair of permutations is $n-1$, and the total number of cyclic descents is n. From this we can conclude that D_i has average value $\frac{1}{2}$ for all $i \in [n]$. Furthermore, we have the following

$$\text{des} = \sum_{i=1}^{n-1} D_i = \frac{n-1}{2} \cdot \#\mathcal{O}$$
$$\text{cdes} = \sum_{i=1}^{n} D_i = \frac{n}{2} \cdot \#\mathcal{O}$$
$$\text{maj} = \sum_{i=1}^{n-1} iD_i = \frac{n(n-1)}{4} \cdot \#\mathcal{O}$$

It follows that des has average value $\frac{n-1}{2}$, cdes has average value $\frac{n}{2}$, and maj has average value $\frac{n(n-1)}{4}$.

□

2.5 Future Directions

2.5.1 Remaining Conjectures

The largest remaining open question of this chapter is the fixed point homomesy for general permutations in complement-inversion orbits as originally conjectured by La Croix and Roby.

Conjecture 2.5.1 ([LR20]). *The statistic* Fix *is* 1*-mesic for all orbits of complement-inversion.*

There seems to be a correlation between reversal-inversion orbit sizes and complement-inversion orbit sizes, which led to an investigation of reversal-inversion orbits lurking inside complement-inversion ones in a similar way to those seen in Section 2.3.

Consider the example shown in Figure 2.2. As in Section 2.3 we focus on the boxed elements of a complement-inversion orbit as a sort of 'suborbit'. (This is not really a suborbit, but we are using that term to describe the interior alternating sequence detailed in previous sections.) Recall that in the southeast direction, the action is $CFIF^{-1}C$, and in the southwest direction it is FIF^{-1}.

The example appears to reveal a reversal-inversion orbit (with repeated permutations) lurking inside a complement-inversion orbit. While many orbits, especially for $\mathfrak{S}_n$ when $n < 8$ behave this way, not all do, prohibiting the use of an inductive argument based on the size of the orbit. However if we could somehow realize all orbits as a version of this embedded idea, fixed point homomesy could be extended.

In addition to this conjecture, the fixed point homomesy phenomenon observed in the reversal-inversion and complement-inversion maps described in Section 2.3 also appear to hold for both the complement-rotation and the reversal-rotation maps.

Conjecture 2.5.2. *The statistic* Fix *is* 1*-mesic for orbits of reversal-rotation*

Conjecture 2.5.3. *The statistic* Fix *is* 1*-mesic for orbits of complement-rotation*

The complement-rotation case was originally conjectured by La Croix and Roby in [LR20]. It seems likely that one may be able to extend a proof of the reversal-rotation case to the complement-rotation case, as was done in this thesis.

Of the other homomesic statistics we looked at in the Foatic maps, a few remain open. Conjecture 2.5.6 has not received much attention and may not be difficult to prove.

Conjecture 2.5.4. *The statistic* $\underleftarrow{\max} - \underleftarrow{\min}$ *is* 0*-mesic for the rotation-complement map.*

Figure 2.2: 'Suborbit' within a complement-inversion orbit

395821 674 ↘ CFIF*C

FIF* ↙ 385921 476

3859 674 12 ↘ CFIF*C

FIF* ↙ 3 476 95812

3 476 92185 ↘ CFIF*C

FIF* ↙ 39 674 2158

398512 476 ↘ CFIF*C

FIF* ↙ 35891 674 2

35892 476 1 ↘ CFIF*C

FIF* ↙ 39852 674 1

391 476 258 ↘ CFIF*C

FIF* ↙ 391852 674

39 476 2581 ↘ CFIF*C

FIF* ↙ 3 674 92851

3 476 91582 ↘ CFIF*C

39 476 1285

Conjecture 2.5.5. *The statistic* $D_1 - D_{n-1}$ *is* 0*-mesic for the rotation-reversal map.*

Conjecture 2.5.6. *The statistic* $\mathrm{Fix}_1 - \mathrm{Fix}_n$ *is* 0*-mesic for the following four maps: rotated inversion-complement, complement-rotated inversion, rotated inversion-reversal, and reversal-rotated inversion.*

It appears to be the case that in orbits of these maps the fixed points (n) and (1) appear two iterations away from each other, and likely this could be shown in a way similar to the proof given in Section 2.2.1.

2.5.2 Other Statistics

As mentioned previously, we investigated 25 maps created out of intertwinings of the Foata map, its inverse and five dihedral involutions. The statistics we searched for included the following: cyc (the number of cycles), inv, wexc, exc, $\underrightarrow{\max}, \underleftarrow{\max}, \underrightarrow{\min}, \underleftarrow{\min}, D_i$ for all $i \in [n]$ and Fix_i for all $i \in [n]$. Additionally we looked at all linear combinations of these statistics, including $\mathrm{Fix} = \sum_{i=1}^{n} \mathrm{Fix}_i, \mathrm{exc} = \mathrm{wexc} - \mathrm{Fix}, \mathrm{des} = \sum_{i=1}^{n-1} D_i, \mathrm{cdes} = \sum_{i=1}^{n-1} D_i, \mathrm{maj} = \sum_{i=1}^{n-1} iD_i, \mathrm{asc} = n - 1 - \mathrm{des}$. There are certainly more statistics that could be investigated in these maps, such as pinnacles, peaks and valleys, [DNPT] and others.

In the following chapter we will investigate a second map, also attributed to Foata, and a similar intertwining which yields homomesic permutation statistics.

Chapter 3

The Foata–Schützenberger Map

3.1 Introduction to the Foata–Schützenberger Map

There is a second map, also attributed to Foata, which has an entirely different definition then the one in the previous chapter. In this chapter we define this map, explore its properties, and describe similar intertwinings with dihedral involutions. This map comes from work of Dominique Foata and Marcel-Paul Schützenberger, [FS78] and as such we will refer to it here as the Foata–Schützenberger (F–S) map.

The dynamics of the more straightforward action of simply iterating the F–S map on $\mathfrak{S}_n$ has been studied in unpublished work of Amdeberhan [A]. He is interested in the orbit structure and corresponding partition that the map induces on $\mathfrak{S}_n$. (Each orbit must lie entirely within an inverse descent set by Theorem 3.1.12.) He states some conjectures concerning the generating functions that count the number of orbits of different sizes.

The Foata–Schützenberger map was initially defined as a way of providing a bijective proof showing the equidistribution across $\mathfrak{S}_n$ of the **inversion number** (Section 1.3) and the **major index** (Section 2.1), whose definitions we now recall.

Definition 3.1.1. Let $w = w_1 w_2 \cdots w_n \in \mathfrak{S}_n$. The **inversion number**, denoted $\text{inv}(w)$, is given by $\text{inv}(w) = \sum_{i<j, w_i > w_j} 1$, in other words it counts the number of pairs (w_i, w_j) which are "out of order". The **major index**, denoted $\text{maj}(w)$, is defined as the sum of the descents of w, i.e., $\text{maj}(w) = \sum_{i, w_i > w_{i+1}} i$. Note that n is never a descent.

The major index was introduced first by Major Percy MacMahon [Mac16] who proved that it is equidistributed with the inversion number over $\mathfrak{S}_n$. It was named "major index" by Foata, in honor of MacMahon's military rank. In the language of generating functions, this means:

$$\sum_{w \in \mathfrak{S}_n} q^{\text{inv}(w)} = \sum_{w \in \mathfrak{S}_n} q^{\text{maj}(w)} \tag{3.1}$$

We defined q-analogues of combinatorial numbers in Definition 1.3.7 and saw an example there. Here we note

Proposition 3.1.2. $\sum_{w \in \mathfrak{S}_n} q^{\text{inv}(w)} = [n]_q! = (1+q)(1+q+q^2)\cdots(1+q+q^2+\cdots+q^{n-1})$.

Definition 3.1.3. Let a_k but the number of inversions (w_i, w_j) in w with $w_j = k$. The **inversion table** of w is defined to be the n-tuple $I(w) = (a_1, \ldots, a_n)$.

We can recover w from its inversion table by the following algorithm. Given $I(w) = (a_1, \ldots, a_n)$, we successively place the numbers n through 1, where each $n-i$ is inserted into w (expressed in one-line notation), so that it has a_{n-i} elements to its left. This gives a bijection $\mathfrak{S}_n \leftrightarrow [0, n-1] \times [0, n-2] \times \cdots \times [0, 0]$.

Example 3.1.4. Let $I(w) = (a_1, a_2, \ldots, a_6) = (3, 0, 1, 2, 1, 0)$. So we begin with $n = 6$, which has $a_n = 0$ elements to its left. Next we insert 5 with $a_5 = 1$ element (the 6) to its left. Then 4 with $a_4 = 2$ elements to its left, placing it at the end. We continue like this to reconstruct w as follows:

6
65
654
6354
26354
263154

Proof. [Stan11, Proposition 1.3.12, 1.3.13] Certainly a permutation gives a unique inversion table, and the algorithm given in Definition 3.1.3 gives a map from an inversion table to a unique permutation; thus, we have a bijection between permutations and $\mathcal{T}_n \in [0, n-1] \times [0, n-2] \times \cdots \times [0, 0]$, the set of all possible inversion tables.

To see why Equation 3.1.2 is the generating function for the number of inversions, note that if $I(w) = (a_1, a_2, \ldots, a_n)$, then $\text{inv}(w) = a_1 + \cdots + a_n$. It follows that

$$\begin{aligned}
\sum_{w \in \mathfrak{S}_n} q^{\text{inv}(w)} &= \sum_{a_1=0}^{n-1} \sum_{a_2=0}^{n-2} \cdots \sum_{a_n=0}^{0} q^{a_1+a_2+\cdots+a_n} \\
&= \left(\sum_{a_1=0}^{n-1} q^{a_1} \right) \left(\sum_{a_2=0}^{n-2} q^{a_2} \right) \cdots \left(\sum_{a_n=0}^{0} q^{a_n} \right) \\
&= [n]_q [n-1]_q \cdots [1]_q = [n]_q!
\end{aligned}$$

□

MacMahon's proof via generating functions of the equidistribution of inv and maj left open the question of finding a canonical bijection $\varphi : \mathfrak{S}_n \to \mathfrak{S}_n$ where $\mathrm{maj}(w) = \mathrm{inv}(\varphi(w))$. The solution, found by Foata and Schützenberger, is the map we will discuss here.

Definition 3.1.5. [FS78] The Foata–Schützenberger map $\varphi : \mathfrak{S}_n \to \mathfrak{S}_n$ is defined inductively as follows. Let $w = w_1 w_2 \dots w_n$ be a permutation in one-line notation.

i) Define $\gamma_1 = w_1$; assume that γ_k, a partial permutation on $[n]$ of length k, has been defined for some $k \in [n]$, then

ii) if the last letter of γ_k is greater (respectively smaller) than w_{k+1}, split γ_k into "subwords" after each letter greater (respectively smaller) than w_{k+1}; then

iii) in each compartment of γ_k determined by the splits move the last letter to the beginning; obtain γ_{k+1} by appending w_{k+1} to the end of the transformed word;

iv) while $k < n$ repeat with step $(ii) - (iii)$, if $k = n$, then $\varphi(w) = \gamma_k$.

Example 3.1.6. Let $w = 492675138 \in \mathfrak{S}_9$. The algorithm gives us the following

$$\begin{aligned}
\gamma_1 &= 4 \to 4 \\
\gamma_2 &= 4|9 \to 49 \\
\gamma_3 &= 4|9|2 \to 492 \\
\gamma_4 &= 4|92|6 \to 4296 \\
\gamma_5 &= 4|2|96|7 \to 42697 \\
\gamma_6 &= 426|9|7|5 \to 642975 \\
\gamma_7 &= 6|4|2|9|7|5|1 \to 6429751 \\
\gamma_8 &= 642|9751|3 \to 26419753 \\
\gamma_9 &= 2|6|4|1|97|5|3|8 \to 264179538 \\
\varphi(w) &= 264179538
\end{aligned}$$

Note that the descents of $w = 492675138$ occur at indices 2, 5 and 6 so $\text{maj}(w) = 2 + 5 + 6 = 13$. Also the inversion pairs for $\varphi(w)$ are $(2,1)$, $(6,4)$, $(6,1)$, $(6,5)$, $(6,3)$, $(4,1)$, $(4,3)$, $(7,5)$, $(7,3)$, $(9,5)$, $(9,3)$, $(9,8)$, and $(5,3)$ so $\text{inv}(\varphi(w)) = 13$ as desired.

To see that the Foata–Schützenberger map is in fact a bijection, we define an inverse map.

Definition 3.1.7. [FS78] Let $v = v_1v_2\ldots v_n$; to obtain $w = w_1w_2\ldots w_n = \varphi^{-1}(v)$ apply the following procedure to v;

i) Put $\delta_{n-1} = v_1 v_2 \cdots v_{n-1}$ and $w_n = v_n$; assume that the word δ_k and the integers $w_{k+1}, w_{k+2}, \ldots w_n$ have been defined for some k with $1 \leq k < n$;

ii) if the first letter δ_k is greater (respectively smaller) than w_{k+1}, split δ_k *before* each letter greater (respectively smaller) than w_{k+1};

iii) in each compartment of δ_k determined by the splits move the first letter to the end; to obtain δ_{k-1} delete the last letter of the transformed word; furthermore, put w_k equal to that deleted letter;

iv) if $k = 1$, then $\varphi^{-1}(v) = w_1 w_2 \ldots w_n$; if not, replace k by $k - 1$ and return to instructions $(ii) - (iii)$.

Example 3.1.8. Let $v = 385491726 \in \mathfrak{S}_9$. The reverse algorithm proceeds as follows

$$\delta_9 = |38|5|49|17|2 \cdot 6 \to 83594712$$

$$\delta_8 = |8|3|5|9|4|71 \cdot 2 \to 8359417$$

$$\delta_7 = |835|941 \cdot 7 \to 358419$$

$$\delta_6 = |3|5|8|4|1 \cdot 9 \to 35841$$

$$\delta_5 = |3|5|8|4 \cdot 1 \to 3584$$

$$\delta_4 = |358 \cdot 4 \to 583$$

$$\delta_3 = |5|8 \cdot 3 \to 58$$

$$\delta_2 = 5 \cdot 8 \to 5$$

$$\delta_1 = 5$$

$$w = \varphi^{-1}(v) = 583419726$$

Theorem 3.1.9 ([Stan11, Proposition 1.4.6]). *The bijection* $\varphi : \mathfrak{S}_n \to \mathfrak{S}_n$ *transforms* maj *to* inv*, namely* $\operatorname{maj}(w) = \operatorname{inv}(\varphi(w))$.

Proof. Given $w = w_1 w_2 \dots w_n \in \mathfrak{S}_n$, set $\eta_k = w_1 \dots w_k$. We will show by induction on k that $\operatorname{inv}(\gamma_k) = \operatorname{maj}(\eta_k)$, and the proof will follow for the case where $k = n$.

As a base case, note that if $k = 1$, certainly $\operatorname{inv}(\gamma_1) = \operatorname{maj}(\eta_1) = 0$. So assume that for some $k < n$, $\operatorname{inv}(\gamma_k) = \operatorname{maj}(\eta_k)$. Now, suppose that the last letter w_k of γ_k is greater than w_{k+1}. Therefore k is a descent of w, and it will add k to $\operatorname{maj}(\eta_k)$. We need to show that $\operatorname{inv}(\gamma_{k+1}) = k + \operatorname{inv}(\gamma_k)$.

Since we are dealing with the case where $w_k > w_{k+1}$, we split γ_k into compartments after each letter greater than w_{k+1}. So the last letter of each compartment C is the largest letter in that compartment and each compartment contains exactly one letter larger than w_{k+1}. Now, when we cyclically shift the compartment, we are moving the largest letter to the front, giving us $\#C - 1$ new inversions. Also, appending w_{k+1} gives one new inversion for each compartment. So if there are m compartments, we have that the added number of inversions will be $\sum_C (\#C - 1) + m = k$, as desired. The case where $w_k < w_{k+1}$ is shown similarly. □

We can make an even stronger statement about the inv and maj statistics on $\mathfrak{S}_n$, which is that they have a **symmetric joint distribution**.

Definition 3.1.10. [Stan11, Equation 1.44] Two statistics $f, g : S \to \mathbb{N}$ on a set S have a **symmetric joint distribution** if for all $j, k \in \mathbb{N}$ we have

$$\#\{x \in S : f(x) = j, g(x) = k\} = \#\{x \in S : f(x) = k, g(x) = j\}$$

In terms of generating functions, we can state this as

$$\sum_{x \in S} q^{f(x)} t^{g(x)} = \sum_{x \in S} q^{g(x)} t^{f(x)}$$

To see why inv and maj have symmetric joint distribution, we begin with a definition.

Definition 3.1.11. For $w \in \mathfrak{S}_n$, the **inverse descent set** $\mathrm{ID}(w)$ is the descent set of the inverse of w, $\mathrm{ID}(w) = D(w^{-1})$.

Another way of thinking about the inverse descent set is via the "reading set" of w: read the numbers $1, 2, \ldots, n$ from left to right in their standard order in w, returning to the beginning of w if necessary. We then take the cumulative number of elements in these reading sequences (excluding the last) to form the reading set.

Considering our permutation from Example 3.1.6 where $w = 492675138$. The reading sequences would be read off as "1", "2, 3", "4, 5", "6, 7, 8", and "9". So the total numbers read at each stage (excluding the last) are $1, 2, 2, 3$ making $\mathrm{ID}(w) = \{1, 3, 5, 8\}$. However, we also note that applying the F–S bijection to w gave $\varphi(w) = 264179538$ and we observe that $\mathrm{ID}(\varphi(w)) = \{1, 3, 5, 8\}$ as well.

Theorem 3.1.12 ([Stan11, Theorem 1.4.8]). *Let φ be the bijection given in Definition 3.1.5. Then for all $w \in \mathfrak{S}_n$, $\mathrm{ID}(w) = \mathrm{ID}(\varphi(w))$. In other words, φ preserves the inverse descent set.*

Proof. As in Definition 3.1.5, we will define $w = w_1 w_1 \ldots w_n \in \mathfrak{S}_n$, $\eta_k = w_1 w_2 \ldots w_k$, and γ_k to be the permutation of η_k at step k of the F–S bijection.

We can show by induction on k that $ID(\gamma_k) = ID(\eta_k)$. As a base case it is clear that $\gamma_1 = \eta_1$ so $ID(\gamma_1) = ID(\eta_1)$.

For our inductive step, assume that $ID(\gamma_k)$, the reading set of γ_k, is the same as $ID(\eta_k)$, the reading set of η_k. Now consider the addition of w_{k+1} where $w_{k+1} < w_k$. (The case where $w_{k+1} > w_k$ is analogous.)

Since $w_{k+1} < w_k$ we will insert a divider after each letter larger than w_{k+1}, so the last letter of each compartment will be greater than w_{k+1} and each of these is the only letter in the compartment bigger than w_{k+1}. Now, when reading $ID(\gamma_{k+1})$, we will proceed along the same lines of the reading sequences for γ_k, until we come to the largest number less than w_{k+1}. Then we will read w_{k+1}, and then proceed back to the beginning.

For example, consider the step from γ_5 to γ_6 in Example 3.1.6, where we have $\gamma_5 = 426|9|7|5$. Here $w_{k+1} = 5 < w_k = 7$. The compartments are split after all the numbers greater than 5, namely $6, 9$ and 7. These numbers are the largest in each compartment, and are the only numbers greater than 5. So the reading sequences of γ_5 will be "2", "4,6,7" and "9" while the reading sequences of $\gamma_6 = 642975$ will be "2", "4,5", and then back to the beginning for "6, 7" and then "9".

Now consider the reading sequences of η_{k+1}. Again, we follow the reading sequences of η_k until we come to the largest number less than w_{k+1}; then we will read w_{k+1}, then return to the beginning of η_{k+1}. By the inductive hypothesis, the reading sequences of both η_{k+1} and γ_{k+1} will be the same up to the reading of w_{k+1}.

In our example $\eta_5 = 49267$ and $\eta_6 = 492675$. Note that the reading sequences of η_6 are "2", "4,6,7" and "9" , which are the same as γ_5. The addition of the 5 affects the reading sequences in that we begin "2" but then we come to the largest number less than $w_{k+1} = 5$, which is 4 and we read "$4, 5$", then back to the beginning for "$6, 7$" and "9".

Now, we claim that the reading of the numbers larger then w_{k+1} will also proceed in the same order. But note that the numbers larger than w_{k+1} are all in separate compartments, so the cycling that happens when applying the F–S map does not change the order in which these numbers appear; thus, they will be read in the same order as they were for γ_k. In our example this is precisely noting that the $6, 7$ and 9 appear in different compartments, and therefore their order does not change after the rotations. Of course they also appear in the same order in η_{k+1} as η_k since η_{k+1} just appends w_{k+1}, so they will also be read in the same order.

We conclude that $ID(\gamma_{k+1}) = ID(\eta_{k+1})$ and thus that $ID(\varphi(w)) = ID(\gamma_n) = ID(\eta_n) = ID(w)$.

$\square$

Corollary 3.1.13. *Define* $\operatorname{imaj}(w) = \operatorname{maj}(w^{-1}) = \sum_{i \in \mathrm{ID}(w)} i$. *The three pairs of statistics* $(\operatorname{inv}, \operatorname{maj})$, $(\operatorname{inv}, \operatorname{imaj})$ *and* $(\operatorname{maj}, \operatorname{imaj})$ *all have symmetric joint distributions.*

Proof. Let f be a statistic on $\mathfrak{S}_n$ and define $g(w) = f(w^{-1})$; then f and g have symmetric joint distribution. So $(\operatorname{maj}, \operatorname{imaj})$ is just a special case of this. Also, we know that φ converts maj to inv and preserves imaj, so it follows that $(\operatorname{inv}, \operatorname{imaj})$ has symmetric joint distribution. Lastly, since $\operatorname{inv}(w) = \operatorname{inv}(w^{-1})$ [Stan11], we can conclude that $(\operatorname{inv}, \operatorname{maj})$ has symmetric joint distribution as well.

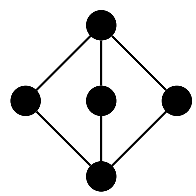
$\square$

3.2 Permutation Statistics and Linear Extensions of Posets

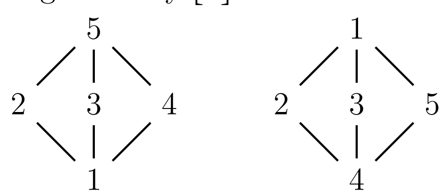
In their paper "Permutation Statistics and Linear Extensions of Posets" [BW91] Björner and Wachs explore classes of permutations which have invariant statistics under the F–S bijection. They specifically look at permutations which are linear extensions of labeled posets. We refer the reader to Definition 1.4.16 for the definition of a poset.

Definition 3.2.1. A **labeled poset** (P, ω) is a finite, partially ordered set P together with a bijection $\omega : P \to [p]$ where p is the cardinality of P. A labeling ω is called **natural** if ω is an order-preserving bijection from P to the natural total order on $[p]$. In other words, our labeling is natural if $\omega(x) < \omega(y)$ whenever $x <_p y$ where $<_p$ denotes the order relation on P. Natural labelings are also known as **linear extensions**.

Example 3.2.2. Consider the following poset



The following are labelings of P by $[5]$. The first is a natural labeling.

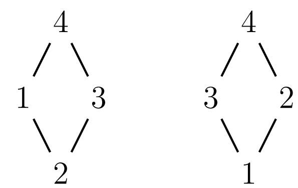


An order ideal (Definition 1.4.20) of a poset P is called **principal** if it is generated by a single element $x \in P$. There is another type of labeling that is useful in understanding certain invariant poset statistics.

Definition 3.2.3. A labeling ω of a poset P is **recursive** if every principal order ideal of P is labeled with a consecutive sequence of labels.

Example 3.2.4. In the examples below the poset on the left is a recursively labeled poset since its three principal order ideals are labeled by $\{1,2\}, \{2,3\}, \{2\}, \varnothing$ and

$\{1, 2, 3, 4\}$ which are all sets of consecutive numbers. Whereas the poset on the right has a principal order ideal labeled $\{1, 3\}$.



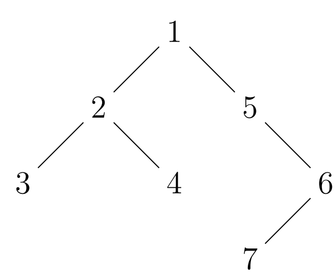
The following three definitions illustrate three different types of recursive labelings.

Definition 3.2.5. A **postorder** is a natural recursive labelling.

Definition 3.2.6. A labelling is **strict** if it is a natural labelling for the dual order. In other words, if the label of each node is less than that of its children.

Definition 3.2.7. A **preorder** is a strict recursive labeling.

Example 3.2.8. In Definition 1.2.6 we defined binary trees. The following is an example of a binary tree with a preordered labelling.



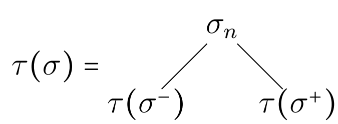
Note that the preorder is an example of a non natural recursive labeling.

Definition 3.2.9. An **inorder** is defined only for binary trees. It is a recursive labelling in which the label of a node is greater than that of its left child and smaller than that of its right child. In the following theorem and definition we will see an example of inordered binary trees.

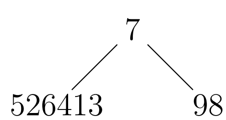
Theorem 3.2.10 ([BW91, Proposition 5.1])**.** *Let B_n be the set of inordered labeled binary trees with n vertices. For every $\sigma \in \mathfrak{S}_n$, there is a unique $T \in B_n$ such that σ is a linear extension of T.*

For the proof of this proposition, Björner and Wachs introduce the following map τ, which helps reveal properties of the Foata–Schützenberger map.

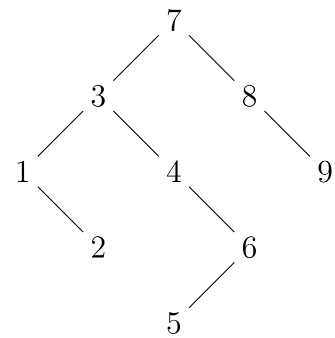
Definition 3.2.11. Let θ denote both the empty word and the empty tree. Let $l \in [n]$ where σ_l is the last letter of the word σ and let σ^- and σ^+ be the subwords of σ consisting of all letters less than σ_l and all letters greater than σ_l, respectively. Then set $\tau : \mathfrak{S}_n \to B_n$ to be the map defined recursively by $\tau(\theta) = \theta$ and



Example 3.2.12. Let $\sigma = 526419837$. So $\sigma_n = 7$, $\sigma^- = 526413$ and $\sigma^+ = 98$ and our first iteration of the binary tree looks like



Now for $\tau(\sigma^-)$, we get that our new $\sigma_l = 3$, $(\sigma^-)^- = 21$ and $(\sigma^-)^+ = 564$. Continuing like this we arrive at the following binary tree.

$\tau(\sigma) =$ 

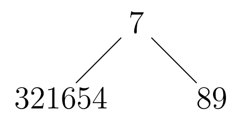
Proof (of Theorem 3.2.10). As we saw in Definition 3.2.9, an inordered labeled binary tree is a recursively labeled binary tree that has the property that every label of a node is greater than the label of its left child and smaller than that of its right child. Let B_n be the set of inordered labeled binary trees with n nodes. We are trying to show that there is a unique $T \in B_n$ such that $\sigma \in \mathcal{L}(T)$, the set of linear extensions of T. So there is a unique inordered labeled binary tree that has σ as its linear extension.

A linear extension is just an order-preserving bijection. So in order to show that σ is a linear extension of $\tau(\sigma)$, we just need to show that if a node is covered by another node, then that number appears earlier in σ. But this follows directly from the definition of $\tau(\sigma)$. By definition, the elements of σ^- are those that appear before σ_l that are less than σ_l and σ^+ are those that appear before σ_l that are greater. So since this is the recursive definition for every parent node, each time the parent appears to the right of its children. Thus, we have an order-preserving bijection.

To see that $T = \tau(\sigma)$ is the unique, inordered labeled binary tree with σ as a linear extension, we will reason by example.

Example 3.2.13. Let $\sigma = 389216547$. We need the tree to be inordered. So the left child of any node needs to have a smaller label and the right child has a larger label. So put $\sigma_n = 7$ on top, because if it is a binary tree, it can only have one "top node" so that needs to be the last number in σ.

Now, put all the options for the left node (all the lower numbers) on the left, and all the options for the right node (all the greater numbers) on the right.



Since we need this to be a linear extension, make the node the last number, this way it will be order-preserving since both of its children will appear before it in σ. So for our example, 4 needs to be the parent node on the left and 9 needs to be the parent node on the right.

Continuing inductively this gives precisely the tree $T = \tau(\sigma)$ that we defined before. So, in fact this is the unique inordered labeled binary tree with σ as a linear extension. We note here that this is not the only σ which is a linear extension of T.

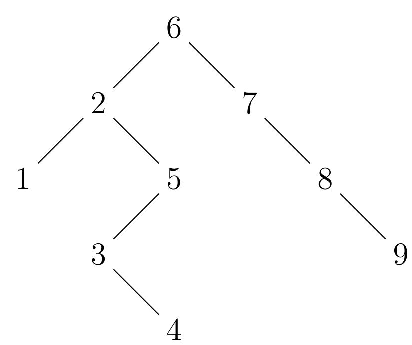
$\square$

The key usefulness of the linear extension given in Definition 3.2.11 is that it has the property of being **Foata-invariant**, meaning that it remains unchanged under the F–S bijection. The following theorem comes from Björner and Wach's first paper on the q-Hook length formula.

Theorem 3.2.14 ([BW89, Theorem 2.2]). *Let (P, w) be a labeled poset with $\#P = n$, and let $\varphi : \mathfrak{S}_n \to \mathfrak{S}_n$ be the F–S bijection. If ω is a recursive labeling, and $\mathcal{L}(P, \omega)$*

is the set of linear extensions of (P, ω)*, then* $\varphi(\mathcal{L}(P, \omega)) = \mathcal{L}(P, \omega)$*. In other words,* $\mathcal{L}(P, \omega)$ *is a Foata-invariant.*

Example 3.2.15. We will use the following example to illustrate the proof. Let $\sigma = 498315276$ be a linear extension corresponding to the following inordered labeled binary tree, obtained by applying the τ map given in Definition 3.2.11.

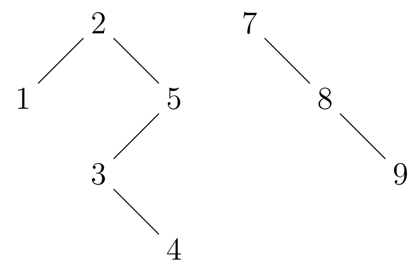


Proof. We will accompany the proof by tracing an example, denoted by $\bullet$. We will show this by induction on the size of the poset P. For the base case, if $\#P = 1$, then $\#\mathcal{L}(P, \omega) = 1$.

Now assume this assertion is true for all posets of size $n-1$, and consider a poset P of size n. Let $\sigma = \sigma_1\sigma_2\ldots\sigma_n$ be a linear extension of P corresponding to nodes $x_1, x_2, x_3, \ldots x_n$ where $\omega(x_i) = \sigma_i$.

To begin, we will remove a (possibly the) maximal element x_n, leaving us with the subposet P' with labeling ω'. Since ω is a recursive labelling, it follows that ω' is as well. Thus we can apply the inductive hypothesis to get that $\varphi(\mathcal{L}(P', \omega')) = \mathcal{L}(P', \omega')$. It remains to show that $\varphi(\sigma) \in \mathcal{L}(P, \omega)$.

$\bullet$ In our example this step corresponds to removing the parent node labeled 6, leaving us with the following subposet P'.



Note that if $\alpha \in \mathcal{L}(P', \omega')$ and we concatenate σ_n, then the resulting $\alpha \cdot \sigma_n \in \mathcal{L}(P, \omega)$. Furthermore, since σ_n is the last element, it continues to be the last element after applying φ. Recall that the last step of the Foata map is denoted as γ_n. So if we can show that $\gamma_n(\alpha) \in \mathcal{L}(P', \omega')$ whenever $\alpha \in \mathcal{L}(P', \omega')$, it will follow that $\varphi(\sigma) \in \mathcal{L}(P, \omega)$.

Assume for a contradiction that $\alpha \in \mathcal{L}(P', \omega')$, but $\gamma_n(\alpha) \notin \mathcal{L}(P', \omega')$. Now, in the last step of the Foata bijection, the letter σ_n will induce splits that factor the word into compartments as $\alpha = \alpha_1\alpha_2\cdots\alpha_k$. Each α_i corresponds to a labeled subposet of (P', ω'), call it (P_i, ω_i), where $\alpha_i \in \mathcal{L}(P_i, \omega_i)$.

• In our example, when applying φ to $\sigma = 498315276$, the step γ_9 will take the subword 43958127 and induce the factorization 439|58|127| since $7 > 6$, so $a_1 = 439$, $a_2 = 58$ and $a_3 = 127$.

If $\gamma_n(\alpha) \notin \mathcal{L}(P', \omega')$, then it must be that one of the α_i's, after the rotation step is not a linear extension of its subposet (P_i, ω_i). Denote by $\tilde{\alpha}_i$ the subword after the rotation of the last letter to the beginning. So, if the last letter of α_i is z, then the first letter of $\tilde{\alpha}_i$ is z. This is the only difference between α_i and $\tilde{\alpha}_i$.

• In our running example, the rotation step yields 439|58|127| → 943|85|712|.

So if $\tilde{\alpha}_i \notin \mathcal{L}(P', \omega')$, then the issue must be the relationship between z and some

$x \in P_i$. Since z appears after the other letters in α_i, the issue must be that the movement of z to the front of $\tilde{\alpha}_i$, causes a problem. Namely it must be that $z >_P x$.

• So for example consider $\alpha_3 = 127$, the corresponding scenario would be if the node labeled 7 was covering the node labeled 2.

The breaking up of α into $\alpha_1\alpha_2\cdots\alpha_k$ was induced by the concatenation of $\sigma_n = w(x_n)$. So we have one of two cases (i) either $\sigma_n = w(x_n)$ is less than $w(z)$ but greater than the other letters in α_i, namely $w(x_n) > w(x)$, giving us that $w(x) < w(x_n) < w(z)$; or (ii) the reverse, giving us $w(x) > w(x_n) > w(z)$.

• In our running example, we have that $x_n = 6 < 7 = z$ and also $6 < 1$ and $6 < 2$ (the two other letters in the subword α_3).

But if $x <_P z$, since ω is a recursive labeling, every principle order ideal must contain a consecutive sequence of numbers, so it must also be true that $x_n <_P z$. But this contradicts the maximality of x_n. Thus, $\gamma_n(\alpha) \in \mathcal{L}(P', \omega')$ as desired.

• In our example, if the node labeled 7 covers the node labeled 2, and we know that $7 > 6 > 2$. Since ω is recursive, the node labeled 7 must also cover the node labeled 6, otherwise the principal order ideal induced by the node labeled 7 wouldn't have all the consecutive numbers 7 through 2. Of course this contradicts the fact that the node labeled 6 was supposed to be our maximal node.

□

Definition 3.2.16. A permutation statistic $s : \mathfrak{S}_n \to \mathbf{N}$ is a **tree-dependent statistic** if there is some tree statistic $S : B_n \to \mathbf{N}$ such that $s(\sigma) = S(\tau(\sigma))$ for all $\sigma \in \mathfrak{S}_n$ where τ is the map from Definition 3.2.11.

The visual representation of a permutation given in Definition 3.2.11 lends itself to a few tree-dependent statistics that we will look at in a moment. First, we extend the notion of a descent (Definition 2.2.6) to k-descents.

Definition 3.2.17. A k**-descent** of a permutation is an index i where $\sigma_i > \sigma_{i+1}$, σ_{i+2}, … σ_{i+k}. In other words i is an index where σ_i is greater than the k elements to its right. The k**-descent set** of a permutation $\sigma \in \mathfrak{S}_n$ is defined as follows

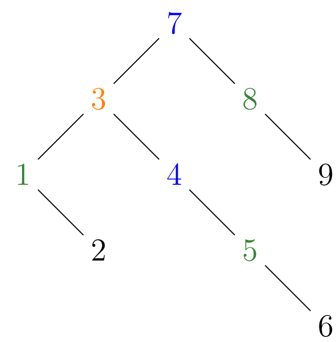
$$\mathrm{Des}_k(\sigma) = \{i = 1, 2, \ldots, n-k : \sigma_i > \sigma_{i+1}, \sigma_{i+2}, \ldots, \sigma_{i+k}\}$$

So $\mathrm{Des}_1(\sigma)$ is the ordinary descent set of σ, set $\mathrm{des}_k(w) = \#\,\mathrm{Des}_k(w)$.

Example 3.2.18. Let $w = 528431976$, so we have the following.

$$\mathrm{Des}_1(\omega) = \{1, 3, 4, 5, 7, 8\}$$
$$\mathrm{Des}_2(\omega) = \{3, 4, 7\}$$
$$\mathrm{Des}_3(\omega) = \{3\}$$

Clearly, $\mathrm{Des}_i(w) \supseteq \mathrm{Des}_{i+1}(w)$ for $i \geq 1$. The set of labeled nodes of $\tau(\sigma)$ whose right subtrees have at least k nodes is $\mathrm{Des}_k(\sigma^{-1})$. Note that $\omega = 528431976$ is the inverse of $\sigma = 625491837$ and thus we can see each $\mathrm{Des}_k(\sigma^{-1})$ by looking at the tree in Example 3.2.12. We have $\mathrm{Des}_3(\sigma^{-1})$ in orange, $\mathrm{Des}_2(\sigma^{-1})$ in blue and orange $\mathrm{Des}_1(\sigma^{-1})$ in blue , orange and green.



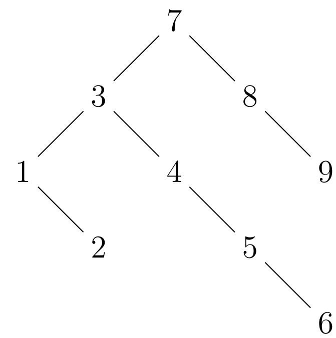
We also recall the definitions of left-to-right maxima and minima.

Definition 3.2.19. Let $w = w_1 w_2 \cdots w_n \in \mathfrak{S}_n$. We call w_i a **left-to-right maximum** if $w_i > w_j$ for all $j \in [i-1]$. In other words, when reading from left to right, w_i is the largest number that has been read thus far. The terms **right-to-left maximum**, **left-to-right minimum** and **right-to-left minimum** are defined similarly. We recall that $\underrightarrow{\mathrm{Max}}(w)$ is the set of left-to-right maxima, $\underrightarrow{\mathrm{max}}(w)$ is the number of left-to-right maxima, and $\underleftarrow{\mathrm{Min}}(w)$, $\underleftarrow{\mathrm{min}}$, $\underleftarrow{\mathrm{Max}}(w), \underleftarrow{\mathrm{max}}(w), \underrightarrow{\mathrm{Min}}(w)$, and $\underrightarrow{\mathrm{min}}(w)$ are defined in the same fashion.

Lemma 3.2.20. *The quantities* $\underleftarrow{\mathrm{max}}(w)$ *and* $\underleftarrow{\mathrm{min}}(w)$ *are also tree-dependent statistics for a permutation* $w \in \mathfrak{S}_n$.

Proof. This is because $\underleftarrow{\mathrm{Max}}(w)$ is the set of labels on the rightmost branch of $\tau(w)$ and $\underleftarrow{\mathrm{Min}}(w)$ is the set of labels on the leftmost branch of $\tau(w)$. The root of the tree is w_n, which would of course be an element of both $\underleftarrow{\mathrm{Max}}(w)$ and $\underleftarrow{\mathrm{Min}}(w)$. The right child of the root is a node labeled with a number larger than w_n since it is in w^- and it is the farthest to the right by definition of τ. Continuing this reasoning we see that $\underleftarrow{\mathrm{Max}}(w)$ is the set of labels on the rightmost branch; the reasoning for $\underleftarrow{\mathrm{Min}}(w)$ is analogous. $\square$

Example 3.2.21. For our running example $w = 62549837$, $\underleftarrow{\text{Max}}(w) = \{7,8,9\}$ which is the set of labels on the rightmost branch of $\tau(w)$, and $\underleftarrow{\text{Min}}(w) = \{7,3,1\}$ is the set of labels on the leftmost branch of $\tau(w)$.



3.3 Homomesies and the Foata–Schützenberger Map

As in Chapter 2, we explore compositions of the Foata–Schützenberger map with dihedral involutions. Because of the way φ is defined, it is straightforward to see that $\varphi^{-1} \circ \mathcal{C} \circ \varphi$ is equivalent to $\mathcal{C}$. Thus, we will leave out compositions that involve the complement map. There are sixteen remaining maps to check, many of which show at least conjectural homomesies. A list of all conjectured and proved homomesies can be seen in Appendix B. As in Chapter 2, we abbreviate the names of the maps to just the dihedral involutions. So the map $\mathcal{I} \circ \varphi^{-1} \circ \mathcal{R} \circ \varphi$ would just be called the reversal-inversion map.

3.3.1 Maxima and Minima

Homomesies for minima and maxima rely on reasoning similar to that which proves the following theorem of Björner and Wachs.

Theorem 3.3.1 ([BW91, Theorem 5.6])**.** *The F–S involution* $\Psi = \mathcal{I} \circ \varphi \circ \mathcal{I} \circ \varphi^{-1} \circ \mathcal{I} : \mathfrak{S}_n \to \mathfrak{S}_n$ *preserves* $\underleftarrow{\max}(w)$ *for* $w \in \mathfrak{S}_n$ *and exchanges* inv *and* maj*. Dually, there exists an involution* Ψ' *which preserves* $\underleftarrow{\min}(w)$ *and exchanges* inv *and* maj*.*

Proof. To begin, we have that $w_j \in \underleftarrow{\mathrm{Max}}(w)$ if and only if $j \in \underleftarrow{\mathrm{Max}}(w^{-1})$. This implies that $\underleftarrow{\max}(w) = \underleftarrow{\max}(w^{-1})$. Next, since $\underleftarrow{\max}(w)$ is tree-dependent by Lemma 3.2.20, it is also preserved by φ. Thus, Ψ preserves $\underleftarrow{\max}(w)$. For the dual statement, let $\Psi' = \mathcal{C} \circ \Psi \circ \mathcal{C}$.

□

Example 3.3.2. Consider the permutation, written here in 2-line notation for ease of understanding the argument, $w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 6 & 9 & 4 & 1 & 3 & 8 & 2 & 5 & 7 \end{pmatrix}$. The elements of $\underleftarrow{\mathrm{Max}}(w)$, namely $w_j = 7, 8, 9$ are shown in red and their inverses, $j = 2, 6$, and 9, are shown in blue. We have $w^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 7 & 5 & 3 & 8 & 1 & 9 & 6 & 2 \end{pmatrix}$ where we can see that $9, 6, 2 \in \underleftarrow{\mathrm{Max}}(w^{-1})$. This correspondence implies that $\underleftarrow{\max}(w) = \underleftarrow{\max}(w^{-1})$. We also have that $\varphi(w) = 164329587$ and we can see that $7, 8, 9 \in \underleftarrow{\mathrm{Max}}(\varphi(w))$ (shown in violet), illustrating that $\underrightarrow{\max}(w)$ is preserved by φ.

This brings us to the main results of this chapter, homomesy for maxima and minima related statistics.

Theorem 3.3.3. *The inversion-reversal map is* 0*-mesic for the statistic* $\underrightarrow{\max} - \underleftarrow{\max}$*. Inversion-rotation is* 0*-mesic for* $\underrightarrow{\min} - \underleftarrow{\max}$ *and inversion-inversion is* 0*-mesic for* $\underrightarrow{\max} - \underleftarrow{\min}$.

Proof. In Theorem 3.3.1, we saw that the map $\Psi = \mathcal{I} \circ \varphi \circ \mathcal{I} \circ \varphi^{-1} \circ \mathcal{I} : \mathfrak{S}_n \to \mathfrak{S}_n$ preserves $\underleftarrow{\max}(w)$ for $w \in \mathfrak{S}_n$. In the proof we also saw that $\underleftarrow{\max}(w) = \underleftarrow{\max}(w^{-1})$. Since φ is a bijection, we can conclude that $\varphi^{-1} \circ \mathcal{I} \circ \varphi$ preserves $\underleftarrow{\max}$.

The reversal map has the property that $\underleftarrow{\mathrm{Max}}(w)$ turns into $\underrightarrow{\mathrm{Max}}(\mathcal{R}(w))$, since $\mathcal{R}(w)$ is the same permutation in reverse order, as illustrated with the following example.

Example 3.3.4. Here $\underleftarrow{\mathrm{Max}}(w)$ are in red and $\underrightarrow{\mathrm{Max}}(\mathcal{R}(w))$ are in blue.

$$694138257 \xrightarrow{\mathcal{R}} 752831496$$

Given some $w \in \mathfrak{S}_n$, $\underleftarrow{\max}(w) = \underrightarrow{\max}(\mathcal{R} \circ \phi^{-1} \circ \mathcal{I} \circ \phi(w))$, so the map is 0-mesic for the statistic $\underrightarrow{\max} - \underleftarrow{\max}$ as previously claimed.

Similarly, we recall that $\mathcal{Q}^2 = \mathcal{R} \circ \mathcal{C}$. So rotation takes $\underleftarrow{\mathrm{Max}}(w)$ to $\underrightarrow{\mathrm{Min}}(\mathcal{Q}^2(w))$ as seen in the following example.

Example 3.3.5. In the process of applying $\mathcal{Q}^2 = \mathcal{R} \circ \mathcal{C}$, the reversal map converts $\underleftarrow{\mathrm{Max}}(w)$ (shown in red) to $\underrightarrow{\mathrm{Max}}(\mathcal{R}(w))$ and the complement map converts $\underrightarrow{\mathrm{Max}}(\mathcal{R}(w))$ to their complements which are $\underrightarrow{\mathrm{Min}}(\mathcal{C}(\mathcal{R}(w)))$ (shown in blue).

$$694138257 \xrightarrow{\mathcal{Q}^2} 358279614$$

So given some $w \in \mathfrak{S}_n$, $\underrightarrow{\min}(w) = \underleftarrow{\max}(\mathcal{Q}^2 \circ \phi^{-1} \circ \mathcal{I} \circ \phi(w))$; thus the map is 0-mesic for the statistic $\underrightarrow{\min} - \underleftarrow{\max}$ as previously claimed.

In the case of the inversion-inversion map, we note, and illustrate by example, that inversion converts the indices of the elements of $\underleftarrow{\mathrm{Min}}(w)$ to elements of $\underrightarrow{\mathrm{Max}}(w^{-1})$. So if $w_i \in \underleftarrow{\mathrm{Min}}(w)$, then $i \in \underrightarrow{\mathrm{Max}}(w^{-1})$ and it follows that $\underrightarrow{\max} - \underleftarrow{\min}$ is 0-mesic.

Example 3.3.6. Let $w = 64\textcolor{red}{13}2\textcolor{red}{57}$ and note that the $\underrightarrow{\mathrm{Min}}(w) = \{7, 5, 2, 1\}$ (shown in red) with indices $\{7, 6, 5, 3\}$ respectively. Observing that $w^{-1} = \textcolor{blue}{35}42\textcolor{blue}{6}1\textcolor{blue}{7}$ and $\underleftarrow{\mathrm{Max}}(w^{-1}) = \{3, 5, 6, 7\}$ (shown in blue).

In conclusion we can see the 0-mesies for each of these three maps.

□

3.3.2 Descents

Lemma 3.3.7. *For a permutation $w \in \mathfrak{S}_n$, if $n-1$ is a descent, then $\varphi(w)_1 > \varphi(w)_n$.*

Example 3.3.8. For example, if $w = 529731864$, then $n - 1 = 8$ is a descent since $w_{n-1} = 6 > 4 = w_n$. Note that $\varphi(w) = 957862314$ and $\varphi(w)_1 = 9 > 4 = \varphi(w)_n$.

Proof. To see why this is true, we consider the last step of φ, γ_n, on w. Since $n - 1$ is a descent, and in step γ_{n-1} the number w_{n-1} was added in, it follows that the last number in γ_{n-1} is greater than w_n, thus the splits in γ_n are placed after every number which is greater than w_n. So in our example we would have

$$9|5|7|8|2316|4$$

If, at this stage, the first number, $(\gamma_{n-1})_1$, is less than w_n, then there would not be a split after $(\gamma_{n-1})_1$, but in fact after the next $(\gamma_{n-1})_k$ to the right of $(\gamma_{n-1})_1$ such that $(\gamma_{n-1})_k > w_n$. But then during the rotation of numbers in the cell, $(\gamma_{n-1})_k$ would become the first number, and so $\varphi(w)_1 = (\gamma_{n-1})_k$ and $\varphi(w)_1 > w_n = \varphi(w)_n$.

□

Lemma 3.3.9. *For a permutation $w \in \mathfrak{S}_n$, if $w_n > w_1$, $n-1$ is an ascent of $\varphi^{-1}(w)$.*

Proof. We consider the first step of φ^{-1} on w. If $w_n > w_1$, then the split is placed before every number that is smaller than w_n, and then this number is rotated to appear at the end of its cell. Thus, after the rotation, the number in position $(\delta_n)_{n-1}$ is smaller than w_n. Since this number will become $\varphi^{-1}(w)_{n-1}$, we conclude that $\varphi^{-1}(w)_{n-1} < \varphi^{-1}(w)_n$.

□

Theorem 3.3.10. *The reversal-rotation and rotation-reversal maps are* 1*-mesic for the statistic $D_1 + D_{n-1}$.*

Facts 3.3.11. The following facts are straightforward and will be shown by example. We illustrate with the permutation $w = 492675138$

1. If $w_1 < w_n$, then after applying the reversal map, $\mathcal{R}(w)_1 > \mathcal{R}(w)_n$ (and vice versa).

   In our example $w_1 = 4 < w_n = 8$ and for $R(w) = 831576294$, $R(w)_1 = 8 > 4 = R(w)_n$.

2. The reversal map exchanges ascents and descents in positions 1 and $n-1$.

In our example there is an ascent in position 1 where $w_1 = 4$ and an ascent in position $n-1$ where $w_{n-1} = 3$. After applying the reversal map, there is a descent in position 1 where $R(w)_1 = 8$ and in position $n-1$ where $R(w)_{n-1} = 9$.

3. For the rotation map, if $w_1 < w_n$, then after applying the rotation map, $\mathcal{Q}^2(w)_1 < \mathcal{Q}^2(w)_n$ (and vice versa).

   In our example $w_1 = 4 < w_n = 8$, and $\mathcal{Q}^2(w) = 279434816$ so $\mathcal{Q}^2(w)_1 = 2 < 6 = \mathcal{Q}^2(w)_n$.

4. If 1 is an ascent (similarly a descent) for w, then $n-1$ is an ascent (similarly a descent) for $\mathcal{Q}^2(w)$. Analogously, an ascent (or descent) in position $n-1$ becomes an ascent (or descent) in position 1 after applying $\mathcal{Q}^2$.

   In our example there are ascents in positions 1 and $n-1$ where $w_1 = 4$ and $w_{n-1} = 3$. After applying the rotation map, there are ascents in positions 1 and $n-1$ where $\mathcal{Q}^2(w)_1 = 2$ and $\mathcal{Q}^2(w)_{n-1} = 1$.

Proof (of Theorem 3.3.10). Employing the statements in Facts 3.3.11, we begin with the reversal-rotation map.

From Lemma 3.3.7, we know that if $n-1$ is a descent of w, then $\varphi(w)_1 > \varphi(w)_n$. Now, from Fact 1, it follows that $\mathcal{R}(\varphi(w))_n > \mathcal{R}(\varphi(w))_1$. From Lemma 3.3.9, $n-1$ is an ascent for $\varphi^{-1}(\mathcal{R}(\varphi(w)))$. Finally, from Fact 4, 1 is an ascent for $\mathcal{Q}^2(\varphi^{-1}(\mathcal{R}(\varphi(w))))$.

Now, we have shown that the reversal-rotation map takes a descent for w_{n-1} and converts it to an ascent for $\mathcal{Q}^2(\varphi^{-1}(\mathcal{R}(\varphi(w))))_1$. So the number of descents in position $n-1$ is equivalent to the number of ascents in position 1. For an orbit of size m, if there are k descents in position $n-1$, there are $m-k$ descents in position 1, so that average number per orbit size is $\frac{k+(m-k)}{m} = 1$ and so $D_1 + D_{n-1}$ is 1-mesic for orbits of reversal-rotation.

For the rotation-reversal map, we can use similar reasoning. We begin with an ascent in position $n-1$, so $w_{n-1} < w_n$ and by Lemma 3.3.7 we have $\varphi(w)_1 < \varphi(w)_n$. Fact 3 gives us that $\mathcal{Q}^2(\varphi(w))_1 < \mathcal{Q}^2(\varphi(w))_n$ and Lemma 3.3.9 gives us that $n-1$ is an ascent for $\varphi^{-1}(\mathcal{Q}^2(\varphi(w)))$. Lastly Fact 2 gives us the result that 1 is a descent of $\mathcal{R}(\varphi^{-1}(\mathcal{Q}^2(\varphi(w))))$ is a descent. Thus the average number for the statistic $D_1 + D_{n-1}$ will be 1 and we have a homomesy as stated.

□

By almost identical reasoning, we have the following difference homomesies.

Theorem 3.3.12. *The following maps are homomesic for the statistic $D_1 - D_{n-1}$: reversal-reversal, rotation-rotation, rotation-inversion.*

3.4 Future Directions

Recall that D_i is an indicator function where $D_i = 1$ if i is the index of a descent and 0 otherwise. The following conjectures on linear combinations of descents remain open:

Conjecture 3.4.1. *The statistic $D_k - D_{n-k}$ for $1 < k < n$ is homomesic for the maps reversal-reversal, rotation-rotation, and rotation-inversion.*

Conjecture 3.4.2. *The statistic $D_k + D_{n-k}$ for $1 < k < n$ is homomesic for the maps reversal-rotation and rotation-reversal*

We know that the descents are preserved under $\mathcal{I} \circ \varphi \circ \mathcal{I}$ (see Example 3.2.18), which may be a useful fact in determining some of these more general cases.

In addition there are open problems related to linear combinations of maxima and minima in relation to the rotated inverse map. It is possible that these could be proven using similar reasoning that that given in Section 3.3.1.

Conjecture 3.4.3. *The following statistics are* 0*-mesic:*

Map	**Homomesic Statistic(s)**
Rotated Inverse-Inversion	$\underrightarrow{\max} - \underleftarrow{\min}$
Rotated Inverse-Rotation	$\underrightarrow{\max} - \underleftarrow{\min}$
Rotated Inverse-Reversal	$\underrightarrow{\min} - \underleftarrow{\min}$
Inversion-Rotated Inverse	$\underleftarrow{\max} - \underleftarrow{\min}$
Rotated Inverse-Rotated Inverse	$\underrightarrow{\max} - \underleftarrow{\min}$

Lastly, there appears to be a handful of homomesies involving fixed points, similar to those explored in Section 2.2.1.

Conjecture 3.4.4. *The following maps are* 0*-mesic for the statistic* $\mathrm{Fix}_1 - \mathrm{Fix}_n$*: inversion-rotated inverse, rotated inverse-rotated inverse, inversion-rotation, and inversion-inversion.*

Unlike the work seen in Section 2.4, we did not come across any near-homomesic statistics for these maps. For future research, we also saw that the k-descent set, as seen in Definition 3.2.17, is a tree-dependent statistic. There may be homomesies related to this or other permutation statistics that we have not looked into here. As mentioned in Section 2.5.2 considering peaks, valleys and pinnacles, [DNPT] for example, may be a fruitful path. Additionally, another possible path for homomesy hunting may be in looking at permutation statistics for the alternating subgroup, $\mathfrak{A}_n$ of the symmetric group.

Appendix A

Homomesies for the Rényi–Foata Map

Map	**Homomesic Statistic(s)**	**c-value(s)**
Complement-Rotation	$\mathrm{Fix}_1 - \mathrm{Fix}_n$, Fix*	$0, 1$
Complement-Inversion	Fix*	1
Complement-Rotated Inverse	$\mathrm{Fix}_1 - \mathrm{Fix}_n$*, wexc	$0, \frac{n+1}{2}$
Complement-Complement	$\mathrm{Fix}_1 - \mathrm{Fix}_n$	0
Reversal-Rotation	$\mathrm{Fix}_1 - \mathrm{Fix}_n$, Fix*	$0, 1$
Reversal-Inversion	Fix	1
Reversal-Rotated Inverse	$\mathrm{Fix}_1 - \mathrm{Fix}_n$* , wexc	$0, \frac{n+1}{2}$
Reversal-Reversal	$\mathrm{Fix}_1 - \mathrm{Fix}_n$	0
Rotation-Complement	$\mathrm{Fix}_1 - \mathrm{Fix}_n$, $\underleftarrow{\max} - \underleftarrow{\min}$*	$0, 0$
Rotation-Reversal	$\mathrm{Fix}_1 - \mathrm{Fix}_n$, $D_1 - D_{n-1}$*	$0, 0$
Rotation-Inversion	$\mathrm{wexc} + \mathrm{exc}$, $\underrightarrow{\max} - \underleftarrow{\min}$	$n, 0$
Rotation-Rotation	$\mathrm{wexc} + \mathrm{exc}$, $\underrightarrow{\max} - \underleftarrow{\min}$	$n, 0$
Inversion-Complement	$\underrightarrow{\max} - \underrightarrow{\min}$, $\underleftarrow{\max} - \underleftarrow{\min}$	$0, 0$
Inversion-Reversal	$\underrightarrow{\max} - \underleftarrow{\max}$, $\underrightarrow{\min} - \underleftarrow{\min}$	$0, 0$
Inversion-Rotation	$\mathrm{Fix}_1 - \mathrm{Fix}_n$	0
Inversion-Rotated Inverse	$\mathrm{Fix}_1 - \mathrm{Fix}_n$	0
Inversion-Inversion	$\mathrm{wexc} + \mathrm{exc}$	n
Rotated Inverse-Complement	$\mathrm{Fix}_1 - \mathrm{Fix}_n$*	0
Rotated Inverse-Reversal	$\mathrm{Fix}_1 - \mathrm{Fix}_n$*	0
Rotated Inverse-Inversion	$\mathrm{Fix}_1 - \mathrm{Fix}_n$	0

Appendix B

Homomesies for the Foata–Schützenberger Map

Map	**Homomesic Statistic(s)**	**c-value(s)**
Reversal-Rotation	$D_1 + D_{n-1},\ D_k + D_{n-k}$* for $1 < k < n$	$1, 1$
Reversal-Reversal	$D_1 - D_{n-1},\ D_k - D_{n-k}$* for $1 < k < n$	$0, 0$
Rotation-Reversal	$D_1 + D_{n-1},\ D_k + D_{n-k}$* for $1 < k < n$	$1, 1$
Rotation-Inversion	$D_1 - D_{n-1},\ D_k - D_{n-k}$* for $1 < k < n$	$0, 0$
Rotation-Rotation	$D_1 - D_{n-1},\ D_k - D_{n-k}$* for $1 < k < n$	$0, 0$
Inversion-Reversal	$\underrightarrow{\text{max}} - \underleftarrow{\text{max}}$	0
Inversion-Rotation	$\underrightarrow{\text{min}} - \underleftarrow{\text{max}}$, $\text{Fix}_1 - \text{Fix}_n$*	$0, 0$
Inversion-Rotated Inverse	$\text{Fix}_1 - \text{Fix}_n$*, $\underleftarrow{\text{max}} - \underleftarrow{\text{min}}$*	$0, 0$
Inversion-Inversion	$\underrightarrow{\text{max}} - \underleftarrow{\text{min}}$, $\text{Fix}_1 - \text{Fix}_n$*	$0, 0$
Rotated Inverse-Reversal	$\underrightarrow{\text{min}} - \underleftarrow{\text{min}}$*	0
Rotated Inverse-Rotation	$\underrightarrow{\text{max}} - \underleftarrow{\text{min}}$*	0
Rotated Inverse-Inversion	$\underrightarrow{\text{max}} - \underleftarrow{\text{min}}$*	0
Rotated Inverse-Rotated Inverse	$\text{Fix}_1 - \text{Fix}_n$*, $\underrightarrow{\text{max}} - \underleftarrow{\text{min}}$*	$0, 0$

Bibliography

[A] T. Amdeberhan, *Dynamical Combinatorics à la Foata*, personal communication (2015).

[AST13] D. Armstrong, C. Stump, and H. Thomas, *A Uniform Bijection between Nonnesting and Noncrossing Partitions*, Trans. Amer. Math. Soc. **365** (2013), no. 8, 4121–4151, `arXiv:1101.1277v2`.

[BB05] A. Björner and F. Brenti, *Combinatorics of Coxeter Groups*, Graduate Texts in Mathematics, Springer, (2005).

[BPS13] J. Bloom, O. Pechenik, and D. Saracino, *Proofs and Generalizations of a Homomesy Conjecture of Propp and Roby*, (2013), `arXiv:1308.0546v3`.

[BR] R. Behrend and T. Roby, *Dihedral Homomesy for Alternating Sign Matrices*, in preparation.

[BR05] D. Bernstein and A. Regev, *A Foata Bijection for the Alternating Group and for q-Analogues*, (2005), `arXiv:0503112v1`.

[BW89] A. Björner and M. Wachs, *q-Hook Length Formulas for Forests*, J. Combin. Theory Ser. A **52** (1989), 165–187.

[BW91] A. Björner and M. Wachs, *Permutation Statistics and Linear Extensions of Posets*, J. Combin. Theory Ser. A **58** (1991), 85–114.

[DNPT] R. Davis, S. Nelson, T. Kyle Peterson, B. Tenner, *The Pinnacle Set of a Permutation*, (2020), `arXiv:1704.05494v2`.

[FS78] D. Foata and M.P. Schützenberger, *Major Index and Inversion Number of Permutations*, Math. Nachr. **83** (1978), 143–159.

[Gans80] E. Gansner, *On the Equality of Two Plane Partition Correspondences*, Discrete Math. **30** (1980), 121–132.

[Gar83] M. Gardner, *Mathematical Games: Tasks you cannot help finishing no matter how hard you try to block finishing them*, Scientific American **249** (1983), 12–21.

[Hop12] B. Hopkins, *30 years of Bulgarian solitaire*, College Math. J., **43** (2012), #2, 135–140.

[JR18] M. Joseph and T. Roby, *Toggling Independent Sets of a Path Graph*, Electron. J. Combin., **25(1)** (2018), 1–18.

[LR20] M. La Croix and T. Roby, *Foatic Actions of the Symmetric Group and Fixed-Point Homomesy*, (2020), `arXiv:2008.03292`.

[Mac16] P. A. MacMahon, *Two Applications of General Theorems in Combinatory Analysis*, Proc. London Math. Soc., **15** (1916), 314–321.

[Pan09] D. Panyushev, *On Orbits of Antichains of Positive Roots*, Europ. J. Combin. **30(2)** (2009), 586–594, `arXiv:0711.3353v2`.

[PR15] J. Propp and T. Roby, *Homomesy in Products of Two Chains*, Electronic J. Combin. **22(3)** (2015), #P3.4

[Rho10] B. Rhoades, *Cyclic Sieving, Promotion, and Representation Theory*, J. Combin. Theory Ser. A **117** (2010), 38–76; `arXiv:1005.2568v1`.

[Rob16] T. Roby, *Dynamical Algebraic Combinatorics and the Homomesy Phenomenon*, Recent Trends in Combinatorics, Springer, (2016), 619–652. Also available at
`http://www.math.uconn.edu/~troby/homomesyIMA2015Revised.pdf`.

[RSW04] V. Reiner, D. Stanton, and D. White, *The Cyclic Sieving Phenomenon*, J. Combin. Theory Ser. A **108** (2004), no. 1, 17–50.

[RSW14] V. Reiner, D. Stanton, and D. White, *What is... Cyclic Sieving*, Notices Amer. Math. Soc **61** (2014), no. 2, 169–171.

[STWW15] A. Schilling, N. Thiéry, G. White, and N. Williams, *Braid Moves in Commutation Classes of the Symmetric Group* (2015), `arXiv:1507.00656v1`.

[Stan11] R. Stanley, *Enumerative Combinatorics, volume 1, 2nd edition*, Cambridge University Press, (2011). Also available at
`http://math.mit.edu/~rstan/ec/ec1/`.

[Stan99] R. Stanley, *Enumerative Combinatorics, volume 2*, Cambridge University Press, (1999).

[Stem94] J. Stembridge, *Some Hidden Relations Involving the Ten Symmetry Classes of Plane partitions*, J. Combin. Theory Ser. A **68** (1994), no. 2, 372–409.