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Resolvent Estimates and Discrete Maximal Parabolic Regularity for Galerkin Finite Element Methods

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ABSTRACT

We study space-time fully discrete maximal parabolic regularity for second order advection-diffusion operators. These estimates have many applications, including in the establishment of optimal a priori estimates in non Hilbert space norms. For time discretization, we use discontinuous Galerkin finite element methods that, in the simplest case of piecewise constant approximating functions, are equivalent to a modified backwards Euler time-stepping scheme. For discretization of the spatial variable, we analyze both continuous Galerkin (cG) and discontinuous Galerkin finite element methods (dG). Discontinuous Galerkin methods in space are analyzed because of our particular interest in the case where advection dominates diffusion, where stablized methods are needed.

Resolvent Estimates and Discrete Maximal Parabolic Regularity for Galerkin Finite Element Methods

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Resolvent Estimates and Discrete Maximal Parabolic Regularity for Galerkin Finite Element Methods

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Contents

Chapter 1

Introduction

1.1 Model Problem

We consider the following parabolic PDE. Let Ω be a convex polygonal domain in $\mathbb{R}^n$ for $n = 2$ and let $J = (0, T]$ be some time interval. Let $A = -\epsilon\Delta u + \vec{\beta} \cdot \nabla u$ be an elliptic second order differential operator, where $0 \leq \epsilon \leq 1$ is a constant, and $\vec{\beta}$ is a constant vector. Given $f \in L^s(J; L^p(\Omega))$ and $u_0 \in L^p(\Omega)$, find $u(t, x)$ such that,

$$u_t(t,x) + Au(t,x) = f(t,x), \quad (t,x) \in J \times \Omega \tag{1.1.1}$$

$$u(t,x) = 0, \qquad (t,x) \in J \times \partial\Omega \tag{1.1.2}$$

$$u(0,x) = u_0(x), \qquad x \in \Omega\,. \tag{1.1.3}$$

Maximal parabolic regularity is an important analytical tool that has a number of applications, especially to nonlinear problems and/or optimal control problems where sharp regularity results are required.

Theorem 1.1.1 (Maximal Parabolic Regularity). *Let* $f \in L^s(J; L^p(\Omega))$ *and* $u_0 = 0$. *Then* $u_t \in L^s(J; L^p(\Omega))$ *and* $Au \in L^s(J; L^p(\Omega))$. *Moreover, the following a priori estimate holds:* $\exists C > 0$ *such that*

$$\|u_t\|_{L^s(J;L^p(\Omega))} + \|Au\|_{L^s(J;L^p(\Omega))} \leq C\|f\|_{L^s(J;L^p(\Omega))}$$

for all $1 < p, s < \infty$.

Maximal parabolic regularity was first investigated by Lions in 1961 [9] for the case $p = 2$. Since then, much work has been done. The case when the operator A is independent of time, i.e. autonomous, is well understood. Maximal parabolic regularity for the non-autonomous case, $A = A(t)$, is far less understood, and many open questions remain.

In comparison to the continuous case mentioned above, much less is known about discrete versions of maximal parabolic regularity. Recently, there has been growing interest in establishing these results for various discrete methods. There are many applications of discrete maximal parabolic regularity. For example, discrete maximal parabolic regularity has been used to prove pointwise best approximation estimates for fully discrete Galerkin solutions.

The goal of this dissertation is to extend maximal parabolic regularity to advection-diffusion equations. In addition, we are interested in tracing the dependence of our constants on ϵ and $|\vec{\beta}|$. In [8], Leykekhman and Vexler established maximal parabolic regularity for a family of discontinuous in time and continuous in space Galerkin finite element approximations to the heat equation. In this dissertation, we extend these results for operators containing an advection term, $Au = -\epsilon\Delta u + \vec{\beta} \cdot \nabla u$. There is a number of challenges associated with advection-diffusion operators. Most importantly when A is advection-dominated, there are issues with numerical stability. Because of this, we also analyze discontinuous Galerkin methods and show maximal parabolic regularity for time discontinuous and spatially discontinuous Galerkin approximations.

1.2 Variational Formulations

Before we obtain the time semi-discrete and space-time fully discrete approximations to the equation (1.1.1), we lay out the framework of some much needed PDE theory. First, is the idea of a *variational formulation* of a PDE. A variational formulation is an alternate way to view a PDE in an integral sense. For example, let $f \in L^2(\Omega)$, and suppose we would like to solve the following Dirichlet boundary value problem,

$$-\Delta u = f,\ x \in \Omega \tag{1.2.1}$$

$$u = 0,\ x \in \partial\Omega. \tag{1.2.2}$$

Classically, this problem may not be able to be solved analytically. Instead, we seek to find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \; dx = \int_\Omega fv \; dx \;, \forall v \in H_0^1(\Omega). \tag{1.2.3}$$

This weak form, as is usually the case, was obtained by multiplying the original PDE by some "test" function $v \in H_0^1(\Omega)$, integrating both sides over Ω, and integrating by parts on the left hand side. We note the right hand side is a linear functional on

$H_0^1(\Omega)$ and the left hand side is a *bilinear form*.

Using more general notation for (1.2.3), we search to find $u \in H_0^1(\Omega)$ such that

$$B(u,v) = L(v), \quad \forall v \in H_0^1(\Omega), \tag{1.2.4}$$

where $B(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx$ and $L(v) = \int_\Omega fv \, dx$.

When the PDE is viewed through the variational formulation, existence and uniqueness of "weak" solutions can be easily guaranteed through the Lax-Milgram theorem. Most importantly, the variational formulation gives rise to finite element methods, which are used to effectively approximate solutions to PDEs through the variational formulation.

1.2.1 Lax-Milgram Theorem

Theorem 1.2.1. *Let V be a Hilbert space and let $B : V \times V \to \mathbb{R}$ be a bilinear form. Let $L(v)$ be a continuous linear functional on V. Assume there exists $C_1, C_2 > 0$ such that the following conditions hold:*

(i) $B(u,u) \geq C_1 \|u\|_V^2, \forall u \in V$,

(ii) $|B(u,v)| \leq C_2 \|u\|_V \|v\|_V \ \forall u,v \in V$.

Then there exists a unique $u \in V$ such that $B(u,v) = L(v)$ for all $v \in V$.

Proof. See [4] section 6.2.1. ∎

1.2.2 The Finite Element Method

The idea of the *finite element method* is to project the variational problem onto a finite dimensional "approximation space". For conformal finite element methods, we seek to find $u_h \in V_h \subset V$ such that $B(u_h, v_h) = L(v_h)$, $\forall v_h \in V_h$. Since V_h is a finite dimensional space, all functions in V_h can be written as finite linear combinations of basis functions in V_h. Thus, the variational problem projected onto the finite element space V_h boils down to solving a matrix equation for the coefficients of the basis functions.

By the definition of the finite element problem, we immediately get that the error between the true variational solution u and the finite element approximation u_h has a property called *Galerkin orthogonality*. That is,

$$B(u - u_h, v_h) = 0, \forall v_h \in V_h.$$

This basically means that the error between the finite element approximation and true solution is orthogonal to the subspace V_h. In other words, u_h is the closest approximation to u in V_h.

Because the model problem depends on both time and space, we will need to use a finite element method on the temporal and spatial variables in order to obtain a fully discrete approximation. In Chapter 2, we introduce the temporal discretization technique that is used on our model problem throughout the rest of this dissertation. In Chapters 3 and 4, we discuss two separate finite element methods on the spatial variable and resolvent estimates of the resulting discrete operators.

In Chapter 3, we consider the classical conformal finite element methods, where $V_h \subset H_0^1(\Omega)$. That is, the approximation space consists of continuous functions on Ω that are polynomials when restricted to a single element. In Chapter 4, we analyze non-conformal finite element methods. That is, our approximating functions are allowed to have discontinuities across element boundaries. Thus, the approximation space is not a subset of $H_0^1(\Omega)$. These non-conformal methods are considered due to their nice stability properties in the advection-dominated case.

Chapter 2

Temporal Discretization

In this section, we introduce the discontinuous Galerkin (dG) discretization method in time. Discontinuous Galerkin methods are attractive for several reasons. For example, dG methods have good stability properties, and since the temporal trial space allows for discontinuity at the interval endpoints, different spatial discretizations can be used for each time step.

We begin the time discretization by taking a finite number of points (nodes) in J, $0 = t_0 < t_1 < ... < t_M = T$, and splitting up the time interval J into subinvervals $J_m = (t_{m-1}, t_m]$, $m = 1, 2, ..., M$, where $k_m = t_m - t_{m-1}$ is the length of the m^{th} time step and k is the maximum time step. We use polynomials of degree q in time on each subinterval J_m, and allow for discontinuity across the nodes. That is, the semi-discrete in time finite element approximation space is defined as

$$X_k^q = \{u : (0, T] \mapsto H_0^1(\Omega) \mid u_{|_{J_m}} \in \mathcal{P}^q(J; H_0^1(\Omega))\},$$

where $\mathcal{P}^q(H_0^1(\Omega))$, consists of polynomials of degree q in time with coefficients in $H_0^1(\Omega)$. Due to the discontinuity at the nodes, we denote the limits from the left and the right of the nodes, as well as the jumps at the nodes as follows:

$$u_{k,m}^- = \lim_{t\to t_m^-} u_k(t)\ ,\ u_{k,m}^+ = \lim_{t\to t_m^+} u_k(t), \text{ and } [u_k]_m = u_{k,m}^+ - u_{k,m}^-.$$

2.1 Semi-Discrete in Time Formulation

The semi-discrete problem in time is obtained by multiplying the continuous problem by a test function $v_k \in X_k^q$, integrating both sides in time over the interval J, then integrating by parts over each subinterval J_m. When integrating by parts over the subintervals, we evaluate the test function from the right hand side of the node t_m, $v_{k,m}^+$. That is, we seek to find $u_k \in X_k$ such that $\forall v_k \in X_k^q$,

$$\sum_{m=1}^{M}\int_{J_m} u_k' v_k\, dt + \int_J A u_k v_k\, dt + \sum_{m=1}^{M-1}[u_k]_m v_{k,m}^+ + u_{k,0}^+ v_{k,0}^+ = u_0 v_{k,0}^+ + \int_J f v_k\, dt. \quad (2.1.1)$$

We note that the above equation is in the variational sense in space as well. That is, for example, by $\int_J A u_k v_k\, dt$, we mean $\int_J (A u_k, v_k)_\Omega\, dt$. We will use (2.1.1) going forward to ease the burden of notation.

2.2 Piecewise Constant in Time - dG(0)

In this subsection, we focus on the case where approximating functions in time are piecewise constant. Because of the structure of the dG(0) space X_k^0, basis functions can be chosen very simply. For each interval J_m, we choose a local basis function $\phi_m(t)$.

$$\phi_m(t) = \begin{cases} 1 & t \in J_m \\ 0 & t \notin J_m. \end{cases}$$

Thus, any function u_k in X_k^0 can be written as a linear combination of the basis functions with coefficients $c_m(x) \in H_0^1(\Omega)$,

$$u_k = \sum_{m=1}^{M} c_m(x)\phi_m(t). \tag{2.2.1}$$

We also note that $u_{k|_{J_m}} = c_m(x)$. Furthermore, because u_k is constant in time on each subinterval J_m, its value across J_m can be determined from its value at the time t_m. Thus, we have the following for the dG(0) case:

$$u_{k,m} = u_{k|J_m},\ u^+_{k,m-1} = u_{k,m}, \text{ and } u^-_{k,m} = u_{k,m}.$$

To solve the dG(0) problem, we need to find u_k where (2.1.1) holds for each basis function ϕ_m. Because each basis function is only supported on one interval, the problem boils down to an iterative method.

We begin by testing (2.1.1) with $v_k = \phi_1$. Noting $u_k' = 0$, supp $\phi_1 = J_1$, and that Au_k is constant in time on J_1,

$$u_{k,1} + k_1 A u_{k,1} = u_0 + \int_{J_1} f dt. \tag{2.2.2}$$

To obtain the time semi-discrete approximation on J_1, i.e. the approximate solution at time t_1, we solve the above elliptic problem for $u_{k,1}$.

For each time step, $m = 1, 2, .., M$, we solve a similar elliptic problem,

$$u_{k,m} + k_m A u_{k,m} = u_{m-1} + \int_{J_m} f dt. \tag{2.2.3}$$

We see that in the homogeneous case, $f \equiv 0$, the dG(0) method is equivalent to the implicit Euler time stepping method. That is, in equation 1.1.1, u_t is replaced by the discrete time derivative $\frac{u_{k,m} - u_{k,m-1}}{k_m}$, and Au is evaluated at time t_m, $Au_{k,m}$. For the non homogeneous case, it results in a modified implicit Euler method, where the average value of f on J_m is used in lieu of $f(t_m)$.

2.2.1 Homogeneous Time Semi-Discrete dG(0), Approximation

In this section, we consider the case where $f = 0$ and $u_0 \in L^p(\Omega)$. Solving the elliptic problem for $u_{k,1}$ in (2.2.2), we have

$$u_{k,1} = (I + k_1 A)^{-1} u_0. \tag{2.2.4}$$

We update our solution on each subsequent time step by solving the elliptic problem (2.2.3) for $u_{k,m}$. That is,

$$u_{k,m} = (I + k_m A)^{-1} u_{k,m-1}. \tag{2.2.5}$$

Thus,

$$u_{k,m} = \prod_{j=1}^{m} (I + k_j A)^{-1} u_0. \tag{2.2.6}$$

We define a complex valued rational function $r_m : \mathbb{C} \to \mathbb{C}$,

$$r(z) = (1+z)^{-1}. \tag{2.2.7}$$

Abusing notation, we write

$$u_{k,m} = \prod_{j=1}^{m} r(k_j A) u_0. \tag{2.2.8}$$

2.2.2 Non-Homogeneous Time Semi-Discrete dG(0), Approximation

In this section, we consider the case where $f \in L^s(J; L^p(\Omega))$ and $u_0 = 0$. Similar to the homogeneous case, to obtain the time semi-discrete non-homogeneous dG(0) approximation, one must solve an elliptic problem at each time step. Solving (2.2.2), the approximate solution on the first interval J_1 is given by

$$u_{k,1} = (I + k_1 A)^{-1} f_1, \tag{2.2.9}$$

where $f_m = \int_{J_m} f dt$. On the m^{th} time step, solving (2.2.3) yields,

$$u_{k,m} = (I + k_m A)^{-1} f_m + (I + k_m A)^{-1} u_{k,m-1}. \tag{2.2.10}$$

Thus, we can write the non-homogeneous time semi-discrete dG(0) approximation on J_m as

$$u_{k,m} = \sum_{i=1}^{m} k_i \left(\prod_{j=1}^{m-i+1} (I + k_{m-j+1} A)^{-1} \right) f_i. \tag{2.2.11}$$

Using the notation in (2.2.8), we can write the non-homogeneous dG(0) solution in the following way,

$$u_{k,m} = \sum_{i=1}^{m} k_i \prod_{j=1}^{m-i+1} r(k_{m-j+1}A) f_i. \tag{2.2.12}$$

We note that each term in the sum of the right hand side of equation (2.2.12) can be viewed as a homogeneous dG(0) solution with initial condition f_i at $t = t_{i-1}$.

2.3 Piecewise Polynomials of Degree q- dG(q)

In this section, we focus on generalizing the time semi-discrete dG approximation using polynomials of degree q, dG(q). Similar to the dG(0) case, basis functions for the time semi-discrete dG(q) approximation space can be chosen locally on each interval. However, since approximating functions are polynomials of degree q in time on each subinterval, there are $q+1$ basis functions for each subinterval. Thus, we define local basis functions on each J_m using standard Lagrange basis polynomials of degree q in time. If $q = 1$, local linear Lagrange basis functions would be placed at both the endpoints of each interval, t_{m-1} and t_m. If $q > 1$, we will need more nodes on each interval for the localized basis functions. We can add nodes to each interval by inserting a node at $t_j^m = t_{m-1} + \frac{jk_m}{q}$, for $j = 1, ..., q-1$. Thus, we define standard local Lagrange basis polynomials of degree q on each interval as Φ_j^m, $j = 0, 1, ...q$, where Φ_j^m is the Lagrange polynomial of degree q on J_m at the node t_j^m. We note for each $j = 0, .., q$, supp $\Phi_j^m(t) = J_m$. Using the notation for the localized basis functions, we can represent a function in X_k^q restricted to one interval as a linear combination of the local basis functions,

$$u_{k|_{J_m}} = \sum_{j=0}^{q} U_j^m(x)\Phi_j^m(t), \tag{2.3.1}$$

where $U_j^m(x) \in H_0^1(\Omega)$ and is independent of time. Using this notation, we have the left hand and right hand limits at the nodes t_m,

$$u_{k,m}^+ = U_0^{m+1} \text{ and } u_{k,m}^- = U_q^m.$$

Using the ideas from [3], on an interval J_m, we have for $j = 0, 1, .., q$,

$$U_j^1 = \widehat{r}_j(k_1 A)u_0, \ m = 1, \tag{2.3.2}$$

$$U_j^m = \widehat{r}_j(k_m A)U_q^{m-1}, \ m = 2, 3, ..., M, \tag{2.3.3}$$

where $\widehat{r}_j : \mathbb{C} \to \mathbb{C}$ are complex valued rational polynomials of the form

$$\widehat{r}_j(z) = \frac{p_j(z)}{p(z)},$$

where $p_j(z)$ is a polynomial of order q, and $p(z)$ is a polynomial of order $q+1$ with no roots on the right hand side of the complex plane. Thus, $\widehat{r}_j(z)$ is analytic on the right hand half of the complex plane. We shall make use of this fact later.

Chapter 3

Continuous Galerkin Resolvent Estimates

In this section, we show an L^∞ resolvent estimate for the continuous Galerkin operator. To do this, we build on an L^2 resolvent estimate from [6], and use a weighted norm technique. We will use the usual space of conforming finite elements. That is, we let $V_h \subset H_0^1(\Omega)$, where V_h consists of continuous functions across Ω that are polynomials of degree r when restricted to each element τ. We consider the following problem. Let $z \in \mathbf{C}$. Find $u_h \in \mathbf{V_h} = V_h + iV_h$ such that for all $v_h \in \mathbf{V_h}$,

$$z(u_h, v_h) - A_h^{cG}(u_h, v_h) = (f, v_h), \tag{3.0.1}$$

where

$$A_h^{cG}(u_h, v_h) = \epsilon(\nabla u_h, \nabla v_h) - (u_h, \vec{\beta} \cdot \nabla v_h).$$

We note that since we are dealing with complex valued functions, our inner product is defined as

$$(u, v) = \int_\Omega u\bar{v}\, dx.$$

As in [11], we define a weight function,

$$\sigma(x) = \sqrt{|x - x_0|^2 + h^2}. \tag{3.0.2}$$

We will use the fact that this weight function compensates for the essential singularity of the *discrete delta function*, δ_h, where δ_h is defined as $(v_h, \delta_h) = v_h(x_0)$ for all $v_h \in \mathbf{V_h}$. This is shown in Lemma 6.3 in [11].

Lemma 3.0.1. *For $x_0 \in \Omega$, there exists $C > 0$ such that $\|\sigma\delta_h\| \leq C$.*

In addition to the lemma above, we will make use of the following properties of σ, where $l_h = |\log(h)|$:

$$|\nabla\sigma| \leq C, \tag{3.0.3}$$

$$\left\|\sigma^{-1}\right\| \leq C l_h^{\frac{1}{2}}, \tag{3.0.4}$$

$$\max \sigma_{|\tau} \leq C \min \sigma_{|\tau}, \forall\tau. \tag{3.0.5}$$

We will also need the following superapproximation results for the weighted resolvent estimate, as shown in [7].We let P_h be the L^2 projection operator, $P_h : L^2(\Omega) \to \mathbf{V_h}$ defined by $(u, v_h) = (P_h u, v_h)\ \forall v_h \in \mathbf{V_h}$.

Lemma 3.0.2. *Let $v_h \in \mathbf{V_h}$. Then there exists $C > 0$ such that*

$$\left\|\sigma^{-1}(\sigma^2 v_h - P_h(\sigma^2 v_h))\right\| + h\left\|\sigma^{-1}(\nabla(\sigma^2 v_h - P_h(\sigma^2 v_h)))\right\| \leq Ch\left\|v_h\right\|. \tag{3.0.6}$$

We define a set in the complex plane $\Sigma_\theta = \{z \in \mathbb{C} : |\arg z| \leq \theta\}$. The following L^2 discrete resolvent estimate was shown in [6], Lemma 3.1.

Theorem 3.0.3 (L^2 Resolvent Estimate)**.** *Let u_h solve* (3.0.1)*. Then there exists $C \approx \frac{|\vec{\beta}|}{\epsilon}$ and $\theta \in (\theta_0, \frac{\pi}{2})$, where $\theta_0 \approx \arctan(|\vec{\beta}|/\epsilon)$, such that*

$$\|u_h\| \leq C\frac{\|f\|}{|z|}$$

for all $z \in \mathbb{C} - \Sigma_\theta$.

We will prove a modification of this result that will lead us to the weighted resolvent estimate. In doing so, we use a discrete Sobolev inequality, as shown in Lemma 6.4 [11].

Theorem 3.0.4. *There exists $C > 0$ such that*

$$\|v_h\|_{L^\infty(\Omega)} \leq C l_h^{\frac{1}{2}} \|\nabla v_h\| \tag{3.0.7}$$

for all $v_h \in \mathbf{V_h}$.

We now prove a weighted modification of the L^2 resolvent estimate.

Lemma 3.0.5. *Let u_h solve* (3.0.1)*. Then there exist $C > 0$, $k \approx \frac{|\vec{\beta}|}{\epsilon}$, and $\theta \in (\theta_0, \frac{\pi}{2})$, where $\theta_0 \approx \arctan(|\vec{\beta}|/\epsilon)$, such that for $M = k + 1$,*

$$\|u_h\|^2 \leq C\frac{M^2 l_h^2}{|\vec{\beta}|}\frac{\|\sigma f\|^2}{|z|},$$

$$\|\nabla u_h\| \leq C\frac{M l_h}{|\vec{\beta}|}\|\sigma f\|,$$

for all $z \in \mathbb{C} - \Sigma_\theta$*, where* $\theta = \arctan(2M)$*.*

Proof. We test (3.0.1) with $v_h = -u_h$, resulting in

$$-z\|u_h\|^2 + \epsilon\|\nabla u_h\|^2 + (u_h, \vec{\beta}\cdot\nabla u_h) = -(f, u_h). \tag{3.0.8}$$

Noticing that $\mathrm{Re}(u_h, \vec{\beta}\cdot\nabla u_h) = 0$, and taking the real parts of the equation, we have

$$\mathrm{Re}(-z)\|u_h\|^2 + \epsilon\|\nabla u_h\|^2 \leq |(f_h, u_h)|. \tag{3.0.9}$$

Using the discrete Sobolev inequality, and property (3.0.4), we obtain

$$\begin{aligned}\mathrm{Re}(-z)\|u_h\|^2 + \epsilon\|\nabla u_h\|^2 &\leq |(f_h, u_h)| \\ &\leq \|f_h\|_{L^1}\|u_h\|_{L^\infty} \\ &\leq \|\sigma f_h\|\left\|\sigma^{-1}\right\|\|u_h\|_{L^\infty(\Omega)} \\ &\leq Cl_h\|\sigma f_h\|\|\nabla u_h\|.\end{aligned}$$

Now, we consider the imaginary parts of the equation. Using the Poincaré inequality on the advection term and the same analysis as in the real part of the equation, we have

$$|\operatorname{Im}(-z)| \|u_h\|^2 \leq C_p|\vec{\beta}| \|\nabla u_h\|^2 + Cl_h \|\sigma f_h\| \|\nabla u_h\| . \tag{3.0.10}$$

Now, we multiplying (4.0.9) by $k = \frac{2C_p|\vec{\beta}|}{\epsilon}$, and combining the resulting inequality with (4.0.10), we have

$$\left(k \operatorname{Re}(-z) + |\operatorname{Im}(-z)|\right) \|u_h\|^2 + C_p|\vec{\beta}| \|\nabla u_h\|^2 \leq C(k+1)l_h \|\sigma f_h\| \|\nabla u_h\| . \tag{3.0.11}$$

Thus,

$$\|\nabla u_h\| \leq \frac{C}{|\vec{\beta}|}(k+1)l_h \|\sigma f_h\| .$$

Using the Young's inequality in (4.0.11) and kicking back $C_p|\vec{\beta}|$, we also have

$$\left(k \operatorname{Re}(-z) + |\operatorname{Im}(-z)|\right) \|u_h\|^2 \leq \frac{C}{|\vec{\beta}|}(k+1)^2 l_h^2 \|\sigma f_h\|^2 . \tag{3.0.12}$$

From analysis in [6], we have our result,

$$\|u_h\|^2 \leq C \frac{M^2 l_h^2}{|\vec{\beta}|} \frac{\|\sigma f\|^2}{|z|}.$$

■

Theorem 3.0.6. *Let u_h solve* (3.0.1)*. Then there exists a $C > 0$ and $\theta \in (\theta_0, \frac{\pi}{2})$, where $\theta_0 \approx \arctan(|\vec{\beta}|/\epsilon)$, such that*

$$\|\sigma u_h\| \leq C l_h \frac{|\vec{\beta}|}{\epsilon} \frac{\|\sigma f_h\|}{|z|} \tag{3.0.13}$$

for all $z \in \mathbb{C} - \Sigma_\theta$.

Proof. We test (3.0.1) with $v_h = -P_h(\sigma^2 u_h)$, then add and subtract $A_h^{cG}(u_h, \sigma^2 u_h)$, resulting in

$$-z\left\|\sigma u_h\right\|^2 + A_h^{cG}(u_h, \sigma^2 u_h) = -(f_h, \sigma^2 u_h) + A_h^{cG}(u_h, \sigma^2 u_h - P_h(\sigma^2 u_h)). \quad (3.0.14)$$

Thus,

$$-z\left\|\sigma u_h\right\|^2 + \epsilon\left\|\sigma\nabla u_h\right\|^2 = F, \quad (3.0.15)$$

where $F = A_h^{cG}(u_h, \sigma^2 u_h - P_h(\sigma^2 u_h)) - (\sigma f_h, \sigma u_h) - 2\epsilon(\nabla u_h, u_h\sigma\nabla\sigma) + (u_h, \vec{\beta}\cdot\nabla(\sigma^2 u_h))$. Hence, the above equation is of the form

$$e^{i\alpha}a + b = F, \ a, b > 0,$$

where $0 \leq |\alpha| \leq \pi - \theta$. Multiplying by $e^{-i\frac{\alpha}{2}}$, and taking the real parts of the equation, we have

$$a + b \leq |F|\big(\cos(\alpha/2)\big)^{-1} \leq |F|\big(\sin(\theta/2)\big)^{-1} \leq C|F|,$$

for all $z \in \mathbb{C} - \Sigma_\theta$.

That is,

$$|z|\left\|\sigma u_h\right\|^2 + \epsilon\left\|\sigma\nabla u_h\right\|^2 \leq C|F|. \quad (3.0.16)$$

Now, we bound each term in F. Using the Cauchy-Schwarz inequality, property (3.0.3), and the Young's inequality, we have

$$\begin{aligned}|2C\epsilon(\nabla u_h, u_h\sigma\nabla\sigma)| \leq & \ C\epsilon \left\|\sigma\nabla u_h\right\| \left\|u_h\right\| \\ \leq & \ \frac{\epsilon}{4} \left\|\sigma\nabla u_h\right\|^2 + C\epsilon \left\|u_h\right\|^2 .\end{aligned}$$

Using the Cauchy-Schwarz inequality and the Young's inequality, we have

$$\begin{aligned}|C(u_h, \vec{\beta}\cdot\nabla(\sigma^2 u_h))| &= |-C(\vec{\beta}\cdot\nabla u_h, \sigma^2 u_h)| \\ &\leq C|\vec{\beta}| \left\|\nabla u_h\right\| \left\|\sigma u_h\right\| \\ &\leq \frac{|z|}{4} \left\|\sigma u_h\right\|^2 + C\frac{|\vec{\beta}|^2}{|z|} \left\|\nabla u_h\right\|^2 .\end{aligned}$$

Using the Cauchy-Schwarz inequality, superapproximation, $\sigma \leq C$, and the inverse inequality, we have

$$\begin{aligned}|C(u_h, \vec{\beta}\cdot\nabla(\sigma^2 u_h - P_h(\sigma^2 u_h)))| &= |-C(\vec{\beta}\cdot\nabla u_h, \sigma^2 u_h - P_h(\sigma^2 u_h))| \\ &\leq C|\vec{\beta}| \left\|\sigma\nabla u_h\right\| \left\|\sigma^{-1}(\sigma^2 u_h - P_h(\sigma^2 u_h))\right\| \\ &\leq C|\vec{\beta}|h \left\|\nabla u_h\right\| \left\|u_h\right\| \\ &\leq C|\vec{\beta}| \left\|u_h\right\|^2 .\end{aligned}$$

Using the Cauchy-Schwarz inequality, superapproximation, and the Young's inequality, we have

$$
\begin{aligned}
|C\epsilon(\nabla u_h, \nabla(\sigma^2 u_h - P_h(\sigma^2 u_h)))| &= |C\epsilon(\sigma \nabla u_h, \sigma^{-1}\nabla(\sigma^2 u_h - P_h(\sigma^2 u_h)))| \\
&\leq C\epsilon \|\sigma \nabla u_h\| \|u_h\| \\
&\leq \frac{\epsilon}{4} \|\sigma \nabla u_h\|^2 + C\epsilon \|u_h\|^2 .
\end{aligned}
$$

Finally, using the Cauchy-Schwarz inequality and the Young's inequality,

$$
|C(f_h, \sigma^2 u_h)| \leq \frac{|z|}{4} \|\sigma u_h\|^2 + \frac{C}{|z|} \|\sigma f_h\|^2 .
$$

Kicking back, we arrive at

$$
\frac{|z|}{2} \|\sigma u_h\|^2 \leq C|\vec{\beta}| \|u_h\|^2 + C\frac{|\vec{\beta}|^2}{|z|} \|\nabla u_h\|^2 + \frac{C}{|z|} \|\sigma f_h\|^2 . \tag{3.0.17}
$$

Applying the results from Lemma 3.0.5,

$$
\frac{|z|}{2} \|\sigma u_h\|^2 \leq CM^2 l_h^2 \frac{\|\sigma f_h\|^2}{|z|} + CM^2 l_h^2 \frac{\|\sigma f_h\|^2}{|z|} + \frac{C}{|z|} \|\sigma f_h\|^2 . \tag{3.0.18}
$$

Dividing both sides by $|z|$ and taking square roots, we arrive at our result

$$
\|\sigma u_h\| \leq CMl_h \frac{\|\sigma f_h\|}{|z|} = Cl_h \frac{|\vec{\beta}|}{\epsilon} \frac{\|\sigma f_h\|}{|z|}. \tag{3.0.19}
$$

■

We note that the weighted resolvent estimate holds for the adjoint operator A_h^{cG*} by the exact same analysis as above. We can now prove the main result of this chapter- the L^∞ continuous Galerkin resolvent estimate.

Theorem 3.0.7. *There exist $C > 0$ and $\theta \in (\theta_0, \frac{\pi}{2})$, where $\theta_0 \approx \arctan(|\vec{\beta}|/\epsilon)$, such that for any $v_h \in \mathbf{V_h}$*

$$\left\|(z - A_h^{cG})^{-1} v_h\right\|_{L^\infty} \leq C l_h^{\frac{3}{2}} \frac{|\vec{\beta}|}{\epsilon |z|} \left\|v_h\right\|_{L^\infty},$$

for all $z \in \mathbb{C} - \Sigma_\theta$.

Proof. Let $x_0 \in \Omega$, and let A_h^{cG*} be the adjoint operator of A_h^{cG}. Let $z \in \mathbb{C} - \Sigma_\theta$ from 3.0.13, and let G_h be defined by $(z - A_h^{cG*})G_h = \delta_h$, or

$$(G_h, v_h) = \big((z - A_h^{cG*})^{-1}\delta_h, v_h\big), \ \forall v_h \in \mathbf{V_h}.$$

Thus,

$$\begin{aligned} |(G_h, v_h)| &= |\big((z - A_h^{cG*})^{-1}\delta_h, v_h\big)| \\ &= |\big(\delta_h, (\bar{z} - A_h^{cG})^{-1} v_h\big)| \\ &= |(\bar{z} - A_h^{cG})^{-1} v_h(x_0)|. \end{aligned}$$

Also, using (3.0.4), we have

$$\begin{aligned} |(G_h, v_h)| &\leq \left\|G_h\right\|_{L^1(\Omega)} \left\|v_h\right\|_{L^\infty(\Omega)} \\ &\leq \left\|\sigma G_h\right\| \left\|\sigma^{-1}\right\| \left\|v_h\right\|_{L^\infty(\Omega)} \\ &\leq C l_h^{\frac{1}{2}} \left\|\sigma G_h\right\| \left\|v_h\right\|_{L^\infty(\Omega)}. \end{aligned}$$

Using the estimate shown in 3.0.13, along with 3.0.1, and noting that $\forall z \in \mathbb{C} - \Sigma_\theta, \bar{z} \in \mathbb{C} - \Sigma_\theta$,

$$\|\sigma G_h\| \leq Cl_h \frac{|\vec{\beta}|}{\epsilon|z|} \|\sigma \delta_h\| \leq Cl_h \frac{|\vec{\beta}|}{\epsilon|z|}.$$

Thus, we have that

$$|(\bar{z} - A_h^{cG})^{-1} v_h(x_0)| \leq Cl_h^{\frac{3}{2}} \frac{|\vec{\beta}|}{\epsilon|z|} \|v_h\|_{L^\infty(\Omega)} .$$

Since the above estimate holds for all $x_0 \in \Omega$, we have shown that for all $z \in \mathbb{C} - \Sigma_\theta$,

$$\left\|(z - A_h^{cG})^{-1} v_h\right\|_{L^\infty} \leq Cl_h^{\frac{3}{2}} \frac{|\vec{\beta}|}{\epsilon|z|} \|v_h\|_{L^\infty} .$$

■

Using a duality argument, we obtain the L^1 resolvent estimate.

Corollary 3.0.8. *There exist $C > 0$ and $\theta \in (\theta_0, \frac{\pi}{2})$, where $\theta_0 \approx \arctan(|\vec{\beta}|/\epsilon)$, such that for any $v_h \in \mathbf{V_h}$*

$$\left\|(z - A_h^{cG})^{-1} v_h\right\|_{L^1} \leq Cl_h^{\frac{3}{2}} \frac{|\vec{\beta}|}{\epsilon|z|} \|v_h\|_{L^1} ,$$

for all $z \in \mathbb{C} - \Sigma_\theta$.

Proof.

$$\begin{aligned}
\left\|(z-A_h^{cG})^{-1}v_h\right\|_{L^1} &= \sup_{\chi\in\mathbf{V_h},\|\chi\|_{L^\infty}=1} |((z-A_h^{cG})^{-1}v_h,\chi)| \\
&= \sup_{\chi\in\mathbf{V_h},\|\chi\|_{L^\infty}=1} |(v_h,(\bar{z}-A_h^{cG*})^{-1}\chi)| \\
&\leq \sup_{\chi\in\mathbf{V_h},\|\chi\|_{L^\infty}=1} \|v_h\|_{L^1}\left\|(\bar{z}-A_h^{cG*})^{-1}\chi\right\|_{L^\infty} \\
&\leq Cl_h^{\frac{3}{2}}\frac{|\vec{\beta}|}{\epsilon|z|}\|v_h\|_{L^1}\,.
\end{aligned}$$

■

Chapter 4

Discontinuous Galerkin Resolvent Estimates

In this chapter, we consider a discontinuous Galerkin scheme in space. Discontinuous Galerkin (dG) methods are attractive because of their good stability properties. The choice of a method with exceptional stability properties is extremely important in the advection dominated case. That is, when the advection field $\vec{\beta}$ dominates the diffusion constant (ϵ), $|\epsilon| << |\vec{\beta}|$. In the advection dominated case, the solution develops layers, which are small regions in space where the gradient becomes very large. For the discretization of the diffusion term, i.e. the second order term, we use a symmetric interior penalty Galerkin method (SIPG). For discretization of the advection term, i.e. the first order term, we use an upwind discontinuous Galerkin method. As was the case with Chapter 3, the ultimate goal of this chapter is to show an L^∞ resolvent estimate for the dG disrete advection-diffusion operator.

4.1 Discrete Formulation

We begin the spatial discretization by triangulating $\Omega = \cup\tau_i$ We consider the same operator as before, $Au = -\epsilon\Delta u + \vec{\beta}\cdot\nabla u$. Our approximation space, S_h consists of polynomials of degree r when restricted to each element τ. That is $S_h = \{u_h : u_{h|_{\tau_i}} \in P^r(\tau_i)\}$. Since functions in S_h are allowed to be discontinuous across element boundaries, we define the average and jump of a scalar valued function u_h and vector valued function ∇u_h in S_h on a common edge e as follows:

$$\{u_h\}_e = \frac{1}{2}(u_{h|_{\tau_1}} + u_{h|_{\tau_2}}), \quad [u_h]_e = u_{h|_{\tau_1}} - u_{h|_{\tau_2}},$$

$$\{\nabla u_h\}_e = \frac{1}{2}(\nabla u_{h|_{\tau_1}} + \nabla u_{h|_{\tau_2}}), \quad [\nabla u_h]_e = (\nabla u_{h|_{\tau_1}} - \nabla u_{h|_{\tau_2}})\cdot n_e,$$

where n_e is the outward pointing normal vector to τ_1 on a given edge e such that $\vec{\beta}\cdot n_e > 0$. We define the upwind direction of u_h as $u^+_{h_e} = u_{h|_{\tau_1}}$. Using this notation, we define the discrete variational formulation, A_h, of A to be,

$$A_h(u,v) = a_h(u,v) + b_h(u,v),$$

where a_h and b_h are defined as follows:

$$a_h(u,v) = \epsilon\sum_\tau \big(\nabla u, \nabla v\big)_\tau - \epsilon\sum_e \big(\{\nabla u\},[v]\big)_e - \epsilon\sum_e \big([u],\{\nabla v\}\big)_e + \epsilon\lambda h^{-1}\sum_e \big([u],[v]\big)_e,$$

$$b_h(u,v) = -\sum_\tau \big(u, \vec{\beta}\cdot\nabla v\big)_\tau + \sum_e \big(u^+,[v]|\vec{\beta}\cdot n_e|\big)_e,$$

where λ is a penalty parameter chosen to enforce stability of the method.

Once again, we will be considering solving the problem $(z-A_h)u_h = f_h$, where $z \in \mathbb{C}$. Thus, we once again consider the complex L^2 inner product and finite element space $\mathbf{S_h} = S_h + iS_h$.

4.2 dG Inequalities

Before we begin the proofs of the resolvent estimates, we state analogues of inequalities used in Chapter 3 applied to functions in $\mathbf{S_h}$. For *dG* functions, we define the L^2 projection onto $\mathbf{S_h}$ locally. That is, $P_h : L^2(\Omega) \to \mathbf{S_h}$ defined on each τ, $(v,\chi)_\tau = (P_h v, \chi)_\tau$ for all $\chi \in P^k(\tau)$.

Theorem 4.2.1 (Broken Poincaré Inequality)**.** *Let $v_h \in \mathbf{S_h}$. There exists a constant $C_P > 0$ such that*

$$\|v_h\| \leq C_P \left(\sum_\tau \|\nabla v_h\|_\tau^2 + \sum_e h_e^{-1} \|[v_h]\|_e^2 \right)^{\frac{1}{2}}$$

Proof. A proof can be found in [1] by Brenner. ∎

Lemma 4.2.2 (Broken Sobolev Inequality)**.** *Let $v_h \in \mathbf{S_h}$. Then*

$$\|v_h\|_{L^\infty(\Omega)} \leq C|\log(h)|^{\frac{1}{2}} \left(\sum_\tau \|\nabla v_h\|_\tau^2 + \sum_e h_e^{-1} \|[v_h]\|_e^2 \right)^{\frac{1}{2}}.$$

Proof. A proof can be found in [5]. ∎

Lemma 4.2.3 (Edge Superapproximation)**.** *Let $v_h \in \mathbf{S_h}$ and let P_h be the L^2 projection onto $\mathbf{S_h}$. Then there exists $C > 0$ such that the following hold:*

$$\left(\sum_e \left\|\sigma^{-1}(\sigma^2 v_h - P_h(\sigma^2 v_h))\right\|_e^2\right)^{\frac{1}{2}} \leq Ch^{\frac{1}{2}} \|v_h\|, \tag{4.2.1}$$

$$\left(\sum_e \left\|\sigma^{-1}\nabla(\sigma^2 v_h - P_h(\sigma^2 v_h))\right\|_e^2\right)^{\frac{1}{2}} \leq Ch^{-\frac{1}{2}} \|v_h\|. \tag{4.2.2}$$

Proof. Using the usual trace inequality, property (4.0.5), and superapproximation results from Chapter 3, we have

$$\begin{aligned}
\left\|\sigma^{-1}(\sigma^2 u_h - P_h(\sigma^2 u_h))\right\|_e^2 &\leq Ch_e^{-1} \left\|\sigma^{-1}\right\|_{L^\infty(\tau_e)}^2 \left\|\sigma^2 u_h - P_h(\sigma^2 u_h)\right\|_{\tau_e}^2 \\
&\quad + Ch_e \left\|\sigma^{-1}\right\|_{L^\infty(\tau_e)}^2 \left\|\nabla(\sigma^2 u_h - P_h(\sigma^2 u_h))\right\|_{\tau_e} \\
&\leq Ch_e^{-1} \left\|\sigma^{-1}(\sigma^2 u_h - P_h(\sigma^2 u_h))\right\|_{\tau_e}^2 \\
&\quad + Ch_e \left\|\sigma^{-1}\nabla(\sigma^2 u_h - P_h(\sigma^2 u_h))\right\|_{\tau_e} \\
&\leq Ch_e \|u_h\|_{\tau_e}^2 .
\end{aligned}$$

∎

4.3 dG Resolvent Estimates

We take the same path towards the L^∞ resolvent estimate as in Chapter 3. First, we show an L^2 resolvent estimate for the dG operator A_h. Second, we use a modification of this L^2 estimate to show a weighted L^2 estimate. Finally, the weighted estimate leads us to the L^∞ estimate.

Theorem 4.3.1 (L^2 Resolvent)**.** *Let* u_h *solve* $z\big(u_h, v_h\big)_\Omega - A_h(u_h, v_h) = \big(f_h, v_h\big)_\Omega$.

Then there exist $\theta \in (\theta_0, \frac{\pi}{2})$ *and* $C > 0$, $\theta_0 \approx \arctan(|\vec{\beta}|/\epsilon)$, *such that*

$$\|u_h\| \leq C\frac{|\vec{\beta}|\,\|f\|}{\epsilon|z|},\ \forall z \in \mathbb{C} - \Sigma_\theta.$$

Proof. We let $v_h = -u_h$, resulting in

$$-z\,\|u_h\|^2 + \epsilon\sum_\tau \|\nabla u_h\|_\tau^2 + \epsilon\lambda\sum_e h_e^{-1}\,\|[u_h]\|_e^2 = F, \tag{4.3.1}$$

where

$$F = \sum_\tau \left(u_h, \vec{\beta}\cdot\nabla u_h\right)_\tau + 2\,\mathrm{Re}\left(\epsilon\sum_e \left(\{\nabla u_h\}, [u_h]\right)_e\right) - \sum_e \left(u_h^+|\vec{\beta}\cdot\vec{n_e}|, [u_h]\right)_e - (f, u_h)_\Omega.$$

First, we claim

$$-\mathrm{Re}\,b(u_h,u_h) = \mathrm{Re}\left(\sum_\tau \left(u_h, \vec{\beta}\cdot\nabla u_h\right)_\tau - \sum_e \left(u_h^+|\vec{\beta}\cdot\vec{n_e}|, [u_h]\right)_e\right) \leq 0.$$

Indeed, integrating by parts on each triangle, we hasve

$$\sum_\tau (u_h, \vec{\beta}\cdot\nabla u_h)_\tau = -\sum_\tau \left(\vec{\beta}\cdot\nabla u_h, u_h\right)_\tau + \sum_e \int_e (|u_h^+|^2 - |u_h^-|^2)|\vec{\beta}\cdot n_e|.$$

This implies,

$$
\begin{aligned}
-\operatorname{Re} b(u_h, u_h) &= \sum_e \int_e \frac{1}{2}(|u_h^+|^2 - |u_h^-|^2)|\vec{\beta} \cdot n_e| - (|u_h^+|^2 - 2\operatorname{Re} u_h^+ \overline{u_h^-} - |u_h^-|^2)|\vec{\beta} \cdot n_e| \\
&= \sum_e \int_e -\frac{1}{2}|u_h^+ - u_h^-|^2 |\vec{\beta} \cdot n_e| \\
&\le 0.
\end{aligned}
$$

Thus, taking the real parts of equation (5.3.1), we have

$$
\operatorname{Re}(-z) \|u_h\|^2 + \epsilon \sum_\tau \|\nabla u_h\|_\tau^2 + \epsilon\lambda \sum_e h_e^{-1} \|[u_h]\|_e^2 \le 2\epsilon \Big| \sum_e \big(\{\nabla u_h\}, [u_h]\big)_e \Big| + |(f, u_h)|.
$$

Using the Cauchy-Schwarz inequality, discrete trace inequality, Young's inequality, and summing over the τ_e, where τ_e is a triangle with edge e, we have

$$
\begin{aligned}
2\epsilon \sum_e \big| \big(\{\nabla u_h\}, [u_h]\big)_e \big| &\le 2\epsilon \sum_e C_T h_e^{-\frac{1}{2}} \|\nabla u_h\|_{\tau_e} \|[u_h]\|_e \\
&\le \frac{\epsilon}{2} \sum_\tau \|\nabla u_h\|_\tau^2 + 4\epsilon C_T^2 \sum_e h_e^{-1} \|[u_h]\|_e^2 .
\end{aligned}
$$

Thus, choosing λ such that $\lambda - 4C_T^2 > \frac{1}{2}$, we have

$$
\operatorname{Re}(-z) \|u_h\|^2 + \frac{\epsilon}{2} \sum_\tau \|\nabla u_h\|_\tau^2 + \frac{\epsilon}{2} \sum_e h_e^{-1} \|[u_h]\|_e^2 \le \big|(f, u_h)\big|. \qquad (4.3.2)
$$

Now we consider the imaginary parts of equation (5.2.1):

$$|\operatorname{Im}(-z)| \, \|u_h\|^2 \leq \left|(f, u_h)\right| + \Big|\sum_{\tau} \left(u_h, \vec{\beta} \cdot \nabla u_h\right)_\tau\Big| + \Big|\sum_{e} \left(u^+|\vec{\beta} \cdot \vec{n_e}|, [u_h]\right)_e\Big|. \quad (4.3.3)$$

Using the Cauchy-Schwarz inequality, Young's inequality, and the broken discrete Poincare inequality, we have

$$\begin{aligned}
\sum_{\tau} \left|\left(u_h, \vec{\beta} \cdot \nabla u_h\right)_\tau\right| &\leq |\vec{\beta}| \sum_{\tau} \|u_h\|_\tau \, \|\nabla u_h\|_\tau \\
&\leq \frac{|\vec{\beta}|}{2} \|u_h\|^2 + \frac{|\vec{\beta}|}{2} \sum_{\tau} \|\nabla u_h\|_\tau^2 \\
&\leq C|\vec{\beta}| \sum_{\tau} \|\nabla u_h\|_\tau^2 + C|\vec{\beta}| \sum_{e} h_e^{-1} \|[u_h]\|_e^2 .
\end{aligned}$$

Using the Cauchy-Schwarz ineqaulity, a discrete trace inequality, Young's inequality, and the broken discrete Poincare inequality, we have

$$\begin{aligned}
\sum_{e} \left|\left(u_h^+|\vec{\beta} \cdot \vec{n_e}|, [u_h]\right)_e\right| &\leq |\vec{\beta}| \sum_{e} C_T h^{-\frac{1}{2}} \|[u_h]\|_e \left\|u_h^+\right\|_\tau \\
&\leq \frac{C|\vec{\beta}|}{2} \|u_h\|^2 + \frac{C|\vec{\beta}|}{2} \sum_{e} h_e^{-1} \|[u_h]\|_e^2 \\
&\leq C|\vec{\beta}| \sum_{\tau} \|\nabla u_h\|_\tau^2 + C|\vec{\beta}| \sum_{e} h_e^{-1} \|[u_h]\|_e^2 .
\end{aligned}$$

Thus,

$$|\operatorname{Im}(-z)| \, \|u_h\|^2 \leq |(f, u_h)| + 2C|\vec{\beta}| \sum_{\tau} \|\nabla u_h\|_\tau^2 + 2C|\vec{\beta}| \sum_{e} h_e^{-1} \|[u_h]\|_e^2 . \qquad (4.3.4)$$

Multiplying (5.2.2) by $k = \frac{6C|\vec{\beta}|}{\epsilon}$ and adding this to (5.2.3) results in

$$\left(k\operatorname{Re}(-z)+|\operatorname{Im}(-z)|\right)\|u_h\|^2+C|\vec{\beta}|\sum_\tau \|\nabla u_h\|_\tau^2+C|\vec{\beta}|\sum_e h_e^{-1}\|u_h\|_e^2 \leq (k+1)|(f,u_h)|.$$

$$\|u_h\|^2 \leq \frac{k+1}{\operatorname{Re}(-z)+|\operatorname{Im}(-z)|}|(f,u_h)| \leq \frac{k+1}{\operatorname{Re}(-z)+|\operatorname{Im}(-z)|}\|f\|\,\|u_h\|.$$

Letting $\theta = \arctan(1+2k)$ and following the techniques in [6], for any $z \in \mathbb{C} - \Sigma_\theta$

$$\frac{k+1}{\operatorname{Re}(-z)+|\operatorname{Im}(-z)|} \leq C\frac{|\vec{\beta}|}{\epsilon|z|}.$$

Thus, $\|u_h\| \leq C\frac{|\vec{\beta}|\|f\|}{\epsilon|z|}$.

■

The following is a weighted modification of the result above that we will use to prove the weighted resolvent estimate.

Corollary 4.3.2. *Let u_h solve $z\big(u_h, v_h\big)_\Omega - A_h(u_h, v_h) = \big(f_h, v_h\big)_\Omega$. Then there exist $C > 0$, $k \approx \frac{|\vec{\beta}|}{\epsilon}$, and $\theta \in (\theta_0, \frac{\pi}{2})$, where $\theta_0 \approx \arctan(|\vec{\beta}|/\epsilon)$, such that for $M = k+1$,*

$$\|u_h\|^2 \le C\frac{M^2 l_h^2}{|\vec{\beta}|}\frac{\|\sigma f\|^2}{|z|},$$

$$\sum_\tau \|\nabla u_h\|_\tau^2 \le C\frac{M^2 l_h^2}{|\vec{\beta}|^2}\|\sigma f\|^2,$$

$$\sum_e h_e^{-1}\|[u_h]\|_e^2 \le C\frac{M^2 l_h^2}{|\vec{\beta}|^2}\|\sigma f\|^2,$$

for all $z \in \mathbb{C} - \Sigma_\theta$, *where* $\theta = \arctan(2M)$.

Proof. From Theorem 5.3.1, we have

$$\big(k\operatorname{Re}(-z)+|\operatorname{Im}(-z)|\big)\|u_h\|^2 + C_1|\vec{\beta}|\sum_\tau \|\nabla u_h\|_\tau^2 + C_1|\vec{\beta}|\sum_e h_e^{-1}\|u_h\|_e^2 \le (k+1)|(f_h, u_h)|. \tag{4.3.5}$$

Also, using (4.0.4) and the broken discrete Sobolev inequality, we have

$$\begin{aligned}
(k+1)|(f_h,u_h)| &\le (k+1)\|f_h\|_{L^1}\|u_h\|_{L^\infty} \\
&\le M\|\sigma f_h\|\left\|\sigma^{-1}\right\|\|u_h\|_{L^\infty} \\
&\le M l_h^{\frac{1}{2}}\|\sigma f_h\|\|u_h\|_{L^\infty} \\
&\le CMl_h\|\sigma f_h\|\Big(\sum_\tau \|\nabla u_h\|_\tau^2 + \sum_e h_e^{-1}\|[u_h]\|_e^2\Big)^{\frac{1}{2}} \\
&\le CM^2 l_h^2\frac{\|\sigma f_h\|^2}{|\vec{\beta}|} + \frac{C_1|\vec{\beta}|}{2}\sum_\tau \|\nabla u_h\|_\tau^2 + \frac{C_1|\vec{\beta}|}{2}\sum_e h_e^{-1}\|[u_h]\|_e^2.
\end{aligned}$$

Kicking back the gradient and jump terms, we have

$$\big(k\,\mathrm{Re}(-z)+|\,\mathrm{Im}(-z)|\big)\,\|u_h\|^2+\frac{C_1|\vec{\beta}|}{2}\sum_{\tau}\|\nabla u_h\|_{\tau}^2+\frac{C_1|\vec{\beta}|}{2}\sum_{e}h_e^{-1}\,\|[u_h]\|_e^2 \le CM^2 l_h^2\frac{\|\sigma f_h\|^2}{|\vec{\beta}|}.$$

Finally, using the fact that

$$\frac{M}{k\,\mathrm{Re}(-z)+|\,\mathrm{Im}(-z)|}\le C\frac{M}{|z|}$$

for all $z\in\mathbb{C}-\Sigma_\theta$, we arrive at our result. ∎

We can now proceed with the weighted dG resolvent estimate.

Theorem 4.3.3. *Let u_h solve $z\big(u_h,v_h\big)_\Omega - A_h(u_h,v_h) = \big(f_h,v_h\big)_\Omega$. Then there exists a $C>0$ and $\theta\in(\theta_0,\frac{\pi}{2})$, where $\theta_0\approx\arctan(|\vec{\beta}|/\epsilon)$, such that*

$$\|\sigma u_h\|\le Cl_h\frac{|\vec{\beta}|}{\epsilon}\frac{\|\sigma f_h\|}{|z|} \tag{4.3.6}$$

for all $z\in\mathbb{C}-\Sigma_\theta$.

Proof. We choose $v_h=-P_h(\sigma^2u_h)$, resulting in

$$-z\,\|\sigma u_h\|^2 = -(f,\sigma^2u_h) - A_h(u_h,P_h(\sigma^2u_h)).$$

Adding $A_h(u_h,\sigma^2u_h)$ to both sides of the above equation, and defining $e_h=\sigma^2u_h - P_h(\sigma^2u_h)$ gives us

$$-z\,\|\sigma u_h\|^2+\epsilon\sum_{\tau}\|\sigma\nabla u_h\|_{\tau}^2+\epsilon\lambda\sum_{e}h_e^{-1}\,\|[\sigma u_h]\|_e^2 = F,$$

$$F = A_h(u_h, e_h) - (f_h, \sigma^2 u_h) - 2\epsilon \sum_{\tau} \big(\nabla u_h, \sigma \nabla \sigma u_h\big)_{\tau}$$
$$+ \sum_{\tau} \big(u_h, \vec{\beta} \cdot \nabla(\sigma^2 u_h)\big)_{\tau} - \sum_{e} \big(u_h^+ |\vec{\beta} \cdot n_e|, [\sigma^2 u_h]\big)_{e}.$$

Hence, the above equation is of the form

$$e^{i\alpha} a + b = F,\ a, b > 0,$$

where $0 \leq |\alpha| \leq \pi - \theta$. Multiplying by $e^{-i\frac{\alpha}{2}}$, and taking the real parts of the equation, we have

$$a + b \leq |F| \big(\cos(\alpha/2)\big)^{-1} \leq |F| \big(\sin(\theta/2)\big)^{-1} \leq C|F|,$$

for all $z \in \mathbb{C} - \Sigma_\theta$.

That is $\forall z \in \mathbb{C} - \Sigma_\theta$, we have

$$|z| \, \|\sigma u_h\|^2 + \epsilon \sum_{\tau} \|\sigma \nabla u_h\|_{\tau}^2 + \epsilon \lambda \sum_{e} h_e^{-1} \, \|[\sigma u_h]\|_e^2 \leq C|F|.$$

We now bound each term on the right hand side. Using the Cauchy-Schwarz inequality, Young's inequality and property (4.0.3), we have

$$|2C\epsilon \sum_{\tau} \big(\nabla u_h, \sigma \nabla \sigma u_h\big)_{\tau}| \leq \frac{\epsilon}{4} \sum_{\tau} \|\sigma \nabla u_h\|_{\tau}^2 + C\epsilon \, \|u_h\|^2 .$$

Using the Cauchy-Schwarz inequality, Young's inequality, property (4.0.3)and $|\sigma| \leq C$, we have

$$
\begin{aligned}
|C\sum_\tau \big(u_h, \vec{\beta}\cdot\nabla(\sigma^2 u_h)\big)_\tau| &= |C\sum_\tau \big(u_h, \sigma^2\vec{\beta}\cdot\nabla u_h\big)_\tau + 2C\sum_\tau \big(u_h, \sigma\vec{\beta}\cdot\nabla\sigma u_h\big)_\tau| \\
&\le \frac{|z|}{4}\|\sigma u_h\|^2 + C\frac{|\vec{\beta}|^2}{|z|}\sum_\tau \|\nabla u_h\|_\tau^2 + C|\vec{\beta}|\,\|u_h\|^2 .
\end{aligned}
$$

Using the Cauchy-Schwarz inequality, $|\sigma| \le C$, property (4.0.5), discrete trace inequality, and Young's inequality, we have

$$
\begin{aligned}
|C\sum_e \big(u_h^+|\vec{\beta}\cdot n_e|, [\sigma^2 u_h]\big)_e| &\le C|\vec{\beta}|\sum_e \|\sigma u_h\|_e \|[u_h]\|_e \\
&\le C|\vec{\beta}|\sum_e C_T h_e^{-\frac{1}{2}} \|\sigma u_h\|_{\tau_e} \|[u_h]\|_e \\
&\le \frac{|z|}{4}\|\sigma u_h\|^2 + \frac{C|\vec{\beta}|^2}{|z|}\sum_e h_e^{-1}\|[u_h]\|_e^2 .
\end{aligned}
$$

Using the Cauchy-Schwarz inequality, edge superapproximation, and Young's inequality, we have

$$
\begin{aligned}
|C\epsilon\lambda\sum_e h_e^{-1}\big([u_h],[e_h]\big)_e| &\le C\epsilon\lambda\sum_e h_e^{-1}\|[\sigma u_h]\|_e \left\|\sigma^{-1}e_h\right\|_e \\
&\le C\epsilon\|u_h\|^2 + \epsilon\frac{\lambda}{2}\sum_e h_e^{-1}\|[\sigma u_h]\|_e^2 .
\end{aligned}
$$

Using the Cauchy-Schwarz inequality, edge superapproximation, and Young's inequality, we have

$$\begin{aligned}|C\epsilon \sum_{e} \left([u_h], \{\nabla u_h\}\right)_e| &= C\epsilon \sum_{e} \left\|[\sigma u_h]\right\|_e \left\|\sigma^{-1} e_h\right\|_e \\ &\leq C\epsilon \left\|u_h\right\|^2 + \frac{\epsilon}{2} \sum_{e} h_e^{-1} \left\|[\sigma u_h]\right\|_e^2 ,\end{aligned}$$

where τ_e is a triangle with edge e.

Using the Cauchy-Schwarz inequality, discrete trace inequality, property (4.0.5), and edge superapproximation, we have

$$\begin{aligned}|C\epsilon \sum_{e} \left(\{\nabla u_h\}, [e_h]\right)_e| &\leq C\epsilon \sum_{e} C_T h_e^{-\frac{1}{2}} \left\|\sigma \nabla u_h\right\|_{\tau_e} \\ &\leq C\epsilon \left\|u_h\right\|^2 + \frac{\epsilon}{4} \sum_{\tau} \left\|\sigma \nabla u_h\right\|_{\tau}^2 .\end{aligned}$$

Using the Cauchy-Schwarz inequality, superapproximation, and $|\sigma| \leq C$,

$$\begin{aligned}| - C \sum_{\tau} \left(u_h, \vec{\beta} \cdot \nabla e_h\right)_\tau| &\leq C|\vec{\beta}| \sum_{\tau} \left\|\sigma u_h\right\|_\tau \left\|\sigma^{-1} \nabla e_h\right\|_\tau \\ &\leq C|\vec{\beta}| \left\|u_h\right\|^2 .\end{aligned}$$

Similarly, using the Cauchy-Schwarz inequality, the discrete trace inequality, property (4.0.5), edge superapproximation, and $|\sigma| \leq C$, we have

$$|C \sum_{e} \left(u_h^{+} |\vec{\beta} \cdot n_e|, [e_h]\right)_e| \leq C|\vec{\beta}| \left\|u_h\right\|^2 .$$

Finally, using Young's inequality,

$$|(\sigma f_h, \sigma u_h)| \leq \frac{|z|}{4} \|\sigma u_h\|^2 + C \frac{\|\sigma f_h\|^2}{|z|}.$$

Thus, kicking back, we have

$$\frac{|z|}{4} \|\sigma u_h\|^2 \leq C|\vec{\beta}| \|u_h\|^2 + C \frac{|\vec{\beta}|^2}{|z|} \sum_{\tau} \|\nabla u_h\|_{\tau}^2 + C \frac{|\vec{\beta}|^2}{|z|} \sum_{e} h_e^{-1} \|[u_h]\|_e^2 + C \frac{\|\sigma f_h\|^2}{|z|}.$$

Using the results from Corollary 5.3.2,

$$\frac{|z|}{4} \|\sigma u_h\|^2 \leq C l_h^2 M^2 \frac{\|\sigma f_h\|^2}{|z|}, \ \forall z \in \mathbb{C} - \Sigma_{\theta}.$$

Dividing both sides by $|z|$ and taking square roots gives us our desired estimate.

∎

We note that the weighted resolvent estimate holds for the adjoint operator A_h^* by the exact same analysis as above. We can now prove the main result of this chapter- the L^∞ discontinuous Galerkin resolvent estimate.

Theorem 4.3.4. *There exist $C > 0$ and $\theta \in (\theta_0, \frac{\pi}{2})$, where $\theta_0 \approx \arctan(|\vec{\beta}|/\epsilon)$, such that for all $v_h \in \mathbf{S_h}$,*

$$\left\|(z - A_h)^{-1} v_h\right\|_{L^\infty} \leq C l_h^{\frac{3}{2}} \frac{|\vec{\beta}|}{\epsilon |z|} \|v_h\|_{L^\infty},$$

for all $z \in \mathbb{C} - \Sigma_{\theta}$.

Proof. The proof follows from the analysis in Theorem 4.0.7, using the dG weighted resolvent estimate from this chapter.

■

Remark 4.3.5. Using a duality argument, we also obtain a resolvent estimate in the L^1 norm.

Theorem 4.3.6. *There exist $C > 0$ and $\theta \in (\theta_0, \frac{\pi}{2})$, where $\theta_0 \approx \arctan(|\vec{\beta}|/\epsilon)$, such that for all $v_h \in \mathbf{S_h}$,*

$$\left\|(z - A_h)^{-1} v_h\right\|_{L^\infty} \leq C l_h^{\frac{3}{2}} \frac{|\vec{\beta}|}{\epsilon|z|} \left\|v_h\right\|_{L^1},$$

for all $z \in \mathbb{C} - \Sigma_\theta$.

Chapter 5

Fully Discrete dG(q)cG(r) and dG(q)dG(r) Maximal Parabolic Regularity

With the resolvent estimates behind us for both the continuous Galerkin and discontinuous Galerkin operators, A_h^{cG} and A_h (dG operator), we can now show maximal parabolic regularity for the discontinuous in time and continuous in space, dG(q)cG(r), approximation, as well as for the discontinuous in time and discontinuous in space, dG(q)dG(r), approximations. To ease some notation, we define

$$\kappa = l_h^{\frac{3}{2}} \frac{|\vec{\beta}|}{\epsilon}.$$

to be the constant appearing in Chapter 4. The final analytical tool we will use is the *Dunford-Taylor integral representation.* The Dunford-Taylor integral represents the time-stepping formula from Chapter 2 as a contour integral in the right half of the complex plane. The following result was shown in [2].

Theorem 5.0.1 (Dunford-Taylor Integral Representation). *Let A be a closed, linear operator. Let $U \subset \mathbb{C}$ open containing the spectrum of A. If $r : \mathbb{C} \to \mathbb{C}$ is analytic on $\overline{U}$, then we have,*

$$r(A) = r(\infty)I + \frac{1}{2\pi i}\int_{\partial U}(z - A)^{-1}r(z)\,dz.$$

As in [3], we will combine this integral representation with the resolvent estimates shown in Chapter 4 and 5 to show maximal parabolic regularity.

5.1 dG(q)cG(r) Maximal Parabolic Regularity

Let V_h be the standard continuous Galerkin finite element space described in Chapter 3. Then the space-time fully discrete approximation space is defined as

$$X_{k,h}^{q,r} = \{u : (0, T] \mapsto H_0^1(\Omega) \mid u_{|_{J_m}} \in \mathcal{P}^q(V_h)\},$$

where $\mathcal{P}^q(V_h)$ consists of polynomials of degree q in time with coefficients in V_h. We let A_h^{cG} be the discrete operator defined in Chapter 3. That is,

$$A_h^{cG}(u_h, v_h) = \epsilon(\nabla u_h, \nabla v_h) - (u_h, \vec{\beta} \cdot \nabla v_h),\ \forall v_h \in V_h.$$

5.1.1 dG(0)cG(r) Maximal Parabolic Regularity

We begin this chapter by showing maximal parabolic regularity for the least technical temporal approximation, $dG(0)cG(r)$. Recall from (2.2.8) that the time semi-discrete homogeneous $dG(0)$ approximation was obtained by solving the elliptic problem,

$$u_{k,m} = \prod_{j=1}^{m} (I + k_j A)^{-1} u_0.$$

Thus, the fully discrete homogeneous $dG(0)cG(r)$ approximation is obtained by solving

$$u_{kh,m} = \prod_{j=1}^{m} (I + k_j A_h^{cG})^{-1} P_h u_0.$$

As in Chapter 2, we define a complex valued rational function $r : \mathbb{C} \to \mathbb{C}$,

$$r(z) = (1 + z)^{-1}. \tag{5.1.1}$$

Abusing notation, we write

$$u_{kh,m} = \prod_{j=1}^{m} r(k_j A_h^{cG}) P_h u_0. \tag{5.1.2}$$

Theorem 5.1.1 (L^∞ Homogeneous Smoothing Estimate)**.** *Let $f = 0$, $u_0 \in L^\infty(\Omega)$, and let $u_{kh,m}$ be the $dG(0)cG(r)$ solution. Then there exists $C > 0$ such that for each $m = 1, 2, .., M$,*

$$\left\| A_h^{cG} u_{kh,m} \right\|_{L^\infty(\Omega)} \le \frac{C}{t_m} \kappa \left\| u_0 \right\|_{L^\infty(\Omega)}.$$

Proof. Using (5.1.2), we write

$$u_{kh,m} = \prod_{j=1}^{m} r(k_j A_h^{cG}) P_h u_0.$$

Let θ and Σ_θ be as in Theorem 3.0.7. In addition, define $\Gamma \subset \mathbb{C}$ by

$$\Gamma = \{z \in \mathbb{C} : z = |x| + icx, \forall x \in \mathbb{R}\},$$

with $c > 0$ chosen large enough such that the sector created by Γ contains the spectrum of A_h^{cG}, and $\Gamma \subset \mathbb{C} - \Sigma_\theta$. We note that $r(z)$ is analytic on the right half of the complex plane, and in particular, in said sector. Thus, applying the Dunford-Taylor integral formula, we can write the fully-discrete approximate solution at time t_m as

$$u_{kh,m} = \frac{1}{2\pi i} \int_\Gamma \prod_{j=1}^{m} r(k_j z)(z - A_h^{cG})^{-1} \, dz \; P_h u_0.$$

Moreover,

$$A_h^{cG} u_{kh,m} = \frac{1}{2\pi i} \int_\Gamma \prod_{j=1}^{m} r(k_j z) A_h^{cG} (z - A_h^{cG})^{-1} \, dz \; P_h u_0.$$

Using the identity

$$-A_h^{cG}(z - A_h^{cG})^{-1} = I - z(z - A_h^{cG})^{-1},$$

as well as the resolvent estimate, Theorem 3.0.7, we have for all $z \in \Gamma$,

$$\left\| -A_h^{cG}(z - A_h^{cG})^{-1} \right\|_{L^\infty(\Omega) \to L^\infty(\Omega)} \le C\kappa.$$

Thus, using the above estimate, we have

$$\left\|A_h^{cG}u_{kh,m}\right\|_{L^\infty(\Omega)} \leq C\int_\Gamma \prod_{j=1}^{m}\left|r(k_jz)\right| \left\|A_h^{cG}(z-A_h^{cG})^{-1}\right\|_{L^\infty(\Omega)\to L^\infty(\Omega)}\ dz\ \ \|P_hu_0\|_{L^\infty(\Omega)}$$
$$\leq C\kappa\,\|P_hu_0\|_{L^\infty(\Omega)}\int_\Gamma \prod_{j=1}^{m}\left|r(k_jz)\right|\ dz.$$

To show our desired result, we must show

$$\int_\Gamma \prod_{j=1}^{m}|r(k_jz)| \leq \frac{C}{t_m}.$$

Note that, for any $z\in\Gamma$,

$$\begin{aligned}|1+k_jz| &= |1+k_j(|x|+cix)|\\ &= ((1+k_j|x|)^2+c^2x^2)^{\frac{1}{2}}\\ &\geq 1+k_j|x|.\end{aligned}$$

Thus,

$$\int_\Gamma \prod_{j=1}^{m}|r(k_jz)|\ dz \leq C\int_0^\infty \prod_{j=1}^{m}\frac{1}{1+k_jx}\ dx.$$

Now, using the assumption $k \leq \frac{T}{2}$, we have

$$\begin{aligned}\prod_{j=1}^{m}(1+k_j) &\geq 1 + t_m x + \frac{1}{2}x^2 \sum_{i \neq j} k_i k_j \\ &= 1 + t_m x + \frac{1}{2}x^2 (t_m^2 - \sum_{j=1}^{m} k_j^2) \\ &\geq 1 + t_m x + \frac{1}{2}x^2 (t_m^2 - \frac{t_m}{2}\sum_{j=1}^{m} k_j) \\ &\geq 1 + t_m x + \frac{1}{4}t_m^2 x^2 \\ &\geq C(1 + t_m^2 x^2).\end{aligned}$$

Hence,

$$\begin{aligned}\int_\Gamma \prod_{j=1}^{m} |r(k_j z)| \, dz &\leq C \int_0^\infty \frac{1}{1+t_m^2 x^2} \, dx \\ &\leq \frac{C}{t_m}.\end{aligned}$$

This implies, together with $\|P_h u_0\|_{L^\infty(\Omega)} \leq C \|u_0\|_{L^\infty(\Omega)}$, that

$$\left\| A_h^{cG} u_{kh,m} \right\|_{L^\infty(\Omega)} \leq \frac{C}{t_m} \kappa \|u_0\|_{L^\infty(\Omega)}.$$

■

An immediate corollary is a smoothing result for the jumps.

Corollary 5.1.2 (L^∞ Jump Smoothing Estimate)**.** *Let $f = 0$, $u_0 \in L^\infty(\Omega)$, and let $u_{kh,m}$ be the dG(0)cG(r) solution. Then there exists $C > 0$ such that for each $m = 1, 2, .., M$,*

$$\left\|\frac{[u_{kh}]_m}{k_m}\right\|_{L^\infty(\Omega)} \leq \frac{C}{t_m}\kappa \left\|u_0\right\|_{L^\infty(\Omega)}.$$

Proof. By the $dG(0)$ equation,

$$u_{kh,m} + k_m A_h^{cG} u_{kh,m} = u_{kh,m-1}.$$

Rearranging the equation by subtracting and dividing by k_m, we have

$$\frac{[u_{kh}]_{m-1}}{k_m} = -A_h u_{kh,m}.$$

Applying the previous theorem, we have our result.

∎

We note that the above smoothing estimates hold for any $1 \leq p \leq \infty$. For the case $p = 2$, the logarithmic factor in κ can be removed.

Corollary 5.1.3 (L^2 Homogeneous Smoothing Estimate)**.** *Let $f = 0$, $u_0 \in L^2(\Omega)$, and let $u_{kh,m}$ be the $dG(0)cG(r)$ solution. Then there exists $C > 0$ such that for each $m = 1, 2, .., M$,*

$$\left\|A_h^{cG} u_{kh,m}\right\|_{L^2(\Omega)} \leq \frac{C}{t_m}\frac{|\vec{\beta}|}{\epsilon}\left\|u_0\right\|_{L^2(\Omega)}.$$

Proof. Identical analysis from the L^∞ case is used. However, the constant in the L^2 resolvent estimate, Theorem 3.0.3, does not contain any dependence on h. ∎

Using the homogeneous smoothing estimates as above, we can now show fully discrete maximal parabolic regularity in the $L^\infty(J; L^\infty(\Omega))$ norm.

Theorem 5.1.4 (Fully Discrete dG(0)cG(r) Maximal Parabolic Regularity). *Let* $u_0 = 0$. *There exists* $C > 0$ *independent of* k *and* h *such that for any* $f \in L^\infty(J; L^\infty(\Omega))$ *the* $dG(0)cG(r)$ *solution* u_{kh} *satisfies*

$$\left\|A_h^{cG} u_{kh}\right\|_{L^\infty(J;L^\infty(\Omega))} \leq C\kappa \ln\left(\frac{T}{k}\right) \|f\|_{L^\infty(J;L^\infty(\Omega))}.$$

Proof. Recall from Chapter 2, the semi-discrete in time dG(0) non-homogeneous solution at time t_m can be written as

$$u_{k,m} = \sum_{i=1}^{m} k_i \prod_{j=1}^{m-i+1} r(k_{m-j+1}A) f_i, \tag{5.1.3}$$

where $f_i = \frac{1}{k_i}\int_{J_i} f\, dt$ and $r(z) = \frac{1}{1+z}$. Thus, the fully discrete dG(0)cG(r) solution at time t_m can we written as

$$u_{kh,m} = \sum_{i=1}^{m} k_i \prod_{j=1}^{m-i+1} r(k_{m-j+1}A_h^{cG}) f_{h,i}. \tag{5.1.4}$$

We notice that each term in the sum on the right hand side of the equation is simply a homogeneous dG(0)cG(r) solution with initial value $f_{h,i}$ and $t = t_{i-1}$.

Using this fact, we can apply the homogeneous smoothing estimate on the each of these terms in the sum to see

$$\left\|A_h^{cG} u_{kh,m}\right\|_{L^\infty(\Omega)} \leq \sum_{i=1}^{m} k_i \prod_{j=1}^{m-i+1} \left\|A_h^{cG} r(k_{m-j+1}A_h^{cG}) f_{h,i}\right\|_{L^\infty(\Omega)} \tag{5.1.5}$$

$$\leq C\kappa \sum_{i=1}^{m} \frac{k_i}{t_m - t_{i-1}} \|f_{h,i}\|_{L^\infty(\Omega)}. \tag{5.1.6}$$

Taking the $L^\infty(J)$ norm on both sides, and using the fact that

$$\max_i \|f_{h,i}\|_{L^\infty(\Omega)} \le \|f_h\|_{L^\infty(J;L^\infty(\Omega))},$$

$$\left\|A_h^{cG} u_{kh,m}\right\|_{L^\infty(J;L^\infty(\Omega))} \le C\kappa \|f_h\|_{L^\infty(J;L^\infty(\Omega))} \max_m \sum_{i=1}^{m} \frac{k_i}{t_m - t_{i-1}}.$$

Finally, we note that $\sum_{i=1}^{m-1} \frac{k_i}{t_m - t_{i-1}}$ is a left hand Riemann sum approximation to the integral

$$\int_0^{t_{m-1}} \frac{dt}{t_m - t} = C \ln\left(\frac{t_m}{k_m}\right) \le C \ln\left(\frac{T}{k}\right).$$

Thus, we conclude

$$\left\|A_h^{cG} u_{kh}\right\|_{L^\infty(J;L^\infty(\Omega))} \le C\kappa \ln\left(\frac{T}{k}\right) \|f_h\|_{L^\infty(J;L^\infty(\Omega))}.$$

■

Remark 5.1.5. Following [8], the result also holds in the $L^1(J; L^\infty(\Omega))$ norm. Thus, for all $1 \le p, s \le \infty$ the theorem holds in the $L^s(J; L^p(\Omega))$ norm.

Corollary 5.1.6. *Let $u_0 = 0$. There exists $C > 0$ independent of k and h such that for any $f \in L^\infty(J; L^\infty(\Omega))$ the dG(0)cG(r) solution u_{kh} satisfies*

$$\max_{1\le m\le M} \left\|\frac{[u_{kh}]_m}{k_m}\right\|_{L^\infty(\Omega)} \le \frac{C}{\kappa} \ln\left(\frac{T}{k}\right) \|f\|_{L^\infty(J;L^\infty(\Omega)}.$$

Proof. The proof follows from the fact that

$$\left\|\frac{[u_{kh}]_m}{k_m}\right\|_{L^\infty(\Omega)} = \|f_{h,m} - A_h u_{kh,m}\|_{L^\infty(\Omega)}.$$

Then we use the previous theorem. ■

5.1.2 dG(q)cG(r), q ≥1 Maximal Parabolic Regularity

In this section, we generalize our results for dG(q)cG(r), approximation, where $q > 0$. We recall from Chapter 2, the semi-discrete dG(q) approximation restricted to an interval J_m is given as

$$u_{k|_{J_m}} = \sum_{j=0}^{q} U_j^m(x)\Phi_j^m(t). \tag{5.1.7}$$

Moreover,

$$U_j^1 = \widehat{r}_j(k_1 A)u_0,\ m = 1, \tag{5.1.8}$$

$$U_j^m = \widehat{r}_j(k_m A)U_q^{m-1},\ m = 2, 3, ..., M, \tag{5.1.9}$$

where $\widehat{r}_j : \mathbb{C} \to \mathbb{C}$ are complex valued rational polynomials of the form

$$\widehat{r}_j(z) = \frac{p_j(z)}{p(z)},$$

where $p_j(z)$ is a polynomial of order q, and $p(z)$ is a polynomial of order $q + 1$ with no roots on the right half of the complex plane. Thus, $\widehat{r}_j$ is analytic on the right half of the complex plane.

Using the above representation for the semi-discrete dG(q) approximation, we can write the fully-discrete dG(q)cG(r) restricted to the J_m as

$$u_{k|_{J_m}} = \sum_{j=0}^{q} U_j^m(x)\Phi_j^m(t), \tag{5.1.10}$$

where

$$U_j^1 = \widehat{r}_j(k_1 A_h^{cG})u_0,\ m = 1 \tag{5.1.11}$$

$$U_j^m = \widehat{r}_j(k_m A_h^{cG})U_q^{m-1},\ m = 2, 3, ..., M. \tag{5.1.12}$$

Theorem 5.1.7 (Fully Discrete dG(q)cG(r) Maximal Parabolic Regularity)**.** *Let* $u_0 = 0$*. There exists* $C > 0$ *independent of* k *and* h *such that for any* $f \in L^\infty(J; L^\infty(\Omega))$ *the dG(q)cG(r) solution* u_{kh} *satisfies*

$$\left\| A_h^{cG} u_{kh} \right\|_{L^\infty(J;L^\infty(\Omega))} \leq C\kappa \ln\left(\frac{T}{k}\right) \|f\|_{L^\infty(J;L^\infty(\Omega))}.$$

Proof. We refer the reader to Section 4 in [8], where the authors proved this result for $A_h = -\Delta_h$. The idea of their proof is to rewrite each $\widehat{r}_j$ using a partial fraction decomposition, then as in the dG(0) case, apply the Dunford-Taylor integral representation, along with the resolvent estimates for Δ_h. We can use the exact same analysis, with the resolvent estimates we proved for A_h^{cG} in Chapter 3. ■

Corollary 5.1.8. *Let* $u_0 = 0$*. There exists* $C > 0$ *independent of* k *and* h *such that for any* $f \in L^\infty(J; L^\infty(\Omega))$ *the dG(q)cG(r) solution* u_{kh} *satisfies*

$$\max_{1 \leq m \leq M} \left\| \frac{[u_{kh}]_m}{k_m} \right\|_{L^\infty(\Omega)} \leq C\kappa \ln\left(\frac{T}{k}\right) \|f\|_{L^\infty(J;L^\infty(\Omega))}.$$

Proof. Once again, we follow the proof in [8]. ■

Since u_{kh} is order $q > 1$ in time, there is a non zero time derivative. This final result states that time derivative is bounded by the right hand side in the $L^\infty(J; L^\infty(\Omega))$ norm. Once again, we refer the reader to [8] for proof.

Corollary 5.1.9. *Let $u_0 = 0$. There exists $C > 0$ independent of k and h such that for any $f \in L^\infty(J; L^\infty(\Omega))$ the dG(q)cG(r) solution u_{kh} satisfies*

$$\max_{1\le m\le M} \left\| u'_{kh,m} \right\|_{L^\infty(\Omega)} \le C\kappa \ln\left(\frac{T}{k}\right) \|f\|_{L^\infty(J;L^\infty(\Omega)}\,,$$

where $u'_{kh,m}$ is the time derivative of $u_{kh,m}$.

5.2 dG(q)dG(r) Maximal Parabolic Regularity

We can use the exact same analysis as in the dG(q)cG(r) subsection using the discontinuous Galerkin resolvent estimates from Chapter 4 to obtain the same results for the dG(q)dG(r) approximation. Here, A_h is the discontinuous Galerkin operator from Chapter 4.

Theorem 5.2.1 (Fully Discrete dG(0)dG(r) Maximal Parabolic Regularity)**.** *Let $u_0 = 0$. There exists $C > 0$ independent of k and h such that for any $f \in L^\infty(J; L^\infty(\Omega))$ the dG(0)dG(r) solution $\widehat{u}_{kh}$ satisfies*

$$\|A_h \widehat{u}_{kh}\|_{L^\infty(J;L^\infty(\Omega))} \le C\kappa \ln\left(\frac{T}{k}\right) \|f\|_{L^\infty(J;L^\infty(\Omega))}\,.$$

Corollary 5.2.2. *Let $u_0 = 0$. There exists $C > 0$ independent of k and h such that for any $f \in L^\infty(J; L^\infty(\Omega))$ the dG(0)dG(r) solution $\widehat{u}_{kh}$ satisfies*

$$\max_{1\le m\le M} \left\| \frac{[\widehat{u}_{kh}]_m}{k_m} \right\|_{L^\infty(\Omega)} \le C\kappa \ln\left(\frac{T}{k}\right) \|f\|_{L^\infty(J;L^\infty(\Omega)}\,.$$

5.3 Application to best approximation Error Estimates

In this section, we use fully discrete maximal parabolic regularity to prove pointwise best error estimates for the dG(q)cG(r) approximation.

Theorem 5.3.1. *Let $u \in C(\bar{J} \times \bar{\Omega}) \cap C(\bar{J}; H_0^1(\Omega))$ be the solution to the model problem. Let u_{kh} be the dG(q)cG(r) solution. Then there exists $C > 0$ independent of k and h such that*

$$\|u - u_{kh}\|_{L^\infty(J;L^\infty(\Omega))} \le C\kappa \ln\left(\frac{T}{k}\right) \inf_{\chi \in X_{k,h}^{q,r}} \|u - \chi\|_{L^\infty(J;L^\infty(\Omega))}.$$

Proof. Let $\widetilde{t} \in J$ and $x_0 \in \Omega$. Without loss of generality, we assume $\widetilde{t} \in (t_{M-1}, T]$. We consider two cases. First, assume $\widetilde{t} = T$. To show a bound on $u_{kh}(T, x_0)$, we consider a backwards problem.

$$-g_t + Ag = f, \quad (t,x) \in J \times \Omega \tag{5.3.1}$$

$$g = 0, \quad (t,x) \in J \times \partial\Omega \tag{5.3.2}$$

$$g(T,x) = \widetilde{\delta^{x_0}}, \quad x \in \Omega\,. \tag{5.3.3}$$

Let g_{kh} be the solution to the problem

$$B(\psi_{kh}, g_{kh}) = \psi_{kh}(T, x_0), \ \forall \psi_{kh} \in X_{k,h}^{q,r}. \tag{5.3.4}$$

Using the Galerkin orthogonality

$$\begin{aligned}
u_{kh}(T,x_0) &= B(u_{kh},g_{kh})\\
&= B(u,g_{kh})\\
&= \sum_{m=1}^{M}(u,g_{kh,t})_{J_m\times\Omega} + (\nabla u,\nabla g_{kh})_{J\times\Omega} + (\vec{\beta}\cdot\nabla u, g_{kh})_{J\times\Omega}\\
&\quad - \sum_{m=1}^{M}(u_m,[g_{kh}]_m)_\Omega + (u(T),g_{kh,m}^{-})_\Omega.
\end{aligned}$$

By the Hölder inequality, we have

$$\sum_{m=1}^{M}(u,g_{kh,t})_{J_m\times\Omega} \le \|u\|_{L^\infty(J;L^\infty(\Omega))}\sum_{m=1}^{M}\|g_{kh,t}\|_{L^1(J_m;L^1(\Omega))}.$$

Using the stability of the elliptic Galerkin projection R_h, see [10], and the Hölder inequality, we have

$$\begin{aligned}
(\nabla u,\nabla g_{kh})_{J\times\Omega} + (\vec{\beta}\cdot\nabla u, g_{kh})_{J\times\Omega} &= (\nabla R_h u,\nabla g_{kh})_{J\times\Omega} + (\vec{\beta}\cdot\nabla R_h u, g_{kh})_{J\times\Omega}\\
&= (R_h u, A_h^{cG} g_{kh})_{J\times\Omega}\\
&\le C\,\|u\|_{L^\infty(J;L^\infty(\Omega))}\left\|A_h^{cG} g_{kh}\right\|_{L^1(J;L^1(\Omega))}.
\end{aligned}$$

In addition, using the Hölder inequality, we have

$$\sum_{m=1}^{M}(u_m,[g_{kh}]_m)_\Omega + (u(T),g_{kh,m}^{-})_\Omega \le \|u\|_{L^\infty(J;L^\infty(\Omega))}\sum_{m=1}^{M}\|[g_{kh}]_m\|_{L^1(\Omega)}.$$

Thus, using a direct consequence of the homogeneous smoothing estimate in the $L^1(\Omega)$ norm, as in Lemma 5.2 in [7], we have

$$|u_{kh,}(T,x_0)| \leq C\kappa \ln\left(\frac{T}{k}\right) \left\|P_h\widetilde{\delta^{x_0}}\right\|_{L^1(\Omega)} \|u\|_{L^\infty(J;L^\infty(\Omega))}.$$

Finally using the fact $\left\|P_h\widetilde{\delta^{x_0}}\right\|_{L^1(\Omega)} \leq \|\delta^{x_0}\|_{L^1(\Omega)} \leq C$, we have completed the case $\widetilde{t} = T$.

Now we consider the case $\widetilde{t} \in (t_{M-1}, T)$. For this, we let $\Theta(t)$ be the regularized delta function in time and $\widetilde{\delta^{x_0}}$ be as before. We note that $\text{supp}(\Theta) \in J_m$ and $(\Theta, \psi_k)_J = \psi_k(\widetilde{t})$ for all $\psi_k \in P^q(J_m)$. We consider the following problem. Find G such that

$$\begin{aligned} -G_t + AG = \Theta\widetilde{\delta^{x_0}},\ & (t,x) \in J\times\Omega \\ G = 0,\ & (t,x) \in J\times\partial\Omega \\ G(T,x) = 0,\ & x \in\ \Omega\,. \end{aligned}$$

We let G_{kh} be the dG(q)cG(r) approximation of G. By Galerkin orthogonality, we have

$$
\begin{aligned}
u_{kh}(\widetilde{t}, x_0) &= (u_{kh}, \Theta\widetilde{\delta^{x_0}})_{J\times\Omega} \\
&= B(u_{kh}, G) \\
&= B(u_{kh}, G_{kh}) \\
&= B(u, G_{kh}) \\
&= \sum_{m=1}^{M} (u, G_{kh,t})_{J_m\times\Omega} + (\nabla u, \nabla G_{kh})_{J\times\Omega} + \\
&\quad (\vec{\beta}\cdot\nabla u, G_{kh})_{J\times\Omega} - \sum_{m=1}^{M} (u_m, [g_{kh}]_m)_{\Omega}.
\end{aligned}
$$

As in the first case, we use the elliptic Galerkin projection $R_h G$ and similar estimates. Since G_{kh} is an approximation to the non homogeneous problem, however, we use $L^1(J; L^1(\Omega))$ fully-discrete maximal parabolic regularity instead of the homogeneous smoothing estimate. Hence,

$$
|u_{kh}(\widetilde{t}, x_0)| \leq C\kappa \ln\left(\frac{T}{k}\right) \|u\|_{L^\infty(J;L^\infty(\Omega))}. \tag{5.3.5}
$$

Finally using the fact that the dG(q)cG(r) method is invariant on $X^{q,r}_{k,h}$, replacing u and u_{kh} with $u - \chi$ and $u_{kh} - \chi$ and using the triangle inequality, we have

$$
\|u - u_{kh}\|_{L^\infty(J;L^\infty(\Omega))} \leq C\kappa \ln\left(\frac{T}{k}\right) \inf_{\chi\in X^{q,r}_{k,h}} \|u - \chi\|_{L^\infty(J;L^\infty(\Omega))}.
$$

■

Remark 5.3.2. We note that we have a similar result for the dG(q)dG(r) approximation, provided that there is stability for the elliptic discontinuous Galerkin projection.

Chapter 6

Numerical Experiments

We consider the initial value boundary value problem,

$$u_t - \epsilon\Delta + \vec{\beta} \cdot \nabla u = 1, \quad (t, x) \in J \times \Omega$$

$$u = 0, \qquad (t, x) \in J \times \partial\Omega$$

$$u(0, \cdot) = u_0(x), \qquad x \in \Omega \ ,$$

where $J = (0, 1)$, $\vec{\beta} = (1, 1)^T$, and $\Omega = (0, 1) \times (0, 1)$. We use a dG(0)cG(1) approximation in FreeFem with a uniform time step $k = 0.1$. For various values of epsilon, we observed the following scaling for the jumps:

ϵ	$\max \frac{[u_{kh}]_m}{k}$
1000	0.001
100	0.01
10	0.13
1	0.67
0.1	0.99
0.01	0.999

Based on these numerical results, we believe that our analysis of maximal parabolic regularity for jumps is a sharp bound in the diffusion dominated case. That is, when $|\vec{\beta}| \leq \epsilon$. However, we see that we lose sharpness in the advection dominated case. We believe that the advection term provides a stabilizing effect for small epsilon.

Preliminary numerical results for the estimates on Au for the fully discrete approximation indicate that our analysis does not result in a sharp bound. We believe this may be due to the fact that ϵ is multiplying the stiffness matrix A_h. We plan to investigate this further.

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