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Topics on Asymptotically Hyperbolic Manifolds in General Relativity

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ABSTRACT

Asymptotically hyperbolic manifolds are natural objects to be considered in certain physical circumstances. This dissertation particularly focuses on construction and rigidity of such manifolds, which includes the following three main results.

Firstly, we obtain a new construction of a 3-dimensional asymptotically hyperbolic manifold from a 2-sphere by using a solution of the Ricci flow as a foliation. These asymptotically hyperbolic manifolds provide examples of 'admissible extensions' in the context of an asymptotically hyperbolic analogue of the Bartnik mass.

Secondly, we prove the equality case of the positive mass theorem for asymptotically hyperbolic manifolds without a spin assumption. This is the last piece necessary to complete the proof of the positive mass theorem in the asymptotically hyperbolic setting, which was a long standing open problem in the area.

Lastly, we establish some warped product splitting theorems with a scalar curvature lower bound. The imposed conditions are motivated by the notion of outer-trapped surfaces, which is used to study black holes using local geometry. Moreover, the resulting warped product serves a model space for asymptotically locally hyperbolic manifolds.

Topics on Asymptotically Hyperbolic Manifolds in General Relativity

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Contents

Chapter 1

Introduction

Asymptotically hyperbolic manifolds, whose geometry approaches hyperbolic space near infinity, naturally arise in the study of initial data sets in general relativity. This dissertation consists of three main results concerned with construction and rigidity of such manifolds.

First of all, we construct an asymptotically hyperbolic 3-manifold using Ricci flow foliation method and investigate properties of the metric. The main theorem of the paper is the following.

Theorem 1.0.1. *[57]Let (Σ, g_0) be a surface diffeomorphic to $\mathbb{S}^2$ with area 4π and let N be the product manifold $[1,\infty) \times \Sigma$. Suppose also that the Gauss curvature $K(g_0) > -3$. And let $H \in C^\infty(\Sigma)$ with $H > 0$ and $\bar{R} \in C^\infty(N)$ satisfying $\bar{R} = -6 + O(r^{-5}) \geq -6$ be given. Then there exist a function $u \in C^\infty(N)$ and an asymptotically hyperbolic 3-metric $\bar{g}$ on N of the form*

$$\bar{g} = \frac{u^2}{1+r^2} dr^2 + r^2 \, g(r), \tag{1.0.1}$$

where $g(r)$ is the solution of the modified Ricci flow starting from (Σ, g_0), such that $R_{\bar{g}} = \bar{R}$ and $H_\Sigma = H$ where $R_{\bar{g}}$ is the scalar curvature of the metric $\bar{g}$ and H_Σ is the mean curvature in direction ∂_r on $\{1\} \times \Sigma$.

In addition, we prove that the metric constructed by this method satisfies some rigid properties in terms of the Hawking mass. This particular construction is motivated by the work C. -Y. Lin [64] and C. -Y. Lin and C. Sormani [65], which constructed asymptotically flat 3-manifolds via Ricci flow.

These asymptotically hyperbolic manifolds can be used to study the notion of an asymptotically hyperbolic analogue of the Bartnik mass. We will briefly discuss about this topic at the end of Chapter 3.

Secondly, we prove the borderline case of the positive mass theorem for asymptotically hyperbolic manifolds without spin assumption. This is the last piece necessary to complete the proof of the positive mass theorem in the asymptotically hyperbolic setting, which was a long standing open problem in the area.

Theorem 1.0.2. *[54] Let $n \geq 3$ and (M, g) be an n-dimensional asymptotically hyperbolic manifold with scalar curvature $R_g \geq -n(n-1)$ and with equality $p_0 = \sqrt{p_1^2 + \cdots + p_n^2}$, where $(p_0, p_1, \ldots, p_n)$ is the mass of g. Suppose the following holds:*

- *($\star$) There is an open neighborhood $\mathcal{M}$ of g in the space of asymptotically hyperbolic manifolds such that the inequality $p_0(\gamma) \geq \sqrt{(p_1(\gamma))^2 + \cdots + (p_n(\gamma))^2}$ holds if $\gamma \in \mathcal{M}$ and the scalar curvature satisfies $R_\gamma = R_g$.*

Then (M, g) is isometric to hyperbolic space.

By [30], the assumption ($\star$) can be dropped and thus we reach the following conclusion.

Theorem 1.0.3. *Let $n \geq 3$ and (M, g) an n-dimensional asymptotically hyperbolic manifold with scalar curvature $R_g \geq -n(n-1)$ and with the equality $p_0 = \sqrt{p_1^2 + \cdots + p_n^2}$. Then (M, g) is isometric to hyperbolic space.*

Lastly, we prove several splitting results for Riemannian manifolds (M^n, g) with scalar curvature $R_g \geq -n(n-1)$ (or $R_g \geq 0$), and having compact boundary N satisfying a related mean curvature inequality. The main theorem is the following:

Theorem 1.0.4. *Let (M, g) be a complete, noncompact n-dimensional ($n \geq 3$) Riemannian manifold with compact boundary N. Assume:*

1. *M has scalar curvature $R_g \geq -\varepsilon n(n-1)$, where $\varepsilon = 0$ or 1.*

2. *N has mean curvature $H_N \leq -\varepsilon(n-1)$.*

3. *N does not carry a metric of positive scalar curvature and is weakly outermost.*

Then (M, g) is isometric to $[0, \infty) \times N$, with (warped) product metric $dt^2 + e^{-2\varepsilon t}h$, where (N, h) is Ricci flat.

The proofs make use of results on marginally outer trapped surfaces applied to appropriate initial data sets: we say that N is *weakly outermost* if there does not exist a compact hypersurface $\Sigma \subset M \setminus N$ cobordant to N (which means that $\Sigma \cup N$ is the boundary of a compact n-submanifold in M) satisfying the *strict* mean curvature inequality, $H_\Sigma < -\varepsilon(n-1)$.

One of the results involves an analysis of Obata's equation on manifolds with boundary. This result is relevant to the ongoing work with Lan-Hsuan Huang concerning the rigidity of asymptotically locally hyperbolic manifolds with zero mass.

Chapter 2

General Relativity and Asymptotically hyperbolic manifolds

In this chapter, we present the relevant background in mathematical general relativity and certain topics related to asymptotically hyperbolic manifolds in order to put the results into the context. We begin with the Einstein Equations, and the Einstein Constraint equations for initial data sets. In this context, it will be explained how asymptotically hyperbolic manifolds arise, and examples will also be presented. Then, we will discuss the mass in general relativity for both asymptotically flat and hyperbolic settings.

2.1 Basic notions from General relativity

2.1.1 Einstein Equations

The fundamental idea of general relativity is to incorporate gravitational effects in the geometry of a spacetime as follows: consider a Lorentzian manifold $(\mathcal{M}^{n+1}, \mathfrak{g}), n \geq 3$ which satisfies the *Einstein equations*

$$\mathrm{Ric}_{\mathfrak{g}} - \frac{1}{2} R_{\mathfrak{g}}\, \mathfrak{g} + \Lambda \mathfrak{g} = T \tag{2.1.1}$$

where $\mathrm{Ric}_{\mathfrak{g}}, R_{\mathfrak{g}}$ are the Ricci tensor and the scalar curvature with respect to the metric $\mathfrak{g}$, respectively, $\Lambda \in \mathbb{R}$ is the *cosmological constant*, and T is a divergence free $(0,2)$-tensor called the *stress-energy tensor*. The stress-energy tensor T represents the matter distribution in the spacetime, thus a solution of the Einstein Equations encodes the physical effects by the matter presence. In particular, we call the *vacuum Einstein Equations* when $T = 0$: equivalently, the vacuum Einstein Equations (VEE) can be expressed as the following simpler form:

$$\mathrm{Ric}_{\mathfrak{g}} = \frac{2\Lambda}{n-1} \mathfrak{g}. \tag{2.1.2}$$

The conventional choices of Λ are either $\Lambda = 0, \frac{n(n-1)}{2}$, and $-\frac{n(n-1)}{2}$. While the case of $\Lambda > 0$ has been studied significantly, we will focus on $\Lambda = 0$ or $\Lambda < 0$. Here, we present a few well-known examples of a solution of the VEE.

Example 2.1.1. The Minkowski spacetime, which is defined as $\mathbb{R}^{n+1}$ equipped with the metric

$$-dt^2 + \sum_{i=1}^{n} (dx^i)^2, (t, x) \in \mathbb{R}^{n+1},$$

solves the VEE with $\Lambda = 0$.

The anti-de Sitter (AdS) spacetime is defined on $\{(t, r, \omega) \in \mathbb{R} \times \mathbb{R} \times \mathbb{S}^{n-1}\}$ equipped with the metric

$$-(1 + r^2)\, dt^2 + \frac{1}{1 + r^2}\, dr^2 + r^2 g_{\mathbb{S}^{n-1}},$$

and this is a solution of the VEE with $\Lambda = -\frac{n(n-1)}{2}$.

These are the simplest examples regarded as the "empty" universes with the corresponding Λ.

Example 2.1.2. The next example is the *Schwarzschild spacetime*: for $m > 0$, a manifold $\{(t, r, \omega) \in \mathbb{R} \times \mathbb{R} \times \mathbb{S}^{n-1} : r > (2m)^{\frac{1}{n-2}}\}$ equipped with the metric

$$-\left(1 - \frac{2m}{r^{n-2}}\right) dt^2 + \left(1 - \frac{2m}{r^{n-2}}\right)^{-1} dr^2 + r^2\, g_{\mathbb{S}^{n-1}}.$$

This is a solution of the VEE with $\Lambda = 0$. The Schwarzschild spacetime with the given coordinates describes the gravitational field outside a spherical mass. In fact, one can consider the same metric on $\{(t, r, \omega) \in \mathbb{R} \times \mathbb{R} \times \mathbb{S}^{n-1} : 0 < r < 2m\}$ that represent the black hole region. In this thesis, we will not concern about the geometry of the black hole region, so we refer the interested reader to [74, Chapter 31].

The corresponding example with $\Lambda = -\frac{n(n-1)}{2}$ is called the *AdS-Schwarzschild spacetime*, which is defined as for $m > 0$, a manifold $\{(t, r, \omega) \in \mathbb{R} \times \mathbb{R} \times \mathbb{S}^{n-1} : r > r_0\}$, where r_0 is the largest zero of $r^n + r^{n-2} - 2m$, equipped with the metric

$$-\left(1 + r^2 - \frac{2m}{r^{n-2}}\right) dt^2 + \left(1 + r^2 - \frac{2m}{r^{n-2}}\right)^{-1} dr^2 + r^2 g_{\mathbb{S}^{n-1}}.$$

This spacetime is a solution of the VEE with $\Lambda = -\frac{n(n-1)}{2}$.

If one choose the harmonic (or wave) coordinates $\{x^i\}_{i=0}^n$ with $\mathfrak{g}(\partial_{x^0}, \partial_{x^0}) < 0$, then the Ricci tensor is expressed as

$$(\mathrm{Ric}_{\mathfrak{g}})_{ij} := \mathrm{Ric}_{\mathfrak{g}}(\partial_{x^i}, \partial_{x^j}) = -\frac{1}{2}\Delta_{\mathfrak{g}}(\mathfrak{g}_{ij}) + \text{lower order terms.}$$

Since the Laplace operator on a Lorentzian manifold behaves like the wave operator, the Einstein equations can be viewed as a system of quasi-linear hyperbolic partial differential equations. Therefore, it is natural to approach the evolution problem from a certain initial data for the Einstein Equations.

2.1.2 Initial data sets and Einstein constraint equations

A triple (M^n, g, k) is called an *initial data set* where (M^n, g) is a Riemannian manifold and k is a symmetric two-covariant tensor. To solve the evolution problem, there must be compatible conditions on an initial data set. A hypersurface embedded in a Lorentzian manfiold $(\mathcal{M}^{n+1}, \mathfrak{g})$ is said to be spacelike if the induced metric of $\mathfrak{g}$ is Riemannian. If we assume that (M^n, g) is a spacelike hypersurface in a solution $(\mathcal{M}^{n+1}, \mathfrak{g})$ of the Einstein Equations and k is the second fundamental form of M, then one can derive the *Einstein constraint equations* from the Gauss-Codazzi equation:

$$\begin{aligned} R_g + (\mathrm{tr}_g k)^2 - |k|^2 - 2\Lambda &= 2\mu, \\ \mathrm{div}_g k - d(\mathrm{tr}_g k) &= J, \end{aligned} \tag{2.1.3}$$

where $\mu = T(\nu, \nu)$ is the energy density and $J = T(\nu, \cdot)$ is the momentum density for ν is the future-pointing normal vector of M. In particular, if $(\mathcal{M}^{n+1}, \mathfrak{g})$ is a solution of VEE, then the *vacuum constraint equations* can be written as

$$\begin{aligned} R_g &= 2\Lambda - (\mathrm{tr}_g k)^2 + |k|^2, \\ \mathrm{div}_g k - d(\mathrm{tr}_g k) &= 0. \end{aligned} \tag{2.1.4}$$

In 1952, Choquet-Bruhat [41] proved the remarkable theorem that for a given initial data set (M, g, k) satisfying the vacuum constraint equations, there exists a Lorentzian manifold $(\mathcal{M}, \mathfrak{g})$ solving VEE such that the initial data set is embedded as a spacelike hypersurface. Motivated by this strking result of Choquet-Bruhat, many mathematicians have developed various methods to construct an initial data set satisfying the constraint equations. See the survey [34] for this topic.

Example 2.1.3. The special case with the second fundamental form k being identically zero is called a *time-symmetric initial data*. It is straightforward that the hypersurface $\{t = 0\}$ in the Minkowski, AdS, Schwarzschild, and AdS Schwazschild spacetime is a spacelike and time-symmetric hypersurface. The hypersurface $\{t = 0\}$ in the Minkowski spacetimes (resp. the AdS spacetime) is Euclidean space (resp. hyperbolic space).

As a nontrivial example, the hypersurface $\{t = 0\}$ in the Schwarzschild spacetime is called the *Riemmannian Schwarzschild manifold* $\{(r, \omega) \in \mathbb{R} \times \mathbb{S}^{n-1} : r \geq (2m)^{\frac{1}{n-2}}\}$ with the metric

$$g = \left(1 - \frac{2m}{r^{n-2}}\right)^{-1} dr^2 + r^2 g_{\mathbb{S}^{n-1}}.$$

In addition, the hypersurface $\{t = 0\}$ in the AdS Schwarzschild spacetime is called the *Riemannian AdS Schwarzschild manifold* $\{(r, \omega) \in \mathbb{R} \times \mathbb{S}^{n-1} : r \geq r_0\}$ with the metric

$$g = \left(1 + r^2 - \frac{2m}{r^{n-2}}\right)^{-1} dr^2 + r^2 g_{\mathbb{S}^{n-1}}.$$

Here, r_0 is the same as in Example 2.1.2.

Remark 2.1.4. From physical motivation, we mainly consider initial data sets satisfying the *dominant energy condition*: $\mu \geq |J|_g$. This condition implies that $R_g \geq 2\Lambda$ for a time-symmetric initial data $(M, g, 0)$, which motivates the specific scalar curvature lower bound.

Note that the Riemannian Schwarzschild (resp. AdS Schwarzschild) manifold resembles Euclidean metric (resp. hyperbolic metric) as r approaches infinity. In fact, these are examples for *asymptotically flat* or *hyperbolic* manifolds, which we will define the next section.

2.2 The mass in general relativity

We briefly introduce the concept of the total mass for asymptotically flat and hyperbolic manifolds. In fact, there are various notions of the mass or energy in the literature, with different motivations. We will not attempt to explain all here. Instead, we refer to the excellent lecture notes [33] by P. Chruściel. We also restrict ourselves to the time-symmetric case, hence Riemannian manifolds (M, g). We will discuss the positive mass theorem for asymptotically hyperbolic case with more detail and motivate the main results in this thesis.

2.2.1 Asymptotically flat and hyperbolic Riemannian manifolds

When modeling isolated gravitational systems, one may naturally consider spacetimes that solves Einstein Equation where the metric approaches the model metrics for 'empty universe.' Accordingly, an isolated system at the level of an initial data set may be modeled by a Riemannian manifold which approaches Euclidean space ($\Lambda = 0$) or hyperbolic space ($\Lambda < 0$) near infinity.

For the case $\Lambda = 0$, we consider a class of asymptotically flat manifolds defined as below.

Definition 2.2.1. A Riemannian manifold (M^n, g) is said to be *asymptotically flat of order* q for some $q > \frac{n-2}{2}$ if there exists a compact set K and a diffeomorphism $\Phi : M \setminus K \to \mathbb{R}^n \setminus \overline{B_1}(0)$ such that in a coordinate chart Φ (called the *exterior coordinate chart*), we have

$$\partial_x^\alpha (g_{ij} - \delta_{ij}) = O(|x|^{-|\alpha|-q})$$

for $|\alpha| \leq 2$. Here, δ is the Euclidean metric on $\mathbb{R}^n$. We also require the scalar curvature R_g to be integrable over M.

Example 2.2.2. The Riemannian Schwarzschild manifold from Example 2.1.3 is asymptotically flat of order $n-2$: one way to see it is to use the transformation $r = \rho \left(1 + \frac{m}{2\rho^{n-2}}\right)^{\frac{2}{n-2}}$, the metric can be written as

$$g = \left(1 + \frac{m}{2\rho^{n-2}}\right)^{\frac{4}{n-2}} \delta,$$

which is defined on $\{(\rho,\omega)\in\mathbb{R}\times\mathbb{S}^{n-1} : \rho > \left(\frac{m}{2}\right)^{\frac{1}{n-2}}\}$. This implies that the metric in the rectangular coordinates on $\mathbb{R}^n \setminus \overline{B\left(0, \left(\frac{m}{2}\right)^{\frac{1}{n-2}}\right)}$ with $|x| = \rho$ has the asymptotic expansion

$$g_{ij}(x) = \left(1 + \frac{2m}{n-2}|x|^{2-n}\right)\delta_{ij}(x) + O_2(|x|^{1-n}).$$

Here, the notation $f(x) = O_i(|x|^{-p})$ means that $\partial_x^\alpha f(x) = O(|x|^{-|\alpha|-p})$ for $i \in \mathbb{N}$ and $|\alpha| \le i$. The coordinates used here are called *isotropic coordinates* for the Schwarzschild metric.

We now discuss a class of manifolds whose geometry approaches hyperbolic space near infinity, which can be regarded as a time-symmetric initial data for $\Lambda < 0$. There are two different ways in the literature to define asymptotically hyperbolic manifolds: the conformally compact approach and the chart-at-infinity approach. Although we will mainly use the second approach, we briefly introduce both in this section. See more detailed discussions in [51].

The conformally compact approach can be related to the ball model of hyperbolic space: the n-dimensional (open) unit ball B^n with the metric

$$g_B = \left(\frac{2}{1-|y|^2}\right)^2 |dy|^2,$$

where $y^1, \ldots, y^n$ are the coordinates on B^n and $|dy|^2 = \sum_i (dy^i)^2$. From this setting, we can observe that a conformally deformed metric $\bar{g} = \left((1-|y|^2)/2\right)^2 g_B$ extends up to the boundary of the ball. Moreover, the induced metric $\bar{g}|_{\partial B^n}$ is the standard unit sphere metric. Hence, the round conformal sphere (since the conformal factor can vary) represents the boundary at infinity of hyperbolic space in a certain way. The following definition introduced by X. Wang [99] is motivated by this property of hyperbolic space:

Definition 2.2.3 (Conformally compact approach)**.** A Riemannian manifold (X, g) is called *conformally compact* if there exists a compact smooth Riemannian manifold with boundary $(\overline{X}, \bar{g})$ such that X is the interior of $\overline{X}$ and $\bar{g} = \phi^2 g$ on X where ϕ is a defining function for $\partial\overline{X}$, i.e., $\phi \geq 0, \phi^{-1}(0) = \partial\overline{X}$, and $d\phi$ is nonvanishing on $\partial\overline{X}$.

A conformally compact manifold (X, g) is said to be *conformally compact asymptotically hyperbolic* if the metric g on a neighborhood of $\partial\overline{X}$ has the expansion

$$g = \frac{1}{(\sinh\rho)^2}\left(d\rho^2 + g_{\mathbb{S}^{n-1}} + \frac{\rho^n}{n}\kappa + O(\rho^{n+1})\right), \tag{2.2.1}$$

where ρ is a defining function and κ is a symmetric $(0, 2)$-tensor on $\mathbb{S}^{n-1}$, which is called the *mass aspect tensor*.

This approach is more relevant to the aspect of AdS-CFT correspondence than the context of general relativity, which emphasizes the link between Einstein metrics on complete manifolds and conformal geometry on compact manifolds. See [46, 38, 19].

The second way to define asymptotically hyperbolic manifolds is using the exterior coordinate chart as in the asymptotically flat case. Consider the hyperboloidal model of hyperbolic space: $\mathbb{R}^n$ with the metric in spherical coordinates

$$g_{\mathbb{H}^n} = \frac{dr^2}{1 + r^2} + r^2\, g_{\mathbb{S}^{n-1}},$$

where h is the standard unit sphere metric on $\mathbb{S}^{n-1}$.

Definition 2.2.4 (Chart-at-infinity approach)**.** A Riemannian manifold (M^n, g) is said to be *asymptotically hyperbolic of order* q for some $q \in \left(\frac{n}{2}, n\right]$ if there exists a compact set K and a diffeomorphism $\Phi : M \setminus K \to \mathbb{R}^n \setminus \overline{B_1}(0)$ such that in a coordinate chart Φ, we have

$$\partial_x^\alpha (g_{ij} - (g_{\mathbb{H}^n})_{ij}) = O(|x|^{-q}) \tag{2.2.2}$$

fpr $|\alpha| \leq 2$. We also require the scalar curvature $R_g = -n(n-1) + O(|x|^{-n})$.

In [51], it is pointed out that a conformally compact asymptotically hyperbolic manifold satisfies the definition of asymptotically hyperbolic manifold in the chart-at-infinity approach provided it has certain regularity of the conformal extension up to the boundary. For some technical reasons, we will rephrase this definition by using the weighted Hölder space in Chapter 4.

Example 2.2.5. The Riemannian AdS Schwarzschild manifold is an asymptotically hyperbolic manifold that satisfies both definitions: it is straightforward that it fits the chart-at-infinity approach from how it is defined. For the conformally compact approach, we set the following ODE to change coordinates

$$\begin{cases} \dfrac{dr}{d\rho} = -\dfrac{\sqrt{1 + r^2 - \frac{2m}{r^{n-2}}}}{\sinh \rho}, \\ r \to +\infty \text{ as } \rho \to 0^+. \end{cases}$$

The negative sign on the right hand side from the first equation reflects on the behavior that r increases as ρ decreases. Then the metric can be expressed as

$$g = \frac{1}{(\sinh \rho)^2} \left(d\rho^2 + u^2 g_{\mathbb{S}^{n-1}} \right)$$

where $u(\rho) = r(\rho) \sinh \rho$. By routine computation, one can show that u has the following asymptotics near $\rho = 0$:

$$u(\rho) = 1 + \frac{m}{n}\rho^n + O(\rho^{n+1}).$$

2.2.2 The mass of asymptotically flat and hyperbolic manifolds

Arnowitt, Deser, and Misner [14] introduced the notion of the mass of a given asymptotically flat manifold, called the *ADM mass*, which is defined as the following:

Definition 2.2.6. Let (M^n, g) be a smooth, asymptotically flat manifold. The ADM mass $m_{ADM}(M, g)$ is defined by

$$\begin{aligned} m_{ADM}(M,g) &= \lim_{r\to\infty} \frac{1}{c_n} \int_{S_r} \sum_{i,j=1}^{n} (g_{ij,i} - g_{ii,j}) \frac{x^j}{|x|} d\sigma_{S_r} \\ &= \lim_{r\to\infty} \frac{1}{c_n} \int_{S_r} \big(\mathring{\operatorname{div}}\, g - d(\mathring{\operatorname{tr}}\, g)\big)(\nu_0)\, d\sigma_{S_r}. \end{aligned} \tag{2.2.3}$$

Here, c_n is the constant depending on its dimension, the quantities with indices are evaluated with the exterior coordinate chart, and S_r and $d\sigma_{S_r}$ are the coordinate sphere of radius r, the volume form of the standard sphere metric of radius r, respectively. In the second equality, $\mathring{\operatorname{div}}, \mathring{\operatorname{tr}}$, and ν_0 are the divergence, trace, and outward normal vector with respect to the flat metric.

One can show that the ADM mass of an asymptotically flat manifold exists and is finite. In fact, the integrability of the scalar curvature is necessary for the ADM mass to be well-defined and finite. Note that the ADM mass is a geometric invariant, i.e., even though the ADM mass is computed in a specific exterior coordinate chart, it does not depend on the choice of the exterior coordinate chart. See [15, 28].

Example 2.2.7. The ADM mass of the Riemannian Schwarzschild manifold is equal to the parameter m up to a constant multiple. Recall the Schwarzschild metric in isotropic coordinates:

$$g = \left(1 + \frac{m}{2\rho^{n-2}}\right)^{\frac{4}{n-2}} \delta.$$

We can compute the ADM mass as

$$\begin{aligned}
m_{ADM} &= \lim_{\rho\to\infty} \frac{1}{c_n} \int_{S_\rho} \sum_{i,j=1}^{n} (g_{ij,i} - g_{ii,j}) \frac{x^j}{|x|} d\sigma_{S_\rho} \\
&= \lim_{\rho\to\infty} \frac{1}{c_n} \int_{\mathbb{S}^{n-1}} \left[\frac{2(n-1)m}{\rho^{n-1}} \left(1 + \frac{m}{2\rho^{n-2}}\right)^{\frac{4}{n-2}-1} \right] \rho^{n-1} d\sigma_{\mathbb{S}^{n-1}} \\
&= \lim_{\rho\to\infty} \frac{1}{c_n} \int_{\mathbb{S}^{n-1}} 2(n-1)m + O(\rho^{-1})\, d\sigma_{\mathbb{S}^{n-1}} \\
&= \frac{2(n-1)\omega_{n-1}}{c_n} m,
\end{aligned}$$

where ω_{n-1} is the volume of the unit $(n-1)$-sphere. In fact, one may choose the normalizing constant c_n to be equal to $2(n-1)\omega_{n-1}$ so that the parameter m is precisely the mass of the Schwarzschild metric.

We now turn to the mass of asymptotically hyperbolic manifolds. Unlike the asymptotically flat case, it is defined as $(n+1)$-tuple, or the linear functional on the $(n+1)$-dimensional vector space. The reason why the mass should be a linear functional is explained in [51, Section 3.2] and references therein via Hamiltonian analysis of general relativity. On the other hand, one can still find a suitable scalar quantity derived from the mass functional (see e.g. [27, Section 3], [80, Section 2.2]), which is a geometric invariant of the given space.

There are two ways to define the mass as the definition of asymptotically hyperbolic manifolds, which are introduced by X. Wang [100] and P. Chruściel and M.

Herzlich [27] separately. First, we define the mass using the chart-at-infinity approach as in [27].

Let $b = g_{\mathbb{H}^n}$ be the hyperbolic metric and define

$$\mathcal{N}_b = \{V \in C^\infty(\mathbb{H}^n) : \mathrm{Hess}_b V = V b)\}.$$

It is well-known (see [32, Appendix B]) that $\mathcal{N}_b$ is the $(n+1)$-dimensional vector space that is spanned by

$$V_0 = \sqrt{r^2+1}, \qquad V_i = x^i \quad \text{for } i = 1, 2, \dots, n,$$

where $\{x^i\}_{i=0}^n$ is the coordinates of the hyperboloidal model and $r = |x|$.

Definition 2.2.8. Let (M^n, g) be an asymptotically hyperbolic manifold with an exterior coordinate chart $\Phi : \mathbb{R}^n \setminus \overline{B_1(0)} \to M \setminus K$. We define the mass integral

$$H_g(V) = \lim_{r\to\infty} \int_{S_r} \left[V\big(\mathring{\mathrm{div}}\, e - d(\mathring{\mathrm{tr}}\, e)\big)(\nu_0) + (\mathring{\mathrm{tr}}\, e) dV(\nu_0) - e(\mathring{\nabla} V, \nu_0)\right] d\sigma_b, \quad (2.2.4)$$

where $e = \Phi^* g - b$, ν_0 is the outward unit normal vector to $S_r = \{|x| = r\}$, and $\mathring{\mathrm{div}}, \mathring{\mathrm{tr}}, \mathring{\nabla}$, are all with respect to b. The volume form $d\sigma_b$ is the restriction of the volume form of b on S_r. The *mass vector* (also called energy-momentum vector) $(p_0(g), p_1(g), \dots, p_n(g))$ is defined by

$$p_0(g) = H_g(\sqrt{1+r^2}) \quad \text{and} \quad p_i(g) = H_g(x^i) \text{ for } i = 1, \dots, n.$$

It is shown in [27] that the number

$$m(g) = \sqrt{p_0(g)^2 - \sum_{i=1}^{n} p_i(g)^2}$$

is a geometric invariant provided $p_0(g)^2 - \sum_{i=1}^{n} p_i(g)^2 \geq 0$. We call this $m(g)$ the *total (hyperbolic) mass.*

Example 2.2.9. We compute the mass vector of the Riemannian AdS Schwarzschild manifold. Since $e = \frac{2m}{r^{n-2}(1+r^2)\left(1+r^2-\frac{2m}{r^{n-2}}\right)}dr^2$, by direct computation, we have

$$\begin{aligned}
\mathring{\operatorname{div}}\, e(\nu_0) &= 2m\sqrt{1+r^2}\left[\frac{d}{dr}\left(\frac{1}{r^{n-2}\left(1+r^2-\frac{2m}{r^{n-2}}\right)}\right) + \frac{n-1}{r^{n-1}\left(1+r^2-\frac{2m}{r^{n-2}}\right)}\right], \\
d(\mathring{\operatorname{tr}}\, e)(\nu_0) &= 2m\sqrt{1+r^2}\cdot\frac{d}{dr}\left(\frac{1}{r^{n-2}\left(1+r^2-\frac{2m}{r^{n-2}}\right)}\right), \\
(\mathring{\operatorname{tr}}\, e)dV_i(\nu_0) &- e(\mathring{\nabla} V_i, \nu_0) = 0 \text{ for } i = 0, 1, \dots, n.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
p_0(g) &= \lim_{r\to\infty}\int_{\mathbb{S}^{n-1}} \frac{2m(1+r^2)(n-1)}{r^{n-1}\left(1+r^2-\frac{2m}{r^{n-2}}\right)}\cdot r^{n-1}d\sigma_{\mathbb{S}^{n-1}} \\
&= \lim_{r\to\infty}\int_{\mathbb{S}^{n-1}} 2m(n-1) + O(r^{-2})\, d\sigma_{\mathbb{S}^{n-1}} = 2m(n-1)\omega_{n-1}, \\
p_i(g) &= \lim_{r\to\infty}\int_{S_r} x^i\big(\mathring{\operatorname{div}}\, e - d(\mathring{\operatorname{tr}}\, e)\big)(\nu_0)\, d\sigma_b = 0 \text{ for } i = 1, 2, \dots, n.
\end{aligned}$$

The last integral above vanishes due to the symmetry of x^i on the sphere S_r.

For the conformally compact approach, the definition of the mass is much simpler.

Definition 2.2.10. Let (M^n, g) be a conformally compact asymptotically hyperbolic manifold. The mass vector $(p_0(g), p_1(g), \ldots, p_n(g))$ is defined by

$$p_0(g) = \int_{\mathbb{S}^{n-1}} \mathrm{tr}_{g_{\mathbb{S}^{n-1}}} \kappa \, d\sigma_{g_{\mathbb{S}^{n-1}}}, \qquad p_i(g) = \int_{\mathbb{S}^{n-1}} x^i \mathrm{tr}_{g_{\mathbb{S}^{n-1}}} \kappa \, d\sigma_{g_{\mathbb{S}^{n-1}}} \quad \text{for } i = 1, 2, \ldots, n,$$

where x^i's are the rectangular coordinates of $\mathbb{R}^n$ restricted on the unit sphere $\mathbb{S}^{n-1}$.

Example 2.2.11. Recall that the Riemannian AdS Schwarzschild metric is conformally compact asymptotically hyperbolic with the expansion from Example 2.2.5

$$g = \frac{1}{(\sinh \rho)^2} \left(d\rho^2 + g_{\mathbb{S}^{n-1}} + \frac{\rho^n}{n} (2m g_{\mathbb{S}^{n-1}}) + O(\rho^{n+1}) \right),$$

hence the mass aspect tensor is equal to $2m g_{\mathbb{S}^{n-1}}$. Thus, we have

$$p_0(g) = \int_{\mathbb{S}^{n-1}} 2m(n-1) \, d\sigma_{g_{\mathbb{S}^{n-1}}} = 2m(n-1)\omega_{n-1}$$
$$p_i(g) = 0 \text{ for } i = 1, 2, \ldots, n.$$

As we have seen from the AdS Schwarzschild case, it is well-known that the two mass vectors are the same up to a constant multiple provided both are well-defined on (M, g). The more detailed comparison between these approaches can be found in [51].

2.2.3 Positive mass theorem for asymptotically hyperbolic manifolds

One of the most influential works in mathematical relativity is the positive mass theorem for asymptotically flat manifolds.

Theorem 2.2.12. *[88, 89, 102, 90] Let (M^n, g) be an asymptotically flat manifold* *with $R_g \geq 0$. Then the ADM mass is non-negative. Moreover, the ADM mass is zero if and only if the manifold is isometric to Euclidean space.*

Note that $R_g \geq 0$ is from the dominant energy condition for time-symmetric data. The conclusion of this theorem has two parts: that ADM mass is non-negative and that ADM mass being zero implies the manifold isometric to Euclidean space. The second part characterizes Euclidean space in a sense that Euclidean space is the unique asymptotically flat manifold with nonnegative scalar curvature whose ADM mass is zero. We will not discuss about this subject any further. For more broad context of the positive mass theorem for asymptotically flat manifolds, we recommend the book [61] written by D. Lee.

As introduced in the previous section, the mass for asymptotically hyperbolic manifolds is defined as an $(n+1)$-tuple. Due to the fact that its Lorentzian inner product $p_0(g)^2 - (p_1(g)^2 + \cdots + p_n(g)^2)$ is a geometric invariant, the positive mass conjecture for asymptotically hyperbolic manifolds is stated in terms of that invariant quantity.

Conjecture 2.2.13. *Let $n \geq 3$ and (M^n, g) be an asymptotically hyperbolic manifold with $R_g \geq -n(n-1)$. Then the mass satisfies the inequality*

$$p_0(g) \geq \sqrt{p_1(g)^2 + \cdots + p_n(g)^2}$$

with equality only if (M^n, g) is isometric to hyperbolic space.

Based on the spinor approach, X. Wang [99] established the positive mass theorem for conformally compact asymptotically hyperbolic spin manifolds, and P. Chruściel and M. Herzlich [31] proved for the ones defined by the chart-at-infinity approach with the spin assumption.

One of the remarkable results toward removing the spin assumption is the paper [12] by L. Andersson, M. Cai, and G. Galloway. They first obtained the following scalar curvature rigidity of hyperbolic space.

Theorem 2.2.14. *Suppose $(M^n, g), 3 \leq n \leq 7$, has scalar curvature R_g satisfying $R_g \geq -n(n-1)$, and is isometric to hyperbolic space outside a compact set. Then (M, g) is globally isometric to hyperbolic space.*

Assuming (M, g) is spin, this theorem was first proved by Min-Oo [73] (see also [13, 36]). By using this theorem, they proved that for dimensions $3 \leq n \leq 7$ the mass aspect function $\mathrm{tr}_{g_{\mathbb{S}^{n-1}}} \kappa$ in (2.2.1) cannot be pointwise negative on $\mathbb{S}^{n-1}$. Combining it with the novel deformation argument, they obtain the following version of the positive mass theorem:

Theorem 2.2.15. *[12] Let $(M^n, g), 3 \leq n \leq 7$, be an asymptotically hyperbolic manifold with scalar curvature $R_g \geq -n(n-1)$. Assume that the mass aspect function $\mathrm{tr}_{g_{\mathbb{S}^{n-1}}} \kappa$ does not change sign, i.e., that it is either negative, zero, or positive. Then the mass of (M^n, g) is positive, or (M^n, g) is isometric to hyperbolic space.*

In their recent paper [30], Chruściel and Delay proved the inequality statement by a gluing argument in general dimensions without spin assumption. Also, A. Sakovich [81] proved an approach using the Jang equation to the positivity of mass in three dimensions. However, their proofs do not provide the equality case. In Chapter 4, we will present the complete proof of this case.

2.2.4 Quasi-local mass in general relativity

There are various existing notions of *quasi-local* mass in the context of general relativity. Although the term 'quasi-local mass' is extensively used in the literature, there does not exist a precise mathematical definition. It is a descriptive notion that is supposed to be some quantity of mass contained in a small (spacelike) region, say Ω. In particular, it may be desirable that this notion depends only on the first order geometry of $\partial\Omega$, i.e. the induced metric and the second fundamental form on $\partial\Omega$. For the succinct introduction to the concept of quasi-local mass, we refer the reader to the lecture notes by M.-T. Wang [97] and P. Chruściel [33]. Here, we present the definitions of two significant examples of quasi-local mass, which will be used in Chapter 3.

The first example is proposed by S. Hawking [50]. The following definition is adopted from [67].

Definition 2.2.16. Let Σ^2 be a closed spacelike surface in a spacetime $(\mathcal{M}^4, \mathfrak{g})$. The Hawking mass, denoted by $m_H(\Sigma)$, is defined as

$$m_H(\Sigma) = \sqrt{\frac{|\Sigma|_{\mathfrak{g}}}{16\pi}}\left(\frac{\chi(\Sigma)}{2} - \frac{1}{16\pi}\int_\Sigma \left(|\vec{H}|^2_{\mathfrak{g}} + \frac{4}{3}\Lambda\right) d\sigma_{\mathfrak{g}}\right) \tag{2.2.5}$$

where $|\Sigma|_{\mathfrak{g}}$ is the area, $\chi(\Sigma)$ is the Euler characteristic of Σ, $\vec{H}$ is the mean curvature vector of Σ, and Λ is the cosmological constant.

The most common setting for the Hawking mass is that Σ is embedded in a time-symmetric initial data (M^3, g) and diffeomorphic to a sphere. In this setting, the

Hawking mass comes down to the following:

$$\begin{cases} m_H^{AF}(\Sigma) = \sqrt{\dfrac{|\Sigma|_g}{16\pi}} \left(1 - \dfrac{1}{16\pi} \displaystyle\int_\Sigma H^2 \, d\sigma_g\right) \text{ if } (M^3, g) \text{ is asymptotically flat,} \\ m_H^{AH}(\Sigma) = \sqrt{\dfrac{|\Sigma|_g}{16\pi}} \left(1 - \dfrac{1}{16\pi} \displaystyle\int_\Sigma H^2 \, d\sigma_g + \dfrac{|\Sigma|_g}{4\pi}\right) \\ \qquad\qquad\qquad\qquad \text{if } (M^3, g) \text{ is asymptotically hyperbolic,} \end{cases} \tag{2.2.6}$$

where H is the mean curvature of Σ in (M, g). For more geometric understanding of the Hawking mass, see [61, Section 4.2.1].

The Hawking mass has appeared in many geometric and physical applications. Arguably, the most famous one is for the proof of Penrose inequality in [56] and [21]. There are also several interesting rigidity results on surfaces by using the Hawking mass, see [68, 94, 91].

The second example of quasi-local mass is due to Bartnik [16]. The idea of this notion is well-explained in [61, Section 6.1], we try to summarize the discussion as follows: Let (Ω, g) be a compact Riemannian manifold with nonempty boundary and $R_g \geq 0$. One natural way to define a local notion of mass is to take the infimum of the total masses of all possible extensions from (Ω, g). To make this idea work properly, some geometric restrictions for the extensions should be imposed. There are many different versions of Bartnik mass in the literature. We refer the interested reader to the articles of J. Jauregui [58] and S. McCormick [69] for further discussions about various definitions.

The following version of the definition is adapted from [76].

Definition 2.2.17. Let (Σ^2, γ) be a compact surface equipped with a nonnegative function η. The triple (Σ, γ, η) is called a *Bartnik data*. We say that (M^3, g) an *admissible extension* of (Σ, γ, η) if the following hold:

(1) (M, g) is a complete, asymptotically flat manifold with boundary ∂M identified with Σ.

(2) $R_g \geq 0$ in the interior of M.

(3) $(\partial M, g|_{\partial M}) \cong (\Sigma, \gamma)$ with mean curvature $H = \eta$.

(4) ∂M is outward-minimizing.

Let $\mathcal{A}(\Sigma, \gamma, \eta)$ be the set of all admissible extensions of (Σ, γ, η). The *Bartnik mass* of (Σ, γ, η) is defined as

$$m_B(\Sigma, \gamma, \eta) = \inf_{\mathcal{A}(\Sigma,\gamma,\eta)} m_{ADM}(M, g).$$

We say any element of $\mathcal{A}(\Sigma, \gamma, \eta)$ achieving this infimum a *Bartnik minimizer*.

Bartnik conjectured that if $\eta > 0$, then there always exists a Bartnik minimizer of $(\Sigma, \gamma.\eta)$, and such a minimizer is asymptotically flat solution to the static vacuum Einstein Equations. Recently, M. Andersson and J. Jauregui in [6] showed that the second statement is true, but the first statement does not hold in general.

Apparently, due to its nature, computing the Bartnik mass of given Bartnik data is extremely difficult. A natural approach is to estimate an upper bound for it by constructing a certain extension. See the survey article [77] related to this topic.

While the usual Bartnik mass used asymptotically flat extensions, there is a natural analogue of the Bartnik mass for asymptotically hyperbolic manifolds. The formulation of the hyperbolic Bartnik mass first appeared in [22] as the following:

Definition 2.2.18. We say that (M^3, g) an *asymptotically hyperbolic admissible extension* of a Bartnik data (Σ, γ, η) if the following hold:

(1) (M, g) is a complete, asymptotically hyperbolic manifold with boundary ∂M identified with Σ.

(2) $R_g \geq -6$ in the interior of M.

(3) $(\partial M, g|_{\partial M}) \cong (\Sigma, \gamma)$ with mean curvature $H = \eta$.

(4) ∂M is outward-minimizing.

Let $\mathcal{A}_{AH}(\Sigma, \gamma, \eta)$ be the set of all asymptotically hyperbolic admissible extensions of (Σ, γ, η). The *hyperbolic Bartnik mass* of (Σ, γ, η) is defined as

$$m_B^{AH}(\Sigma, \gamma, \eta) = \inf\{m(g) : (M, g) \in \mathcal{A}_{AH}(\Sigma, \gamma, \eta)\}.$$

where $m(g)$ is the total hyperbolic mass.

The hyperbolic analogue of the Bartnik mass has been relatively less studied than the usual Bartnik mass. The recent papers by A. Cabrera Pacheco, C. Cederbaum, and S. McCormick [22] and P. Miao, Y. Wang, and N. Xie [71] obtained upper bounds by a gluing technique.

Chapter 3

Asymptotically hyperbolic extension via Ricci flow

3.1 Introduction

The construction of asymptotically flat solutions to the Einstein constraint equations, which provide Cauchy data for Einstein equation with $\Lambda = 0$, has been studied extensively. From physical motivation, the dominant energy condition requires the scalar curvature of such metrics to be nonnegative. In 1993, R. Bartnik [17] introduced a construction of 3-metrics with prescribed scalar curvature by considering 3-manifolds foliated by round spheres. There have been other interesting results inspired by this foliation construction. See [92, 93, 64]. In particular, C. -Y. Lin [64] used Hamilton's modified Ricci flow on surfaces as foliation to construct an asymptotically flat end. Let (Σ, g) be a surface diffeomorphic to $\mathbb{S}^2$ whose area is 4π. Recall that Hamilton's modified Ricci flow in [49] is defined as the family $(\Sigma, g(t))$ satisfying

$$\begin{cases} \dfrac{\partial}{\partial t} g_{ij} = 2M_{ij}, \\ g(1) = g. \end{cases} \tag{3.1.1}$$

Here, the symmetric $(0,2)$-tensor M_{ij} is defined by

$$M_{ij} = \frac{1}{2}(r - R)g_{ij} + D_i D_j f,$$

where $R = R(t)$ is the scalar curvature of $g(t)$ on Σ and

$$r = \frac{1}{|\Sigma_t|}\int_\Sigma R(t)\, d\mu_t = 2,$$

where $|\Sigma_t|$ is the area of Σ with respect to $g(t)$ and $f = f(t,x)$ is the Ricci potential satisfying the equation

$$\Delta f = R - r.$$

Note that this flow converges to a metric of constant curvature exponentially fast in any C^k-norm (see [4, Appendix B]). Consider a metric $\overline{g}$ on $N = [1,\infty) \times \Sigma$ of the form

$$\overline{g} = u^2 dt^2 + t^2 g(t).$$

The unknown function u on N with prescribed scalar curvature $\overline{R}$ satisfies a quasi-linear second order parabolic equation derived from the Gauss equation for each slice $\{t\}\times\Sigma$. Therefore, Lin obtained asymptotically flat 3-metrics by solving this equation with some conditions for $\overline{R}$. Moreover, C. Sormani and Lin [65] studied the class of asymptotically flat three-dimensional Riemannian manifolds foliated by Hamilton's modified Ricci flow, and they used these manifolds to estimate the Bartnik mass. In addition, they showed rigidity and monotonicity of the Hawking mass of level sets given in the foliation (see [65, Theorem 5]).

In this paper, we construct an asymptotically hyperbolic 3-metric using Ricci flow foliation method and investigate properties of the metric.

In section 3.2, we derive the quasilinear parabolic equation for a function u from prescribed scalar curvature $\overline{R}$ on N, and prove the existence of a solution.

In section 3.3, we construct an asymptotically hyperbolic 3-metric by the result in section 3.2.

Theorem 3.1.1. *Let* (Σ, g) *be a surface which is diffeomorphic to* $\mathbb{S}^2$ *with area* 4π *and let* N *be the product manifold* $[1, \infty) \times \Sigma$. *Then for any* $H \in C^\infty(\Sigma)$ *with* $H > 0$, *there exists an asymptotically hyperbolic* 3*-metric on* N *of the form*

$$\overline{g} = \frac{u^2}{1+t^2} dt^2 + t^2 g(t), \tag{3.1.2}$$

with the scalar curvature $\overline{R} \equiv -6$ *where* $u \in C^\infty(N)$ *is positive everywhere, and* $g(t)$ *is the solution to Hamilton's modified Ricci flow (3.1.1). Here* H *is the mean curvature in direction* ∂_t *on* $\{1\} \times \Sigma$.

As in [64], the crucial step here is to verify when the solution of the equation in section 3.2 exists. In fact, this construction is obtainable with a more general condition on the prescribed scalar curvature, such as the follwing holds:

$$\overline{R} = -6 + O(t^{-5}) \geq -6.$$

The dominant energy condition requires that $\overline{R} \geq -6$, and the decay is needed to control the behavior of u near infinity. See Theorem 3.3.1.

In section 3.4, we prove the corresponding rigidity and monotonicity result of the Hawking mass as in [65, Theorem 5]. To study the rigid case, we use the following result about the Hawking mass and the mass vector

$$p_0 = \lim_{r\to\infty} m_H^{AH}(S_r) \tag{3.1.3}$$

proved by P. Miao, L. -F. Tam, and N. Xie [70].

Theorem 3.1.2. *Let (Σ, g_1) be a surface diffeomorphic to $\mathbb{S}^2$ with positive mean curvature (not necessarily constant) and let $N = [1,\infty) \times \Sigma$ be an asymptotically hyperbolic extension obtained in Theorem 3.3.1. Then $m_H^{AH}(\Sigma_t)$ is nondecreasing, where $\Sigma_t = \{t\} \times \Sigma$. Furthermore, if*

$$p_0 = m_H^{AH}(\Sigma) \tag{3.1.4}$$

then $\overline{R} = -6$ everywhere, Σ is isometric to the standard sphere, and N is rotationally symmetric. If $m_H^{AH}(\Sigma) = 0$ then N is isometric to a rotationally symmetric region in hyperbolic space. If $m_H^{AH}(\Sigma) = m > 0$ then N is isometric to a rotationally symmetric region in anti-de Sitter Schwarzschild space of mass m.

3.2 Parabolic equation with Ricci flow foliation

In this section, we will derive the equation for prescribed scalar curvature $\overline{R}$ on N from (3.1.2) and obtain a priori estimates for a solution u. The argument is slightly modified from [17] to be suitable for the derived equation.

From the Gauss equation for each slice $\{t\} \times \Sigma$, we have

$$\overline{R} = R_t + 2\overline{\mathrm{Ric}}\left(\frac{\sqrt{1+t^2}}{u}\partial_t, \frac{\sqrt{1+t^2}}{u}\partial_t\right) + ||h||^2 - H^2,$$

where R_t is the scalar curvature on $\{t\} \times \Sigma$ with the induced metric $t^2 g(t)$, h is the second fundamental form, and H is the mean curvature in direction ∂_t. By direct computation, we have

$$\begin{aligned}
\overline{\Gamma}^0_{ij} = \frac{1}{2}\overline{g}^{0l}(\overline{g}_{lj,i} + \overline{g}_{il,j} - \overline{g}_{ij,l}) &= \frac{1+t^2}{2u^2}\left(-\frac{\partial}{\partial t}(t^2 g(t)_{ij})\right) \\
&= \frac{1+t^2}{u^2}\left(-2tg_{ij} - 2t^2 M_{ij}\right), \\
h_{ij} = -\overline{g}\left(\overline{\nabla}_{\partial_i}\partial_j, \frac{\sqrt{1+t^2}}{u}\partial_0\right) &= -\frac{\sqrt{1+t^2}}{u}\overline{g}(\overline{\Gamma}^0_{ij}\partial_0, \partial_0) \\
&= \frac{\sqrt{1+t^2}}{2u}(2tg_{ij} + 2t^2 M_{ij}),
\end{aligned}$$

$$\begin{aligned}
H = \overline{g}^{ij}h_{ij} = t^{-2}g^{ij}\left(\frac{\sqrt{1+t^2}}{u}tg_{ij}\right) &= \frac{2\sqrt{1+t^2}}{tu}, \\
||h||^2 = \frac{2(1+t^2)}{t^2u^2} + \frac{1+t^2}{u^2}|M|^2_{g(t)}.
\end{aligned}$$

Then, we obtain

$$\overline{\mathrm{Ric}}\left(\frac{\sqrt{1+t^2}}{u}\partial_t, \frac{\sqrt{1+t^2}}{u}\partial_t\right) = -\frac{1}{u}\Delta_{\overline{g}|\Sigma_t}u + \frac{\sqrt{1+t^2}}{u}\frac{\partial H}{\partial t} - ||h||^2,$$

$$\begin{aligned}
h_{ij} &= \frac{\sqrt{1+t^2}}{u}(tg_{ij} + t^2 M_{ij}), \\
H &= \frac{2\sqrt{1+t^2}}{tu}, \quad ||h||^2 = \frac{2(1+t^2)}{t^2u^2} + \frac{1+t^2}{u^2}|M|^2_{g(t)}.
\end{aligned}$$

Thus we get the following quasilinear second order parabolic equation

$$t(1+t^2)\frac{\partial u}{\partial t} = \frac{u^2\Delta_{g(t)}u}{2} - \frac{u^3}{4}(R_{g(t)} - t^2\overline{R}) \\ + u\left(\frac{1+3t^2}{2} + \frac{t^2(1+t^2)|M|^2_{g(t)}}{4}\right). \tag{3.2.1}$$

Here $\Delta_{g(t)}$ and $R_{g(t)}$ are the Laplace operator and the scalar curvature on $(\{t\}\times\Sigma, g(t))$ respectively. For any interval $I \subset \mathbb{R}^+$, let $A_I = I \times \Sigma$. For sake of convenience, we will use the following notations: $\Delta = \Delta_{g(t)}$, $\nabla = \nabla^{g(t)}$, for any $f \in C^0(A_I)$, $f^*, f_* : I \to \mathbb{R}$ are defined by

$$f_*(t) = \inf\{f(t,x) : x \in \Sigma\}, \qquad f^*(t) = \sup\{f(t,x) : x \in \Sigma\}.$$

Now from the parabolicity of (3.2.1), the local existence can be obtained by standard Schauder theory [3, Theorem 8.2].

Proposition 3.2.1. *Let $I = [t_0, t_1], 1 \le t_0 < t_1 < \infty$, and let $\overline{R} \in C^{\alpha,\alpha/2}(A_I)$. Then for any initial condition*

$$u(t_0, x) = \varphi(x), \quad x \in \Sigma, \tag{3.2.2}$$

where $\varphi \in C^{2,\alpha}(\Sigma)$ satisfies

$$0 < \delta_0 \le \varphi^{-2}(x) \le \delta_0^{-1}, \quad x \in \Sigma, \tag{3.2.3}$$

for some constant $\delta_0 > 0$, the parabolic equation (3.2.1) with the initial condition (3.2.2) has a unique solution $u \in C^{2+\alpha,1+\alpha/2}(A_{[t_0,t_0+T]})$ for some $T > 0$. Here T depends on $\delta_0, t_0, ||R||_{\alpha,\alpha/2;A_I}, ||M||_{\alpha,\alpha/2;A_I}$, $||\overline{R}||_{\alpha,\alpha/2;A_I}$ and $||\varphi||_{2,\alpha}$.

To state the existence of global solution, we need the following a priori C^0 estimates for the solution u which control the parabolicity and prevent the finite-time blow up.

Proposition 3.2.2. *Suppose $u \in C^{2+\alpha,1+\alpha/2}(A_{[t_0,t_1]}), 1 \leq t_0 < t_1$, is a positive solution to (3.2.2). If we further assume that $\overline{R}$ is defined on $A_{[1,\infty)}$ such that the functions*

$$\delta_*(t) = \frac{1}{t(1+t^2)} \int_1^t \frac{\left(R_{g(s)} - s^2\overline{R}\right)_*}{2} \exp\left(-\int_s^t \frac{\tau(|M|^*)^2}{2} d\tau\right) ds \tag{3.2.4}$$

and

$$\delta^*(t) = \frac{1}{t(1+t^2)} \int_1^t \frac{\left(R_{g(s)} - s^2\overline{R}\right)^*}{2} \exp\left(-\int_s^t \frac{\tau(|M|_*)^2}{2} d\tau\right) ds \tag{3.2.5}$$

are defined and finite for all $t \in [t_0, \infty)$, then for $t_0 \leq t \leq t_1$, we have

$$u^{-2}(t,x) \geq \delta_*(t) + \frac{t_0(1+t_0^2)}{t(1+t^2)}(u^*(t_0)^{-2} - \delta_*(t_0)) \exp\left(-\int_{t_0}^t \frac{\tau(|M|^*)^2}{2} d\tau\right) \tag{3.2.6}$$

and

$$u^{-2}(t,x) \leq \delta^*(t) + \frac{t_0(1+t_0^2)}{t(1+t^2)}(u_*(t_0)^{-2} - \delta^*(t_0)) \exp\left(-\int_{t_0}^t \frac{\tau(|M|_*)^2}{2} d\tau\right). \tag{3.2.7}$$

Proof. Let $w = u^{-2}$ then we have

$$\Delta u = -\frac{3}{2} w^{-1} \nabla u \cdot \nabla w - \frac{1}{2} w^{-\frac{3}{2}} \Delta w.$$

Substituting the Laplace term in (3.2.1), we obtain

$$\frac{\partial w}{\partial t} = \frac{1}{t(1+t^2)}\left[\frac{3}{2}u\nabla u\cdot\nabla w + \frac{1}{2w}\Delta w + \frac{1}{2}(R_{g(t)} - t^2\overline{R}) \right. \\ \left. -w\left(1+3t^2+\frac{t^2(1+t^2)|M|^2}{2}\right)\right]. \tag{3.2.8}$$

By applying the maximum principle, we have

$$t\frac{dw_*}{dt} \geq \frac{1}{1+t^2}\left(-w_*\left(1+3t^2+\frac{t^2(1+t^2)\left(|M|^*\right)^2}{2}\right)+\frac{1}{2}(R_{g(t)}-t^2\overline{R})_*\right)$$

at the maximum of $u(t,x)$. We solve the following ODE

$$t\frac{dw_*}{dt} = -w_*\left(1+t\left(\frac{2t}{1+t^2}+\frac{t\left(|M|^*\right)^2}{2}\right)\right)+\frac{(R_{g(t)}-t^2\overline{R})_*}{2(1+t)^2}. \tag{3.2.9}$$

Using the integrating factor method, we let

$$\varphi(t) = \exp\left(\int_1^t \frac{2s}{1+s^2}+\frac{s\left(|M|^*\right)^2}{2}ds\right).$$

Then we have

$$\frac{d(t\varphi(t)w_*(t))}{dt} = \frac{(R_{g(t)}-t^2\overline{R})_*}{2(1+t^2)}\varphi(t).$$

Integrating to solve $t\varphi(t)w_*(t)$ and noting $u^{-2}(t,x) = w(t,x) \geq w_*(t)$, we derive

$$\begin{aligned} u^{-2} \geq{}& \frac{1}{t}\int_{t_0}^t \frac{\left(R_{g(s)}-s^2\overline{R}\right)_*}{2(1+s^2)}\exp\left(-\int_s^t\left(\frac{2\tau}{1+\tau^2}+\frac{\tau(|M|^*)^2}{2}\right)d\tau\right)ds \\ &+\frac{t_0}{t}\exp\left(-\int_{t_0}^t\left(\frac{2\tau}{1+\tau^2}+\frac{\tau(|M|^*)^2}{2}\right)d\tau\right)w_*(t_0) \\ ={}& \delta_*(t)+\frac{t_0(1+t_0^2)}{t(1+t^2)}(w_*(t_0)-\delta_*(t_0))\exp\left(-\int_{t_0}^t\left(\frac{\tau(|M|^*)^2}{2}\right)d\tau\right). \end{aligned}$$

Similarly, applying the maximum principle to w^*, we get the upper bound of u^{-2}. $\square$

Remark 3.2.3. In proof of Proposition 3.2.2, the idea of substitution as $w = u^{-2}$ first appeared in Bartnik's work [17, Proposition 3.3]. The advantage of this is that we can simplify the coefficient of the term $R_{g(t)} - t^2\overline{R}$ as in (3.2.8) so that we can find the explicit solution of the equation (3.2.9) when applying the maximum principle. This will be used not only to prove the global existence of solution but also to show that $\overline{g} = \frac{u^2}{1+t^2}dt^2 + t^2 g(t)$ is asymptotically hyperbolic with a certain initial condition on $\overline{R}$.

With Propositions 3.2.1 and 3.2.2, we can prove the global existence of solution as the following.

Theorem 3.2.4. *Assume that $\overline{R} \in C^{\alpha,\alpha/2}(N)$ and the constant K is defined by*

$$K = \sup_{1 \leq t < \infty} \left\{ -\int_1^t \frac{\left(R_{g(s)} - s^2\overline{R}\right)_*}{4} \exp\left(\int_1^s \frac{\tau(|M|^*)^2}{2} d\tau \right) ds \right\} < \infty. \tag{3.2.10}$$

Then for every $\varphi \in C^{2,\alpha}(\Sigma)$ such that

$$0 < \varphi(x) < \frac{1}{\sqrt{K}} \text{ for all } x \in \Sigma, \tag{3.2.11}$$

there is a unique positive solution $u \in C^{2+\alpha,1+\alpha/2}(N)$ with the initial condition

$$u(1, \cdot) = \varphi(\cdot). \tag{3.2.12}$$

Remark 3.2.5. One can impose the condition on $\overline{R}$ to guarantee $K < \infty$ by estimating the lower bound of the integral: roughly, we can estimate

$$\begin{aligned}&\int_1^t \frac{\left(R_{g(s)} - s^2\overline{R}\right)_*}{4} \exp\left(\int_1^s \frac{\tau(|M|^*)^2}{2} d\tau\right) ds \\ &\quad \geq \int_1^t \frac{(R_{g(s)})_* - s^2\overline{R}^*}{4} ds\end{aligned}$$

thus it follows that K is finite by assuming that there exists a positive constant C such that the following inequality holds

$$\overline{R}(x,t) \leq \frac{C}{t^4} \text{ for any } (x,t) \in N.$$

Proof of Theorem 6. By considering (3.2.6) and (3.2.10) simultaneously, we have

$$\begin{aligned}\left(u^{-2}\right)_*(t) &> \delta_*(t) + \frac{2K}{t(1+t^2)} \exp\left(-\int_1^t \frac{\tau\left(|M|^*\right)^2}{2} d\tau\right) \\ &\geq \frac{2}{t(1+t^2)} \exp\left(-\int_1^t \frac{\tau\left(|M|^*\right)^2}{2} d\tau\right) \\ &\qquad \times \left(\int_1^t \frac{\left(R_{g(s)} - s^2\overline{R}\right)_*}{4} \exp\left(\int_1^s \frac{\tau(|M|^*)^2}{2} d\tau\right) ds + K\right) \\ &\geq 0\end{aligned}$$

for all $t \geq 1$. Hence it follows from Proposition 3.2.2 that u does not blow up for all $t \geq 1$. Combining this and Proposition 3.2.1 which states the local existence, we get the desired result. □

3.3 Asymptotically hyperbolic 3-metric with Ricci flow foliation

Using Theorem 3.2.4, we can construct a metric with prescribed scalar curvature $\overline{R}$ along the Ricci flow foliation. By assuming the approximate decay for $\overline{R}$, we prove that the metric is asymptotically hyperbolic.

Theorem 3.3.1. *Let (Σ, g) be a 2-manifold which is diffeomorphic to $\mathbb{S}^2$ with area 4π. Let N be the product manifold $[1, \infty) \times \Sigma$. Assume that $\overline{R} \in C^\infty(N)$ satisfies*

$$\overline{R} = -6 + O(t^{-5}) \geq -6. \tag{3.3.1}$$

Then for any $H \in C^\infty(\Sigma)$ with the condition

$$H > 2\sqrt{2K} \tag{3.3.2}$$

where K is defined as (3.2.10), there exists an asymptotically hyperbolic 3-metric on N of the form

$$\overline{g} = \frac{u^2}{1+t^2}dt^2 + t^2 g(t). \tag{3.3.3}$$

Here $g(t)$ is the solution to Hamilton's modified Ricci flow (3.1.1), such that $\overline{R}$ and H are the scalar curvature on $(N, \overline{g})$ and the mean curvature in direction ∂_t on $\{1\} \times \Sigma$, respectively.

From Remark 3.2.5, the constant K is guaranteed to be finite in the above condition on $\overline{R}$. To prove the theorem we need the following lemma which investigates the decay of u by the assumption (3.3.1). The method is similar to the proof of Y. Shi and L. -F. Tam in [92].

Lemma 3.3.2. *Let u be the solution of (3.2.1) with the initial condition $u(1, x) = \varphi(x)$, where $\varphi(x)$ satisfies (3.2.11). Then for sufficiently large t, we have the estimate*

$$\left| \left(\frac{\partial}{\partial t}\right)^k \left(\frac{\partial^{|\beta|}}{\partial x^\beta}\right) (u-1) \right| \leq \frac{C}{t^3} \tag{3.3.4}$$

where β is a multi-index.

Proof of Lemma 3.3.2. First we need to verify the C^0 bounds. Since $|R_{g(s)} - 2|$ is bounded we have

$$\begin{aligned} \int_1^t \frac{(R_{g(s)} - 2)^*}{2} \exp\left(-\int_s^t \frac{\tau(|M|_*)^2}{2} d\tau\right) ds &\leq C_1, \\ \int_1^t \frac{(R_{g(s)} - 2)_*}{2} \exp\left(-\int_s^t \frac{\tau(|M|^*)^2}{2} d\tau\right) ds &\geq C_2. \end{aligned}$$

From the scalar curvature condition (3.3.1), we obtain

$$\begin{aligned} \delta^*(t) &= \frac{1}{t(1+t^2)} \int_1^t \frac{\left(R_{g(s)} - s^2\overline{R}\right)^*}{2} \exp\left(-\int_s^t \frac{\tau(|M|_*)^2}{2} d\tau\right) ds \\ &= \frac{1}{t(1+t^2)} \left[\int_1^t \exp\left(-\int_s^t \frac{\tau(|M|_*)^2}{2} d\tau\right) ds \right. \\ &\quad \left. + \int_1^t \frac{\left(R_{g(s)} - 2 + 6s^2 + O(s^{-3})\right)^*}{2} \exp\left(-\int_s^t \frac{\tau(|M|_*)^2}{2} d\tau\right) ds \right]. \end{aligned}$$

Since $\exp\left(-\int_s^t \frac{\tau(|M|_*)^2}{2} d\tau\right) \leq 1$, we have

$$\int_1^t \exp\left(-\int_s^t \frac{\tau(|M|_*)^2}{2} d\tau\right) \leq t - 1.$$

Thus we get

$$\begin{aligned}\delta^*(t) &\leq \frac{1}{1+t^2} + \frac{C_1 - 1}{t(1+t^2)} + 1 - \frac{1}{1+t^2} + O(t^{-3}) \\ &\leq 1 + \frac{C_3}{t^3}.\end{aligned}$$

To get the lower bound estimate, similarly we have

$$\begin{aligned}\delta_*(t) &= \frac{1}{t(1+t^2)} \int_1^t \frac{\left(R_{g(s)} - s^2\overline{R}\right)_*}{2} \exp\left(-\int_s^t \frac{\tau(|M|^*)^2}{2} d\tau\right) ds \\ &= \frac{1}{t(1+t^2)} \left[\int_1^t \exp\left(-\int_s^t \frac{\tau(|M|^*)^2}{2} d\tau\right) ds \right. \\ &\quad \left. + \int_1^t \frac{\left(R_{g(s)} - 2 + 6s^2 + O(s^{-3})\right)_*}{2} \exp\left(-\int_s^t \frac{\tau(|M|^*)^2}{2} d\tau\right) ds \right].\end{aligned}$$

Let $t_0 \geq 1$ such that

$$\int_{t_0}^{\infty} \frac{\tau(|M|^*)^2}{2} d\tau \leq 1.$$

Then, from [17, Lemma 4.1], we have

$$1 - \frac{C_4}{t} \leq \frac{1}{t} \int_1^t \exp\left(-\int_s^t \frac{\tau(|M|^*)^2}{2} d\tau\right) \leq 1 + \frac{C_4}{t}.$$

Then we get

$$\begin{aligned}
\delta_*(t) \geq \frac{1}{1+t^2} - \frac{C_4}{t(1+t^2)} + \frac{C_2}{t(1+t^2)} \\
+ \frac{1}{t(1+t^2)} \int_1^t 3s^2 \exp\left(-\int_s^t \frac{\tau(|M|^*)^2}{2} d\tau\right) ds + O(t^{-3}) \\
\geq \frac{1}{1+t^2} + \frac{1}{t(1+t^2)} \left[\int_1^{t_0} 3s^2 \exp\left(-\int_s^t \frac{\tau(|M|^*)^2}{2} d\tau\right) ds \right. \\
\left. + t^3 - t_0^3 + \int_{t_0}^t 3s^2 \left(\exp\left(-\int_s^t \frac{\tau(|M|^*)^2}{2} d\tau\right) - 1\right) ds\right] + O(t^{-3}) \\
\geq 1 - \frac{1}{t(1+t^2)} \left[\int_{t_0}^t 3s^2 \int_s^t \tau(|M|^*)^2 d\tau ds\right] + O(t^{-3}).
\end{aligned}$$

We used the fact that $e^\eta - 1 \geq -2|\eta|$ for $|\eta| \leq 1$ to get the third inequality. It follows from the fact that

$$|M|^2 \leq C_5 e^{-ct}$$

that we have

$$\begin{aligned}
\int_{t_0}^t 3s^2 \int_s^t \tau(|M|^*)^2 d\tau ds &\leq \int_{t_0}^t 3s^2 \int_s^t \tau C_5 e^{-c\tau} d\tau ds \\
&\leq C_5 \int_{t_0}^t 3s^2 \int_s^t \tau e^{-c\tau} d\tau ds \\
&\leq O(e^{-ct}) + C_5 \int_{t_0}^t 3s^2 \left(\frac{1}{c} s e^{-cs} + \frac{1}{c^2} e^{-cs}\right) ds \\
&= O(e^{-ct}).
\end{aligned}$$

Hence we obtain

$$\delta_*(t) \geq 1 - \frac{C_6}{t^3},$$

and thus

$$|u(t,x)-1| \le \frac{C}{t^3}.$$

Now we find the estimate for derivatives. Write the equation (3.2.1) as follows

$$\begin{aligned}
&t(1+t^2)\frac{\partial u}{\partial t}\\
&\quad= \frac{\partial}{\partial x^i}\left(\frac{1}{2}u^2 g^{ij}\frac{\partial u}{\partial x^j}\right) - \left[\frac{\partial}{\partial x^i}\left(\frac{u^2}{2\sqrt{|g|}}\right)\right]\left(\sqrt{|g|}g^{ij}\frac{\partial u}{\partial x^j}\right)\\
&\qquad - \frac{u^3}{4}(R_{g(t)} - t^2\overline{R}) + u\left(\frac{1+3t^2}{2} + \frac{t^2(1+t^2)|M|^2}{4}\right)\\
&\quad= \frac{\partial}{\partial x^i}\left(\frac{1}{2}u^2 g^{ij}\frac{\partial u}{\partial x^j}\right) - \left(u g^{ij}\frac{\partial u}{\partial x^i}\frac{\partial u}{\partial x^j}\right)\\
&\qquad - \left[\frac{\partial}{\partial x^i}\left(\frac{1}{2\sqrt{|g|}}\right)\right]\left(u^2\sqrt{|g|}g^{ij}\frac{\partial u}{\partial x^j}\right) - \frac{u^3}{4}(R_{g(t)} - t^2\overline{R})\\
&\qquad + u\left(\frac{1+3t^2}{2} + \frac{t^2(1+t^2)|M|^2}{4}\right).
\end{aligned}$$

Then by letting $s = \log\left(\frac{t}{\sqrt{1+t^2}}\right) + 1$, we get the following form

$$\frac{\partial u}{\partial s} = \frac{\partial}{\partial x^i}a_i(x,s,u,Du) - a(x,s,u,Du) \tag{3.3.5}$$

where

$$a_i(x,s,u,p) = \frac{1}{2}u^2 g^{ij}p_j,$$

$$\begin{aligned}
a(x,s,u,p) = u g^{ij}p_i p_j &+ \left[\frac{\partial}{\partial x^i}\left(\frac{1}{2\sqrt{|g|}}\right)\right]\left(u^2\sqrt{|g|}g^{ij}p_j\right)\\
&+ \frac{u^3}{4}(R_{g(t)} - t^2\overline{R}) - u\left(\frac{1+3t^2}{2} + \frac{t^2(1+t^2)|M|^2}{4}\right).
\end{aligned}$$

It follows from the C^0 estimate of u that

$$a_i p_i \geq C|p|^2, \quad |a_i| \leq C|p|, \quad |a| \leq C(1+|p|^2),$$

where C is independent of s. By [1, Theorem V.1.1], for any $s_0, s_1 \in [1 - \frac{1}{2}\log 2, 1)$ with $s_0 < s_1$, there are constants $\beta > 0$ and $C_1 > 0$ independent of s_0, s_1, such that

$$||u||_{\beta,\beta/2;A_{[s_0,s_1]}} \leq C_1.$$

Now consider the function $v = u - 1$, we have the linear parabolic equation in terms of v

$$\begin{aligned}
\frac{\partial v}{\partial s} &= \frac{u^2}{2} g^{ij} \frac{\partial^2 v}{\partial x^i \partial x^j} - \frac{u^2}{2} \frac{\partial}{\partial x^i}\left(\sqrt{|g|} g^{ij}\right) \frac{\partial v}{\partial x^j} \\
&\quad - \frac{u^3}{4}(R_{g(t)} - t^2 \overline{R}) + u\left(\frac{1+3t^2}{2} + \frac{t^2(1+t^2)|M|^2}{4}\right) \\
&:= Lv - \frac{1}{4}(R_{g(t)} - t^2\overline{R}) + \frac{1+3t^2}{2} + \frac{t^2(1+t^2)|M|^2}{4} \\
&= Lv + f,
\end{aligned}$$

where $f(x,t) = -\frac{R_{g(t)}-2}{4} + \frac{t^2(1+t^2)|M|^2}{4} + O(t^{-3}) = O(t^{-3})$, since $|R_{g(t)} - 2|$ and $|M|$ converge to 0 exponentially fast. Therefore the usual Schauder interior estimates [1, Theorem IV.10.1] and bootstrap argument give the desired result. $\square$

Proof of Theorem 3.3.1. It suffices to show that the metric $\overline{g}$ obtained from Theorem 3.2.4 is asymptotically hyperbolic. From Lemma 3.3.2, we have the following expression of the metric $\overline{g}$:

$$\begin{aligned}\overline{g} &= \frac{u^2}{1+t^2}dt^2 + t^2 g(t) \\ &= \frac{dt^2}{1+t^2} + O(t^{-5})dt^2 + t^2 g(t).\end{aligned}$$

This implies that

$$\begin{aligned}\overline{g}_{tt} &= \frac{1}{t^2} - \frac{1}{t^4} + \frac{\overline{g}_{tt}^{(-5)}}{t^5} + \frac{\overline{g}_{tt}^{(-6)}}{t^6} + O(t^{-7}), \\ \overline{g}_{ij} &= t^2\sigma_{ij} + O(e^{-ct})\end{aligned}$$

where σ_{ij} is the standard metric on the sphere S^2 and $\overline{g}_{tt}^{(-5)}, \overline{g}_{tt}^{(-6)} \in C^\infty(\Sigma)$. By change of coordinates as $t = \sinh r$, we conclude that the metric $\overline{g}$ is asymptotically hyperbolic in Definition 2.2.4. □

Corollary 3.3.3. *Let (Σ, σ) be the 2-sphere with the standard metric, and fix any $0 < m < 1$. Then by prescribing the scalar curvature $\overline{R} \equiv -6$ on $N = [1, \infty) \times \Sigma$, the metric $\overline{g}$ obtained from Theorem 3.3.1, with the initial condition for the constant mean curvature H on $\{1\} \times \Sigma$ as*

$$H \equiv \sqrt{8(1-m)},$$

is the anti-de Sitter Riemannian Schwarzschild metric with the mass m.

Proof. Note that from the initial metric (Σ, σ) the solution to Hamilton's modified Ricci flow is constant, i.e., $|M| \equiv 0$ and $R_{g(t)} \equiv 2$. Then from (3.2.10), we have

$$K = \sup_{1 \le t < \infty} \left\{ -\int_1^t \frac{2+6s^2}{4} ds \right\} = 0.$$

Thus by Theorem 3.3.1, there exists an asymptotically hyperbolic metric $\overline{g}$ with mean curvature $H = \sqrt{8(1-m)}$ on $\{1\} \times \Sigma$. It is easy to see that from Proposition 3.2.2 we have

$$u^{-2}(t,x) = 1 - \frac{1}{t(1+t^2)}\left(2 - \frac{H^2}{4}\right),$$

and hence the metric on N we obtained is

$$\begin{aligned}\overline{g} &= \left(1 + t^2 - \frac{1}{t}\left(2 - \frac{H^2}{4}\right)\right)^{-1} dt^2 + t^2\sigma \\ &= \left(1 + t^2 - \frac{2m}{t}\right)^{-1} dt^2 + t^2\sigma\end{aligned}$$

Notice that the boundary at $t = 1$ is not totally geodesic. However, once we obtain the explicit form, we can extend this metric on $N = [1, \infty) \times \Sigma$ up to the totally geodesic boundary as $\overline{N} = [t_0, \infty) \times \Sigma$ where t_0 is the largest zero of the polynomial $t^3 + t - 2m$. □

3.4 Rigidity and Monotonicity of the Hawking Mass

In this section we prove Theorem 3.1.2 regarding rigidity and monotonicity of the Hawking mass with the foliation we used in previous sections. The proof basically follows an argument in [65, Theorem 5].

Proof of Theorem 3.1.2. Consider $N = [1, \infty) \times \Sigma$ equipped with the metric

$$\overline{g} = \frac{u^2}{1+t^2} dt^2 + t^2 g(t)$$

where $g(t)$ is the solution of the modified Ricci flow. From (3.1.3) we have

$$p_0 = \lim_{t\to\infty} m_H^{AH}(\Sigma_t).$$

We compute the Hawking mass of $\Sigma_t = \{t\} \times \Sigma$

$$\begin{aligned} m_H^{AH}(\Sigma_t) &= \sqrt{\frac{|\Sigma_t|}{16\pi}}\left(1 - \frac{1}{16\pi}\int_{\Sigma_t} H^2\, d\sigma_t + \frac{|\Sigma_t|}{4\pi}\right) \\ &= \sqrt{\frac{4\pi t^2}{16\pi}}\left(1 - \frac{1}{16\pi}\int_{\Sigma} \frac{4(1+t^2)}{t^2u^2} t^2\, d\sigma + \frac{4\pi t^2}{4\pi}\right) \\ &= \frac{t(1+t^2)}{2}\left(1 - \frac{1}{4\pi}\int_\Sigma u^{-2} d\sigma\right) \\ &= \frac{1}{4\pi}\int_\Sigma \frac{t(1+t^2)}{2}(1-u^{-2})d\sigma. \end{aligned} \tag{3.4.1}$$

Hence we have the following equality

$$p_0 = \lim_{t\to\infty} \frac{1}{4\pi}\int_\Sigma \frac{t(1+t^2)}{2}(1-u^{-2})d\sigma. \tag{3.4.2}$$

Now by Gauss-Bonnet theorem, we have

$$\begin{aligned} \frac{d}{dt} m_H^{AH}(\Sigma_t) &= \frac{1}{4\pi}\int_\Sigma \frac{3t^2+1}{2}(1-u^{-2}) + \frac{t(1+t^2)}{2}2u^{-3}\frac{\partial u}{\partial t} d\sigma \\ &= \frac{1}{4\pi}\int_\Sigma \frac{3t^2+1}{2} + \frac{u^{-1}\Delta u}{2} - \frac{R}{4} + \frac{t^2\overline{R}}{4} + \frac{t^2(1+t^2)}{4u^2}|M|^2 d\sigma \\ &= \frac{1}{4\pi}\int_\Sigma \frac{(\overline{R}+6)t^2}{4} + \frac{u^{-1}\Delta u}{2} + \frac{t^2(1+t^2)}{4u^2}|M|^2 d\sigma \\ &= \frac{1}{8\pi}\int_\Sigma \frac{(\overline{R}+6)t^2}{2} + \frac{|\nabla u|^2}{u^2} + \frac{t^2(1+t^2)}{2u^2}|M|^2 d\sigma \geq 0 \end{aligned}$$

given $\overline{R} \geq -6$. Thus the condition $p_0 = m_H^{AH}(\Sigma)$ implies that $\frac{d}{dt} m_H^{AH}(\Sigma_t) = 0$, that is, $\overline{R} = -6, |M| = 0$, and $\nabla u = 0$. It follows from $|M| = 0$ that (Σ, g_1) is isometric to a standard sphere. Since $\nabla u = 0$, N is rotationally symmetric. From the result [82, Theorem 3.3] by Sakovich and Sormani, if $m_H^{AH}(\Sigma) = 0$ then N is isometric to a hyperbolic space or if $m_H^{AH}(\Sigma) = m > 0$ then N is isometric to a Riemannian anti-de Sitter Schwarzschild manifold of mass m. $\square$

3.4.1 Admissible extension for the hyperbolic Bartnik mass

In this subsection, we remark that this construction of asymptotically hyperbolic 3-metrics can be used to study a hyperbolic analogue of the Bartnik mass. Recall from Definition 2.2.18 that the hyperbolic Bartnik mass is defined as the infimum of the total mass of all admissible extensions. It follows from that $H_{\Sigma_t} = \frac{2\sqrt{1+t^2}}{tu} > 0$ that the extension we obtained from the previous section is admissible.

Here, we attempt the rough estimate of the component p_0 for the Ricci flow extension. As in the previous section, we will use the limit of the Hawking mass to estimate p_0.

Suppose that (Σ, g_1) satisfies the condition on Theorem 3.3.1, H is a positive constant, and $\overline{R} \equiv -6$. For convenience, we adopt the following notation from [65]:

$$E(s,t) := \exp\left(- \int_s^t \frac{\tau(|M|^*)^2}{2} d\tau \right)$$

By using the lower bound in (3.2.6), we have

$$\begin{aligned} u^{-2}(t,x) \geq & \frac{1}{t(1+t^2)} \int_1^t \frac{(R_{g(s)} + 6s^2)_*}{2} E(s,t) ds \\ & + \frac{2}{t(1+t^2)} u(1)^{-2} E(1,t). \end{aligned}$$

Hence, it follows that

$$\begin{aligned}
m_H^{AH}(\Sigma_t) &= \frac{1}{4\pi}\int_\Sigma \frac{t(1+t^2)}{2}(1-u^{-2})d\sigma \\
&\leq -\frac{1}{2}\int_1^t \frac{(R_{g(s)}+6s^2)_*}{2}E(s,t)ds - u^{-2}(1)E(1,t) + \frac{t(1+t^2)}{2} \\
&\leq -\frac{1}{2}\int_1^t \frac{(R_{g(s)})_*}{2}E(s,t)ds + m_H^{AH}(\Sigma_1)E(1,t) - \frac{t^3+1}{2}E(1,t) + \frac{t(1+t^2)}{2} \\
&= \frac{1}{2}\int_1^t 1 - \frac{(R_{g(s)})_*}{2}E(s,t)ds + m_H^{AH}(\Sigma_1)E(1,t) + \frac{t^3+1}{2}(1-E(1,t)).
\end{aligned}$$

It is proved in [65] that the first term of the last line converges as $t \to \infty$. Unfortunately, the above estimate is too crude in a sense that it approaches infinity as $t \to \infty$ unless the last term $\frac{t^3+1}{2}(1-E(1,t))$ converges. Nevertheless, it might still be possible to get a nontrivial upper bound for the hyperbolic Bartnik mass by improving the estimate. Note that if the initial surface (Σ, g_1) is isometric to the standard unit sphere metric, then

$$p_0(\bar{g}) = \lim_{t\to\infty} m_H^{AH}(\Sigma_t) \leq m_H^{AH}(\Sigma_1).$$

Thus, by Theorem 3.1.2, the extension $(N, \bar{g})$ must be isometric to a rotationally symmetric region either in hyperbolic space or AdS Schwarzschild space.

Chapter 4

Mass rigidity for asymptotically hyperbolic manifolds

4.1 Introduction

As mentioned in Section 2.2.3, the positivity of mass for asymptotically hyperbolic manifolds without the spin assumption has been proved in [81, 30]. However, their proofs involve a contradiction argument, which are not applicable to the equality case. In this chapter, our main goal is the following rigidity statement for the equality case. The technical terms are defined in Section 4.2. We also refer to Definition 4.2.3 for the precise definition of asymptotically hyperbolic manifolds, which includes a technical assumption that $g \in C^{\infty}_{\text{loc}}$. See Remark 4.2.5.

Theorem 4.1.1. *Let $n \geq 3$ and (M, g) be an n-dimensional asymptotically hyperbolic manifold with scalar curvature $R_g \geq -n(n-1)$ and with equality $p_0 = \sqrt{p_1^2 + \cdots + p_n^2}$, where $(p_0, p_1, \ldots, p_n)$ is the mass of g. Suppose the following holds:*

($\star$) *There is an open neighborhood $\mathcal{M}$ of g in the space of asymptotically hyperbolic metrics on M such that the inequality $p_0(\gamma) \geq \sqrt{(p_1(\gamma))^2 + \cdots + (p_n(\gamma))^2}$ holds if $\gamma \in \mathcal{M}$ and the scalar curvature satisfies $R_\gamma = R_g$.*

Then (M, g) is isometric to hyperbolic space.

Using positivity of mass proven in [30], the assumption ($\star$) can be dropped and thus we arrive at the following result.

Theorem 4.1.2. *Let $n \geq 3$ and (M, g) an n-dimensional asymptotically hyperbolic manifold with scalar curvature $R_g \geq -n(n-1)$ and with the equality $p_0 = \sqrt{p_1^2 + \cdots + p_n^2}$. Then (M, g) is isometric to hyperbolic space.*

We outline the proof of Theorem 4.1.1, which is included in Section 4.4. We show that a metric that realizes the mass equality is a minimizer of a functional $\mathcal{F}$, defined by (4.4.5), subject to a scalar curvature constraint. By studying the first variation of this functional, we show that such a metric must be static and, in fact, possess a static potential with certain asymptotics. The desired characterization of hyperbolic space follows from proving a static uniqueness result.

We remark that the approach is motivated by a constrained minimization scheme proposed by R. Bartnik [18] for his quasi-local mass program. The connection between the constrained minimization and mass rigidity was recently employed by D. Lee and the first named author in their proof to the rigidity conjecture of the spacetime positive mass theorem [55].

In our proof of Theorem 4.1.1, it is essential to analyze the scalar curvature map and to derive the following result.

Theorem 4.1.3. *Let (M, g) be an n-dimensional asymptotically hyperbolic manifold. For $k \geq 2$ and $s \in (-1, n)$, the linearized scalar curvature map*

$$L_g : C^{k,\alpha}_{-s}(M) \to C^{k-2,\alpha}_{-s}(M)$$

is surjective. As a consequence, the scalar curvature map is locally surjective at g. Namely, there are constants $\epsilon, C > 0$ such that if $\|\phi - R_g\|_{C^{k-2,\alpha}_{-s}(M)} < \epsilon$, then there is a metric γ with $\|\gamma - g\|_{C^{k,\alpha}_{-s}(M)} \leq C\epsilon$ that realizes the scalar curvature $R_\gamma = R_g + \phi$.

Theorem 4.1.3 is also of independent interest from the perspective of scalar curvature deformation. For example, it produces infinitely many asymptotically hyperbolic metrics with scalar curvature greater than $-n(n-1)$ by perturbation.

We remark that the weighted Hölder space is chosen as our analytical framework because the known results on the positivity of mass require that regularity. It is shown that the Einstein constraint map is surjective among the appropriate weighted *Sobolev* spaces by E. Delay and J. Fougeirol [37]. However, it does not seem to imply Theorem 4.1.3. In fact, our proof relies on a different argument. One difficulty is that the dual space $(C^{k-2,\alpha}_{-s})^*$ is not well-understood. Efforts are made to analyze the kernel of the adjoint operator L^*_g on $(C^{k-2,\alpha}_{-s})^*$ without assuming the kernel elements to decay at infinity. See Section 4.3 and more specifically, Theorem 4.3.5.

Finally, we remark that the proof of Theorem 4.1.3 uses the assumption that an asymptotically hyperbolic manifold is complete without boundary (see Definition 4.2.3). For manifolds with compact boundary, while the same argument still works if one imposes either Dirichlet or Neumann type condition on the metrics, we need the surjectivity to hold for tensors with stronger vanishing condition at the boundary to establish the mass rigidity. In a forthcoming paper, we use a different argument and extend Theorem 4.1.3 for metrics that coincide with g of infinite order at the boundary. It enables us to prove the mass rigidity for asymptotically *locally* hyperbolic manifolds. In that setting, the model spaces that we consider have compact boundary with natural geometric boundary conditions.

4.2 Preliminaries

4.2.1 Weighted Hölder spaces and asymptotically hyperbolic manifolds

Denote by $\mathbb{H}^n$ the n-dimensional hyperbolic space with scalar curvature $-n(n-1)$. As our model for hyperbolic space, we consider the upper-sheet of the hyperboloid in Minkowski space $(\mathbb{R}^{n,1}, -dt^2 + dx_1^2 + \cdots + dx_n^2)$, defined by

$$\mathbb{H}^n = \left\{(x,t) = (x_1, \ldots, x_n, t) \in \mathbb{R}^{n,1} : t = \sqrt{1 + x_1^2 + \cdots + x_n^2}\right\}.$$

The restriction of the Minkowski metric to the upper-sheet hyperboloid is hyperbolic space and can be expressed in spherical coordinates as

$$b = \frac{1}{1+r^2}\, dr^2 + r^2 h, \tag{4.2.1}$$

where $r = |x| := \sqrt{x_1^2 + \cdots + x_n^2}$ is the radial coordinate, and h is the standard metric on the round unit $(n-1)$-sphere. We refer $(\mathbb{R}^n, b)$ as the *hyperboloid model* of hyperbolic space.

The volume form of b is $d\mu_b = \frac{r^{n-1}}{\sqrt{1+r^2}}\, dr\, d\omega$, where $d\omega$ is the volume form on the round unit $(n-1)$-sphere. By the co-area formula, it is direct to see that the induced volume form on $S_r = \{|x| = r\}$ of the hyperbolic metric b is the same as the standard volume form on the round $(n-1)$-sphere of radius r.

Let B be an open ball in $\mathbb{R}^n$ centered at the origin. Denote $\mathbb{H}^n \setminus B = (\mathbb{R}^n \setminus B, b)$. We fix an orthonormal frame $\{e_1, \dots, e_n\}$ on $\mathbb{H}^n \setminus B$ defined by, with respect to the spherical coordinates $\{r, \theta_1, \dots, \theta_{n-1}\}$,

$$e_1 = \sqrt{1+r^2}\frac{\partial}{\partial r}, \quad e_2 = r^{-1}\frac{\partial}{\partial \theta_1}, \quad \dots, \quad e_n = (r\sin(\theta_1)\dots\sin(\theta_{n-2}))^{-1}\frac{\partial}{\partial \theta_{n-1}}. \quad (4.2.2)$$

Definition 4.2.1. For $k = 0, 1, 2, \dots$, $\alpha \in (0,1)$, and $q \in \mathbb{R}$, we define the *weighted Hölder spaces* $C^{k,\alpha}_{-q}(\mathbb{H}^n \setminus B)$ as the collection of $C^{k,\alpha}_{\text{loc}}(\mathbb{H}^n \setminus B)$ functions f on $\mathbb{H}^n \setminus B$ that satisfy

$$\|f\|_{C^{k,\alpha}_{-q}(\mathbb{H}^n\setminus B)} := \sum_{\ell=0,1,\dots,k} \sup_{x\in\mathbb{H}^n\setminus B} |x|^q |\mathring{\nabla}^\ell f(x)|_b + \sup_{x\in\mathbb{H}^n\setminus B} |x|^q [\mathring{\nabla}^k f]_{\alpha;B_1(x)} < \infty,$$

where $\mathring{\nabla}$ is the covariant derivative with respect to b,

$$[\mathring{\nabla}^k f]_{\alpha;B_1(x)} = \sup_{1\le i_1,\dots,i_k\le n} \sup_{y\neq z\in B_1(x)} \frac{|e_{i_1}\cdots e_{i_k}(f)(y) - e_{i_1}\cdots e_{ik}(f)(z)|}{(d_b(y,z))^\alpha},$$

and $B_1(x)$ is the unit ball centered at x intersecting with $\mathbb{H}^n \setminus B$. We extend the definition to tensors of arbitrary types: a tensor $h \in C^{k,\alpha}_{-q}(\mathbb{H}^n \setminus B)$ if and only if each tensor component with respect to the orthonormal frame lies in $C^{k,\alpha}_{-q}(\mathbb{H}^n \setminus B)$.

Let M be a smooth manifold covered by an atlas that consists of a non-compact chart $\Phi : M \setminus K \cong \mathbb{H}^n \setminus B$ and finitely many compact charts. We define the *weighted Hölder norm* $\|f\|_{C^{k,\alpha}_{-q}(M)}$ (for a function or tensor) to be the sum of the weighted norm $\|\Phi_* f\|_{C^{k,\alpha}_{-q}(\mathbb{H}^n\setminus B)}$ and the usual $C^{k,\alpha}$ norms on compact charts. Denote by $C^{k,\alpha}_{-q}(M)$

the completion of $C_c^{k,\alpha}(M)$ functions with respect to the weighted Hölder norm. We often suppress M when the context is clear.

Notation. Throughout the paper, we use the notation $O^{k,\alpha}(r^{-q})$ to denote a function or tensor, that belongs to the corresponding weighted space $C^{k,\alpha}_{-q}(M)$. We simply write $O(r^{-q})$ in place of $O^0(r^{-q})$.

We collect the following basic facts about the weighted Hölder spaces.

Lemma 4.2.2. *Let $k = 0, 1, 2, \dots,$ $\alpha \in (0, 1)$, and $q, s \in \mathbb{R}$.*

1. $|x|^{-q} \in C^{k,\alpha}_{-q}(M \setminus K)$.
2. $f \in C^{k,\alpha}_{-q}(M \setminus K)$ *if and only if* $|x|^s f \in C^{k,\alpha}_{s-q}(M \setminus K)$.
3. *If* $f \in C^{k,\alpha}_{-s}, g \in C^{k,\alpha}_{-q}$, *then* $fg \in C^{k,\alpha}_{-s-q}$ *and there is a constant* $C > 0$ *such that*
$$\|fg\|_{C^{k,\alpha}_{-s-q}} \leq C\|f\|_{C^{k,\alpha}_{-s}}\|g\|_{C^{k,\alpha}_{-q}}.$$
4. *The inclusion* $C^{k,\alpha}_{-s}(M) \subset C^{k,\beta}_{-s+\epsilon}(M)$ *is compact for any* $\epsilon > 0$ *and* $\beta < \alpha$.

Proof. The first three statements follow directly from the definition. The last statement is standard compact embedding for weighted norms. While similar statements can be found in [63, Lemma 3.6] and [36, Proposition 8], we include the proof for completeness as the weighted norms are defined with slight variations in the literature. Let $\{u_i\}$ be a sequence of functions in $C^{k,\alpha}_{-s}$ with $\|u_i\|_{C^{k,\alpha}_{-s}} = 1$. Applying Arzela-Ascoli on a sequence of compact sets that exhaust M and by a diagonal sequence argument, there is a subsequence of $\{u_i\}$ (which we still denote by $\{u_i\}$, without loss of generality) and a function $u \in C^{k,\alpha}_{\text{loc}}$ so that u_i converges to u locally uniformly in $C^{k,\beta}$. That is, for $\epsilon > 0$ and a compact subset Ω, there is an integer I (depending on ϵ and

Ω) such that $\|u - u_i\|_{C^{k,\beta}(\Omega)} < \epsilon$ for all $i \geq I$. In fact, $u \in C^{k,\beta}_{-s}$ because, for each compact set Ω,

$$\|u\|_{C^{k,\beta}_{-s}(\Omega)} = \lim_{i\to\infty} \|u_i\|_{C^{k,\beta}_{-s}(\Omega)} \leq 1.$$

Let B_r be the coordinate ball of radius r. Using $\|u_i - u\|_{C^{k,\beta}_{-s+\epsilon}(M\setminus B_r)} \leq r^{-\epsilon}(\|u_i\|_{C^{k,\beta}_{-s}(M)} + \|u\|_{C^{k,\beta}_{-s}(M)})$, we have that u_i converges to u in $C^{k,\beta}_{-s+\epsilon}(M)$. □

Definition 4.2.3. Let $n \geq 3$ and $q \in \left(\frac{n}{2}, n\right)$. Let M be an n-dimensional, connected, complete manifold without boundary endowed with a Riemannian metric $g \in C^\infty_{\text{loc}}$. We say that (M, g) is *asymptotically hyperbolic* (of order q) if the following holds:

1. There exists a diffeomorphism $M\backslash K \cong \mathbb{H}^n\backslash B$ for some compact subset $K \subset M$. We call the induced coordinate chart as *the chart at infinity.*

2. With respect to the chart at infinity, $g - b \in C^{2,\alpha}_{-q}(M \setminus K)$.

3. The scalar curvature satisfies $R_g + n(n-1) \in C^{0,\alpha}_{-n-\epsilon}(M)$ for some $\epsilon > 0$.

Remark 4.2.4. By direct computation, the assumption (2) implies that the Ricci curvature of g satisfies $\text{Ric}_g = -(n-1)g + O^{0,\alpha}(r^{-q})$.

Remark 4.2.5. Note the assumption $g \in C^\infty_{\text{loc}}$. We add this technical assumption to employ elliptic interior regularity for distribution solutions. Namely, if a distribution solution u weakly solves $a_{ij}\partial^2_{ij} + b_i\partial_i u + cu = f$ and if $f \in C^{k-2,\alpha}_{\text{loc}}$, then $u \in C^{k,\alpha}_{\text{loc}}$, provided that the coefficients a_{ij}, b_i, c are locally smooth. This elliptic regularity is only used in the proofs of Theorem 4.1.3 and Theorem 4.4.3. If the regularity statement holds for coefficients that are just Hölder regular, then that technical assumption may be dropped.

To compare Definition 4.2.3 with various notions of asymptotically hyperbolic manifolds in the existing literature, we express the assumption (2) in Definition 4.2.3 in coordinates. It appears that our asymptotic assumption is more general than (2.2.1).

Lemma 4.2.6. *A* $(0,2)$*-tensor* g *satisfies* $g - b \in C^{2,\alpha}_{-q}(M \setminus K)$ *if and only if the tensor components have the following asymptotics in spherical coordinates:*

$$g = \left(\frac{1}{1+r^2} + O^{2,\alpha}(r^{-2-q}) \right) dr^2 + O^{2,\alpha}(r^{-q}) dr d\theta_j + (r^2 h_{j\ell} + O^{2,\alpha}(r^{2-q})) d\theta_j d\theta_\ell \qquad \text{as } r \to \infty.$$

By changing the coordinate $r = \frac{1}{\sinh \rho}$*, we can express* g *as*

$$g = \frac{1}{(\sinh \rho)^2} \left[(1 + O(\rho^q)) d\rho^2 + O(\rho^q) d\rho d\theta_i + (h_{j\ell} + O(\rho^q)) d\theta_j d\theta_\ell \right] \qquad \text{as } \rho \to 0,$$

where we slightly abuse the O*-notation in the previous expression and write* $u = O(\rho^q)$ *if* $\frac{u}{\rho^q}$ *is bounded as* $\rho \to 0$.

Proof. Via the diffeomorphism on the chart at infinity, it suffices to prove the result for tensors defined on $\mathbb{H}^n \setminus B$. Express g in the spherical coordinates as follows:

$$g = A\, dr^2 + 2 \sum_j B_j\, dr\, d\theta_j + \sum_{j,\ell} C_{j\ell}\, d\theta_j\, d\theta_\ell. \tag{4.2.3}$$

By definition, $\kappa := g - b$ belongs to $C^{k,\alpha}_{-q}(\mathbb{H}^n \setminus B)$ if and only if each tensor component $\kappa(e_i, e_j) \in C^{k,\alpha}_{-q}(\mathbb{H}^n \setminus B)$. By (4.2.2) and (4.2.3), we have

$$\kappa(e_1, e_1) = (1 + r^2) A, \quad \kappa(e_1, e_{j+1}) = \sqrt{1 + r^2}\, r^{-1} B_j, \quad \text{and} \quad \kappa(e_{j+1}, e_{\ell+1}) = r^{-2} C_{j\ell}.$$

Thus, $\kappa \in C^{k,\alpha}_{-q}(\mathbb{H}^n \setminus B)$ if and only if the tensor components satisfy

$$A \in C^{k,\alpha}_{-2-q}, \quad B_j \in C^{k,\alpha}_{-q}, \quad \text{and} \quad C_{j\ell} \in C^{k,\alpha}_{2-q}.$$

□

4.2.2 Wang-Chruściel-Herzlich mass, and an alternative definition

X. Wang [99] defined the mass for asymptotically hyperbolic manifolds that are conformally compact. For the class of asymptotically hyperbolic manifolds adopted in the current paper, we use the following more general definition of P. Chruściel and M. Herzlich [31].

Definition 4.2.7. Let (M, g) be an asymptotically hyperbolic manifold. Given a function $V \in C^1(M \setminus K)$, we define the mass integral

$$H_g(V) = \lim_{r\to\infty} \int_{S_r} \Big[V\big(\mathring{\operatorname{div}}\, h - d(\mathring{\operatorname{tr}}\, h)\big)(\nu_0) + (\mathring{\operatorname{tr}}\, h) dV(\nu_0) - h(\mathring{\nabla} V, \nu_0)\Big] \, d\sigma_b, \quad (4.2.4)$$

where $h = g - b$, ν_0 is the outward unit normal vector to $S_r = \{|x| = r\}$, and $\mathring{\operatorname{div}}$, $\mathring{\operatorname{tr}}$, $\mathring{\nabla}$, are all with respect to b. The volume form $d\sigma_b$ is the restriction of the volume form of b on S_r. The *mass of Wang-Chruściel-Herzlich* is defined by

$$p_0(g) = H_g(\sqrt{1 + r^2}) \quad \text{and} \quad p_i(g) = H_g(x_i) \text{ for } i = 1, \dots, n.$$

We may omit g and simply write the mass $(p_0, p_1, \dots, p_n)$ when the context is clear.

Remark 4.2.8. In the above definition, we can replace the functions $\sqrt{1+r^2}$ and x_i by $\sqrt{1+r^2} + O^2(r^{1-q})$ and $x_i + O^2(r^{1-q})$ respectively, since the differences in the corresponding mass integrals go to zero in the limit. For the same reason, we may also replace $\nu_0, \mathring{\operatorname{div}}, \mathring{\operatorname{tr}}, \mathring{\nabla}$, and $d\sigma_b$ in (4.2.4) by the corresponding objects with respect to another asymptotically hyperbolic metric and still obtain the same limit.

Remark 4.2.9. The quantity $(p_0, p_1, \ldots, p_n)$ is a geometric invariant among an appropriate class of charts at infinity (see [31], also [51]). We denote the functions appearing in the above definition by

$$V_0 = \sqrt{1+r^2} \quad \text{and} \quad V_i = x_i \quad \text{for } i = 1, \ldots, n.$$

In $\mathbb{H}^n$, these functions satisfy the differential equation $\mathring{\nabla}^2 V_i = V_i b$, for $i = 0, 1, \ldots, n$. They are the so-called *static potentials*. We will discuss general properties of static potentials in an asymptotically hyperbolic manifold in Section 4.3.

We recall an equivalent definition of mass, which will be used in the proof of the main theorem. This formula is known to the experts and is stated in [52, Theorem 3.3], whose proof is similar to the analogous formula for asymptotically flat manifolds.

Proposition 4.2.10. *Let (M, g) be an asymptotically hyperbolic manifold. If $V \in C^2(M \setminus K)$ satisfies*

$$\mathring{\nabla}^2 V = V b,$$

then

$$\lim_{r\to\infty} \int_{S_r} (\operatorname{Ric}_g + (n-1)g)(\mathring{\nabla} V, \nu_0)\, d\sigma_b = -\tfrac{n-2}{2} H(V),$$

provided the quantity on either side of the equation converges.

4.2.3 Operators asymptotic to $\Delta - n$

To analyze the scalar curvature operator on an asymptotically hyperbolic manifold, the following class of operators naturally appears.

Definition 4.2.11. Let (M, g) be asymptotically hyperbolic. Let Δ be the Laplace-Beltrami operator of g, which is the trace of the covariant Hessian. For $k \geq 2$, we say that the differential operator $T : C^{k,\alpha}_{-s} \to C^{k-2,\alpha}_{-s}$ defined by $Tu = \Delta u + \xi \cdot \nabla u + \eta u$ is *asymptotic to* $\Delta - n$ if there is a number $\epsilon > 0$ such that the vector field $\xi \in C^{k-2,\alpha}_{-\epsilon}$ and the function $\eta + n \in C^{k-2,\alpha}_{-\epsilon}$.

We recall the following classical result on isomorphism.

Lemma 4.2.12. *Let (M, g) be an n-dimensional asymptotically hyperbolic manifold and $s \in (-1, n)$. The operator $T_0 : C^{k,\alpha}_{-s}(M) \to C^{k-2,\alpha}_{-s}(M)$ defined by $T_0 u = \Delta u - nu$ is an isomorphism.*

Proof. The isomorphism result is proven for asymptotically hyperbolic manifolds that are conformally compact in [62, Proposition 3.3] (based on the argument of [47, Section 3]; see also [63] for a general class of operators.) It is clear that the proof can be adapted for our class of asymptotically hyperbolic manifolds. □

We also need the following standard Fredholm property for our class of operators. Note similar statements under greater generality can be found in [63], but we include a proof more specific to our setting for completeness.

Proposition 4.2.13. *Let (M, g) be an n-dimensional asymptotically hyperbolic manifold and $s \in (-1, n)$. Let $T : C^{k,\alpha}_{-s} \to C^{k-2,\alpha}_{-s}$ be asymptotic to $\Delta - n$. Then T is Fredholm.*

Proof. We write $Tu = T_0 u + \xi \cdot \nabla u + (\eta + n)u$. Note T_0 is an isomorphism by Lemma 4.2.12, and hence Fredholm. To show that T is Fredholm, it suffices to show that the map $T - T_0 : C^{k,\alpha}_{-s} \to C^{k-2,\alpha}_{-s}$ is compact.

Let $\{u_i\}$ be a sequence of functions in $C^{k,\alpha}_{-s}$ with $\|u_i\|_{C^{k,\alpha}_{-s}} = 1$. We show that $\{(T - T_0)u_i\}$ has a convergent subsequence in $C^{k-2,\alpha}_{-s}$. By Lemma 4.2.2, $C^{k,\alpha}_{-s} \subset C^k_{-s+\epsilon}$ is compact for $\epsilon > 0$, so there is a subsequence (still denoted by $\{u_i\}$ without loss of generality) that converges to u in $C^k_{-s+\epsilon}$. Observe the sequence $\{(T - T_0)u_i\}$ converges in $C^{k-2,\alpha}_{-s}$ because

$$\begin{aligned}
\|(T - T_0)(u_i - u)\|_{C^{k-2,\alpha}_{-s}} &= \|\xi \cdot \nabla(u_i - u) + (\eta + n)(u_i - u)\|_{C^{k-2,\alpha}_{-s}} \\
&\leq C \Big[\|\xi\|_{C^{k-2,\alpha}_{-\epsilon}} \|\nabla(u_i - u)\|_{C^{k-2,\alpha}_{-s+\epsilon}} \\
&\qquad\qquad + \|\eta + n\|_{C^{k-2,\alpha}_{-\epsilon}} \|u_i - u\|_{C^{k-2,\alpha}_{-s+\epsilon}} \Big] \\
&\leq C \|u_i - u\|_{C^{k-1,\alpha}_{-s+\epsilon}} \\
&\leq C \|u_i - u\|_{C^{k}_{-s+\epsilon}} \to 0 \qquad \text{as } i \to \infty.
\end{aligned}$$

□

4.3 Surjectivity of the linearized scalar curvature map

Let (Ω, g) be a Riemannian manifold. The linearization L_g of the scalar curvature map at g acts on a symmetric $(0,2)$-tensor $h \in C^2_{\text{loc}}$ by the formula

$$L_g h = -\Delta(\operatorname{tr} h) + \operatorname{div}\operatorname{div} h - h \cdot \operatorname{Ric}_g, \tag{4.3.1}$$

and the formal L^2-adjoint operator L_g^* is given by, for a function $V \in C^2_{\mathrm{loc}}$,

$$L_g^* V = -(\Delta V) g + \nabla^2 V - V \operatorname{Ric}_g. \tag{4.3.2}$$

Here div, tr, $\cdot$, Δ, and ∇ are all taken with respect to g.

We say that (Ω, g) is *static* if it admits a function V, not identically zero, that satisfies the static equation

$$L_g^* V = 0. \tag{4.3.3}$$

We call a solution V to this equation a *static potential*. Equation (4.3.3) is equivalent to the following equation:

$$\nabla^2 V = \Big(\operatorname{Ric}_g - \tfrac{1}{n-1} R_g \, g \Big) V.$$

Example 4.3.1. It is well-known that a static manifold has constant scalar curvature on each connected component [39], so a static asymptotically hyperbolic manifold (which is assumed to be connected in Definition 4.2.3) must have constant scalar curvature $-n(n-1)$. Thus, (4.3.3) implies

$$\begin{aligned} \nabla^2 V &= (\operatorname{Ric}_g + ng) V \\ \Delta V &= nV. \end{aligned} \tag{4.3.4}$$

The prototype of a static asymptotically hyperbolic manifold is hyperbolic space. Recall in Remark 4.2.9, the space of static potentials is an $(n+1)$-dimensional real vector space spanned by the functions $\sqrt{1+r^2}, x_1, \ldots, x_n$ with respect to the coordinates of the hyperboloid model. They come from the restriction of the Minkowski coordinate functions $t, x_1, \dots, x_n$ to the hyperboloid.

The goal of this section is to analyze the growth rate of V solving $L_g^* V = \tau$ on an asymptotically hyperbolic manifold (M, g) where $\tau \in C^0_{1-q}$. Specifically, we show in Theorem 4.3.5 below that such V must either grow linearly in a cone region or go to zero at infinity. As an application (the case $\tau = 0$), at the end of this section we prove Theorem 4.1.3 that L_g is surjective between the appropriate weighted Hölder spaces. We also use Theorem 4.3.5 (the case $\tau \neq 0$) in the proof of Theorem 4.4.3 in the next section.

We remark that it is possible to obtain more detailed asymptotics of V (at least for the case $\tau = 0$) as discussed in [29, Remark A.3] by their analysis. Here we establish elementary properties for a class of inhomogeneous, second-order linear ODEs that suffice for our purpose.

We analyze the asymptotic behavior of a static potential, by studying the static equation along geodesic rays. Note $\nabla^2 V = (\mathrm{Ric}_g + ng)V = \left(g + O^{0,\alpha}(r^{-q})\right) V$ by the asymptotically hyperbolic assumption. The corresponding equation along a geodesic ray is asymptotic to $u'' = u$. We prove in the next three technical lemmas that the solutions to a large class of ODEs share similar properties as the solutions to $u'' = u$, which are generated by e^t, e^{-t}.

Lemma 4.3.2. *Let $P(t), Q(t) \in C^{0,\alpha}([0,\infty))$ and $Q > 0$. Consider the ODE given by*

$$u'' = Pu' + Qu. \tag{4.3.5}$$

Then the following holds:

1. *A solution u has at most one zero, unless u is identically zero.*

2. *If u and v are two solutions satisfying the initial condition $u(0) \geq v(0)$ and $u'(0) \geq v'(0)$, then $u(t) > v(t)$ and $u'(t) > v'(t)$ for all $t > 0$, unless u is identical to v.*

3. *There is a solution u with $u(t) > 0$ and $u'(t) < 0$, for all t.*

Proof. Let $K(t) = \exp\big(-\int_0^t P(s)\,ds\big) > 0$. Then

$$(Ku')' = KQu. \tag{4.3.6}$$

To see (1), suppose that u is not identically zero and, to give a contradiction, that u has two or more zeros. Let $t_1 < t_2$ be two adjacent zeros. We may without loss of generality assume that $u > 0$ on (t_1, t_2). This implies that $u'(t_1) \geq 0$ and $u'(t_2) \leq 0$. In fact, both inequalities are strict; otherwise u is identically zero by uniqueness of solutions. However, this contradicts the fact that Ku' is increasing on $[t_1, t_2]$ by (4.3.6). For (2), by linearity it suffices to show that if u is a solution satisfying the initial condition $u(0) \geq 0$ and $u'(0) \geq 0$, then $u(t) > 0$ and $u'(t) > 0$ for all $t > 0$, unless u is identically zero. The desired statement in (2) follows from (4.3.6) and by observing that if $u \geq 0$ then Ku' is increasing.

We now prove (3) by constructing a compact family of solutions. For an integer $j > 0$, let u_j be the solution that satisfies $u_j(0) = 1$ and $u_j(j) = 0$. By (1) and (2), we have $0 \leq u_j < u_{j+1} < u_{j+2} < \cdots < 1$ and $u_j' < u_{j+1}' < u_{j+2}' < \cdots < 0$ for $t \in (0, j]$. Using (4.3.5) to bound the higher derivatives, we see that u_j is locally uniformly bounded in $C^{2,\alpha}$. By Arzela-Ascoli, a subsequence locally uniformly converges to a solution u in $C^2([0,\infty))$ that satisfies $u(0) = 1$ and $0 \leq u(t) \leq 1, u' \leq 0$ for all t. It is straightforward to verify that the inequalities are strict: $u(t) > 0$ and $u'(t) < 0$ for all t.

□

Lemma 4.3.3. *Let $P(t), Q(t) \in C^{0,\alpha}([0,\infty))$. Suppose $1 + Q > 0$ and that there are constants $d, C_0 > 0$ such that $|P(t)|, |Q(t)| \le C_0 e^{-dt}$. Then there are two linearly independent solutions u_1 and u_2 to the homogeneous equation*

$$u'' = Pu' + (1+Q)u,$$

and u_1, u_2 satisfy the following: there is a constant $C > 0$ such that, for all t,

$$\begin{aligned} C^{-1}e^{t} \le u_1(t) \le Ce^{t}, \qquad & C^{-1}e^{t} \le u_1'(t) \le Ce^{t}, \\ C^{-1}e^{-t} \le u_2(t) \le Ce^{-t}, \qquad & C^{-1}e^{-t} \le -u_2'(t) \le Ce^{-t}. \end{aligned} \tag{4.3.7}$$

Proof. Let u_1 be a solution with the initial condition $u_1(0) = 1$ and $u_1'(0) > 0$. By (2) in Lemma 4.3.2, we have $u_1 > 0$ and $u_1' > 0$ for all t. Let $w(t) = u_1(t) + u_1'(t)$. Then $w > 0$ satisfies

$$w' = (1+P)u_1' + (1+Q)u_1.$$

This implies the following differential inequality for w:

$$(1 - |P| - |Q|)w \le w' \le (1 + |P| + |Q|)w.$$

Integrating the inequality gives

$$w(0)\exp\left(\int_0^t (1-|P(s)|-|Q(s)|)\,ds\right) \le w(t) \le w(0)\exp\left(\int_0^t (1+|P(s)|+|Q(s)|)\,ds\right).$$

That is, there is a constant $C_1 > 0$ (depending only on $w(0)$, $\|P\|_{L^1}$, and $\|Q\|_{L^1}$) such that

$$C_1^{-1}e^t \le u_1(t) + u_1'(t) \le C_1 e^t. \tag{4.3.8}$$

This gives the upper bound for u_1, u_1' in (4.3.7). To derive the lower bound for u_1, u_1', we set $z(t) = u_1(t) - u_1'(t)$. Then $z' = -z - Pu_1' - Qu_1$ and $|z' + z| \le 2C_0C_1e^{(1-d)t}$. Solving the differential inequality gives $|z| \le C_2(e^{(1-d)t} + e^{-t} + te^{-t})$ for some constant $C_2 > 0$. For t sufficiently large, we derive $|u_1(t) - u_1'(t)| \le \frac{1}{2}C_1^{-1}e^t$. Together with (4.3.8), we obtain the desired estimate (4.3.7) for u_1, u_1'.

By (3) of Lemma 4.3.2, there is a solution u_2 so that $u_2(t) > 0$ and $u_2'(t) < 0$ for all t. Set $h(t) = u_2(t) - u_2'(t)$. Then $h > 0$ satisfies

$$h' = (1-P)u_2' - (1+Q)u_2,$$

and hence $(-1-|Q|-|P|)h \le h' \le (-1+|Q|+|P|)h$. Just as computing above, we have $C^{-1}e^{-t} \le u_2(t) - u_2'(t) \le Ce^{-t}$, which gives the upper bound for u_2, u_2' in (4.3.7). Similarly, by estimating the differential inequality for $u_2 + u_2'$, we derive the desired lower bound.

Lastly, we note that the two solutions u_1, u_2 are linearly independent because their

Wronskian is not zero and furthermore, by (4.3.7),

$$\det \begin{bmatrix} u_1 & u_2 \\ u_1' & u_2' \end{bmatrix} = u_1 u_2' - u_2 u_1' \leq -2C^{-2} \qquad \text{for all } t. \tag{4.3.9}$$

□

Lemma 4.3.4. *Let $P(t), Q(t) \in C^{0,\alpha}([0,\infty))$ and $f(t) \in C^0([0,\infty))$. Suppose $1+Q > 0$ and that there are constants $d, C_0 > 0$ such that $|P(t)|, |Q(t)|, |f(t)| \leq C_0 e^{-dt}$. Let u solve*

$$u'' = Pu' + (1+Q)u + f. \tag{4.3.10}$$

Then there are constants $C > 0$ and c_1, c_2 such that, for all $t \geq a$,

$$|u(t) - \big(c_1 u_1(t) + c_2 u_2(t)\big)| \leq \begin{cases} Ce^{-dt} & \text{for } d \neq 1 \\ Cte^{-t} & \text{for } d = 1 \end{cases}. \tag{4.3.11}$$

Proof. Let u_p be a particular solution to (4.3.10). Notice that $u - u_p$ satisfies the homogeneous equation, and hence is a linear combination of u_1 and u_2, where $\{u_1, u_2\}$ is the set of fundamental solutions from Lemma 4.3.3. It suffices to show that the estimate (4.3.11) holds for u_p.

By the method of variation of parameters, we can choose u_p to be

$$u_p = \alpha_1 u_1 + \alpha_2 u_2,$$

where the functions α_1, α_2 are defined by

$$\alpha_1(t) = -\int_0^t \frac{u_2(s)f(s)}{u_1(s)u_2'(s) - u_2(s)u_1'(s)}\, ds$$
$$\alpha_2(t) = \int_0^t \frac{u_1(s)f(s)}{u_1(s)u_2'(s) - u_2(s)u_1'(s)}\, ds.$$

If $d > 1$, using (4.3.7), (4.3.9), and the assumption on f, we see that both integrals converge as $t \to \infty$. Let $A_i = \lim_{t\to\infty} \alpha_i(t)$ for $i = 1, 2$. There is a constant $C > 0$ so that

$$|\alpha_1(t) - A_1| \leq \int_t^\infty \left| \frac{u_2(s)f(s)}{u_1(s)u_2'(s) - u_2(s)u_1'(s)} \right| ds \leq C \int_t^\infty e^{-s}|f(s)|\, ds \leq Ce^{-(d+1)t}$$
$$|\alpha_2(t) - A_2| \leq \int_t^\infty \left| \frac{u_1(s)f(s)}{u_1(s)u_2'(s) - u_2(s)u_1'(s)} \right| ds \leq C \int_t^\infty e^{s}|f(s)|\, ds \leq Ce^{-(d-1)t}.$$

It implies that

$$|u_p - A_1u_1 - A_2u_2| \leq |\alpha_1 - A_1|u_1 + |\alpha_2 - A_2|u_2 \leq Ce^{-dt}.$$

If $0 < d \leq 1$, then $\lim_{t\to\infty} \alpha_2(t)$ may not converge. Nevertheless, there is a constant $C > 0$ such that $|\alpha_2| \leq Ce^{(1-d)t}$ if $d \neq 1$ and $|\alpha_2| \leq Ct$ if $d = 1$. Together with the above estimate for α_1, we obtain

$$|u_p - A_1u_1| \leq |\alpha_1 - A_1|u_1 + |\alpha_2|u_2 \leq \begin{cases} Ce^{-dt} & \text{for } d \neq 1 \\ Cte^{-t} & \text{for } d = 1 \end{cases}.$$

□

We proceed to discuss the asymptotics of a function that solves the static equation up to an error term. We define a *cone* U as an unbounded open subset in $M \setminus K$ that consists of points in spherical coordinates such that, for some $r_0 > 0$ and a non-empty open subset Θ in the domain of the angular coordinates on S^{n-1}:

$$U = \{(r, \theta_1, \cdots, \theta_{n-1}) \in M \setminus K : r > r_0 \text{ and } (\theta_1, \ldots, \theta_{n-1}) \in \Theta\}.$$

Theorem 4.3.5. *Let (M, g) be an asymptotically hyperbolic manifold, and $V \in C^2_{\text{loc}}(M \setminus K)$ satisfy*

$$L_g^* V = \tau \tag{4.3.12}$$

where $\tau \in C^0_{1-q}(M \setminus K)$ is a symmetric $(0, 2)$-tensor. Then V satisfies precisely one of the following:

1. *There is a cone $U \subset M \setminus K$ and a constant $C > 0$ such that*

$$C^{-1}|x| \leq |V(x)| \leq C|x| \qquad \text{for all } x \in U.$$

2. *There are constants $C > 0$ and $0 < d \leq 1$ such that*

$$|V(x)| \leq C|x|^{-d} \qquad \text{for all } x \in M \setminus K.$$

Proof. Let B_r be a large coordinate ball in M that contains K. It suffices to prove the theorem on $M \setminus B_r$. Note that any point $x \in M \setminus B_r$ could be reached by a geodesic emanating from ∂B_r with the initial velocity ∂_r. Let $\gamma(t)$, $0 \leq t < \infty$, be the geodesic emanating from a point $p \in \partial B_r$ with $\gamma'(0) = \partial_r$, parametrized by the arc length parameter t, i.e.

$$t = d_g(p, \gamma(t)).$$

With respect to the hyperbolic metric b on $M \setminus B_r$ (pull back by the diffeomorphism that gives the chart at infinity) and letting o be the origin of $\mathbb{H}^n$, we have $d_b(o, \gamma(t)) = \sinh^{-1}(|\gamma(t)|)$ and hence $|d_b(p, \gamma(t)) - \sinh^{-1}(|\gamma(t)|)| \leq d_b(o, p)$ by the triangle inequality, where $|\gamma(t)|$ denotes the radial coordinate of the point $\gamma(t)$. Since the distance in g is comparable to the distance in b by the asymptotically hyperbolic assumption, there is a constant $C > 0$ such that $|t - \sinh^{-1}(|\gamma(t)|)| \leq C$ for all t. Thus, there is a constant $C > 0$ such that

$$C^{-1} e^t \leq |\gamma(t)| \leq C e^t. \tag{4.3.13}$$

By (4.3.12) and the assumption on τ, we have

$$\begin{aligned} \nabla^2 V &= \left(\mathrm{Ric}_g - \tfrac{1}{n-1} R_g \, g\right) V + \tau - \left(\tfrac{1}{n-1} \mathrm{tr}\, \tau\right) g \\ &= \left(g + O^{0,\alpha}(r^{-q})\right) V + O(r^{1-q}). \end{aligned} \tag{4.3.14}$$

Let $u(t) = V \circ \gamma(t)$. The equation (4.3.14) implies that u satisfies the following ODE:

$$\begin{aligned} u'' &= \nabla^2 V(\gamma'(t), \gamma'(t)) + \nabla V(\nabla_{\gamma'(t)} \gamma'(t)) \\ &= \nabla^2 V(\gamma'(t), \gamma'(t)) \\ &= (1 + Q(t)) u + f, \end{aligned}$$

where $|Q(t)| \leq C e^{-qt}$ and $|f(t)| \leq C e^{(1-q)t}$ by (4.3.14) and (4.3.13). By Lemma 4.3.4, there is a constant $C > 0$ and $d \in (0, 1]$ such that V satisfies

1. either $C^{-1}e^t \le |V(\gamma(t))| \le Ce^t$ for all t

2. or $|V(\gamma(t))| \le Ce^{-dt}$ for all t.

If (1) holds for some geodesic γ, by continuous dependence of ODE solutions on the initial conditions, the estimate $C^{-1}|x| \le |V(x)| \le C|x|$ holds in a cone, where we use (4.3.13) to replace e^t with $|\gamma(t)|$ and enlarge the constant C if needed. If (1) does not hold for any geodesic γ, then (2) holds for all $\gamma(t)$, with a uniform constant C by compactness of ∂B_r. Using (4.3.13) and enlarging C if necessary, we have $|V(x)| \le C|x|^{-d}$ for all $x \in M \setminus K$. □

Corollary 4.3.6. *Let (M, g) be an asymptotically hyperbolic manifold, and $V \in C^2_{\mathrm{loc}}$ solve $L_g^* V = 0$ in M. If V is not identically zero, then there is a cone $U \subset M \setminus K$ and a constant $C > 0$ such that V satisfies*

$$C^{-1}|x| \le |V(x)| \le C|x| \qquad \text{for all } x \in U.$$

Proof. Recall $\Delta V = nV$ in (4.3.4). By letting $\tau = 0$ in Theorem 4.3.5, we have that either the desired estimate holds, or there are constants $d, C > 0$ such that $|V(x)| \le C|x|^{-d}$ for all $x \in M \setminus K$. However, the latter case implies that V is identically zero by maximum principle. □

We now prove the main result in this section.

Proof of Theorem 4.1.3. It suffices to show that the linearized scalar curvature map is surjective. Local surjectivity of the scalar curvature map follows from standard functional analysis.

We first show that the range of L_g is closed. Define the operator $T(u) := L_g(ug)$ for functions $u \in C^{k,\alpha}_{-s}(M)$. Then $\frac{1}{1-n}T(u) = \Delta u + \frac{1}{n-1}R_g u$ is asymptotic to $\Delta - n$

and hence Fredholm by Proposition 4.2.13. In particular, the range of T has finite codimension, and so does the range of L_g. It implies that the range of L_g is closed.

To see surjectivity of L_g, we show that the adjoint operator $L_g^* : (C^{k-2,\alpha}_{-s})^* \to (C^{k,\alpha}_{-s})^*$ has a trivial kernel. Let $u \in (C^{k-2,\alpha}_{-s})^*$ weakly solve $L_g^* u = 0$. Note that since C_c^∞ is dense in $C^{k-2,\alpha}_{-s}$, u is, in particular, a distribution. Taking the trace of $L_g^* u = 0$ implies that u weakly solves an elliptic PDE, whose coefficients are locally smooth by the hypothesis $g \in C^\infty_{\text{loc}}$. Applying elliptic regularity for distribution solutions (see, e.g. [2, Theorem 6.33]), we have $u \in C^{k,\alpha}_{\text{loc}}$ with the duality given by

$$u(\phi) = \int_M u\phi \, d\mu_g, \qquad \text{for all } \phi \in C_c^\infty. \tag{4.3.15}$$

Suppose, to give a contradiction, that u is not identically zero. We shall show that the above pairing is not bounded for some $\phi \in C^{k-2,\alpha}_{-s}$. By Corollary 4.3.6, there is a constant $C > 0$ such that $|u(x)| \geq C|x|$ in a nonempty cone $U \subset M \setminus K$. We may without loss of generality assume $u > 0$ and hence $u(x) \geq C|x|$ on U. Let $\phi(x)$ be a non-negative function in $C^{k-2,\alpha}_{-s}$ so that $\phi(x) = |x|^{-s}$ in a smaller cone $U' \subset U \subset M \setminus K$ and $\phi \equiv 0$ outside U. Let $\phi_i \in C_c^\infty(U)$ be a monotone sequence of non-negative functions that converge to ϕ in $C^{k-2,\alpha}_{-s}$ (for example, let $\phi_i = \chi_i \phi$ where χ_i is a monotone sequence of bump functions uniformly bounded in C^∞). Then

$$u(\phi) = \lim_{i\to\infty} u(\phi_i) = \lim_{i\to\infty} \int_M u\phi_i \, d\mu_g = \int_M u\phi \, d\mu_g,$$

where the first equality is from continuity of u as a functional, the second equality is by (4.3.15), and the last equality is by monotone convergence theorem. However, since $s \le n$ and $d\mu_g = \left(\frac{r^{n-1}}{\sqrt{1+r^2}} + O(r^{n-2-q})\right) dr d\omega$, the last integral diverges to infinity:

$$\int_M \phi u \, d\mu_g \ge C \int_{U'} r^{1-s} \, d\mu_g = \infty.$$

$\square$

4.4 Mass minimizer and static uniqueness

Let (M, g) be an n-dimensional asymptotically hyperbolic manifold. Consider the following Banach (affine) space of symmetric $(0, 2)$-tensors:

$$\begin{aligned} \mathcal{B} &= \{g + h : h \in C^{2,\alpha}_{-q}(M) \text{ is a symmetric } (0,2)\text{-tensor}\} \\ \mathcal{M} &\subset \mathcal{B} \text{ is an open neighborhood of } g \text{ containing positive definite tensors.} \end{aligned} \tag{4.4.1}$$

Suppose $f \in C^{2,\alpha}_{\text{loc}}(M)$ satisfies the following asymptotics, for some $a_0, a_1, \dots, a_n \in \mathbb{R}$,

$$f(x) = a_0\sqrt{1+r^2} - (a_1 x_1 + \cdots + a_n x_n) + O^{2,\alpha}(|x|^{1-q}). \tag{4.4.2}$$

By direct computation,

$$\begin{aligned} \nabla^2\sqrt{1+r^2} &= \sqrt{1+r^2}\, g + O^{0,\alpha}(r^{1-q}) \\ \nabla^2 x_i &= x_i\, g + O^{0,\alpha}(r^{1-q}) \quad \text{for } i = 1, \dots, n. \end{aligned} \tag{4.4.3}$$

Therefore, we have

$$L_g^* f = -(\Delta f)g + \nabla^2 f - f\mathrm{Ric}_g = O^{0,\alpha}(r^{1-q}). \tag{4.4.4}$$

We define the corresponding functional $\mathcal{F}$ on $\mathcal{M}$ by

$$\mathcal{F}(\gamma) = a_0 p_0(\gamma) - (a_1 p_1(\gamma) + \cdots + a_n p_n(\gamma)) - \int_M (R(\gamma) + n(n-1))\, f\, d\mu_g \tag{4.4.5}$$

where $R : \mathcal{M} \to C^{0,\alpha}_{-q}$ is the scalar curvature map and recall $(p_0(\gamma), \ldots, p_n(\gamma))$ denotes the mass of γ.

It may not be immediately obvious that $\mathcal{F}(\gamma)$ is finite for $\gamma \in \mathcal{M}$. Since γ is not assumed to satisfy the scalar curvature assumption (3) of Definition 4.2.3, either term in the definition of $\mathcal{F}$ may diverge. In the next lemma, we give an alternative expression for $\mathcal{F}$ and show that $\mathcal{F}$ is well-defined. We also compute its first variation.

Lemma 4.4.1. *Let $f \in C^{2,\alpha}_{\mathrm{loc}}(M)$ satisfy the asymptotics*

$$f(x) = a_0\sqrt{1+r^2} - (a_1 x_1 + \cdots + a_n x_n) + O^{2,\alpha}(|x|^{1-q}).$$

Then the corresponding functional $\mathcal{F} : \mathcal{M} \to \mathbb{R}$ can be expressed as

$$\mathcal{F}(\gamma) = \int_M \left([L_g(\gamma - \mathfrak{b}) - (R(\gamma) + n(n-1))]\, f - (\gamma - \mathfrak{b}) \cdot L_g^* f \right) d\mu_g, \tag{4.4.6}$$

where $\mathfrak{b}$ is any fixed smooth symmetric $(0,2)$-tensor in M that coincides with the hyperbolic metric b in the chart at infinity.

As a consequence, the linearization $D\mathcal{F}|_g : C^{2,\alpha}_{-q} \to \mathbb{R}$ *at* g *is given by*

$$D\mathcal{F}|_g(h) = -\int_M h \cdot L_g^* f \, d\mu_g.$$

Proof. We recall the formulas for L_g and L_g^* in (4.3.1) and (4.3.2) for the following computations.

Let $e = \gamma - \mathfrak{b}$. By Definition 4.2.7 and Remark 4.2.8, we have

$$\begin{aligned}
\mathcal{F}(\gamma) &= \lim_{r\to\infty} \int_{S_r} \big(f \left(\operatorname{div} e - d(\operatorname{tr} e)\right)(\nu) + (\operatorname{tr} e)\, df(\nu) - e(\nabla f, \nu) \big) \, d\sigma_g \\
&\qquad\qquad\qquad\qquad - \int_M \big(R(\gamma) + n(n-1) \big) f \, d\mu_g \\
&= \int_M \operatorname{div} \left[f \left(\operatorname{div} e - d(\operatorname{tr} e)\right) + (\operatorname{tr} e)\, df - e(\nabla f, \cdot) \right] \, d\mu_g - \int_M \big(R(\gamma) + n(n-1) \big) f \, d\mu_g \\
&= \int_M \left[\operatorname{div}\operatorname{div} e - \Delta(\operatorname{tr} e) - R(\gamma) - n(n-1) \right] f \, d\mu_g - \int_M \big(-(\Delta f) g + \nabla^2 f \big) \cdot e \, d\mu_g \\
&= \int_M \left[L_g(e) - \left(R(\gamma) + n(n-1) \right) \right] f \, d\mu_g - \int_M \big(-(\Delta f) g + \nabla^2 f - f \operatorname{Ric}_g \big) \cdot e \, d\mu_g.
\end{aligned}$$

Note (4.4.4) and $R(\gamma) + n(n-1) = L_g(e) + O(r^{-2q})$ by Taylor expansion. Both integrals converge by routine computations. □

So far, we have considered the functional $\mathcal{F}$ defined by an arbitrary function f satisfying the asymptotics (4.4.2). In what follows, we will choose specifically f which is an eigenfunction $\Delta f = nf$.

Lemma 4.4.2 ([78, Lemma 3.3])**.** *Let* (M, g) *be an asymptotically hyperbolic manifold. There are functions* $f_0, f_1, \ldots, f_n \in C^{2,\alpha}_{\mathrm{loc}}(M)$ *satisfying* $\Delta f_0 = nf_0$ *and* $\Delta f_i =$

nf_i for $i = 1, \dots, n$ with the asymptotics

$$f_0(x) = \sqrt{1+r^2} + O^{2,\alpha}(r^{1-q})$$
$$f_i(x) = x_i + O^{2,\alpha}(r^{1-q}).$$

Proof. Taking the trace of equations in (4.4.3) yields

$$\Delta\sqrt{1+r^2} = n\sqrt{1+r^2} + O^{0,\alpha}(r^{1-q})$$
$$\Delta x_i = nx_i + O^{0,\alpha}(r^{1-q}).$$

Note that the operator $\Delta - n : C^{2,\alpha}_{1-q} \to C^{0,\alpha}_{1-q}$ is an isomorphism by Lemma 4.2.12. There is a unique $v \in C^{2,\alpha}_{1-q}$ that solves $\Delta v - nv = -\Delta\sqrt{1+r^2} + n\sqrt{1+r^2}$. We set $f_0 = \sqrt{1+r^2} + v$. Other eigenfunctions f_i are obtained similarly. □

Theorem 4.4.3. *Let (M, g) be an asymptotically hyperbolic manifold with scalar curvature $R_g \geq -n(n-1)$ and with the equality $p_0 = \sqrt{p_1^2 + \cdots + p_n^2}$, where $(p_0, p_1, \dots, p_n)$ is the mass of g. Suppose the following holds:*

($\star$) There is an open neighborhood $\mathcal{M}$ of g in $\mathcal{B}$ such that for any $\gamma \in \mathcal{M}$ with $R(\gamma) = R_g$, the inequality $p_0(\gamma) \geq \sqrt{(p_1(\gamma))^2 + \cdots + (p_n(\gamma))^2}$ holds.

Then (M, g) is static with a static potential $f > 0$ satisfying the asymptotics:

$$f = \begin{cases} p_0\sqrt{1+r^2} - (p_1x_1 + \cdots + p_nx_n) + O^{2,\alpha}(r^{1-q}) & \text{if } p_0 > 0 \\ \sqrt{1+r^2} + O^{2,\alpha}(r^{1-q}) & \text{if } p_0 = 0 \end{cases}. \tag{4.4.7}$$

Proof. **Case 1: $p_0 > 0$.** Let $f_0, f_1, \dots, f_n$ be from Lemma 4.4.2. Define

$$f = p_0 f_0 - (p_1 f_1 + \dots + p_n f_n),$$

where $(p_0, p_1, \dots, p_n)$ is the mass of g. Note $\Delta f = nf$. Since $f > 0$ outside a large compact set, it follows from the maximum principle that f is everywhere positive. We claim that f is a static potential on M.

Consider the functional $\mathcal{F} : \mathcal{M} \to \mathbb{R}$ defined by (4.4.5) corresponding to this particular choice of f with the coefficients $a_k = p_k$ for all $k = 0, 1, \dots, n$. Let $R : \mathcal{M} \to C^{0,\alpha}_{-q}$ be the scalar curvature map that sends γ to the scalar curvature of γ. Define $\mathcal{C}_g = \{\gamma \in \mathcal{M} : R(\gamma) = R_g\}$. By hypothesis ($\star$), for $\gamma \in \mathcal{C}_g$, we have

$$p_0(\gamma) \geq \sqrt{\big(p_1(\gamma)\big)^2 + \dots + \big(p_n(\gamma)\big)^2}.$$

We compute that the functional $\mathcal{F}$ achieves a local minimum at g among the constraint set $\mathcal{C}_g$:

$$\begin{aligned}
\mathcal{F}(\gamma) - \mathcal{F}(g) &= p_0 p_0(\gamma) - \big(p_1 p_1(\gamma) + \dots + p_n p_n(\gamma)\big) \\
&\geq p_0 p_0(\gamma) - \sqrt{p_1^2 + \dots + p_n^2}\sqrt{\big(p_1(\gamma)\big)^2 + \dots + \big(p_n(\gamma)\big)^2} \\
&= p_0 \left(p_0(\gamma) - \sqrt{(p_1(\gamma))^2 + \dots + (p_n(\gamma))^2}\right) \\
&\geq 0
\end{aligned}$$

with equalities realized at $\gamma = g$.

By Theorem 4.1.3, $L_g : C^{2,\alpha}_{-q} \to C^{0,\alpha}_{-q}$ is surjective, so we can apply the method of Lagrange Multipliers (see, for example, [55, Theorem C.1]) to obtain $\lambda \in (C^{0,\alpha}_{-q})^*$

that satisfies

$$D\mathcal{F}|_g(h) = \lambda(L_g(h)) \qquad \text{for all } h \in C^{2,\alpha}_{-q}.$$

We substitute the left-hand side above by the first variation formula in Lemma 4.4.1 and get

$$-\int_M h \cdot L_g^*(f)\, d\mu_g = \lambda(L_g(h)) \qquad \text{for all } h \in C^{2,\alpha}_{-q}. \tag{4.4.8}$$

Considering $h \in C^\infty_c$ in the above identity implies that λ, as a distribution, is a weak solution to $-L_g^* f = L_g^* \lambda$. Taking the trace of the previous equation implies that λ weakly solves an elliptic PDE with locally smooth coefficients, by the hypothesis $g \in C^\infty_{\text{loc}}$. By elliptic interior regularity for distribution solutions (see, for example, [2, Theorem 6.33]), $\lambda \in C^{2,\alpha}_{\text{loc}}(M)$ with the duality given by

$$\lambda(L_g(h)) = \int_M \lambda L_g(h)\, d\mu_g \qquad \text{for } h \in C^\infty_c(M).$$

Together with (4.4.8), λ solves $L_g^*\lambda = -L_g^* f$ in the classical sense.

We recall $L_g^* f \in C^{0,\alpha}_{1-q}$. Applying Theorem 4.3.5 yields that there are numbers $d, C > 0$ such that either $|\lambda(x)| \geq C|x|$ in a nonempty cone $U \subset M \setminus K$, or $|\lambda(x)| \leq C|x|^{-d}$ in $M \setminus K$. Since λ is a bounded functional on $C^{0,\alpha}_{-q}$, the first case does not occur, by the same argument as in the last paragraph in the proof of Theorem 4.1.3. Therefore, we must have $|\lambda(x)| \leq C|x|^{-d}$ in $M \setminus K$; in particular, $\lambda(x) \to 0$ as $|x| \to \infty$. Taking the trace of $L_g^*\lambda = -L_g^* f$ gives that

$$\Delta\lambda - n\lambda = -(\Delta f - nf) = 0.$$

We conclude λ is identically zero by the maximum principle. We conclude that f is a static potential.

Case 2: $p_0 = 0$. We let $f = f_0$ where f_0 is from Lemma 4.4.2. That is, $f = \sqrt{1+r^2} + O^{2,\alpha}(r^{1-q})$ and $\Delta f = nf$. Note $f > 0$ by maximum principle. We will show that f satisfies the static equation. Let $\mathcal{F} : \mathcal{M} \to \mathbb{R}$ be the functional defined by (4.4.5) corresponding to this particular choice of f with $a_0 = 1$ and $a_1 = \cdots = a_n = 0$. Specifically,

$$\mathcal{F}(\gamma) = p_0(\gamma) - \int_M (R(\gamma) + n(n-1))\, f\, d\mu_g.$$

Recall $\mathcal{C}_g$ defined above. Among the constraint $\gamma \in \mathcal{C}_g$, we have $\mathcal{F}(\gamma) - \mathcal{F}(g) = p_0(\gamma) - p_0(g) \geq 0$ by hypothesis $(\star)$ and thus $\mathcal{F}$ attains the minimum at $\gamma = g$. Now, we can apply the method of the Lagrange multipliers and argue that f is a static potential as above.

□

We have shown that a metric g that locally minimizes the functional $\mathcal{F}$ possesses a static potential with specific asymptotics. To conclude the proof of Theorem 4.1.1, we establish static uniqueness and show isometry to hyperbolic space. (In particular, the case $p_0 > 0$ in (4.4.7) cannot happen.)

Lemma 4.4.4. *Let (M, g) be an asymptotically hyperbolic manifold that admits a positive static potential f with the asymptotics* (4.4.7)*. Then on any large coordinate ball B_r, the following identity holds*

$$\int_{B_r} f^{-1} |\nabla^2 f - fg|^2\, d\mu_g = \int_{\partial B_r} \big(\mathrm{Ric}_g + (n-1)g\big)(\nabla f, \nu)\, d\sigma_g$$

where $|\cdot|$ *is the norm taken with respect to* g *and* ν *is the outward unit normal vector on* ∂B_r.

Proof. The following identity is due to X. Wang [101]. Set $S = \mathrm{Ric}_g + (n-1)g$. By the static equation, $S = f^{-1}\nabla^2 f - g$ and S is both trace and divergence free. We compute

$$\begin{aligned} f^{-1}|\nabla^2 f - fg|^2 &= f|S|^2 \\ &= fg(f^{-1}\nabla^2 f, S) \quad (S \text{ is trace-free}) \\ &= g(\nabla^2 f, S) \\ &= \mathrm{div}(S(\nabla f)) \quad (S \text{ is divergence-free}). \end{aligned}$$

The lemma follows by integrating the identity on B_r and applying the divergence theorem. □

We are ready to prove Theorem 4.1.1. We restate the assumption ($\star$) using the precise Banach spaces defined earlier in (4.4.1).

Theorem 4.4.5. *Let* $n \geq 3$ *and* (M, g) *be an* n*-dimensional asymptotically hyperbolic manifold with scalar curvature* $R_g \geq -n(n-1)$ *and with the equality* $p_0 = \sqrt{p_1^2 + \cdots + p_n^2}$, *where* $(p_0, p_1, \ldots, p_n)$ *is the mass of* g. *Suppose the following holds:*

- ($\star$) *There is an open neighborhood* $\mathcal{M}$ *of* g *in* $\mathcal{B}$ *such that any* $\gamma \in \mathcal{M}$ *with* $R(\gamma) = R_g$, *the inequality* $p_0(\gamma) \geq \sqrt{(p_1(\gamma))^2 + \cdots + (p_n(\gamma))^2}$ *holds.*

Then (M, g) *is isometric to hyperbolic space.*

Proof. By Theorem 4.4.3, M admits a positive static potential f of asymptotics (4.4.7). Using Lemma 4.4.4 and Proposition 4.2.10, in either the case $p_0 > 0$ or

$p_0 = 0$, we have the following identity

$$\int_M f^{-1}|\nabla^2 f - fg|^2 \, d\mu_g = \lim_{r\to\infty} \int_{\partial B_r} \Big(\mathrm{Ric}_g + (n-1)g\Big)(\mathring{\nabla} f, \nu_0)\, d\sigma_0 = -\tfrac{n-2}{2} H(f) = 0.$$

This implies $\nabla^2 f = fg$, which characterizes hyperbolic space by an elementary argument, which we present in Proposition 4.4.6 below.

Alternatively, we could use again that f satisfies the static equation by Theorem 4.4.3 to see that g is Einstein with $\mathrm{Ric}_g = -(n-1)g$. Then M is isometric to hyperbolic space by Bishop-Gromov volume comparison. □

Proposition 4.4.6. *Let (M, g) be asymptotically hyperbolic. If there is a non-zero function $f \in C^2_{\mathrm{loc}}(M)$ satisfying $f > -c$ for some real number c and the following equation on M:*

$$\nabla^2 f = fg, \tag{4.4.9}$$

then (M, g) is isometric to hyperbolic space.

Proof. If f has at least one critical point, the result is classical (see [96] and also [60, Theorem C] and [79, Lemma 3.3]).

We now assume that f has no critical point in M, i.e. ∇f is never zero. We compute the first and second covariant derivatives of $\nabla^2 f = fg$ at a point $p \in M$ with respect to a geodesic normal coordinate chart:

$$\begin{aligned} 0 &= f_{;ijk} - f_{;ikj} - R_{kj\ell i} f^\ell = f_k g_{ij} - f_j g_{ik} - R_{kj\ell i} f^\ell \\ 0 &= f(g_{km} g_{ij} - g_{jm} g_{ik}) - R_{kj\ell i;m} f^\ell - R_{kjmi} f, \end{aligned}$$

where $R_{kj\ell i} = g(\nabla_{\partial_k}\nabla_{\partial_j}\partial_\ell - \nabla_{\partial_j}\nabla_{\partial_k}\partial_\ell, \partial_i)$ in our convention. We than obtain the following formulas, for any vector fields W, X, Y, Z,

$$\begin{aligned} R(X, Y, \nabla f, Z) &= g(\nabla f, X)g(Y, Z) - g(\nabla f, Y)g(X, Z) \\ (\nabla_Z R)(X, Y, \nabla f, W) &= -f\Big(R(X, Y, Z, W) - g(X, Z)g(Y, W) + g(Y, Z)g(X, W)\Big). \end{aligned} \tag{4.4.10}$$

Let $\gamma : (-\infty, \infty) \to M$ be the integral curve of $\frac{\nabla f}{|\nabla f|}$ through a point $p \in M$, i.e. $\gamma'(t) = \frac{\nabla f(\gamma(t))}{|\nabla f|(\gamma(t))}$. By direct computation using (4.4.9), we have $\nabla_{\gamma'}\gamma' = 0$ and thus γ is a geodesic parametrized by arc length. We compute

$$\begin{aligned} \frac{d}{dt} f(\gamma(t)) &= g(\nabla f, \gamma'(t)) = |\nabla f(\gamma(t))| > 0 \\ \frac{d^2}{dt^2} f(\gamma(t)) &= \nabla^2 f(\gamma'(t), \gamma'(t)) = f(\gamma(t)). \end{aligned} \tag{4.4.11}$$

Solving the ODE yields that, for $t \in (-\infty, \infty)$,

$$f(\gamma(t)) = C_1 e^t + C_2 e^{-t}, \tag{4.4.12}$$

where $C_1 \geq 0 \geq C_2$ and C_1, C_2 are not both zero.

Let X, Y be two orthonormal vector fields perpendicular to γ' and parallel along γ. The sectional curvature $K(X \wedge \gamma') = R(X, \gamma', \gamma', X) = -1$ along γ by (4.4.10). Next, we compute that the sectional curvature $K(t) := K(X \wedge Y) = R(X, Y, Y, X)$ along $\gamma(t)$. In what follows, we slightly abuse the notation and denote $f(t) = f(\gamma(t))$

and $|\nabla f|(t) = |\nabla f|(\gamma(t))$. We compute, for all $t \in (-\infty, \infty)$,

$$\begin{aligned} K'(t) &= \gamma'(R(X,Y,Y,X)) = (\nabla_{\gamma'} R)(X,Y,Y,X) \\ &= -(\nabla_Y R)(X,Y,X,\gamma') - (\nabla_X R)(X,Y,\gamma',Y) \qquad \text{(by the second Bianchi identity)} \\ &= -2\frac{f(t)}{|\nabla f|(t)}(K(t)+1) \end{aligned}$$

where in the last equation we use the second equation in (4.4.10). We would like to show that $K(t) + 1 \equiv 0$ for all t. Suppose, to give a contradiction, that $K(t) + 1$ is not identically zero. Then $K(t) + 1$ has no zeros, and we can divide the equation of $K'(t)$ by $K(t) + 1$ to achieve

$$\frac{d}{dt} \log |K(t) + 1| = -2\frac{f(t)}{|\nabla f|(t)}. \tag{4.4.13}$$

Note that $\frac{f(t)}{|\nabla f|(t)}$ satisfies the following ODE on $\gamma(t)$ by direct computation:

$$\frac{d}{dt}\left(\frac{f(t)}{|\nabla f|(t)}\right) = 1 - \left(\frac{f(t)}{|\nabla f|(t)}\right)^2 \qquad \text{for } t \in (-\infty, \infty).$$

Solving this ODE yields that either (1) $\frac{|f|}{|\nabla f|} \equiv 1$ on $\gamma(t)$, or (2) there is a constant $C > 0$ such that $\frac{f}{|\nabla f|} = 1 - \frac{2}{Ce^{2t}+1}$ on $\gamma(t)$. For Case (1), we find $K(t) + 1$ to be either Be^{-2t} or Be^{2t} for some nonzero constant B for $t \in (-\infty, \infty)$ by (4.4.13), which contradicts the asymptotically hyperbolic assumption. For Case (2), $f(t)$ has a zero and hence $C_2 < 0$ in (4.4.12), which contradicts the assumption $f > -c$.

Varying $p \in M$, we conclude that the sectional curvature of M is identically -1, which implies the universal cover of M is hyperbolic space. Together with the asymptotically hyperbolic assumption, M is isometric to hyperbolic space. $\square$

Remark 4.4.7. We thank Piotr Chruściel for pointing out an example that demonstrates the necessity of the hypothesis $f \geq -c$. Let (Σ, h) be a complete $(n-1)$-dimensional manifold (either closed or unbounded) with bounded sectional curvature. Consider the product $M = (-\infty, \infty) \times \Sigma$ endowed with the warped product metric $g = dt^2 + (\cosh t)^2 h$. One can directly check that the sectional curvature of (M, g) approaches -1 as $t \to \pm\infty$ and that $f(t) = \sinh t$ satisfies $\nabla^2 f = fg$. (In particular, $\frac{f}{|\nabla f|} = 1 - \frac{2}{e^{2t}+1}$ realizes Case (2) above.) If we further specify $\Sigma = \mathbb{R}^{n-1}$ endowed with a metric h whose sectional curvature is identically -1 outside a compact set of Σ, the sectional curvature of the resulting metric g approaches -1 toward the infinity of M. However, (M, g) is of constant sectional curvature -1 if and only if h is of constant sectional curvature -1 everywhere on Σ.

Chapter 5

Splitting theorems with scalar curvature lower bounds

5.1 Introduction

In the recent development of differential geometry, it has been of increasing interest to understand the geometry of Riemannian manifolds with lower bounds on their scalar curvature. In [83, 85], R. Schoen and S.-T. Yau proved the milestone result that the n-dimensional torus T^n, for $3 \leq n \leq 7$, does not admit a metric of positive scalar curvature by using minimal surface techniques. In more recent work, Schoen and Yau [87] have been able to use the minimal surface method to prove this for all dimensions $n \geq 3$. This had been proved by M. Gromov and H. B. Lawson [48] using spinor methods. The key observation made by Schoen and Yau in [85] is the following.

Proposition 5.1.1. *Let $(M^n, g), n \geq 3$ be a Riemannian manifold with positive scalar curvature, $S > 0$. If N^{n-1} is a stable, two-sided closed minimal hypersurface in M^n then N^{n-1} admits a metric of positive scalar curvature.*

Moreover, by refinements of the arguments in [85], one obtains the rigidity statement that if $S \geq 0$, and N^{n-1} does not admit a metric of positive scalar curvature then N^{n-1} is totally geodesic and Ricci flat, and $S = 0$ along N^{n-1} (cf. [40, 43]). In [23], M. Cai proved the following splitting theorem by assuming N is area-minimizing instead of being only stable (see also [43] for a simplified proof).

Proposition 5.1.2. *Let $(M^n, g), n \geq 3$ be a Riemannian manifold with nonnegative scalar curvature, $S \geq 0$, and suppose N^{n-1} is a two-sided closed minimal hypersurface which locally minimizes area. If N does not admit a metric of positive scalar curvature then there exists a neighborhood V of N such that $(V, g|_V)$ is isometric to $(-\delta, \delta) \times N$ with product metric $dt^2 + h$, where $h = g|_N$, and (N, h) is Ricci flat.*

This result extends to higher dimensions the torus splitting result in [24] for 3-manifolds of nonnegative scalar curvature. For some related rigidity results in three dimensions under different assumptions on the ambient scalar curvature and the topology of the minimal surface, see for example, [20, 75, 72, 5, 25, 26].

The minimal surface techniques introduced in [83, 85] also played an important role for the proof of the celebrated positive mass theorem for asymptotically flat manifolds by Schoen and Yau in [84, 86], which they have now extended to arbitrary dimension $n \geq 3$ in [87]. These results include the rigidity statement that the mass vanishes if and only if the manifold is isometric to Euclidean space. Somewhat more relevant for the present work are results concerning asymptotically hyperbolic manifolds. A proof of the positivity of mass in this setting was obtained by X. Wang [98] for spin manifolds, with improvements by P. Chruściel and M. Herzlich [27]. In the paper [7], L. Andersson, M. Cai, and the first author proved a positive mass result without spin assumption in dimensions n, $3 \leq n \leq 7$, for asymptotically hyperbolic manifolds, assuming a sign on the mass aspect. As an element in the proof, a splitting result analogous to Proposition 5.1.2 was obtained in [7, Section 2.2], whereby the 'brane' functional takes the place of the area functional and the scalar curvature satisfies $S \geq -n(n-1)$. Recently, making use of work of Lohkamp [66], Chruściel and Delay in [30] have established the nonnegativity of the mass for asymptotically hyperbolic manifolds, without spin assumption and in arbitrary dimension $n \geq 3$. The rigidity statement, when the mass vanishes, has been proved by L.-H. Huang, D. Martin and the second author in [54].

The aim of the present paper is to obtain splitting theorems for manifolds with compact boundary satisfying the scalar curvature inequality, $S \geq -\varepsilon n(n-1)$ (where $\varepsilon = 0$ or 1). An initial motivation for this paper comes from recent work of L.-H. Huang and the second author [53] concerning the rigidity of asymptotically locally hyperbolic manifolds of zero mass.

For our splitting results, we will use a condition that replaces the least area (or brane minimization) assumption. Let (M, g) be a Riemannian manifold with compact boundary N having mean curvature $H_N \leq H_0$, $H_0 \in \mathbb{R}$.[1] To set sign conventions, the mean curvature H_N is defined as the divergence of the inward pointing unit normal. We say that N is *weakly outermost* if there does not exist a compact hypersurface $\Sigma \subset M \backslash N$ cobordant to N satisfying the (strict) mean curvature inequality, $H_\Sigma < H_0$. We further define, in order to state the local version of our results, that N is *locally weakly outermost* provided that there is a neighborhood U of N such that N is weakly

[1] For simplicity we always assume M and N are connected.

outermost in $(U, g|_U)$. In our results, in addition to a weakly outermost condition, we will also require that the boundary N not admit a metric of positive scalar curvature, as in Proposition 5.1.2. We will discuss the necessity of these assumptions in Remark 5.1.4.

We now state our main result.

Theorem 5.1.3. *Let (M, g) be an n-dimensional ($n \geq 3$) Riemannian manifold with compact boundary N. Assume:*

1. *M has scalar curvature $S \geq -\varepsilon n(n-1)$, where $\varepsilon = 0$ or 1.*

2. *N has mean curvature $H_N \leq \varepsilon(n-1)$.*

3. *N does not carry a metric of positive scalar curvature and is locally weakly outermost.*

Then there exists a neighborhood V of N such that $(V, g|_V)$ is isometric to $[0, \delta) \times N$, with (warped) product metric $dt^2 + e^{2\varepsilon t}h$, where (N, h) is Ricci flat.

If we assume N is (globally) weakly outermost, one can obtain the global splitting result, as stated in Theorem 5.3.2. In the case $\varepsilon = 0$, the conclusion is that a neighborhood V of N splits as a product, which can be viewed as a variation of Proposition 5.1.2. On the other hand, in the case $\varepsilon = 1$, V splits as a warped product. Note that if h is flat, the manifold $([0, \infty) \times N, dt^2 + e^{2t}h)$ is of constant sectional curvature -1, and serves as a model space to define an asymptotically locally hyperbolic manifold.

Remark 5.1.4. The assumption in point 3 that the boundary N is weakly outermost is not sufficient to obtain the desired rigidity. For the case $\varepsilon = 0$, consider the *spatial Schwarzschild manifold*: $M = \mathbb{R}^n \setminus \{r < (\frac{m}{2})^{\frac{1}{n-2}}\}$, with metric (in isotropic coordinates),

$$g = \left(1 + \frac{m}{2r^{n-2}}\right)^{\frac{4}{n-2}} g_E\,,$$

where g_E is the Euclidean metric and $r = \sqrt{\sum_{i=1}^{n} x_i^2}$. M has vanishing scalar curvature, $S = 0$, and the boundary $N : r = (\frac{m}{2})^{\frac{1}{n-2}}$ is minimal, $H_N = 0$. Moreover, it follows from the maximum principle for hypersurfaces that N is weakly outermost. However, the conclusion of Theorem 5.1.3 does not hold

For the case $\varepsilon = 1$, the *AdS Schwarzschild manifold* is a further example illustrating the need for the scalar curvature assumption on N: $M = [r_m, \infty) \times S^{n-1}$ with metric

$$g = \left(1 + r^2 - \frac{2m}{r^{n-2}}\right)^{-1} dr^2 + r^2 g_{S^{n-1}}$$

where $r_m = (2m)^{\frac{1}{n-2}}$ and $g_{S^{n-1}}$ is the standard unit sphere metric. In this case, (M, g) has constant scalar curvature $S = -n(n-1)$ and the mean curvature of its boundary $N = \{r_m\} \times S^{n-1}$ is equal to $n - 1$. Also, N is weakly outermost but N carries a metric of positive scalar curvature.

As a final example, consider the toroidal Kottler metrics with $m > 0$: $M = [r_0, \infty) \times T^{n-1}$ with metric

$$g = \left(r^2 - \frac{2m}{r^{n-2}}\right)^{-1} dr^2 + r^2 h$$

where $r_0 = (2m)^{\frac{1}{n}}$ and h is a flat metric on T^{n-1}. One can easily check this example satisfies the conditions 1, 2 (with $\varepsilon = 1$). T^{n-1} does not carry a metric of positive scalar curvature, but the boundary $N = \{r_0\} \times T^{n-1}$ is not weakly outermost. This example shows that the boundary N being weakly outermost is needed as well, in addition to N not admitting a metric of positive scalar curvature.

By similar arguments one also obtains the following variation of Theorem 5.1.3.

Theorem 5.1.5. *Let (M,g) be an n-dimensional ($n \geq 3$) Riemannian manifold with compact boundary N. Assume:*

1. *M has scalar curvature $S \geq -\varepsilon n(n-1)$, where $\varepsilon = 0$ or 1.*

2. *N has mean curvature $H_N \leq -\varepsilon(n-1)$.*

3. *N does not carry a metric of positive scalar curvature and is locally weakly outermost.*

Then there exists a neighborhood V of N such that $(V, g|_V)$ is isometric to $[0,\delta) \times N$, with (warped) product metric $dt^2 + e^{-2\varepsilon t}h$, where (N,h) is Ricci flat.

The above variation can be roughly regarded as a scalar curvature version of a (warped product) splitting theorem proved by Croke and Kleiner in [35], in which they assume the corresponding lower bound on Ricci curvature, but do not require a scalar curvature condition on the boundary N. In this Ricci curvature case, the condition of being weakly outermost is implicit in their assumptions.

In addition to the above mentioned results, we prove a global splitting result by using Obata's equation, $\nabla^2 f = fg$.

Theorem 5.1.6. *Let (M,g) be an n-dimensional ($n \geq 3$) complete, noncompact Riemannian manifold with compact boundary N. Let $h = g|_N$. Suppose that*

1. *$S \geq -n(n-1)$ in a neighborhood of N.*

2. *N has mean curvature $H_N \leq \delta(n-1)$, where $\delta = 1$ or -1.*

3. *N does not carry a metric of positive scalar curvature and is locally weakly outermost.*

4. *There exists a nonzero function f satisfying $\nabla^2 f = fg$.*

Then (M, g) is isometric to $[0, \infty) \times N$, with warped product metric $dt^2 + e^{2\delta t}h$ where (N, h) is Ricci flat.

Here we only require N to be locally weakly outermost. Instead, we can extend the local splitting result globally by assuming the existence of a nontrivial solution to Obata's equation. Note that the resulting warped product corresponds to an unbounded portion of the hyperbolic cusp: it contains either an expanding end when $\delta = 1$ or a shrinking end when $\delta = -1$. This result plays a role in the recent work of Lan-Hsuan Huang and the second author [53] mentioned above.

As discussed in the next section, which includes relevant background, the proofs of Theorems 5.1.3 and 5.1.5 make use of results on *marginally outer trapped surfaces*, applied to specific *initial data sets*. The proof of Theorem 5.1.3, and its globalization are presented in Section 3. The proof of Theorem 5.1.6 is presented in Section 4.

5.2 Marginally outer trapped surfaces

For the proof of Theorem 5.1.3, we will make use of the theory of marginally outer trapped surfaces. Such surfaces play an important role in the theory of black holes, and, as indicated below, may be viewed as spacetime analogues of minimal surfaces in Riemannian geometry. For further background on marginally trapped surfaces, including their connection to minimal surfaces, we refer the reader to the survey article [8].

We begin by recalling some basic definitions and properties. By an initial data set, we mean a triple (M, g, K), where M is a smooth manifold, g is a Riemannian metric on M and K is a symmetric covariant 2-tensor on M. In general relatvity, an initial data set (M, g, K) corresponds to a spacelike hypersurface M with induced metric g and second fundamental form K, embedded in a spacetime (time-oriented Lorenzian manifold) $(\mathcal{M}, \mathfrak{g})$.

Let (M, g, K) be an initial data set. For convenience, we may assume, without loss of generality, that this initial data set is embedded in a spacetime $(\bar{M}, \bar{g})$ (see e.g. [10, Section 3.2]). While the definition of various quantities is more natural when expressed with respect to an ambient spacetime, all the relevant quantities we introduce depend solely on the initial data set. With respect to the spacetime $(\bar{M}, \bar{g})$, the tensor K becomes the second fundamental form of M: $K(X, Y) = \bar{g}(\bar{\nabla}_X u, Y)$ for all $X, Y \in T_pM$, where u is the future directed unit normal field to M in $\bar{M}$.

Let Σ be a closed (compact without boundary) two-sided hypersurface in M. Then Σ admits a smooth unit normal field ν in M, unique up to sign. By convention, refer to such a choice as outward pointing. Then $\ell = u + \nu$ is a future directed outward pointing null normal vector field along Σ. Associated to ℓ is the *null second fundamental form*, χ defined as,

$$\chi : T_p\Sigma \times T_p\Sigma \to \mathbb{R}, \quad \chi(X, Y) = \bar{g}(\bar{\nabla}_X \ell, Y). \tag{5.2.1}$$

In terms of the intial data,

$$\chi = K|_{T\Sigma} + A \tag{5.2.2}$$

where A is the second fundamental form of $\Sigma \subset M$ with respect to the outward unit normal ν. The *null expansion scalar* (or *null mean curvature*) θ of Σ is obtained by tracing χ with respect to the induced metric h on Σ,

$$\theta = \mathrm{tr}_h \chi = h^{AB}\chi_{AB} = \mathrm{div}_{\,\Sigma}\ell \,. \tag{5.2.3}$$

Physically, θ measures the divergence of the outgoing light rays emanating from Σ. In terms of the initial data (M, g, K),

$$\theta = \mathrm{tr}_h K + H \,, \tag{5.2.4}$$

where H is the mean curvature of Σ within M (given by the divergence of ν along Σ).

We say that Σ is outer trapped (resp. weakly outer trapped) if $\theta < 0$ (resp. $\theta \leq 0$) on Σ. If θ vanishes identically along Σ then we say that Σ is a marginally outer trapped surface, or MOTS for short. Note that in the so-called time-symmetric case, in which $K = 0$, a MOTS is simply a minimal ($H = 0$) surface in M, as follows from (5.2.4). It is in this sense that MOTS are a spacetime generalization of minimal surfaces in Riemannian geometry.

5.2.1 Stability of MOTS.

Unlike minimal surfaces, MOTS in general do not admit a variational characterization. Nevertheless, they admit an important notion of stability which we now describe; cf., [9, 8]. Let Σ be a MOTS in the initial data set (M, g, K) with outward unit normal ν. Consider a normal variation of Σ in M, i.e., a variation $t \to \Sigma_t$ of $\Sigma = \Sigma_0$ with variation vector field $V = \frac{\partial}{\partial t}|_{t=0} = \phi\nu$, $\phi \in C^\infty(\Sigma)$. Let $\theta(t)$ denote the null expansion of Σ_t with respect to $l_t = u + \nu_t$, where u is the future directed timelike unit normal to M and ν_t is the outer unit normal to Σ_t in M. A computation shows,

$$\left.\frac{\partial\theta}{\partial t}\right|_{t=0} = L(\phi) \, , \tag{5.2.5}$$

where $L : C^\infty(\Sigma) \to C^\infty(\Sigma)$ is the operator [9],

$$L(\phi) = -\Delta\phi + 2\langle X, \nabla\phi\rangle + \left(\frac{1}{2}S_\Sigma - (\mu + J(\nu)) - \frac{1}{2}|\chi|^2 + \operatorname{div} X - |X|^2\right)\phi \, . \tag{5.2.6}$$

In the above, Δ, ∇ and div are the Laplacian, gradient and divergence operator, respectively, on Σ, S_Σ is the scalar curvature of Σ, X is the vector field on Σ dual to the one form $X^\flat = K(\nu, \cdot)|_\Sigma$, $\langle\, , \rangle = h$ is the induced metric on Σ, and μ and J are defined in terms of the Einstein tensor $G = \mathrm{Ric}_{\bar{M}} - \frac{1}{2}R_{\bar{M}}\bar{g}$: $\mu = G(u, u)$, $J = G(u, \cdot)$. When the Einstein equations are assume to hold, μ and J represent the energy density and linear momentum density along M. As a consequence of the Gauss-Codazzi equations, the quantities μ and J can be expressed solely in terms of initial data,

$$\mu = \frac{1}{2}\left(S + (\operatorname{tr} K)^2 - |K|^2\right) \quad \text{and} \quad J = \operatorname{div} K - d(\operatorname{tr} K) \, , \tag{5.2.7}$$

where S is the scalar curvature on M.

An initial data set (M, g, K) is said to satisfy the *dominant energy condition*, provided the inequality,

$$\mu \geq |J| \tag{5.2.8}$$

holds along M. When one assumes the Einstein equations hold, this leads to an inequality on the energy-momentum tensor that is satisfied by most classical matter fields. Note that in the time-symmetric case ($K = 0$), the dominant energy condition reduces to $S \geq 0$, and hence the importance of manifolds of nonnegative curvature in general relativity.

In the time-symmetric case, the operator L reduces to the classical stability (or Jacobi) operator of minimal surface theory. As shown in [9], although L is not in general self-adjoint, the eigenvalue $\lambda_1(L)$ of L with the smallest real part, which is referred to as the principal eigenvalue of L, is necessarily real. Moreover there exists an associated eigenfunction ϕ which is strictly positive. The MOTS Σ is then said to be stable if $\lambda_1(L) \geq 0$.

A basic criterion for stability is the following. We say that a MOTS Σ is *weakly outermost* provided there are no outer trapped ($\theta < 0$) surfaces outside of, and cobordant to, Σ. Weakly outermost MOTS are necessarily stable. Indeed, if $\lambda_1(L) < 0$, Equation (5.2.5), with ϕ a positive eigenfunction ($L(\phi) = \lambda_1(L)\phi$) would then imply that Σ could be deformed outward to an outer trapped surface.

5.2.2 Rigidity of MOTS

The proof of Theorem 5.1.3 will be based on two rigidity results for MOTS. The following result was proved by R. Schoen and the first author in [45].

Theorem 5.2.1 (infinitesimal rigidity)**.** *Let (M, g, K) be an initial data set that satisfies the dominant energy condition (DEC)* (5.2.8)*, $\mu \geq |J|$. If Σ is a stable MOTS in M that does not admit a metric of positive scalar curvature then*

1. *Σ is Ricci flat.*

2. *$\chi = 0$ and $\mu + J(\nu) = 0$ along Σ.*

By strengthening the stability assumption, namely by requiring the MOTS Σ to be weakly outermost, as defined at the end of Section 5.2.1, we obtain additional rigidity. The following was proved in [44].

Theorem 5.2.2. *Let (M, g, K), be an initial data set satisfying the DEC. Suppose Σ is a weakly outermost MOTS in M that does not admit a metric of positive scalar curvature. Then there exists an outer neighborhood $U \approx [0, \delta) \times \Sigma$ of Σ in M such that each slice $\Sigma_t = \{t\} \times \Sigma$, $t \in [0, \delta)$ is a MOTS.*

Remark 5.2.3. It follows again from the discussion at the end of Section 5.2.1 that, in the theorem above, each MOTS Σ_t is stable, as otherwise Σ would not be weakly outermost.

The proofs of both rigidity results rely on the *MOTS stability inequality* obtained in [45] (see Equation 2.12). To prove Theorem 5.1.3, we will apply these results to the initial data set $(M, g, K = -\varepsilon g)$. The proof of Theorem 5.1.5, is quite similar, where now one uses the initial data set $(M, g, K = \varepsilon g)$.

5.3 Proof of Theorem 5.1.3

Proof of Theorem 5.1.3. Let (M, g) satisfy the assumptions of the theorem, and consider the initial data set (M, g, K) where $K = -\varepsilon g$.

We first observe that, with respect to this initial data set, the DEC (5.2.8) holds. Inserting $K = -\varepsilon g$ into the expression for μ in (5.2.7) leads to

$$\mu = \frac{1}{2}(S + \varepsilon^2 n(n-1)) = \frac{1}{2}(S + \varepsilon n(n-1)) \,. \tag{5.3.1}$$

Hence, by property 1 of Theorem 5.1.3, $\mu \geq 0$. Further, $K = -\varepsilon g$ implies $J = 0$, so that $\mu + |J| \geq 0$, and the DEC is satisfied.

Next, let's consider the null expansion of N. Equation (5.2.4) implies that N has null expansion,

$$\theta = -\varepsilon(n-1) + H_N\,. \tag{5.3.2}$$

Hence by property 2 of Theorem 5.1.3, $\theta \leq 0$, i.e. N is weakly outer trapped. In fact one must have $\theta \equiv 0$. Otherwise, it follows from [10, Lemma 5.2], that, by a small perturbation of N, there would exist a strictly outer trapped ($\theta < 0$) compact hypersurface $N' \subset U$ outside of, and cobordant to N, thereby contradicting the assumption that N is weakly outermost in U.

Hence, N is a weakly outermost MOTS in U. So, by Theorem 5.2.2, we can introduce coordinates (t, x^i) on a neighborhood $V = [0,\delta) \times N$ of N in U, so that g in these coordinates may be written as,

$$g = \psi^2 dt^2 + h_{ij} dx^i dx^j\,, \tag{5.3.3}$$

where $\psi = \psi(t, x^i)$ is positive, $h_t = h_{ij}(t, x^i)dx^i dx^j$ is the induced metric on $N_t = \{t\} \times N$, and N_t is a MOTS, $\theta(t) = 0$.

A computation similar to that leading to (5.2.5) (but where for the moment we do not assume assume $\theta = \theta(t)$ vanishes) leads to the following 'evolution equation' for $\theta = \theta(t, x^i)$ ([11, 42]),

$$\frac{\partial\theta}{\partial t} = -\Delta\psi + 2\langle X_t, \nabla\psi\rangle + \left(Q_t - \frac{1}{2}\theta^2 + \theta\,\mathrm{tr}K + \mathrm{div}\,X_t - |X_t|^2\right)\phi\,, \tag{5.3.4}$$

$$Q_t = \frac{1}{2}S_{N_t} - (\mu + J(\nu)) - \frac{1}{2}|\chi_t|^2\,, \tag{5.3.5}$$

where it is understood that, for each t, the above terms live on Σ_t, e.g., $\Delta = \Delta_t$ is the Laplacian on N_t, $\langle , \rangle = h_t$, $X_t^\flat = K(\nu_t, \cdot)|_{N_t}$, etc.

Note from the form of K, $X_t = 0$. Setting $\theta = 0$ and $X_t = 0$ in (5.3.4), and using (5.3.5), we obtain,

$$\Delta\psi + ((\mu + J(\nu)) + \frac{1}{2}|\chi_t|^2 - \frac{1}{2}S_{N_t})\psi = 0\,. \tag{5.3.6}$$

By Remark 5.2.3, each N_t is a stable MOTS. Hence, by Theorem 5.2.1,

$$N_t \text{ is Ricci flat, } \chi_t = 0, \text{ and } \mu + J(\nu) = 0\,. \tag{5.3.7}$$

Equation (5.3.6) then becomes,

$$\Delta\psi = 0\,,$$

and, hence, ψ is constant along each N_t, $\psi = \psi(t)$. By a simple change of variable, we thus may assume $\psi = 1$, and so (5.3.3) becomes,

$$g = dt^2 + h_{ij}dx^i dx^j\,. \tag{5.3.8}$$

From (5.2.2), $\chi_t = K|_{TN_t} + A_t = -\varepsilon h_t + A_t$ where A_t is the second fundamental form of N_t. Then, from the second equation in (5.3.7), $A_t = \varepsilon h_t$, which becomes, in the coordinate expression (5.3.8), $\frac{\partial h_{ij}}{\partial t} = 2\varepsilon h_{ij}$. Integrating gives, $h_{ij}(t,x) = e^{2\varepsilon t} h_{ij}(0,x)$. Thus, up to isometry, we have $V = [0,\delta) \times N$, $g|_V = dt^2 + e^{2\varepsilon t} h$. □

Remark 5.3.1. Theorem 5.1.3 has the following consequence. Let (M,g) be an n-dimensional, $3 \le n \le 7$, asymptotically flat manifold with compact minimal boundary N, and with nonnegative scalar curvature, $S \ge 0$. Suppose, further, that N is an outermost minimal surface, i.e. suppose that there are no minimal surfaces in $M \setminus N$ homologous to N. Then N necessarily carries a metric of positive scalar curvature. For, suppose not. To apply Theorem 5.1.3 in the case $\varepsilon = 0$, it is sufficient to show that N is locally weakly outermost. If that were not the case, there would exist a compact hypersurface N_1 cobordant to N with mean curvature $H_1 < 0$. On the other hand sufficient far out on the asymptotically flat end there exists a compact hypersurface N_2 cobordant to N_1 with mean curvature $H_2 > 0$. N_1 and N_2 bound a region W. Basic existence results for minimal surfaces (or for MOTS [8]), guarantee the existence of a minimal surface in W homologous to N, contrary to assumption. Hence N is weakly outermost. Theorem 5.1.3 then implies that (M,g) locally splits near N, contrary to N being an outermost minimal surface. The same consequence holds for an n dimensional, $3 \le n \le 7$, asymptotically hyperbolic manifold (M,g) with compact boundary N of constant mean curvature $n-1$, and with scalar curvature $S \ge -n(n-1)$ in the following sense: Consider the initial data set $(M,g,-g)$, with (M,g) as just described, and suppose N is an outermost MOTS. Then N necessarily carries a metric of positive scalar curvature.

Theorem 5.1.3 globalizes in a straight-forward way, as follows.

Theorem 5.3.2. *Let (M,g) be a complete, noncompact n-dimensional ($n \ge 3$) Riemannian manifold with compact boundary N. Assume:*

1. *M has scalar curvature $S \geq -\varepsilon n(n-1)$, where $\varepsilon = 0$ or 1.*

2. *N has mean curvature $H_N \leq \varepsilon(n-1)$.*

3. *N does not carry a metric of positive scalar curvature and is weakly outermost.*

Then (M, g) is isometric to $[0, \infty) \times N$, with (warped) product metric $dt^2 + e^{2\varepsilon t}h$, where (N, h) is Ricci flat.

Proof of Theorem 5.3.2. By Theorem 5.1.3, there exists a neighborhood V of N such that $(V, g|_V)$ is isometric to $([0, \delta) \times N, dt^2 + e^{2\varepsilon t}h)$. By the completeness assumption, it is clear that this warped product structure extends to $t = \delta$. From the fact that N is weakly outermost, it follows that $N_\delta = \{\delta\} \times N$ is weakly outermost. Theorem 5.1.3 then implies that the warped product structure extends beyond $t = \delta$. By a continuation argument, it follows that the warped product structure exists for all $t \in [0, \infty)$. □

Proof of Theorem 5.1.5. The proof of Theorem 5.1.5 is very similar to the proof of 5.1.3, except that now one works with the initial data set $(M, g, K = \varepsilon g)$. We leave the details to the interested reader.

Similar to Theorem 5.1.3, Theorem 5.1.5 implies the following global result.

Theorem 5.3.3. *Let (M, g) be a complete, noncompact n-dimensional ($n \geq 3$) Riemannian manifold with compact boundary N. Assume:*

1. *M has scalar curvature $S \geq -\varepsilon n(n-1)$, where $\varepsilon = 0$ or 1.*

2. *N has mean curvature $H_N \leq -\varepsilon(n-1)$.*

3. *N does not carry a metric of positive scalar curvature and is weakly outermost.*

Then (M, g) is isometric to $[0, \infty) \times N$, with (warped) product metric $dt^2 + e^{-2\varepsilon t}h$, where (N, h) is Ricci flat.

5.4 Warped product splitting and Obata's equation

The main aim of this section is to prove Theorem 5.1.6 stated in the introduction. Obata's equation in the form $\nabla^2 f = fg$ has been studied previously in the literature; see e.g. [59, 95]. In addition to Theorems 5.1.3 and 5.1.5, the proof of Theorem 5.1.6 will make use of the following result, which extends to manifolds with boundary certain results in [59].

Proposition 5.4.1. *Let (M^n, g) be a complete connected Riemannian manifold with compact connected boundary N $(n \geq 3)$. Suppose there exists a nonzero function f that satisfies*

$$\nabla^2 f = fg, \tag{5.4.1}$$

and N is a regular hypersurface $f^{-1}(a)$ for $a \in \mathbb{R}$. Then the following hold:

1. *If M is compact, then (M, g) is isometric to a hyperbolic cap $[0, R] \times \mathbb{S}^{n-1}$ equipped with the metric*

   $$dt^2 + (\sinh t)^2 g_{\mathbb{S}^{n-1}}$$

   where $g_{\mathbb{S}^{n-1}}$ is the standard unit sphere metric and $R = d_g(p, N)$ for $p \in M \setminus N$ which is a critical point of f.

2. *If M is noncompact, then (M, g) is isometric to a manifold $[0, \infty) \times N$ with*

(warped) product metric of the form

$$dt^2 + \xi(t)^2 g|_N,$$

where $\xi : [0,\infty) \to \mathbb{R}$ *is the solution to the following ODE*

$$\begin{cases} \xi'' - \xi = 0 \ \ on \ [0,\infty), \\ \xi(0) = 1 \ \ and \ \xi'(0) = \dfrac{a}{|\nabla f|_N} . \end{cases} \tag{5.4.2}$$

(We note, as follows from (5.4.1)*, that* $|\nabla f|_N$ *is constant.)*

Proof of Proposition 5.4.1. First we claim that f has a critical point on the interior of M if and only if M is compact (with boundary).

Suppose that f has a critical point p in M. Consider a unit speed geodesic $\gamma : [0,\infty) \to M$ emanating from p. It follows that

$$\frac{d^2}{dr^2} f(\gamma(r)) - f(\gamma(r)) = 0$$

thus $f(\gamma(r)) = c(e^r + e^{-r})$ and $\frac{d}{dr} f(\gamma(r)) = c(e^r - e^{-r})$, where $c \neq 0$ (as otherwise f would vanish identically). Observe that f depends only on the geodesic distance from the point p, which implies that γ' is parallel to ∇f. Moreover, there cannot be any other critical point of f. Let $R = d_g(p, N)$. Then it follows that $N = \exp_p(S_R)$, and from this that $\exp_p : \overline{B_R} \to M$ is bijective. By continuity of the exponential map, this implies that M must be compact.

Suppose, conversely, M is compact. For contradiction, suppose also that f has no critical points. Without loss of generality, we may assume that ∇f points inward on N. Let $\nu = \nabla f / |\nabla f|$, and consider the integral curve γ of ν emanating from a point

$p \in N$, i.e., $\gamma(0) = p$. It is straightforward that γ is a geodesic parametrized by arc length, and we also have

$$f \circ \gamma(t) = c_1 e^t + c_2 e^{-t}$$

as we observed before. Since f has no critical point, γ can be extended to $[0, \infty)$, which implies that γ is an injective infinite length geodesic. This contradicts the condition that M is compact, hence f must have a critical point on the interior of M.

We now show the first case of the proposition: assume that M is compact. From the previous argument, there is a critical point p such that $\exp_p : \overline{B_R} \to M$ is bijective where $R = d_g(p, N)$. Now we show that it is a diffeomorphism. Let J be a Jacobi field along γ such that $J(0) = 0$ and $|J'(0)| = 1$ and $g(J', \gamma') = 0$. Then we have for $r > 0$,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt}\bigg|_{t=r} g(J, J) &= g(J, \nabla_{\gamma'} J)|_{t=r} \\ &= A(J, J)|_{t=r} = \frac{f}{|\nabla f|} g(J, J)\bigg|_{t=r}, \end{aligned}$$

where $A = \nabla^2 f / |\nabla f|$ is the second fundamental form of the geodesic spheres. Thus for $r > r_0 > 0$,

$$|J|^2(r) = \left(\frac{e^r - e^{-r}}{e^{r_0} - e^{-r_0}} \right)^2 |J|^2(r_0) \neq 0$$

where r_0 is sufficiently small that $|J(r_0)| \neq 0$. This implies that there is no conjugate point from p thus $\exp_p : \overline{B_R} \to M$ is a diffeomorphism. Furthermore, by using geodesic polar coordinates, we can write the metric g on M diffeomorphic to $[0, R] \times$

$\mathbb{S}^{n-1}$ as

$$g = dt^2 + (\sinh t)^2 g_{\mathbb{S}^{n-1}}.$$

We turn to the second case: assume that M is noncompact. Let $h = g|_N$. We will construct an isometry between (M, g) and the manifold

$$([0, \infty) \times N, dt^2 + \xi^2 h)$$

where ξ is given in (5.4.2). Without loss of generality, we may assume that ∇f points inward on N.

Let φ be the flow generated by $\nu = \nabla f / |\nabla f|$, and define the map $\psi : [0, \infty) \times N \to M$ by

$$\psi(t, \bar{p}) = \varphi_t(\bar{p})$$

for $\bar{p} \in N$ and $t \in [0, \infty)$. Since f has no critical points, it is clear that ψ is a diffeomorphism.

As we observed before, we have the general solution

$$f \circ \psi(\bar{p}, t) = c_1 e^t + c_2 e^{-t}, \quad \bar{p} \in N, t \in [0, \infty)\,,$$

where the constants c_1 and c_2 are determined by the conditions on f at N; specifically, $c_1 + c_2 = a$ and $c_1 - c_2 = |\nabla f|_N$. In terms of these constants, the solution to the ODE (5.4.2) is given by, $\xi(t) = \frac{c_1 e^t - c_2 e^{-t}}{c_1 - c_2}$.

Now we prove that ψ is the desired isometry from $([0, \infty) \times N, dt^2 + \xi(t)^2 h)$ onto

(M, g). Using ψ as a coordinate chart, we can write the metric

$$g = dt^2 + g_{ij}(t, \bar{p})\, dx^i dx^j$$

where $\{x^i\}_{i=1}^{n-1}$ are local coordinates near $\bar{p}$ on N and $g_{ij}(t, \bar{p}) = g_{(t,\bar{p})}(\partial_i, \partial_j)$ for $t \in [0, \infty)$. Then we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\bigg|_{t=\tau} g_{ij}(t, \bar{p}) &= g_{(\tau,\bar{p})}(\partial_i, \nabla_\nu \partial_j) = A_{(\tau,\bar{p})}(\partial_i, \partial_j) \\ &= \frac{f(\tau)}{|\nabla f|(\tau)} g_{ij}(\tau, \bar{p}) = \frac{c_1 e^{\tau} + c_2 e^{-\tau}}{c_1 e^{\tau} - c_2 e^{-\tau}} g_{ij}(\tau, \bar{p}) \end{aligned}$$

where $A_{(\tau,\bar{p})}$ is the second fundamental form of the hypersurface $f^{-1}(f \circ \gamma(\tau))$. Thus we obtain

$$g_{ij}(\tau, \bar{p}) = \xi(\tau)^2 g_{ij}(0, \bar{p}) = \xi(\tau)^2 h_{ij}(\bar{p})\,,$$

and by varing $(\tau, \bar{p}) \in M$ it proves that ψ is the desired isometry. $\square$

Proof of Theorem 5.1.6. We will only prove $\delta = 1$ since the proof of the other case is almost identical.

By Theorem 5.1.3 (with $\varepsilon = 1$), we have the local splitting near N, that is, there exists a neighborhood U of N such that U is isometric to $[0, b) \times N$ for some $b > 0$ with the metric $dt^2 + e^{2t}h$. To use Proposition 5.4.1, we show that N is the level set $f^{-1}(a)$ for some $a \in \mathbb{R}$.

Let $\{x^i\}_{i=1}^{n-1}$ be local coordinates on N. This gives rise to local coordinates $\{t = x_0, x_1, ..., x_{n-1}\}$ on $[0, b) \times N$ in the obvious manner. Then, by direct computation, we have

$$0 = \nabla_{\partial_t}\nabla_{\partial_i} f = \partial_t\partial_i f - \sum_{k=0}^{n-1} \Gamma_{ti}^k \partial_k f$$
$$= \partial_t\partial_i f - \partial_i f, \tag{5.4.3}$$
$$f = \nabla_{\partial_t}\nabla_{\partial_t} f = \partial_t^2 f, \tag{5.4.4}$$
$$e^{2t} f\, h_{ij} = \nabla_{\partial_i}\nabla_{\partial_j} f = \partial_i\partial_j f + e^{2t} h_{ij}\partial_t f - \sum_{l=1}^{n-1} \bar{\Gamma}_{ij}^l \partial_l f. \tag{5.4.5}$$

where $\bar{\Gamma}$ is the Christoffel symbol with respect to h. Denote $f = f(p,t)$ on U for $p \in N$ and $t \in [0,b)$. Then from the above computations we have

$$\partial_t^2 f - f = 0 \Rightarrow f(p,t) = c_1(p)e^t + c_2(p)e^{-t}, \tag{5.4.6}$$
$$\partial_t(\partial_i f) - \partial_i f = 0 \Rightarrow \partial_i f(p,t) = c_3(p)e^t, \tag{5.4.7}$$

It follows from (5.4.6) and (5.4.7) that $c_2(p)$ is constant on N, hence we can write

$$f(p,t) = c_1(p)e^t + c_2 e^{-t}, \text{ and } \partial_t f - f = -2c_2 e^{-t}.$$

Now we shall show that $c_1(p)$ is constant on N. By (5.4.5), we have

$$\partial_i\partial_j f + e^{2t} h_{ij}(\partial_t f - f) - \sum_{l=1}^{n-1} \bar{\Gamma}_{ij}^l \partial_l f = 0$$
$$\Rightarrow e^t \left(\partial_i\partial_j(c_1(p)) - 2c_2 h_{ij} - \sum_{l=1}^{n-1} \bar{\Gamma}_{ij}^l \partial_l c_1(p) \right) = 0$$
$$\Rightarrow \nabla^N_{\partial_i}\nabla^N_{\partial_j} c_1 = 2c_2 h_{ij} \Rightarrow \Delta_N c_1 = 2(n-1)c_2\,. \tag{5.4.8}$$

Since N is compact without boundary, we have

$$0 = \int_N \Delta_N c_1 = 2(n-1)c_2|N|$$

where $|N|$ is the area of N. This implies that $c_2 = 0$, and hence $c_1(p)$ is harmonic on N. Therefore $c_1(p)$ is constant on N so $N = f^{-1}(c_1)$.

By Proposition 5.4.1, (M, g) is isometric to $[0, \infty) \times N$ with the metric $dt^2 + \xi(t)^2 h$. In particular, one can see that the warping factor is $\xi(t) = e^t$. □

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