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Modular Functions and Asymptotic Geometry on Punctured Riemann Spheres

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ABSTRACT

In the first chapter, we derive a precise asymptotic expansion of the complete Kähler-Einstein metric on the punctured Riemann sphere with three or more omitting points. This new technique is at the intersection of analysis and algebra. By using Schwarzian derivative, we prove that the coefficients of the expansion are polynomials on the two parameters which are uniquely determined by the omitting points. Futhermore, we use the modular form and Schwarzian derivative to explicitly determine the coefficients in the expansion of the complete Kähler-Einstein metric for punctured Riemann sphere with $3, 4, 6$ or 12 omitting points.

The second chapter gives an explicit formula of the asymptotic expansion of the Kobayashi-Royden metric on the punctured sphere $\mathbb{CP}^1\backslash\{0, 1, \infty\}$ in terms of the exponential Bell polynomials. We prove a local quantitative version of the Little Picard's theorem as an application of the asymptotic expansion. Furthermore, the explicit formula of the metric and the conclusion regarding the coefficients apply to more general cases of $\mathbb{CP}^1\backslash\{a_1, \dots, a_n\}$, $n \geq 3$ as well, and the metric on $\mathbb{CP}^1\backslash\{0, \frac{1}{3}, -\frac{1}{6} \pm \frac{\sqrt{3}}{6}i\}$ will be given as a concrete example of our results.

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Contents

Chapter 1

Hyperbolic Metric, punctured Riemann sphere and Modular functions

1.1 Introduction

The main object in this paper is the punctured Riemann sphere $\mathbb{CP}^1 \setminus \{a_1 = 0, \ldots, a_n\}$, where $\mathbb{CP}^1$ is the projective space over $\mathbb{C}$ of dimension one, and $\{a_1, \ldots, a_n\}$ are n different points that are omitting from $\mathbb{CP}^1$. It is well-known that there exists a unique complete Kähler-Einstein metric on $\mathbb{CP}^1 \setminus \{a_1, \ldots, a_n\}$, $n \geq 3$, with negative constant Gauss curvature. However, an explicit asymptotic expansion of the metric near a_j, $j = 1, \ldots, n$, remains unknown. More precisely, the coefficients in the expansion are difficult to determine. In this article, we derive a precise asymptotic formula for the Kähler-Einstein metric whose coefficients are polynomials on the first two parameters, which are determined by the punctures $\{a_1, \ldots, a_n\}$. Furhtermore, all coefficient polynomials can be explicitly written down when $n = 3, 4, 6, 12$.

The asymptotic expansion of the complete Kähler-Einstein metric on the quasi-projective manifold $M = \overline{M} \setminus D$ was proposed by Yau [Yau00, p. 377], where D is a normal crossing divisor in the project manifold $\overline{M}$ such that $K_{\overline{M}}+D > 0$. The leading order term has been known to him since the late 1970s (see [Yau78]). Several people have worked on the asymptotic expansion, see for example [Sch98, Wu06, RZ12], using techniques from partial differential equations. Another important class of the quasi-projective manifolds is that M is the quotient of Siegel space $\mathcal{S}_g/\Gamma$ by an arithmetic subgroup Γ, $g \geq 2$, and $\overline{M}$ is Mumford's toroidal compactification. In this case $K_{\overline{M}} + D$ is big and nef. The complete Kähler-Einstein volume form on M has been written down in [Wan93, YZ14].

The open Riemann surface $\mathbb{CP}^1 \setminus \{a_1, \ldots, a_n\}$, $n \geq 3$, is only a one-dimensional example of $\overline{M} \setminus D$ with $K_{\overline{M}}+[D] > 0$. It is nevertheless the building block of a general complete quasi-projective manifold with negative holomorphic sectional curvature. Indeed, given a projective manifold X, for any point $x \in X$, there exists a Zariski neighborhood $U = X \setminus Z$ (where Z is an algebraic subvariety of X) of x such that U can be embedded into the product

$$S_1 \times \cdots \times S_N$$

as a closed algebraic submanifold, in which N is some positive integer, and each S_j is of the form $\mathbb{CP}^1 \setminus \{a_1, \ldots, a_n\}$ for $n \geq 3$ (see [Gri71, p. 25, Lemma 2.3]). Take the complete Kähler-Einstein metric on each S_j. Then, the product metric restricted to U gives a complete Kähler metric ω on U of finite volume and negatively pinched holomorphic sectional curvature. Furthermore, by a recent result [WY20, Theorem

3], the quasi-projective manifold U possesses a complete Kähler-Einstein metric which is uniformly equivalent to the metric ω; see also [WY18, Example 5.4].

To derive the expansion of Kähler-Einstein metric on $\mathbb{CP}^1 \setminus \{a_1, \ldots, a_n\}$, we make use of the modular forms in number theory. This technique enables us to obtain the precise expansion with explicit coefficients, which were obscure in the literature by the abstract series expansion or the kernel of local linear operators. Recall the well-known uniformization theorem (see [Hub06]) which indicates that the punctured Riemann sphere has the unit disk $\mathbb{D}$ as its universal covering space, or equivalently, the upper half plane $\mathbb{H}$. The idea is to find this covering map, then induce the Poincaré metric from $\mathbb{H}$ to our object.

F. Klein and R. Fricke had included classic theories regard automorphic function in their lecture notes (see [FKRF17]). Later, the Teichmüller space was introduced by O. Teichmüller and developed by L. Ahlfors and L. Bers (see [Ber70, Ber73, Ahl80]). One of the methods that turns out very useful is the application of Schwarzian derivative. Let f be a covering map for the universal covering space $\mathbb{H} \to \mathbb{CP}^1 \setminus \{a_1 = 0, \ldots, a_n\}$, then f uniquely determines its Schwarzian derivative

$$\{f, \tau\} = 2(f_{\tau\tau}/f_\tau)_\tau - (f_{\tau\tau}/f_\tau)^2, \quad \text{where } f = f(\tau),\ \tau \in \mathbb{H}.$$

On the other hand, the Schwarzian derivative of the inverse of f is well-defined and uniquely determined by the following equation

$$\{\tau, f\} = \sum_{j=1}^{n} \left(\frac{1}{(f - a_j)^2} + \frac{2\beta_j}{\alpha_j} \frac{1}{f - a_j} \right), \tag{1.1.1}$$

where all a_j are the omitting points from the punctured Riemann sphere, and β_j, α_j

are some constants, $j = 1, \ldots, n$, which will be discussed in section 1.4. By an argument from its geometric property, there are $(n-3)$ ratios from the set $\{\frac{\beta_1}{\alpha_1}, \ldots, \frac{\beta_n}{\alpha_n}\}$ that are independent to each other, these constants are often called the accessory parameters. Determining the accessory parameters is a way to figure out the automorphic function. This approach has been studied by I. Kra in [Kra89] and A. B. Venkov in [Ven83]. However, despite the results from automorphic function and Teichmüller theory, such covering map $f : \mathbb{H} \to \mathbb{CP}^1 \backslash \{a_1 = 0, \ldots, a_n\}$ is still quite difficult to write out.

On the other hand, the congruence subgroup in $\mathrm{SL}_2(\mathbb{Z})$ is a great collection of Fuchsian groups with nice properties. J. McKay and A. Sebbar discovered a connection between modular forms and Schwarzian derivative of automorphic functions in [MS00], which will be mentioned in section 1.7. Furthermore, there are several interesting connections between modular curve, Schwarzian derivative and graph theory that are mentioned by A. Sebbar in [Seb01, Seb02].

The main result of this article is given by the following theorem.

Theorem 1. *A universal covering map $f : \mathbb{H} \to \mathbb{CP}^1 \setminus \{a_1 = 0, a_2, \ldots, a_n\}$ which vanishes at infinity can be given by the following expansion*

$$f = f(\tau) = A\left(q_{_k} + \frac{B}{A}q_{_k}^2 + \mathrm{C}_3(A, \frac{B}{A})q_{_k}^3 + \sum_{m=4}^{\infty} \mathrm{C}_m(A, \frac{B}{A})q_{_k}^m\right)$$

in $q_{_k} = \exp\{\frac{2\pi i}{k}\tau\}$, $\tau \in \mathbb{H}$, for some real number k, where A, B are constants depending only on $a_2, \ldots, a_n$, and the coefficient term $\mathrm{C}_m(A, \frac{B}{A})$ is a polynomial on A and $\frac{B}{A}$ for $m \geq 3$. In particular, $\mathrm{C}_3(A, \frac{B}{A}) = \frac{1}{16}\left[19\frac{B^2}{A^2} - A^2\sum_{j=2}^{n}\left(\frac{1}{a_j^2} - \frac{1}{a_j}\frac{2\beta_j}{\alpha_j}\right)\right]$, where α_j, β_j are the constants in equation (1.1.1)*. Consequently, the complete Kähler-*

Einstein metric can be given by the following asymptotic expansion

$$|ds| = \frac{1}{|f|\log|\frac{f}{A}|}\left|1 - \left(\frac{B}{A}\frac{f}{A} - \frac{\mathrm{Re}(\frac{B}{A}\frac{f}{A})}{\log|\frac{f}{A}|}\right) + \sum_{m=2}^{\infty} R_m(A, \frac{B}{A}, \frac{f}{A}, \frac{f^s\overline{f^{m-s}}}{A^s\overline{A^{m-s}}\log^j|\frac{f}{A}|})\right| |df| \tag{1.1.2}$$

at the cusp 0*, where* $f \in \mathbb{CP}^1 \setminus \{a_1 = 0, a_2, \ldots, a_n\}$ *and* $R_m(A, \frac{B}{A}, \frac{f}{A}, \frac{f^s\overline{f^{m-s}}}{A^s\overline{A^{m-s}}\log^j|\frac{f}{A}|})$ *is a polynomial in* $A, \frac{B}{A}, \frac{f}{A}, \frac{f^s\overline{f^{m-s}}}{A^s\overline{A^{m-s}}\log^j|\frac{f}{A}|}$*,* $s, j = 0, 1, \ldots, m$*, with constant coefficients for* $m \geq 2$*.*

To illustrate the second result, we define the following values for $n(N)$,

$$n(2) = 3, \quad n(3) = 4, \quad n(4) = 6, \quad n(5) = 12. \tag{1.1.3}$$

The second result is given by the following theorem and corollary.

Theorem 2. *Let* $n(N)$ *be the values in equation* (1.1.3)*, and let* $f_N : \mathbb{H} \to \mathbb{CP}^1 \setminus \{a_1 = 0, a_2, \ldots, a_{n(N)}\}$ *be a universal covering space with deck transformation group* $\mathrm{Aut}(f_N) = \Gamma(N)$*-the principal congruence subgroup-which vanishes at infinity,* $N = 2, 3, 4, 5$*. Then the map* f_N *can be given as the following expansion*

$$f_N = f_N(\tau) = A\left(q_{_N} + \frac{B}{A}q_{_N}^2 + \sum_{m=3}^{\infty}\mathrm{C}_m(\frac{B}{A})q_{_N}^m\right) \tag{1.1.4}$$

in $q_{_N} = \exp\{\frac{2\pi}{N}i\tau\}, \tau \in \mathbb{H}$*, where the constants* $A, B \in \mathbb{C}$ *are uniquely determined by the set of values of the punctured points* $\{a_1 = 0, a_2, \ldots, a_{n(N)}\}$*, and the coefficient term* $\mathrm{C}_m(\frac{B}{A})$ *is a polynomial in* $\frac{B}{A}$ *with constant coefficients for* $m \geq 3$*.*

Remark 1. Explicit formulas of $\mathrm{C}_m(A, \frac{B}{A})$ are given in [CQ20] in terms of Bell polynomials.

In particular, when $n = 2$, $n(2) = 3$, it is a triple punctured Riemann sphere, the covering map $\mathbb{H} \to \mathbb{CP}^1 \setminus \{a_1 = 0, a_2, a_3\}$ for arbitrary a_2, a_3 is given in the following example.

Example 1.1.1. The covering map $f_2 : \mathbb{H} \to \mathbb{CP}^1 \setminus \{a_1 = 0, a_2, a_3\}$ which vanishes at infinity has the following expansion

$$f_2(\tau) = \frac{16a_2a_3}{a_3 - a_2}\left[q_{_2} - \left(8 + \frac{16a_2a_3}{a_3 - a_2}\right) q_{_2}^2 + \sum_{m=3}^{\infty} \mathrm{C}_m \left(8 + \frac{16a_2a_3}{a_3 - a_2}\right) q_{_2}^m\right],$$

in $q_{_2} = \exp\{\pi i \tau\}$, $\tau \in \mathbb{H}$, where $\mathrm{C}_m\left(-8 - \frac{16a_2a_3}{a_3-a_2}\right)$ is a polynomial in the term $\left(-8 - \frac{16a_2a_3}{a_3-a_2}\right)$ for $m \geq 3$. In particular, $\mathrm{C}_3 = \left(-8 - \frac{16a_2a_3}{a_3-a_2}\right)^2 - 20$. In this case, the constants A, B in Theorem 2 are given by the following

$$A = \frac{16a_2a_3}{a_3 - a_2}, \qquad B = \left(\frac{16a_2a_3}{a_3 - a_2}\right) \cdot \left[-\left(8 + \frac{16a_2a_3}{a_3 - a_2}\right)\right].$$

Corollary 3. *As a consequence of Theorem 2, the complete Kähler-Einstein metric on $\mathbb{CP}^1 \setminus \{a_1 = 0, a_2, \ldots, a_{n(N)}\}$ has the following asymptotic expansion*

$$|ds| = \frac{1}{|f| \log|\frac{f}{A}|}\left|1 - \left(\frac{B}{A}\frac{f}{A} - \frac{\mathrm{Re}(\frac{B}{A}\frac{f}{A})}{\log|\frac{f}{A}|}\right) + \sum_{m=2}^{\infty} R_m\left(\frac{B}{A}, \frac{f}{A}, \frac{f^s \overline{f^{m-s}}}{A^s \overline{A^{m-s}} \log^j |\frac{f}{A}|}\right)\right| |df| \tag{1.1.5}$$

at the cusp $a_1 = 0$, where $R_m(\frac{B}{A}, \frac{f}{A}, \frac{f^s\overline{f^{m-s}}}{A^s\overline{A^{m-s}}\log^j|\frac{f}{A}|})$ is a polynomial in $\frac{B}{A}, \frac{f}{A}, \frac{f^s\overline{f^{m-s}}}{A^s\overline{A^{m-s}}\log^j|\frac{f}{A}|}$, $s, j = 0, 1, \ldots, m$, with constant coefficients for $m \geq 2$.

As an application of our result, several examples are given in the section 1.8.2. For example, we give the complete Kähler-Einstein metric of $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$ in the following example.

Example 1.1.2. The covering map $f_2 : \mathbb{H} \to \mathbb{CP}^1 \setminus \{0, 1, \infty\}$ has the following expansion

$$f = f_2(\tau) = 16q_2 - 128q_2^2 + 704q_2^3 - 3072q_2^4 + 11488q_2^5 + O(q_2^6) \tag{1.1.6}$$

in $q_2 = \exp\{\pi i \tau\}$, $\tau \in \mathbb{H}$, and we write $f = f_2$ for convenience. In this case, the constants A, B in Theorem 2 take values $A = 16, B = -128$. Therefore the coefficients of $f = f_2$ in equation (1.1.4) are givien by $B/A = \frac{-128}{16} = -8$, $\mathrm{C}_3 = (B/A)^2 - 20 = (-8)^2 - 20 = 44$. From Corollary 3, the complete Kähler-Einstein metric has the following explicit expansion

$$\begin{aligned}|ds| = \frac{1}{|f| \log |f/16|} \Bigg| 1 + \frac{1}{2} \left(f - \frac{\operatorname{Re} f}{\log |f/16|} \right) \\ - \left[\frac{51}{32} f^2 - \frac{1}{4} \frac{f \operatorname{Re} f}{\log |f/16|} + \frac{51}{64} \frac{\operatorname{Re}(f^2)}{\log |f/16|} + \frac{1}{4} \frac{(\operatorname{Re} f)^2}{\log^2 |f/16|} \right] + O(f^3) \Bigg| |df|\end{aligned}$$

at the cusp 0, where $f \in \mathbb{CP}^1 \setminus \{0, 1, \infty\}$.

Remark 2. The Kähler-Einstein metric on $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$ was known by S. Agard. He gave a globle explicit formula in terms of a double integral (see equation (2.6) in [Aga68]).

Remark 3. The argument of this article can derive a general case of a_1, a_2, a_3, please see Corollaries 1.8.8 and 1.8.9.

In this article, we start with studying the fundamental group π_1 of the punctured Riemann sphere, its topology indicates that every element from $\pi_1(\mathbb{CP}^1 \setminus \{a_1 = 0, \ldots, a_n\})$ corresponds to an automorphism of its universal cover, more precisely an element in $\mathrm{SL}_2(\mathbb{R})$. Two similar arguments from its monodromy action imply Theorem 1.3.9, which indicates that each generator is of *parabolic* type. This fact allows us to interpret the covering map f as an expansion at the corresponding cusp, so as the complete Kähler-Einstein metric. In the next section, section 1.4, we introduce and discuss the Schwarzian derivative in the analytic sense, and derive Theorem 1.4.5 at the end, which plays an important role to our main result. In section 1.5, we will focus on the action of a discrete group on $\mathbb{H}$. Couple propositions are introduced in section 1.6 in order to target the congruence subgroups that are of genus *zero* without *elliptic* point. In section 1.7, we introduce the modular forms and discuss its relation with the Schwarzian derivative. The main results and couple examples are provided in the last section.

1.2 Construction of Universal Covering Space

We will sketch the process of the universal covering from $\mathbb{H}$ to $\mathbb{CP}^1\backslash\{a_1 = 0, \ldots, a_n\}$ in the graphic way. The general procedure of constructing the universal covering space is mentioned in [Nev70, p. 8-12, section 2]. I will describe the most straightforward case in this article, which will be sufficient and helpful for readers to understand the topic. To cover the punctured Riemann sphere $\mathbb{CP}^1\backslash\{a_1 = 0, \ldots, a_n\}$ from the unit disk $\mathbb{D}$, or equivalently, the upper half plane $\mathbb{H}$, we first connect the n punctures such that it is a polygon without self-crossing, rename the vertexes as $a_1, a_2, \ldots, a_n$ counter-clockwisely. Then, let us pick n points randomly on the unit circle, and name them $c_1, \ldots, c_n$ counter-clockwisely which correspond to $a_1, \ldots, a_n$ respectively, and let $c_1 = 1, c_n = -1$. The collection of the n points all lie on the upper half unit circle. Connecting the two points that are right next to each other by arcs. In particular, c_1 and c_n are connected by the diameter, and we denote this circular polygon as

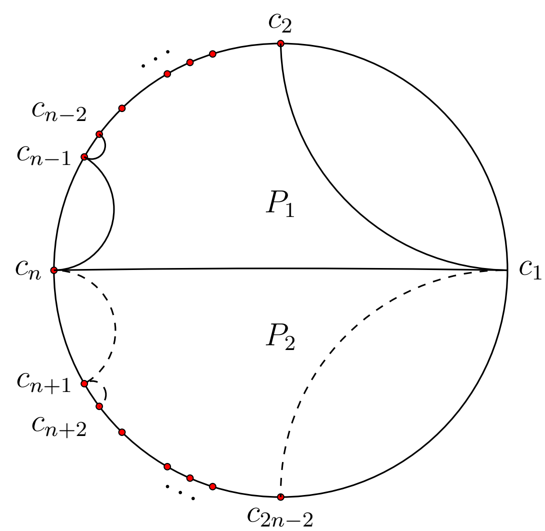


FIGURE 1.1: Polygon on Disk

P_1. Reflect P_1 with respect to the side c_1c_n, it will result in $(n-2)$ points from the reflections of $\{c_2, \ldots, c_{n-1}\}$ on the lower half circle, we mark those points counter-clockwisely by $c_{n+1}, \ldots, c_{2n-2}$, and mark the reflected polygon by P_2. Then we call the combined polygon $P = P_1 + P_2$ a fundamental polygon of this covering (see Figure 1.1).

Next we construct the reproduction of the covering process. Let V be the set of vertexes with either only odd index or even index. For a vertex $c_m \in V$, reflect P with respect to an adjacent side $c_{m-1}c_m$, it results in a new polygon $\tilde{P} = \tilde{P}_1 + \tilde{P}_2$, where $\tilde{P}_i$ is the reflecting image of P_i, $i = 1, 2$. Notice that $\tilde{P}_1$ is the image of P_1 from reflections of odd times. $\tilde{P}_2$ is the image of P_1 from reflections of even times since $\tilde{P}_2$ is the reflection of P_2 and P_2 is the reflection of P_1. Next we reflect P with respect to c_mc_{m+1}, which is the other adjacent side of c_m. We replicate such reflections on both sides of each vertex in the set V, it will result in a new polygon $P^{(1)}$ on $\mathbb{D}$, where $P^{(1)}$ is composed by $2 \times (n-1) + 1 = 2n - 1$ copies of $P^{(0)} = P$, and it has

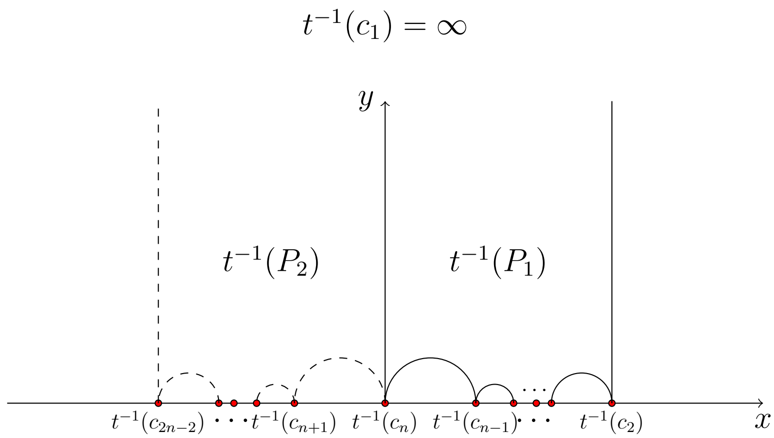


FIGURE 1.2: Polygon on $\mathbb{H}$

$2 \times (2n-3) \times (n-1)$ sides and vertexes. Keep repeating this process on every $P^{(i)}$, it will cover the unit disk $\mathbb{D}$ as i approaches ∞.

If we change the universal covering space to the upper half plane $\mathbb{H}$, the construction will be the same through the Cayley transformation,

$$t : \mathbb{H} \to \mathbb{D}, \qquad z \mapsto t(z) = \frac{z-i}{z+i}. \tag{1.2.1}$$

For example, Figure 1.2 shows the corresponding fundamental polygon on $\mathbb{H}$.

1.3 Deck Transformation Group and Expansion in q_k

1.3.1 Deck Transformation Group and Generators

Let $x \in \mathbb{CP}^1 \backslash \{a_1 = 0, \ldots, a_n\}$ be a base point and $\gamma : [0,1] \to \mathbb{CP}^1 \backslash \{a_1 = 0, \ldots, a_n\}$ be a non-trivial loop with $\gamma(0) = \gamma(1) = x$. Assume $f : \mathbb{H} \to \mathbb{CP}^1 \backslash \{a_1 = 0, \ldots, a_n\}$ is a universal covering map. For an arbitrary pre-image $\tau_1 \in f^{-1}(x) \subseteq \mathbb{H}$, the end point τ_2 of the lift $\tilde{\gamma}$ with initial point τ_1 is uniquely determined by γ and the choice of τ_1. The uniqueness implies that the non-trivial loop γ induces a permutation on points of $f^{-1}(x)$, and such permutation induces an isomorphism of its covering space $\mathbb{H}$.

Definition 1.3.1. We use $\mathrm{Aut}(f)$ to denote the deck transformation group of the universal covering $f : \mathbb{H} \to \mathbb{CP}^1 \backslash \{a_1 = 0, \ldots, a_n\}$. Equivalently, we have $f \circ \gamma(\tau) = f(\tau)$ for every element $\gamma \in \mathrm{Aut}(f) \subseteq \mathrm{Aut}(\mathbb{H})$ and every point $\tau \in \mathbb{H}$.

There is a one-to-one correspondence between the deck transformation group $\mathrm{Aut}(f)$ and its fundamental group:

$$\mathrm{Aut}(f) \leftrightarrow \pi_1(\mathbb{CP}^1 \backslash \{a_1 = 0, \ldots, a_n\}), \qquad h \leftrightarrow \text{loop}.$$

Notice that the fundamental group $\pi_1(\mathbb{CP}^1 \backslash \{a_1 = 0, \ldots, a_n\})$ is a free group generated by the $(n-1)$ loops, each of which circles around only one punctured point $a \in \{a_1, \cdots, a_n\}$. Therefore the corresponding relation implies that $\mathrm{Aut}(f)$ is generated by $(n-1)$ elements from $\mathrm{Aut}(\mathbb{H})$. We state the conclusion as the following statement.

Theorem 1.3.2. *The deck transformation group* $\mathrm{Aut}(f)$ *of the universal covering* $f : \mathbb{H} \to \mathbb{CP}^1 \backslash \{a_1 = 0, \ldots, a_n\}$ *is a free group generated by* $(n-1)$ *elements from* $\mathrm{Aut}(\mathbb{H})$*, where each generator corresponds to a single loop in* $\pi_1(\mathbb{CP}^1 \backslash \{a_1 = 0, \ldots, a_n\})$.

Next we will discuss the properties of $\mathrm{Aut}(f)$. Let us recall some elementary definitions.

Definition 1.3.3. An element $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$ acts on $[z_0 : z_1] \in \mathbb{CP}^1$ in the following sense

$$\gamma([z_0 : z_1]) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} = \begin{pmatrix} az_0 + b \\ cz_1 + d \end{pmatrix} = [az_0 + b : cz_1 + d].$$

Remark 4. Let $\gamma, \gamma_1, \gamma_2 \in \mathrm{GL}_2(\mathbb{C})$, and $[z_0 : z_1] \in \mathbb{CP}^1$.

1. $\gamma_2(\gamma_1([z_0 : z_1])) = (\gamma_2 \cdot \gamma_1)([z_0 : z_1])$ because of the associativity of matrix multiplication.

2. If we consider $z = \frac{z_0}{z_1}$ when $z_1 \neq 0$, and $z = \infty$ when $z_1 = 0$, then we have $\gamma(z) = \frac{az+b}{cz+d}$ and $\gamma(\infty) = \frac{a}{c}$.

Definition 1.3.4. Let $z \in \mathbb{CP}^1$ and an element $\gamma \in \mathrm{GL}_2(\mathbb{C})$ such that $\gamma(z) = z$, we say that z is fixed under the action of γ, or, equivalently, z is invariant under γ.

Elements in $\mathrm{Aut}(\mathbb{H})$ are the ones that we are interested in. On one side, we know that $\mathrm{Aut}(\mathbb{CP}^1) \subseteq \mathrm{SL}_2(\mathbb{C})$, therefore we have $\mathrm{Aut}(\mathbb{H}) \subseteq \mathrm{SL}_2(\mathbb{C})$. On the other side, the isomorphism of $\mathbb{H}$ keeps its boundary $\mathbb{R}$ invariant, this implies that $\mathrm{Aut}(\mathbb{H}) = \mathrm{SL}_2(\mathbb{R})$. We will focus on the group $\mathrm{SL}_2(\mathbb{R})$ from now on.

Proposition 1.3.5. *Let ω be a linear transformation in $SL_2(\mathbb{R})$, and $\tau \in \mathbb{H}$.*

1. *If $0, \infty$ are invariant under the action of ω, i.e., $\omega(0) = 0$ and $\omega(\infty) = \infty$, then $\omega(\tau) = \lambda\tau$ for some $0 < \lambda \in \mathbb{R}$.*

2. *If ω has only one invariant point ∞, i.e., $\omega(\infty) = \infty$, then $\omega(\tau) = \tau + k$ for some $k \in \mathbb{R}$.*

3. *If ω has only one invariant point $i = \sqrt{-1}$, i.e., $\omega(i) = i$, then $\frac{\omega(\tau)-i}{\omega(\tau)+i} = e^{i\theta}\frac{\tau-i}{\tau+i}$ for some $\theta \in \mathbb{R}$.*

Proof. Assume $\omega = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$. The condition $\omega(\infty) = \infty$ implies that $\frac{a}{c} = \infty$, so we get $c = 0$ and $a \neq 0$ since $ad - bc = 1$. And $\omega(0) = 0$ implies $\frac{b}{d} = 0$, so $b = 0$ and $d \neq 0$.

1. If both of 0 and ∞ are invariant under ω, we have $\omega = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ and $\omega(z) = \frac{a}{d}z$, specially $\lambda = \frac{a}{d} > 0$ since $ad = 1 > 0$.

2. If ∞ is the only invariant point under the action of ω, we have $\omega = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ where $b \neq 0$. Any point other than ∞ are not invariant means that the equation $\omega(z) = \frac{a}{d}z + \frac{b}{d} = z$ has no solution, so $\frac{a}{d} = 1$. Let $\frac{b}{d} = k$, we conclude $\omega(z) = z + k$.

3. For the last case, the composition $t \circ \omega \circ t^{-1}$ is an automorphism of the unit disk $\mathbb{D}$ with 0 being fixed, where t denotes the Cayley transformation (1.2.1). Therefore we have relation $t \circ \omega \circ t^{-1}(\tilde{z}) = e^{i\theta}\tilde{z}$ for $\tilde{z} \in \mathbb{D}$. Let $\tilde{z} = t(z)$, the following equation

$$t \circ \omega \circ t^{-1}(\tilde{z}) = t \circ \omega \circ t^{-1} \circ t(z) = t \circ \omega(z) = e^{t\theta}t(z)$$

holds, where the last equality implies $\frac{\omega(\tau)-i}{\omega(\tau)+i} = e^{i\theta}\frac{\tau-i}{\tau+i}$.

□

Proposition 1.3.6. *For an element $\omega \in SL_2(\mathbb{C})$, it fits in one of the following three situations.*

1. *If ω has two different invariant points on the boundary $\mathbb{R} \cup \{\infty\}$, it is called of hyperbolic type. Furthermore, if $r_1 < r_2 \in \mathbb{R}$ are the two invariant points, then the transformation has expression*

$$\frac{\omega(z) - r_2}{\omega(z) - r_1} = \lambda \frac{z - r_2}{z - r_1}, \quad \text{for some } 0 < \lambda \in \mathbb{R}.$$

2. *If ω has only one invariant point on the boundary $\mathbb{R} \cup \{\infty\}$, it is called of parabolic type. Furthermore, when the invariant point $r \neq \infty$, it has expression*

$$-\frac{1}{\omega(z) - r} = -\frac{1}{z - r} + k, \quad \text{for some } k \in \mathbb{R}.$$

3. *If ω has only one invariant point $r \in \mathbb{H}$ in the interior of $\mathbb{H}$, it is called of elliptic type. It has expression*

$$\frac{\omega(z) - r}{\omega(z) - \overline{r}} = e^{i\theta} \frac{z - r}{z - \overline{r}}, \quad \text{for some } \theta \in \mathbb{R}.$$

Proof. Assume $\omega = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$, if $z \in \mathbb{C}$ is invariant under ω, i.e., $\omega(z) = z$, we have the following equation:

$$z = \frac{az + b}{cz + d} \quad \text{implies} \quad cz^2 + (d - a)z - b = 0.$$

Its determinant $\Delta = (d - a)^2 + 4bc$ shows that when $\Delta > 0$, there are two real solutions, it is of *hyperbolic* type; when $\Delta = 0$, it has one and only real solution, it is of *parabolic* type; and when $\Delta < 0$, it has two non-real solutions, we say it is of *elliptic* type. We consider them separately.

1. *Hyperbolic* type. $\Delta > 0$, we assume the two different real solutions are r_1 and r_2, without loss of generality, let us also assume $r_1 < r_2 \in \mathbb{R}$. Consider the map $\begin{pmatrix} 1 & -r_1 \\ 1 & -r_2 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R}) \subseteq \mathrm{Aut}(\mathbb{H})$ which maps the following way

$$z \mapsto \frac{z - r_2}{z - r_1}, \qquad r_2 \mapsto 0, \qquad r_1 \mapsto \infty,$$

it shows that the transformation between $\frac{\omega(z)-r_2}{\omega(z)-r_1}$ and $\frac{z-r_2}{z-r_1}$ is an automorphism of $\mathbb{H}$ that keeps 0 and ∞ invariant. From Proposition 1.3.5, this automorphism has the form $\bullet \mapsto \frac{a}{d}\bullet = \lambda \cdot \bullet$ for some $\lambda \in \mathbb{R}^+$. Therefore, if ω fixes two points $r_1 < r_2$ on the real line $\mathbb{R}$, there is a positive real number λ such that should have the following relation:

$$\frac{\omega(z) - r_2}{\omega(z) - r_1} = \lambda \frac{z - r_2}{z - r_1}, \qquad \text{for some } \lambda \in \mathbb{R}^+. \tag{1.3.1}$$

2. *Parabolic* type. $\Delta = 0$, we assume $r \in \mathbb{R}$ is the only invariant point under ω. Consider the map $z \mapsto -\frac{1}{z-r}$ in $\mathrm{Aut}(\mathbb{H})$ which sends r to ∞. Similarly, the transformation between $-\frac{1}{\omega(z)-r}$ and $-\frac{1}{z-r}$ is an automorphism of $\mathbb{H}$ that only fixes ∞. Proposition 1.3.5 implies that such transformation has the form $\bullet \mapsto \bullet + \frac{b}{d} = \bullet + k$ for some $k \in \mathbb{R}$. We conclude the following relation:

$$-\frac{1}{\omega(z) - r} = -\frac{1}{z - r} + k, \qquad \text{for some } k \in \mathbb{R}. \tag{1.3.2}$$

3. *Elliptic type.* If $\Delta < 0$, the equation $cz^2 + (d-a)z - b = 0$ has two non-real solutions that are conjugate to each other, assume they are $r \in \mathbb{H}$ and $\overline{r}$. The transformation $\frac{z-r}{z-\overline{r}}$ maps $\mathbb{H} \to \mathbb{D}$ with $r \mapsto 0$. By a similar argument, Proposition 1.3.5 implies the relation

$$\frac{\omega(z) - r}{\omega(z) - \overline{r}} = e^{i\theta} \frac{z - r}{z - \overline{r}}, \qquad \text{for some } \theta \in \mathbb{R}. \tag{1.3.3}$$

□

Remark 1.3.1. The proof directly follows from Proposition 1.3.5. Also see [Shi71, p. 7, Proposition 1.13] and [Nev70, p. 8, section 1.6].

Next, we will show that each generator in Theorem 1.3.2 has to be of *parabolic* type. For a universal covering space $f : \mathbb{H} \to \mathbb{CP}^1 \backslash \{a_1 = 0, \dots, a_n\}$, it is equivalent to say that f is biholomorphic. Suppose an element $\gamma \in \mathrm{Aut}(f)$ corresponds to a generating loop l in $\pi_1(\mathbb{CP}^1 \backslash \{a_1 = 0, \dots, a_n\})$, where $l : [0,1] \to \mathbb{CP}^1 \backslash \{a_1 = 0, \dots, a_n\}$ and l isolates only one puncture $a \in \{a_1, \dots, a_n\}$. Fix a point τ_0 in the fiber $f^{-1}(l(0))$, the lift $\tilde{l}(t)$ of $l(t)$ with initial point τ_0 is uniquely determined and so as its terminal point $\tilde{l}(1) = \tau_1$. Similarly, the lift $\widetilde{l^n}$ of $l^n = l * \cdots * l$, is uniquely determined by the fixed initial point τ_0, where $*$ means $(l * l)(t) = \begin{cases} l(2t), & 0 \leq t < \frac{1}{2}, \\ l(2t-1), & \frac{1}{2} \leq t \leq 1. \end{cases}$ Denote the terminal point $\tau_n = \widetilde{l^n}(1)$, meanwhile $\tau_n \in f^{-1}(l(0))$. By the corresponding relation between l and γ, we have $\tau_n = (l * \cdots * l)^{\sim}(1) = \gamma \circ \cdots \circ \gamma(\tau_0) = \gamma^n(\tau_0)$. We will prove that this γ can not be of *hyperbolic* type by analyzing its monodormy behavior. Similarly, we can conclude that γ can not be of *elliptic* type either. The proofs of the following propositions are elementary, but it is helpful to understand the uniformizing function at singularities. For this purpose, I will include one of the proofs.

Proposition 1.3.7. *The generator is not of hyperbolic type.*

Proof. Suppose l is a loop in $\mathbb{CP}^1\backslash\{a_1 = 0, \ldots, a_n\}$ which isolates a puncture $a \in \{a_1, \ldots, a_n\}$. γ is the corresponding element in $\mathrm{Aut}(f)$, where f is the universal covering map. If γ is of *hyperbolic* type, then there exists $r_1 < r_2 \in \mathbb{R}$ which are the two invariant points of γ, and let $\lambda \in \mathbb{R}^+$ be the constant in Proposition 1.3.6 such that the relation

$$\frac{\gamma(\tau) - r_2}{\gamma(\tau) - r_1} = \lambda \frac{\tau - r_2}{\tau - r_1}, \qquad \tau \in \mathbb{H}$$

is satisfied. Therefore the following equality

$$\frac{\tau_{n+1} - r_2}{\tau_{n+1} - r_1} = \frac{\gamma(\tau_n) - r_2}{\gamma(\tau_n) - r_1} = \lambda \frac{\tau_n - r_2}{\tau_n - r_1}$$

implies relation

$$\frac{\widetilde{l^{n+1}}(1) - r_2}{\widetilde{l^{n+1}}(1) - r_1} = \lambda \frac{\widetilde{l^n}(1) - r_2}{\widetilde{l^n}(1) - r_1} = \lambda^n \frac{\widetilde{l}(1) - r_2}{\widetilde{l}(1) - r_1},$$

which leads to the following equation

$$\frac{\tau_n - r_2}{\tau_n - r_1} = \lambda^n \frac{\tau_0 - r_2}{\tau_0 - r_1}. \tag{1.3.4}$$

On the branch $0 < \arg\left(\frac{\tau - r_2}{\tau - r_1}\right) < \pi$, the following calculation

$$\begin{aligned} \exp\{\frac{2\pi i}{\log\lambda} \cdot \log \frac{\tau_n - r_2}{\tau_n - r_1}\} &= \exp\{\frac{2\pi i}{\log\lambda} \cdot \log\left(\lambda^n \cdot \frac{\tau_0 - r_2}{\tau_0 - r_1}\right)\}, \\ &= \exp\{\frac{2\pi i}{\log\lambda} \cdot n\log\lambda\} \cdot \exp\{\frac{2\pi i}{\log\lambda} \cdot \log \frac{\tau_0 - r_2}{\tau_0 - r_1}\}, \\ &= \exp\{\frac{2\pi i}{\log\lambda} \cdot \log \frac{\tau_0 - r_2}{\tau_0 - r_1}\} \end{aligned} \tag{1.3.5}$$

holds since $\lambda \in \mathbb{R}^+$. Define a function

$$\Phi(x) = \exp\left\{\frac{2\pi i}{\log\lambda}\cdot\log\left(\frac{f^{-1}(x)-r_2}{f^{-1}(x)-r_1}\right)\right\}, \quad U_l - \{a\} \to \mathbb{C},$$

where U_l is the open set bounded by l with a inside. Notice that the value defined by the expression $\exp\left\{\frac{2\pi i}{\log\lambda}\cdot\log\left(\frac{f^{-1}\circ l(0)-r_2}{f^{-1}\circ l(0)-r_1}\right)\right\}$ is single valued over the fiber of $l(0)$ due to equation (1.3.5). Applying the same argument on every point $x \in U_l - \{a\}$, we conclude that $\Phi(x)$ is a well-defined single valued function on $U_l - \{a\}$ over the branch $0 < \arg\frac{f^{-1}(x)-r_2}{f^{-1}(x)-r_1} < \pi$. We estimate $|\Phi(x)|$ in $U_l - \{a\}$ as the following,

$$\begin{aligned}
|\Phi(x)| &= \left|\exp\left\{\frac{2\pi i}{\log\lambda}\cdot\log\left(\frac{f^{-1}(x)-r_2}{f^{-1}(x)-r_1}\right)\right\}\right|, \\
&= \left|\exp\left\{\frac{2\pi i}{\log\lambda}\right\}\cdot\log\left|\frac{f^{-1}(x)-r_2}{f^{-1}(x)-r_2}\right|\right|\cdot\exp\left\{-\frac{2\pi}{\log\lambda}\cdot\arg\frac{f^{-1}(x)-r_2}{f^{-1}(x)-r_1}\right\}, \\
&= \exp\left\{-\frac{2\pi}{\log\lambda}\cdot\arg\frac{f^{-1}(x)-r_2}{f^{-1}(x)-r_1}\right\}. \qquad (1.3.6)
\end{aligned}$$

Over the branch $0 < \arg\frac{f^{-1}(x)-r_2}{f^{-1}(x)-r_1} < \pi$, equation (1.3.6) implies that $|\Phi(x)|$ is bounded between two positive values $\exp\left\{-\frac{2\pi^2}{\log\lambda}\right\}$ and $1 = \exp\{-\frac{2\pi}{\log\lambda}\cdot 0\}$. By Cauchy-Riemann theorem, a is a removable singularity and $\Phi(a) \neq 0$, thus the exponential of $\Phi(x)$ will be regular at $x = a$, as well as $f^{-1}(x)$. This contradicts with the fact that a is an essential singularity. □

Proposition 1.3.8. *The generator is not of elliptic type.*

Proof. It is similar to the proof of Proposition 1.3.7 (see [Nev70, p. 16-17, section 3.2]). □

From Propositions 1.3.7 and 1.3.8, we have the following conclusion.

Theorem 1.3.9. *The deck transformation group* $\mathrm{Aut}(f)$ *of a universal covering space* $f : \mathbb{H} \to \mathbb{CP}^1 \backslash \{a_1 = 0, \ldots, a_n\}$ *is generated by* $(n-1)$ *parabolic elements in* $SL_2(\mathbb{R})$*.*

1.3.2 The Expansion and Metric Formula

Suppose $\gamma_1, \ldots, \gamma_{n-1}$ are the $(n-1)$ *parabolic* generators corresponding to the $(n-1)$ generating loops of $\pi_1(\mathbb{CP}^1 \backslash \{a_1 = 0, \ldots, a_n\})$. Theorem 1.3.9 shows that each of them satisfies the following relation

$$-\frac{1}{\gamma_j(\tau) - r_j} = -\frac{1}{\tau - r_j} + k_j, \qquad \text{when } r_j \neq \infty,$$

or

$$\gamma_j(\tau) = \tau + k_j, \qquad \text{when } r_j = \infty,$$

where r_j and k_j are the constants corresponding to γ_j, $j = 1, \ldots, n-1$, in Proposition 1.3.6. Let a_j be the singularity that corresponds to γ_j, the function

$$\Phi_j(x) = \begin{cases} e^{-\frac{2\pi i}{k_j}\frac{1}{f^{-1}(x) - r_j}}, & \text{when } r_j \neq \infty, \\ e^{\frac{2\pi i}{k_j} f^{-1}(x)}, & \text{when } r_j = \infty, \end{cases} \qquad U_l - \{a_j\} \to \mathbb{CP}^1 \backslash \{a_1 = 0, \ldots, a_n\}$$

is biholomorphic, where $U_l - \{a_j\}$ is a punctured neighborhood that is defined similar to the one in Proposition 1.3.7. Such function $\Phi_j(x)$ is single valued on the neighborhood $U_l - \{a_j\}$ around the singularity a_j. We call this function a uniformizing function at a_j.

Without loss of generality, we will focus on the situation $\infty \in \{r_1, \ldots, r_{n-1}\}$ for convenience. Let $f : \mathbb{H} \to \mathbb{CP}^1 \backslash \{a_1 = 0, \ldots, a_n\}$ be a universal covering map, assume $r_1 = \infty$, and let γ_1 be the corresponding generator and a_1 be the corresponding singularity. The uniformizing function $\Phi_1(x)$ is defined as the following

$$\Phi_1(x) = e^{\frac{2\pi i}{k_1} f^{-1}(x)}, \qquad U_l - \{a_1\} \to \mathbb{CP}^1 \backslash \{a_1 = 0, \dots, a_n\}.$$

If $\gamma_1 \in \mathrm{SL}_2(\mathbb{R})$ is a generator of $\mathrm{Aut}(f)$ that corresponds to r_1, so γ_1 fixes ∞, we have the following periodic property,

$$f(x) = f \circ \gamma_1(x) = f(x + k_1).$$

Fundamental Fourier analysis implies the following proposition.

Proposition 1.3.10. *Let f be a covering map $f : \mathbb{H} \to \mathbb{CP}^1 \backslash \{a_1 = 0, \dots, a_n\}$ such that a generator of* $\mathrm{Aut}(f)$ *fixes ∞. Then the map f has the following global expansion*

$$f(\tau) = f(q_{_k}) = c_0 + Aq_{_k} + Bq_{_k}^2 + \sum_{m=3}^{\infty} c_m q_{_k}^m \tag{1.3.7}$$

in $q_{_k} = q_{_k}(\tau) = \exp\{\frac{2\pi i}{k}\tau\}$ for any $\tau \in \mathbb{H}$, where $k \in \mathbb{R}$ is the constant that corresponds to the generator in Proposition 1.3.6, and the constant $A \neq 0$.

Proof. If ∞ is invariant under a generator $\gamma \in \mathrm{Aut}(f)$, then there is a constant $k \in \mathbb{R}$ such that $\gamma(\tau) = \tau + k$, which implies that f has the following periodic property

$$f(\tau + k) = f(\tau), \qquad \text{for any } \tau \in \mathbb{H}.$$

Consider $f(\tau) = f(x + iy) = f_y(x)$ as a function of x on $\mathbb{R}$. Its Fourier series

$$f(x + iy) = f_y(x) = \sum_{n=-\infty}^{\infty} c_n(y) e^{\frac{2\pi}{k} inx}, \tag{1.3.8}$$

converges uniformly on x due to Corollaries 2.3 and 2.4 in [SS03, p. 41-42], where the coefficients are given by the following equation

$$c_n(y) = \frac{1}{k}\int_0^k f_y(x)e^{-\frac{2\pi}{k}inx}dx.$$

The Cauchy-Riemann equation implies the following equation

$$i\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} \tag{1.3.9}$$

holds since $f(\tau)$ is holomorphic. Calculating the left hand side, due to the uniform convergence on x, we have equation

$$\begin{aligned} i\frac{\partial f}{\partial x} &= i\frac{\partial}{\partial x}\left(\sum_{n=-\infty}^{\infty} c_n(y)e^{\frac{2\pi}{k}inx}\right) = i\left(\sum_{n=-\infty}^{\infty} c_n(y)e^{\frac{2\pi}{k_1}inx}\cdot\frac{2\pi}{k}in\right), \\ &= \sum_{n=-\infty}^{\infty} -\frac{2n\pi}{k}c_n(y)e^{\frac{2\pi}{k}inx}. \end{aligned}$$

Also calculating the right hand side, we have equation

$$\frac{\partial f}{\partial y} = \sum_{n=-\infty}^{\infty} c_n'(y)e^{\frac{2\pi}{k}inx}.$$

Matching the coefficients, the differential equation

$$\frac{2n\pi}{k}c_n(y) + c_n'(y) = 0$$

holds, which has solution

$$c_n(y) = C_n e^{-\frac{2\pi}{k}ny}, \quad \text{for some constant } C_n \in \mathbb{C}.$$

Substituting the solution set in equation (1.3.8), we have equation

$$f(\tau) = f(x+iy) = \sum_{n=-\infty}^{+\infty} C_n e^{-\frac{2\pi}{k}ny} e^{\frac{2\pi}{k}inx} = \sum_{n=-\infty}^{+\infty} C_n e^{\frac{2\pi}{k}in\tau},$$

the last equality holds because $inx - ny = in(x+iy) = in\tau$. On the other hand, the uniformizing function $q_{_k} = e^{\frac{2\pi i}{k}\tau}$ is a local coordinate on a neighborhood of the corresponding singularity $a \in \{a_1, \ldots, a_n\}$ on $\mathbb{CP}^1 \backslash \{a_1 = 0, \ldots, a_n\}$, so f has a Taylor expansion

$$f(\tau) = c_0 + c_1 q_{_k} + c_2 q_{_k}^2 + \sum_{m=3}^{\infty} c_m q_{_k}^m \tag{1.3.10}$$

in $q_{_k}$ on a neighborhood of a, which will coincide with its Fourier series, thus equation (1.3.10) holds globally. For convenience in later sections, we write $A = c_1, B = c_2$,

$$f(\tau) = c_0 + A q_{_k} + B q_{_k}^2 + \sum_{j=3}^{\infty} c_j q_{_k}^j.$$

To see $A \neq 0$, remember that $f(\tau)$ is a covering map without ramification point, this implies that $f_{q_{_k}}|_{q_{_k}=0} \neq 0$, so $A \neq 0$. □

The condition $A \neq 0$ in equation (1.3.7) allows us to find an inversion series $q_{_k}(f)$ in f such that $q_{_k}(f(q_{_k})) = q_{_k}$. We assume the inversion series $q_{_k}(f)$ has expression

$$q_{_k}(f) = \tilde{c}_0 + \tilde{A} f + \tilde{B} f^2 + \sum_{j=3}^{\infty} \tilde{c}_j f^j. \tag{1.3.11}$$

Theorem 1.3.11. *Let* $f : \mathbb{H} \to \mathbb{CP}^1 \backslash \{a_1 = 0, \ldots, a_n\}$ *is a covering map for the universal covering space, and* $\mathrm{Aut}(f)$ *has a generator* γ *which fixes infinity, then there* *exists a positive constant $k \in \mathbb{R}$ such that f can be given as the following expansion*

$$f(\tau) = f(q_{_k}) = c_0 + Aq_{_k} + Bq_{_k}^2 + c_3 q_{_k}^3 + \sum_{m=4}^{\infty} c_m q_{_k}^m$$

in $q_{_k} = \exp\{\frac{2\pi i}{k}\tau\}, \tau \in \mathbb{H}$ and $A \neq 0$. Furthermore, such covering map induces a complete Kähler-Einstein metric on $\mathbb{CP}^1 \backslash \{a_1 = 0, \ldots, a_n\}$ from $\mathbb{H}$, it can be given by the following equation

$$ds^2 = \frac{|q'_{_k}(f)|^2}{|q_{_k}(f)|^2 \log^2 |q_{_k}(f)|} |df|^2, \quad f \in \mathbb{CP}^1 \backslash \{a_1 = 0, \ldots, a_n\}, \tag{1.3.12}$$

where $q_{_k}(f)$ is the inversion series (1.3.11).

Proof. The first statement is directly from Proposition 1.3.10, we only need to show the second part. For any point $x_0 \in \mathbb{CP}^1 \backslash \{a_1 = 0, \ldots, a_n\}$, there is an evenly covered neighborhood U of x_0 that is simply connected. Let $\tilde{U}$ and $\tilde{U}'$ are two different pre-images of U, i.e., $\tilde{U} \neq \tilde{U}' \in f^{-1}(U)$. Notice that $\tilde{U}$ and $\tilde{U}'$ are both in $\mathbb{H}$, we can define logarithms of $\exp\{\frac{2\pi i}{k}\tau\} = q_{_k} = q_{_k}(f)$ on different branches as the following

$$\begin{aligned} \tau = \tau(f) &= \frac{k}{2\pi i} \log q_{_k}(f), \qquad \text{on } \tilde{U}, \\ \tau' = \tau'(f) &= \frac{k}{2\pi i} \log q_{_k}(f), \qquad \text{on } \tilde{U}'. \end{aligned}$$

Recall that $\tilde{U}$ and $\tilde{U}'$ are homeomorphic through a cover transformation $h \in \mathrm{Aut}(f) \subseteq \mathrm{SL}_2(\mathbb{R})$, i.e., $h \circ \tau(f) = \tau'(f)$. Since the Poincaré metric is invariant under actions of $\mathrm{GL}_2(\mathbb{R})$, therefore we have equality

$$ds^2 = \frac{-4}{(\tau - \overline{\tau})^2} |d\tau|^2 = \frac{-4}{(\tau' - \overline{\tau'})^2} |d\tau'|^2, \tag{1.3.13}$$

Thus the general logarithm property

$$\frac{d}{df}(\log q_{_k}(f)) = q_{_k}'(f)/q_{_k}(f)df$$

holds since $\log q_{_k}(f)$ is biholomorphic on the branch $\tilde{U}$, . The metric can be induced on $\tilde{U}$ by the following calculation

$$\begin{aligned}
ds^2 &= \frac{-4}{(\tau - \overline{\tau})^2}|d\tau|^2, \\
&= \frac{-4}{((\frac{k}{2\pi i}\log q_{_k}(f) - \overline{\frac{k}{2\pi i}\log q_{_k}(f)})^2}|d(\frac{k}{2\pi i}\log q_{_k}(f))|^2, \\
&= \frac{-4\cdot(2\pi i)^2}{k^2(\log q_{_k}(f) + \overline{\log q_{_k}(f)})^2}(\frac{k}{2\pi})^2\left|\frac{q_{_k}'(f)}{q_{_k}(f)}df\right|^2, \\
&= \frac{|q_{_k}'(f)|^2}{|q_{_k}(f)|^2\log^2|q_{_k}(f)|}|df|^2.
\end{aligned} \tag{1.3.14}$$

Furthermore, the invariant property in equation (1.3.13) implies that the expression (1.3.14) is independent of the choice of branches. Therefore the complete Kähler-Einstein metric ds^2 can be defined globally by equation (1.3.14) on $\mathbb{CP}^1\backslash\{a_1 = 0, \ldots, a_n\}$. □

One thing that is worth to mention is that Theorems 1.3.11 and 1.4.5, which will be included in the later section, together imply the main result Theorem 1.

1.4 Schwarzian Derivative

1.4.1 Introduction

The Schwarzian derivative is invariant under linear transformations, let $\tau = \tau(f) = f^{-1}$ be the inverse of the covering map $f : \mathbb{H} \to \mathbb{CP}^1 \backslash \{a_1 = 0, \ldots, a_n\}$, the Schwarzian derivative $\{\tau, f\}$ is actually well-defined due to the invariance. This fact builds a connection between $\{f, \tau\}$ and $\{\tau, f\}$, which will lead to the main result of this article.

Definition 1.4.1. If a function f is locally biholomorphically defined on the complex $z-$plane, the Schwarzian derivative of f with respect to z is defined as the following:

$$\{f, z\} = 2\left(\frac{f_{zz}}{f_z}\right)_z - \left(\frac{f_{zz}}{f_z}\right)^2,$$

where we write $f_z = df/dz$ for convenience.

In Definition 1.4.1, we require f being locally biholomorphic so that we are able to talk about the Schwarzian derivative of its inverse function $z = f^{-1}$. We simply write $\{z, f\}$ to denote the Schwarzian derivative of $z = z(f)$ with respect to f. If w is also a locally biholomorphic function defined on the complex plane, then it allows us to consider the Schwarzian derivative of the composition $f \circ w(z)$. Since the functions f, w are biholomorphic, we will simply write the inverse f^{-1} as $z(f)$ and the inverse $w^{-1} = z(w)$.

Proposition 1.4.2. *Assume f, w are two locally biholomorphic functions defined on the complex complex plane, then we have the following properties:*

1. $\{\frac{af+b}{cf+d}, z\} = \{f, z\}$ *for any* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C})$,

2. $\{f, z\} = \{f, w\}w_z^2 + \{w, z\}$ *if* $f = f \circ w(z)$. *In particular,* $\{z, f\} = -\{f, z\}z_f^2$.

Proof. The proof is obvious. □

Recall Theorem 1.3.11 in section 1.3.2, the covering map f can be expressed as an expansion in q_k, let us denote this expansion as $f(q_k)$. Notice that the three functions $f(\tau)$ and $f(q_k)$ and $q_k(\tau)$ are all locally biholomorphic, it makes sense to consider the relation of the Schwarzian derivatives among the three of them.

Proposition 1.4.3. *Let* $f : \mathbb{H} \to \mathbb{CP}^1 \backslash \{a_1 = 0, \ldots, a_n\}, \tau \mapsto f(\tau)$ *be a covering map and* $q_k(\tau) = \exp\{\frac{2\pi i}{k}\tau\}$, *then we have the following relation*

$$\{f, \tau\} = \frac{4\pi^2}{k^2}(1 - q_k^2\{f, q_k\}). \tag{1.4.1}$$

Proof. Assume $w = q_k = e^{\frac{2\pi}{k}i\tau}$ in Proposition 1.4.2, we have equalities

$$(q_k)_\tau = \frac{2\pi i}{k}q_k, \qquad (q_k)_{\tau\tau} = -(\frac{2\pi}{k})^2 q_k, \qquad (q_k)_\tau^2 = -(\frac{2\pi}{k})^2 q_k^2,$$

and

$$\{q_k, \tau\} = 0 - (\frac{2\pi i}{k})^2 = \frac{4\pi^2}{k^2}.$$

Direct calculation leads to the conclusion of this proposition,

$$\{f, \tau\} = -(\frac{2\pi}{k})^2 q_k^2\{f, q_k\} + (\frac{2\pi}{k})^2 = \frac{4\pi^2}{k^2}(1 - q_k^2\{f, q_k\}).$$

$\square$

It is beneficial to have a closer look at the expansion of $\{f, q_k\}$ in q_k.

Proposition 1.4.4. *If a covering map $f : \mathbb{H} \to \mathbb{CP}^1 \backslash \{a_1 = 0, \ldots, a_n\}$ has the following expansion*

$$f(\tau) = f(q_k) = Aq_k + Bq_k^2 + c_3 q_k^3 + c_4 q_k^4 + \sum_{m=5}^{\infty} c_m q_k^m \tag{1.4.2}$$

in $q_k = \exp\{\frac{2\pi i}{k}\tau\}$, where $\tau \in \mathbb{H}$. Then the Schwarzian derivative of $f(q_k)$ with respect to q_k has the following expansion

$$\{f, q_k\} = P_0(\mathrm{B}, \mathrm{C}_3) + \sum_{m=1}^{\infty} P_m(\mathrm{B}, \mathrm{C}_3, \ldots, \mathrm{C}_{m+3}) q_k^m \tag{1.4.3}$$

in q_k, where $\mathrm{B} = \frac{B}{A}$, $\mathrm{C}_{m+3} = \frac{c_{m+3}}{A}$ and the coefficient term $P_m(\mathrm{B}, \mathrm{C}_3, \ldots, \mathrm{C}_{m+3})$ is a polynomial in $\mathrm{B}, \mathrm{C}_3, \ldots, \mathrm{C}_{m+3}$ with degree 1 in C_{m+3}, and degree $m+2$ in B, $m \geq 0$. Specially $P_0(\mathrm{B}, \mathrm{C}_3) = 12(\mathrm{C}_3 - \mathrm{B}^2)$.

Proof. We will write $f'(q_k) = \frac{\mathrm{d} f(q_k)}{\mathrm{d} q_k}$ and $f''(q_k) = \frac{\mathrm{d}}{\mathrm{d} q_k}\left(\frac{\mathrm{d} f}{\mathrm{d} q_k}\right)$ for convenience. Direct calculation gives the following equations

$$f'(q_k) = A + 2Bq_k + 3c_3 q_k^2 + \sum_{m=4}^{\infty} m c_m q_k^{m-1} = A + A \cdot \left(2\mathrm{B} q_k + \sum_{m=2}^{\infty} Q_m^{(1)}(\mathrm{C}_{m+1}) q_k^m\right),$$

$$f''(q_k) = 2B + 6c_3 q_k + \sum_{m=4}^{\infty} m(m-1) c_m q_k^{m-2} = A\left(2\mathrm{B} + \sum_{m=1}^{\infty} Q_m^{(2)}(\mathrm{C}_{m+2}) q_k^m\right),$$

where $Q_m^{(1)}(\mathrm{C}_{m+1})$ is a polynomial in C_{m+1} with degree 1, and $Q_m^{(2)}(\mathrm{C}_{m+2})$ is a polynomial in C_{m+2} with degree 1 as well. We have the following calculation

$$\begin{aligned} 1/f'(q_{_k}) &= \frac{1}{A}\cdot\frac{1}{1+\left(2Bq_{_k}+\sum_{m=2}^{\infty}Q_m^{(1)}(C_{m+1})q_{_k}^m\right)}, \\ &= \frac{1}{A}\cdot\left[1-2Bq_{_k}+\left(4B^2-3C_3\right)q_{_k}^2+\sum_{m=3}^{\infty}Q_m^{(3)}(B,C_3,\ldots,C_{m+1})q_{_k}^m\right], \end{aligned} \tag{1.4.4}$$

where the coefficients $Q_m^{(3)}(B, C_3, \ldots, C_{m+1})$ are polynomials in $B, C_3, \ldots, C_{m+1}$, which has degree 1 in C_{m+1} with constant coefficient. We applied the expansion $\frac{1}{1-z} = 1+z+z^2+\sum_{m=3}^{\infty} z^m$ in the above calculation. The series (1.4.4) converges since that $f'(q_{_k})$ is still analytic on the unit disk $\mathbb{D}$ (see [Ahl78, p. 179-182]). Then we have the following equation

$$f''(q_{_k})/f'(q_{_k}) = 2B + (6C_3 - 4B^2)q_{_k} + \sum_{m=2}^{\infty} Q_m^{(4)}(B, C_3, \ldots, C_{m+2})q_{_k}^2,$$

where $Q_m^{(4)}$ are polynomials in $B, C_3, \ldots, C_{m+2}$ with degree 1 in C_{m+2} with constant coefficient. Continue computing the terms in $\{f, q_{_k}\} = 2\left(\frac{f''(q_k)}{f'(q_k)}\right)' - \left(\frac{f''(q_k)}{f'(q_k)}\right)^2$, the Schwarzian derivative can be given by the following expansion

$$\begin{aligned} \{f, q_{_k}\} &= \sum_{m=0}^{\infty} Q_m^{(5)}(B, C_3, \ldots, C_{m+3})q_{_k}^m - \sum_{m=0}^{\infty} Q_m^{(6)}(B, C_3, \ldots, C_{m+2})q_{_k}^m, \\ &= 12(C_3 - B^2) + \sum_{m=1}^{\infty} P_m(B, C_3, \ldots, C_{m+3})q_{_k}^m \end{aligned}$$

in $q_{_k}$, where $Q_m^{(5)}$ is a polynomial in $B, C_3, \ldots, C_{m+3}$ which has degree 1 in C_{m+3} with constant coefficient; $Q_m^{(6)}$ is a polynomial in $B, C_3, \ldots, C_{m+2}$ that has degree 1 in C_{m+2} but with coefficient in terms of B. Therefore the coefficient $P_m(B, C_3, \ldots, C_{m+2}) =$

$Q_m^{(5)} - Q_m^{(6)}$ of q_k^m is a polynomial in $\mathrm{B}, \mathrm{C}_3, \ldots, \mathrm{C}_{m+2}$, which has degree 1 in C_{m+3} with constant coefficient. □

1.4.2 Differential Equation

Recall section 1.3.1, the branches of the inverse $\tau(f)$ are related by linear transformations, Proposition 1.4.2 implies that $\{\tau(f), f\} = \{\tau, f\}$ is well-defined. In order to discuss its analytic property, we consider the following uniformizing function near each singularity $a \in \{a_1, \ldots, a_n\}$,

$$\Phi(f) = \exp\{-\frac{2\pi i}{k}\frac{1}{\tau(f) - r}\}, \quad \text{when } r \neq \infty, \tag{1.4.5}$$

or

$$\Phi(f) = \exp\{\frac{2\pi i}{k}\tau(f)\}, \quad \text{when } r = \infty, \tag{1.4.6}$$

where r and k are the constants with respect to the parabolic transformation γ in Propositions 1.3.5 and 1.3.6, and γ is the generator corresponding to the singularity a. On the other side, from section 1.3.1, the uniformizing functions are single valued in a neighborhood of a. So the uniformizing function $\Phi(f)$ has the following expansion near a,

$$\Phi(f) = \alpha_a(f - a) + \beta_a(f - a)^2 + (f - a)^2\phi(f), \quad a \neq \infty, \alpha_a \neq 0, \tag{1.4.7}$$

$$\Phi(f) = \alpha_\infty f + \beta_\infty + \varphi(\frac{1}{f}), \quad a = \infty, \alpha_\infty \neq 0, \tag{1.4.8}$$

where $\phi(f)$ and $\varphi(\frac{1}{f})$ are regular at a and satisfying

$$\begin{aligned} \phi(a) &= 0, \qquad \text{when } a \neq \infty, \\ \varphi(\frac{1}{a}) = \varphi(0) &= 0, \qquad \text{when } a = \infty. \end{aligned}$$

Assume one of the singularities is 0, and the parabolic transformation γ that associates to this singularity $a = 0$ fixes $r = \infty$, and k is the constnat in Proposition 1.3.6, then we have the following analytic expansion of the uniformizing function $\Phi(f)$:

$$\exp\{\frac{2\pi i}{k}\tau(f)\} = \alpha_{(a=0)} f + \beta_{(a=0)} f^2 + f^2\phi(f). \tag{1.4.9}$$

We use equation (1.4.9) to calculate its Schwarzian derivative $\{\tau, f\}$. Notice that differentiation will make the branch problem no longer an issue if we take the derivative of its logarithm, we write $\alpha = \alpha_{(a=0)}$ and $\beta = \beta_{(a=0)}$ for convenience,

$$\begin{aligned} \frac{2\pi i}{k}\tau(f) &= \log\alpha + \log f + \log(1 + \frac{\beta}{\alpha}f + \frac{1}{\alpha}f\phi(f)), \\ \frac{2\pi i}{k}\tau_f &= \frac{1}{f} + \frac{\frac{\beta}{\alpha} + \frac{1}{\alpha}(\phi(f) + f\phi'(f))}{1 + \frac{\beta}{\alpha}f + \frac{1}{\alpha}f\phi(f)} = \frac{1}{f} + \frac{\beta}{\alpha} + \phi_1(f), \\ \log\frac{2\pi i}{k} + \log\tau_f &= \log\frac{1}{f} + \log\left(1 + \frac{\beta}{\alpha}f + f\phi_1(f)\right), \\ \frac{\tau_{ff}}{\tau_f} &= -\frac{1}{f} + \frac{\frac{\beta}{\alpha} + \phi_2(f)}{1 + \frac{\beta}{\alpha}f + f\phi_1(f)} = -\frac{1}{f} + \frac{\beta}{\alpha} + \phi_3(f), \end{aligned}$$

where each $\phi_l(f)$, $l = 1, 2, 3$, is regular at $a = 0$, and $\phi_l(0) = 0$. Therefore the Schwarzian derivative has expansion around $a = 0$ as the following equation

$$\begin{aligned} \{\tau, f\} &= 2\left(\frac{\tau_{ff}}{\tau_f}\right)_f - \left(\frac{\tau_{ff}}{\tau_f}\right)^2, \\ &= \frac{1}{f^2} + \frac{2\beta}{\alpha}\frac{1}{f} + \phi_4(f), \end{aligned} \tag{1.4.10}$$

where $\phi_4(f)$ is regular at $a = 0$. Recall that $\{\tau, f\}$ is invariant under any linear transformation on τ, so we have the following equality

$$\{\tau(f), f\} = \{\frac{1}{\tau(f) - r}, f\}.$$

For the special case $a = \infty$, equation (1.4.8) implies that $\{\tau, f\}$ does not have a singularity at $a = \infty$. Therefore, due to Liouville's theorem, we have the following conclusion from equation (1.4.10) (see [Neh75, p. 201]),

$$\{\tau, f\} = \begin{cases} \displaystyle\sum_{j=1}^{n} \frac{1}{(f - a_j)^2} + \frac{2\beta_j/\alpha_j}{f - a_j}, & \text{if } \infty \notin \{a_1, \dots, a_n\}, \\ \displaystyle\sum_{j=1}^{n-1} \frac{1}{(f - a_j)^2} + \frac{2\beta_j/\alpha_j}{f - a_j}, & \text{if } a_n = \infty. \end{cases} \tag{1.4.11}$$

From now on, we will fix one of the singularities to be 0. Assume $f : \mathbb{H} \to \mathbb{CP}^1 \setminus \{a_1 = 0, a_2, \dots, a_n\}, \tau \mapsto f(\tau)$ is a covering map, then we have the following equation

$$\{\tau, f\} = \frac{1}{f^2} + \frac{2\beta}{\alpha}\frac{1}{f} + \sum_{j=2}^{n}\left[\frac{1}{(f - a_j)^2} + \frac{2\beta_j}{\alpha_j}\frac{1}{f - a_j}\right], \tag{1.4.12}$$

where $\alpha = \alpha_{(a_1=0)}, \beta = \beta_{(a_1=0)}$, and $\alpha_j = \alpha_{a_j}, \beta_j = \beta_{a_j}$, $j = 2, \ldots, n$, are the coefficients of the first and second order leading terms in the analytic expansion (1.4.5) of the unifomizing function at the singularity a_j. It is worth to mention that α_j, β_j are uniquely determined by the given covering map f due to its analytic property.

Let us recall Propositions 1.4.2 and 1.4.3, we have the following equation

$$\{\tau, f\} = -\frac{4\pi^2}{k^2}(1 - q_{_k}^2\{f, q_{_k}\})\tau_f^2$$

and equation

$$\{\tau, f\} = \frac{1}{f_{q_k}^2}(\frac{1}{q_{_k}^2} - \{f, q_{_k}\}) \tag{1.4.13}$$

due to $\tau_f = \frac{k}{2\pi i} q_{_k}^{-1} f_{q_k}^{-1}$ from chain rule. Therefore, we have the following relation that is connected by $\{\tau, f\}$,

$$\frac{1}{f_{q_k}^2}(\frac{1}{q_{_k}^2} - \{f, q_{_k}\}) = \frac{1}{f^2} + \frac{2\beta}{\alpha}\frac{1}{f} + \sum_{j=2}^{n}\left[\frac{1}{(f-a_j)^2} + \frac{2\beta_j}{\alpha_j}\frac{1}{f-a_j}\right]. \tag{1.4.14}$$

From equations (1.4.3) and (1.4.4), we have the following expansion of the left hand side of equation (1.4.14),

$$\frac{1}{A^2}\left[\frac{1}{q_k^2} - 4\mathrm{B}\frac{1}{q_k} + (24\mathrm{B}^2 - 18\mathrm{C}_3) + \sum_{m=1}^{\infty}\tilde{P}_m(\mathrm{B}, \mathrm{C}_3, \ldots, \mathrm{C}_{m+3})q_{_k}^m\right], \tag{1.4.15}$$

where the coefficient term $\tilde{P}_m(\mathrm{B}, \mathrm{C}_3, \ldots, \mathrm{C}_{m+3})$ is a polynomial in $\mathrm{B}, \mathrm{C}_3, \ldots, \mathrm{C}_{m+3}$ which has degree 1 in C_{m+3} for $m \geq 1$. For convenience, we denote the constant term $\tilde{P}_0(\mathrm{B}, \mathrm{C}_3) = (24\mathrm{B}^2 - 18\mathrm{C}_3)$.

On the punctured sphere $\mathbb{CP}^1 \setminus \{a_1 = 0, a_2, \ldots, a_n\}$, the singularities are discrete since it is a set of finite points. Take a neighborhood U which contains the one and only singularity $a_1 = 0$, then we have the following expansion in U,

$$\frac{1}{f} = \frac{1}{A}\frac{1}{q_k}[1 - \mathrm{B}q_k + \sum_{m=2}^{\infty} Q_m^{(7)}(\mathrm{B}, \mathrm{C}_3, \ldots, \mathrm{C}_{m+1})q_k^m], \tag{1.4.16}$$

where $Q_m^{(7)}$ is a polynomial in $\mathrm{B}, \mathrm{C}_3, \ldots, \mathrm{C}_{m+1}$ which has degree 1 in C_{m+1} with constant coefficient. And for other terms $\frac{1}{f-a_j}$, $j = 2, \ldots, n$, it has expansion

$$\frac{1}{f - a_j} = -\frac{1}{a_j} \cdot \left[1 + \frac{A}{a_j}q_k + \sum_{m=2}^{\infty} Q_m^{(8)}(\frac{A}{a_j}, \frac{c_2}{a_j}, \ldots, \frac{c_m}{a_j})q_k^m\right], \tag{1.4.17}$$

where $c_2 = B$, $Q_m^{(8)}$ is a polynomial in $\frac{A}{a_j}, \frac{c_2}{a_j}, \ldots, \frac{c_m}{a_j}$ which has degree 1 in $\frac{c_m}{a_j}$ with constant coefficient. Therefore equation (1.4.12) on the right hand side of equation (1.4.14) has expansion in q_k as the following

$$\begin{aligned}\frac{1}{A^2}\Bigg\{\frac{1}{q_k^2} - 4\mathrm{B}\frac{1}{q_k} + \left[5\mathrm{B}^2 - 2\mathrm{C}_3 + A^2\sum_{j=2}^{n}\left(\frac{1}{a_j^2} - \frac{1}{a_j}\frac{2\beta_j}{\alpha_j}\right)\right] \\ + \sum_{m=1}^{\infty}\left[Q_m^{(9)}(\mathrm{B}, \mathrm{C}_3, \ldots, \mathrm{C}_{m+3}) + \sum_{j=2}^{n} Q_{j_m}^{(10)}(\frac{1}{a_j}, c_1, \ldots, c_m)\right] q_k^m\Bigg\},\end{aligned} \tag{1.4.18}$$

where $c_1 = A, c_2 = B$, and $Q_m^{(9)}(\mathrm{B}, \mathrm{C}_3, \ldots, \mathrm{C}_{m+3})$ is a polynomial in $\mathrm{B}, \mathrm{C}_3, \ldots, \mathrm{C}_{m+3}$ that has degree 1 in C_{m+3} with constant coefficient; $Q_{j_m}^{(10)}(\frac{1}{a_j}, c_1, \ldots, c_m)$ is a polynomial in $\frac{1}{a_j}, c_1, \ldots, c_m$ which has degree 1 in c_m with constant coefficient. If we consider each $c_m = \mathrm{C}_m \cdot A$ for $m \geq 1$, then $Q_{j_m}^{(10)}$ is a polynomial $Q_{j_m}^{(10)'}$ in $A, \mathrm{B}, \mathrm{C}_3, \ldots, \mathrm{C}_m$. We identify equation (1.4.18) with equation (1.4.15) to get a set of equations

$$\tilde{P}_m(\mathrm{B}, \mathrm{C}_3, \ldots, \mathrm{C}_{m+3}) = Q_m^{(9)}(\mathrm{B}, \mathrm{C}_3, \ldots, \mathrm{C}_{m+3}) + \sum_{j=2}^{n} Q_{j_m}^{(10)'}(A, \mathrm{B}, \mathrm{C}_3, \ldots, \mathrm{C}_m), \tag{1.4.19}$$

where $m \geq 1$. And the constant term gives equation

$$24\mathrm{B}^2 - 18\mathrm{C}_3 = 5\mathrm{B}^2 - 2\mathrm{C}_3 + A^2 \sum_{j=2}^{n} \left(\frac{1}{a_j^2} - \frac{1}{a_j} \frac{2\beta_j}{\alpha_j} \right), \tag{1.4.20}$$

thus the solution for C_3 is given by the following equation

$$\mathrm{C}_3 = \mathrm{C}_3(A, \mathrm{B}) = \frac{1}{16} \left[19\mathrm{B}^2 - A^2 \sum_{j=2}^{n} \left(\frac{1}{a_j^2} - \frac{1}{a_j} \frac{2\beta_j}{\alpha_j} \right) \right]. \tag{1.4.21}$$

Notice that the left hand side of equation (1.4.19) is a polynomial in $\mathrm{B}, \mathrm{C}_3, \ldots, \mathrm{C}_{m+3}$ that has degree 1 in C_{m+3} with constant coefficient, and the right hand side of equation (1.4.19) is a polynomial in $A, \mathrm{B}, \mathrm{C}_3, \ldots, \mathrm{C}_{m+3}$ that also has degree 1 in C_{m+3} with constant coefficient. Let us start with $m = 1$, equation (1.4.19) has one and the only one unknown term C_4, we can solve for C_4 and the solution is a polynomial in C_3, A and B. The solution $\mathrm{C}_4(\mathrm{C}_3, A, \mathrm{B})$ can be expressed as a polynomial $\mathrm{C}_4(A, \mathrm{B})$ in A, B with constant coefficients since $\mathrm{C}_3 = \mathrm{C}_3(A, \mathrm{B})$. By induction, it is easy to conclude that C_m can be solved as a polynomial $\mathrm{C}_m(A, \mathrm{B})$ in A, B with constant coefficients for $m \geq 3$, i.e.,

$$\mathrm{C}_m = \mathrm{C}_m(A, \mathrm{B}, a_2, \ldots, a_n) = \mathrm{C}_m(A, \mathrm{B}), \tag{1.4.22}$$

where $a_2, \ldots, a_n$ are the singularities. Therefore we can conclude the following theorem.

Theorem 1.4.5. *Let $f : \mathbb{H} \to \mathbb{CP}^1 \setminus \{a_1 = 0, a_2, \ldots, a_n\}$ be a covering map, and the parabolic generator corresponding to $a_1 = 0$ fixes infinity, then the map f can be uniquely determined up to the coefficients A, B of the first two order leading terms of* *its expansion* (1.4.2), *i.e.,*

$$f/A = q_{_k} + \mathrm{B}q_{_k}^2 + \mathrm{C}_3(A,\mathrm{B})q_{_k}^3 + \sum_{m=4}^{\infty} \mathrm{C}_m(A,\mathrm{B})q_{_k}^m, \tag{1.4.23}$$

where $q_{_k} = \exp\{\frac{2\pi i}{k}\tau\}$, *and* $\mathrm{C}_m(A,\mathrm{B})$ *are polynomials in* $A, \mathrm{B} = \frac{B}{A}$ *with constant coefficients for* $m \geq 3$.

Recall Theorem 1.3.11 in section 1.3.2 and Theorem 1.4.5, they together imply our main result Theorem 1, which will be concluded in the last section.

1.5 Ramification Point

1.5.1 Introduction

In section 1.3.1, we concluded that the deck transformation group $\mathrm{Aut}(f)$ is generated by $(n-1)$ *parabolic* transformations from $\mathrm{Aut}(\mathbb{H})$. On the other hand, we will discover properties of such subgroups of $\mathrm{GL}_2(\mathbb{R})$. Notice that $\gamma(\tau) = (\frac{\gamma}{\det\gamma})(\tau)$, so we will only consider $\mathrm{SL}_2(\mathbb{R})$ from now. Consider $\mathrm{SL}_2(\mathbb{R})$ with the normal $\mathbb{R}^4$ topology. The general matrix multiplication

$$\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R}), \quad (\gamma_1, \gamma_2) \mapsto \gamma_1\gamma_2$$

and inversion

$$\mathrm{SL}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R}), \quad \gamma \mapsto \gamma^{-1}$$

are continuous maps.

Definition 1.5.1. We say that $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{H}$ continuously if the following map

$$\mathrm{SL}_2(\mathbb{R}) \times \mathbb{H} \to \mathbb{H}, \qquad (\gamma, \tau) \mapsto \gamma(\tau)$$

is continuous.

In general, if G is a subgroup of $\mathrm{SL}_2(\mathbb{R})$. An orbit of a point $\tau \in \mathbb{H}$ is the set of images of τ under actions in G, i.e.,

$$\mathrm{Orbit}_G(\tau) = \{g(\tau) \,|\, g \in G\}.$$

The stabilizer of a point $\tau \in \mathbb{H}$ are the elements in G that fixes τ, i.e.,

$$\mathrm{Stab}_G(\tau) = \{g \,|\, g(\tau) = \tau, g \in G\}.$$

We say that two points $\tau, \tau' \in \mathbb{H}$ are equivalent under the action of G if $\tau' \in \mathrm{Orbit}_G(\tau)\tau$, i.e.,

$$\tau \sim \tau' \qquad \text{if and only if} \qquad \tau' = g(\tau), \text{ for some } g \in G.$$

The set $\mathrm{Orbit}_G(\tau)$ is the set of points that are equivalent to τ under the action of G.

Definition 1.5.2. If G is a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$, define the quotient space $\mathbb{H}/G$ to be the space of equivalent classes of $\mathbb{H}$ under the action of G, i.e.,

$$\mathbb{H}/G = \mathbb{H}/\sim,$$

equipped the induced topology through the quotient map.

Let $\mathrm{SL}_2(\mathbb{R})/G$ to be the left cosets of G in $\mathrm{SL}_2(\mathbb{R})$, i.e.,

$$\mathrm{SL}_2(\mathbb{R})/G = \bigcup_{h \in H} hG,$$

where H is the left coset representation of G in $\mathrm{SL}_2(\mathbb{R})$. The topology on $\mathrm{SL}_2(\mathbb{R})/G$ is induced from the left group action.

Definition 1.5.3. A point $P \in \mathbb{H}$ is *elliptic* of a discrete subgroup $G \subset \mathrm{SL}_2(\mathbb{R})$ if it is invariant under the action of an *elliptic* element in G.

The goal for this section is to show that a point $P \in \mathbb{H}/G$ is *elliptic* if and only if the point $P \in \mathbb{H}/G$ is a ramification point.

1.5.2 Discrete Group, Quotient Space and Ramification Point

All the conclusions can be found in [Shi71, p. 2-20, Chapter 1]. For the completion of this article, I still state them.

Lemma 1.5.4. *$SO_2(\mathbb{R})$ is compact in $SL_2(\mathbb{R})$. Furthermore, the map*

$$\tilde{\alpha} : SL_2(\mathbb{R})/SO_2(\mathbb{R}) \to \mathbb{H}, \qquad [\gamma] \mapsto \tilde{\alpha}([\gamma]) = \gamma(i) \tag{1.5.1}$$

is a homeomorphism.

Proof. To show the compactness, define a map

$$\begin{array}{rccc} \varphi : & S^1 & \to & \mathrm{SL}_2(\mathbb{R}), \\ & e^{i\theta} & \mapsto & \varphi(e^{i\theta}) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \end{array}$$

Notice that φ is continuous and S^1 is compact, so its image $\varphi(S^1) = \mathrm{SO}_2(\mathbb{R})$ is also compact. To show $\tilde{\alpha}$ is a homeomorphism, it is equivalent to show that it is bijective and continuous and open. We define a map α on $\mathrm{SL}_2(\mathbb{R})$ as the following,

$$\alpha : \mathrm{SL}_2(\mathbb{R}) \to \mathbb{H}, \qquad \gamma \mapsto \alpha(\gamma) = \gamma(i).$$

Since $\mathrm{SL}_2(\mathbb{R})$ acts transitively on $\mathbb{H}$, i.e., $\mathrm{Orbit}_{\mathrm{SL}_2(\mathbb{R})}(\tau) = \mathbb{H}$ for any $\tau \in \mathbb{H}$, so the map α is onto, which implies that $\tilde{\alpha}$ is surjective. For any $\gamma_1, \gamma_2 \in \mathrm{SL}_2(\mathbb{R})$, the condition $\tilde{\alpha}([\gamma_1]) = \tilde{\alpha}([\gamma_2])$ holds if and only if the equation $\gamma_1(i) = \gamma_2(i)$ holds. If $\gamma_1(i) = \gamma_2(i)$ is true, then we have equality

$$\gamma_2^{-1}(\gamma_1(i)) = (\gamma_2^{-1}\gamma_1)(i) = i,$$

which implies the following relation

$$(\gamma_2^{-1}\gamma_1) \in \mathrm{Stab}_{\mathrm{SL}_1(\mathbb{R})}(i) = \mathrm{SO}_2(\mathbb{R}), \quad \text{equivalently} \quad [\gamma_1] = [\gamma_2],$$

thus α is injective. Therefore $\tilde{\alpha}$ is a bijection. The continuity of $\tilde{\alpha}$ follows from the continuity of α.

Now we only need to show it is open. Observe that if $x+yi \in \mathbb{H}$ such that $\alpha(\gamma) = x+yi$ for some $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$, then we have the following conclusion

$$ad - bc = 1, \quad \frac{ai+b}{ci+d} = x + yi, \qquad \text{implies} \qquad (x^2+y^2)d - bx - ay = 0,$$

since $y \neq 0, c = \frac{dx-b}{y}$. Suppose $\Omega \subseteq \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$ is open, we will show that any point $x_0 + y_0 i \in \tilde{\alpha}(\Omega)$ is an interior point. Notice that $x_0 + y_0 i$ is in the image $\tilde{\alpha}(\Omega)$, so there exists an element $\gamma_0 = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ such that $[\gamma_0] \in \Omega$ and $\gamma_0(i) = x_0 + y_0 i$, we have the following equation

$$(x_0^2 + y_0^2)d_0 - b_0 x_0 - a_0 y_0 = 0.$$

Define a function

$$\begin{aligned} s: \quad \mathbb{R}^4 \times \mathbb{R} &\to \mathbb{R}, \\ (x, y, a, b, d) &\mapsto (x^2 + y^2)d - bx - ay. \end{aligned}$$

Notice that $\partial s / \partial d = x^2 + y^2 > 0$ since $y > 0$. By implicit function theorem, there is a continuous differentiable function $g(x, y, a, b) = d$ on a neighborhood Ω_0 of (x_0, y_0, a_0, b_0) in $\mathbb{R}^4$ with $s(x, y, a, b, d) = 0$. Notice that $d = g(x_0, y_0, a, b_0) = \frac{y_0}{x_0^2+y_0^2} a + \frac{x_0}{x_0^2+y_0^2} b_0$ is linearly dependent on a, thus there is $\{x_0\} \times \{y_0\} \times (a_0 - \varepsilon, a_0 + \varepsilon) \times \{b_0\} \subseteq \Omega_0$ for some $\varepsilon > 0$ such that $(d_0 - \varepsilon', d_0 + \varepsilon') = U_{d_0} \subseteq g(\Omega_0)$ for some $\varepsilon' > 0$. Recall the fact that $\Omega \subseteq \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$ is open, without loss of generality, we assume $\{a_0\} \times \{b_0\} \times \{c_0\} \times U_{d_0} \subseteq \Omega$. Therefore the solution set $g^{-1}(U_{d_0}) \subseteq \mathbb{R}^4$ is open, so there is an neighborhood $U_{x_0} \times U_{y_0} \times \{a_0\} \times \{b_0\} \subseteq g^{-1}(U_{d_0})$ such that $U_{x_0} \times U_{y_0}$ is an open neighborhood of (x_0, y_0), thus it is an interior point. We can conclude that α is a homeomorphism. $\square$

Remark 1.5.1. Other proofs see [Shi71, p. 2, Theorem 1.1] or [Mil17, p. 25, Proposition 2.1 (d)].

Lemma 1.5.5. *Assume Γ is a discrete subgroup of $SL_2(\mathbb{R})$, if V_1, V_2 are any compact subsets in $\mathbb{H}$, then the set $\{\gamma \in \Gamma | \gamma(V_1) \cap V_2 \neq \emptyset\}$ is finite.*

Proof. Assume $V_1 \subseteq \mathbb{H}$ is compact, we will show that $\alpha^{-1}(V_1)$ is also compact in $\mathrm{SL}_2(\mathbb{R})$, where α is the same map defined in lemma 1.5.4. Since $\mathrm{SL}_2(\mathbb{R})$ has the normal topology in $\mathbb{R}^4$, we can consider the collection of balls $B_i = \{(a,b,c,d) \mid |(a,b,c,d)| < i\} \subseteq \mathrm{SL}_2(\mathbb{R})$ which forms an open cover of $\mathbb{R}^4$, we have the following conclusion

$$\mathrm{SL}_2(\mathbb{R}) \subseteq \cup_{i=1}^{\infty} B_i \quad \text{implies} \quad V_1 \subseteq \cup_{i=1}^{\infty} \alpha(B_i).$$

Thus there exists a finite number $m \in \mathbb{Z}$ such that $V_1 \subseteq \alpha(B_m)$ since V_1 is compact. Therefore the image of the closure $\overline{B_m}$ of B_m covers V_1, i.e., $V_1 \subseteq \alpha(\overline{B_m}) \subseteq \mathbb{H}$. Consider the composition with the inverse of α,

$$\alpha^{-1}(V_1) \subseteq \alpha^{-1}(\alpha(\overline{B_m})),$$

we conclude that $\alpha^{-1}(V_1)$ is closed because α is continuous and V_1 is compact so it is closed. Consider the right hand side $\alpha^{-1}(\alpha(\overline{B_m})) = \tilde{\alpha}^{-1}(\alpha(\overline{B_m})) \times \mathrm{SO}_2(\mathbb{R})$, $\alpha(\overline{B_m})$ is compact since $\overline{B_m}$ is compact, and $\tilde{\alpha}^{-1}(\alpha(\overline{B_m}))$ is compact since $\tilde{\alpha}$ is a homeomorphism. Therefore the set $\tilde{\alpha}^{-1}(\alpha(\overline{B_m})) \times \mathrm{SO}_2(\mathbb{R})$ is compact since $\mathrm{SO}_2(\mathbb{R})$ is compact. The pre-image $\alpha^{-1}(V_1)$ of V_1 is a closed subset in a compact subset, so it is also compact. Similarly, $\alpha^{-1}(V_2)$ is also compact. The fact that the intersection of a discrete set with a compact set $\Gamma \cap (\alpha^{-1}(V_1) \cup \alpha^{-1}(V_2))$ is finite, so there is only finitely γ such that $\gamma(V_1) \cap V_2 \neq \emptyset$. □

Remark 1.5.2. See [Shi71, p. 3, Proposition 1.6] or [Mil17, p. 26, Proposition 2.4] for other proofs.

Proposition 1.5.6. *If $\Gamma \in SL_2(\mathbb{R})$ is a discrete subgroup, then for any $\tau \in \mathbb{H}$, there is a neighborhood U of τ such that if $\gamma \in \Gamma$ and $U \cap \gamma(U) \neq \emptyset$, then $\gamma(\tau) = \tau$.*

Proof. For any $\tau \in \mathbb{H}$, let V be a compact neighborhood of τ in $\mathbb{H}$. By lemma 1.5.5, we assume there is a finite subset $\{\gamma_1, \ldots, \gamma_l\} \subseteq \Gamma$ such that $\gamma(V) \cap V \neq \emptyset$. Suppose $\{\gamma_1, \ldots, \gamma_s\}$ are the ones that fixes τ, i.e., $\gamma_i(\tau) = \tau$ for $1 \leq i \leq s$. And for any element γ_i which is from the rest of the set $\{\gamma_{s+1}, \ldots, \gamma_l\}$ does not fix τ, i.e., $\gamma_i(\tau) \neq \tau (s+1 \leq i \leq l)$. Take some disjoint neighborhoods $U_i^{(1)}$ of τ and $U_i^{(2)}$ of $\gamma_i(\tau)$ in V, consider the intersection

$$U = \cap_{i=s+1}^{l} (U_i^1 \cap \gamma_i^{-1}(U_i^2)),$$

then U is the neighborhood we are looking for. □

Remark 1.5.3. Also see [Shi71, p. 3, Proposition 1.7] or [Mil17, p. 27, Proposition 2.5 (b)].

Next, we are able to show that if Γ is a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$, a point $[\tau] = [\tau_0] \in \mathbb{H}/\Gamma$ is ramified if and only if following condition

$$\mathrm{Stab}_\Gamma(\tau) - \{I\} \neq \emptyset \tag{1.5.2}$$

holds, for any $\tau \in [\tau_0] \subset \mathbb{H}$. Recall Definition 1.5.3, a point P satisfying condition (1.5.2) if and only if P is an *elliptic* point. The following proposition can be concluded from [Shi71, p. 8, Proposition 1.18], we will present proof here since it is not obvious.

Proposition 1.5.7. *Let Γ be a discrete subgroup of $SL_2(\mathbb{R})$. Then on the quotient space $\mathbb{H}/\Gamma$, a point $\tau \in \mathbb{H}$ is ramified if and only if it is an elliptic point of Γ.*

Proof. If $\tau \in \mathbb{H}$ is not an elliptic point, then by Proposition 1.5.6, there is a neighborhood U of τ such that $\gamma(U) \cap U = \emptyset$, for any $\gamma \in \Gamma$. It implies that τ is not ramified. If $\tau \in \mathbb{H}$ is an elliptic point of Γ, assume there is an elliptic element $\gamma \in \Gamma$ such that $\gamma(\tau) = \tau$. Let $\sigma \in \mathrm{SL}_2(\mathbb{R})$ such that $\sigma(i) = \tau$, then the composition $\sigma^{-1} \circ \gamma \circ \sigma \in \mathrm{Stab}(i) = \mathrm{SO}_2(\mathbb{R})$, so $< \sigma^{-1} \circ \gamma \circ \sigma >= \sigma^{-1} \circ < \gamma > \circ \sigma$ is a subgroup of the conjugate $\sigma^{-1}\Gamma\sigma$, which is still a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$. Also we have the following correspondence relation

$$< \gamma > \cong \sigma^{-1} < \gamma > \sigma = \mathrm{SO}_2(\mathbb{R}) \bigcap \sigma^{-1}\Gamma\sigma,$$

the right hand side is an intersection of a compact set and a discrete set, so it has to be finite. Therefore if $m = \#\mathrm{Stab}_\Gamma(\tau)$, τ is a ramification point of index m. □

1.6 Modular Group

1.6.1 The Full Modular Group $\Gamma(1)$

In this section, we will start with introducing the full modular group.

Definition 1.6.1. The full modular group $\Gamma(1)$ is defined to be the image of $\mathrm{SL}_2(\mathbb{Z})$ by identifying $+\gamma$ and $-\gamma$ for any element $\gamma \in \mathrm{SL}_2(\mathbb{R})$. Equivalently, it is the same as the following definition:

$$\Gamma(1) = \mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\pm I.$$

In this article, we focus on the principal congruence subgroups of $\Gamma(1)$.

Definition 1.6.2. The principle congruence subgroup of level N is defined as the following:

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod N \right\} / \pm I.$$

The goal is to determine all the principal congruence subgroups $\Gamma(N)$ such that they are candidates for $\mathrm{Aut}(f)$, i.e., $\mathbb{H}/\mathrm{Aut}(f) = \mathbb{H}/\Gamma(N) \cong \mathbb{CP}^1 \backslash \{a_1 = 0, \ldots, a_n\}$ for some $n \geq 3$. Notice that $\mathbb{CP}^1 \backslash \{a_1 = 0, \ldots, a_n\}$ does not have any ramification point, and the compactification of $\mathbb{CP}^1 \backslash \{a_1 = 0, \ldots, a_n\}$ is the Riemann sphere, which is of genus *zero*. We will use these two properties to determine all suitable $\Gamma(N)$. First, we will use the *fundamental domain* of $\Gamma(1)$ to locate all ramification points of $\mathbb{H}/\Gamma(1)$.

Definition 1.6.3. The *fundamental domain* for a discrete subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{R})$ is a connected open subset D of $\mathbb{H}$ such that every pair of points in D are inequivalent under Γ, and meanwhile $\mathbb{H} \subseteq \bigcup_{\gamma \in \Gamma} [\gamma(\overline{D})]$.

These conditions are equivalent to $D \to \mathbb{H}/\Gamma$ is injective and $\overline{D} \to \mathbb{H}/\Gamma$ is surjective. Now we recall the *fundamental domain* of $\Gamma(1)$ (see [Shi71, p. 16], [Bum97, p. 19, Proposition 1.2.2] or [Ser73, p. 78-79]).

Theorem 1.6.4. *The fundamental domain of* $\Gamma(1)$ *is the set*

$$D = \{z = x + yi \,|\, |z| > 1, |x| < \frac{1}{2}, y > 0\}. \tag{1.6.1}$$

Proposition 1.6.5. *Let* p *be the quotient map*

$$p : \mathbb{H} \to \mathbb{H}/\Gamma(1), \qquad \tau \mapsto [p(\tau)].$$

The elliptic points in $\mathbb{H}/\Gamma(1)$ *are either* $[i]$ *or* $[\rho]$*, i.e.,*

$$\{\textit{ramification points of } \mathbb{H} \to \mathbb{H}/\Gamma(1)\} = p^{-1}(i) \bigcup p^{-1}(\rho),$$

where $\rho = e^{\frac{\pi}{3}i} = \frac{1+\sqrt{3}i}{2}$*, the root of* 1 *of order* 6*.*

Proof. Assume $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an *elliptic* element in $\Gamma(1)$, so $a + d = 0$ or ± 1. By solving a quadratic equation, an *elliptic* point z_0 satisfies the following equation

$$z_0 = \frac{a-d}{2c} + \frac{\sqrt{(a+d)^2 - 4}}{2c}, \qquad \text{we assume } c > 0 \text{ from now on.}$$

Therefore z_0 takes values from the following choices

1. $z_0 = \frac{a}{c} + \frac{i}{c}$, or
2. $z_0 = \frac{2a \pm 1}{2c} + \frac{1}{c}\frac{\sqrt{3}i}{2}$,

where $c \in \mathbb{Z}^+, a \in \mathbb{Z}$. On the other side, consider the *fundamental domain* D of $\Gamma(1)$ in theorem 1.6.4 and property in remark **??**. It implies that $\overline{D}$ contains all representative points of equivalent classes of *elliptic* points, so we only need to identity the *elliptic* points in $\overline{D}$. If it is case (1), $\text{Im}(z_0) = \frac{i}{c} \in \{i, \frac{i}{2}, \frac{i}{3}, \cdots\}$, the only choice for $z_0 \in \overline{D}$ is $\text{Im}(z_0) = i$, $c = 1$, and accordingly $a = 0$. Take $b = -1, d = 0$, the point $z_0 = i$ is invariant under the *elliptic* matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. For case (2), $\text{Im}(z_0) = \{\frac{\sqrt{3}i}{2}, \frac{\sqrt{3}i}{4}, \frac{\sqrt{3}i}{6}, \cdots\}$, the only choice is $\text{Im}(z_0) = \frac{\sqrt{3}i}{2}$ when $c = 1$, then $a = 0$, $z_0 = \rho$ or ρ^2. $\rho = \frac{1+\sqrt{3}i}{2}$ is invariant under $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$, and $\rho^2 = \frac{-1+\sqrt{3}i}{2}$ is invariant under

$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$. Notice that ρ and ρ^2 are equivalent in $\Gamma(1)$ by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Therefore we can conclude that every *elliptic* point of $\Gamma(1)$ is either equivalent to i or ρ.

□

Remark 1.6.1. See [Shi71, p. 14-15, section 1.4].

1.6.2 Genus Formula for Subgroups of Modular Group

We consider a subgroup Γ of $\Gamma(1)$ with finite index, then the quotient space $\mathbb{H}/\Gamma$ is composed by copies of $\mathbb{H}/\Gamma(1)$. The genus of $\mathbb{H}/\Gamma$ can be computed by applying the Riemann-Roch Theorem and the Riemann-Hurwitz Formula.

Proposition 1.6.6. *If Γ is a subgroup of $\Gamma(1)$ with finite index m, then the genus g of $\mathbb{H}/\Gamma$ is given by the following formula*

$$g = 1 + m/12 - \nu_2/4 - \nu_3/3 - \nu_\infty/2, \tag{1.6.2}$$

where ν_2 is the number of ramification points of order 2, ν_3 is the number of ramification points of order 3, ν_∞ is the number of cusps and $m = [\Gamma(1) : \Gamma]$.

Remark 1.6.2. See [Shi71, p. 23, Proposition 1.40] or [Mil17, p. 37-38, Theorem 2.22] for further references.

we need the following conclusions. (see [**?**])

Theorem 1.6.7 (Riemann-Roch Theorem)**.** *If X is a compact Riemann surface, K is a canonical divisor, define g =genus(X). The following equality*

$$l(D) = 1 - g + deg(D) + l(K - D), \tag{1.6.3}$$

holds for any divisor D on X,

Corollary 1.6.8. *If K is a canonical divisor on X, then the following equalities*

1. $l(K) = g$,

2. $deg(K) = 2g - 2$,

are true.

Proof. For case 1, let $D = 0$. then $l(D) = 0$, $\deg(D) = 0$ and $l(K - D) = l(K)$, by theorem 1.6.7 we have equation

$$l(K) = g - 1 + l(D) + \deg(D) = g.$$

To prove case 2, let $D = K$, apply theorem 1.6.7 again, we have $l(K) = g$ and $l(K - D) = l(0) = 1$, therefore $l(K) = 1 - g + \deg(K) + l(0)$ holds, we have conclusion

$$\deg(K) = 2g - 2.$$

□

Theorem 1.6.9 (Riemann-Hurwitz Formula). *Let X, Y be Riemann surfaces, assume $s : Y \to X$ is a biholomorphic covering map of degree m with finite ramification points. Let R be the set of ramification points on Y, e_P be the ramification index at $P \in R$, then the following formula*

$$2g(Y) - 2 = m(2g(X) - 2) + \sum_{P \in R} (e_P - 1), \tag{1.6.4}$$

holds.

Proof. Let ω be a canonical divisor on X such that it does not have zeros or poles at any ramification point $P \in R$. On X, we have equation

$$\deg(\omega) = 2g(X) - 2.$$

Consider the pull-back $1-$form $s^*\omega$ on Y, we will calculate its degree. On un-ramified points, the degree satisfies relation

$$m \cdot \deg(\omega) = m \cdot (2g(X) - 2)$$

since it is $m : 1$ on un-ramified points. On a ramified point $P \in R$, let e_P be the multiplicity of P in the fiber of s, then locally we have coordinate

$$X \to Y, \quad z \mapsto z^{e_P} = z'.$$

so we have $dz' = d(z^{e_P}) = e_P z^{e_P - 1} dz$, therefore the total degree on R is given by the summation

$$\Sigma_{P \in R}(e_P - 1).$$

Therefore we conclude the formula

$$\deg(s^*\omega) = 2g(Y) - 2 = m(2g(X) - 2) + \sum_{P \in R}(e_P - 1).$$

□

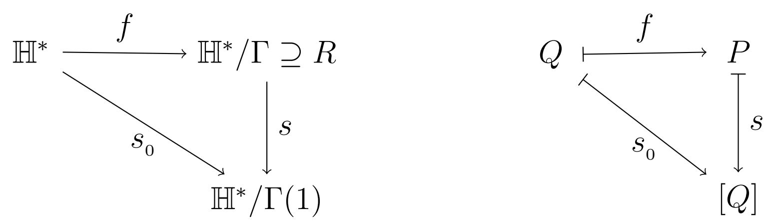
Now, we are able to prove proposition 1.6.6.

Proof of Proposition 1.6.6. Let $Y = \mathbb{H}^*/\Gamma$ and $X = \mathbb{H}^*/\Gamma(1)$ in theorem 1.6.9. Since

$g(X) = g(\mathbb{H}^*/\Gamma(1)) = 0$, we have following equation

$$g(\mathbb{H}^*/\Gamma) = g(Y) = \frac{1}{2}m(0-2) + \frac{1}{2}\sum_{P\in R}(e_P - 1) = 1 - m + \frac{1}{2}\sum_{P\in R}(e_P - 1).$$

We will count e^P for each $P \in R$. Write the maps s, s_0, f in the following chart



Let $e_s(P/[Q])$ be the ramification index of P in s, and let $e_{s_0}(Q/[Q])$ and $e_f(Q/P)$ be the index of Q in s_0 and f respectively. Since above diagram commute, we have relation

$$e_{s_0}(Q/[Q]) = e_f(Q/P) \cdot e_s(P/[Q]).$$

Since $[i], [\rho]$ are the only ramification points of s_0, so $R \subseteq s^{-1}([i]) \cup s^{-1}([\rho])$. For any $P \in R$, $s(P)$ is either $[i]$, $[\rho]$ or $[\infty]$, consider each case separately

1. $s(P) = [i]$, $e_f(Q/P) \cdot e_s(P/[i]) = e_{s_0}(Q/[i]) = 2$, so $e_s(P/[i]) = 1$ or 2.

   (a) $e_s(P/[i]) = 1$, $e_f(Q/P) = 2$. P is not ramified in s, but P is ramified in f, let ν_2 be the number of this kind points in $\mathbb{H}^*/\Gamma$.

   (b) $e_s(P/[i]) = 2$, $e_f(Q/P) = 1$. P is ramified with index 2 in s, but P is not ramified in f. $\#(s^{-1}([i])) = m$ since s is $(m:1)$, so there are a number of $\frac{1}{2}(m - \nu_2)$ for this type of points.

2. $s(P) = [\rho]$, $e_f(Q/P) \cdot e_s(P/[\rho]) = e_{s_0}(Q/[\rho]) = 3$, thus $e_s(P/[\rho]) = 1$ or 3.

(a) $e_s(P/[\rho]) = 1$, $e_f(Q/P) = 3$. P is not ramified in s, but P is ramified in f, let ν_3 be the number of this kind points in $\mathbb{H}^*/\Gamma$.

(b) $e_s(P/[\rho]) = 3$, $e_f(Q/P) = 1$. P is ramified with index 3 in s, there are $\frac{1}{3}(m - \nu_3)$ points of this kind.

3. $s(P) = [\infty]$, $e_f(Q/P) = e_{s_0}(Q/[\infty]) = \infty$, if P is ramified, then by definition we have the following equation

$$\sum_{P \in s^{-1}([\infty])} e_s(P/[\infty]) = m.$$

Let ν_∞ be the number of cusps of $\mathbb{H}^*/\Gamma$, i.e., the number of points P such that $e_s(P/[\infty]) > 1$.

Therefore the sum is given by the following calculation

$$\begin{aligned}
\sum_{P \in R}(e_s(P) - 1) &= \sum_{e_s(P/[i])=2}(2-1) + \sum_{e_s(P)/[\rho]}(3-1) + \sum_{e_s(P/[\infty])=m}(e_s(P/[\infty]) - 1), \\
&= \frac{1}{2}(m - \nu_2) + \frac{2}{3}(m - \nu_3) + (m - \nu_\infty), \\
&= \frac{11}{6}m - \frac{1}{2}\nu_2 - \frac{2}{3}\nu_3 - \nu_\infty.
\end{aligned}$$

The genus formula is completed as the following

$$\begin{aligned}
g(Y) &= 1 - m + \frac{1}{2}[\frac{11}{6}m - \frac{1}{2}\nu_2 - \frac{2}{3}\nu_3 - \nu_\infty], \\
&= 1 + \frac{1}{12}m - \frac{1}{4}\nu_2 - \frac{1}{3}\nu_3 - \frac{1}{2}\nu_\infty.
\end{aligned}$$

□

1.6.3 Principle Congruence Subgroups

Recall that our subject is the finitely punctured Riemann sphere, if we consider it as a quotient space $\mathbb{H}/\Gamma(N)$ for some suitable $N \in \mathbb{Z}$, the fact that the punctured Riemann sphere has genus *zero* implies $g(\Gamma(N)) = 0 < 1$. The following theorem will show that $N = 2, 3, 4, 5$ are the only values such that $g(\Gamma(N)) = 0$.

Theorem 1.6.10. *The genus of* $\mathbb{H}/\Gamma(N)$ *is given by the following formula:*

$$g(\Gamma(N)) = \begin{cases} 0, & N = 2, \\ 1 + \frac{N-6}{24} \cdot N^2 \prod_{p|N}(1 - p^{-2}), & N \geq 3. \end{cases} \tag{1.6.5}$$

Recall that if $N > 1$, $\Gamma(N)$ does not have *elliptic* element, which is equivalent to the fact that $\mathbb{H}/\Gamma(N)$ does not contain any elliptic point, so it does not have any ramification point as well. Therefore $\nu_2 = \nu_3 = 0$ in equation (1.6.2), the genus is given by the following simplified formula

$$g(\Gamma(N)) = 1 + m/12 - \nu_\infty/2, \qquad n \geq 2. \tag{1.6.6}$$

By the properties of elements in $\Gamma(N)$, the index m of $\Gamma(N)$ in $\Gamma(1)$ is given by the following proposition.

Proposition 1.6.11. *The index* m *of a principle congruence group* $\Gamma(N)$ *in the full modular group is given by the formula*

$$m = (\Gamma(1) : \Gamma(N)) = \begin{cases} 6, & N = 2, \\ \frac{1}{2}\#(SL_2(\mathbb{Z}/N\mathbb{Z})) = \frac{1}{2}N^3 \prod_{p|N}(1 - p^{-2}), & N \geq 3, \end{cases} \tag{1.6.7}$$

where p *are the prime divisors of* N.

Proof. See [Shi71, p. 22, Equation (1.62)] (where $\tilde{\Gamma}(N)$ is in our notation $\Gamma(N)$). □

Consider the stabilizer of ∞ in $\Gamma(1)$ and the stabilizer of ∞ in $\Gamma(N)$, we have the following proposition.

Proposition 1.6.12. *The number of cusps of* $\Gamma(N)$ *is* $\nu_\infty = m/N$*, where* $N \geq 2$*.*

Proof. See [Shi71, p. 22-23]. □

Therefore equation (1.6.6) becomes

$$g(\Gamma(N)) = 1 + \frac{m}{12N}(N-6), \tag{1.6.8}$$

equation (1.6.8) and (1.6.10) imply equation (1.6.5). Therefore the number of cusps of $\Gamma(N)$ is given by the formula:

$$\nu_\infty(\Gamma(N)) = m/N = \begin{cases} 3, & N = 2, \\ \frac{1}{2}N^2\Pi_{p|N}(1-p^{-2}), & N \geq 3. \end{cases}$$

We list the number of cusps for each $\Gamma(N)$, $N = 2, 3, 4, 5$,

$$\begin{aligned} \nu_\infty(\Gamma(2)) &= \frac{m(\Gamma(2))}{2} = \frac{6}{2} = 3, & \nu_\infty(\Gamma(3)) &= \frac{1}{2} \cdot 9 \cdot \frac{8}{9} = 4, \\ \nu_\infty(\Gamma(4)) &= \frac{1}{2} \cdot 16 \cdot \frac{3}{4} = 6, & \nu_\infty(\Gamma(5)) &= \frac{1}{2} \cdot 25 \cdot \frac{24}{25} = 12. \end{aligned}$$

From now on, we will use $n(N)$ to denote the number of cusps of $\Gamma(N)$. More precisely,

$$n(2) = 3, \quad n(3) = 4, \quad n(4) = 6, \quad n(5) = 12. \tag{1.6.9}$$

Let us conclude the following theorem.

Theorem 1.6.13. *Let us define* $n(N)$, $N = 2, 3, 4, 5$, *as above. If we have a covering space*

$$f : \mathbb{H} \to \mathbb{CP}^1 \backslash \{a_1, a_2, \ldots, a_{n(N)}\} = \mathbb{H}/\Gamma(N)$$

for approporiate choices of $\{a_1, a_2, \ldots, a_{n(N)}\} \subset \mathbb{CP}^1$, *then* $\mathrm{Aut}(f) = \Gamma(N)$.

Proof. The definition of $\Gamma(N)$ implies that $\Gamma(N)$ does not contain elliptic elements. From Propositions 1.5.6 and 1.5.7, for an arbitrary point $x \in \mathbb{H}$, there exists a neighborhood $x \in U$ such that $\gamma_1(U) \cap \gamma_2(U) = \emptyset$ for any $\gamma_1 \neq \gamma_2 \in \Gamma(N)$. The conclusion directly follows from [Hat02, p. 72, Proposition 1.40 (b)]. □

Proposition 1.6.14. *The number of cusps of* $\Gamma(N)$ *is* $\nu_\infty = m/N$ $(N \geq 2)$.

Proof. Let $\mathbb{Q}^* = \mathbb{Q} \cup \{\infty\}$, and $\mathbb{Q}^*/\Gamma(N)$ is the quotient space of $\mathbb{Q}^*$ under the action of $\Gamma(N)$. The quotient space $\Gamma(1)/\Gamma(N)$ is well defined since $\Gamma(N)$ is normal in $\Gamma(1)$. We define the following short exact sequence,

$$0 \overset{\sigma_1}{\to} \mathrm{Stab}_{\Gamma(1)}(\infty)/\mathrm{Stab}_{\Gamma(N)}(\infty) \overset{\sigma_2}{\to} \Gamma(1)/\Gamma(N) \xrightarrow{\sigma_3} \mathbb{Q}^*/\Gamma(N) \xrightarrow{\sigma_4} 0$$

$$[\gamma^\infty] \xmapsto{\quad\sigma_2\quad} [\gamma^\infty]$$

$$[\gamma] \xmapsto{\quad\sigma_3\quad} [\gamma(\infty)]$$

which satisfies the following properties and facts

1. σ_1, σ_4 are trivial maps;

2. σ_2 is well-defined. If $[\gamma_1^\infty] = [\gamma_2^\infty]$ in $\mathrm{Stab}_{\Gamma(1)}(\infty)/\mathrm{Stab}_{\Gamma(N)}(\infty)$ for some $\gamma_1^\infty \neq \gamma_2^\infty \in \mathrm{Stab}_{\Gamma(1)}(\infty)$, then $(\gamma_2^\infty) \circ \gamma_1^\infty \in \mathrm{Stab}_{\Gamma(N)}(\infty) \subseteq \Gamma(N)$. It implies that $\sigma_2([\gamma_1^\infty]) = \sigma_2([\gamma_2^\infty])$ in $\Gamma(1)/\Gamma(N)$;

3. $(\mathbb{Q}^*/\Gamma(N), \odot)$ is a group with identity $[\infty]$. The fact that $\Gamma(1)$ acts transitively on $\mathbb{Q}^*$ allows us to find $\gamma_i \in \Gamma(1)$ for any $b_i \in \mathbb{Q}^*(i = 1, 2)$ such that $b_i = \gamma_i(\infty)$.

For $[b_1], [b_2] \in \mathbb{Q}^*/\Gamma(N)$ and $\gamma_2 \in \Gamma(1)$ such that $b_2 = \gamma_2(\infty)$. We define operation $\odot$:

$$[b_1] \odot [b_2] = [\gamma_1 \circ \gamma_2(\infty)] \qquad \text{on } \mathbb{Q}^*/\Gamma(N).$$

If $[b_1] = [b_1']$, there is $\gamma_N \in \Gamma(N)$ such that $\gamma_N(b_1') = b_1$, and an element $\gamma_1' \in \Gamma(1)$ such that $\gamma_1'(\infty) = b_1'$, so $b_1 = \gamma_N \circ \gamma_1'(\infty) = \gamma_1(\infty)$, then the following equation

$$[b_1'] \odot [b_2] = [\gamma_1' \circ \gamma_2(\infty)] = [\gamma_N \circ \gamma_1' \circ \gamma_2(\infty)] = [b_1] \cdot [b_2],$$

holds. If $[b_2] = [b_2']$, there is $\gamma_N \in \Gamma(N)$ such that $\gamma_N(b_2') = b_2$, and $\gamma_2' \in \Gamma(1)$ such that $\gamma_2'(\infty) = b_2'$, we have $\gamma_2(\infty) = \gamma_N \circ \gamma_2'(\infty)$. On the other hand $\Gamma(N)$ is normal in $\Gamma(1)$, there is $\gamma_N' \in \Gamma(N)$ such that $\gamma_N' \circ \gamma_1 = \gamma_1 \circ \gamma_N$, therefore the following equation

$$\begin{aligned} [b_1] \odot [b_2'] &= [\gamma_1 \circ \gamma_2'(\infty)] = [\gamma_1 \circ \gamma_N \circ \gamma_2(\infty)] \\ &= [\gamma_N' \circ \gamma_1 \circ \gamma_2(\infty)] = [b_1] \odot [b_2], \end{aligned}$$

holds. The associativity, identity and inverse conditions are trivial.

4. σ_2, σ_3 are group homomorphisms;

5. $\mathrm{im}\sigma_2 = \ker \sigma_3$. For $[\gamma] \in \Gamma(1)/\Gamma(N)$, we have the following equivalent relations

$$\begin{aligned} & \sigma_3([\gamma]) = [\gamma(\infty)] = [\infty] \in \mathbb{Q}^*/\Gamma(N), \\ \text{if and only if} \quad & \gamma(\infty) = \gamma_N(\infty), \\ \text{if and only if} \quad & \gamma_N^{-1} \circ \gamma(\infty) = \infty, \quad \text{exists } \gamma_N \in \Gamma(N), \end{aligned}$$

the last equation implies relation

$$\gamma_N^{-1} \circ \gamma \in \mathrm{Stab}_{\Gamma(1)}(\infty) \subseteq \Gamma(1),$$

which is equivalent to $[\gamma_N^{-1} \circ \gamma] \in \mathrm{Stab}_{\Gamma(1)}(\infty)/\mathrm{Stab}_{\Gamma(N)}(\infty)$, thus $\mathrm{im}\sigma_2 = \ker \sigma_3$.

This exact short sequence implies relation

$$\begin{aligned}\mathbb{Q}^*/\Gamma(N) \cong (\Gamma(1)/\Gamma(N))/\ker \sigma_3 =& (\Gamma(1)/\Gamma(N))/\mathrm{im}\sigma_2,\\ =& (\Gamma(1)/\Gamma(N))/(\mathrm{Stab}_{\Gamma(1)}(\infty)/\mathrm{Stab}_{\Gamma(N)}(\infty)).\end{aligned}$$

Counting their cardinalities,

$$\begin{aligned}\nu_\infty(\Gamma(N)) = \#(\mathbb{Q}^*/\Gamma(N)) =& \# \big((\Gamma(1)/\Gamma(N))/(\mathrm{Stab}_{\Gamma(1)}(\infty)/\mathrm{Stab}_{\Gamma(N)}(\infty))\big)\\ =& \#(\Gamma(1)/\Gamma(N))/\#(\mathrm{Stab}_{\Gamma(1)}(\infty)/\mathrm{Stab}_{\Gamma(N)}(\infty)),\end{aligned}$$

where $(\Gamma(1) : \Gamma(N)) = \#(\Gamma(1)/\Gamma(N)) = m$. Direct calculation shows the following facts

$$\mathrm{Stab}_{\Gamma(1)}(\infty) =< \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} >, \qquad \mathrm{Stab}_{\Gamma(N)}(\infty) =< \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix} >,$$

therefore $\#(\mathrm{Stab}_{\Gamma(1)}(\infty)/\mathrm{Stab}_{\Gamma(N)}(\infty)) = N$. We proved the conclusion

$$\nu_\infty(\Gamma(N)) = m/N.$$

□

Proposition 1.6.15. *The index* m *of principle congruence group in the full modular group is given by the formula*

$$m = (\Gamma(1) : \Gamma(N)) = \begin{cases} 6, & N = 2, \\ \frac{1}{2}\#(SL_2(\mathbb{Z}/N\mathbb{Z})) = \frac{1}{2}N^3 \prod_{p|N}(1-p^{-2}), & N \geq 3, \end{cases} \tag{1.6.10}$$

where p *are the prime divisors of* N*.*

Proof. First we show the equation on the left in equation (1.6.10). Consider the following group homomorphism

$$\begin{aligned} \phi : \quad \mathrm{SL}_2(\mathbb{Z}) &\rightarrow \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}), \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\mapsto \begin{pmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{pmatrix}, \end{aligned}$$

where $\overline{a}$ is the image of a in $\mathbb{Z}/N\mathbb{Z}$, so as $\overline{b}, \overline{c}$ and $\overline{d}$. The identity in $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ is $\begin{pmatrix} \overline{1} & \overline{0} \\ \overline{0} & \overline{1} \end{pmatrix}$, recall the fact

$$\Gamma(N) \cong \phi^{-1}\begin{pmatrix} \overline{1} & \overline{0} \\ \overline{0} & \overline{1} \end{pmatrix} = \ker\phi.$$

On the other hand,

$$\mathrm{SL}_2(\mathbb{Z})/\ker\phi \cong \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}),$$

therefore the following relation

$$\Gamma(1)/\Gamma(N) \cong (\mathrm{SL}_2(\mathbb{Z})/\pm I)/\ker\phi \cong \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\pm I,$$

holds, which implies equation

$$(\Gamma(1):\Gamma(N)) = \#(\mathrm{PSL}_2(\mathbb{Z}))/\ker\phi = \frac{1}{2}\#(\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})), \qquad N \geq 3,$$

and equation

$$(\Gamma(1):\Gamma(2)) = \#(\mathrm{PSL}_2(\mathbb{Z}))/\ker\phi = \#(\mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z}),$$

since $\overline{I} = \overline{-I}$ in $\mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z})$.

Claim 1. We have the following

$$\#\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}) = \Pi_{i=1}^{m} p_i^3(1-p_r^{-2}) = N^3\Pi_{p|N}(1-p^{-2}).$$

With the help of Claim 1, we conclude 1.6.10. □

Proof of Claim 1. Write the integer N as the following

$$N = p_1^{r_1}\dots p_m^{r_m},$$

where $p_i (i = 1,\dots,m)$ are different prime numbers. By Chinese remainder theorem, we have the following isomorphism

$$\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) \cong \mathrm{GL}_2(\mathbb{Z}/p_1^{r_1}\mathbb{Z}) \times \ldots \times \mathrm{GL}_2((\mathbb{Z}/p_m^{r_m}\mathbb{Z})),$$

and isomorphism

$$\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}) \cong \mathrm{SL}_2(\mathbb{Z}/p_1^{r_1}\mathbb{Z}) \times \ldots \times \mathrm{SL}_2((\mathbb{Z}/p_m^{r_m}\mathbb{Z})).$$

The first one holds because of the following equivalent relations

$$\begin{aligned} &\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) \\ \text{if and only if }\ &\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in (\mathbb{Z}/N\mathbb{Z})^{\times} \\ \text{if and only if }\ &\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in (\mathbb{Z}/p_i^{r_i}\mathbb{Z})^{\times} \\ \text{if and only if }\ &\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}/p_i^{r_i}\mathbb{Z}), \quad i = 1, \ldots, m. \end{aligned}$$

The second one holds because of the equivalence relation

$$a \equiv \overline{1} \in (\mathbb{Z}/N\mathbb{Z})^{\times} \quad \text{if and only if} \quad a \equiv \overline{1} \in (\mathbb{Z}/p_i^{r_i}\mathbb{Z})^{\times} \quad i = 1, \ldots, m.$$

It will be enough to calculate $\#(\mathrm{SL}_2(\mathbb{Z}/p_i^{r_i}\mathbb{Z}))$. Claim

$$\#(\mathrm{SL}_2(\mathbb{Z}/p_i^{r_i}\mathbb{Z})) = \#(\mathrm{GL}_2(\mathbb{Z}/p_i^{r_i}\mathbb{Z}))/\#(\mathbb{Z}/p_i^{r_i}\mathbb{Z})^{\times}. \tag{1.6.11}$$

To show the claim, consider the homomorphism map det:

$$\begin{aligned} \det : \mathrm{GL}_2(\mathbb{Z}/p_i^{r_i}\mathbb{Z}) &\to (\mathbb{Z}/p_i^{r_i}\mathbb{Z})^{\times}, \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\mapsto \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \overline{ad - bc}. \end{aligned}$$

The kernel is $\det^{-1}(\overline{1}) = \mathrm{SL}_2(\mathbb{Z}/p_i^{r_i}\mathbb{Z})$, it induces an isomorphism

$$\mathrm{GL}_2(\mathbb{Z}/p_i^{r_i}\mathbb{Z})/\mathrm{SL}_2(\mathbb{Z}/p_i^{r_i}\mathbb{Z}) \cong (\mathbb{Z}/p_i^{r_i}\mathbb{Z})^\times,$$

which implies the claim. We have $\#(\mathbb{Z}/p_i^{r_i}\mathbb{Z})^\times = \varphi(p_i^{r_i}) = p_i^{r_i} - p_i^{r_i-1}$. Now let us compute the order of $\mathrm{GL}_2(\mathbb{Z}/p_i^{r_i}\mathbb{Z})$. Consider the natural homomorphism

$$h : \mathrm{GL}_2(\mathbb{Z}/p_i^{r_i}\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{Z}/p_i\mathbb{Z}).$$

It is an onto map, so $\#\mathrm{GL}_2(\mathbb{Z}/p_i^{r_i}\mathbb{Z}) = \#\mathrm{GL}_2(\mathbb{Z}/p_i\mathbb{Z}) \cdot \# \ker h$. Claim the following equation

$$\#\mathrm{GL}_2(\mathbb{Z}/p_i\mathbb{Z}) = (p_i^2 - 1)(p_i^2 - p_i).$$

Because p_i is prime, we only need the first and second row to be linearly independent. There are $p_i^2 - 1$ choices for first row, and then $p_i^2 - p_i$ choices for the second row, so the claim holds. For $\ker h$, the pre-image of $\begin{pmatrix} \overline{1} & \overline{0} \\ \overline{0} & \overline{1} \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}/p_i\mathbb{Z})$ has the following form

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + p_i \begin{pmatrix} * & * \\ * & * \end{pmatrix},$$

where $\begin{pmatrix} * & * \\ * & * \end{pmatrix}$ is any matrix with elements in $\mathbb{Z}/p_i^{r_i-1}\mathbb{Z}$. The cardinality is $(p_i^{r_i-1})^4 = \# \ker h$. Therefore we have the following calculations

$$\#\mathrm{GL}_2(\mathbb{Z}/p_i^{r_i}\mathbb{Z}) = (p_i^2 - 1)(p_i^2 - p_i)(p_i^{r_i-1})^4,$$

$$\#\mathrm{SL}_2(\mathbb{Z}/p_i^{r_i}\mathbb{Z}) = (p_i^2-1)(p_i^2-p_i)(p_i^{r_i-1})^4/(p_i^{r_i}-p_i^{r_i-1}),$$
$$= (p_i^2-1)p_i^{3r_i-2} = p_i^3(1-p_r^{-2}),$$

and we conclude the claim

$$\#\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}) = \Pi_{i=1}^m p_i^3(1-p_r^{-2}) = N^3\Pi_{p|N}(1-p^{-2}).$$

□

1.7 Modular Forms and Modular Functions

Definition 1.7.1. A function $f:\mathbb{H}\to\mathbb{C}$ is called a modular function of weight k if f satisfies the following properties:

1. The function f is meromorphic on $\mathbb{H}$;
2. The equation $f\circ\gamma(\tau) = (c\tau+d)^k f(\tau)$ holds for any $\tau\in\mathbb{H}$ and any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$, where $\Gamma(1)$ is the modular group;
3. The function f is meromorphic at infinity.

Furthermore, if f is a modular function of weight k and holomorphic on $\mathbb{H}\cup\{\infty\}$, we call f a modular form of weight k.

As a matter of fact, the space of weight 4 modular form for $\Gamma(1)$ is one dimensional and generated by the Eisenstein series $E_4(\tau)$. This is a classic result, see [Ser73, p. 88, Theorem 4 (ii) and p. 93, Examples] (where E_2 is in our notation E_4).

Theorem 1.7.2. *The weight 4 modular form is a dimension one vector space generated by the Eisenstein series $E_4(\tau)$. $E_4(\tau)$ has the following expansion*

$$E_4(\tau) = 1 + 240 \sum_{m=1}^{\infty} \sigma_3(m) q^m \tag{1.7.1}$$

in $q = \exp\{2\pi i \tau\}$, $\tau \in \mathbb{H}$, where $\sigma_3(m) = \sum_{d|m} d^3$-the sum of the cubes of all positive divisors of m.

1.7.1 Schwarzian Derivative and Modular Function and Modular Form

For convenience, we introduce the following definition.

Definition 1.7.3. We say that a function $f : \mathbb{H} \to \mathbb{C}$ is an automorphic function for a discrete group $\Gamma' \subseteq \mathrm{SL}_2(\mathbb{R})$ if f is meromorphic on $\mathbb{H} \cup \{\infty\}$ and satisfies property

$$f \circ \gamma(\tau) = f(\tau), \qquad \text{for all } \tau \in \mathbb{H} \text{ and } \gamma \in \Gamma'.$$

Remark 5. Modular functions and modular forms are automorphic functions and automorphic forms of weight *zero* for the modular group $\Gamma(1)$, respectively. Notice that the covering map $f : \mathbb{H} \to \mathbb{CP}^1 \backslash \{a_1 = 0, \ldots, a_n\}$ is an automorphic function, it is a generator of the function field over $\mathbb{H}/\mathrm{Aut}(f)$ which has a traditional name *Hauptmodul*. We shall refer the covering map $f : \mathbb{H} \to \mathbb{CP}^1 \backslash \{a_1 = 0, \ldots, a_n\}$ as a Hauptmodul for the group $\mathrm{Aut}(f)$ in the future.

The following lemma is mentioned in [MS00, Proposition 3.2], for the completion of this article, I will restate and proof it.

Lemma 1.7.4. *Let f be a Hauptmodul for a genus zero discrete group* $\mathrm{Aut}(f) \subset SL_2(\mathbb{R})$*. For every γ that normalizes* $\mathrm{Aut}(f)$ *in* $SL_2(\mathbb{R})$*, there exists a corresponding matrix* $\eta = \begin{pmatrix} \eta_1 & \eta_2 \\ \eta_3 & \eta_4 \end{pmatrix} \in SL_2(\mathbb{C})$ *such that the following equation*

$$f \circ \gamma(\tau) = \eta \circ f(\tau)$$

is true for any $\tau \in \mathbb{H}$.

Proof. Notice that γ normalizes $Aut(f)$ in $\mathrm{SL}_2(\mathbb{R})$, i.e., $\gamma \in \mathbb{H} = \mathrm{SL}_2(\mathbb{R})$. Therefore the composition $f \circ \gamma(\tau)$ is also an automorphic function for $\mathrm{Aut}(f)$. The compactification of $\mathbb{CP}^1 \backslash \{a_1 = 0, \ldots, a_n\} \cong \mathbb{H}/\mathrm{Aut}(f)$ is the Riemann sphere $\mathbb{CP}^1$ which has genus zero, which implies that $\mathbb{H}^*/\mathrm{Aut}(f)$ has transcendental degree zero. Thus $f \circ \gamma(\tau)$ and $f(\tau)$ are related by an automorphism of $\mathbb{CP}^1$, i.e., an element η in $\mathrm{SL}_2(\mathbb{C})$. More precisely, assume $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ and $\eta = \begin{pmatrix} \eta_1 & \eta_2 \\ \eta_3 & \eta_4 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$, we have the conclusion

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = \frac{\eta_1 f(\tau) + \eta_2}{\eta_3 f(\tau) + \eta_4}$$

holds for every $\tau \in \mathbb{H} \cup \{\infty\}$. $\square$

The following theorem plays an important role in determining the automorphic function for $\Gamma(N)$, $N = 2, 3, 4, 5$, it is also mentioned in [MS00, Proposition 3.1], I restate and proof it here for the completion of this article. We will apply Proposition 1.4.2 and Lemma 1.7.4 to prove the following theorem.

Theorem 1.7.5. *Let f be a Hauptmodul for a genus zero discrete group* $\Gamma' \subseteq SL_2(\mathbb{R})$*, then* $\{f, \tau\}$ *is a weight* 4 *automorphic form for* Γ'.

Proof. First we show that $\{f, \tau\}$ is holomorphic on $\mathbb{H}$ and also holomorphic at infinity. From the assumption that $f(\tau)$ is a covering map, $f(\tau)$ is locally biholomorphic at any point $\tau_0 \in \mathbb{H}$ with $f'(\tau_0) \neq 0$. The definition of Schwarzian derivate

$$\{f, \tau\} = 2\left(\frac{f_{\tau\tau}}{f_\tau}\right)_\tau - \left(\frac{f_{\tau\tau}}{f_\tau}\right)^2$$

shows the analyticity of $\{f, \tau\}$ at τ_0. Recall equations (1.4.1) and (1.4.3), the following equation

$$\begin{aligned}\{f, \tau\} &= \frac{4\pi^2}{k^2}(1 - q_k^2\{f, q_k\}), \\ &= \frac{4\pi^2}{k^2}(1 - \sum_{m=0}^{\infty} P_m(\mathrm{B}, \mathrm{C}_3, \ldots, \mathrm{C}_{m+3}) q_k^{m+2}),\end{aligned}$$

implies that $\{f, \tau\}$ is analytic at $\tau = \infty$. Therefore, $\{f, \tau\}$ is holomorphic on $\mathbb{H} \cup \{\infty\}$. To show $\{f, \tau\}$ is a weight 4 modular form for Γ' we only need to show equation

$$\{f(\gamma(\tau)), \gamma(\tau)\} = (c\tau + d)^4 \{f(\tau), \tau\} \tag{1.7.2}$$

holds for every element $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma'$. On one hand, we know that $f \circ \gamma(\tau)$ and $f(\tau)$ are related by a linear transformation from Lemma 1.7.4, and Proposition 1.4.2 implies equation

$$\begin{aligned}\{f(\tau), \tau\} = \{f(\tau), \gamma(\tau)\}(\gamma_\tau)^2 + \{\gamma(\tau), \tau\} &= \{f(\tau), \gamma(\tau)\}(\gamma_\tau)^2 + \{\tau, \tau\}, \\ = \{f(\tau), \gamma(\tau)\} \cdot (c\tau + d)^4, & \qquad (1.7.3)\end{aligned}$$

where $\{\tau, \tau\} = 0$ since $\tau_{\tau\tau} = (1)_\tau = 0$. Therefore equation (1.7.2) holds, i.e., $\{f, \tau\}$ is a weight 4 modular form for Γ'. □

Next we have the following conclusion as a corollary of Theorem 1.7.5.

Corollary 1.7.6. *ILet f be a Hauptmodul for a genus zero discrete group $\Gamma' \subseteq SL_2(\mathbb{R})$. Then $\{f, \tau\}$ is a weight 4 automorphic form for the normalizer of Γ' in $SL_2(\mathbb{R})$.*

Proof. Assume $\mu \in \mathrm{SL}_2(\mathbb{R})$ normalizes Γ'. By Lemma 1.7.4, there exists an matrix $\eta = \begin{pmatrix} \eta_1 & \eta_2 \\ \eta_3 & \eta_4 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$ such that $f(\tau) = \eta \circ f(\tau)$. Therefore we have the following equalities

$$\begin{aligned} \{f(\mu(\tau)), \mu(\tau)\} &= \{\frac{\eta_1 f(\tau) + \eta_2}{\eta_3 f(\tau) + \eta_4}, \mu(\tau)\} = \{f(\tau), \mu(\tau)\}, \\ &= (c\tau + d)^4 \{f, \tau\}, \end{aligned}$$

where $\mu = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Aut}(f)$, and the last equation comes from direct calculation. This shows that $\{f, \tau\}$ is a weight 4 auromorphic form for the normalizer of Γ' in $\mathrm{SL}_2(\mathbb{R})$. □

Lemma 1.7.7. $\Gamma(1)$ *normalizes* $\Gamma(N)$ *in* $SL_2(\mathbb{R})$*. Consequently,* $\Gamma(N)$ *is normal in* $\Gamma(1)$*.*

Proof. The proof is elementary. □

1.8 Main Result and Examples

1.8.1 Main Result

In section 1.3.2, we mentioned the inversion series (1.3.11). Now let us consider the situation that one of the singularities is $a_1 = 0$, and the corresponding parabolic generator fixes infinity. It is equivalent to say that a covering map $f : \mathbb{H} \to \mathbb{CP}^1 \backslash \{a_1 = 0, \ldots, a_n\}$ has expansion

$$f = Aq_{_k} + Bq_{_k}^2 + c_3 q_{_k}^3 + \sum_{m=4}^{\infty} c_m q_{_k}^m$$

in $q_{_k} = \exp\{\frac{2\pi i}{k}\tau\}$, $\tau \in \mathbb{H}$, for some real constant k with $A \neq 0$. Let us denote $\mathbf{f} = \frac{f}{A}$, $f \in \mathbb{CP}^1 \backslash \{a_1 = 0, \ldots, a_n\}$, for convenience, recall equation (1.4.23) in Theorem 1.4.5,

$$\mathbf{f} = f/A = q_{_k} + \mathrm{B}q_{_k}^2 + \mathrm{C}_3(A, \mathrm{B})q_{_k}^3 + \sum_{m=4}^{\infty} \mathrm{C}_m(A, \mathrm{B})q_{_k}^m,$$

where $\mathrm{B} = \frac{B}{A}$, $\mathrm{C}_m = \frac{c_m}{A}$ for $m \geq 3$. It is not hard to see that $q_{_k}$ has the following expansion

$$q_{_k}(\mathbf{f}) = \mathbf{f} + \tilde{B}(\mathrm{B})\mathbf{f}^2 + \tilde{c}_3(\mathrm{B}, \mathrm{C}_3)\mathbf{f}^3 + \sum_{m=4}^{\infty} \tilde{c}_m(\mathrm{B}, \mathrm{C}_3, \ldots, \mathrm{C}_m)\mathbf{f}^m \tag{1.8.1}$$

in $\mathbf{f}$, where $\tilde{B}(\mathrm{B}) = -\mathrm{B}$ and $\tilde{c}_m(\mathrm{B}, \mathrm{C}_3, \ldots, \mathrm{C}_m)$ are polynomials in $\mathrm{B}, \mathrm{C}_3, \ldots, \mathrm{C}_m$ which has degree 1 in C_m with constant coefficient. Let us restate the main result, Theorem 1, and proof it here.

Theorem 1.8.1. *Let $f : \mathbb{H} \to \mathbb{CP}^1 \setminus \{a_1 = 0, a_2, \ldots, a_n\}$ be a covering map with* *expression*

$$f = f(\tau) = Aq_{_k} + Bq_{_k}^2 + c_3q_{_k}^3 + \sum_{m=4}^{\infty} c_m q_{_k}^m.$$

Then the complete Kähler-Einstein metric has asymptotic expansion

$$|ds| = \frac{1}{|A||\mathbf{f}|\log|\mathbf{f}|}\left|1 - \left(\mathrm{B}\mathbf{f} - \frac{\mathrm{Re}(\mathrm{B}\mathbf{f})}{\log|\mathbf{f}|}\right) + \sum_{m=2}^{\infty} R_m(A, \mathrm{B}, \mathbf{f}, \frac{\mathbf{f}^s\overline{\mathbf{f}^{m-s}}}{\log^j|\mathbf{f}|})\right| |df| \qquad (1.8.2)$$

at the cusp 0*, where* $\mathbf{f} = \frac{f}{A}$ *for* $f \in \mathbb{CP}^1 \setminus \{a_1 = 0, a_2, \ldots, a_n\}$*,* $\mathrm{B} = \frac{B}{A}$*, and* $R_m(A, \mathrm{B}, \mathbf{f}, \frac{\mathbf{f}^s\overline{\mathbf{f}^{m-s}}}{\log^j|\mathbf{f}|})$ *is a polynomial in* $A, \mathrm{B}, \mathbf{f}, \frac{\mathbf{f}^s\overline{\mathbf{f}^{m-s}}}{\log^j|\mathbf{f}|}$*,* $s, j = 0, 1, \ldots, m$*, with constant coefficients for* $m \geq 2$*.*

Proof. For convenience, we will simply write $q = q_{_k}$ in this proof. Recall the proof of Theorem 1.3.11, we apply the substitution $\mathbf{f} = \frac{f}{A}$,

$$\begin{aligned} ds^2 &= \frac{-4}{\left(\frac{k}{2\pi i}\log q(\mathbf{f}) - \overline{\frac{k}{2\pi i}\log q(\mathbf{f})}\right)^2}\left|d\left(\frac{k}{2\pi i}\right)\log q(\mathbf{f})\right|^2, \\ &= \frac{|q_{\mathbf{f}}(\mathbf{f})|^2}{|q(\mathbf{f})|^2\log^2|q(\mathbf{f})|}\left|\frac{d\mathbf{f}}{df}\right||df|^2, \\ &= \frac{|q_{\mathbf{f}}(\mathbf{f})|^2}{A^2|q(\mathbf{f})|^2\log^2|q(\mathbf{f})|}|df|^2. \end{aligned} \qquad (1.8.3)$$

We calculate the expansion of each terms in equation (1.8.3) by (1.8.1),

$$\begin{aligned} \frac{q_{\mathbf{f}}(\mathbf{f})}{q(\mathbf{f})} &= (\log|q(\mathbf{f})|)_{\mathbf{f}} = (\log|\mathbf{f}| + \log|1 + \tilde{B}\mathbf{f} + \sum_{m=2}^{\infty}\tilde{c}_{m+1}\mathbf{f}^m|)_{\mathbf{f}}, \\ &= \frac{1}{\mathbf{f}} + \tilde{B} + \sum_{m=1}^{\infty}\tilde{Q}_m^{(11)}(\tilde{B}, \tilde{c}_3, \ldots, \tilde{c}_{m+2})\mathbf{f}^m, \end{aligned} \qquad (1.8.4)$$

where $\tilde{Q}_m^{(11)}(\tilde{B}, \tilde{c}_3, \ldots, \tilde{c}_{m+2})$ is a polynomial in $\tilde{B}, \tilde{c}_3, \ldots, \tilde{c}_{m+2}$ which has degree 1 in $\tilde{c}_{m+2}$ with constant coefficients for $m \geq 1$. Next we calculate the expansion of $\frac{1}{\log|q(\mathbf{f})|}$

in $\mathbf{f}$ and $\log|\mathbf{f}|$,

$$\frac{1}{\log|q(\mathbf{f})|} = \frac{1}{\log|\mathbf{f}|}\left(1+\frac{\log|1+\tilde{B}\mathbf{f}+\sum_{m=2}^{\infty}\tilde{c}_{m+1}\mathbf{f}^m|}{\log|\mathbf{f}|}\right)^{-1},$$
$$= \frac{1}{\log|\mathbf{f}|}\left[1-\frac{\log|1+\tilde{B}\mathbf{f}+\sum_{m=2}^{\infty}\tilde{c}_{m+1}\mathbf{f}^m|}{\log|\mathbf{f}|}+\sum_{l=2}^{\infty}(-1)^l\left(\frac{\log|1+\tilde{B}\mathbf{f}+\sum_{m=2}^{\infty}\tilde{c}_{m+1}\mathbf{f}^m|}{\log|\mathbf{f}|}\right)^l\right] \tag{1.8.5}$$

Let us use the expansion $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \sum_{n\geq 4}(-1)^{n+1}\frac{x^n}{n}$ and write $\tilde{c}_1 = 1$ and $\tilde{c}_2 = \tilde{B}$ for convenience,

$$\log|1+\tilde{B}\mathbf{f}+\sum_{m=2}^{\infty}\tilde{c}_{m+1}\mathbf{f}^m| = \frac{1}{2}\log|1+\tilde{B}\mathbf{f}+\tilde{c}_3\mathbf{f}^2+\sum_{m=2}^{\infty}\tilde{c}_{m+1}\mathbf{f}^m|^2,$$
$$= \frac{1}{2}\log\left[1+2\,\mathrm{Re}(\tilde{B}\mathbf{f})+\sum_{m=2}^{\infty}\left(\sum_{s+j=m,s,j\geq 0}\tilde{c}_{s+1}\overline{\tilde{c}_{j+1}}\mathbf{f}^s\overline{\mathbf{f}^j}\right)\right],$$
$$= \frac{1}{2}\left\{\left[2\,\mathrm{Re}(\tilde{B}\mathbf{f})+\sum_{m=2}^{\infty}\left(\sum_{s+j=m,s,j\geq 0}\tilde{c}_{s+1}\overline{\tilde{c}_{j+1}}\mathbf{f}^s\overline{\mathbf{f}^j}\right)\right]\right.$$
$$\left.+\sum_{l=2}^{\infty}(-1)^{l+1}\frac{1}{l}\left[2\,\mathrm{Re}(\tilde{B}\mathbf{f})+\sum_{m=2}^{\infty}\left(\sum_{s+j=m,s,j\geq 0}\tilde{c}_{s+1}\overline{\tilde{c}_{j+1}}\mathbf{f}^s\overline{\mathbf{f}^j}\right)\right]^l\right\},$$
$$= \frac{1}{2}\left[2\,\mathrm{Re}(\tilde{B}\mathbf{f})+\sum_{m=2}^{\infty}R_m^{(1)}(\mathbf{f}^m,\mathbf{f}^{m-1}\overline{\mathbf{f}},\ldots,\mathbf{f}\overline{\mathbf{f}^{m-1}},\overline{\mathbf{f}^m})\right],$$

where $R_m^{(1)}(\mathbf{f}^m,\mathbf{f}^{m-1}\overline{\mathbf{f}},\ldots,\mathbf{f}\overline{\mathbf{f}^{m-1}},\overline{\mathbf{f}^m})$ is a polynomial in $\mathbf{f}^m,\mathbf{f}^{m-1}\overline{\mathbf{f}},\ldots,\mathbf{f}\overline{\mathbf{f}^{m-1}},\overline{\mathbf{f}^m}$ with coefficients in $\tilde{B}, \tilde{c}_3, \ldots, \tilde{c}_{m+1}$. Therefore equation (1.8.5) has expansion

$$\begin{aligned}\frac{1}{\log|q(\mathbf{f})|} &= \frac{1}{\log|\mathbf{f}|}\left(1 - \frac{1}{2}\frac{1}{\log|\mathbf{f}|}\left[2\,\mathrm{Re}(\tilde{B}\mathbf{f}) + \sum_{m=2}^{\infty} R_m^{(1)}(\mathbf{f}^m, \mathbf{f}^{m-1}\overline{\mathbf{f}}, \ldots, \mathbf{f}\overline{\mathbf{f}^{m-1}}, \overline{\mathbf{f}^m})\right]\right. \\ &\qquad \left. + \sum_{l=2}^{\infty}\frac{(-1)^l}{2^l \log^l|\mathbf{f}|}\left[2\,\mathrm{Re}(\tilde{B}\mathbf{f}) + \sum_{m=2}^{\infty} R_m^{(1)}(\mathbf{f}^m, \mathbf{f}^{m-1}\overline{\mathbf{f}}, \ldots, \mathbf{f}\overline{\mathbf{f}^{m-1}}, \overline{\mathbf{f}^m})\right]^l\right), \\ &= \frac{1}{\log|\mathbf{f}|}\left[1 - \frac{\mathrm{Re}(\tilde{B}\mathbf{f})}{\log|\mathbf{f}|} + \sum_{m=2}^{\infty} R_m^{(2)}\left(\frac{\mathbf{f}^s\overline{\mathbf{f}^{m-s}}}{\log^j|\mathbf{f}|}\right)\right], \qquad (1.8.6)\end{aligned}$$

where $R_m^{(2)}(\frac{\mathbf{f}^s\overline{\mathbf{f}^{m-s}}}{\log^j|\mathbf{f}|})$ is a polynomial in $\frac{\mathbf{f}^s\overline{\mathbf{f}^{m-s}}}{\log^j|\mathbf{f}|}$, $s = 0, 1, \ldots, m$ and $j = 1, \ldots, m$, which has coefficients in $\tilde{B}, \tilde{c}_3, \ldots, \tilde{c}_{m+1}$ for $m \geq 2$. Therefore equation (1.8.4) and (1.8.6) implies that the metric $|ds|$ defined by equation (1.8.3) has expression

$$\begin{aligned}|ds| &= \frac{1}{|A||\mathbf{f}|\log|\mathbf{f}|}\left|1 + \tilde{B}\mathbf{f} + \sum_{m=1}^{\infty}\tilde{Q}^{(11)}(\tilde{B}, \tilde{c}_3, \ldots, \tilde{c}_{m+2})\mathbf{f}^{m+1}\right| \cdot \left[1 - \frac{\mathrm{Re}(\tilde{B}\mathbf{f})}{\log|\mathbf{f}|} + \sum_{m=2}^{\infty} R_m^{(2)}\left(\frac{\mathbf{f}^s\overline{\mathbf{f}^{m-s}}}{\log^j|\mathbf{f}|}\right)\right], \\ &= \frac{1}{|A||\mathbf{f}|\log|\mathbf{f}|}\left|1 + \left(\tilde{B}\mathbf{f} - \frac{\mathrm{Re}(\tilde{B}\mathbf{f})}{\log|\mathbf{f}|}\right) + \sum_{m=2}^{\infty}\tilde{R}_m\left(\mathbf{f}, \frac{\mathbf{f}^s\overline{\mathbf{f}^{m-s}}}{\log^j|\mathbf{f}|}\right)\right||df|, \qquad (1.8.7)\end{aligned}$$

where $\tilde{R}_m(\mathbf{f}, \frac{\mathbf{f}^s\overline{\mathbf{f}^{m-s}}}{\log^j|\mathbf{f}|})$ is a polynomial in the variable set $\frac{\mathbf{f}^s\overline{\mathbf{f}^{m-s}}}{\log^j|\mathbf{f}|}$, $s, j = 0, 1, \ldots, m$, with coefficients in $\tilde{B}, \tilde{c}_2, \ldots, \tilde{c}_{m+1}$ for $m \geq 2$. Due to equation (1.8.1) and Theorem 1.4.5, each term $\tilde{B}, \tilde{c}_2, \ldots, \tilde{c}_{m+1}$ can be solved as a polynomial in A, B, so the coefficients in $\tilde{R}_m(\mathbf{f}, \frac{\mathbf{f}^s\overline{\mathbf{f}^{m-s}}}{\log^j|\mathbf{f}|})$ are polynomials in A, B. Let us write $\tilde{R}_m(\mathbf{f}, \frac{\mathbf{f}^s\overline{\mathbf{f}^{m-s}}}{\log^j|\mathbf{f}|})$ as $R_m(A, \mathrm{B}, \mathbf{f}, \frac{\mathbf{f}^s\overline{\mathbf{f}^{m-s}}}{\log^j|\mathbf{f}|})$, which denotes a polynomial in $A, \mathrm{B}, \mathbf{f}, \frac{\mathbf{f}^s\overline{\mathbf{f}^{m-s}}}{\log^j|\mathbf{f}|}$ with constant coefficients. Recall that $\tilde{B} = -\mathrm{B}$, we have conclusion that the metric $|ds|$ is given by the following expression

$$|ds| = \frac{1}{|A||\mathbf{f}|\log|\mathbf{f}|}\left|1 - \left(\mathrm{B}\mathbf{f} - \frac{\mathrm{Re}(\mathrm{B}\mathbf{f})}{\log|\mathbf{f}|}\right) + \sum_{m=2}^{\infty} R_m\left(A, \mathrm{B}, \mathbf{f}, \frac{\mathbf{f}^s\overline{\mathbf{f}^{m-s}}}{\log^j|\mathbf{f}|}\right)\right||df|, \qquad (1.8.8)$$

and, consequently, $|ds|$ is uniquely determined up to a choice of A, B. $\square$

Recall Theorem 1.6.13, now let us focus on the case that the deck transformation group is $\Gamma(N)$, $N = 2, 3, 4, 5$.

Theorem 1.8.2. *Let $n(N)$ be the numbers that is defined by equation (1.1.3), and let $f_N : \mathbb{H} \to \mathbb{CP}^1 \setminus \{a_1 = 0, a_2, \ldots, a_{n(N)}\}$ be a universal covering with deck transformation group $\mathrm{Aut}(f_N) = \Gamma(N)$, $N = 2, 3, 4, 5$, satisfying that f_N vanishes at infinity, i.e., $f_N(\infty) = 0$. Then f_N can be given by the following expansion*

$$f_N(\tau) = Aq_{_N} + Bq_{_N}^2 + \sum_{m=3}^{\infty} A \cdot \mathrm{C}_m(\mathrm{B})q_{_N}^m \tag{1.8.9}$$

in $q_{_N} = \exp\{\frac{2\pi}{N} i\tau\}, \tau \in \mathbb{H}$, where the constants $A, B \in \mathbb{C}$ are uniquely determined by the set of values of the punctured points $\{a_1 = 0, a_2, \ldots, a_{n(N)}\}$, and the term $\mathrm{C}_m(\mathrm{B})$ in the coefficient is a polynomial in B with constant coefficients for $m \geq 3$.

Proof. By Proposition 1.4.3, the following identity

$$\{f_N, \tau\} = \frac{4\pi^2}{N^2}(1 - q_{_N}^2\{f_N, q_{_N}\})$$

holds. Corollary 1.7.6 and Lemma 1.7.7 imply that $\{f_N, \tau\}$ is a weight 4 modular form for $\Gamma(1)$. Recall Theorem 1.7.2, there is a constant κ such that the following equation

$$E_4(\tau) = \kappa\{f_N, \tau\} = \kappa\frac{4\pi^2}{N^2}(1 - q_{_N}^2\{f_N, q_{_N}\}) \tag{1.8.10}$$

holds. Equations (1.7.1) and (1.4.3) imply the following identity

$$1 + 240\sum_{m=1}^{\infty}\sigma_3(m)q^m = \frac{4\pi^2}{N^2}\kappa[1 - \sum_{m=0}^{\infty} P_m(\mathrm{B}, \mathrm{C}_3, \ldots, \mathrm{C}_{m+3})q_{_N}^{m+2}]. \tag{1.8.11}$$

Matching the constant terms in equation (1.8.11),

$$1 = \frac{4\pi^2}{N^2}\kappa \quad \text{implies} \quad \kappa = \frac{N^2}{4\pi^2}.$$

Therefore we have the following identity

$$1 + 240\sum_{m=1}^{\infty}\sigma_3(m)q^m = 1 - P_0(\mathrm{B}, \mathrm{C}_3)q_{_N}^2 - \sum_{m=1}^{\infty}P_m(\mathrm{B}, \mathrm{C}_3, \ldots, \mathrm{C}_{m+3})q_{_N}^{m+2}$$

holds. Notice the identity $q_{_N}^N = q$ by their definitions, we match the coefficient of $q_{_N}^{lN}$ in equation (1.8.11) with the coefficient of q^l in equation (1.7.1), and let other coefficients in equation (1.8.11) be 0. We get the following set of equations,

$$P_m(\mathrm{B}, \mathrm{C}_3, \ldots, \mathrm{C}_{m+3}) = \begin{cases} 240\sigma_3(l), & \text{if } m = lN - 2 \text{ for } l = 1, 2, \ldots, \\ 0, & \text{otherwise,} \end{cases} \tag{1.8.12}$$

where $m \geq 0$. Therefore C_3 can be solved in terms of B when $m = 0$, and notice that every coefficient $P_m(\mathrm{B}, \mathrm{C}_3, \ldots, \mathrm{C}_{m+3})$ is a polynomial of degree 1 in C_{m+3} with constant coefficient. By induction, we can conclude that C_m can be solved as a polynomial in B. Equation (1.8.9) holds since $c_m = A \cdot \mathrm{C}_m$, $m \geq 3$. □

Corollary 1.8.3. *Under the same assumption in Theorem 1.8.2, the complete Kähler-Einstein metric has asymptotic expansion*

$$|ds| = \frac{1}{|A||\mathbf{f}|\log|\mathbf{f}|}\left|1 - \left(\mathrm{B}\mathbf{f} - \frac{\mathrm{Re}(\mathrm{B}\mathbf{f})}{\log|\mathbf{f}|}\right) + \sum_{m=2}^{\infty}R_m(\mathrm{B}, \mathbf{f}, \frac{\mathbf{f}^s\overline{\mathbf{f}^{m-s}}}{\log^j|\mathbf{f}|})\right||df| \tag{1.8.13}$$

at the cusp 0*, where* $R_m(\mathrm{B}, \mathbf{f}, \frac{\mathbf{f}^s\overline{\mathbf{f}^{m-s}}}{\log^j|\mathbf{f}|})$ *is a polynomial in* $\mathrm{B}, \mathbf{f}, \frac{\mathbf{f}^s\overline{\mathbf{f}^{m-s}}}{\log^j|\mathbf{f}|}$, $s, j = 0, 1, \ldots, m$, *with constant coefficients for* $m \geq 2$.

Proof. It directly follows from Theorems 1.8.1 and 1.8.2. □

1.8.2 Examples

We will see some examples by applying the main theorems.

Example 1.8.4 (The covering space $\mathbb{CP}^1 \setminus \{a_1 = 0, a_2, \ldots, a_{12}\} \cong \mathbb{H}/\Gamma(5)$)**.** In this case, $N = 5$, $q_5^5 = q$, so the coefficients of q_5^2, q_5^3, q_5^4 are all 0. We have the following,

$$\begin{cases} 12(\mathrm{C}_3 - \mathrm{B}^2) = 0, \\ 48(\mathrm{C}_4 - 2\mathrm{C}_3\mathrm{B} + \mathrm{B}^3) = 0, \\ 24(5\mathrm{C}_5 - 10\mathrm{C}_4\mathrm{B} + 17\mathrm{C}_3\mathrm{B}^2 - 6\mathrm{C}_3^2 - 6\mathrm{B}^4) = 0, \\ \ldots, \end{cases} \quad \text{implies} \quad \begin{cases} \mathrm{C}_3 = \mathrm{B}^2, \\ \mathrm{C}_4 = \mathrm{B}^3, \\ \mathrm{C}_5 = \mathrm{B}^4, \\ \ldots. \end{cases}$$

Therefore the covering map $f_5 : \mathbb{H} \to \mathbb{CP}^1 \setminus \{a_1 = 0, a_2, \ldots, a_{12}\}$ with deck transformation group $\Gamma(5)$ can be given by the following expansion

$$\mathbf{f}_5 = \frac{f_5(\tau)}{A} = q_5 + \mathrm{B}q_5^2 + \mathrm{B}^2q_5^3 + \mathrm{B}^3q_5^4 + \mathrm{B}^4q_5^5 + \sum_{m=6}^{\infty} P_m(\mathrm{B})q_5^m, \tag{1.8.14}$$

where the value of B is up to the collection of punctures $\{a_1 = 0, a_2, \ldots, a_{12}\}$. Therefore the complete Kähler-Einstein metric at the cusp $a_1 = 0$ is given by the following equation

$$|ds| = \frac{1}{|A||\mathbf{f}|\log|\mathbf{f}|}\left|1 - \left(\mathrm{B}\mathbf{f} - \frac{\mathrm{Re}(\mathrm{B}\mathbf{f})}{\log|\mathbf{f}|}\right) + \sum_{m=2}^{\infty} R_m(\mathrm{B}, \mathbf{f}, \frac{\mathbf{f}^s\overline{\mathbf{f}^{m-s}}}{\log^j|\mathbf{f}|})\right||df|,$$

where $\mathbf{f} = \mathbf{f}_5 = \frac{f_5}{A}$ for convenience, $f_5 \in \mathbb{CP}^1 \setminus \{a_1 = 0, a_2, \ldots, a_{12}\}$.

The following example is the case of the triple punctured Riemann sphere, which was mentioned in Example 1.1.1.

Example 1.8.5 (The covering space of $\mathbb{H}/\Gamma(2) \cong \mathbb{CP}^1 \setminus \{a_1 = 0, a_2, a_3\}$)**.** In this example, $N = 2$, $q_2^2 = q$, we have the following,

$$\begin{cases} 12(\mathrm{C}_3 - \mathrm{B}^2) = -240, \\ 48(\mathrm{C}_4 - 2\mathrm{C}_3\mathrm{B} + \mathrm{B}^3) = 0, \\ 24(5\mathrm{C}_5 - 10\mathrm{C}_4\mathrm{B} + 17\mathrm{C}_3\mathrm{B}^2 - 6\mathrm{C}_3^2 - 6\mathrm{B}^4) = -2160, \\ \dots, \end{cases} \quad \text{implies} \quad \begin{cases} \mathrm{C}_3 = \mathrm{B}^2 - 20, \\ \mathrm{C}_4 = \mathrm{B}^3 - 40\mathrm{B}, \\ \mathrm{C}_5 = \mathrm{B}^4 - 60\mathrm{B}^2 + 462, \\ \dots. \end{cases}$$

Consequently, the covering map $f_2 : \mathbb{H} \to \mathbb{CP}^1 \setminus \{a_1 = 0, a_2, a_3\}$ with deck transformation group $\Gamma(2)$ has expression

$$\mathbf{f}_2 = \frac{f_2(\tau)}{A} = q_2 + \mathrm{B}q_2^2 + (\mathrm{B}^2 - 20)q_2^3 + \sum_{m=4}^{\infty} P_m(\mathrm{B})q_2^m. \tag{1.8.15}$$

The metric equation (1.8.2) have the following expression

$$|ds| = \frac{1}{|A||\mathbf{f}|\log|\mathbf{f}|}\left|1 - \left(\mathrm{B}\mathbf{f} - \frac{\mathrm{Re}(\mathrm{B}\mathbf{f})}{\log|\mathbf{f}|}\right) + \sum_{m=2}^{\infty} R_m(\mathrm{B}, \mathbf{f}, \frac{\mathbf{f}^s\overline{\mathbf{f}^{m-s}}}{\log^j|\mathbf{f}|})\right| |df|,$$

where $\mathbf{f} = \mathbf{f}_2 = \frac{f_2}{A}$, $f_2 \in \mathbb{CP}^1 \setminus \{0, a_2, a_3\}$, for convenience, and $\mathrm{B} = \frac{B}{A}$ is the only free parameter which is uniquely determined by the two punctures a_2 and a_3.

Next example is a special case of the triple punctured Riemann sphere, which was mentioned as Example 1.1.2 in section 1.1. The explicit metric formula is also given by S. Agard from a different approach in [Aga68].

Example 1.8.6 (The quotient space $\mathbb{H}/\Gamma(2) \cong \mathbb{CP}^1 \setminus \{0, 1, \infty\}$)**.** When $A = 16, B =$

-128, equation (1.8.15) is the covering map given by the modular lambda function

$$f_2 = f_2(\tau) = \lambda(\tau) = 16q_{_2} - 128q_{_2}^2 + 704q_{_2}^3 - 3072q_{_2}^4 + O(q_{_2}^5), \tag{1.8.16}$$

where $f_2 = \lambda(\tau)$ is the covering map of $\mathbb{H} \to \mathbb{CP}^1 \setminus \{0, \infty, 1\}$ with values at the cusps as below

$$\infty \mapsto 0, \qquad 0 \mapsto 1, \qquad 1 \mapsto \infty.$$

In this case, $\mathrm{B} = \frac{-128}{16} = -8$, $\mathrm{C}_3 = \frac{704}{16} = 44 = (-8)^2 - 20$, the metric is given by the following

$$\begin{aligned}
|ds| = &\frac{1}{|16||\mathbf{f}|\log|\mathbf{f}|}\Bigg|1 + 8\left(\mathbf{f} - \frac{\operatorname{Re}\mathbf{f}}{\log|\mathbf{f}|}\right) \\
&- \left[(2\cdot 44 + 5\cdot 64)\mathbf{f}^2 - 64\frac{\mathbf{f}\operatorname{Re}\mathbf{f}}{\log|\mathbf{f}|} + (44 + \frac{5}{2}\cdot 64)\frac{\operatorname{Re}(\mathbf{f}^2)}{\log|\mathbf{f}|} + 64\frac{(\operatorname{Re}\mathbf{f})^2}{\log^2|\mathbf{f}|}\right] + O(\mathbf{f}^3)\Bigg|\,|df|, \\
= &\frac{1}{|f|\log|f/16|}\Bigg|1 + \frac{1}{2}\left(f - \frac{\operatorname{Re}f}{\log|f/16|}\right) \\
&- \left[\frac{51}{32}f^2 - \frac{1}{4}\frac{f\operatorname{Re}f}{\log|f/16|} + \frac{51}{64}\frac{\operatorname{Re}(f^2)}{\log|f/16|} + \frac{1}{4}\frac{(\operatorname{Re}f)^2}{\log^2|f/16|}\right] + O(f^3)\Bigg|\,|df|,
\end{aligned}$$

where $f = f_2$ and $\mathbf{f} = \mathbf{f}_2 = \frac{f_2}{16}$, $f = f_2 \in \mathbb{CP}^1 \setminus \{0, 1, \infty\}$.

Example 1.8.7 (The punctured Riemann sphere $\mathbb{CP}^1 \setminus \{a_1, a_2, a_3\}$ for arbitrary a_1, a_2, a_3)**.** Assume a_1, a_2 and a_3 are three different points on $\mathbb{CP}^1$, the Möbius transformation

$$\lambda \mapsto \frac{a_1(a_2 - a_3) - a_3(a_2 - a_1)\lambda}{(a_2 - a_3) - (a_2 - a_1)\lambda} = \tilde{\lambda}$$

maps $\{0, 1, \infty\}$ to $\{a_2, a_3, a_1\}$ respectively. Direct calculation indicates the following conclusion.

Corollary 1.8.8. *A covering map $f : \mathbb{H} \to \mathbb{CP}^1 \setminus \{a_1, a_2, a_3\}$ for any three different* *points a_1, a_2, a_3 can be uniquely determined by the following equation*

$$f(\tau) = \frac{a_1(a_2 - a_3) - a_3(a_2 - a_1)\lambda(\tau)}{(a_2 - a_3) - (a_2 - a_1)\lambda(\tau)}. \tag{1.8.17}$$

Furthermore, $f(\tau) = \tilde{\lambda}(\tau)$ can be given by the following expansion,

$$f(\tau) = a_1 + 16\frac{(a_1 - a_3)(a_2 - a_1)}{a_2 - a_3}q_2 + 128(a_1 - a_3)\left[2\frac{(a_2 - a_1)^2}{(a_2 - a_3)^2} - \frac{a_2 - a_1}{a_2 - a_3}\right]q_2^2 + O(q_2^3).$$

Therefore the metric expansion at a_1 can be given by the following corollary.

Corollary 1.8.9. *The complete Kähler-Einstein metric on $\mathbb{CP}^1 \setminus \{a_1, a_2, a_3\}$ at cusp a_1 has the following asymptotic expansion*

$$|ds| = \frac{|a_2 - a_3|}{16|a_1 - a_3||a_2 - a_1|}\frac{1}{\mathfrak{f}\log|\mathfrak{f}|}\left\{1 - 8\left[(2\frac{a_2 - a_1}{a_2 - a_3} - 1)\mathfrak{f} - \frac{\mathrm{Re}((2\frac{a_2 - a_1}{a_2 - a_3} - 1)\mathfrak{f})}{\log|\mathfrak{f}|}\right] + O(\mathfrak{f}^2)\right\}|df|$$

in $\mathfrak{f} = \frac{(a_2 - a_3)}{16(a_1 - a_3)(a_2 - a_1)}(f - a_1)$, $f \in \mathbb{CP}^1 \setminus \{a_1, a_2, a_3\}$.

Proof. Direct calculation gives the value of coefficients A, B,

$$A = 16\frac{(a_1 - a_3)(a_2 - a_1)}{a_2 - a_3}, \; B = 128(a_1 - a_3)\left[2\frac{(a_2 - a_1)^2}{(a_2 - a_3)^2} - \frac{a_2 - a_1}{a_2 - a_3}\right].$$

It implies the following value

$$\mathrm{B} = \frac{B}{A} = 8\left(2\frac{a_2 - a_1}{a_2 - a_3} - 1\right).$$

Then the result follows directly from Corollary 1.8.3. ☐

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Chapter 2

The Kobayashi-Royden metric on punctured spheres

2.1 Introduction

In this paper, we investigate the explicit formula of the Kobayashi-Royden metric on the one-dimensional open Riemann surface $\mathbb{CP}^1\backslash\{a_1, \ldots, a_n\}$, $n \geq 3$ and its application. Especially, we provide an explicit formula of the Kobayashi-Royden metric on $\mathbb{CP}^1\backslash\{0, 1, \infty\}$ by using the exponential Bell polynomials. Furthermore, the result will also apply to general cases and we prove a local quantitative version of the Little Picard's theorem as an application.

In a broader point of view, given any projective manifold X, for any point $x \in X$, there exists a Zariski neighborhood $U = X\backslash Z$ of x such that U can be embedded into the product

$$M_1 \times \cdots \times M_k$$

as a closed algebraic submanifold, where k is some positive integer and each $M_r = \mathbb{CP}^1 \backslash \{a_1, \ldots, a_{n_r}\}$, $n_r \geq 3$ (see [Gri71, p. 25, Lemma 2.3]). Hence the Kobayashi-Royden metric on $\mathbb{CP}^1 \backslash \{a_1, \ldots, a_{n_r}\}$ combining with the well-known formula of the Kobayashi-Royden metric on product manifolds provides the lower bound of the Kobayashi-Royden metric on U.

On the other side, a recent result shows that a complete Kähler manifold (M, ω) with negatively pinched holomorphic sectional curvature range, the base Kähler metric ω is uniformly equivalent to the complete Kähler-Einstein metric and the Kobayashi-Royden metric (see [WY20, Theorems 2 and 3]). Hence it is interesting to ask the class of Kähler manifolds on which the two metrics exactly coincide as Finsler metrics (for example, see [Cho19]). We will show that the two metrics coincide on puncture spheres while other classical invariant metrics like the Carathéodory-Reiffen metric and the Bergman metric vanish everywhere.

A lower bound of the Kobayashi-Royden metric on $\mathbb{CP}^1 \backslash \{a_1, a_2, a_3\}$ can be established by constructing a volume form (see [CG72]). In [KLZ14], H. Kang, L. Lee and C. Zeager estimated the derivative of the modular lambda function $\lambda(\tau)$ to obtain the boundary behavior of the Kobayashi-Royden metric on $\mathbb{CP}^1 \backslash \{0, 1, \infty\}$. Specifically, the Kobayashi-Royden metric at the point p in the direction $v = 1$ has the following estimate

$$\chi_{\mathbb{CP}^1 \backslash \{\infty, 0, 1\}}(p; v) \approx \frac{1}{\delta \log(1/\delta)} \qquad \text{as} \qquad \delta \to 0^+,$$

where $\delta = \mathrm{dist}(p, 0)$. As an improvement of the above estimate, we derive a precise asymptotic expansion of the Kobayashi-Royden metric on $\mathbb{CP}^1 \backslash \{a_1 = 0, a_2, \ldots, a_n\}$ from the asymptotic expansion of the complete Kähler-Einstein metric in [Qia19]. Furthermore, we enhance the formula in [Qia19] by using the exponential Bell polynomials.

Let us denote the Kobayashi-Royden metric on a complex manifold N at $p \in N$ in the direction $v \in T_pN$ by $\chi_{_N}(p;v)$, and sometime we will write $M = \mathbb{CP}^1 \backslash \{a_1 = 0, a_2, \ldots, a_n\}$ for convenience. Let $f : \mathbb{H} \to \mathbb{CP}^1 \backslash \{a_1 = 0, a_2, \ldots, a_n\}$ be a covering map with $f(\infty) = 0$, where $\mathbb{H}$ is the upper half plane in $\mathbb{C}$, then we have the following expansion of f

$$f(q_{_k}) = \sum_{m=1}^{\infty} c_m q_{_k}^m, \tag{2.1.1}$$

and the inversion series $q_{_k} = q_{_k}(f)$ around $f = 0$,

$$q_{_k}(f) = \sum_{m=1}^{\infty} b_m f^m, \tag{2.1.2}$$

where $q_{_k} = \exp\{\frac{2\pi i}{k}\tau\}$, $\tau \in \mathbb{H}$. The exponential Bell polynomial is defined as the following

$$\mathrm{B}_{n,k}(t_1, t_2, \ldots, t_{n-k+1}) = \sum \frac{n! t_1^{r_1} \cdots t_{n-k+1}^{r_{n-k+1}}}{r_1! \cdots r_{n-k+1}! (1!)^{r_1} \cdots ((n-k+1)!)^{r_{n-k+1}}},$$

here, the summation takes place over all integers $r_1, \ldots, r_{n-k+1} \geq 0$ such that

$$r_1 + r_2 + \cdots + r_{n-k+1} = k,$$
$$r_1 + 2r_2 + \cdots + (n-k+1) r_{n-k+1} = n.$$

The followings are the results of this paper.

Theorem 2.1.1. *For a covering space $\mathbb{H} \to \mathbb{CP}^1 \backslash \{a_1 = 0, a_2, \ldots, a_n\}$, let $f(\tau)$ be the universal covering map given by equation* (2.1.1). *Then the Kobayashi-Royden metric at point $p \in \mathbb{CP}^1 \backslash \{a_1 = 0, a_2, \ldots, a_n\}$ near the boundary* 0 *in the direction v has the* *following asymptotic expansion*

$$\chi_{M}(p;v)=\frac{1}{|p||\log|b_1p||}\cdot\left|1+\sum_{m=1}^{\infty}\left(\sum_{k=1}^{m}\frac{l_k \mathrm{C}_{m-k}(p)p^k}{(k-1)!(m-k)!}+\frac{\mathrm{C}_m(p)}{m!}\right)\right|\cdot||v||, \quad (2.1.3)$$

where

$$\begin{aligned}
\mathrm{C}_0(p) &= 0,\\
\mathrm{C}_m(p) &= \sum_{k=1}^{m}\frac{(-1)^k k!}{\log^k|b_1p|}\mathrm{B}_{m,k}(\mathrm{Re}(l_1p),\ldots,\mathrm{Re}(l_{m-k+1}p^{m-k+1})), && \textit{for } m\geq 1,\\
l_m &= \sum_{k=1}^{m}\frac{(-1)^{k-1}(k-1)!}{b_1^k}\mathrm{B}_{m,k}(1!b_2,2!b_3,\ldots,(m-k+1)!b_{n-k+2}), && \textit{for } m\geq 1,
\end{aligned}$$

and $b_1, b_2, \ldots,$ *are the coefficients in equation* (2.1.2).

Consequently, we show the the following local quantitative version of the Little Picard's theorem as an application of Theorem 2.1.1.

Theorem 2.1.2. *Let* $R > 0$ *be the maximal radius of the existence of a holomorphic map* $\varphi : \mathbb{D}_R \to \mathbb{CP}^1\backslash\{0, 1, \infty\}$ *satisfying* $\varphi(0) = p$ *and* $\varphi'(0) = 1$*, i.e.,*

$$R=\sup\{R_0:\varphi\in \mathrm{Hol}(\mathbb{D}_{R_0},\mathbb{CP}^1\backslash\{0,1,\infty\}),\varphi(0)=p,\varphi'(0)=1\},$$

where $p \in \mathbb{CP}^1\backslash\{0, 1, \infty\}$ *is a point near* 0*, and* $\mathbb{D}_R$ *is the disk* $\{z \in \mathbb{C} : |z| < R\}$*. Then the maximal radius* R *satisfies*

$$R<\left|p\log|p/16|+\frac{1}{2}p\,\mathrm{Re}(p)-p^2\log|p/16|+O(|p|^3)\right|, \qquad \textit{as } p\to 0. \qquad (2.1.4)$$

Specially, such $R \to 0$ *when* $p \to 0$*.*

2.2 The Kobayashi-Royden metric on punctured spheres

Let N be a complex manifold and let $\mathbb{D}$ be the unit disk in $\mathbb{C}$. For any point $z \in N$ and a tangent vector $v \in T_z N$, the Kobayashi-Royden metric is defined as

$$\chi_{N}(z; v) = \inf\{\frac{1}{\alpha} : \alpha > 0, f \in \mathrm{Hol}(\mathbb{D}, N), f(0) = z, f'(0) = \alpha v\}. \tag{2.2.1}$$

The Kobayashi-Royden metric χ_N is one of the effective invariant metrics to the study of holomorphic maps; however, it is not easy to compute for arbitrary complex manifolds (see [Wu93]). One natural way to compute χ_N is to consider the case when N admits a holomorphic covering $\pi : \tilde{N} \to N$, where $\tilde{N}$ is a complex manifold. Actually, the holomorphic covering π becomes an isometry between $\tilde{N}$ and N with respect to the Kobayashi-Royden metric due to the standard lifting argument of the covering map π (see Theorem 7.3.1 in [GKK11] and Exercise 3.9.8 in [JP13]). We have the following theorem which shows the relation between the Kobayashi-Royden metric and the Kähler-Einstein metric.

Proposition 2.2.1. *Let $\pi : \tilde{N} \to N$ be a holomorphic covering between two complex manifolds $\tilde{N}$ and N. Assume $\tilde{N}$ possesses the Kähler-Einstein metric $\omega_{\tilde{N}}$ of negative Ricci curvature, the Kobayashi-Royden metric $\chi_{\tilde{N}}$, and these two metrics coincide in the following sense*

$$\sqrt{\omega_{\tilde{N}}(v, v)} = \chi_{\tilde{N}}(p; v), \quad \textit{for any } v \in T_{\tilde{p}}\tilde{N},\ \tilde{p} \in \tilde{N}.$$

Then N also possesses the Kähler-Einstein metric ω_N and the Kobayashi-Royden metric χ_N. Furthermore, the two metrics coincide on N as well, and they satisfy the *following relation*

$$\sqrt{\omega_{_N}(v,v)} = \chi_{_N}(p;v) = \chi_{_{\tilde N}}(\tilde p;v)\frac{||v||}{|\pi'(\tilde p)|}, \qquad v \in T_pN,$$

where $\pi(\tilde p) = p$, $||v||$ denotes the Euclidean norm of the vector v, and π' is the differential map of π.

Proof. For $\dim_{\mathbb{C}} N \geq 2$, it follows directly from [GKK11, p.207, Corollary 7.6.4]. For the case of $\dim_{\mathbb{C}} N = 1$, it follows from [JP13, p.122, Theorem 3.3.7] (see, also [KLZ14, Lemma 1]). □

Let f be a covering map from $\mathbb{H}$ to $\mathbb{CP}^1\backslash\{a_1 = 0, a_2, \ldots, a_n\}$ with $f(\infty) = 0$, then f can be written as the following expansion

$$f(\tau) = f(q_{_k}) = c_1 q_{_k} + c_2 q_{_k}^2 + \sum_{m=3}^{\infty} c_m q_{_k}^m, \tag{2.2.2}$$

where $q_{_k} = \exp\{\frac{2\pi i}{k}\tau\}$, $\tau \in \mathbb{H}$, k is a real constant and it holds for any $\tau \in \mathbb{H}$ ([Qia19, Proposition 3.10]). Let us write $q_{_k}$ as a series in f,

$$q_{_k}(f) = b_1 f + b_2 f^2 + \sum_{m=3}^{\infty} b_m f^m, \tag{2.2.3}$$

where $b_1 = 1/c_1$ and

$$b_m = \frac{1}{m!}\sum_{k=1}^{m-1}\frac{(-1)^k}{c_1^{m+k}}\mathrm{B}_{m+k-1,k}(0, 2!c_2, 3!c_3, \ldots, m!c_m), \qquad m \geq 2, \tag{2.2.4}$$

where $\mathrm{B}_{n,k}$ is the exponential Bell polynomial (see [Com74, p. 134 and p. 151] and [Bel34]) defined as the following

$$\mathrm{B}_{n,k}(t_1, t_2, \dots, t_{n-k+1}) = \sum \frac{n! t_1^{r_1} \cdots t_{n-k+1}^{r_{n-k+1}}}{r_1! \cdots r_{n-k+1}! (1!)^{r_1} \cdots ((n-k+1)!)^{r_{n-k+1}}}, \qquad (2.2.5)$$

the summation takes place over all integers $r_1, \dots, r_{n-k+1} \geq 0$ such that

$$r_1 + r_2 + \cdots + r_{n-k+1} = k,$$
$$r_1 + 2r_2 + \cdots + (n-k+1) r_{n-k+1} = n.$$

Theorem 2.2.1. *For a covering space* $\mathbb{H} \to \mathbb{CP}^1 \backslash \{a_1 = 0, a_2, \dots, a_n\}$*, let* $f(\tau)$ *be the universal covering map given by equation* (2.2.2)*. Then the Kobayashi-Royden metric at* $p \in \mathbb{CP}^1 \backslash \{a_1 = 0, a_2, \dots, a_n\}$ *near* 0 *in the direction* v *is given as the following*

$$\chi_{_M}(p; v) = \frac{1}{|p| |\log |b_1 p||} \cdot \left| 1 + \sum_{m=1}^{\infty} \left(\sum_{k=1}^{m} \frac{l_k \mathrm{C}_{m-k}(p) p^k}{(k-1)!(m-k)!} + \frac{\mathrm{C}_m(p)}{m!} \right) \right| \cdot ||v||, \quad (2.2.6)$$

where

$$\mathrm{C}_m(p) = \sum_{k=1}^{m} \frac{(-1)^k k!}{\log^k |b_1 p|} \mathrm{B}_{m,k}(\mathrm{Re}(l_1 p), \dots, \mathrm{Re}(l_{m-k+1} p^{m-k+1})), \quad \mathrm{C}_0(p) = 0,$$
$$l_m = \sum_{k=1}^{m} \frac{(-1)^{k-1} (k-1)!}{b_1^k} \mathrm{B}_{m,k}(1! b_2, 2! b_3, \dots, (m-k+1)! b_{m-k+2}).$$

Proof. From Theorem 2.2.1 and [Qia19, Theorem 3.11], we have

$$\chi_{_M}(p; v) = |ds| = \frac{|q'(p)|}{|q(p)| \log |q(p)|} ||v||. \qquad (2.2.7)$$

Let us consider $q_{_f}/q$ as $(\log q)_{_f}$, which is well-defined due to the differentiation. Let us fix a branch and use equation (2.2.3), we have the following logarithm expansion

([Com74, p. 141])

$$\log q = \log(b_1 f) + \log\left(1 + \sum_{m=1}^{\infty} \frac{b_{m+1}}{b_1} f^m\right)$$
$$= \log(b_1 f) + \sum_{m=1}^{\infty} \frac{l_m}{m!} f^m, \tag{2.2.8}$$

where

$$l_m = \sum_{k=1}^{m} \frac{(-1)^{k-1}(k-1)!}{b_1^k} \mathrm{B}_{m,k}(1!b_2, 2!b_3, \ldots, (m-k+1)!b_{m-k+2}). \tag{2.2.9}$$

Therefore we have

$$(\log q(f))' = \frac{1}{f} + \sum_{m=1}^{\infty} \frac{l_m}{(m-1)!} f^{m-1} = \frac{1}{f} + \frac{b_2}{b_1} + \sum_{m=1}^{\infty} \frac{l_{m+1}}{m!} f^m.$$

On the other side, we have

$$\log|1+z| = \frac{1}{2}\log|1+z|^2 = \frac{1}{2}\log(1+z)(1+\overline{z}),$$

let us consider both of $\log(1+z)$ and $\log(1+\overline{z})$ on the principal branch,

$$2\log|1+z| = \log(1+z) + \log(1+\overline{z})$$
$$= \sum_{m=1}^{\infty}(-1)^{m+1}\frac{z^m}{m} + \sum_{m=1}^{\infty}(-1)^{m+1}\frac{\overline{z}^m}{m}$$
$$= 2\sum_{m=1}^{\infty}(-1)^{m+1}\frac{\mathrm{Re}(z^m)}{m}.$$

The asymptotic expansion of the metric can be given as the following

$$\begin{aligned}
\frac{|q'|}{|q||\log|q||} &= |(\log(q))'|\cdot|\log|q||^{-1} \\
&= |(\log(q))'|\cdot|\log|b_1 f||^{-1}\cdot\left|1+\sum_{m=1}^{\infty}\frac{1}{m!\log|b_1 f|}\operatorname{Re}(l_m f^m)\right|^{-1} \\
&= \left|\frac{(\log q)'}{\log|b_1 f|}\right|\cdot\left|1+\sum_{m=1}^{\infty}\frac{1}{m!}\mathrm{C}_m(f)\right| \\
&= \frac{1}{|f||\log|b_1 f||}\cdot\left|1+\sum_{m=1}^{\infty}\frac{ml_m}{m!}f^m\right|\cdot\left|1+\sum_{m=1}^{\infty}\frac{C_m(f)}{m!}\right| \\
&= \frac{1}{|f||\log|b_1 f||}\cdot\left|1+\sum_{m=1}^{\infty}\mathrm{R}_m(f)\right|, \qquad (2.2.10)
\end{aligned}$$

where $\mathrm{C}_m(f)$ and $\mathrm{R}_m(f)$ are defined as

$$\begin{aligned}
\mathrm{C}_m(f) &= \sum_{k=1}^{m}\frac{(-1)^k k!}{\log^k|b_1 f|}\mathrm{B}_{m,k}(\operatorname{Re}(l_1 f),\ldots,\operatorname{Re}(l_{m-k+1}f^{m-k+1})), \quad \mathrm{C}_0(f)=0, \\
\mathrm{R}_m(f) &= \sum_{k=1}^{m}\frac{l_k\mathrm{C}_{m-k}(f)f^k}{(k-1)!(m-k)!}+\frac{\mathrm{C}_m(f)}{m!}. \qquad (2.2.11)
\end{aligned}$$

□

For some special cases, we are able to obtain more information on the coefficients of the metric expansion.

Theorem 2.2.2. *The Kobayashi-Royden metric on* $\mathbb{CP}^1\backslash\{a_1=0,a_2,\ldots,a_n\}$, $n=3,4,6,12$, *has the following asymptotic expansion*

$$\chi_{_M}(p;v)=\frac{1}{|p||\log|b_1 p||}\cdot\left|1+\sum_{m=1}^{\infty}\mathrm{R}_m(p)\right|\cdot||v||, \qquad (2.2.12)$$

where R_m *is defined by equation* (2.2.11), *and the coefficients* l_m *in* $\mathrm{R}_m(p)$ *satisfy*

$l_m \in \mathbb{Q}(c_1, c_2)$.

Proof. Notice that we have the following relation

$$1 + 240 \sum_{m=1}^{\infty} \sigma_3(m) q^m = 1 - q_{_N}^2 \{f, q_{_N}\}, \quad N = 2, 3, 4, 5, \tag{2.2.13}$$

where $\sigma_3(m) = \sum_{d|m} d^3$ ([Qia19, Theorem 8.2]), and $\{f, q_{_N}\}$ is the Schwarzian derivative defined as the following

$$\{f, q_{_N}\} = 2 \left(\frac{f_{q_N q_N}}{f_{q_N}} \right)_{q_N} - \left(\frac{f_{q_N q_N}}{f_{q_N}} \right)^2. \tag{2.2.14}$$

We write $q = q_{_N}$ for convenience. Let us calculate f_{qq}/f_q by finding the series expansion of $(\log f_q)_q$, the branch choice is no longer an issue due to the differentiation,

$$\begin{aligned} (\log(f_q))_q &= \left[\log \left(c_1 + \sum_{m=1}^{\infty} \frac{(m+1)! c_{m+1}}{m!} q^m \right) \right]_q \\ &= \left(\log(c_1) + \sum_{m=1}^{\infty} \frac{\tilde{l}_m}{m!} q^m \right)_q \\ &= \tilde{l}_1 + \sum_{m=1}^{\infty} \frac{\tilde{l}_{m+1}}{m!} q^m, \end{aligned}$$

where

$$\tilde{l}_m = \sum_{k=1}^{m} \frac{(-1)^{k-1}(k-1)!}{c_1^k} \mathrm{B}_{m,k}(2! c_2, \ldots, (m-k+2)! c_{m-k+2}). \tag{2.2.15}$$

We have the following series expansion

$$\{f, q\} = \sum_{m=0}^{\infty} \left(\frac{2\tilde{l}_{m+2}}{m!} - \sum_{k=0}^{m} \frac{\tilde{l}_{k+1}\tilde{l}_{m-k+1}}{k!(m-k)!} \right) q^m \tag{2.2.16}$$

for the Schwarzian derivative. We can solve for each c_m from the following relation

$$1 + 240 \sum_{m=1}^{\infty} \sigma_3(m) q_{_N}^m = 1 - \sum_{m=0}^{\infty} \left(\frac{2\tilde{l}_{m+2}}{m!} - \sum_{k=0}^{m} \frac{\tilde{l}_{k+1}\tilde{l}_{m-k+1}}{k!(m-k)!} \right) q_{_N}^{m+2}. \tag{2.2.17}$$

Specifically, for $m \geq 0$, we have equations

$$\begin{cases} \dfrac{2\tilde{l}_{m+2}}{m!} - \displaystyle\sum_{k=0}^{m} \frac{\tilde{l}_{k+1}\tilde{l}_{m-k+1}}{k!(m-k)!} = -240\sigma_3(\frac{m+2}{N}), & \text{if } N|m+2, \\ \dfrac{2\tilde{l}_{m+2}}{m!} - \displaystyle\sum_{k=0}^{m} \frac{\tilde{l}_{k+1}\tilde{l}_{m-k+1}}{k!(m-k)!} = 0, & \text{otherwise.} \end{cases} \tag{2.2.18}$$

Each c_{m+2} can be solved from $\tilde{l}_{m+2}$, and $\tilde{l}_{m+2}$ can be solved in terms of $\tilde{l}_1, \ldots, \tilde{l}_{m+1}$. Relation (2.2.18) implies that $c_{m+2} \in \mathbb{Q}(c_1, c_2)$, consequently, $b_m \in \mathbb{Q}(c_1, c_2)$ due to equation (2.2.4), so as $l_m \in \mathbb{Q}(c_1, c_2)$ due to equation (2.2.9). □

Corollary 2.2.2. *The Kobayashi-Royden metric on* $\mathbb{CP}^1 \backslash \{0, 1, \infty\}$ *has the following asymptotic expansion*

$$\chi_{_M}(p, v) = \frac{1}{|p| \log |\frac{p}{16}|} \left\{ 1 + \frac{1}{2} \left(p - \frac{\operatorname{Re} p}{\log |\frac{p}{16}|} \right) + O(|p|^2) \right\} ||v||, \qquad \textit{as } p \to 0.$$

Proof. It directly follows from Theorem 2.2.2, and the fact that $c_1 = 16$, $c_2 = -128$ in equation (2.2.2). □

Remark 2.2.3. For $M = \mathbb{CP}^1 \backslash \{a_1 = 0, a_2, \ldots, a_n = \infty\}$, $n \geq 3$, we can identify M with $\mathbb{C}^1 \backslash \{a_1 = 0, a_2, \ldots, a_{n-1}\}$. Hence M is a quasi-projective algebraic manifold which is biholomorphic to

$$\{(x, y) \in \mathbb{C}^2 : y(x - a_1) \cdots (x - a_{n-1}) - 1 = 0\}.$$

Therefore our result is a concrete example of the Kobayashi-Royden pseudometric on a quasi-projective algebraic manifold Z which satisfies

$$\chi_Z(p;v) = \inf_C \chi_C(p;v),$$

where C runs over all (possibly singular) closed algebraic curves in Z that is tangent to v (see Corollary 1.3 in [DLS94]).

2.3 An application of the Kobayashi-Royden metric on $\mathbb{CP}^1\backslash\{0,1,\infty\}$

The Little Picard's theorem is one of the classic theorems in complex analysis, it states that there is no non-trivial holomorphic map from $\mathbb{C}$ to $\mathbb{CP}^1\backslash\{0,1,\infty\}$. Hence it is natural to ask the maximal radius $R > 0$ of the existence of a non-trivial holomorphic function $\varphi : \mathbb{D}_R \to \mathbb{CP}^1\backslash\{0,1,\infty\}$, where $\mathbb{D}_R = \{z \in \mathbb{C} : |z| < R\}$. For this question, the Kobayashi-Royden metric gives a local quantitative information which can be used to determine R. Recall the definition of the Kobayashi-Royden metric at a point $p \in N$ in the direction of $v \in T_pN$ in equation (2.2.1),

$$\chi_N(p;v) = \inf\{\frac{1}{\alpha} : \alpha > 0, f \in \mathrm{Hol}(\mathbb{D}, N), f(0) = p, f'(0) = \alpha v\}.$$

Let us consider the collection of holomorphic maps defined on $\mathbb{D}_R$ instead of the maps defined on the unit disk $\mathbb{D}$, and take the tangent vector $v = 1$, we have the following alternative definition

$$\chi_{_N}(p;1) = \inf\{\frac{1}{R} : \alpha > 0, g \in \mathrm{Hol}(\mathbb{D}_R, N), g(0) = 0, g'(0) = 1\}. \tag{2.3.1}$$

We use the asymptotic expansion of the Kobayashi-Royden metric on $\mathbb{CP}^1\backslash\{\infty, 0, 1\}$ to obtain the following local quantitative version of the Little Picard's theorem.

Theorem 2.3.1. *Let $R > 0$ be the maximal radius of the existence of a holomorphic map $\varphi : \mathbb{D}_R \to \mathbb{CP}^1\backslash\{0, 1, \infty\}$ satisfying $\varphi(0) = p$ and $\varphi'(0) = 1$, i.e.,*

$$R = \sup\{R_0 : \varphi \in \mathrm{Hol}(\mathbb{D}_{R_0}, \mathbb{CP}^1\backslash\{0, 1, \infty\}), \varphi(0) = p, \varphi'(0) = 1\},$$

where $p \in \mathbb{CP}^1\backslash\{0, 1, \infty\}$ is a point near 0. Then the maximal radius R satisfies

$$R < \left| p \log |p/16| + \frac{1}{2} p \,\mathrm{Re}(p) - p^2 \log |p/16| + O(|p|^3) \right|, \qquad \text{as } p \to 0. \tag{2.3.2}$$

Specially, such $R \to 0$ when $p \to 0$.

Proof. Let us take $v = 1$, equations (2.2.7) and (2.3.1) imply the following equation

$$\chi_{_{\mathbb{CP}^1\backslash\{\infty,0,1\}}}(p;1) = \inf \frac{1}{R_0} = R.$$

More precisely, the upper bound of the radius can be approximated as the following

$$
\begin{aligned}
R &< \frac{1}{\chi_{\mathbb{CP}^1\setminus\{\infty,0,1\}}(p;1)} \qquad (2.3.3)\\
&= \frac{|\log|q||}{|(\log q)'|}\\
&= |\log|q|| \cdot |f| \cdot \left|1 + \sum_{m=1}^{\infty} \frac{m l_m}{m!} f^m\right|^{-1}\\
&= |\log|q|| \cdot |f| \cdot \left|1 + \sum_{m=1}^{\infty} \frac{\tilde{c}_m}{m!} f^m\right|\\
&= |f||\log|b_1 f|| \cdot \left|1 + \sum_{m=1}^{\infty} \frac{1}{m!\log|b_1 f|} \operatorname{Re}(l_m f^m)\right| \cdot \left|1 + \sum_{m=1}^{\infty} \frac{\tilde{c}_m}{m!} f^m\right|\\
&= |f||\log|b_1 f|| \cdot \left|1 + \sum_{m=1}^{\infty} \left(\frac{\tilde{c}_m f^m}{m!} + \sum_{k=1}^{m} \frac{\tilde{c}_{m-k} f^{m-k} \operatorname{Re}(l_k f^k)}{(m-k)!k!\log|b_1 f|}\right)\right|, \qquad (2.3.4)
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{c}_m &= \sum_{k=1}^{m} (-1)^k k! \mathrm{B}_{m,k}(l_1, 2l_2, \ldots, (m-k+1)l_{m-k+1}),\\
l_m &= \sum_{k=1}^{m} \frac{(-1)^{k-1}(k-1)!}{b_1^k} \mathrm{B}_{m,k}(1!b_2, 2!b_3, \ldots, (m-k+1)!b_{m-k+2}).
\end{aligned}
$$

Notice that $b_1 = \frac{1}{16}$ and $b_2 = \frac{1}{32}$. For $m = 1$, we have

$$
\frac{\tilde{c}_1 f^1}{1!} + \sum_{k=1}^{1} \frac{\tilde{c}_{1-k} f^{1-k} \operatorname{Re}(l_k f^k)}{(1-k)!k!\log|f/16|} = \frac{1}{2}\left(f \operatorname{Re} f - f^2 \log|f/16|\right).
$$

For $m \geq 2$, we have

$$
\frac{\tilde{c}_m f^m}{m!} + \sum_{k=1}^{m} \frac{\tilde{c}_{m-k} f^{m-k} \operatorname{Re}(l_k f^k)}{(m-k)!k!\log|f/16|} = O(f^2) \qquad \text{as} \qquad f \to 0.
$$

The conclusion follows directly. □

Remark 2.3.2. The estimation of the maximal radius R is sharp when the point p is close to 0. It does not work for a point p if $|p|$ is large. The reason is that the asymptotic expansion of the Kobayashi-Royden metric works locally since the inversion series (2.2.3) converges locally.

2.4 Example

Examples 2.4.1. For the punctured sphere $\mathbb{CP}^1\backslash\{0, \frac{1}{3}, -\frac{1}{6} \pm \frac{\sqrt{3}}{6}i\}$, a universal covering map can be given by

$$f(\tau) = \left(\frac{\eta(3\tau)}{\eta(\frac{\tau}{3})}\right)^3 = q_3 + 3q_3^2 + \sum_{m=3}^{\infty} c_m q_3^m,$$

and the corresponding deck transformation group is $\Gamma(3)$ (see [Seb02]), where $\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n)$ is the Dedekind eta function, here $q = \exp\{2\pi i\tau\}$ and $\tau \in \mathbb{H}$. In this case, we have

$$c_1 = 1 \quad \text{and} \quad c_2 = 3.$$

The first couple of coefficients in $q = \sum_{m=1}^{\infty} b_m f$ follows from Theorems 2.2.1 and 2.2.2,

$$c_3 = 9, b_1 = 1,\ b_2 = -3,\ b_3 = 9,$$

so we have

$$\chi_M(p; v) = \frac{1}{|p|\log|p|}\left|1 - 3\left(p - \frac{\operatorname{Re} p}{\log|p|}\right)\right.$$
$$\left. +9\left(p^2 - \frac{p\operatorname{Re} p}{\log|p|} - \frac{1}{2}\frac{\operatorname{Re}(p^2)}{\log|p|} + \frac{(\operatorname{Re} p)^2}{\log^2|p|}\right) + O(|p|^3)\right| ||v||,$$

as $p \to 0$, where $M = \mathbb{CP}^1 \backslash \{0, \frac{1}{3}, -\frac{1}{6} \pm \frac{\sqrt{3}}{6} i\}$, $p \in M$ and $v \in T_p M$.

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