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Log-Sobolev Inequality for the Wright-Fisher Diffusion and Optimal Investment with Random Endowments Under Anticipation

Berend Coster
University of Connecticut - Storrs, berend.coster@uconn.edu

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Berend Johannes Coster, Ph.D.

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ABSTRACT

We prove a Log-Sobolev inequality for the one-dimensional Wright-Fisher diffusion by proving a Γ_2 lower bound for this diffusion. The result is extended to the two-dimensional case. In subsequent chapters an explicit formula is derived for the value of weak information in a discrete time model that works for a wide range of utility functions including the logarithmic and power utility. We assume a market with a finite number of assets and a finite number of possible outcomes. Results are given for complete and incomplete markets with random endowments. Explicit calculations are performed for a binomial model with two assets. Results for the continuous time case are also reviewed and discussed.

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Berend Johannes Coster

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APPROVAL PAGE

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Presented by

Berend Johannes Coster, B.S., M.S.

Major Advisor ______________________
Fabrice Baudoin

Associate Advisor ______________________
Oleksii Mostovyi

Associate Advisor ______________________
Ambar Sengupta

University of Connecticut, 2019

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Contents

Chapter 1

Log-Sobolev inequality and convergence to equilibrium for the Wright-Fisher diffusion

1.1 Introduction

1.1.1 Motivation

My work on functional inequalities is motivated by recent work on degenerate diffusions by C. Villani and F. Baudoin (see [23] and [3]). Degenerate diffusions arise as infinite population limits of the Wright-Fisher models in population genetics. (see [9]) These describe the prevalence of a mutant allele, in a population of fixed size, under the effects of genetic drift, mutation, migration and selection. The formal generator of the infinite population limit acts on functions defined on $[0, 1]$ (the space of frequencies) and is given by

$$L_{WF} = x(1-x)\partial_x^2 + \big[b_0(1-x) - b_1 x + sx(1-x)\big]\partial_x$$

Processes defined by such operators were studied by Feller in the early 1950s and used to great effect by Kimura, et al. in the 1960s and 70s to give quantitative answers to a wide range of questions in population genetics.

1.1.2 Goal

Our goal is to say more about convergence to equilibrium for the following diffusion:

$$L = \frac{1}{2}x(1-x)\partial_x^2 + \big[b_0(1-x) - b_1 x\big]\partial_x \tag{1.1.1}$$

on $[0,1]$ and the following two-dimensional extension:

$$L = \frac{1}{2}\sum_{i,j=1}^{2}\left(\delta_{ij}x_i - x_i x_j\right)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{k=1}^{n}\left(b_k - x_k\sum_{l=1}^{3} b_l\right)\frac{\partial}{\partial x_k}. \tag{1.1.2}$$

for $x \in \{(x_1, x_2) | x_1, x_2 \geq 0, x_1 + x_2 \leq 1\}$.

In particular we are able to show convergence in entropy which is much stronger than the L^2- convergence that was already established by [21](Proposition 3.3), at least in the one-dimensional case.

1.1.3 Method

Following [2] we assume μ is a probability measure. We define the entropy of a positive function f to be

$$Ent(f) = \int (f \log f) d\mu - \int f d\mu \log \int f d\mu.$$

Then the (tight) Log-Sobolev inequality

$$Ent(f^2) \leq C \int \Gamma(f, f) d\mu$$

for all $f \in C^2(\mathbb{R}^n)$ implies convergence to equilibrium with rate $\frac{4}{C}$:

Proposition 1.1.1. *The tight logarithmic Sobolev inequality holds with a constant C if and only if, for any integrable positive function f,*

$$Ent(P_t f) \leq e^{-4t/C} Ent(f).$$

We proceed with the development of a so-called Γ-Calculus. This Γ-Calculus will allow us to prove the Log-Sobolev inequality above.

Γ-Calculus

Consider $\mathbb{R}^n$ and diffusion operator L. Let $P_t = e^{tL}$ be the heat semigroup generated by L. Recall that for every $f \in C_0^\infty(\mathbb{R}^n)$, the function $u(t, x) = P_t(f)(x)$ is a solution to the the heat equation associated to L,

$$\partial_t u(t, x) = Lu(t, x) \quad , u(0, x) = f(x).$$

The "length of the gradient" of a smooth function $f : M \to \mathbb{R}$ is given by the *carré du champ* operator for L

$$\Gamma(f,f) = \Gamma(f) := \frac{1}{2}\left(Lf^2 - 2fLf\right).$$

We can also define a bilinear form:

$$\Gamma_2(f,f) = \Gamma_2(f) := \frac{1}{2}\left(L\Gamma(f,f) - 2\Gamma\left(f, Lf\right)\right).$$

We have that (see [2] Theorem 3.2(2))

Proposition 1.1.2. *If*

$$\Gamma_2(f,f) \geq \rho\Gamma(f,f) \text{ for all } f \in C_c^\infty(\mathbb{R}^n)$$

for some $\rho > 0$ the invariant measure is finite and, for the invariant probability, the tight logarithmic Sobolev inequality holds with a constant C bounded above by $\frac{2}{\rho}$.

1.2 The one-dimensional Wright-Fisher model

We consider the one-dimensional diffusion operator on $[0,1]$,

$$L = \frac{1}{2}x(1-x)\frac{\partial^2}{\partial x^2} + \left(b_0(1-x) - b_1 x\right)\frac{\partial}{\partial x}$$

where $b_0, b_1 \geq 0$. This diffusion operator corresponds to a stochastic process (X_t) that is characterized by the following stochastic differential equation:

$$dX_t = \big(b_0(1 - X_t) - b_1 X_t\big)\, dt + \sqrt{X_t(1 - X_t)} dB_t$$

where (B_t) is a standard Brownian motion.

We can write L in the form

$$L = \frac{1}{m} \frac{\partial}{\partial x} \left(m\sigma^2 \frac{\partial}{\partial x} \right),$$

where

$$\sigma^2(x) = \frac{1}{2} x(1 - x)$$

and

$$m(x) = x^{2b_0 - 1}(1 - x)^{2b_1 - 1}.$$

The operator L is symmetric for the measure $d\mu(x) = m(x)dx$. We define

$$\Gamma(f) = \frac{1}{2} L(f^2) - fLf.$$

A straightforward computation shows that

$$\Gamma(f) = \frac{1}{2} x(1 - x) \left(\frac{\partial f}{\partial x} \right)^2.$$

Consider now

$$\Gamma_2(f) = \frac{1}{2} L\Gamma(f) - \Gamma(f, Lf).$$

We compute:

Lemma 1.2.1.

$$\Gamma_2(f) = \left(\frac{1}{2}x(1-x)\frac{\partial^2 f}{\partial x^2} + \frac{1}{4}\left(1-2x\right)\frac{\partial f}{\partial x}\right)^2 + \frac{1}{4}\left(b_1 x + b_0(1-x) - \frac{1}{4}\right)\left(\frac{\partial f}{\partial x}\right)^2$$

Proof. We write the operator L using arbitrary coefficients $a(x)$ and $b(x)$. We have that

$$\begin{aligned}
Lf &= a(x)f_{xx} + b(x)f_x, \\
\Gamma(f,g) &= a(x)f_x g_x, \\
L\Gamma(f) &= aa_{xx}(f_x)^2 + 4aa_x f_x f_{xx} + 2a^2(f_{xx})^2 + 2a^2 f_x f_{xxx} + ba_x(f_x)^2 + 2abf_x f_{xx}, \\
\Gamma(f,Lf) &= aa_x f_x f_{xx} + a^2 f_x f_{xxx} + ab_x(f_x)^2 + abf_x f_{xx}, \\
\Gamma_2(f) &= \frac{1}{2}L\Gamma(f) - \Gamma(f,Lf) \\
&= \left(\frac{1}{2}aa_{xx} + \frac{1}{2}ba_x - ab_x\right)(f_x)^2 + aa_x f_x f_{xx} + a^2(f_{xx})^2 \\
&= \left(\frac{1}{2}aa_{xx} + \frac{1}{2}ba_x - ab_x - \frac{1}{4}a_x^2\right)(f_x)^2 + \left(\frac{1}{2}a_x f_x + af_{xx}\right)^2
\end{aligned}$$

Denote by $\varphi(x)$ the coefficient of $(f_x)^2$ on the right-hand side of the preceding inequality. We have

$$\begin{aligned}
&a(x) = \frac{1}{2}x(1-x), \quad a_x(x) = \frac{1}{2}(1-2x), \quad a_{xx}(x) = -1, \\
&b(x) = b_0(1-x) - b_1 x, \quad b_x(x) = -(b_0 + b_1),
\end{aligned}$$

and so we obtain

$$\varphi(x) = \frac{1}{4}\left(b_1 x + b_0(1-x) - \frac{1}{4}\right).$$

■

From this we have:

Corollary 1.2.2. *Assume* $b_0, b_1 \geq \frac{1}{4}$*. Then*

$$\Gamma_2(f) \geq \rho\Gamma(f), \tag{1.2.1}$$

with

$$\rho = \frac{1}{2}\left(b_0 + b_1 - \frac{1}{2}\right) + \sqrt{\left(b_0 - \frac{1}{4}\right)\left(b_1 - \frac{1}{4}\right)}$$

.

Proof.

$$\Gamma_2(f) = \left(\frac{1}{2}x(1-x)\frac{\partial^2 f}{\partial x^2} + \frac{1}{4}(1-2x)\frac{\partial f}{\partial x}\right)^2 + \frac{1}{4}\left(b_1 x + b_0(1-x) - \frac{1}{4}\right)\left(\frac{\partial f}{\partial x}\right)^2$$

$$\geq \frac{1}{4}\left(b_1 x + b_0(1-x) - \frac{1}{4}\right)\left(\frac{\partial f}{\partial x}\right)^2$$

$$\geq \rho\Gamma(f).$$

with

$$\rho = \frac{1}{2}\left(b_0 + b_1 - \frac{1}{2}\right) + \sqrt{\left(b_0 - \frac{1}{4}\right)\left(b_1 - \frac{1}{4}\right)}$$

. ■

The main result for the one-dimensional Wright-Fisher diffusion is given by:

Theorem 1.2.3. *Let* P_t *be the semigroup corresponding to the diffusion with infinitesimal generator given by equation 1.1.1 and assume* $b_0, b_1 \geq \frac{1}{4}$*. Then for any integrable positive function* f*,*

$$Ent(P_t f) \leq e^{-At} Ent(f)$$

where $A \geq (b_0 + b_1 - \frac{1}{2}) + 2\sqrt{(b_0 - \frac{1}{4})(b_1 - \frac{1}{4})}$.

Proof. Combining corollary (1.2.2) with proposition (1.1.2) we obtain a tight Log-Sobolev inequality with $C \leq \frac{2}{\rho}$. The theorem follows then from proposition (1.1.1). ■

Remark 1 The case $b_0 = b_1 = 1/4$ is interesting (critical case for which $\rho = 0$). Itô's formula shows that the diffusion process with generator L is

$$X_t = \frac{1}{2}(1 + \cos B_t)$$

where B is a Brownian motion.

1.3 The two-dimensional Wright-Fisher diffusion

We now consider the 2-dimensional diffusion operator

$$L = \frac{1}{2} \sum_{i,j=1}^{2} \left(\delta_{ij} x_i - x_i x_j\right) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{k=1}^{2} \left(b_k - x_k \sum_{l=1}^{n+1} b_l \right) \frac{\partial}{\partial x_k}.$$

For all $i, j = 1, \ldots, 2$, we denote for brevity:

$$a_{ij}(x) = \frac{1}{2}(\delta_{ij} x_i - x_i x_j), \quad \text{and} \quad c_i(x) = b_i - x_i \sum_{l=1}^{3} b_l. \tag{1.3.1}$$

Lemma 1.3.1. *For all $f \in C^2(\mathbb{R}^2)$, we have*

$$\begin{aligned}\Gamma_2(f) = \frac{1}{2}\sum_{i,j,k,l=1}^{2} a_{ij}\partial_{x_i x_j} a_{lk} f_{x_l} f_{x_k} + \frac{1}{2}\sum_{i,j,k=1}^{2} c_i \partial_{x_i} a_{lk} f_{x_l} f_{x_k} - \sum_{i,j,k=1}^{2} a_{ij}\partial_{x_j} c_k f_{x_i} f_{x_k} \\ + \sum_{i,j,k,l=1}^{2} a_{ij}\partial_{x_i} a_{lk}\left(f_{x_j x_l} f_{x_k} + f_{x_j x_k} f_{x_l} - f_{x_l x_k} f_{x_j}\right) \\ + \sum_{i,j,k,l=1}^{2} a_{ij} a_{lk} f_{x_l x_i} f_{x_k x_j}.\end{aligned} \tag{1.3.2}$$

Proof. Identity (1.3.2) is obtained by direct calculations. We have

$$\Gamma(f,g) = \sum_{i,j=1}^{2} a_{ij} f_{x_i} g_{x_j}, \tag{1.3.3}$$

which gives us

$$\begin{aligned}\partial_{x_i}\Gamma(f) &= \sum_{l,k=1}^{2} (\partial_{x_i} a_{lk}) f_{x_l} f_{x_k} + \sum_{l,k=1}^{2} a_{lk}\left(f_{x_l x_i} f_{x_k} + f_{x_l} f_{x_k x_i}\right), \\ \partial_{x_i x_j}\Gamma(f) &= \sum_{l,k=1}^{2} \partial_{x_i x_j} a_{lk} f_{x_l} f_{x_k} + \sum_{l,k=1}^{2} \partial_{x_i} a_{lk}\left(f_{x_l x_j} f_{x_k} + f_{x_l} f_{x_k x_j}\right) \\ &\quad + \sum_{l,k=1}^{2} \partial_{x_j} a_{lk}\left(f_{x_l x_i} f_{x_k} + f_{x_l} f_{x_k x_i}\right) \\ &\quad + \sum_{l,k=1}^{2} a_{lk}\left(f_{x_l x_i x_j} f_{x_k} + f_{x_l x_i} f_{x_k x_j} + f_{x_l x_j} f_{x_k x_i} + f_{x_l} f_{x_k x_i x_j}\right).\end{aligned}$$

Combining the preceding two identities we obtain that

$$\begin{aligned} L(\Gamma(f)) = & \sum_{i,j,k,l=1}^{2} a_{ij}\partial_{x_i x_j} a_{lk} f_{x_l} f_{x_k} + 2 \sum_{i,j,k,l=1}^{2} a_{ij}\partial_{x_i} a_{lk} \left(f_{x_l x_j} f_{x_k} + f_{x_l} f_{x_k x_j} \right) \\ & + 2 \sum_{i,j,l,k=1}^{2} a_{ij} a_{lk} \left(f_{x_l x_i x_j} f_{x_k} + f_{x_l x_i} f_{x_k x_j} \right) \\ & + \sum_{i,l,k=1}^{2} c_i \partial_{x_i} a_{lk} f_{x_l} f_{x_k} + 2 \sum_{i,l,k=1}^{2} c_i a_{lk} f_{x_l x_i} f_{x_k}. \end{aligned} \tag{1.3.4}$$

We also have

$$\partial_{x_j} Lf = \sum_{l,k=1}^{2} \partial_{x_j} a_{lk} f_{x_l x_k} + \sum_{l,k=1}^{2} a_{lk} f_{x_l x_k x_j} + \sum_{l=1}^{2} \partial_{x_j} c_l f_{x_l} + \sum_{l,k=1}^{2} c_l f_{x_l x_j}.$$

which gives us

$$\begin{aligned} \Gamma(f, Lf) = & \sum_{i,j,l,k=1}^{2} a_{ij}\partial_{x_j} a_{lk} f_{x_l x_k} f_{x_i} + \sum_{i,j,l,k=1}^{2} a_{ij} a_{lk} f_{x_l x_k x_j} f_{x_i} \\ & + \sum_{i,jl=1}^{2} a_{ij} \partial_{x_j} c_l f_{x_l} f_{x_i} + \sum_{l,k=1}^{2} a_{ij} c_l f_{x_l x_j} f_{x_i}. \end{aligned} \tag{1.3.5}$$

Combining (1.3.5) and (1.3.4), we obtain identity (1.3.2). ■

In the two dimensional case we can prove the following lower bound for Γ_2:

Lemma 1.3.2. *If $b_1, b_2, b_3 \geq 0$ are such that*

$$b_1 + b_2 + b_3 > 1$$

and

$$\frac{b_1 + b_2}{b_1 + b_2 + b_3 - 1} < \frac{\min_{1 \leq i \leq 2} b_i - 3}{\min_{1 \leq i \leq 2} b_i}$$

then there is a positive constant ρ *such that*

$$\Gamma_2(f) \geq \rho\Gamma(f). \tag{1.3.6}$$

The optimal ρ *for the inequality above is given by*

$$\rho = \frac{(b_3 - 1)\min_{1\leq i\leq 2} b_i}{12(b_1 + b_2 + b_3 - 1)} - \frac{1}{4}$$

Proof. Using the form of the coefficients $a_{ij}(x)$ and $c_i(x)$ (see equation (1.3.1)), we have

$$\begin{aligned}
\partial_{x_l} a_{ij} &= \frac{1}{2}\delta_{il}\delta_{jl} - \frac{1}{2}\delta_{il}x_j - \frac{1}{2}\delta_{jl}x_i, \\
\partial_{x_l x_k} a_{ij} &= -\frac{1}{2}\delta_{il}\delta_{jk} - \frac{1}{2}\delta_{ik}\delta_{jl}, \\
\partial_{x_l} c_i &= -\delta_{il}\sum_{k=1}^{3} b_k.
\end{aligned}$$

and so, we obtain a simplified form of $\Gamma_2(f)$:

$$\begin{aligned}
\Gamma_2(f) = &\sum_{i=1}^{2}\left[-\frac{1}{2}a_{ii} + \frac{1}{4}c_i(1 - 2x_i) + a_{ii}\sum_{k=1}^{3} b_k\right]\left(f_{x_i}\right)^2 \\
&+ \sum_{i<j}\left[-a_{ij} - \frac{1}{2}c_i x_j - \frac{1}{2}c_j x_i + 2a_{ij}\sum_{k=1}^{3} b_k\right](f_{x_i} f_{x_j}) \\
&+ \sum_{i,j,k,l=1}^{2} a_{ij}\partial_{x_i} a_{lk}\left(f_{x_j x_l} f_{x_k} + f_{x_j x_k} f_{x_l} - f_{x_l x_k} f_{x_j}\right) \\
&+ \sum_{i,j,k,l=1}^{2} a_{ij} a_{lk} f_{x_l x_i} f_{x_k x_j}.
\end{aligned} \tag{1.3.7}$$

To obtain the inequality $\Gamma_2(f) \geq \rho\Gamma(f)$, it is necessary to have that

$$\sum_{i,j,k,l=1}^{2} a_{ij}a_{lk}f_{x_lx_i}f_{x_kx_j} \geq 0. \tag{1.3.8}$$

Consider the $(2^2 \times 2^2)$-matrix defined by

$$A_{pq} = a_{ij}a_{lk}, \quad \forall\, p, q = 1, \ldots, 2^2,$$

where the indices i, j, k, l are the unique integers in $\{1, 2, \ldots, 2\}$, chosen such that $p = (l-1)2 + i$ and $q = (k-1)2 + j$. We also consider the 2^2-column vector $X_{(l-1)2+i} := f_{x_lx_i}$. Then we can write

$$\sum_{i,j,k,l=1}^{2} a_{ij}a_{lk}f_{x_lx_i}f_{x_kx_j} = \langle X, AX\rangle\,,$$

and so, to prove inequality (1.3.8), we only need to show that the matrix A is nonnegative definite. Writing the vector $X = (\eta_1, \eta_2)$, where $\eta_i \in \mathbb{R}^2$, for $i = 1, 2$, we have that

$$\langle X, AX\rangle = \sum_{i,j=1}^{2} a_{ij}\left\langle \eta_i, a\eta_j\right\rangle.$$

Because a is a symmetric, nonnegative definite matrix, there is an orthogonal matrix, P, and a diagonal matrix, D, such that $a = P^T DP$. By letting $\zeta_i := \sqrt{D}P\eta_i$, we have that $\left\langle \eta_i, a\eta_j\right\rangle = \left\langle \zeta_i, \zeta_j\right\rangle$, and so we can find real constants, ξ_1, ξ_2, such that $\left\langle \zeta_i, \zeta_j\right\rangle = \xi_i\xi_j$. The preceding three identities together with the fact that a is a nonnegative matrix give us that

$$\langle X, AX\rangle = \sum_{i,j=1}^{2} a_{ij}\xi_i\xi_j \geq 0.$$

Thus inequality (1.3.8) holds.

Since the operator L is defined on the unit simplex $\{x_1, x_2 \geq 0 : x_1 + x_2 \leq 1\}$ $\Gamma(f)$ and $\Gamma_2(f)$ are also defined on this domain. First assume x_1,$x_2 > 0$ and $x_1 + x_2 < 1$. Returning to the simplified expression for $\Gamma_2(f)$ as given in equation (1.3.7) we notice that

$$\begin{aligned} &\sum_{i,j,k,l=1}^{2} a_{ij}(\partial_{x_i} a_{lk})\Big(f_{x_jx_l}f_{x_k} + f_{x_jx_k}f_{x_l} - f_{x_lx_k}f_{x_j}\Big) \\ = &\sum_{j,k,l=1}^{2} f_{x_jx_l}f_{x_k}(a_{jkl} + a_{jlk} - a_{klj}) = \sum_{j,k,l=1}^{2} a_{jkl}f_{x_jx_l}f_{x_k} \end{aligned} \tag{1.3.9}$$

where a_{jkl} denotes $\sum_{i=1}^{2} a_{ij}(\partial_{x_i} a_{lk})$.

Using this notation we write the simplified $\Gamma_2(f)$ as follows:

$$\Gamma_2(f) = a_{11}^2 f_{x_1x_1}^2 + \big(4a_{11}a_{12}f_{x_1x_2} + 2a_{12}^2 f_{x_2x_2} + a_{111}f_{x_1} + a_{121}f_{x_2}\big)f_{x_1x_1} \tag{1.3.10}$$

$$+(2a_{11}a_{22} + 2a_{12}^2)f_{x_1x_2}^2 + 4a_{12}a_{22}f_{x_1x_2}f_{x_2x_2} + a_{22}^2 f_{x_2x_2}^2 + \sum_{j,k,l=1,(j,l)\neq(1,1)}^{2} a_{jkl}f_{x_jx_l}f_{x_k}$$

$$+S_1(a_{11}(x), c_1(x), x_1)f_{x_1}^2 + S_2(a_{22}(x), c_2(x), x_2)f_{x_2}^2 + S_{12}(a_{12}(x), c_1(x), c_2(x), x_1, x_2)f_{x_1}f_{x_2}$$

Completing the square with respect to $f_{x_1x_1}$,$f_{x_1x_2}$ and $f_{x_2x_2}$ we get the previous expression (equation (1.3.10)) is equal to

$$\Big(...\Big)^2+\Big(...\Big)^2+\Big(...\Big)^2-\Big(-\frac{1}{4}x_1f_{x_1}+\frac{\frac{1}{2}x_1x_2-\frac{1}{2}x_1-x_2+\frac{1}{2}}{2(1-x_1)}f_{x_2}\Big)^2 \tag{1.3.11}$$

$$-\Big(-\frac{1}{2}(x_1-\frac{1}{2})f_{x_1}+\frac{\frac{1}{2}x_2(x_1-\frac{1}{2})}{(1-x_1)}f_{x_2}\Big)^2-\frac{x_1x_2(1-x_1-x_2)}{8(1-x_1)^2}f_{x_2}^2$$

$$+S_1(a_{11}(x),c_1(x),x_1)f_{x_1}^2+S_2(a_{22}(x),c_2(x),x_2)f_{x_2}^2+S_{12}(a_{12}(x),c_1(x),c_2(x),x_1,x_2)f_{x_1}f_{x_2}$$

$$\geq\Big(...\Big)^2+\Big(...\Big)^2+\Big(...\Big)^2-\Big(-\frac{1}{4}x_1f_{x_1}+\frac{\frac{1}{2}x_1x_2-\frac{1}{2}x_1-x_2+\frac{1}{2}}{2(1-x_1)}f_{x_2}\Big)^2 \tag{1.3.12}$$

$$-\Big(-\frac{1}{2}(x_1-\frac{1}{2})f_{x_1}+\frac{\frac{1}{2}x_2(x_1-\frac{1}{2})}{(1-x_1)}f_{x_2}\Big)^2-\frac{1}{8}\frac{1}{2}\left\|Df\right\|^2$$

$$+S_1(a_{11}(x),c_1(x),x_1)f_{x_1}^2+S_2(a_{22}(x),c_2(x),x_2)f_{x_2}^2+S_{12}(a_{12}(x),c_1(x),c_2(x),x_1,x_2)f_{x_1}f_{x_2}$$

Now observe that equation (1.3.12) is of the form

$$\Big(...\Big)^2+\Big(...\Big)^2+\Big(...\Big)^2-\Big(\beta_1(x)f_{x_1}+\beta_2(x)f_{x_2}\Big)^2-\Big(\gamma_1(x)f_{x_1}+\gamma_2(x)f_{x_2}\Big)^2$$

$$-\frac{1}{16}\left\|Df\right\|^2+S_1(a_{11}(x),c_1(x),x_1)f_{x_1}^2+S_2(a_{22}(x),c_2(x),x_2)f_{x_2}^2+S_{12}(a_{12}(x),c_1(x),c_2(x),x_1,x_2)f_{x_1}f_{x_2}$$

with

$$\beta_1(x)=-\frac{1}{4}x_1$$

$$\beta_2(x)=\frac{\frac{1}{2}x_1x_2-\frac{1}{2}x_1-x_2+\frac{1}{2}}{2(1-x_1)}$$

$$\gamma_1(x)=-\frac{1}{2}(x_1-\frac{1}{2})$$

$$\gamma_2(x) = \frac{\frac{1}{2}x_2(x_1 - \frac{1}{2})}{(1 - x_1)}$$

Notice that the functions β_1, β_2, γ_1, γ_2 are all bounded by the constant $\frac{1}{4}$. Consequently we have

$$\Gamma_2(f) \geq -3\frac{1}{16}\|Df\|^2 - 3\frac{1}{16}\|Df\|^2 - \frac{1}{16}\|Df\|^2$$

$$+S_1(a_{11}(x), c_1(x), x_1)f_{x_1}^2 + S_2(a_{22}(x), c_2(x), x_2)f_{x_2}^2 + S_{12}(a_{12}(x), c_1(x), c_2(x), x_1, x_2)f_{x_1}f_{x_2}$$

$$= -\frac{7}{16}\|Df\|^2$$

$$+\Big(-\frac{1}{2}a_{11} + \frac{1}{4}c_1(1-2x_1) + a_{11}\sum_{k=1}^{3} b_k\Big)f_{x_1}^2 + \Big(-\frac{1}{2}a_{22} + \frac{1}{4}c_2(1-2x_2) + a_{22}\sum_{k=1}^{3} b_k\Big)f_{x_2}^2$$

$$+\Big(-a_{12} - \frac{1}{2}c_1x_2 - \frac{1}{2}c_2x_1 + 2a_{12}\sum_{k=1}^{3} b_k\Big)f_{x_1}f_{x_2}$$

Using the expression of the coefficients $a_{ij}(x)$ and $c_i(x)$, direct calculations give us

$$\begin{aligned}
&-\frac{1}{2}a_{ii}(x) + \frac{1}{4}c_i(x)(1-2x_i) + a_{ii}(x)\sum_{k=1}^{3} b_k \\
&\quad = \frac{1}{4}x_i^2 + \frac{1}{4}\left(\sum_{k=1}^{3} b_k - 1\right)x_i + \frac{1}{4}b_i(1-2x_i), \\
&-a_{ij}(x) - \frac{1}{2}c_i(x)x_j - \frac{1}{2}c_j(x)x_i + 2a_{ij}(x)\sum_{k=1}^{3} b_k \\
&\quad = \frac{1}{2}x_ix_j - \frac{b_i}{2}x_j - \frac{b_j}{2}x_i.
\end{aligned}$$

For all $\epsilon > 0$ and $\xi \in \mathbb{R}^2$, we use the inequalities

$$-\frac{b_i}{2}x_j\xi_i\xi_j \geq -\frac{\varepsilon}{4}b_ix_j\xi_i^2 - \frac{1}{4\varepsilon}b_ix_j\xi_j^2,$$

$$-\frac{b_j}{2}x_i\xi_i\xi_j \geq -\frac{1}{4\varepsilon}b_jx_i\xi_i^2 - \frac{\varepsilon}{4}b_jx_i\xi_j^2,$$

and we obtain that

$$\sum_{i=1}^{2}\left[-\frac{1}{2}a_{ii}(x) + \frac{1}{4}c_i(x)(1-2x_i) + a_{ii}(x)\sum_{k=1}^{3}b_k\right]f_{x_i}^2$$
$$+\sum_{i<j}\left[-a_{ij} - \frac{1}{2}c_ix_j - \frac{1}{2}c_jx_i + 2a_{ij}\sum_{k=1}^{3}b_k\right]f_{x_i}f_{x_j}$$
$$\geq \frac{1}{4}\left(\sum_{i=1}^{2}x_if_{x_i}\right)^2$$
$$+\sum_{i=1}^{2}\left[\frac{1}{4}\left(b_3 - 1 + \left(1-\frac{1}{\varepsilon}\right)\sum_{\substack{j=1\\ j\neq i}}^{2}b_j - (1-\varepsilon)b_i\right)x_i + \frac{1}{4}b_i\left(1-\varepsilon\sum_{j=1}^{2}x_j\right)\right]f_{x_i}^2.$$

Since $\sum_{j=1}^{2}x_j \leq 1$, we have that

$$\frac{1}{4}b_i\left(1-\varepsilon\sum_{j=1}^{2}x_j\right) \geq \frac{1}{4}b_i(1-\varepsilon), \quad \forall\,\varepsilon\in(0,1).$$

By assumption we have

$$\frac{b_1+b_2}{b_1+b_2+b_3-1} < \frac{\min_{1\leq i\leq 2}b_i - 3}{\min_{1\leq i\leq 2}b_i}.$$

Pick ε such that

$$\frac{b_1+b_2}{b_1+b_2+b_3-1} < \varepsilon < \frac{\min_{1\leq i\leq 2}b_i - 3}{\min_{1\leq i\leq 2}b_i}.$$

$$\varepsilon > \frac{b_1+b_2}{b_1+b_2+b_3-1}$$

implies

$$b_3 - 1 + \left(1 - \frac{1}{\varepsilon}\right) b_1 - (1-\varepsilon) b_2 > b_3 - 1 + \left(1 - \frac{1}{\varepsilon}\right)(b_1 + b_2) > 0$$

and

$$b_3 - 1 + \left(1 - \frac{1}{\varepsilon}\right) b_2 - (1-\varepsilon) b_1 > b_3 - 1 + \left(1 - \frac{1}{\varepsilon}\right)(b_1 + b_2) > 0$$

Therefore we can write

$$\Gamma_2(f) \geq -\frac{3}{4}\|Df\|^2 + \frac{1}{4}\sum_{i=1}^{2} b_i (1-\varepsilon) f_{x_i}^2 \geq -\frac{3}{4}\|Df\|^2 + \frac{1}{4}(1-\varepsilon)\min_{1\leq i\leq 2} b_i \|Df\|^2 .$$

Now $\varepsilon < \frac{\min_{1\leq i\leq 2} b_i - 3}{\min_{1\leq i\leq 2} b_i}$ implies

$$\delta = -\frac{3}{4} + \frac{1}{4}(1-\varepsilon)\min_{1\leq i\leq 2} b_i > 0.$$

Therefore we may conclude as follows:

There exists a $\rho > 0$ such that

$$\Gamma_2(f) \geq \delta \|Df\|^2 \geq \frac{\delta}{3}\Gamma(f) = \rho\Gamma(f).$$

To be precise this ρ is given by

$$\rho = -\frac{1}{4} + \frac{1}{(2n-1)4}(1-\varepsilon)\min_{1\leq i\leq n} b_i = -\frac{1}{4} + \frac{1}{12}(1-\varepsilon)\min_{1\leq i\leq 2} b_i.$$

The optimal ρ (make optimal choice ϵ) is given by

$$\rho = \frac{(b_3 - 1)\min_{1\leq i\leq 2} b_i}{12(b_1 + b_2 + b_3 - 1)} - \frac{1}{4}$$

For the points on the boundary of the domain we also have the desired lower bound for Γ_2. ■

With the help of the lemma above we can prove the following theorem.

Theorem 1.3.3. *Let P_t be the semigroup corresponding to the diffusion with infinitesimal generator given by equation 1.1.2. Assume $b_1 + b_2 + b_3 > 1$ and $\frac{b_1+b_2}{b_1+b_2+b_3-1} < \frac{\min_{1\leq i\leq 2} b_i - 3}{\min_{1\leq i\leq 2} b_i}$. Then for any integrable positive function f,*

$$Ent(P_t f) \leq e^{-At} Ent(f)$$

where $A \geq \frac{(b_3-1)\min_{1\leq i\leq 2} b_i}{6(b_1+b_2+b_3-1)} - \frac{1}{2}$.

Proof. Combining lemma (1.3.2) with proposition (1.1.2) we obtain a tight Log-Sobolev inequality with $C \leq \frac{2}{\rho}$. The theorem follows then from proposition (1.1.1).

■

Chapter 2

Introduction to the value of weak information in incomplete markets

2.1 Set-up

Following [4] and [5] we start with a constant finite time horizon $T > 0$ and a filtered probability space $\left(\Omega, (\mathcal{F}_t)_{0\leq t\leq T}, \tilde{\mathbb{P}}\right)$ which satisfies the usual conditions.($\mathcal{F}_0$ is trivial) Assume the price process of a given contingent claim is a continuous $d-$dimensional and $\mathcal{F}$-adapted square integrable local martingale $(S_t)_{0\leq t\leq T}$ with quadratic covariation matrix denoted by $\langle S\rangle$:

$$\langle S\rangle_t = \left(\left\langle S^i, S^j\right\rangle_t\right)_{1\leq i,j\leq d}$$

We assume $\langle S\rangle$ is almost surely valued in the space of positive matrices so that S is non-degenerate. Since we assumed S to be a local martingale under $\tilde{\mathbb{P}}$, the market just defined has no arbitrage. The no arbitrage condition is discussed in more depth in later in this chapter. In the context described above there is only one measure $\tilde{\mathbb{P}}$ that guarantees S is a local martingale. This situation corresponds to a complete market. More generally we can consider a set of measures $\mathcal{M}$ for which S is a local martingale. This means the market is incomplete.

2.2 Weak information

Let $Y : \Omega \longrightarrow \mathbb{R}^d$ be a $\mathcal{F}_T$-measurable random variable. Let $\mathbb{P}_Y$ denote the law of Y. We consider an insider who is (only) weakly informed on the random variable Y. This means that they have knowledge of the filtration $\mathcal{F}$ and of the law of Y. We proceed by associating a probability measure ν on $\mathbb{R}^d$ to the random variable Y. It is required that ν is equivalent to the law of Y $\mathbb{P}_Y$ with a bounded density. One can interpret the measure ν as the law of Y under the effective probability of the market. Now consider the following probability measure associated with the weak information (Y, ν).

Definition. Define the probability measure $\mathbb{P}^\nu$ defined on $(\Omega, \mathcal{F}_T)$ by:

$$\mathbb{P}^\nu(A) = \int_{\mathbb{R}^d} \tilde{\mathbb{P}}(A|Y \in dy)\nu(dy), A \in \mathcal{F}_T$$

where $\tilde{\mathbb{P}} \in \mathcal{M}$ is one of the risk-neutral probabilities in the (incomplete) market. $\mathbb{P}^\nu$ is called the minimal probability associated with the weak information (Y, ν). $\mathbb{P}^\nu$ is minimal in the sense of the proposition below. Let $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}$ be a given convex function. Denote by $\mathcal{E}^\nu$ the set of probability measures $\mathbb{Q}$ on Ω which are equivalent to $\mathbb{P}$ and such that the law of Y under $\mathbb{Q}$ is ν. Then

Proposition 2.2.1.

$$\min_{\mathbb{Q}\in\mathcal{E}^\nu} \mathbb{E}\Big[\phi\Big(\frac{d\mathbb{Q}}{d\mathbb{P}}\Big)\Big] = \mathbb{E}\Big[\phi\Big(\frac{d\mathbb{P}^\nu}{d\mathbb{P}}\Big)\Big].$$

Remark. Proposition 2.2.1. (over the class of functions ϕ) says that $\mathbb{P}^\nu$ dominates every other measure in $\mathcal{E}^\nu$ in the sense of second-order stochastic dominance:

$$\mathbb{P}^\nu \geq_2 \mathbb{Q}$$

for every $\mathbb{Q} \in \mathcal{E}^\nu$.

We are interested in the financial value of the weak information as described above. We need the following definitions and notations.

Consider an economic agent whose preferences over terminal consumption bundles are represented by a utility function $U : (0, \infty) \to \mathbb{R}$. The function U is assumed to be strictly concave, strictly increasing and continuously differentiable and to satisfy the Inada conditions:

$$U'(0) := \lim_{x\to 0} U'(x) = \infty, U'(\infty) := \lim_{x\to\infty} U'(x) = 0. \tag{2.2.1}$$

Let $\mathcal{A}_{\mathcal{F}}(S)$ be the set of adapted admissible strategies.

Definition The space $\mathcal{A}_{\mathcal{F}}(S)$ of admissible strategies is the space of $\mathbb{R}^d$-valued and $\mathcal{F}$-predictable processes Θ integrable with respect to the price process S, such that

$$\left(\int_0^t \Theta_u \cdot dS_u\right)_{0\le t\le T}$$

is a $(\tilde{\mathbb{P}}, \mathcal{F})$-martingale for all $\tilde{\mathbb{P}} \in \mathcal{M}(S)$.

Remark. Θ_t^i represents the number of shares of the risky asset S_i held by an investor at time t and the wealth process associated with the strategy $\Theta \in \mathcal{A}_{\mathcal{F}}(S)$ with initial capital x is given by

$$V_t = x + \int_0^t \Theta_u \cdot dS_u.$$

In particular, our strategies are self-financing.

Definition We define the financial value of the weak information (Y, ν) as being

$$u(x,\nu) := \inf_{\mathbb{Q} \in \mathcal{E}^\nu} \sup_{\Theta \in \mathcal{A}_{\mathcal{F}}(S)} \mathbb{E}^{\mathbb{Q}}\left(U\left(x + \int_0^T \Theta_u dS_u \right) \right)$$

where $x > 0$ denotes the initial wealth of the investor.

2.3 Weak information and (the absence of) arbitrage opportunities

Since S is a local martingale under $\tilde{\mathbb{P}}$, the market defined in the previous subsections has no arbitrage. What we mean with no arbitrage here is there are no arbitrage opportunities with tame portfolios as discussed in (Leventhal and Skorohod, 1995), Corollary 2. A tame portfolio is defined as follows:

Definition (Tame portfolio) A portfolio Θ is called a tame portfolio if there exists $C > -\infty$ such that $P(X_t > C \ \forall\ 0 \leq t \leq 1) = 1$.

Here (X_t) is the so-called discounted capital gain process (see Definition 2 in (Leventhal and Skorohod, 1995)) associated with the portfolio Θ. Restricting our attention to tame portfolios can be interpreted as putting a limit on borrowing.

Consider the following financial market:

$$\left(\Omega, (\mathcal{H}_t)_{0\leq t\leq T}, (S_t)_{0\leq t\leq T}, \mathbb{Q}\right)$$

where $\mathcal{H}$ is a filtration (right-continuous and $\mathbb{P}$-complete) which contains the natural filtration of S, and $\mathbb{Q}$ a probability measure equivalent to $\mathbb{P}$.

Following [7] we give the following definition concerning arbitrage opportunities.

Definition (No Arbitrage) We say there is no arbitrage on the financial market

$$\left(\Omega, (\mathcal{H}_t)_{0\leq t\leq T}, (S_t)_{0\leq t\leq T}, \mathbb{Q}\right),$$

if there exists a probability measure $\tilde{\mathbb{Q}}$ equivalent to $\mathbb{Q}$ such that $(S_t)_{0\leq t\leq T}$ is an $\mathcal{H}$-adapted local martingale under $\tilde{\mathbb{Q}}$.

Notice that by Corollary 1.2 in [7] this is equivalent to NFLVR. (see [7], [8]) As it turns out there are no arbitrage opportunities in a financial market that has the weak information.

Let $\mathcal{E}^\nu$ be the space of measures $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ such that

$$1.\mathbb{Q} \sim \mathbb{P}$$

$$2.\mathbb{Q}(Y \in dy) = \nu(dy)$$

Observe that the financial market associated with a measure $\mathbb{Q} \in \mathcal{E}^\nu$

$$\left(\Omega, (\mathcal{F}_t)_{0 \le t \le T}, (S_t)_{0 \le t \le T}, \mathbb{Q}\right),$$

is free from arbitrage since there exists a measure $\mathbb{P} \sim \mathbb{Q}$ such that $(S_t)_{0 \le t \le T}$ is an $\mathcal{F}$-adapted local martingale under $\mathbb{P}$ which is equivalent to NFLVR.

Definition.(Space of local martingale measures) The space $\mathcal{M}(S)$ of martingale measures is the set of probability measures $\tilde{\mathbb{P}} \sim \mathbb{P}$ such that $(S_t)_{0 \le t \le T}$ is an $\mathcal{F}$-adapted local martingale under $\tilde{\mathbb{P}}$. Clearly if

$$\mathcal{M}(S) \neq \emptyset \tag{2.3.1}$$

there are no arbitrage opportunities.Now consider the set of cadlag densities of ELMM's

Definition

$$\mathcal{Z} = \left\{ \left(\frac{d\tilde{\mathbb{P}}_t}{d\mathbb{P}_t} \right)_{0 \le t \le T} : \tilde{\mathbb{P}} \in \mathcal{M}(S) \right\}$$

Here $\mathbb{P}$ denotes the so-called historical measure of the price process S not the probability measure $\mathbb{P}$ introduced above.The condition

$$\mathcal{Z} \neq \emptyset \tag{2.3.2}$$

is equivalent to condition (2.3.1). [15] introduced a weaker no arbitrage formulation where the set $\mathcal{Z}$ is enlarged in the following way:

$$\mathcal{Z} := \{Z \geq 0 : Z_0 = 1, (XZ)_{0\leq t\leq T} \textit{ is a supermartingale for every } X \in \mathcal{X}(1)\}.$$

whose elements are supermartingales as $1 \in \mathcal{X}(1)$.

2.4 Review of results on the value of weak information

As it turns out the in a complete market the financial value of weak information is given by the following expression:

Theorem 2.4.1. *Assume that integrals below are convergent. Then for each initial investment $x > 0$,*

$$u(x,\nu) = \sup_{\Theta\in\mathcal{A}_{\mathcal{F}}(S)} \mathbb{E}^{\nu}\left(U\left(x + \int_0^T \Theta_u dS_u\right)\right)$$

$$= \int_{\mathbb{R}^d} U\left(I\left(\frac{\Lambda(x)}{\xi(y)}\right)\right)\nu(dy)$$

where $\Lambda(x)$ is defined by

$$\int_{\mathbb{R}^d} I\left(\frac{\Lambda(x)}{\xi(y)}\right)\mathbb{P}_Y(dy) = x.$$

For a more detailed version of this theorem and details on notation see [4]. By convex duality this theorem is a consequence of proposition 2.2.1. The proof uses classical results on the martingale dual approach in a complete market. (see [13] and [14])For the log and power utility we obtain the following formula's for the value of weak information:

Example(Log utility) Let $U(x) = \ln(x)$. Then

$$u(x,\nu) = \ln(x) + \int_{\mathcal{P}} \frac{d\nu}{d\tilde{\mathbb{P}}_Y}(y) \ln\left(\frac{d\nu}{d\tilde{\mathbb{P}}_Y}\right) \tilde{\mathbb{P}}(Y \in dy).$$

Example(Power utility) Let $\alpha \in (0,1)$ and $U(x) = \frac{x^\alpha}{\alpha}$. Then

$$u(x,\nu) = \frac{x^\alpha}{\alpha} \left[\int_{\mathcal{P}} \left(\frac{d\nu}{d\tilde{\mathbb{P}}_Y}(y)\right)^{\frac{1}{1-\alpha}} \tilde{\mathbb{P}}(Y \in dy) \right]^{1-\alpha}.$$

[4] also has a version of this theorem for incomplete markets (Theorem 20, p. 77). To prove this theorem using classical duality methods we need the additional assumption of asymptotic elasticity of the utility function U:

$$\limsup_{x \to \infty} \frac{xU'(x)}{U(x)} < 1.$$

See [16] for details on this assumption.

Theorem 2.4.2. *For each initial investment $x > 0$,*

$$u(x,\nu) = \inf_{y>0} \left(\left(\inf_{\pi \in \mathcal{D}} \int_{\mathcal{P}} \tilde{U}(y\pi(u))\nu(du) \right) + xy \right)$$

where

$$\mathcal{D} = \{\frac{d\tilde{\mathbb{P}}_Y}{d\nu}, \tilde{\mathbb{P}} \in \mathcal{M}(S)\}.$$

Chapter 3

The financial value of weak information in discrete market models

3.1 Introduction

Suppose an investor is interested in maximizing her or his expected utility from the value of a portfolio at a future time. Assume the investor possesses some information in particular she or he knows the distribution of the prices of stocks in the market at this future time. Because the information concerns the distribution of stock prices it is referred to as weak information. The main question of this chapter is:": *What is the financial value of this information ?*

Notice that we will be looking at expected utility, not expected wealth. This is in line with classical results and allows us to take an individual's attitude towards risk into account. In the previous chapter we discussed some results on utility optimization and the financial value of weak information in a continuous time. The purpose of the current chapter is to describe some of the results recently obtained in a discrete time setting by the author and others in [1]. It should be stressed that the discrete time results cannot be obtained as a consequence of the continuous time results.

In this chapter we assume a complete market with no transaction costs. The incomplete market case is discussed in chapter 5. For an introduction to complete markets we refer the reader to [6]. The main tool for finding the optimal expected utility in the presence of weak information is the martingale method. (see [22]) The problem we discuss is related to robust utility maximization problems. (see [10] and later works by H. Föllmer, A. Gundel and S. Weber)

3.2 Utility Functions

Since we are interested in optimizing expected utility, it is crucial to introduce the notion of a utility function and discuss some assumptions we are making about utility functions. We will denote our utility function by U. A utility function measures an individual's happiness or satisfaction with a certain level of wealth as opposed to using the level of wealth itself as a measure of happiness. This allow us to take an individual's risk aversion into account. Risk aversion of a utility function can be measured in terms of the absolute or relative risk aversion coefficient. (see [17]) We assume that a utility function is strictly concave, strictly increasing and continuously differentiable. In addition we assume that (as in [4])

$$\lim_{x\to 0} U'(x) = +\infty \text{ and } \lim_{x\to\infty} U'(x) = 0. \tag{3.2.1}$$

These conditions guarantee the utility function U exhibits risk aversion, satisfies the law of diminishing marginal utility and an increase in wealth causing an increase in utility. We are interested in three specific types of utility functions:

1. Log Utility: $U(x) = \ln(x), \quad x > 0$

   The relative risk aversion coefficient of this utility function is equal to 1. Under certain conditions this might be interpreted as the proportion of risky assets held in a portfolion being independent of the individuals wealth x.

2. Power Utility: $U(x) = \frac{x^\gamma}{\gamma}, \quad$ for $-\infty < \gamma < 0$ and $0 < \gamma < 1$ and $x > 0$

   Here the relative risk aversion coefficient is also constant but equal to $1 - \gamma$. In particular, γ being between 0 and 1 corresponds to a less risk averse individual compared to an individual having a logarithmic utility function whereas the parameter $\gamma < 0$ corresponds to an individual being more risk averse than an individual having a log utility function. If γ goes to $-\infty$ the individual becomes more and more risk-averse.

3. Exponential Utility: $U(x) = -e^{-\alpha x}$, for $\alpha > 0$ and $x \in \mathbb{R}$

   The absolute risk aversion coefficient for the utility function given above is equal to α. This corresponds to an individual taking a constant amount of risk (independent of their wealth) as opposed to a constant proportion of risk. The exponential utility function does not satisfy the condition (3.2.1) but is mentioned here because the results described in this section also hold true for this utility function.

3.3 Modelling the Financial Value of Weak Information

3.3.1 Setup

Assume we are in a market with d securities. Without loss of generality one of these securities is assumed to be a risk-free asset with (fixed) rate of return r. Let $\Omega_1 = \{\omega_1, .., \omega_M\}$ be the sample space of possible outcomes of security prices after one time period. Assume we have a finite number of time periods with N denoting the final time period. The vector of security prices at time $n \in \{0, 1, .., N\}$ is denoted by $\vec{S}_n \in \mathbb{R}^d$. Consider a probability measure $\mathbb{P}$. Without loss of generality we may assume $\mathbb{P}(\omega_j) > 0, \forall j \in \{1, .., M\}$, since if $\mathbb{P}(\omega_j) = 0$ we simply exclude ω_j from Ω_1. We will assume our market is free from arbitrage and complete. To make this more specific we need to introduce a set of equivalent martingale measures $\mathcal{M}$. The elements of this set are denoted by $\tilde{\mathbb{P}}$. In the current discrete context 'equivalent' means $\tilde{\mathbb{P}}(\omega_j) > 0$ for all ω_j. A probability measure is a martingale measure if and only if discounted stock prices are martingales under $\tilde{\mathbb{P}}$. As it turns out the no arbitrage assumption is equivalent to assuming $\mathcal{M}$ is nonempty. Additionally, in a complete market (with no arbitrage) the set $\mathcal{M}$ is a singleton, $\mathcal{M} = \{\tilde{\mathbb{P}}\}$. More details about no arbitrage, completeness and equivalent martingale methods may be found in [22]. The probability $\tilde{\mathbb{P}}$ may be interpreted as a representation of the knowledge of an uninformed investor. Using Jensen's inequality it may be seen that this is same as having no information at all because it is optimal to invest everything in the risk-free security. The value of an investment portfolio consisting of different quantities of the d securities in the market at time n is denoted by V_n. The initial value of portfolio or initial wealth of the investor V_0 is denoted by v. When comparing investment strategies we consider the set of all self-financing portfolios Ψ^v.

3.3.2 Weak Anticipation

Let Ω denote the path space of the (M-dimensional) security price process $\{\vec{S_n}\}_{1\leq n\leq N}$ and let $\mathcal{A}$ denote the set of possible values of $\vec{S_N}$. Note that since the sample space at every time step Ω_1 is finite $\mathcal{A}$ is a finite set as well. Now suppose an investor has some weak information (weak anticipation) about the security prices at the final time N. In particular she or he knows the distribution of $\vec{S_N}$, denoted by ν. In the setting of the previous chapter we choose $Y = \vec{S_N}$. Similarly to the continuous case, given the probability measure ν we define a probability measure $\mathbb{P}^\nu$ that is minimal in some sense.

Definition : The probability measure $\mathbb{P}^\nu$ defined by

$$\mathbb{P}^\nu(\omega) := \sum_{\vec{x}\in\mathcal{A}} \tilde{\mathbb{P}}(\omega|\vec{S_N} = \vec{x})\nu(\vec{S_N} = \vec{x})$$

is called the minimal probability measure associated with the weak information ν, where $\tilde{\mathbb{P}} \in \mathcal{M}$ is an (remember $\mathcal{M}$ is a singleton in a complete market) equivalent martingale measure.

As it turns out $\mathbb{P}^\nu$ is minimal in the set of probability measures $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that $\mathbb{Q}(\vec{S_N} = \vec{x}) = \nu(\vec{S_N} = \vec{x})$ for all $\vec{x} \in \mathcal{A}$. We denote this set by $\mathcal{E}^\nu$.

Proposition 3.3.1. *Let ϕ be a convex function. Then*

$$\min_{\mathbb{Q}\in\mathcal{E}^{\nu}} \tilde{\mathbb{E}}\left[\phi\left(\frac{d\mathbb{Q}}{d\tilde{\mathbb{P}}}\right)\right] = \tilde{\mathbb{E}}\left[\phi\left(\frac{d\mathbb{P}^{\nu}}{d\tilde{\mathbb{P}}}\right)\right],$$

where $\frac{d\mathbb{Q}}{d\tilde{\mathbb{P}}}$ denotes the Radon-Nikodym derivative of $\mathbb{Q}$ with respect to $\tilde{\mathbb{P}}$.

Proof. Let $\vec{x} \in \mathcal{A}$ and $\mathbb{Q} \in \mathcal{E}^{\nu}$ be given. Then,

$$\tilde{\mathbb{E}}\left[\frac{d\mathbb{Q}}{d\tilde{\mathbb{P}}}|\vec{S}_N = \vec{x}\right] = \frac{\nu(\vec{S}_N = \vec{x})}{\tilde{\mathbb{P}}(\vec{S}_N = \vec{x})}.$$

Let ϕ be a convex function. Then, from conditional Jensen's inequality,

$$\phi\left(\frac{\nu(\vec{S}_N = \vec{x})}{\tilde{\mathbb{P}}(\vec{S}_N = \vec{x})}\right) = \phi\left(\tilde{\mathbb{E}}\left[\frac{d\mathbb{Q}}{d\tilde{\mathbb{P}}}|\vec{S}_N = \vec{x}\right]\right) \leq \tilde{\mathbb{E}}\left[\phi\left(\frac{d\mathbb{Q}}{d\tilde{\mathbb{P}}}\right)|\vec{S}_N = \vec{x}\right].$$

Taking the expected value on both sides, we get

$$\tilde{\mathbb{E}}\left[\phi\left(\frac{\nu(S_N)}{\tilde{\mathbb{P}}(S_N)}\right)\right] = \tilde{\mathbb{E}}\left[\phi\left(\frac{d\mathbb{P}^{\nu}}{d\tilde{\mathbb{P}}}\right)\right] \leq \tilde{\mathbb{E}}\left[\phi\left(\frac{d\mathbb{Q}}{d\tilde{\mathbb{P}}}\right)\right]$$

and the result is proved. ■

3.3.3 The Financial Value of Weak Information

Notice that an investor who knows the distribution of stock prices at the final time is given by ν will follow a different strategy than the uninformed investor and consequently their expected utility from the terminal value of the portfolio will be different. Therefore it is natural to define the financial value of the weak information ν as the lowest expected utility that can be gained from possessing the weak information

ν.

Definition : The **financial value of the weak information** ν to be gained from investing v at time zero is given by

$$u(v,\nu) = \min_{\mathbb{Q}\in\mathcal{E}^\nu} \max_{\psi\in\Psi^v} \mathbb{E}^{\mathbb{Q}}[U(V_N)].$$

Denote $I(x) = (U')^{-1}(x)$. The following theorem characterizes the value of weak information and the corresponding optimal investment strategy in the current context. It is the main theorem in [1].

Theorem 3.3.2. *The financial value of weak information in a complete market is*

$$u(v,\nu) = \max_{\psi\in\Psi^v} \mathbb{E}^\nu[U(V_N)] = \mathbb{E}^\nu\left[U\left(I\left(\frac{\lambda(v)}{(1+r)^N}\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^\nu}\right)\right)\right],$$

where $\lambda(v)$ is uniquely determined by

$$\tilde{\mathbb{E}}\left[\frac{1}{(1+r)^N} I\left(\frac{\lambda(v)}{(1+r)^N}\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^\nu}\right)\right] = v,$$

where $\tilde{\mathbb{P}} \in \mathcal{M}$ is the unique probability measure under which the prices are martingales. Moreover, the optimal wealth at time n $\hat{V}_n$ is given by

$$\hat{V}_n = \frac{1}{(1+r)^{N-n}} \sum_{\omega\in\Omega} I\left(\frac{\lambda(v)}{(1+r)^N}\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^\nu}(\omega)\right)\tilde{\mathbb{P}}(\omega|\vec{S}_n),\ \textit{for } n \in \{0,1,...,N\}.$$

At time n, the optimal amount to purchase of the i^{th} linearly independent asset is

$$\delta_n^i = \sum_{j=1}^{M} (D_{n+1}^{-1})_{i,j} \hat{V}_{n+1}(\omega_j), \ \textit{for } n \in \{0, 1, ..., N-1\},$$

where

$$D_{n+1} = \begin{bmatrix} S_{n+1}^1(\omega_1) & S_{n+1}^2(\omega_1) & . \ . \ . & S_{n+1}^M(\omega_1) \\ S_{n+1}^1(\omega_2) & S_{n+1}^2(\omega_2) & . \ . \ . & S_{n+1}^M(\omega_2) \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ S_{n+1}^1(\omega_M) & S_{n+1}^2(\omega_M) & . \ . \ . & S_{n+1}^M(\omega_M) \end{bmatrix},$$

is the matrix of M linearly independent asset prices at time $n+1$, $(D_{n+1}^{-1})_{i,j}$ represents the element (i, j) of the matrix D_{n+1}^{-1}, and $\hat{V}_{n+1}$ comes from the above equation.

Proof. We will proceed by rewriting $\max\limits_{\psi \in \Psi^v} \mathbb{E}^{\mathbb{Q}}[U(V_N)]$. In order to do this, we need the convex conjugate $\tilde{U}(y) := \max\limits_{x>0}[U(x) - xy]$.(see [12]) We form the Lagrangian for solving $\max\limits_{\psi \in \Psi^v} \mathbb{E}^{\mathbb{Q}}[U(V_N)]$ by

$$\mathcal{L}(\lambda) = \mathbb{E}^{\mathbb{Q}}[U(V_N)] + \lambda \left[v - \mathbb{E}^{\mathbb{Q}} \left[\frac{d\tilde{\mathbb{P}}}{d\mathbb{Q}} \frac{V_N}{(1+r)^N} \right] \right].$$

Now using $\tilde{U}$, substituting in for V_N from the martingale method (see appendix), and doing algebra, we can rewrite our Lagrangian as

$$\mathcal{L}(\lambda) = \lambda v + \tilde{\mathbb{E}} \left[\frac{d\mathbb{Q}}{d\tilde{\mathbb{P}}} \tilde{U} \left(\frac{\lambda}{(1+r)^N} \frac{d\tilde{\mathbb{P}}}{d\mathbb{Q}} \right) \right].$$

Thus, we deduce

$$u(v,\nu) = \min_{\mathbb{Q}\in\mathcal{E}^\nu} \min_{\lambda>0} \left[\lambda v + \tilde{\mathbb{E}} \left[\frac{d\mathbb{Q}}{d\tilde{\mathbb{P}}} \tilde{U} \left(\frac{\lambda}{(1+r)^N} \frac{d\tilde{\mathbb{P}}}{d\mathbb{Q}} \right) \right] \right]$$

$$= \min_{\lambda>0} \left[\lambda v + \min_{\mathbb{Q}\in\mathcal{E}^\nu} \tilde{\mathbb{E}} \left[\frac{d\mathbb{Q}}{d\tilde{\mathbb{P}}} \tilde{U} \left(\frac{\lambda}{(1+r)^N} \frac{d\tilde{\mathbb{P}}}{d\mathbb{Q}} \right) \right] \right].$$

Since the convexity of $\tilde{U}$ implies the function mapping $z \mapsto z\tilde{U}\left(\frac{\lambda}{(1+r)^N z}\right)$ is convex, we can use Proposition 3.3.1 to get

$$u(v,\nu) = \min_{\lambda>0} \left[\lambda v + \tilde{\mathbb{E}} \left[\frac{d\mathbb{P}^\nu}{d\tilde{\mathbb{P}}} \tilde{U} \left(\frac{\lambda}{(1+r)^N} \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^\nu} \right) \right] \right].$$

Taking the derivative now with respect to λ and setting it equal to 0, we find

$$v = \tilde{\mathbb{E}} \left[\frac{1}{(1+r)^N} I \left(\frac{\lambda^*(v)}{(1+r)^N} \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^\nu} \right) \right]$$

where $\lambda^*(v)$ is the minimizer. Now,

$$u(v,\nu) = \lambda^*(v)v + \tilde{\mathbb{E}} \left[\frac{d\mathbb{P}^\nu}{d\tilde{\mathbb{P}}} \tilde{U} \left(\frac{\lambda^*(v)}{(1+r)^N} \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^\nu} \right) \right] = \mathbb{E}^\nu \left[U \left(I \left(\frac{\lambda^*(v)}{(1+r)^N} \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^\nu} \right) \right) \right].$$

Thus, we have shown the first part of the theorem. Now note that discounted optimal wealth process $\{\frac{\hat{V}_n}{(1+r)^n}\}_{0\leq n\leq N}$ is a martingale under $\tilde{\mathbb{P}}$. (see appendix) As a result,

$$\hat{V}_n = \frac{1}{(1+r)^{N-n}} \tilde{\mathbb{E}}[\hat{V}_N | \vec{S}_n] = \frac{1}{(1+r)^{N-n}} \sum_{\omega\in\Omega} I \left(\frac{\lambda(v)}{(1+r)^N} \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^\nu}(\omega) \right) \tilde{\mathbb{P}}(\omega|\vec{S}_n)$$

for all $n \in \{0, 1, ..., N\}$. Further, note that wealth is determined by your portfolio from the previous time period and the current prices. Thus,

$$\hat{V}_{n+1} = D_{n+1}\vec{\delta}_n,$$

so we have

$$\mathrm{D}_{n+1}^{-1}\hat{V}_{n+1} = \vec{\delta}_n.$$

■

Remark. We know from [6] that the matrix of all asset prices in the complete market has rank M. Therefore, we can choose M linearly independent assets to invest in. Further, note that the optimal amount to purchase for each asset is only unique when $M = d$.

Definition : We define the **additional value of weak information** as the extra utility gained from investing with anticipation instead of just putting all of your wealth in the risk-free asset, which we define by

$$F(v, \nu) = u(v, \nu) - U(v(1 + r)^N).$$

Definition : We also define the **ratio of added value to the total value** by

$$\pi(v, \nu) = \frac{F(v, \nu)}{u(v, \nu)} = 1 - \frac{U(v(1 + r)^N)}{u(v, \nu)}$$

Appplying Theorem 3.3.2 to a logarithmic utility function results in an expression for the additional value of weak information that is interesting in its own right.

Corollary 3.3.3. *The additional value of weak information for the log utility function is given by the relative entropy of ν with respect to $\tilde{\mathbb{P}}_{\vec{S}_N}$:*

$$F(v,\nu) = \mathbb{E}^{\nu}\left[\ln\left(\frac{d\nu}{d\tilde{\mathbb{P}}_{\vec{S}_N}}\right)\right].$$

Proof. We first solve for λ.

$$v = \tilde{\mathbb{E}}\left[\frac{1}{(1+r)^N}\cdot I\left(\frac{\lambda}{(1+r)^N}\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^{\nu}}\right)\right]$$
$$v = \tilde{\mathbb{E}}\left[\frac{1}{(1+r)^N}\cdot\frac{(1+r)^N}{\lambda}\cdot\frac{d\mathbb{P}^{\nu}}{d\tilde{\mathbb{P}}}\right]$$
$$\lambda = \frac{1}{v}.$$

Substituting for λ into our value of weak information equation we thus have,

$$\begin{aligned}
u(v,\nu) &= \mathbb{E}^{\nu}\left[U\left(I\left(\frac{\lambda}{(1+r)^N}\cdot\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^{\nu}}\right)\right)\right] \\
&= \mathbb{E}^{\nu}\left[\ln\left(\frac{(1+r)^N}{\frac{1}{v}}\cdot\frac{d\mathbb{P}^{\nu}}{d\tilde{\mathbb{P}}}\right)\right] \\
&= \ln\left(v(1+r)^N\right) + \mathbb{E}^{\nu}\left[\ln\left(\frac{d\mathbb{P}^{\nu}}{d\tilde{\mathbb{P}}}\right)\right]
\end{aligned}$$

This implies the additional value of weak information for log utility is

$$F(v,\nu) = \mathbb{E}^{\nu}\left[\ln\left(\frac{d\mathbb{P}^{\nu}}{d\tilde{\mathbb{P}}}\right)\right] = \mathbb{E}^{\nu}\left[\ln\left(\frac{d\nu}{d\tilde{\mathbb{P}}_{\vec{S}_N}}\right)\right]$$

where the last equality follows from the definition of $\mathbb{P}^\nu$. ■

For the power utility function we obtain the following expression:

Corollary 3.3.4. *The value of weak information for the power utility function is given by*

$$u(v,\nu) = \frac{v^\gamma (1+r)^{N\gamma}}{\gamma \left(\tilde{\mathbb{E}}\left[\left(\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^\nu} \right)^{\frac{1}{\gamma-1}} \right] \right)^{\gamma-1}} \cdot \mathbb{E}^\nu \left[\left(\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^\nu} \right)^{\frac{\gamma}{\gamma-1}} \right].$$

Proof. We now will solve for the value of λ.

$$\tilde{\mathbb{E}}\left[\frac{1}{(1+r)^N} \cdot \left(\frac{\lambda}{(1+r)^N} \cdot \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^\nu} \right)^{\frac{1}{\gamma-1}} \right] = v$$

$$\lambda = \left(\frac{v(1+r)^{\frac{N\gamma}{\gamma-1}}}{\tilde{\mathbb{E}}\left[\left(\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^\nu} \right)^{\frac{1}{\gamma-1}} \right]} \right)^{\gamma-1}.$$

Substituting in for λ we get,

$$\begin{aligned} u(v,\nu) &= \mathbb{E}^\nu \left[U \left(I \left(\frac{\lambda}{(1+r)^N} \cdot \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^\nu} \right) \right) \right] \\ &= \mathbb{E}^\nu \left[\frac{1}{\gamma} \left(\left(\frac{v(1+r)^{\frac{N\gamma}{\gamma-1}}}{\tilde{\mathbb{E}}\left[\left(\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^\nu} \right)^{\frac{1}{\gamma-1}} \right]} \right)^{\gamma-1} \cdot \frac{1}{(1+r)^N} \cdot \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^\nu} \right)^{\frac{\gamma}{\gamma-1}} \right] \end{aligned}$$

$$= \frac{v^\gamma (1+r)^{N\gamma}}{\gamma \left(\tilde{\mathbb{E}} \left[\left(\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^\nu} \right)^{\frac{1}{\gamma-1}} \right] \right)^{\gamma-1}} \cdot \mathbb{E}^\nu \left[\left(\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^\nu} \right)^{\frac{\gamma}{\gamma-1}} \right].$$

■

3.4 Complete Markets: The Binomial Model

3.4.1 Single-Period Binomial Model

In this section we look at the most basic example of a complete market in discrete time: a one-period binomial model. There are two assets in the market, a risk-free asset with payoff $1+r$ and a risky asset with two possible payoffs, $S_0(1+h)$ and $S_0(1-k)$. (stock goes up by $h\%$ or goes down by $k\%$) We assume $S_0 > 0$ and $k < 1$. This two-asset market is free from arbitrage if and only if $h > r > -k$. We assume the market is indeed free from arbitrage. Our portfolio is completely characterized by how many units of the risky asset we own at time n, which is denoted by δ_n, because the portion of our portfolio that is not invested in the risky asset has to be invested in the (unique) risk-free asset.

Figure 3.4.1 graphs the optimal portfolio at time zero δ_0 given the investor knows there is a 50% chance of the stock going up and a 50% chance of the stock going down and given the investor's preferences can be described by the logarithmic utility function. Of course the value of δ_0 depends on the choices of parameter values r, h, k, v and S_0 (also denoted by s).

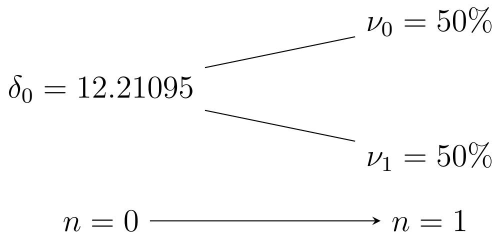


FIGURE 3.4.1 An example of a single-period binomial model using the log utility where where the parameter values are $r = .032, h = .09, k = .019, v = 200.0$, and $s = 20.0$

For the one-period binomial model a simple expression exists for any choice of the parameters r, h, k, v, s and for any weak information vector $\nu = \begin{pmatrix} \nu_0 \\ \nu_1 \end{pmatrix}$ [1] gives the following expressions for the optimal δ_0 assuming a log, power or exponential utility function.

Example 1 Log Utility

When looking at the specific utility functions, in the case of log, we begin by maximizing $\mathbb{E}[U(V_N)]$ with respect to δ. We then are able to obtain our equation for the optimal number of shares with respect to wealth, $\hat{\delta}$, in a one period model.

$$\hat{\delta}_0 = \frac{v(1+r)(\nu_0(h-r) + \nu_1(-k-r))}{-s(h-r)(-k-r)}.$$

Example 2 Power Utility

As in log utility we would solve for our optimal number of shares with respect to wealth, $\hat{\delta}_0$, in a one period model.

$$\hat{\delta}_0 = \frac{((\nu_0(h-r))^{\frac{1}{\gamma-1}} - (\nu_1(-k-r))^{\frac{1}{\gamma-1}})(1+r)v}{(\nu_1(-k-r))^{\frac{1}{\gamma-1}} s(-k-r) - (\nu_0(h-r))^{\frac{1}{\gamma-1}} s(h-r)}.$$

Example 3 Exponential Utility

Similarly to the previously examined utilities we will solve for the optimal number of shares with respect to wealth, $\hat{\delta}$, in a one period model for the exponential utility.

$$\hat{\delta}_0 = \frac{\ln(\nu_0(h-r)) - \ln(-\nu_1(-k-r))}{s(h+k)}.$$

3.4.2 N-Period Binomial Model

Just as for one-period models, for N-period models all the calculations can be done explicitly. The proposition below gives an explicit formula for the transition probabilities of the minimal probability $\mathbb{P}^\nu$. It can be derived using basic probability. The proposition shows $\{S_n\}_{1 \le n \le N}$ is a Markov chain under the measure $\mathbb{P}^\nu$.

Proposition 3.4.1. *Let $l \in \{1, ..., N-1\}$ and $i \in \{0, ..., N-l\}$. Then*

$$\mathbb{P}^\nu(S_{N-l+1} = (1+h)S_{N-l} | S_{N-l} = (1+h^{N-l-i})(1-k)^i S_0)$$

$$= \frac{\sum_{j=0}^{l-1} \binom{l-1}{j} (N-i-j) \dots (N-i-(l-1))(i+1)(i+2) \dots (i+j)\nu_{i+j}}{\sum_{j=0}^{l} \binom{l}{j} (N-i-j) \dots (N-i-(l-1))(i+1)(i+2) \dots (i+j)\nu_{i+j}}$$

and

$$\mathbb{P}^\nu(S_{N-l+1} = (1-k)S_{N-l} | S_{N-l} = (1+h^{N-l-i})(1-k)^i S_0)$$

$$= \frac{\sum_{j=0}^{l-1} \binom{l-1}{j} (N-i-j-1)\dots(N-i-(l-1))(i+1)\dots(i+j+1)\nu_{i+j+1}}{\sum_{j=0}^{l} \binom{l}{j} (N-i-j)\dots(N-i-(l-1))(i+1)(i+2)\dots(i+j)\nu_{i+j}}.$$

FIGURE 3.4.2 $\mathbb{P}^\nu$ for a 3-period binomial model

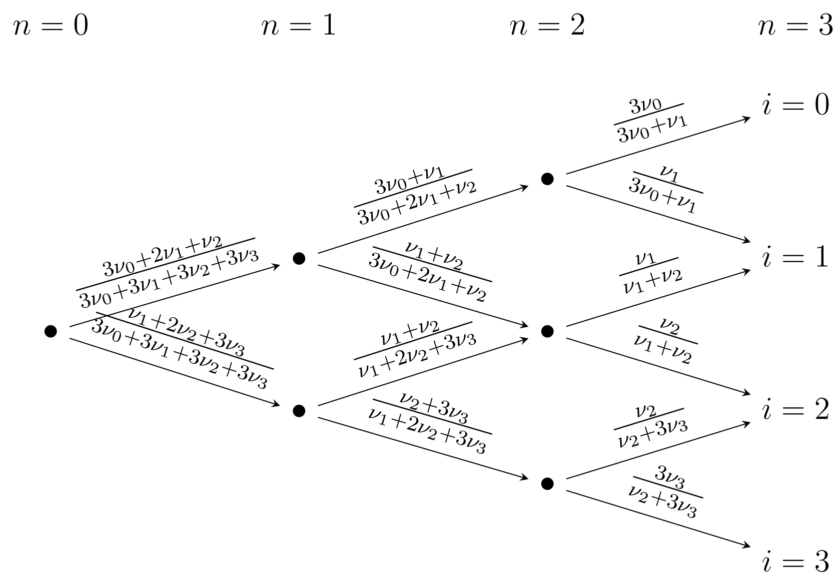
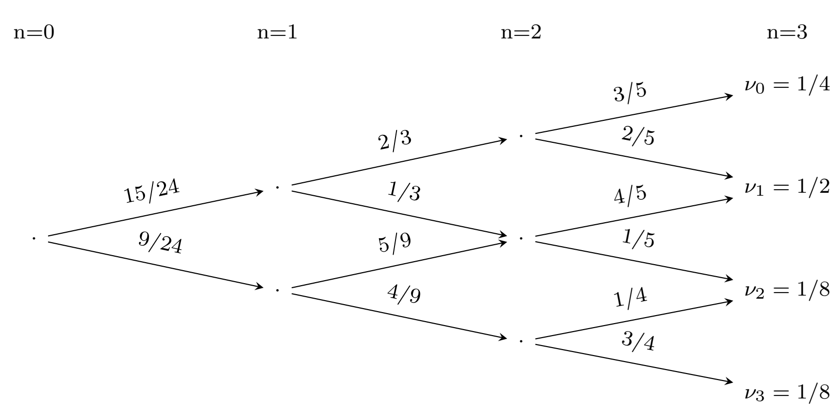


FIGURE 3.4.3 $\mathbb{P}^\nu$ for a 3-period binomial model for a specific choice of ν



Example 1 (Log Utility)

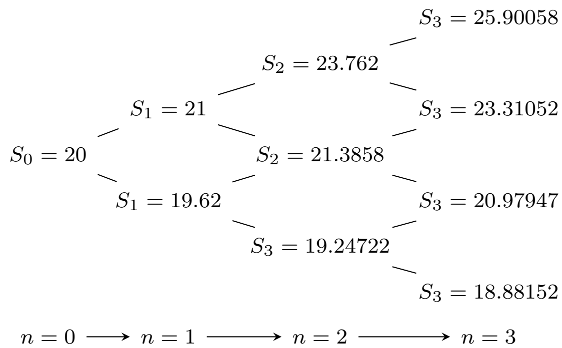


FIGURE 3.4.4 A 3-period binomial tree showing the values of S_n where the parameters are $r = .032, h = .09, k = .019, v = 200.0$, and $s = 20.0$

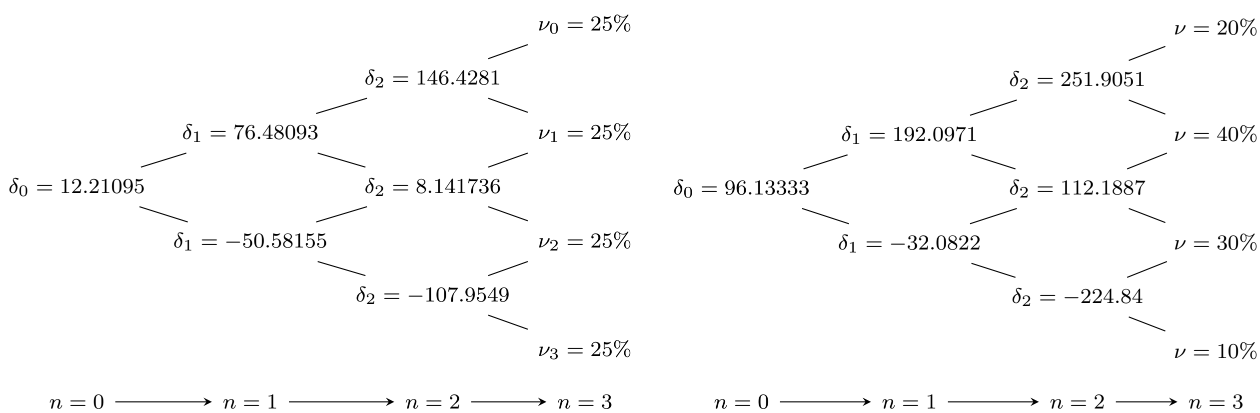


FIGURE 3.4.5 3-period binomial trees showing the values of δ for various anticipations of ν using the log utility where the parameters are $r = .032$, $h = .09$, $k = .019$, $v = 200.0$, and $s = 20.0$

From figure 3.4.5 it can be seen how the optimal investment strategy changes depending on different values of the weak information ν in a 3-period model. Because in the second tree the chances of the stock going to the upper nodes are larger than the chances of going to the lower outcome nodes the optimal strategy has larger $\delta-$ values, which means more investment in the risky asset. Negative values of δ correspond to short-selling the asset. Recall from Corollary 3.3.3 the additional value of weak information for log utility is

$$F(v,\nu) = \mathbb{E}^{\nu}\left[\ln\left(\frac{d\mathbb{P}^{\nu}}{d\tilde{\mathbb{P}}}\right)\right],$$

and the proportion is

$$\pi(v,\nu) = \frac{\mathbb{E}^{\nu}\left[\ln\left(\frac{d\mathbb{P}^{\nu}}{d\tilde{\mathbb{P}}}\right)\right]}{\ln\left(v(1+r)^N\right) + \mathbb{E}^{\nu}\left[\ln\left(\frac{d\mathbb{P}^{\nu}}{d\tilde{\mathbb{P}}}\right)\right]}.$$

Note that $F(v,\nu)$ is only a function of ν, so for any fixed ν, we have that $F(v,\nu)$ is constant. Furthermore, $\pi(v,\nu)$ is a decreasing function of v for any fixed ν. This implies that as the initial wealth of the investor v increases a smaller proportion of utility is gained from possessing weak information. We include some graphs of a 5-period binomial model for the quantities we just discussed assuming the following distributions for the stock prices at the final time:

- Precise: $\{0.01, 0.01, 0.01, 0.95, 0.01, 0.01\}$
- Uniform Distribution: $\{1/6, 1/6, 1/6, 1/6, 1/6, 1/6\}$
- Conservative: $\{0.1, 0.2, 0.2, 0.2, 0.2, 0.1\}$

- Risk-Neutral: $\nu = \tilde{\mathbb{P}}$

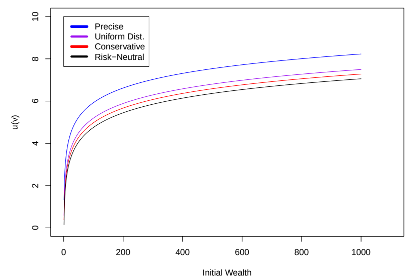


FIGURE 3.4.6 Value of Weak Info. given $r = 3\%$, $h = 8\%$, $k = 4\%$

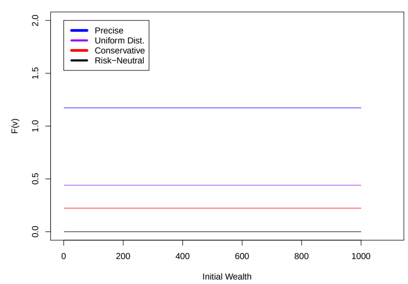


FIGURE 3.4.7 Additional Value of Weak Info. given $r = 3\%$, $h = 8\%$, $k = 4\%$

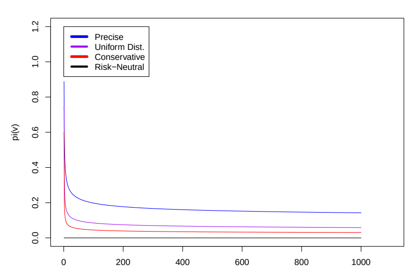


FIGURE 3.4.8 Proportion of Value Added $r = 3\%$, $h = 8\%$, $k = 4\%$

Example 2 (Power Utility)

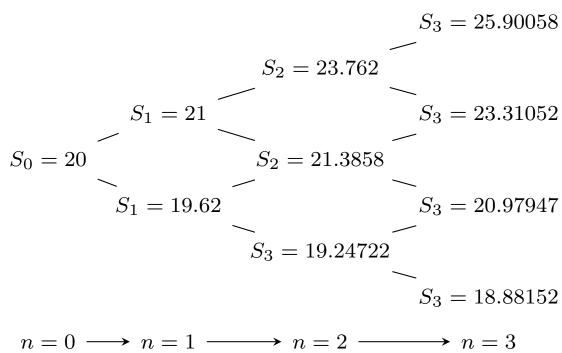
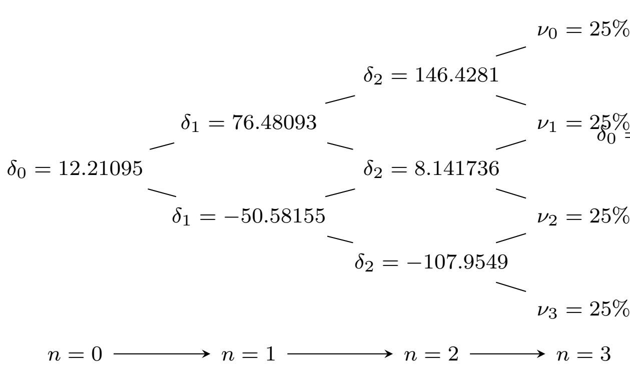


FIGURE 3.4.9 A 3-period binomial tree showing the values of s where the parameters are $r = .032, h = .09, k = .019, v = 200.0$, and $s = 20.0$



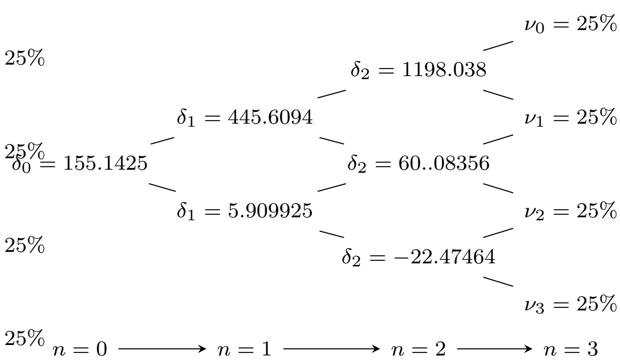


FIGURE 3.4.10 Log Utility

FIGURE 3.4.11 Power Utility

Two different 3-period binomial trees showing the values of δ for equal anticipations of ν using different utility where the constants are the same as Figure 3.4.5. In the power utility model the value of $\gamma = .5$

Figures 3.4.10 and 3.4.11 are included to show the differences between the log and power utility functions. Notice that for $\gamma = 0.5$ (as depicted) the log utility is relatively more risk averse compared to the power utility function. That explains the smaller values of the δ's (in absolute value) for the log utility function.

From Corollary 3.3.4 we have that the additional value for power utility is

$$F(v,\nu) = \frac{v^{\gamma}(1+r)^{N\gamma}}{\gamma\left(\tilde{\mathbb{E}}\left[\left(\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^{\nu}}\right)^{\frac{1}{\gamma-1}}\right]\right)^{\gamma-1}} \cdot \mathbb{E}^{\nu}\left[\left(\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^{\nu}}\right)^{\frac{\gamma}{\gamma-1}}\right] - \frac{v^{\gamma}(1+r)^{N\gamma}}{\gamma},$$

and the proportion is

$$\pi(v,\nu) = 1 - \frac{1}{\mathbb{E}^{\nu}\left[\left(\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^{\nu}}\right)^{\frac{\gamma}{\gamma-1}}\right] \cdot \left(\tilde{\mathbb{E}}\left[\left(\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^{\nu}}\right)^{\frac{1}{\gamma-1}}\right]\right)^{1-\gamma}}.$$

From these expressions it is clear that if ν remains constant the proportion of utility obtained from weak information $\pi(v,\nu)$ is independent of v and the extra utility gained from weak information $F(v,\nu)$ is an increasing function of the initial wealth v.

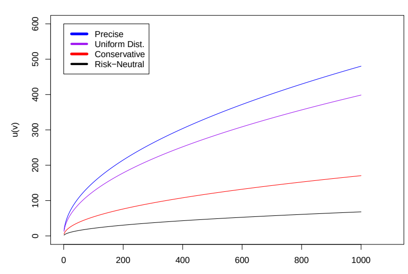


Figure 3.4.12 Value of Weak Info. given $r = 3\%$, $h = 8\%$, $k = 4\%$

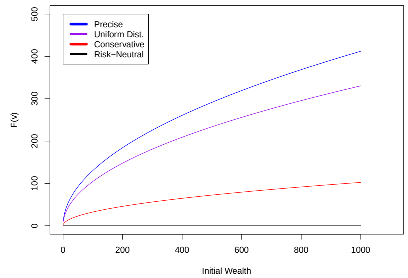


FIGURE 3.4.13 Additional Value of Weak Info. given $r = 3\%$, $h = 8\%$, $k = 4\%$

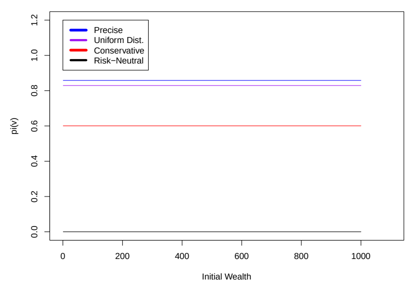


FIGURE 3.4.14 Proportion of Value Added $r = 3\%$, $h = 8\%$, $k = 4\%$

Example 3 (Exponential Utility)

We can also find the financial value of weak information for exponential utility.

$$\mathbb{E}^{\nu}\left[-e^{-a\hat{V}_N}\right] = e^{-v\alpha(1+r)^N - \sum_{i=0}^{N}\binom{N}{i}\tilde{p}^{N-i}\tilde{q}^{i}\ln\left(\binom{N}{i}\frac{\tilde{p}^{n-i}\tilde{q}^{i}}{\nu_i}\right)}.$$

We begin by solving for λ.

$$\tilde{\mathbb{E}}\left[\frac{1}{(1+r)^N}\cdot I\left(\frac{\lambda}{(1+r)^N}\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^\nu}\right)\right]=v$$

We use this equation and then plug in for I.

$$\tilde{\mathbb{E}}\left[\frac{1}{(1+r)^N}\cdot\frac{-1}{\alpha}\cdot\ln\left(\frac{\lambda}{\alpha(1+r)^N}\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^\nu}\right)\right]=v$$

We then solve for λ to be:

$$\lambda=\alpha(1+r)^N e^{-v\alpha(1+r)^N-\mathbb{E}^\nu\left[\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^\nu}\ln\left(\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^\nu}\right)\right]}.$$

Finally we can use our I and our λ to plug in to our equation for the financial value of weak information to solve for the value as it specifically relates to exponential utility.

$$\begin{aligned}
u(v,\nu) &= \mathbb{E}^\nu\left[U\left(I\left(\frac{\lambda}{(1+r)^N}\cdot\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^\nu}\right)\right)\right]\\
&= \mathbb{E}^\nu\left[-e^{-a\frac{-1}{\alpha}\ln\left(\frac{\lambda}{\alpha(1+r)^N}\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^\nu}\right)}\right]\\
&= e^{-v\alpha(1+r)^N-\sum_{i=0}^{N}\binom{N}{i}\tilde{p}^{N-i}\tilde{q}^i\ln\left(\binom{N}{i}\frac{\tilde{p}^{n-i}\tilde{q}^i}{\nu_i}\right)}.
\end{aligned}$$

3.5 Appendix

Recall from section 3.3 Ψ^v denotes the set of self-financing portfolios given initial wealth v.

Theorem 3.5.1. *The discounted wealth process is a martingale under the martingale measure* $\mathbb{Q}$.

Proof. See [20]. ■

Theorem 3.5.2. *Maximizing* $\mathbb{E}[U(V_N)]$ *over the set of self-financing portfolios* Ψ^v *is equivalent to maximizing* $\mathbb{E}[U(V_N)]$ *subject to* $\tilde{\mathbb{E}}[U(V_N)] = v$, *with* $\tilde{\mathbb{P}}$ *being the unique equivalent martingale measure.*

Proof. See [19, Lemma 4.9]. ■

Theorem 3.5.3.

$$\hat{V}_N = I\left(\frac{\lambda}{(1+r)^N}\frac{d\tilde{\mathbb{P}}}{d\mathbb{Q}}\right)$$

More specifically, optimal terminal wealth $\hat{V}_N$ *is attained when* λ *satisfies*

$$v = \tilde{\mathbb{E}}\left[\frac{1}{(1+r)^N} I\left(\frac{\lambda}{(1+r)^N}\frac{d\tilde{\mathbb{P}}}{d\mathbb{Q}}\right)\right].$$

Proof. See [20] p16. ■

Chapter 4

Review of results on optimal investment in incomplete markets with a random endowment

In this chapter we review the results obtained in [11] and [18].

4.1 The optimization problem with random endowment

Consider a utility function $U : (0, \infty) \to \mathbb{R}$. Assume U is strictly concave, strictly increasing and continuously differentiable and satisfies the so-called Inada conditions:

$$U'(0) := \lim_{x \to 0} U'(x) = \infty, U'(\infty) := \lim_{x \to \infty} U'(x) = 0. \tag{4.1.1}$$

Following the notation of [11] we define a self-financing portfolio as a pair (x, H), where $x \in \mathbb{R}$ is the initial wealth and H represents a predictable S-integrable process that gives the number of shares of each stock held in the portfolio. Consequently, the value process of the portfolio (denoted by X) is the stochastic integral of H with respect to S:

$$X_t = x + (H \cdot S)_t = x + \int_0^t H_u dS_u,\ 0 \le t \le T. \tag{4.1.2}$$

Given a non-negative initial wealth $x \ge 0$, let $\mathcal{X}(x)$ be the set of non-negative value processes whose initial value is equal to x

$$\mathcal{X}(x) = \{X \ge 0 : X \text{ satisfies 4.1.2 and } X_0 = x\}.$$

In this chapter, in addition to stock price movements the terminal wealth of our investor also depends on random endowments. Assume that at time 0 in addition to her or his initial capital x the investor has quantities $q = (q_i)_{1\le i\le N}$ of nontraded European contingent claims with maturity T and $\mathcal{F}_T$-measurable payment functions $f = (f_i)_{1\le i\le N}$ in her or his portfolio. The payoff of this portfolio of contingent claims is denoted by

$$\langle q, f\rangle = \sum_{i=1}^{N} q_i f_i.$$

Following [11] we define a set of acceptable processes with initial capital x and quantities q of the nontraded European contingent claims such that the terminal value of the processes dominates the random payoff from the random endowments $-\langle q, f\rangle$:

$$\mathcal{X}(x, q) = \{X : X \text{ is acceptable}, X_0 = x \text{ and } X_T + \langle q, f\rangle \ge 0\}$$

See [11] for more details on acceptable processes.The reader should notice the set $\mathcal{X}(x, q)$ could be empty for certain values of (x, q). This problem is addressed by restricting our attention to the interior of the set of points (x, q) for which $\mathcal{X}(x, q)$ is not empty:

$$\mathcal{K} = int\{(x, q) \in \mathbb{R}^{N+1} : \mathcal{X}(x, q) \neq \emptyset\}.$$

More generally consider the closure of the set $\mathcal{K}$ as in [18] which is characterized by (see lemma 6 in [11])

$$cl\mathcal{K} = \{(x, q) \in \mathbb{R}^{N+1} : \mathcal{X}(x, q) \neq \emptyset\}.$$

In this context, given x and q, the goal of the investor is to maximize the expected utility of his or her terminal wealth. This leads to the following optimization problem:

$$u(x, q) := \sup_{X \in \mathcal{X}(x,q)} \mathbb{E}\Big[U(X_T + \langle q, f \rangle)\Big]$$

for $(x, q) \in cl\mathcal{K}$ where we assume

$$(x, 0) \in \mathcal{K}, x > 0. \tag{4.1.3}$$

This means that the situation without random endowment (see [16]) is a special case of optimization with random endowment that we are describing here.

4.2 Convex duality

To obtain results on the solution of the optimization problem described above we make use of convex duality results. In order to formulate the dual problem we need the following definitions and notations.
Define $V(y) := \sup_{x>0}\Big[U(x) - yx\Big]$. Let us define the set $\mathcal{L}$ which is the relative interior of the polar cone of $-\mathcal{K}$:

$$\mathcal{L} = ri\{(y,r) \in \mathbb{R}^N : xy + qr \geq 0 \text{ for all } (x,q) \in \mathcal{K}\}.$$

and

$$\mathcal{Y}^{\mathbb{Q}}(y) = \{Y \geq 0 : Y_0 = y, XY \text{ is a supermartingale under } \mathbb{Q} \text{ for all } X \in \mathcal{X}(1)\}.$$

Given an arbitrary vector $(y,r) \in \mathcal{L}$, we denote by $\mathcal{Y}^{\mathbb{Q}}(y,r)$ the set of non-negative supermartingales $Y \in \mathcal{Y}^{\mathbb{Q}}(y)$ such that the inequality

$$\mathbb{E}^{\mathbb{Q}}[Y_T(X_T + qf)] \leq xy + qr$$

holds true for all $(x,q) \in \mathcal{K}$ and $X \in \mathcal{X}(x,q)$. Define

$$v^{\mathbb{Q}}(y,r) := \inf_{Y \in \mathcal{Y}^{\mathbb{Q}}(y,r)} E^{\mathbb{Q}}[V(Y_T)], \tag{4.2.1}$$

for $(y,r) \in \mathcal{L}$.

4.3 Results

The following theorem is proved in [11]:

Theorem 4.3.1. *Assume conditions* $\mathcal{M} \neq \emptyset$, (4.1.1) *and* (4.1.3) *hold and*

$$u(x, q) < \infty$$

for some $(x, q) \in \mathcal{K}$. *Then we have:*

(i) The function u *is finitely valued on* $\mathcal{K}$ *and for any* $(y, r) \in \mathcal{L}$ *there exists a constant* $c = c(y, r) > 0$ *such that* $v(cy, cr)$ *is finite. The value functions* u *and* v *are conjugate:*

$$u(x, q) = \inf_{(y,r)\in\mathcal{L}} \{v(y, r) + xy + qr\}, (x, q) \in \mathcal{K}.$$

$$v(y, r) = \sup_{(x,q)\in\mathcal{K}} \{u(x, q) - xy - qr\}, (y, r) \in \mathcal{L}.$$

(ii) The solution $\hat{Y}(y, r)$ *to* (4.2.1) *exists and is unique for all* $(y, r) \in \mathcal{L}$ *such that* $v(y, r) < \infty$.

4.4 The value of weak information in the presence of a random endowment

Let $\mathbb{Q} \in \mathcal{E}^{\nu}$ be given. Define $u^{\mathbb{Q}}(x, q) := \sup_{X\in\mathcal{X}(x,q)} E^{\mathbb{Q}}[U(X_T + qf)]$ Then the value of weak information is defined as follows:

Definition.(Value of Weak Anticipation)

$$u(x, q, \nu) := \inf_{\mathbb{Q}\in\mathcal{E}^{\nu}} u^{\mathbb{Q}}(x, q) \text{ for } (x, q) \in \mathcal{K}.$$

Chapter 5

Discrete market model with random endowment and anticipation

5.1 N-period general discrete market model

5.1.1 Set-up

As in chapter 3 in this chapter we are interested in optimizing utility from terminal wealth. In this chapter however the terminal wealth is no longer solely determined by the value of the investment portfolio but also by the value of a so-called random endowment. Consequently the set-up of this chapter is the same as in chapter 3 with some additional assumptions concerning the random endowment.

Assume that at time 0 the investor has an initial capital of x and q units of a non-traded European contingent claim with maturity T. This claim has a $\mathcal{F}_T$-measurable payment function f such that the investor's payoff at time T will be given by qf.

5.1.2 Results

In a complete market with stocks taking a finite number of possible values and the random endowment f as described above we have the following theorem with regard to the financial value of weak information:

Theorem 5.1.1. *The financial value of weak information in a complete market is*

$$u(v,q,\nu) = \min_{Q\in\mathcal{E}^\nu} \max_{\delta_0,\delta_1,\dots,\delta_{N-1}:V_N+qf\geq 0} \mathbb{E}^{\mathbb{Q}}[U(V_N+qf)] = \mathbb{E}^{\mathbb{P}^\nu}\left[U\left(I\left(\frac{\lambda(v,q)}{(1+r)^N}\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^\nu}\right)\right)\right],$$

where $\lambda(v,q)$ *is determined by*

$$\tilde{\mathbb{E}}\left[\frac{1}{(1+r)^N} I\left(\frac{\lambda(v,q)}{(1+r)^N}\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^\nu}\right)\right] = v + \frac{q\mathbb{E}^{\tilde{\mathbb{P}}}[f]}{(1+r)^N},$$

where $\tilde{\mathbb{P}} \in \mathcal{M}$ *is the unique probability measure under which the prices are martingales.*

Proof. Fix $\mathbb{Q} \in \mathcal{E}^\nu$. Proceed by rewriting $\max_{\delta_0,\delta_1,\dots,\delta_{N-1}:V_N+qf\geq 0} \mathbb{E}^{\mathbb{Q}}[U(V_N+qf)]$. The Lagrangian is given by

$$\mathcal{L}^{RE}(\lambda) = \mathbb{E}^{\mathbb{Q}}[U(V_N+qf)] + \lambda\Big[v - E^{\mathbb{Q}}[\frac{d\tilde{\mathbb{P}}}{d\mathbb{Q}}\frac{V_N}{(1+r)^N}]\Big]$$

The martingale method gives

$$\hat{V}_N + qf = I\big(\frac{\lambda}{(1+r)^N}\frac{d\tilde{\mathbb{P}}}{d\mathbb{Q}}\big)$$

so that

$$\mathcal{L}^{RE}(\lambda) = \mathbb{E}^{\mathbb{Q}}\Big[U(I(\frac{\lambda}{(1+r)^N}\frac{d\tilde{\mathbb{P}}}{d\mathbb{Q}}))\Big]$$

$$+\lambda\Big[v - \mathbb{E}^{\mathbb{Q}}\Big[\frac{d\tilde{\mathbb{P}}}{d\mathbb{Q}}\frac{1}{(1+r)^N}\Big[I(\frac{\lambda}{(1+r)^N}\frac{d\tilde{\mathbb{P}}}{d\mathbb{Q}}) - qf\Big]\Big]\Big]$$

$$= \mathbb{E}^{\mathbb{Q}}\Big[U(I(\frac{\lambda}{(1+r)^N}\frac{d\tilde{\mathbb{P}}}{d\mathbb{Q}}))\Big] + \lambda\Big[v - \mathbb{E}^{\mathbb{Q}}\Big[\frac{d\tilde{\mathbb{P}}}{d\mathbb{Q}}\frac{1}{(1+r)^N}I(\frac{\lambda}{(1+r)^N}\frac{d\tilde{\mathbb{P}}}{d\mathbb{Q}})\Big] + \frac{q\mathbb{E}^{\tilde{\mathbb{P}}}[f]}{(1+R)^N}\Big]$$

so that

$$\mathcal{L}^{RE}(\lambda) = \mathcal{L}(\lambda) + \lambda\frac{q\mathbb{E}^{\tilde{\mathbb{P}}}[f]}{(1+R)^N}$$

where $\mathcal{L}(\lambda)$ is the Lagrangian in the case where there is no random endowment. (see [1]) Consequently

$$u(v,q,\nu) = \min_{\mathbb{Q}\in\mathcal{E}^\nu}\min_{\lambda>0}\Big[\lambda\Big(v + \frac{q\mathbb{E}^{\tilde{\mathbb{P}}}[f]}{(1+R)^N}\Big) + \tilde{\mathbb{E}}\Big[\frac{d\mathbb{Q}}{d\tilde{\mathbb{P}}}\tilde{U}\Big(\frac{\lambda}{(1+r)^N}\frac{d\tilde{\mathbb{P}}}{d\mathbb{Q}}\Big)\Big]\Big]$$

Applying proposition (3.3.1) we have

$$u(v,q,\nu) = \min_{\lambda>0}\Big[\lambda\Big(v + \frac{q\mathbb{E}^{\tilde{\mathbb{P}}}[f]}{(1+R)^N}\Big) + \tilde{\mathbb{E}}\Big[\frac{d\mathbb{P}^\nu}{d\tilde{\mathbb{P}}}\tilde{U}\Big(\frac{\lambda}{(1+r)^N}\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^\nu}\Big)\Big]\Big]$$

take the derivative with respect to λ and setting this equal to zero we obtain

$$v + \frac{q\mathbb{E}^{\tilde{\mathbb{P}}}[f]}{(1+R)^N} = \tilde{\mathbb{E}}\Big[\frac{1}{(1+r)^N}I\Big(\frac{\lambda^*(v,q)}{(1+r)^N}\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^\nu}\Big)\Big]$$

where $\lambda^*(v,q)$ is the minimizer. Now,

$$u(v,q,\nu) = \lambda^*(v,q)\Big(v + \frac{q\mathbb{E}^{\tilde{\mathbb{P}}}[f]}{(1+R)^N}\Big) + \tilde{\mathbb{E}}\Big[\frac{d\mathbb{P}^\nu}{d\tilde{\mathbb{P}}}U\Big(I(\frac{\lambda^*(v,q)}{(1+r)^N}\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^\nu})\Big)\Big]$$

$$-\lambda^*(v,q)\tilde{\mathbb{E}}\Big[\frac{1}{(1+r)^N}I\Big(\frac{\lambda^*(v,q)}{(1+r)^N}\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^\nu}\Big)\Big]$$

$$= \mathbb{E}^{\mathbb{P}^\nu}\left[U\left(I\left(\frac{\lambda^*(v,q)}{(1+r)^N}\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^\nu}\right)\right)\right].$$

■

Example (Log Utility) Let $U(x) = \ln(x)$ for $x > 0$. Then $I(x) := (U')^{-1}(x) = \frac{1}{x}$. Solving

$$v + \frac{q\mathbb{E}^{\tilde{\mathbb{P}}}[f]}{(1+r)^N} = \frac{1}{(1+r)^N}\tilde{\mathbb{E}}\Big[\frac{(1+r)^N}{\lambda(v,q)}\frac{d\mathbb{P}^\nu}{d\tilde{\mathbb{P}}}\Big]$$

for λ gives

$$\lambda(v,q) = \frac{(1+r)^N}{v(1+r)^N + q\mathbb{E}^{\tilde{\mathbb{P}}}[f]}.$$

Then the optimal expected utility is

$$u(v,q,\nu) = \ln\left[v(1+r)^N + q\mathbb{E}^{\tilde{\mathbb{P}}}[f]\right] + \mathbb{E}^{\mathbb{P}^\nu}\left[\ln\left(\frac{\mathbb{P}^\nu}{\tilde{\mathbb{P}}}\right)\right].$$

The second term can be interpreted as the relative entropy of the measures $\mathbb{P}^\nu$ and $\tilde{\mathbb{P}}$.

Example (Power Utility) Let $U(x) = \frac{x^\gamma}{\gamma}$ for $x > 0$ and $-\infty < \gamma < 0$ or $0 < \gamma < 1$. Then $I(x) = x^{\frac{1}{\gamma-1}}$. Solving

$$\tilde{\mathbb{E}}\Big[\frac{1}{(1+R)^N}\Big(\frac{\lambda}{(1+R)^N}\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^\nu}\Big)^{\frac{1}{\gamma-1}}\Big] = v + \frac{q\mathbb{E}^{\tilde{\mathbb{P}}}[f]}{(1+R)^N}$$

for $\lambda := \lambda(v,q)$ we get

$$\lambda(v,q) = \frac{\left[v(1+R)^N + q\tilde{\mathbb{E}}[f]\right]^{\gamma-1}}{\tilde{\mathbb{E}}\left[\left(\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^\nu}\right)^{\frac{1}{\gamma-1}}\right]^{\gamma-1}}(1+R)^N.$$

Consequently

$$u(v,q,\nu) = \frac{1}{\gamma}\left(\frac{v(1+R)^N + q\tilde{\mathbb{E}}[f]}{\tilde{\mathbb{E}}\left[\left(\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^\nu}\right)^{\frac{1}{\gamma-1}}\right]}\right)^{\gamma}\mathbb{E}^{\mathbb{P}^\nu}\left[\left(\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^\nu}\right)^{\frac{\gamma}{\gamma-1}}\right].$$

The theorem in [1] can be generalized to the situation of incomplete markets. In that case the equivalent martingale measure $\tilde{\mathbb{P}} \in \mathcal{M}$ that turns the price process into a martingale is no longer unique.

Theorem 5.1.2. *The financial value of weak information in a incomplete market is*

$$u(v,q,\nu) = \inf_{Q\in\mathcal{E}^\nu} \max_{\delta_0,\delta_1,\ldots,\delta_{N-1}:V_N\geq 0} \mathbb{E}^{\mathbb{Q}}[U(V_N)]$$

$$= \inf_{\tilde{\mathbb{P}}\in\mathcal{M}}\left[\mathbb{E}^{\tilde{\mathbb{P}}^\nu}\left[U\left(I\left(\frac{\lambda(v,q)}{(1+r)^N}\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^\nu}\right)\right)\right]\right]$$

where $\lambda(v,q)$ is determined by

$$\inf_{\tilde{\mathbb{P}}\in\mathcal{M}} \mathbb{E}^{\tilde{\mathbb{P}}}\left[\frac{1}{(1+r)^N}I\left(\frac{\lambda(v,q)}{(1+r)^N}\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^\nu}\right)\right] = v,$$

where $\mathcal{M}$ is the set of probability measures $\tilde{\mathbb{P}}$ under which prices are martingales.

Proof.

$$u(v,q,\nu) = \inf_{\mathbb{Q}\in\mathcal{E}^\nu} \min_{\lambda>0} \left[\lambda v + \inf_{\tilde{\mathbb{P}}\in\mathcal{M}} \mathbb{E}^{\tilde{\mathbb{P}}}\left[\frac{d\mathbb{Q}}{d\tilde{\mathbb{P}}}\tilde{U}\left(\frac{\lambda}{(1+r)^N}\frac{d\tilde{\mathbb{P}}}{d\mathbb{Q}}\right)\right]\right]$$

$$= \min_{\lambda>0} \left[\lambda v + \inf_{\mathbb{Q}\in\mathcal{E}^\nu} \inf_{\tilde{\mathbb{P}}\in\mathcal{M}} \mathbb{E}^{\tilde{\mathbb{P}}}\left[\frac{d\mathbb{Q}}{d\tilde{\mathbb{P}}}\tilde{U}\left(\frac{\lambda}{(1+r)^N}\frac{d\tilde{\mathbb{P}}}{d\mathbb{Q}}\right)\right]\right]$$

$$= \min_{\lambda>0} \left[\lambda v + \inf_{\tilde{\mathbb{P}}\in\mathcal{M}} \inf_{\mathbb{Q}\in\mathcal{E}^\nu} \mathbb{E}^{\tilde{\mathbb{P}}}\left[\frac{d\mathbb{Q}}{d\tilde{\mathbb{P}}}\tilde{U}\left(\frac{\lambda}{(1+r)^N}\frac{d\tilde{\mathbb{P}}}{d\mathbb{Q}}\right)\right]\right]$$

$$= \min_{\lambda>0} \left[\lambda v + \inf_{\tilde{\mathbb{P}}\in\mathcal{M}} \mathbb{E}^{\tilde{\mathbb{P}}}\left[\frac{d\mathbb{P}^\nu}{d\tilde{\mathbb{P}}}\tilde{U}\left(\frac{\lambda}{(1+r)^N}\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^\nu}\right)\right]\right]$$

Taking the derivative wrt λ yields the equation

$$v = \inf_{\tilde{\mathbb{P}}\in\mathcal{M}} \mathbb{E}^{\tilde{\mathbb{P}}}\left[\frac{1}{(1+r)^N} I\left(\frac{\lambda(v,q)}{(1+r)^N}\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^\nu}\right)\right] \tag{5.1.1}$$

Now,

$$u(v,q,\nu) = \lambda(v,q)v + \inf_{\tilde{\mathbb{P}}\in\mathcal{M}} \left[\tilde{\mathbb{E}}\left[\frac{d\mathbb{P}^\nu}{d\tilde{\mathbb{P}}} U\left(I\left(\frac{\lambda(v,q)}{(1+r)^N}\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^\nu}\right)\right)\right]\right.$$

$$\left. -\lambda(v,q)\tilde{\mathbb{E}}\left[\frac{1}{(1+r)^N} I\left(\frac{\lambda(v,q)}{(1+r)^N}\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^\nu}\right)\right]\right]$$

$$= \inf_{\tilde{\mathbb{P}}\in\mathcal{M}} \left[\mathbb{E}^{\tilde{\mathbb{P}}^\nu}\left[U\left(I\left(\frac{\lambda(v,q)}{(1+r)^N}\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^\nu}\right)\right)\right]\right]$$

■

Example. (Power Utility) Assume $U(x) = \frac{x^\gamma}{\gamma}$. Then $I(x) = x^{\frac{1}{\gamma-1}}$. It follows

$$\lambda = \frac{v^{\gamma-1}(1+R)^N[(1+R)^N]^{\gamma-1}}{\left[\inf_{\tilde{\mathbb{P}}}\left(\tilde{\mathbb{E}}\left[\left(\frac{d\tilde{\mathbb{P}}}{\mathbb{P}^\nu}\right)^{\frac{1}{\gamma-1}}\right]\right)\right]^{\gamma-1}}$$

and

$$u(v,\nu) = \frac{1}{\gamma} \frac{[v(1+R)^N]^\gamma}{\left(\inf_{\tilde{\mathbb{P}}} \left(\tilde{\mathbb{E}}\left[\left(\frac{d\tilde{\mathbb{P}}}{\tilde{\mathbb{P}}^\nu}\right)^{\frac{1}{\gamma-1}}\right]\right)\right)^\gamma} \inf_{\tilde{\mathbb{P}}} \mathbb{E}^{\tilde{\mathbb{P}}^\nu}\left[\left(\frac{d\tilde{\mathbb{P}}}{\tilde{\mathbb{P}}^\nu}\right)^{\frac{\gamma}{\gamma-1}}\right].$$

Suppose we are in an incomplete market now with a random endowment. Then we have the following theorem.

Theorem 5.1.3. *The financial value of weak information in a incomplete market is*

$$u(v,q,\nu) = \inf_{Q\in\mathcal{E}^\nu} \max_{\delta_0,\delta_1,\ldots,\delta_{N-1}:V_N+qf\geq 0} \mathbb{E}^{\mathbb{Q}}[U(V_N+qf)] = \inf_{\tilde{\mathbb{P}}\in\mathcal{M}} \left[\mathbb{E}^{\tilde{\mathbb{P}}^\nu}\left[U\left(I\left(\frac{\lambda(v,q)}{(1+r)^N}\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^\nu}\right)\right)\right]\right], \tag{5.1.2}$$

where $\lambda(v,q)$ is determined by

$$-\left(\inf_{\tilde{\mathbb{P}}\in\mathcal{M}}\left[-\tilde{\mathbb{E}}\left[\frac{1}{(1+r)^N} I\left(\frac{\lambda(v,q)}{(1+r)^N}\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^\nu}\right)\right] + \frac{\tilde{\mathbb{E}}[qf]}{(1+r)^N}\right]\right) = v, \tag{5.1.3}$$

where $\mathcal{M}$ is the set of probability measures under which the prices are martingales.

Proof. Let $\mathbb{Q}\in\mathcal{E}^\nu$. Proceed by rewriting $\max_{\delta_0,\delta_1,\ldots,\delta_{N-1}:V_N+qf\geq 0}\mathbb{E}^{\mathbb{Q}}[U(V_N+qf)]$. Proceeding in a similar fashion as in the proof of theorem 5.1.1 we obtain

$$u(v,q,\nu) = \inf_{\mathbb{Q}\in\mathcal{E}^\nu}\min_{\lambda>0}\left[\lambda v + \min_{\tilde{\mathbb{P}}\in\mathcal{M}}\left[\tilde{\mathbb{E}}\left[\frac{d\mathbb{Q}}{d\tilde{\mathbb{P}}}\tilde{U}\left(\frac{\lambda}{(1+r)^N}\frac{d\tilde{\mathbb{P}}}{d\mathbb{Q}}\right) + \frac{\lambda qf}{(1+r)^N}\right]\right]\right]$$

$$= \min_{\lambda>0}\left[\lambda v + \min_{\tilde{\mathbb{P}}\in\mathcal{M}}\left[\inf_{\mathbb{Q}\in\mathcal{E}^\nu}\tilde{\mathbb{E}}\left[\frac{d\mathbb{Q}}{d\tilde{\mathbb{P}}}\tilde{U}\left(\frac{\lambda}{(1+r)^N}\frac{d\tilde{\mathbb{P}}}{d\mathbb{Q}}\right)\right] + \tilde{\mathbb{E}}\left[\frac{\lambda qf}{(1+r)^N}\right]\right]\right]$$

$$= \min_{\lambda>0}\left[\lambda v + \min_{\tilde{\mathbb{P}}\in\mathcal{M}}\left[\tilde{\mathbb{E}}\left[\frac{d\mathbb{P}^\nu}{d\tilde{\mathbb{P}}}\tilde{U}\left(\frac{\lambda}{(1+r)^N}\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^\nu}\right)\right] + \tilde{\mathbb{E}}\left[\frac{\lambda qf}{(1+r)^N}\right]\right]\right]$$

Taking the derivative with respect to λ and setting equal to zero we obtain the equation

$$-\left(\inf_{\tilde{\mathbb{P}}\in\mathcal{M}}\left[-\tilde{\mathbb{E}}\left[\frac{1}{(1+r)^N}I\left(\frac{\lambda(v,q)}{(1+r)^N}\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^\nu}\right)\right]+\frac{\tilde{\mathbb{E}}[qf]}{(1+r)^N}\right]\right)=v$$

for λ. Denote the solution of this equation by λ^*. Then

$$u(v,q,\nu)=\lambda^* v$$

$$+\inf_{\tilde{\mathbb{P}}\in\mathcal{M}}\left[\tilde{\mathbb{E}}\Big[\frac{d\mathbb{P}^\nu}{d\tilde{\mathbb{P}}}U\Big(I\Big(\frac{\lambda^*}{(1+r)^N}\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^\nu}\Big)\Big)\Big]+\tilde{\mathbb{E}}\Big[\frac{\lambda^* qf}{(1+r)^N}\Big]-\frac{\lambda^*}{(1+r)^N}\tilde{\mathbb{E}}\Big[I\Big(\frac{\lambda^*}{(1+r)^N}\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^\nu}\Big)\Big]\right]$$

$$=\inf_{\tilde{\mathbb{P}}\in\mathcal{M}}\left[\mathbb{E}^{\tilde{\mathbb{P}}^\nu}\Big[U\Big(I\Big(\frac{\lambda^*}{(1+r)^N}\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^\nu}\Big)\Big)\Big]\right]$$

■

Example. (Log Utility) Assume $U(x)=\ln(x)$. Then $I(z)=\frac{1}{z}$ and from equation (5.1.3) we find

$$\lambda(v,q)=\frac{1}{v+\frac{\min_{\tilde{\mathbb{P}}\in\mathcal{M}}\tilde{\mathbb{E}}[qf]}{(1+r)^N}}$$

Plugging this into equation (5.1.2) we obtain

$$u(v,q,\nu)=\ln\left[v(1+r)^N+\min_{\tilde{\mathbb{P}}\in\mathcal{M}}\tilde{\mathbb{E}}[qf]\right]+\inf_{\tilde{\mathbb{P}}\in\mathcal{M}}\mathbb{E}^{\tilde{\mathbb{P}}^\nu}\left[\ln\left(\frac{d\mathbb{P}^\nu}{d\tilde{\mathbb{P}}}\right)\right]$$

where the second term can be interpreted as the infinimum over the relative entropies of the probability measures $\tilde{\mathbb{P}}^\nu$ and $\tilde{\mathbb{P}}$. The notation $\tilde{\mathbb{P}}^\nu$ indicates that this measure also depends on $\tilde{\mathbb{P}}$.

Example. (Power Utility) Assume $U(x)=\frac{x^\gamma}{\gamma}$. Then $I(x)=x^{\frac{1}{\gamma-1}}$ and we observe that equation (5.1.3) is equivalent to

$$-\left(\inf_{\tilde{\mathbb{P}}\in\mathcal{M}}\left[\frac{-\lambda^{\frac{1}{\gamma-1}}}{(1+R)^N[(1+R)^N]^{\frac{1}{\gamma-1}}}\tilde{\mathbb{E}}\left[\left(\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^{\nu}}\right)^{\frac{1}{\gamma-1}}\right]+\frac{\tilde{\mathbb{E}}[qf]}{(1+r)^N}\right]\right)=v. \qquad (5.1.4)$$

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