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D-terms in Bosonic and Fermionic Systems

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ABSTRACT

The most fundamental information about a particle is contained in the matrix elements of its energy-momentum tensor (EMT): the mass and spin. But the EMT contains more information than that. Equally important yet far less known is the D-term and with it, the information contained in the spatial components of the EMT. The D-term and the spatial components of the EMT show in detail how the strong forces inside the nucleon balance to form a bound state and provides unique insights on the nucleon structure. The goal of this thesis is to contribute to a better understanding of the physics associated with the D-term. We investigate the EMT form factors of spin-0 and spin 1/2-particles, focusing especially on the unknown particle property D-term.

D-terms in Bosonic and Fermionic Systems

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1 Introduction

This thesis provides the background and the results of new research on the form factors of the energy momentum tensor (EMT) and the *D*-term in spin-0 and spin-$\frac{1}{2}$ systems with the goal to contribute to a better understanding of the physics associated with the *D*-term.

The most fundamental information about a particle is contained in the matrix elements of its EMT: the mass and the spin. But the matrix elements of the EMT which are described in terms of the EMT form factors contain much more information than that. Equally important yet far less known is the *D*-term and with it, the information contained in the spatial components of the EMT. The *D*-term and the spatial components of the EMT show in detail how the strong forces inside the nucleon balance to form a bound state. Studies of the EMT form factors promise therefore to provide unique insights on the structure of the nucleon, nuclei and other hadrons. Experimental studies of high energy reactions in which information about the EMT form factors can be accessed are in the main focus of the experimental programs at the Jefferson National Lab and in the COMPASS experiment at CERN, and where previously studied in the HERMES experiment at DESY.

The goal of this thesis is to contribute to a better understanding of the physics associated with EMT form factors and especially with the *D*-term form factor. For that we will study the EMT form factors of spin-0 and spin 1/2-particles. The outline of this thesis is as follows.

In Sec. 2 the form factors of the EMT are defined, their general properties are reviewed and the interpretation of the form factors in terms of EMT densities is introduced.

In Sec 3 we study the EMT form factors in spin-0 systems. We first calculate the EMT form factors and the *D*-term of a free elementary spin-0 boson as described by the free Klein-Gordon theory. Then we discuss what happens to the *D*-term when interactions are present. For that we consider one weakly interacting case in the Φ^4 theory, as well as one example of a strong-coupling theory, namely QCD albeit only for the Goldstone bosons of chiral symmetry breaking.

In Sec. 4 we study the EMT densities of a point-like particle, and show that they are singular functions but the 3D density description is nevertheless consistent albeit plagued by relativistic corrections which can be removed in the heavy mass limit. For the heavy mass limit it is strictly speaking necessary to introduce an additional scale in a theory which is necessary to formulate the heavy mass limit in a rigorous way. Even in this way the point-like particle is "smeared out by hand," we show that the description of the EMT densities remain consistent.

In Sec. 5 we investigate the question of whether the heuristic introduction of an internal particle structure done by hand in Sec. 4 can be implemented dynamically. For that we construct a microscopic dynamical theory in which a spin-0 particle exhibits an extended structure and whose EMT densities correspond exactly to the "smeared out" densities obtained in Sec. 4. As a by-product of this study we obtain an exactly solvable soliton model in $3+1$ dimensions which is rare to find.

In Sec. 6 we turn our attention to spin-$\frac{1}{2}$ systems. We start with the free theory, and find the unexpected result that a non-interacting point-like fermion has a vanishing *D*-term. We provide heuristic arguments why this result is plausible.

In Sec. 7 we show how interactions in fermionic systems can generate a *D*-term. For that purpose we choose to work with the bag model which thanks to its simplicity provides a lucid framework to show how the *D*-term "emerges" when chiral interactions are "switched on." We will also briefly review the results from a chiral model where it can be shown how the *D*-term "vanishes" if one "switches off" the interactions in that model.

The Appendices A and B contain an overview on the different notations for EMT form factors used in the literature and different definition of EMT form factors for spin$\frac{1}{2}$ particles which is also commonly used.

2 EMT Form Factors

The purpose of this section is to introduce form factors of the energy momentum tensor (EMT), review their general properties, and introduce the interpretation of the form factors in terms of EMT densities.

2.1 Canonical Energy Momentum Tensor

If the action of a system is invariant under a continuous symmetry transformation, then the Noether theorem implies the existence of a corresponding conserved current. In the case of continuous space-time transformations the conserved Noether current is the canonical EMT $T^{\mu\nu}_{\rm can}$.

Let us illustrate the procedure on the example of generic scalar field theory described by the action

$$S = \int d^4x \mathscr{L}(x) \tag{1}$$

where $\mathscr{L}(x) = \mathscr{L}(\partial_\mu\phi, \phi)$, and the field $\phi(x)$ obeys the Euler-Lagrange equations

$$\partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathscr{L}}{\partial \phi} = 0\,. \tag{2}$$

To derive the canonical EMT we follow the procedure in Sec. 1.2.2 of Ref. [1], and consider the transformations $x^\mu \to x'^\mu = x^\mu + \varepsilon^\mu(x)$ which leave the action invariant, $\delta S = 0$, with infinitesimal space-time dependent translations $\varepsilon^\mu(x)$. The fields transform as

$$\phi(x) \to \phi(x') = \phi(x+\delta x) = \phi(x) + \delta\phi(x)\,, \quad \delta\phi(x) = \partial_\nu \phi(x)\,\varepsilon^\nu(x)\,, \tag{3}$$

where higher order terms are neglected. For the variation of the field derivatives we have $\delta\partial_\mu\phi(x) = \partial_\mu\delta\phi(x)$ up to higher order terms. This yields

$$\delta\partial_\mu\phi(x) = (\partial_\mu\partial_\nu\phi)\varepsilon^\nu(x) + (\partial_\nu\phi)(\partial_\mu\varepsilon^\nu(x)) \tag{4}$$

The variation of the action under the transformations is

$$\begin{aligned}\delta S &= \int d^4x\left[(\partial_\nu\mathscr{L})\varepsilon^\nu + \frac{\partial\mathscr{L}}{\partial\phi}\,\delta\phi + \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\delta\partial_\mu\phi\right]\\ &= \int d^4x\left[\left(\partial_\nu\mathscr{L} + \frac{\partial\mathscr{L}}{\partial\phi}\,\partial_\nu\phi + \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}(\partial_\mu\partial_\nu\phi)\right)\varepsilon^\nu + \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\,(\partial_\nu\phi)\partial_\mu\varepsilon^\nu\right].\end{aligned} \tag{5}$$

Performing partial integrations in the second and fourth term on the right-hand side of Eq. (5) and assuming the fields vanish at infinity fast enough we obtain

$$\delta S = \int d^4x\left[\partial_\nu\mathscr{L} + \left\{\frac{\partial\mathscr{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\right)\right\}\partial_\nu\phi - \partial_\mu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\,(\partial_\nu\phi)\right)\right]\varepsilon^\nu\,. \tag{6}$$

Due to the equations of motion, Eq. (2), the curly bracket in Eq. (6) vanishes, and we obtain

$$\begin{aligned}\delta S &= \int d^4x\left[\partial_\nu\mathscr{L} - \partial_\mu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\,\partial_\nu\phi\right)\right]\varepsilon^\nu\\ &= \int d^4x\left[-\partial^\mu\left\{\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\,\partial_\nu\phi - g_{\mu\nu}\,\mathscr{L}\right\}\right]\varepsilon^\nu\,.\end{aligned} \tag{7}$$

Since the action is invariant under the transformations, $\delta S = 0$, the canonical EMT defined by the expression in the curly brackets of Eq. (7),

$$T^{\mu\nu}_{\rm can} = \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)}\,\partial^\nu\phi - g^{\mu\nu}\,\mathscr{L} \tag{8}$$

is a conserved tensor, i.e.

$$\partial_\mu T^{\mu\nu}_{\rm can} = 0\,. \tag{9}$$

2.2 Symmetric Energy Momentum Tensor

The derivation of the canonical EMT in Sec. 2.1 was shown for a scalar field. But this derivation and the definition (8) of the canonical EMT also applies to other Lorentz-invariant theories including Dirac fields or spin-1 fields.

In the case of scalar fields the canonical EMT happens to be automatically symmetric: $T^{\mu\nu}_{\rm can} = T^{\nu\mu}_{\rm can}$. In the following we will work exclusively with the symmetric EMT and denote it simply as $T^{\mu\nu}$.

In the case of higher spin fields the definition (8) in general yields a canonical EMT which is non-symmetric. However, it is always possible to construct a symmetric EMT. This construction can be done in two ways. One way is the so-called Belifante procedure. Here one explores that it is possible to add to $T^{\mu\nu}_{\rm can}$ a total derivative of the type $\partial_\lambda X^{\lambda\mu\nu}$ where the so-called superpotential $X^{\lambda\mu\nu}$ has the property $X^{\lambda\mu\nu} = -X^{\mu\lambda\nu}$. This automatically guarantees that $T^{\mu\nu}_{\rm can} + \partial_\lambda X^{\lambda\mu\nu}$ is conserved: $\partial_\mu(T^{\mu\nu}_{\rm can} + \partial_\lambda X^{\lambda\mu\nu}) = 0$. With an appropriate choice of $X^{\lambda\mu\nu}$ the tensor $T^{\mu\nu} = T^{\mu\nu}_{\rm can} + \partial_\lambda X^{\lambda\mu\nu}$ can be made symmetric.

Another method to derive a symmetric EMT consists in coupling a theory to a weak classical background gravitational field described by a symmetric metric field $g_{\mu\nu}(x)$. For instance the Lagrangian of a scalar field $\mathscr{L}$ can be generalized to a non-flat metric by replacing $(\partial_\mu\phi)(\partial^\mu\phi) \to g_{\mu\nu}(x)(\partial^\mu\phi)(\partial^\nu\phi)$. One obtains the symmetric EMT by varying the action $S_{\rm grav} = \int d^4x\sqrt{-g}\,\mathscr{L}$ with respect to the background field according to

$$\hat{T}_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S_{\rm grav}}{\delta g^{\mu\nu}}\,, \tag{10}$$

where g denotes the determinant of the metric. A pedagogical description of this method can be found in Appendix F of Ref. [2]. In this work we work with the symmetric EMT.

2.3 Importance of the EMT

Integrating Eq. (8) over the volume yields for each index $\nu = 0, 1, 2, 3$ a conserved quantity, namely energy and momentum

$$\int d^3x(\partial_\mu T^{\mu\nu}) = 0 \quad\Rightarrow\quad \frac{d}{dt}P^\nu = \frac{d}{dt}\int d^3x\, T^{0\nu}_{\rm can} = 0\,. \tag{11}$$

Evaluating the matrix elements of the Hamiltonian $H = \int d^3x T^{00}(x)$ in the rest frame of a particle yields the mass of the particle. The matrix elements of the angular momentum operator $J^i = \int d^3x\,\varepsilon^{ijk}x^jT^{0k}(x)$ contain the information about the particle spin.

The Hamiltonian and spin operators are defined in some reference frame (e.g. in the rest frame of a particle). A covariant description of the fundamental properties mass and spin is given in terms of the 2 Casimir operators of the Poincaré group, i.e. the operators which commute with the generators of the Poincaré group, and whose matrix elements have the same value in all inertial frames. One Casimir operator is $\hat{m}^2 = P^\mu P_\mu$ with the eigenvalue mass squared m^2 of the particle. The other Casimir operator is $W^2 = W^\mu W_\mu$ with the eigenvalue $(-m^2)\,S(S+1)$ where $S = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ denotes the particle spin with the Pauli-Lubanski vector $W^\mu = \frac{1}{2}\varepsilon^{\mu\nu\sigma\tau} M_{\nu\sigma} P_\tau$ where $M_{\nu\sigma} = \int d^3x\, J_{0\nu\sigma}(x)$ and $J_{\mu\nu\sigma}(x) = T_{\mu\nu}(x) x_\sigma - T_{\mu\sigma}(x) x_\nu$ [1].

Thus the components $T^{00}(x)$ and $T^{0k}(x)$ of the EMT are ultimately related to the Casimir operators of the Poincaré group and the fundamental properties mass and intrinsic spin of a particle. The spatial components $T^{ij}(x)$ are not related to a Casimir operator of the Poincaré group. However, their matrix elements are nevertheless related to a fundamental property of a particle, namely the D-term.

2.4 Definition of EMT form factors

Let us first state the definition of EMT form factors for a spin-0 particle of mass m. The convention for the covariant normalization of one-particle states is

$$\langle \vec{p}\,'|\vec{p}\,\rangle = 2E\,(2\pi)^3\,\delta^{(3)}(\vec{p}-\vec{p}\,')\,, \quad E = \sqrt{\vec{p}^{\,2}+m^2}\,. \tag{12}$$

We define the kinematic variables

$$P^\mu = p^{\mu\,\prime} + p^\mu\,, \quad \Delta^\mu = p^{\mu\,\prime} - p^\mu\,, \quad t = \Delta^2\,. \tag{13}$$

Many authors use also the average momentum $\overline{P}^\mu = \frac{1}{2}(p^{\mu\,\prime} + p^\mu)$ instead of P^μ. In a theory invariant under parity, charge conjugation, and time reversal the Lorentz structures describing the matrix elements $\langle \vec{p}\,'|\hat{T}^{\mu\nu}(0)|\vec{p}\,\rangle$ can only be constructed from P^μ, Δ^μ and $g^{\mu\nu}$ in such a way that the constraint holds $\langle \vec{p}\,'|\hat{T}^{\mu\nu}(0)|\vec{p}\,\rangle\Delta_\mu = 0$ which reflects EMT conservation. This leaves only two independent symmetric tensors $P^\mu P^\nu$ and $(\Delta^\mu\Delta^\nu - g^{\mu\nu}\Delta^2)$ and hence a spin-0 particle has 2 EMT form factors [3] which we define as

$$\langle \vec{p}\,'|\hat{T}^{\mu\nu}(0)|\vec{p}\,\rangle = \frac{P^\mu P^\nu}{2} A(t) + \frac{\Delta^\mu\Delta^\nu - g^{\mu\nu}\Delta^2}{2} D(t)\,, \tag{14}$$

where $\hat{T}^{\mu\nu}(0)$ denotes the EMT operator at space-time position zero.

In the case of a spin-$\frac{1}{2}$ particle the matrix elements of the EMT operator are described by three form factors [3] as (see App. B for alternative notations)

$$\langle p'|\hat{T}_{\mu\nu}(0)|p\rangle = \bar{u}(p')\biggl[\; A(t)\,\frac{P_\mu P_\nu}{m} + J(t)\,\frac{i(P_\mu\sigma_{\nu\rho} + P_\nu\sigma_{\mu\rho})\Delta^\rho}{2m} + \; D(t)\,\frac{\Delta_\mu\Delta_\nu - g_{\mu\nu}\Delta^2}{4m}\biggr]u(p)\,, \tag{15}$$

where the spinors $u(p) = u_s(p)$ are normalized $\bar{u}_{s'}(p)u_s(p) = 2m\,\delta_{ss'}$.

2.5 Properties of EMT form factors

In a physical scattering process one always has $t < 0$. The point $t = 0$ can only be reached by means of an analytic continuation. The form factor $A(t)$ for a particle which accompanies the Lorentz structure $P^\mu P^\nu$ can be defined for a particle of any spin. Performing the analytic continuation of this form factor to $t \to 0$ yields

$$\lim_{t\to 0} A(t) = A(0) = 1\,. \tag{16}$$

The constraint (16) is explained by recalling that for $\vec{p} \to 0$ and $\vec{p}\,' \to 0$ only the 00-component remains in Eq. (14), and $H = \int d^3x\,\hat{T}_{00}(x)$ is the Hamiltonian of the system with $H\,|\vec{p}\,\rangle = m\,|\vec{p}\,\rangle$ for $\vec{p} \to 0$.

In the spin-$\frac{1}{2}$ case the form factor $J(t)$ appears which is absent in the spin-0 case. In the limit $t \to 0$ this form factor satisfies the constraint

$$\lim_{t \to 0} J(t) = J(0) = \frac{1}{2}, \tag{17}$$

which reflects the fact that the spin of the particle is $\frac{1}{2}$.

Also the form factor $D(t)$ can be defined for particles of any spin like $A(t)$. The analytic continuation of this form factor to $t \to 0$ defines the D-term

$$\lim_{t \to 0} D(t) = D(0) \equiv D. \tag{18}$$

It is important to stress that no constraint exists for the form factor $D(t)$ such that the D-term $D \equiv D(0)$ must be determined from experiment.

Higher spin particles have more EMT form factors, because more Lorentz structures can be constructed using, e.g., in the case of spin-1 particles the polarization vector $\varepsilon^{*\mu}(p')$ and $\varepsilon^{\nu}(p)$ which satisfy $p_\mu \varepsilon^{\mu}(p) = 0$. In this work we will restrict ourselves to the spin-0 and spin-$\frac{1}{2}$ cases.

2.6 The last unknown global property, the *D*-term

The D-term is sometimes referred to as the "last unknown global property" of the nucleon. To understand why, we recall that the structure of strongly interacting particles is typically probed by means of the other fundamental forces: electromagnetic and weak interactions and, at least in principle, the gravity. The particles couple to the interactions via currents $J^{\mu}_{\rm em}$, $J^{\mu}_{\rm weak}$, $T^{\mu\nu}_{\rm grav}$ which are conserved (in the case of weak interactions we deal with the partial conservation of the axial current,

$\langle N'\lvert J^{\mu}_{\rm em}\rvert N\rangle$	$\longrightarrow$	$Q_{\rm prot}$	$=$	$1.602176487(40)\times 10^{-19}$C
		$\mu_{\rm prot}$	$=$	$2.792847356(23)\mu_N$
$\langle N'\lvert J^{\mu}_{\bf weak}\rvert N\rangle$	$\longrightarrow$	g_A	$=$	$1.2694(28)$
		g_p	$=$	$8.06(0.55)$
$\langle N'\lvert T^{\mu\nu}_{\bf grav}\rvert N\rangle$	$\longrightarrow$	$M_{\rm prot}$	$=$	$938.272013(23)\,{\rm MeV}/c^2$
		J	$=$	$\frac{1}{2}$
		D	$=$	**?**

Table 1: The global properties of the nucleon as defined in the text. The experimental values of the known global properties are from the Particle Data Book [4], except for the value of the induced pseudoscalar constant g_p taken from the MuCap experiment [5]. The D-term is the only global property which is still unknown.

PCAC). The matrix elements of these currents are described in terms of form factors which contain a wealth of information on the probed particle. It is true that the most fundamental information corresponds to the form factors at zero momentum transfer. The form factors at $t = 0$ are related to the "global properties:" electric charge Q, magnetic moment μ, axial coupling constant g_A, induced pseudo-scalar coupling constant g_p, mass M, spin J, and D-term D. All these global properties are well-known for the nucleon, and can be looked up e.g. in the particle data book, except for the D-term. The Table **??** gives an overview.

The D-term emerges in this sense as the last unknown global property of the nucleon. This fact alone makes the D-term an interesting property to study.

2.7 Remarks on EMT in QCD

In gauge theories it is possible to define separately gauge invariant EMT operators for fermions and gauge fields. Specifically in QCD one deals with the operators $\hat{T}^{Q}_{\mu\nu}$ and $\hat{T}^{G}_{\mu\nu}$ of quarks and gluons. Here $\hat{T}^{Q}_{\mu\nu} = \sum_q \hat{T}^{q}_{\mu\nu}$ and the operators $\hat{T}^{q}_{\mu\nu}$ for each separate flavor are also gauge invariant each by itself. The matrix elements of these operators define the form factors

$$\langle p'|\hat{T}^Q_{\mu\nu}(0)|p\rangle = \bar{u}(p')\Bigg[\ A^Q(t,\mu)\,\frac{P_\mu P_\nu}{m} + J^Q(t,\mu)\,\frac{i(P_\mu\sigma_{\nu\rho}+P_\nu\sigma_{\mu\rho})\Delta^\rho}{2m} + \ D^Q(t,\mu)\,\frac{\Delta_\mu\Delta_\nu - g_{\mu\nu}\Delta^2}{4m} + \bar{c}^Q(t,\mu)\,m\,g^{\mu\nu}\Bigg]u(p)\,, \quad (19)$$

and analogously for gluons. Here only the spin-$\frac{1}{2}$ case is shown which is relevant for the nucleon, the spin-0 case can be written in an analogous manner. Only the total EMT is conserved, the separate quark and gluon contributions to the EMT are not. As a consequence, a new structure is allowed in the Lorentz decompositions of $\langle p'|\hat{T}^{Q,G}_{\mu\nu}(0)|p\rangle$, namely $\bar{u}(p')\,\bar{c}^{Q,G}(t,\mu)g^{\mu\nu}\,u(p)$. In addition, all form factors acquire a dependence on the renormalization scale μ and the scheme used to renormalize the QCD operators, which drop out in the sums $A(t) = A^Q(t,\mu)+A^G(t,\mu)$, and similarly for $J(t)$ and $D(t)$. The form factors $\bar{c}^{Q,G}(t,\mu)$ have the property $\bar{c}^Q(t,\mu)+\bar{c}^G(t,\mu) = 0$.

The quark and gluon decomposition of the EMT form factors is known only for $A^Q(t)$ and $A^G(t)$ and only at $t = 0$ from parametrizations of parton distribution functions. We have $A^a(0,\mu) = \int dx\,x\,f^a_1(x,\mu)$ where $a = u, d, \ldots g$ denotes the species of the parton. The parton distribution functions $f^a_1(x,\mu)$ can be extracted from cross sections of deep-inelastic lepton-nucleon scattering and other processes. It is $A^Q(0,\mu) \approx 0.54$ and $A^G(0,\mu) \approx 0.46$ at a scale of $\mu^2 = 4\,GeV^2$ according to the parametrization of Ref. [6]. These numbers mean that in an infinite momentum frame, quarks carry about 54% and gluons 46% of the nucleon momentum at a scale of $\mu^2 = 4\,GeV^2$.

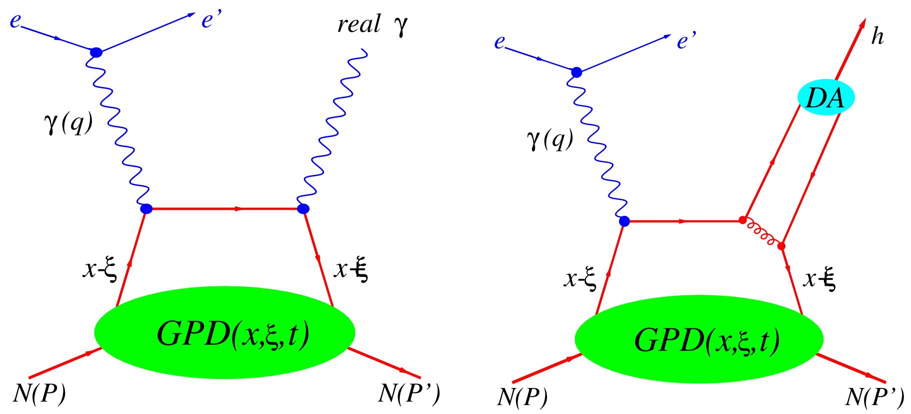
2.8 How can the EMT form factors be measured?

The EMT form factors and the *D*-term have received little attention for a long time as no practical process was known how to measure EMT form factors.

To understand why the *D*-term received so little attention before the 1990s, one must recall that particles couple directly to the EMT through gravity and Einstein's equations. Thus, gravity is the direct way to measure EMT form factors. However, gravity is too weak as the following comparison of the gravitational and electrostatic forces between an electron and proton shows:

$$\frac{|F_{gravity}|}{|F_{coulomb}|} = \frac{GM_p m_e}{r^2} \Big/ \frac{e^2}{4\pi\varepsilon_0}\frac{1}{r^2} = \frac{GM_p m_e}{\alpha\hbar c} \sim 10^{-38}.$$

During the 1990s practical processes to measure the EMT form factors were found. When a high energy virtual photon collides with a proton, in the majority of the events the proton is "dissociated" and jets of hadrons are produced. In about 10% of the cases, however, the collision results in only an outgoing proton and a real photon. This process is called Virtual Compton Scattering, because a virtual photon (emitted by the incoming electron which is scattered) interacts with the proton: $\gamma^* p \to \gamma p'$. In Deeply Virtual Compton Scattering (DVCS), the incoming photon is highly virtual. In this process, in a specific kinematics in the limit when the virtuality of the photon becomes much larger than the mass of the proton, the process is described in terms of the generalized distribution functions (GPDs) of the proton [7, 8, 9, 10] which is sketched in Fig. 1a. Another process where GPDs can be measured is hard-exclusive meson production depicted in Fig. 1b. The EMT form factors are related to the unpolarized GPDs of quarks and gluons [7, 8, 9, 10, 14, 15, 16, 17].



(a) (b)

Figure 1: (a) The leading order ("handbag") diagrams for (a) Deeply Virtual Compton Scattering $eN \to e'N'\gamma$ and (b) Hard Exclusive Meson Production $eN \to e'N'h$ where DA denotes the distribution amplitude describing the production of the meson h. These processes are described in terms of generalized parton distribution functions (GPDs) from which one can extract information on the EMT form factors, and are being studied experimentally for instance at the Jefferson National Lab. Notice that not all leading order diagrams are shown.

More specifically, a spin-$\frac{1}{2}$ particle has two twist-2 unpolarized GPDs, namely $H^a(x,\xi,t)$ and $E^a(x,\xi,t)$ where $q = u,\, d,\, g$ etc. The meaning of the variables is illustrated in Fig. 1: the parton which participates in the hard scattering process carries in the initial state the fraction $(x+\xi)$ of the nucleon momentum, and the fraction $(x-\xi)$ in the final state, while the deflected nucleon absorbs as a whole the momentum transfer t.

The N^{th} Mellin moment of a GPD is defined by $\int dx x^{N-1} H^q(x,\xi,t)$ and analogously for $E^q(x,\xi,t)$. The $N=1$ Mellin moments of the GPDs $H^q(x,\xi,t)$ and

$E^q(x,\xi,t)$ are related to the electromagnetic form factors $F_1^q(t)$ and $F_2^q(t)$ as

$$\int dx H^q(x,\xi,t) = F_1^q(t)\,,$$
$$\int dx E^q(x,\xi,t) = F_2^q(t)\,. \tag{20}$$

The $N=2$ Mellin moments of the GPDs $H^q(x,\xi,t)$ and $E^q(x,\xi,t)$ are related to the EMT form factors as (analogously for gluons)

$$\int dx\, x H^q(x,\xi,t) = A^q(t) + \xi^2 D^q(t)$$
$$\int dx\, x E^q(x,\xi,t) = B^q(t) - \xi^2 D^q(t)\,, \tag{21}$$

where $B^q(t) = 2J^q(t) - A^q(t)$ is an EMT form factor in another notation, see App. B.

Since information on GPDs can be extracted from experiment, this provides in principle a practical process to measure the D-term and other EMT form factors. Hereby it is worth recalling the prominent Ji sum rule: adding up the expressions in Eq. (21), and performing an analytic continuation to $t \to 0$ one obtains

$$\lim_{t\to 0} \int dx\, x \biggl(H^a(x,\xi,t) + E^a(x,\xi,t) \biggr) = 2J^a(0)\,, \tag{22}$$

which tells us which fraction of the nucleon spin is due to the spin and angular momentum contributions of the gluons or the specific quark flavors [8]. This information is currently unknown. What is known with some confidence is that most of the nucleon spin is not due to the quark spin, as one could have naively expected to be the case in non-relativistic models of the nucleon. Deep-inelastic scattering experiments indicate that quark spin accounts for only about 35 %–40 % of the nucleon spin. The contribution of the orbital angular momentum to the nucleon spin is very difficult to define in a gauge theory and experimentally unknown [18]. The perspective to deduce from GPDs the total contribution, spin and orbital angular momentum, of quarks and gluons to the nucleon spin by means of the Ji sum rule (22) and to learn in this way about the nucleon spin decomposition has motivated a lot of theoretical and experimental research.

The *D*-term plays an important role for the phenomenological description of hard-exclusive reactions, but at present it cannot yet be extracted model-independently [108, 109]. After first, vague and model-dependent glimpses on the nucleon *D*-term from the HERMES experiment [19] one may expect more quantitative insights from experiments at Jefferson Lab [20, 21], COMPASS at CERN [22], and the envisioned future Electron-Ion-Collider [23].

The *D*-term is less prominent but nevertheless equally interesting. The specific relation of the *D*-term to GPDs was clarified in [11]. Notice that "our" $D(t)$ corresponds to the leading term in the Gegenbauer expansion of the "*D*-term" as defined in [11]. Further aspects were discussed in [16, 13]. The type of insights that can be learned for the *D*-term will be introduced in the next section.

2.9 3D interpretation of EMT form factors in Breit frame

The form factors of the EMT in Eqs. (14, 15) can be interpreted [12] in analogy to the electromagnetic form factors [24] in the Breit frame characterized by the condition $\Delta^0 = 0$ such that $\Delta^\mu = (0, \vec{\Delta})$ and $t = -\vec{\Delta}^2$. For instance, the electric form factor $G_E(t) = F_1(t) + F_2(t)$ gives information on the 3D charge density $\rho_{el}(r)$ inside the nucleon according to $G_E(t) = \int d^3r \exp(i\vec{\Delta}\vec{r})\, \rho_{el}(r)$ [24]. The total charge of the nucleon $Q_N = \int d^3r\, \rho_{el}(r)$ is known, but the interpretation in terms of the

Fourier transform of the electric form factor is the only known way to learn about the spatial distribution of the electric charge inside the nucleon or in nuclei. It has to be kept in mind that this 3D interpretation is subject to relativistic corrections, as will be discussed in more detail in Sec. 3.

The interpretation of the EMT form factors for the nucleon (spin-$\frac{1}{2}$ case) was worked out in [12] (the case of spin-0 particles will be studied in this work below). In the Breit frame one can define the static energy-momentum tensor as

$$T_{\mu\nu}(\vec{r},\vec{s}) = \int \frac{d^3\vec{\Delta}}{(2\pi)^3 2E}\ \exp(-i\vec{\Delta}\vec{r})\ \langle p',S'|\hat{T}_{\mu\nu}(0)|p,S\rangle \tag{23}$$

with the initial and final polarization vectors of the nucleon S and S' defined such that they are equal to $(0,\vec{s})$ in the respective rest-frame, where we introduce the unit vector $\vec{s}$ denoting the quantization axis for the nucleon spin.

The EMT form factors are related to static EMT $T_{\mu\nu}(\vec{r},\vec{s})$ as follows [12]

$$J(t)+\frac{2t}{3}J'(t) = \int d^3\vec{r}\,e^{-i\vec{r}\vec{\Delta}}\,\varepsilon^{ijk}\,s_i\,r_j\,T_{0k}(\vec{r},\vec{s})\,, \tag{24}$$

$$D(t)+\frac{4t}{3}D'(t)+\frac{4t^2}{15}D''(t) = -\frac{2}{5}\,m\int d^3\vec{r}\,e^{-i\vec{r}\vec{\Delta}}\,T_{ij}(\vec{r})\,\left(r^ir^j-\frac{\vec{r}^{\,2}}{3}\delta^{ij}\right), \tag{25}$$

$$A(t)-\frac{t}{4m^2}\Big(A(t)-2J(t)+D(t)\Big) = \frac{1}{m}\int d^3\vec{r}\,e^{-i\vec{r}\vec{\Delta}}\,T_{00}(\vec{r},\vec{s})\,, \tag{26}$$

where the primes denote derivatives with respect to the argument. For a spin-1/2 particle only the $T^{0\mu}$-components are sensitive to the polarization vector. Let us remark for completeness that Eqs. (24) and (25) hold in QCD separately for quarks and gluons, while Eq. (26) holds only for the quark and gluon sum. In this work we will throughout be concerned with the total EMT.

The constraints (16, 17, 18) on the EMT form factors are expressed in terms of the EMT densities as follows

$$\begin{aligned} A(0) &= \frac{1}{m}\int d^3\vec{r}\, T_{00}(\vec{r},\vec{s}) = 1\,, \\ J(0) &= \int d^3\vec{r}\, \varepsilon^{ijk}\, s_i\, r_j\, T_{0k}(\vec{r},\vec{s}) = \frac{1}{2}\,, \\ D(0) &= -\frac{2}{5}m\int d^3\vec{r}\, T_{ij}(\vec{r})\left(r^i r^j - \frac{\vec{r}^2}{3}\,\delta^{ij}\right) \equiv D\,. \end{aligned} \tag{27}$$

Eqs. (25, 27) show that the D-term is connected to the stress tensor $T_{ij}(\vec{r})$ [12]. For spin-0 and spin-$\frac{1}{2}$ particles, the stress tensor can be uniquely decomposed in a traceless part and a trace (for spin $J \geq 1$ additional contributions from polarization vectors appear). The decomposition of the stress tensor is

$$T_{ij}(\vec{r}) = s(r)\left(\frac{r_i r_j}{r^2} - \frac{1}{3}\,\delta_{ij}\right) + p(r)\,\delta_{ij}\,. \tag{28}$$

Hereby $p(r)$ describes the radial distribution of the "pressure" inside the nucleon, while $s(r)$ is related to the distribution of the "shear forces" [12]. EMT conservation implies for the stress tensor $\nabla^i T_{ij}(\vec{r}) = 0$, which gives rise to a relation between the functions $p(r)$ and $s(r)$ as follows

$$\frac{2}{3}\,\frac{\partial s(r)}{\partial r} + \frac{2s(r)}{r} + \frac{\partial p(r)}{\partial r} = 0\,. \tag{29}$$

Another important property which can be directly derived from the conservation of the EMT is the so-called von Laue condition. Integrating $\int d^3\vec{r}\, r^k(\nabla_i T^{ij}) \equiv 0$ by parts one finds that the pressure $p(r)$ must satisfy the relation

$$\int_0^\infty dr\, r^2 p(r) = 0\,. \tag{30}$$

Further properties related to the conservation of the EMT are discussed in Ref. [25]. Here we only mention that one can express the D-term in terms of $p(r)$ and $s(r)$ as [25]

$$D = -\frac{4}{15}\,m\int d^3\vec{r}\;r^2\,s(r) = m\int d^3\vec{r}\;r^2\,p(r)\,. \tag{31}$$

The possibility to compute the D-term in two different ways provides a powerful check for the internal consistency of model calculations [25].

Due to mechanical stability arguments the densities are expected to comply with the constraints [26]

$$\text{(a)}\quad T_{00}(r)\geq 0\,,\qquad \text{(b)}\quad \frac{2}{3}\,s(r)+p(r)\geq 0\,. \tag{32}$$

We remark that (32b) is equivalent to $\int_0^r dr'\,r'^2 p(r')\geq 0$ which can be derived from Eq. (29).

2.10 Illustration in liquid drop model of nucleus

It is instructive to review the description of the D-term in a simple model of nucleus, the liquid drop model [12]. This will give useful intuition. In a large nucleus with the atomic number A in the liquid drop model the pressure and shear forces are given by $p(r) = p_0\,\theta(R_n - r) - \frac{1}{3}\,p_0 R_n\,\delta(R_n - r)$ and $s(r) = \gamma\,\delta(R_n - r)$ where p_0 is the pressure inside the drop, R_n is the radius of the nucleus with $R_n = R_0 A^{1/3}$, and $\gamma = \frac{1}{2}\,p_0 R_n$ is the surface tension. This yields for the D-term of a nucleus $D_{\text{nucleus}} = -\frac{4}{5}\,(\frac{4\pi}{3})\,m_n\gamma R_n^4$. The D-term is connected to the interaction binding the nucleus, i.e. to the surface tension in this simple model. Finite skin-effects make the Θ- and δ-functions in $p(r)$ and $s(r)$ smooth, see Fig. 2, and the D-term more

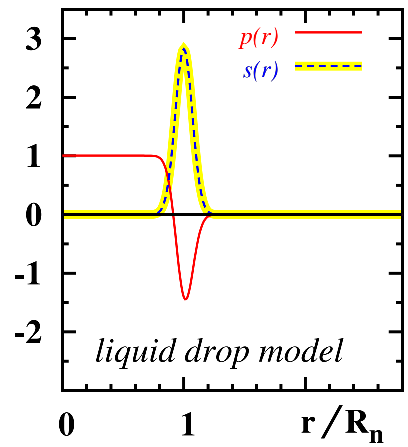


Figure 2: A sketch of the pressure $p(r)$ and shear forces $s(r)$ of a large nucleus as functions of r in the liquid drop model [12]. The $p(r)$ and $s(r)$ are in units of the pressure p_0 inside the drop, and the radius is in units of nuclear radius R_n.

negative [12]. Remarkably the liquid drop model predicts that $D_{\text{nucleus}} \propto A^{7/3}$ since the nuclear masses and radii grow like $m_n \propto A$ and $R_n \propto A^{1/3}$ with the mass number. Calculations in more sophisticated nuclear models support this prediction [27].

2.11 Overview on previous work on the *D*-term

The *D*-term and its appealing connection to the internal forces have attracted interest in theory. The aim of this section is to give a brief overview of previous work which was done before this thesis work was initiated.

D-terms of pions, nucleons, nuclei and other particles were studied in a variety of theoretical approaches. The first (and for a longtime overlooked) calculation of the *D*-term of a free scalar particle was reported by Pagels [3]. The *D*-term of the pion was studied first in [28, 29, 30, 31] and later in Ref. [11] using soft-pion theorems: chiral symmetry makes a unique prediction for the pion *D*-term as we will discuss in Sec. 3.5. Studies of EMT form factors of pions and other Goldstone bosons in chiral perturbation theory were reported in Refs. [32, 33]. The *D*-term of the nucleon was studied in the bag model [34], chiral quark soliton model [35, 36, 37, 38, 25, 39] and in Skyrme model in free space [40, 41] and nuclear medium [42, 43]. The quark contribution to the *D*-term was studied in lattice QCD [44, 45, 46], chiral perturbation theory [47, 48, 49, 50], and using dispersion relations [51]. The *D*-terms of nuclei were studied in the liquid drop model in [12] and in nuclear models in [27, 52, 53]. Studies of pions in chiral models were reported in Refs. [54, 55, 56, 57] and in lattice QCD in [58, 59]. The *D*-term of the photon was investigated in [60]. Interesting insights on EMT densities and the *D*-term were obtained from studies of *Q*-balls, their excitations and *Q*-clouds [61, 62, 63]. The *D*-term of the Δ-resonance was studied [26]. In all theoretical calculations so far the *D*-terms were found to be negative. The negative sign of the *D*-term is likely to be a consequence of stability. No rigorous proof of this conjecture [25] is known, although it was proven from mechanical analogy considerations [26] and in models [25, 61].

A review of *D*-term studies can be found in Ref. [64]. A shorter overview was given in the proceedings [65]. A brief update on the most recent developments will be given in Sec. 8.3.

3 EMT form factors of spin-0 systems

In this section we first calculate the EMT form factors and the D-term of an elementary free spin-0 boson as described by the free Klein-Gordon theory. Then we discuss what happens to the D-term when interactions are present. For that we consider one weakly interacting case in the Φ^4 theory, as well as one example of a strong-coupling regime, namely Goldstone bosons in QCD. (In Sec. 5 we will encounter another example of a strongly interacting theory which can be solved in a semi-classical approximation, namely Q-balls.)

3.1 Free spin-0 field theory

It is instructive to start with the free field case. We consider the Lagrangian of a non-interacting real spin-0 field

$$\mathscr{L} = \frac{1}{2}(\partial_\mu\Phi)(\partial^\mu\Phi) - V_0(\Phi), \quad V_0(\Phi) = \frac{1}{2}m^2\Phi^2 \tag{33}$$

which describes a free spin-0 boson field Φ of mass m obeying the Klein-Gordon equation. We are treating the mass term $\frac{1}{2}m^2\Phi^2$ as a potential.

$$(\Box + m^2)\,\Phi(x) = 0\,. \tag{34}$$

If under parity transformations the field transforms as $\Pi\Phi(x)\Pi^{-1} = \pm\Phi(x)$ then the theory describes scalars (for $+$) or pseudoscalars (for $-$). In theories like (33) the conserved canonical EMT operator is symmetric, and given by

$$\hat{T}^{\mu\nu}(x) = (\partial^\mu\Phi)(\partial^\nu\Phi) - g^{\mu\nu}\mathscr{L}\,, \tag{35}$$

where normal ordering is implied. To evaluate the matrix elements of the EMT we recall that the free field solutions to the equation of motion (34) are given by

$$\Phi(x) = \int \frac{d^3k}{2\,\omega_k(2\pi)^3} \left(\hat{a}(\vec{k})\,e^{-ikx} + \hat{a}^\dagger(\vec{k})\,e^{ikx}\right), \quad \omega_k = \sqrt{\vec{k}^2 + m^2} \tag{36}$$

with creation and annihilation operators satisfying

$$[\hat{a}(k), \hat{a}^\dagger(k')] = 2\,\omega_k\,(2\pi)^3\,\delta^{(3)}(\vec{k} - \vec{k}') \tag{37}$$

in canonical equal-time quantization. The free one-particle states are defined as $|\vec{p}_{\rm free}\rangle = \hat{a}^\dagger(\vec{p})\,|0\rangle$, and are normalized covariantly according to Eq. (12) with the trivial vacuum state normalized as $\langle 0|0\rangle = 1$. The EMT matrix elements can be readily evaluated

$$\langle \vec{p}\,'_{\rm free}|\hat{T}^{\mu\nu}(x)|\vec{p}_{\rm free}\rangle = e^{i(p'-p)x} \times \left\{ p'^\mu p^\nu + p^\mu p'^\nu - g^{\mu\nu}(p'\cdot p - m^2) \right\}. \tag{38}$$

The above result can be reformulated in the notation of Eq. (13). Derived from the definitions of P^μ and Δ^μ, we use the expressions $p'\cdot p - m^2 = -\frac{1}{2}\Delta^2$ and $p'^\mu p^\nu + p^\mu p'^\nu = \frac{1}{2}(P^\mu P^\nu - \Delta^\mu\Delta^\nu)$ such that

$$\langle \vec{p}\,'_{\rm free}|\hat{T}^{\mu\nu}(x)|\vec{p}_{\rm free}\rangle = e^{i(p'-p)x}\,\frac{1}{2}\left\{ P^\mu P^\nu - \Delta^\mu\Delta^\nu + g^{\mu\nu}\Delta^2 \right\}. \tag{39}$$

The dependence on the coordinate x is due to translational invariance $\hat{T}^{\mu\nu}(x) = \exp(i\hat{P}x)\,\hat{T}^{\mu\nu}(0)\exp(-i\hat{P}x)$ where $\hat{P}^\mu = \int d^3x\,\hat{T}^{0\mu}$ denotes the momentum operator. Because this coordinate dependence is trivial, one often defines the EMT form factors using the EMT operator $\hat{T}^{\mu\nu}(0)$ at $x = 0$ as we did in Eq. (14).

Comparing the result (39) with Eq. (14) we see that

$$A(t) = 1, \quad D(t) = -1. \tag{40}$$

Several remarks are in order. First, the form factors are constant functions of t as expected for a free point-like particle. Second, the constraint $A(0) = 1$ in (16) is of course satisfied. Third, the free Klein-Gordon theory makes the unambiguous prediction $D = -1$, and the negative sign is in line with studies in other theoretical frameworks. Fourth, repeating the calculation with a complex Klein-Gordon field reveals that a spin-0 particle and its anti-particle have the same D-term, just as they have the same mass.

It seems to have been largely overlooked in more recent literature that in Ref. [3] not only the notion of EMT form factors was introduced for spin-0 and spin-$\frac{1}{2}$ hadrons and applications were discussed. In addition to that in Ref. [3] also the form factors of a free Klein-Gordon field were quoted. Our result in Eq. (40) agrees with Ref. [3] and was published in [67].

The free Klein-Gordon prediction for the D-term of a spin-0 particle sets a reference point for further studies. It is instructive to examine what happens when one switches on interactions or when the particle is not point-like but extended. We will investigate these topics in the following.

3.2 Naive expectations in weakly interacting theories

Let us introduce in (33) a generic interaction, $V(\Phi) = \frac{1}{2}m^2\Phi^2 + \mathscr{O}(\lambda)$. Here $\mathscr{O}(\lambda)$ denotes an interaction term which is characterized by a coupling constant λ, and we assume λ to be small enough such that it is justified to use perturbative methods to solve the theory. The interesting question is which result one should expect for the *D*-term in such a theory.

If the coupling constant is very small (we may imagine λ infinitesimally small), one would naively expect the *D*-term to deviate only slightly from its free theory value. A natural expectation therefore would be

$$D_{\text{naive interacting}} \stackrel{?}{=} -1 + \mathcal{O}(\lambda), \tag{41}$$

such that one would recover the free theory result in Eq. (40) for $\lambda \to 0$. Of course, limits of this kind are subtle in quantum field theory, and it turns out that the naive expectation (41) is incorrect for two reasons.

First, the EMT is a conserved current. Its anomalous dimension vanishes. It is, therefore, a renormalization scale invariant operator whose matrix elements must not depend on λ. This is because the coupling constant in an interacting quantum field theory acquires a dependence, $\lambda \to \lambda(\mu)$, on the renormalization scale μ governed by the β-function of the theory. Hence the *D*-term must not receive any $\mathcal{O}(\lambda)$-contributions in a perturbative treatment of an interacting theory. As a consequence, no $\mathcal{O}(\lambda)$-contribution is allowed in Eq. (41).

Second, the above argument seems to imply that the *D*-term of a weakly interacting theory would coincide with the free case result (40). However, also this expectation is not true in general. The reason for this is rooted in the renormalization procedure which removes the divergences occurring in loop calculations and renders the results of quantum field theoretical calculations finite and well-defined.

In order to illustrate these points, we use the Φ^4 theory in the next section.

3.3 The D-term in weakly interacting Φ^4 theory

Considerable experience is available with the renormalization of the EMT in the Φ^4-theory. There are difficulties due to the EMT being cut-off dependent, and they are resolved by constructing a new EMT such that the new tensor defines the same four-momentum and Lorentz generators as the conventional tensor but its renormalized matrix elements are finite [69, 70, 71, 72, 73, 74, 75]. The Φ^4 theory is defined by

$$\mathscr{L} = \frac{1}{2}(\partial_\mu\Phi)(\partial^\mu\Phi) - V(\Phi)\,, \quad V(\Phi) = \frac{1}{2}m^2\Phi^2 + \frac{\lambda}{4!}\Phi^4\,. \tag{42}$$

where Φ is the scalar field, m is the mass, and λ is the coupling constant. The "bare EMT operator" of this theory is given by Eq. (35). In this context "bare" means that Eq. (35) gives the "tree level expression" for the EMT operator, which may be modified due to "higher loop contributions." In fact, in order to render the EMT operator a finite operator it is necessary to add a specific term in Eq. (35). The following discussion follows Ref. [75].

According to the general understanding one can add to the EMT operator (35) "any quantity whose divergence is zero and which does not contribute to the Ward identities" [75]. This statement can also be found in many text books, but we shall see soon that in general the statement has to be formulated more carefully.

In principle, there are infinitely many thinkable choices to a "total derivative term" to the EMT operator (35). Among all these choices the following "improvement term" plays a special role [69],

$$T^{\mu\nu}_{\text{improve}} = T^{\mu\nu}_{\text{Eq.(35)}} + \theta^{\mu\nu}_{\text{improve}},$$
$$\theta^{\mu\nu}_{\text{improve}} = -h(\partial^\mu\partial^\nu - g^{\mu\nu}\Box)\,\phi(x)^2\,, \quad h = \frac{1}{4}\left(\frac{n-2}{n-1}\right), \tag{43}$$

where n denotes the number of space-time dimensions. Clearly, the added term (43) is a total derivative and symmetric. It preserves the conservation of the EMT $\partial_\mu T^{\mu\nu}_{\text{improve}} = 0$. It was shown that this term is necessary and sufficient to make the matrix elements of the EMT operator finite in one-, two-, and three-loop calculations [75]. Since we assumed the coupling constant to be small, we can think of the perturbative series to be quickly converging, such that one- or two-loop calculations are more than sufficient for all practical purposes in our weakly interacting case.

The improvement term (43) can be motivated as follows. Let us recall that the coupling of spin-0 fields like (33, 42) to gravity is given by an effective action

$$S_{\text{grav}} = \int \mathrm{d}^4x\,\sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}(\partial_\mu\Phi)(\partial_\nu\Phi) - V(\Phi) - \frac{1}{2}hR\Phi^2\right) \tag{44}$$

where $-\frac{1}{2}hR\Phi^2$ is a non-minimal coupling term, R is the Riemann scalar, g denotes the determinant of the metric, and it is understood that gravity is treated to lowest order. From (44) one obtains the EMT operator via

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\,\frac{\delta S_{\text{grav}}}{\delta g^{\mu\nu}}\,. \tag{45}$$

Omitting the non-minimal term in (44) yields a correct description of a scalar field theory (minimally) coupled to a gravitational background field, and one recovers from (45) the canonical EMT operator (35). Keeping the non-minimal term yields the improved EMT (43). (In flat space the Riemann scalar R vanishes, but its variation with respect to the metric is nevertheless non-zero.)

The improvement term (43) with the particular value for h in (43) can be motivated in classical theory by requiring the kinetic energy in (44) to be conformally invariant. With this improvement term the trace of the EMT operator in Eqs. (43, 43) becomes

$$[T^{\mu}{}_{\mu}]_{\text{improved}} = m^2\Phi(x)^2 \tag{46}$$

which preserves conformal symmetry of the classical theory in the limit where m vanishes. On quantum level, the conformal symmetry is broken, but the improvement term with an insertion of the improved EMT is required to make the Greens functions of the renormalized fields finite (43). More precisely, to make the Greens functions of the renormalized fields finite in dimensional regularization, it is the value for h in (43) that removes the UV divergences up to three-loops in the Greens functions [75].

We will now compute the D-term in Φ^4-theory including loop corrections. Interestingly this can be done without doing explicit loop calculations. In fact, it is sufficient to investigate the effect of the improvement term at tree-level (at the level of no loops.) This is so because loop corrections produce UV divergences which are removed by the improvement term (43) [69, 70, 71, 72, 73, 74, 75]. Of course, after subtracting of the UV divergences in general, some finite parts remain. The finite parts are proportional to the renormalized running coupling constant. At this point it is important to recall that due to the renormalization scale invariance of the EMT operator, the final result must not be altered by $\mathscr{O}(\lambda)$-corrections, as we discussed in Sec. 3.2.

Thus, the tree-level result for the improvement term is the only modification to the free theory result (40) for the D-term. Evaluating the improvement operator at tree-level yields

$$\langle \vec{p}\,'_{\rm free}|\,\hat{\theta}^{\mu\nu}_{\rm improve}(x)\,|\vec{p}_{\rm free}\rangle = 2h\,e^{i(p'-p)x}\left\{\Delta^\mu\Delta^\nu - g^{\mu\nu}\Delta^2\right\}. \tag{47}$$

There is no effect on $A(t)$ which is required by consistency. Since $A(0)=1$ is fixed from general principles and since form factors are constants in free theory (i.e. at tree level), it must be $A(t)=1$ for all t. We already got $A(t)=1$ with no improvement term, see Eq. (40) in Sec. 3.1. The inclusion of the improvement term must not, and does not, spoil this result.

There is, however, an effect on $D(t)$. Interestingly, the improvement term affects $D(t)$. We obtain

$$D_{\rm interacting\ improved} = -1+4\,h = -\frac{1}{3}\,. \tag{48}$$

In very weakly interacting Φ^4 theory, $\lambda \lll 1$, calculations up to one- or two- or three-loop order are certainly more than sufficient. The final result in Eq. (48) follows from the fact that $h=\frac{1}{6}$ in $n=3+1$ dimensions.

This is an unexpected result. Even infinitesimally weak interactions can have a drastic effect on the value of the D-term, as we have seen in Eq. (48). We did our considerations in Φ^4-theory but the conclusions are insightful and of general character, as discussed in the next section.

3.4 Sensitivity of the *D*-term to interactions

Our discussion of the *D*-term in Φ^4 theory gave rise to several important insights which, to the best of our knowledge, have not been discussed in literature before.

First, adding total derivatives to the EMT leaves $P^\mu \equiv \int d^3x T^{0\mu}$ and other Poincaré group generators unaffected, i.e. it does not impact the particle mass or spin. But we see that D in general *is* sensitive to total derivatives, and the *D*-term *is* a physical quantity. This means that in general one *cannot* add total derivatives to the EMT at will, contrary to common belief. If it is necessary to add total derivatives to the EMT, it must be done with due care. We are aware of two examples in quantum field theory where this is indeed needed: *(i)* the Belifante procedure for Dirac particles which renders the canonical EMT a symmetric EMT, see Sec. 2.2; and *(ii)* the Φ^4 theory where an improvement term needs to be added to have a finite renormalized EMT operator, see Sec. 3.3. In such cases it is crucial to establish a unique definition for the improvement term(s). The definition will be dictated by the general properties of the theory. The *D*-term will then be uniquely defined using the unique improvement term(s).

Second, when dealing with a free massive scalar field theory, there is no need for renormalization, and we have no criterion to motivate and uniquely define a specific improvement term. In lack of such a criterion, we conclude that in free scalar theory $D = -1$ as we found in Eq. (40). This is an unambiguous prediction of the free Klein-Gordon theory (minimally coupled to gravity), analogous to the anomalous magnetic moment $g = 2$ predicted from free Dirac theory (minimally coupled to an electromagnetic background field).

Third, in Φ^4 theory we deal with an interacting quantum field theory which has to be renormalized. In this case the unique improvement term (43) ensures that Greens functions with an insertion of the improved EMT are finite. This guarantees the "renormalizability of the combined theory of gravity and matter, with gravity treated to lowest order and the self-interactions of matter [in Φ^4 theory] to all orders" [75]. The inclusion of the improvement term has a drastic effect on the D-term. Assuming even an infinitesimally small coupling constant $\lambda \lll 1$ (such that calculations to three or fewer loops are sufficient) we have $D_{\text{interacting improved}} = -\frac{1}{3}$ instead of the value -1 in the free theory.[1] This clearly demonstrates that the D-term is highly sensitive to the dynamics.

Fourth, the renormalizability of the Φ^4 theory has been studied in weak curved gravitational background fields, and the same improvement term (43) is required [76]. This means that we have $D = -\frac{1}{3}$ in weakly interacting Φ^4 theory also in curved space. As no quantum theory of gravity is known, it is also not known whether the improvement term (43) would ensure renormalizability if quantum gravity effects were included. At this point one might be tempted to think that gravity is far too weak to be of relevance in particle physics. However, we learned that even infinitesimally small interactions in Φ^4 theory can impact the D-term. So why not infinitesimally small gravitational interactions?

Fifth, the D-term emerges to be strongly sensitive to interactions. One must consistently include all forces, perhaps even gravity, to determine the true improve-

[1] For completeness we remark that in the conformally invariant massless free scalar theory, one also has to introduce the improvement term (43) to restore $T^{\mu}{}_{\mu} = m^2\Phi(x)^2 \to 0$ and recover a divergenceless (conserved) conformal current. Thus, in the massless free theory case we also have $D = -\frac{1}{3}$. At this point one may wonder whether the improvement term (43) should not have been added already in the massive free Klein-Gordon theory such that the D-term would exhibit a smooth behavior when m goes to zero. This could certainly be a legitimate step, though there is in general no reason to expect necessarily a smooth behavior of particle properties in a limit such as $m \to 0$. There are also heuristic arguments that support $D = -1$ as the consistent result in the massive free case, see Sec. 4.7.

ment term and the value of the D-term. These issues are beyond the scope of this work as is the question whether a non-trivial Φ^4 theory exists [77].

The above arguments were based on the inspection of the Φ^4 theory which can be solved perturbatively. In the next section we shall discuss an example of a strongly interacting theory, namely QCD.

3.5 D-terms strongly inteacting theory: Goldstone bosons in QCD

In this section we will discuss the D-term in QCD. In general one needs powerful nonperturbative methods to evaluate particle properties in strongly interacting theories, such as lattice QCD. However, there is one notable exception: for pions, kaons and η-meson, the Goldstone bosons of chiral symmetry breaking, one can apply low energy theorems and compute the D-term in the chiral limit.

Chiral symmetry is a symmetry of the QCD Lagrangian associated with the fact that the current masses of the light quark flavors (up, down, strange) are very small compared to the typical energies in the hadronic spectrum. The current masses of up and down quarks are of order few MeV, and the current mass of the strange quark is of order 100 MeV. This is relatively small compared to, for instance, typical masses of baryons which are of order of 1 GeV.

The chiral limit corresponds to the situation that the current masses of light quarks are set to zero. In this limit the Hamiltonian of QCD commutes with the parity operator, and one should observe parity doublets. For example, the nucleon has (by convention) positive parity and should have a negative parity partner of exactly the same mass in the chiral limit. In the real world with non-zero (but small) current masses of the light quarks one should expect the nucleon and its negative parity partner to have approximately the same masses. Instead, the lightest $J^P = \frac{1}{2}^-$ baryon is N(1535) with a Breit-Wigner mass around 1530 MeV (and a Breit-Wigner width of 150 MeV) [4]. Clearly, the chiral symmetry is not realized in the hadronic spectrum.

This situation is referred to as spontaneous breaking of the chiral symmetry. More precisely, in the chiral limit the Lagrangian of QCD has the global symmetry $U(3)_V \otimes U(3)_A$. The symmetry $U(3)_V$ is realized as $U(1)_V \otimes SU(3)_V$ where $U(1)_V$ is associated with the baryon number conservation, and $SU(3)_V$ is realized as an approximate symmetry in the hadronic spectrum with the $J^P = \frac{1}{2}^+$ baryon octet, $J^P = \frac{3}{2}^+$ baryon decuplet, $J^P = 1^-$ meson octet, etc. (This symmetry is approximate because the light quark masses are not zero, and thus are responsible for the observed small mass splittings within the multiplets.)

The symmetry $U(3)_A$ is, however, not realized. The $U(1)_A$ part of this symmetry is explicitly broken by the axial anomaly. The $SU(3)_A$ is spontaneously broken which is accompanied by the emergence of $N^2 - 1$ Goldstone bosons with the quantum number $J^P = 0^-$ corresponding to the quantum numbers of the generators of the broken group. Since there are $N = 3$ flavors, there are the $N^2 - 1 = 8$ Goldstone bosons: π^+, π^0, π^-, K^+, K^0, K^-, $\overline{K}^0$, η. These are the by far lightest hadrons, and would be strictly massless in the chiral limit.

The chiral symmetry imposes strict restrictions on the allowed interactions of the Goldstone bosons. In chiral perturbation theory one explores this fact systematically. The first studies of the EMT of Goldstone bosons were reported already in 1980. The physics of the D-term was not discussed in these works. In the following we will review the results and explore them to compute the D-terms of pions, kaons, and the η-meson.

In Refs. [28, 29] the charmonium decays $\psi' \to J/\psi\,\pi\pi$ were studied. The description of this decay mode of ψ' requires the knowledge of the matrix elements $\langle \pi'\pi|\hat{T}^{\mu\nu}(0)|0\rangle$, or $\langle \pi'|\hat{T}^{\mu\nu}(0)|\pi\rangle$ after applying crossing symmetry.

Let us remark that similar matrix elements enter also the description of the decay of a hypothetical light Higgs boson [78] into two pions. This was discussed at some point in the past in literature when a hypothetical light Higgs boson was not excluded experimentally [79].

Chiral symmetry uniquely determines the interactions of soft pions. In Refs. [28, 29] the following low energy theorem was derived which, in our notation, is given by

$$\langle \pi(\vec{p}\,')|\hat{T}^{\mu\nu}(0)|\pi(\vec{p}\,)\rangle = \frac{1}{2}\biggl(P^{\mu}P^{\nu} - \Delta^{\mu}\Delta^{\nu} + g^{\mu\nu}\Delta^2\biggr) + \mathscr{O}(E^4)\,. \tag{49}$$

Here E is the soft scale associated with the soft momenta of the Goldstone bosons or their masses, i.e. generically $E \sim \mathscr{O}(p, p', m_\pi)$. From (49) we read off (notice the first term on the right-hand side of (49) is already of order E^2)

$$D_h = -1 + \mathscr{O}(E^2)\,, \quad h = \pi,\, K,\, \eta, \tag{50}$$

where we added that the same result is obtained also for kaons and the η-meson. This is a remarkable result. In the soft pion limit chiral symmetry dictates that the form factors of the EMT and the D-term of the light octet mesons coincide (at small values of $-t \sim m_\pi^2 \sim E^2$) with the free-field case in Eq. (40), despite the fact that we are dealing with strongly interacting particles.

Notice that the Goldstone bosons have no internal structure to the considered order in the soft scale in Eqs. (49, 50). The lack of internal structure makes it plausible why the free field value (40) is naturally recovered.

In particular, the results in Eqs. (49, 50) imply that in the chiral limit

$$\lim_{m_h \to 0} D_h = -1\,, \quad h = \pi,\, K,\, \eta\,. \tag{51}$$

This result was derived independently from a soft-pion theorem for pion GPDs in Ref. [11]. At this point one may wonder why no improvement term analogous to (43) was added, which one would expect to be relevant in the massless case, see footnote 1. However, the answer is that such an improvement term is *forbidden* as it violates chiral symmetry [30, 31].

The chiral properties of the EMT form factors $A_i(t)$ and $D_i(t)$ for $i = \pi,\, K,\, \eta$ were studied beyond the chiral limit and evaluated in chiral perturbation theory to order $\mathscr{O}(E^4)$ in Ref. [32]. We quote here only the results for the D-terms [32] which are given by

$$D_\pi = -1 + 16a\frac{m_\pi^2}{F^2} + \frac{m_\pi^2}{F^2}I_\pi - \frac{m_\pi^2}{3F^2}I_\eta + \mathscr{O}(E^4) \tag{52a}$$

$$D_K = -1 + 16a\frac{m_K^2}{F^2} + \frac{2m_K^2}{3F^2}I_\eta + \mathscr{O}(E^4) \tag{52b}$$

$$D_\eta = -1 + 16a\frac{m_\eta^2}{F^2} - \frac{m_\pi^2}{F^2}I_\pi + \frac{8m_K^2}{3F^2}I_K + \frac{4m_\eta^2 - m_\pi^2}{3F^2}I_\eta + \mathscr{O}(E^4) \tag{52c}$$

where

$$a = L_{11}(\mu) - L_{13}(\mu)\,, \quad I_i = \frac{1}{48\pi^2}\left(\log\frac{\mu^2}{m_i^2} - 1\right), \quad i = \pi,\, K,\, \eta\,, \tag{52d}$$

and F denotes the pion decay constant $F \simeq 93\,\text{MeV}$. The expansion parameter in chiral perturbation theory is often associated with the dimensionless ratio $E^2/(4\pi F)^2$ where $(4\pi F)^2 \sim 1\,\text{GeV}^2$. In Eq. (52d) the renormalization scale μ

appears, which is arbitrary because changes in μ are absorbed by appropriate re-definitions of the low energy constants L_{11} and L_{13}. This reflects the fact that the EMT is a renormalization scale invariant operator. Notice also that to the order considered in (52a–52d) which corresponds to $\mathscr{O}(E^6)$ in Eq. (49), the form factors exhibit a t-dependence. The t-dependence signals the fact that Goldstone bosons acquire an internal structure at this order in the chiral expansion.

This allows one to make predictions for the D-terms that are more realistic than the chiral limit prediction (51). The values of the low energy constants were estimated [32] as

$$L_{11}(1\,\text{GeV}) = (1.4\text{–}1.6)\times 10^{-3},$$
$$L_{13}(1\,\text{GeV}) = (0.9\text{–}1.1)\times 10^{-3} \tag{53}$$

using the meson dominance model (for the lower values) and dispersion relation techniques (for the higher values). This yields

$$D_\pi = -0.97\pm 0.01\,, \tag{54}$$
$$D_K = -0.77\pm 0.15\,, \tag{55}$$
$$D_\eta = -0.69\pm 0.19\,, \tag{56}$$

where the uncertainties are due to $\delta L_{11} = \delta L_{13} = 0.2\times 10^{-3}$, the use of the physical value of the pion decay constant $F = 93\,\text{MeV}$ [32] vs chiral limit value $F = 88\,\text{MeV}$ [33], and a heuristic estimate of higher order chiral corrections proportional to $E^4/(4\pi F)^4$ with E the respective meson mass. All these uncertainties are added in quadrature. Chiral interactions modify the soft theorem result $D = -1$, and the modifications are not unexpectedly more sizable for heavier mesons. However, the *D*-terms remain negative.

For completeness we remark that the effects of the electromagnetic interaction on the EMT form factors of charged and neutral pions were investigated in [33]. More recently pion EMT form factors were studied in chiral quark models, where definite predictions for the low energy constants can be made [54]. The quark contribution to pion EMT form factors was also studied in an exploratory lattice QCD study for pion masses in the range $550\,\text{MeV} \leq m_\pi \leq 1090\,\text{MeV}$ for lattice spacings 0.07–0.12 fm and spatial lattice sizes 1.6–2.2 fm [58, 59].

To summarize: for Goldstone bosons of chiral symmetry breaking in QCD one can explore low energy theorems and chiral perturbation theory to predict that $D = -1$ modulo chiral corrections. The corrections are of order $\mathscr{O}(3\,\%)$ for pions, $\mathscr{O}(20\,\%)$ for kaons, and $\mathscr{O}(30\,\%)$ for η-mesons. These corrections make the *D*-term less negative, but do not change its sign.

4 EMT densities of a spin-0 particle

In this section we compute the EMT densities of a point-like particle, and show that they are singular functions but the description is nevertheless consistent. We encounter a manifestation of relativistic corrections and show how they are removed in the heavy mass limit. We then "give by hand" a finite size to the point-like particle, which naturally introduces an additional scale in the theory. This step is necessary to formulate the heavy mass limit in a rigorous way. Even though we "smear out" the point-like particle "by hand" and proceed hereby in a purely heuristic manner, it is remarkable that the description of the EMT densities remains consistent and the property $D = -1$ remains preserved.

The question whether it is possible to "smear out" a point-like particle in a dynamical theory will be addressed in the next section.

4.1 General EMT density formalism for spin-0 particles

The general definition of the static EMT in the Breit frame in Eq. (23) is valid for EMT matrix elements of particles of arbitrary spin [12]. But the formulae in Eqs. (24–26) from Ref. [12] are specific to the spin-$\frac{1}{2}$ case. In this section we derive the explicit expressions for the spin-0 case. This has not been done before in literature, and constitutes a new result which was published in [66].

To derive the explicit expressions for EMT densities of a spin-0 particle we recall that in the Breit frame $P^\mu = (P^0, 0, 0, 0)$ and $\Delta^\mu = (0, \vec{\Delta})$. With this we obtain from Eqs. (14, 23) for the energy density and the stress tensor the results

$$T_{00}(r) = m^2 \int \frac{d^3\Delta}{E(2\pi)^3}\, e^{i\vec{\Delta}\vec{r}} \left[A(t) - \frac{t}{4m^2}(A(t)+D(t))\right] \tag{57}$$

$$T_{ij}(\vec{r}) = \frac{1}{2}\int \frac{d^3\Delta}{2E(2\pi)^3}\, e^{i\vec{\Delta}\vec{r}} \left[\Delta_i\Delta_j - \delta_{ij}\vec{\Delta}^2\right] D(t)\,. \tag{58}$$

The stress tensor $T_{ij}(\vec{r})$ is described in terms of the pressure and shear force distributions, $p(r)$ and $s(r)$, according to Eq. (28). From Eq. (58) we can project out the expressions for the pressure and shear forces, namely

$$p(r) = \frac{1}{3}\int \frac{d^3\Delta}{2E(2\pi)^3}\, e^{i\vec{\Delta}\vec{r}}\, D(t)\left(-\vec{\Delta}^2\right), \tag{59}$$

$$s(r) = \frac{1}{4}\int \frac{d^3\Delta}{2E(2\pi)^3}\, e^{i\vec{\Delta}\vec{r}}\, D(t)\left(-\vec{\Delta}^2 + 3\,(\vec{e}_r\vec{\Delta})^2\right). \tag{60}$$

We choose the coordinates in the Δ-integration such that $\vec{r}$ points along the direction of the Δ_z-axis and define $\vec{e}_r\vec{\Delta} = \cos\theta_\Delta|\vec{\Delta}|$. Recalling that $t = -\vec{\Delta}^2$ in Breit frame we obtain the results

$$p(r) = \frac{1}{3}\int \frac{d^3\Delta}{2E(2\pi)^3}\, e^{i\vec{\Delta}\vec{r}}\, P_0(\cos\theta_\Delta)\Big(t\,D(t)\Big), \tag{61}$$

$$s(r) = \frac{3}{4}\int \frac{d^3\Delta}{2E(2\pi)^3}\, e^{i\vec{\Delta}\vec{r}}\, P_2(\cos\theta_\Delta)\Big(t\,D(t)\Big). \tag{62}$$

The expressions (61, 62) can be further simplified. Using the expansion of a plane wave in spherical Bessel functions and the orthogonality relation of Legendre polynomials,

$$e^{i\vec{\Delta}\vec{r}} = \sum_{l=0}^{\infty} i^l\,(2l+1)\, j_l(|\vec{\Delta}|r)\, P_l(\cos\theta_\Delta)\,, \quad \int_{-1}^{1} dx\, P_l(x)P_k(x) = \frac{2}{2l+1}\,\delta_{lk}\,, \tag{63}$$

yields

$$p(r) = \frac{1}{3}\int \frac{\mathrm{d}^3\Delta}{2E(2\pi)^3}\; j_0(|\vec{\Delta}|r)\Big(t\,D(t)\Big), \tag{64}$$

$$s(r) = -\frac{1}{2}\int \frac{\mathrm{d}^3\Delta}{2E(2\pi)^3}\; j_2(|\vec{\Delta}|r)\Big(t\,D(t)\Big). \tag{65}$$

It is instructive to verify the consistency of these definitions. As a first consistency check we integrate the expression for the energy density in Eq. (57) over the volume

$$\begin{aligned}\int \mathrm{d}^3r\, T_{00}(r) &= m^2\int \mathrm{d}^3r \int \frac{\mathrm{d}^3\Delta}{E(2\pi)^3}\; e^{i\vec{\Delta}\vec{r}}\left[A(t)-\frac{t}{4m^2}(A(t)+D(t))\right]\\ &= m^2\int \frac{\mathrm{d}^3\Delta}{E(2\pi)^3}\left[A(t)-\frac{t}{4m^2}(A(t)+D(t))\right](2\pi)^3\delta^{(3)}(\vec{\Delta})\\ &= \lim_{t\to 0}\frac{m^2}{E}\left[A(t)-\frac{t}{4m^2}(A(t)+D(t))\right] = m \end{aligned} \tag{66}$$

where in the last step we used that $E = m$ for $t = -\vec{\Delta}^2 \to 0$ to yield the desired result. As a second consistency check, we integrate the pressure, as defined in Eq. (59), over the volume. We obtain

$$\begin{aligned}\int \mathrm{d}^3r\, p(r) &= \frac{1}{3}\int \mathrm{d}^3r \int \frac{\mathrm{d}^3\Delta}{2E(2\pi)^3}\; e^{i\vec{\Delta}\vec{r}}\left[t\,D(t)\right]\\ &= \frac{1}{3}\int \frac{\mathrm{d}^3\Delta}{2E(2\pi)^3}\left[t\,D(t)\right](2\pi)^3\delta^{(3)}(\Delta)\\ &= \frac{1}{3}\lim_{t\to 0}\left[\frac{1}{2E}\,t\,D(t)\right] = 0 \end{aligned} \tag{67}$$

which reproduces the von Laue condition (30). As a third consistency test we verify the differential equation (29) connecting the pressure and shear forces. Inserting the expressions (64, 65) into Eq. (29), defining $z = |\vec{\Delta}|r$, recalling that $t = -\vec{\Delta}^2$, and using primes to denote derivatives of a function with respect to its argument, we obtain

$$
\begin{aligned}
&\frac{2}{3}\,\frac{\partial s(r)}{\partial r}+\frac{2s(r)}{r}+\frac{\partial p(r)}{\partial r}\\
&=\int\frac{\mathrm{d}^3\Delta}{2E(2\pi)^3}\left\{\frac{2}{3}\left(-\frac{1}{2}\,j_2'(z)\right)+\frac{2}{z}\left(-\frac{1}{2}\,j_2(z)\right)+\left(\frac{1}{3}\,j_0'(z)\right)\right\}|\vec{\Delta}|\Big[t\,D(t)\Big]\\
&=0 \qquad (68)
\end{aligned}
$$

which vanishes because the expression in the curly brackets is zero due to the identity $j_0'(z)-j_2'(z)-3j_2(z)/z=0$.

4.2 EMT densities of a free spin-0 boson

Let us compute the static EMT densities of a point-like Klein-Gordon particle. In Sec. 3.1 we found for a free spin-0 particle the results $A(t)=1$ and $D(t)=-1$. Inserting these results into the expressions for the energy density, pressure, and shear forces derived in Eqs. (57, 59, 60) yields

$$
\begin{aligned}
T_{00}(\vec{r}) &= m^2\int\frac{\mathrm{d}^3\Delta}{E(2\pi)^3}\,e^{i\vec{\Delta}\vec{r}} = \frac{m^2}{\sqrt{m^2-\vec{\nabla}^2/4}}\;\delta^{(3)}(\vec{r})\,,\\
p(r) &= \frac{1}{3}\int\frac{\mathrm{d}^3\Delta}{2E(2\pi)^3}\,\vec{\Delta}^2\,e^{i\vec{\Delta}\vec{r}} = -\,\frac{\vec{\nabla}^2}{6\,\sqrt{m^2-\vec{\nabla}^2/4}}\;\delta^{(3)}(\vec{r})\,,\\
s(r) &= -\frac{3}{4}\int\frac{\mathrm{d}^3\Delta}{2E(2\pi)^3}\,e^{i\vec{\Delta}\vec{r}}\left((\vec{e}_r\vec{\Delta})^2-\frac{1}{3}\vec{\Delta}^2\right) = \frac{3\,e_r^ie_r^j\,\nabla^i\nabla^j-\vec{\nabla}^2}{8\,\sqrt{m^2-\vec{\nabla}^2/4}}\;\delta^{(3)}(\vec{r})\,.
\end{aligned}
\qquad (69)
$$

It is not surprising that we find the EMT densities of a point-like particle to be given by singular δ-distributions or their derivatives. Notice that in Eq. (69) it is understood that the derivatives act on the δ-functions.

There is an infinite tower of derivatives implicit in the square roots. The infinite tower is a consequence of what is sometimes referred to as "relativistic corrections."

Let us show that, despite these corrections, the expressions are theoretically consistent. For that we assume that the square roots in Eq. (69) can be formally expanded in terms of a series in powers of $\vec{\nabla}^2/(4m^2)$. The derivatives on the δ-functions are handled using

$$\int \mathrm{d}^3 r\, h(\vec{r}) \nabla^i \nabla^j \delta^{(3)}(\vec{r}) = [\nabla^i \nabla^j h(\vec{r})]_{\vec{r}=0} \tag{70}$$

where $h(\vec{r})$ denotes a generic trial function. In the case of the expressions for the mass $m = \int \mathrm{d}^3 r\, T_{00}(r)$ and the von Laue condition $\int \mathrm{d}^3 r\, p(r) = 0$ the trial functions are $h(\vec{r}) = 1$, and we immediately see that $T_{00}(r)$ and $p(r)$ in Eq. (69) satisfy the constraints $m = \int \mathrm{d}^3 r\, T_{00}(r)$ and $\int \mathrm{d}^3 r\, p(r) = 0$.

In order to verify that the two expressions for the D-term in terms of $p(r)$ and $s(r)$ in Eq. (31) both yield the same value $D = -1$, we note that in this case the trial function is $h(\vec{r}) = r^2$. With the identities

$$\nabla^i \nabla^j r^i r^j = 12, \quad \vec{\nabla}^2 r^2 = 6 \tag{71}$$

we see that $D = -\frac{4}{15}\, m \int d^3\vec{r}\, r^2\, s(r)$ and $D = \int d^3\vec{r}\, r^2\, p(r)$ give both the consistent result $D = -1$.

4.3 Relativistic corrections and their remediation in heavy mass limit

The expressions for $T_{00}(r)$, $p(r)$ and $s(r)$ in Eq. (69) can be formally shown to be consistent in the sense described in Sec. 4.2. However, the presence of relativistic corrections artificially mimics an internal structure and leads to unphysical results.

In order to demonstrate this we compute the moments of the energy density, which we define and normalize such that the zeroth moment is unity, the first moment is the mean square radius of $T_{00}(\vec{r})$, etc. With this definition we obtain for the moments of the energy density

$$\begin{aligned} M_k &\equiv \frac{1}{m}\int \mathrm{d}^3r\, r^{2k} T_{00}(\vec{r}) \\ &= \int \mathrm{d}^3r\, r^{2k}\left[\frac{1}{\sqrt{1-\vec{\nabla}^2/(4m^2)}}\,\delta^{(3)}(\vec{r})\right] \\ &= \int \mathrm{d}^3r\, r^{2k}\left[\sum_{j=0}^{\infty} c_j\,(\vec{\nabla}^2)^j\,\delta^{(3)}(\vec{r})\right], \end{aligned} \tag{72}$$

with

$$c_j = (2j-1)!!/[(4m^2)^j\, 2^j\, j!] \tag{73}$$

where $(-1)!! = 1!! = 1$ and $(2j+1)!! = 1\cdot 3\cdot \ldots \cdot (2j-1)\cdot(2j+1)$ for $j > 1$. Performing $2j$ partial integrations in each term of the sum over j and using $[(\vec{\nabla}^2)^j\, r^{2k}]_{r=0} = (2k+1)!\,\delta_{jk}$ in Eq. (72) yields

$$M_k = \frac{(2k+1)!!\,(2k-1)!!}{(4m^2)^k}\,. \tag{74}$$

Let's recall that for a point-like particle one naturally expects $M_k = \delta_{k0}$ and that $M_k \neq 0$ for $k > 0$ would imply an extended structure. In fact, for $k = 1$ we obtain $M_1 = \langle r_E^2\rangle = \frac{3}{4m^2} \neq 0$ as a mean square radius of a point-like(!) particle.

This is an unphysical consequence of relativistic corrections, which is a general limitation of the interpretation of 3D-Fourier transforms of form factors as 3D-densities. The presence of such relativistic corrections has been a "steady companion" of the 3D density formalism since the earliest days, and was already described by Sachs in the context of the electromagnetic 3D densities [24]. A careful discussion can be found in the review article [2], and we will revisit this point in Sec. 4.4 in more detail. It is important to notice that the relativistic corrections set limitations for the *interpretation* of the results, but formally, all theoretical results remain correct and consistent as we have shown earlier in this section.

In order to "switch off" the undesired relativistic corrections and recover well-defined 3D-densities consistent with the notion of a point-like particle, let us assume from now on that we work in the heavy mass limit $m \to \infty$, and retain only the respectively leading terms. Such a description would in principle apply to the (free) Higgs boson, which is the only presently known fundamental spin zero particle. In this way we obtain for a heavy boson

$$\begin{aligned} T_{00}(\vec{r}\,) &= m\,\delta^{(3)}(\vec{r}\,)\,, \\ p(r) &= -\,\frac{\vec{\nabla}^2}{6m}\,\delta^{(3)}(\vec{r}\,)\,, \\ s(r) &= \frac{3\,e_r^i e_r^j\,\nabla^i\nabla^j - \vec{\nabla}^2}{8\,m}\,\delta^{(3)}(\vec{r}\,)\,. \end{aligned} \tag{75}$$

The expressions in (75) are consistent and satisfy the requirements that they lead to the correct results for the mass m, von Laue condition, and the D-term D.

The value m is obviously reproduced from $\int d^3r\,T_{00}(r)$. To show that the von Laue condition (30) is satisfied we explore the contracted version of Eq. (70) with the trial function $h(\vec{r}) = 1$. Finally, the consistent result $D = -1$ for the D-term follows from the two representations in terms of $p(r)$ and $s(r)$ in Eq. (31) by exploring once again the identities in Eqs. (70, 71). Most importantly, the moments of the energy density defined in Eq. (72) satisfy $M_k = \delta_{k0}$. In particular the mean square radius of the energy density $M_1 = \langle r_E^2 \rangle = 0$ as expected for a point like particle. We recall that we have always the density $T^{0k}(\vec{r}) = 0$ for finite m and in the heavy mass limit, which reflects the spin-0 property of our boson.

Thus, the EMT densities in the heavy mass limit defined in Eq. (75) yield a theoretically consistent and physically correct description of a spin-0 boson.

4.4 Heuristic introduction of a particle structure

The crucial question however is: is it legitimate to take the heavy mass limit? This question is ill-posed in a free theory where the only dimensionful parameter is the mass m of the free particle, and hence, the only available length scale is the Compton wave-length of the particle $\lambda_C = 1/m$. The heavy mass limit cannot be formulated, unless we specify with respect to what the mass m is large.

To make the heavy mass limit well-defined, we will introduce a finite size and hence "some internal structure" to our heavy boson. To take into consideration the effects of an internal structure, we proceed here heuristically[2] and replace the δ-functions in the expressions (75) with suitably smeared-out regular and normalized functions $f(r)$,

$$\delta^{(3)}(\vec{r}) \to f(r)\,, \quad I_0 \equiv \int \mathrm{d}^3 r\, f(r) = 1\,. \tag{76}$$

It is understood that $f(r)$ reduces to a δ-function in some well-defined limit.

[2] We postpone here the question how to describe such an "internal structure" in terms of a microscopic dynamical Lagrangian theory. This question will be addressed later.

Let us investigate the effect of such an internal structure on the energy density. We choose (at this point for purely illustrative purposes, cf. footnote 2) the following representation $f_R(r)$ for the δ-function

$$f_R(r) = \frac{1}{\pi^{3/2}R^3}\exp\left(-\frac{r^2}{R^2}\right). \tag{77}$$

We have obviously $\int d^3r\, f_R(r) = 1$ and we recover $f_R(r) \to \delta^{(3)}(\vec{r})$ for $R \to 0$. In the heavy mass limit using the densities in Eq. (75) the "true" first moment of the energy distribution M_1, i.e. mean square radius of the energy density, is given by

$$\langle r_E^2 \rangle \equiv M_1 = \frac{3}{2}R^2\,. \tag{78}$$

Having a specific "(toy) model" for the energy density, we can equally well evaluate the mean square radius of $T_{00}(r)$ using the expression (69) where the relativistic corrections are still present. The result we obtain and the condition required for the interpretation in terms of 3D densities to be applicable are

$$\langle r_E^2 \rangle \equiv M_1 = \frac{3R^2}{2}\left(1+\delta_{\rm rel}\right)\,, \quad \delta_{\rm rel} \equiv \frac{1}{2m^2R^2} \ll 1\,. \tag{79}$$

Thus relativistic corrections $\delta_{\rm rel}$ is negligible when $m^2R^2 \gg 1$, i.e. when the Compton wave-length is small compared to the "size" of the particle: $\lambda_C^2 \ll R^2$. We obtain this condition here in the context of the mean square radius of the energy density, but it can be derived from general considerations [2].

It is instructive to estimate the size of the corrections as defined in Eq. (79) for various particles, see Table 2. For light mesons, like pions, kaons or η the concept of 3D-densities is clearly not applicable. However, for heavier mesons containing charmed quarks the concept makes sense: e.g. for η_c the relativistic corrections are of the order of $\mathcal{O}(4\,\%)$. For nuclei the concept can be safely applied: for instance for ^{4}He, the lightest spin-0 nucleus, the corrections are merely of the order of $\mathcal{O}(0.05\,\%)$ and they diminish quickly for heavier nuclei. This can be understood in the liquid drop model of the nucleus, cf. Sec. 2.10, where a nucleus

particle	J^π	mass [GeV]	size [fm]	$\delta_{\rm rel}$
pion	0^-	0.14	0.67	2.2
kaon	0^-	0.49	0.56	2.5×10^{-1}
η-meson	0^-	0.55	0.68	1.4×10^{-1}
η_c-meson	0^-	2.98	0.26	3.8×10^{-2}
proton	$\frac{1}{2}^+$	0.94	0.89	2.8×10^{-2}
deuteron	1^+	1.88	2.14	1.2×10^{-3}
^{6}Li	1^+	5.60	2.59	9.3×10^{-5}
^{4}He	0^+	3.73	1.68	5.0×10^{-4}
^{12}C	0^+	11.2	2.47	2.6×10^{-5}
^{20}Ne	0^+	18.6	3.01	6.2×10^{-6}
^{32}S	0^+	29.8	3.26	2.1×10^{-6}
^{56}Fe	0^+	52.1	3.74	5.1×10^{-7}
^{132}Xe	0^+	123	4.79	5.6×10^{-8}
^{208}Pb	0^+	194	5.50	1.7×10^{-8}
^{244}Pu	0^+	227	5.89	1.1×10^{-8}

Table 2: Masses, radii, and the sizes of the relativistic corrections $\delta_{\rm rel}$ as defined in Eq. (79) for various spin-0 mesons and for nuclei. The proton, deuteron, ^{6}Li are included for comparison. Masses and mean charge radii of mesons and the proton are from [4] except for the radii of η taken from the estimate [80] and η_c taken from the lattice calculation [81]. Nuclear masses are from [82] and nuclear mean charge radii from [83]. The smaller $\delta_{\rm rel}$ the more safely it is to apply the 3D-density interpretation of form factors.

with mass number A has approximately the mass $\sim A \times 0.93\,\text{GeV}$ and the radius $\sim A^{1/3} \times 1.2\,\text{fm}$ which yields $\delta_{\rm rel} \sim 1.2A^{-8/3}$.

Although they are not spin-0 particles, we have included the proton, deuteron and ^{6}Li in Table 2 for comparison. The concept of 3D-densities is applicable with a reasonable accuracy of the order of $\mathscr{O}(3\,\%)$ even for proton with much smaller corrections for deuteron and ^{6}Li.

As a side remark we notice that in Table 2 the spin-0 mesons have negative parity and the spin-0 nuclei have positive parity, which is the natural J^P assignment for ground states in both systems. This is because mesons are "made of" a $q\bar{q}$-pair, i.e. a fermion and an antifermion which have opposite intrinsic parities. Spin-0 nuclei are made of an even number of protons and an even number of neutrons, which are all fermions and have the same (by convention positive) intrinsic parity. Our discussion of the EMT densities of spin-0 particles applies equally to scalar ($J^P = 0^+$) and pseudoscalar ($J^P = 0^-$) particles.

4.5 Remark on pion "charge radius" and 2D densities

Notice that it is customary to speak about mean square charge radii also for particles like (charged) pions and kaons, even though the concept of 3D-densities cannot be applied for them: the relativistic corrections are $\delta_{\rm rel} = 220\,\%$ for pions and $25\,\%$ for kaons, see Table 2. These "radii" are simply defined by the slopes of the electric form factors. For instance for the pion one defines

$$F_\pi(t) = 1 + \frac{\langle r^2_{\pi,em}\rangle}{6}\,t + \mathscr{O}(t^2), \quad \text{or} \quad \langle r^2_{\pi,em}\rangle = 6F'_\pi(t)\Big|_{t=0}. \tag{80}$$

It is important to remark that it is possible to introduce the concept of the "spatial structure" and "size" for particles of any mass, including light hadrons like pions and kaons, *without* relativistic corrections by working with 2D densities [84, 85, 86, 87, 88]. In that approach the 2D-radius of the particle is still related to the slope of the form factor, but now as $F_\pi(t) = 1 + \frac{1}{4}\langle r^2_{\pi,em,2D}\rangle\, t + \mathcal{O}(t^2)$ (in 3D and in 2D the numerical prefactor is $1/(2d_{\rm space})$ with $d_{\rm space}$ the number of space dimensions in the Fourier-transform).

But the concepts of pressures, shear forces and mechanical stability are inherently 3D. No interpretation exists for the stress tensor in terms of 2D densities. If we wish to learn about the mechanical stability of nucleons and nuclei, we have to pay a prize for that and deal with 3D densities and accept relativistic corrections. However, the relativistic corrections of the 3D densities of the nucleon, about 3 % if we quantify them with the quantity $\delta_{\rm rel}$ seem acceptably small to carry on this program.

It is important to stress the different objectives of the 2D- vs 3D-density interpretations. The 2D-density description is exact and this is indispensable for a rigorous *probabilistic* partonic interpretation. The 3D-density description does not describe partonic probability densities. In our context it describes mechanical response functions of a system. Such response functions are not probability densities but *correlation functions*. As we do not deal with a rigorous probabilistic interpretations, encountering reasonably small relativistic corrections is acceptable. In view of the valuable insights on the internal dynamics obtained in this way, it seems worth it to pay this prize.

Nevertheless it is important to keep in mind the presence of relativistic corrections and the resulting limitations in the 3D density formalism.

4.6 Properties of a "smeared out" point-like boson

Now we investigate the stress tensor of a point-like (heavy) boson which has been given some "internal structure" as described in Sec. 4.4. We continue proceeding heuristically, see Footnote 2, and replace the δ-function in the expressions for $p(r)$ and $s(r)$ in Eq. (75) with a suitable regular normalized function $f(r)$ as given in Eq. (76). We shall assume that $f(r)$ has the following properties:

- (a) it is a radially symmetric function of $\vec{r}$,
- (b) it is at least three times continuously differentiable,
- (c) it satisfies $r^3 f''(r) \to 0$ and $r^2 f'(r) \to 0$ for $r \to 0$, and
- (d) it vanishes at large distances faster than any power of r.

These restrictions will be convenient in the following, even though some of them could be relaxed (e.g. a large-r behavior $\propto 1/r^5$ would be sufficient in all physically relevant situations [25] including the chiral limit).

From Eq. (75) we obtain the results

$$
\begin{aligned}
p(r) &= -\frac{1}{6m}\left(f''(r) + \frac{2}{r} f'(r)\right), \\
s(r) &= \frac{1}{4m}\left(f''(r) - \frac{1}{r} f'(r)\right),
\end{aligned} \tag{81}
$$

where the primes denote derivatives with respect to the argument. It is important that in Eq. (81) we use the same function $f(r)$ in the expressions for $s(r)$ and $p(r)$. This is dictated by the conservation of the EMT, which imposes the differential equation (29). In fact, the relation (29) holds exactly

$$\frac{2}{3}\,\frac{\partial s(r)}{\partial r}+\frac{2s(r)}{r}+\frac{\partial p(r)}{\partial r}=\frac{2}{3}\left(\frac{f'''(r)}{4m}-\frac{f''(r)}{4mr}+\frac{f'(r)}{4mr^2}\right)$$
$$+\frac{2}{r}\left(\frac{f''(r)}{4m}-\frac{f'(r)}{4mr}\right)+\left(-\frac{f'''(r)}{6m}-\frac{2\,f''(r)}{6mr}+\frac{2\,f'(r)}{6mr^2}\right)=0 \qquad (82)$$

for every function $f(r)$ which satisfies the properties a–c. Only here we need that $f(r)$ is 3 times continuously differentiable. For all other purposes 2 times continuously differentiable would be sufficient.

Since Eq. (82) holds for the extended particle and since it is equivalent to the conservation of the EMT, it is clear that all other properties related to the conservation of the EMT are also satisfied.

Let us show this explicitly. The von Laue condition (30) is

$$\int_0^\infty \mathrm{d}r\, r^2 p(r)=\frac{1}{6m}\int_0^\infty \mathrm{d}r\left(r^2 f''(r)+2r\,f'(r)\right)=\frac{1}{6m}\int_0^\infty \mathrm{d}r\,\frac{\partial}{\partial r}\left(r^2 f'(r)\right)=0 \quad (83)$$

for every function $f(r)$ which satisfies the properties a–d. This proves Eq. (30). Finally, for the D-term we obtain in Eq. (29) from the pressure

$$\begin{aligned} D &= m\int \mathrm{d}^3r\, r^2\, p(r) \\ &= -\,4\pi\int_0^\infty \mathrm{d}r\left(r^4\frac{f''(r)}{6}+r^3\frac{f'(r)}{3}\right)=-\left(\frac{4\cdot 3\,I_0}{6}-\frac{3\,I_0}{3}\right)=-\,1\,. \end{aligned} \qquad (84)$$

Using in Eq. (29) the expression in terms of shear forces yields

$$D = -\frac{4m}{15}\int \mathrm{d}^3r\, r^2\, s(r)$$
$$= -4\pi\int_0^\infty \mathrm{d}r\left(r^4\frac{f''(r)}{15} - r^3\,\frac{f'(r)}{15}\right) = -\left(\frac{4\cdot 3\, I_0}{15} + \frac{3\, I_0}{15}\right) = -1\,. \tag{85}$$

In Eqs. (84, 85) we performed respectively one or two partial integrations to express the final results in terms of the integral I_0 introduced in Eq. (76). The conclusion is that the property $D = -1$ holds also for an extended boson, and this is guaranteed by the normalization of the function $f(r)$ in Eq. (76).

We remark that the same regular function $f(r)$ must be chosen to smear out the δ-functions in the expressions for $p(r)$ and $s(r)$ in Eq. (75), because they are connected by Eqs. (29, 82). But one could choose a different $f(r)$ for $T_{00}(r)$. At this point of our considerations, $T_{00}(r)$ is unrelated to $p(r)$ and $s(r)$. This is of course an unphysical feature. The expressions for all EMT densities should all be derived from a Lagrangian of a dynamical theory. A non-trivial question is whether it is possible to construct a dynamical theory in which a "smeared out point-like boson" has a D-term $D = -1$ and its EMT densities reproduce a "point-like particle" in a certain well-defined limit.

Before addressing this question in the next section, we visualize the EMT densities of such an "extended particle." For purely illustrative purposes, we choose the representation $f_R(r)$ for the δ-function defined in Eq. (77). The results are shown in Fig. 3. It is remarkable, that in this way we effortlessly (by smearing out a point-like particle without invoking dynamics) recover the main features of the EMT densities calculated non-perturbatively in dynamical theories of Q-balls

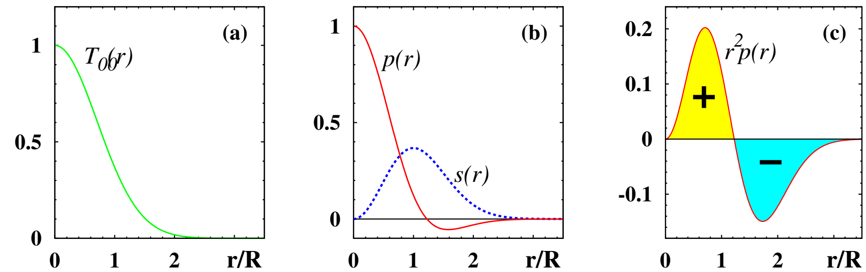


Figure 3: (a) The energy density $T_{00}(r)$ in units of $T_{00}(0)$, and (b) $s(r)$ and $p(r)$ in units of $p_0 = p(0)$ as functions of r of a "point-like" particle in Eq. (75) with the δ-functions "smeared out" in the Gaussian representation (77) (which defines the unit R). (c) The result for $r^2 p(r)$, with the pressure from panel (b), which visualizes how the von Laue condition (30) is realized. In the limit $R \to 0$ (where $T_{00}(0) \to \infty$ and $p_0 \to \infty$) one recovers a point-like particle.

[61, 62, 63], chiral solitons [25, 39], or Skyrmions [40, 41, 42, 43].

4.7 $D = -1$ of heavy bosons from consistency arguments

It is instructive to "rederive" the result $D = -1$ of a free point-like particle using the concept of 3D-densities and consistency considerations. We start with two natural assumptions: (i) the EMT form factors of a free point-like particle are constant, (ii) the energy density of a heavy point-like spin-0 boson must be given by $T_{00}(r) = m\,\delta^{(3)}(\vec{r})$ in Eq. (75).

The constraint $A(0) = 1$ in (16) immediately implies with assumption (i) that $A(t) = 1$ for all t. By the same argument $D(t) = D$ is of course also t-independent, but its value is apriori not known. To determine the value of D we use assumption (ii) which implies that the square bracket in the expression for $T_{00}(r)$ in Eq. (57)

must be a constant,

$$T_{00}(r) = m^2 \int \frac{d^3\Delta}{E(2\pi)^3}\, e^{i\vec{\Delta}\vec{r}}\, \underbrace{\left[A(t) - \frac{t}{4m^2}(A(t)+D(t))\right]}_{=\text{const}}.$$

Clearly, we will recover the desired result if and only if $A(t)+D(t)=0$. As we already established that $A(t)=1$ these considerations immediately lead us to the conclusion that $D(t)=-1$, and in particular

$$D=-1$$

for a point-like heavy particle. Notice that we can obtain this conclusion only from exploring $T_{00}(r)$. Considerations of the stress tensor (58) would not constrain the value of D. In this way and in the abstract mathematical meaning of a point-like particle, we recover $D=-1$ for a free heavy point-like spin-0 particle as a consistency condition of the 3D-density description.

The above arguments do not apply to the massless case discussed in footnote 1 simply because the concepts of 3D densities require a massive particle. These arguments also do not apply to e.g. the Φ^4-theory where the bosons are not free. This explains why one naturally finds $D \neq 1$ in these theories. For Goldstone bosons of chiral symmetry breaking we do have $D=-1$ in the soft pion limit, but this cannot be “explained” in the above way: in chiral limit the Goldstone bosons are massless, and 3D-density concepts are not applicable. The result $D=-1$ for Goldstone bosons is a non-trivial consequence of chiral symmetry breaking and soft pion theorems.

5 Realization of a smeared out spin-0 particle in the Q-ball framework

In the previous section we have shown that the free-theory result $D = -1$ persists even if the point-like spin-0 boson is given an "extended structure." Thereby we "introduced" the internal structure in a heuristic way. The following question emerges: is it possible to construct a microscopic dynamical theory in which a spin-0 particle (a) exhibits an extended structure, (b) its EMT densities reduce to those of a point-like particle in a certain limit, and (c) the property $D = -1$ holds? The answer is yes. In this section we will present one such theory which can be formulated in the theory of Q-balls.

5.1 The Q-ball framework

Q-balls are non-topological solitons in theories with global symmetries and were proposed by Coleman in Ref. [89]. Q-balls have been subject to considerable interest in the literature, because they might have formed in the early universe and are considered to be dark matter candidates. Here we use Q-balls to gain insights on the D-term as an example of a strongly interacting theory of scalar particles [89], see also [90, 91]. In this section we briefly review the Q-ball theory and results on EMT densities and the D-term from [61, 62, 63].

Q-balls are solitons in scalar theories with a global symmetry where a "suitable potential" satisfies certain conditions. What suitable potential means will be stated below. The theory can be formulated in terms of one complex scalar field, or equivalently in terms of two real scalar fields which we shall choose to do here.

The Lagrangian and the equations of motion are given by

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \Phi_1)(\partial^\mu \Phi_1) + \frac{1}{2}(\partial_\mu \Phi_2)(\partial^\mu \Phi_2) - V\,, \quad \left(\Box + \frac{\partial V}{\partial \Phi_i}\right)\Phi_i(x) = 0\,, \quad i = 1, 2. \tag{86}$$

The potential V is such that the theory is invariant under global continuous SO(2) symmetry transformations ($\beta \in \mathbb{R}$)

$$\begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} \to \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}. \tag{87}$$

The global symmetry implies the existence of a conserved Noether current $J^\mu = \Phi_1 \partial^\mu \Phi_2 - \Phi_2 \partial^\mu \Phi_1$. The associated conserved charge $Q = \int \mathrm{d}^3x J^0(x)$ is instrumental for the existence of the soliton solutions.

The soliton solutions are given, in their rest frames, by the expression

$$\begin{pmatrix} \Phi_1(\vec{x},t) \\ \Phi_2(\vec{x},t) \end{pmatrix} = \begin{pmatrix} \cos(\omega t) \\ \sin(\omega t) \end{pmatrix} \phi(r)\,, \tag{88}$$

where $r = |\vec{x}|$ and ω is bound by $\omega_{\text{min}}^2 < \omega^2 < \omega_{\text{max}}^2$. The limiting frequencies are defined in terms of the properties of the potential V, with V understood as a function of the radial field $\phi(r)$, as follows:

$$0 < \omega_{\text{min}}^2 \equiv \min_\phi \left[\frac{2V(\phi)}{\phi^2}\right] < \omega_{\text{max}}^2 = V''(\phi)\Big|_{\phi=0}. \tag{89}$$

Notice that $m = \omega_{\text{max}}$ defines the mass of the elementary quanta of the fields Φ_1

and Φ_2. The solutions satisfying (not satisfying) the equivalent conditions

$$\frac{\mathrm{d}}{\mathrm{d}\omega}\left(\frac{M}{Q}\right) \geq 0 \quad \Leftrightarrow \quad \frac{\mathrm{d}Q}{\mathrm{d}\omega} \leq 0 \quad \Leftrightarrow \quad \frac{\mathrm{d}^2 M}{\mathrm{d}Q^2} \leq 0\,, \tag{90}$$

are stable (unstable) with respect to small fluctuations [90, 91]. The point where the inequalities in (90) become equalities defines the critical frequency ω_c, i.e. for instance $Q'(\omega) = 0$ at $\omega = \omega_c$. The solutions are absolutely stable if $M < mQ$ where m denotes the mass of the elementary fields [91].

5.2 EMT densities and properties of Q-balls

To make this work self-contained, we review here the results for the EMT densities derived in Ref. [61]. From the Euler-Lagrange equations for the fields $\Phi_1(t,\vec{x})$ and $\Phi_2(t,\vec{x})$ in Eq. (86) one finds the following equation for the radial field $\phi(r)$ which describes the Q-ball solution defined in Eq. (88),

$$\begin{aligned} &\phi''(r) + \frac{2}{r}\,\phi'(r) + \omega^2\phi(r) - V'(\phi) = 0\,, \\ &\phi(0) \equiv \phi_0 \neq 0\,, \quad \phi'(0) = 0\,, \quad \phi(r) \to 0 \;\text{ for }\; r \to \infty\,. \end{aligned} \tag{91}$$

Here and in the following primes denote differentiation with respect to the argument. The Noether charge imposed by the SO(2) global symmetry is given by

$$Q = \int \mathrm{d}^3x\, \rho_{\rm ch}(r)\,, \quad \rho_{\rm ch}(r) = \omega\, \phi(r)^2\,, \tag{92}$$

whose sign is determined by ω. Below we choose $\omega > 0$ without loss of generality.

The EMT densities read

$$
\begin{aligned}
T_{00}(r) &= \frac{1}{2}\,\omega^2\phi(r)^2+\frac{1}{2}\,\phi'(r)^2+V, &(93)\\
p(r) &= \frac{1}{2}\,\omega^2\phi(r)^2-\frac{1}{6}\,\phi'(r)^2-V, &(94)\\
s(r) &= \phi'(r)^2. &(95)
\end{aligned}
$$

The Q-ball densities satisfy the relation

$$T_{00}(r)+p(r)=\omega\,\rho_{\rm ch}(r)+\frac{1}{3}\,s(r)\,, \tag{96}$$

which implies the interesting Q-ball specific relation

$$D=\frac{4}{9}\left(\omega\,Q\,M\,\langle r_Q^2\rangle-M^2\,\langle r_E^2\rangle\right), \tag{97}$$

with the Q-ball mass $M=\int d^3x\,T_{00}(r)$ and the mean square radii of energy and charge densities defined as

$$\langle r_E^2\rangle=\frac{1}{M}\int d^3x\,r^2\,T_{00}(r)\,,\;\;\langle r_Q^2\rangle=\frac{1}{Q}\int d^3x\,r^2\,\rho_{\rm ch}(r)\,. \tag{98}$$

In the Q-ball system a rigorous general proof was formulated that $D<0$ for every suitable potential [61]. It was also shown that the numerical values of the D-terms can span orders of magnitude. For that the suitable, often studied (non-renormalizable, effective) theory was used with the sextic potential $V_6=A\,\phi^2-B\,\phi^4+C\,\phi^6$ with $\phi^2=\Phi_1^2+\Phi_2^2$ and positive A,B,C satisfying $0<\zeta\equiv B^2/(4AC)<1$ [89]. For this potential $\omega^2_{\rm min}=2A(1-\zeta)$ and $\omega^2_{\rm max}=2A$. For the parameters $A=1.1,\ B=2.0,\ C=1.0$ it was found $|D|\geq|D_{\rm smallest}|$ with $D_{\rm smallest}=-113.4$

at a frequency ω which was numerically close to the critical frequency $\omega_c = 1.38$ [61]. For ω not in the vicinity of ω_c the D-terms are becoming quickly more and more negative.

In the "Q-ball limit" $\varepsilon_{\rm min} \equiv \sqrt{\omega^2 - \omega_{\rm min}^2} \to 0$ one deals with absolutely stable well-localized solitons [89] characterized by constant charge density in their interior, whose sizes grow as $\varepsilon_{\rm min}^{-4}$, and the masses and charges grow as $\varepsilon_{\rm min}^{-6}$. The most spectacular growth, however, is exhibited by the D-term which behaves as $D \propto \varepsilon_{\rm min}^{-14}$ in this limit [61].

In the opposite "Q-cloud limit" $\varepsilon_{\rm max} \equiv \sqrt{\omega_{\rm max}^2 - \omega^2} \to 0$ [92] the solutions become infinitely dilute, diffuse and disintegrate into a cloud of free Q-quanta. In this limit mass, charge, and mean radii of the solutions diverge as $\varepsilon_{\rm max}^{-1}$. Again, the D-term is the property exhibiting the strongest pattern of divergence with $D \propto \varepsilon_{\rm max}^{-2}$ [63]. Interestingly, in the Q-cloud limit the sextic term in V_6 becomes irrelevant (in the sense of critical phenomena), and after a suitable rescaling one deals with a (complex) Φ^4 theory continued analytically to a negative coupling constant λ [63].

Q-balls can have also excited states (all with spin zero and positive parity as the ground state) which are unstable and have also negative D-terms. The solution $\phi(r)$ of the $N^{\rm th}$ excitation exhibits N nodes (ground state has no node). For a fixed frequency ω the mass and charge of the $N^{\rm th}$ excitation scale as N^3, while the D-term scales as N^8 [62].

The Q-ball system confirms that D-terms of spin-0 particles can deviate significantly from the free-field theory result $D = -1$ though the negative sign of the D-term is preserved. The Q-ball results also strongly support the observation that the D-term is the particle property which is most sensitive to the details of the dynamics of a theory.

5.3 Q-balls with log potential

In order to find a microscopic theory of "smeared out" elementary particles, we consider Q-balls in the logarithmic potential (recall that for the soliton solutions $\Phi_1(\vec{x},t)^2 + \Phi_2(\vec{x},t)^2 = \phi(r)^2$ holds)

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \Phi_1)(\partial^\mu \Phi_1) + \frac{1}{2}(\partial_\mu \Phi_2)(\partial^\mu \Phi_2) - V_{\log}\,, \quad V_{\log} = A\,\phi^2 - B\,\phi^2 \log\left(C\,\phi^2\right). \tag{99}$$

This potential is not bound from below, and it is understood as the limiting case of a well-defined theory. We will discuss this in detail in Sec. 5.6. Notice that actually two parameters are sufficient to define this theory, because we can replace $C \to 1/B$ and $A \to A - B\log(AC)$ without loss of generality which we shall do from now on. For this potential, the equations of motion read

$$\phi''(r) + \frac{2}{r}\,\phi'(r) + \left(\omega^2 - 2A + 2B\right)\phi(r) + 2B\,\phi(r)\,\log\left(\frac{\phi(r)^2}{B}\right) = 0\,. \tag{100}$$

The solution to this differential equation satisfying the boundary conditions (91) can be found analytically

$$\phi(r) = \phi_0\,\exp\left(-B\,r^2\right), \quad \phi_0 = \sqrt{B}\,\exp\left(\frac{2A + 4B - \omega^2}{4B}\right). \tag{101}$$

With the solution (101) all Q-ball properties can be evaluated analytically.

In particular, we obtain for the densities

$$\begin{aligned}
T_{00}(r) &= (\omega^2 - 2B + 4B^2 r^2)\,\phi(r)^2, \qquad (102)\\
p(r) &= (2B - \frac{8}{3}B^2 r^2)\,\phi(r)^2\,, \qquad (103)\\
s(r) &= 4B^2 r^2\,\phi(r)^2\,, \qquad (104)\\
\rho_{\rm ch}(r) &= \omega\,\phi(r)^2\,. \qquad (105)
\end{aligned}$$

The expressions for $s(r)$ and $p(r)$ satisfy the general differential equation (29), $p(r)$ satisfies the von Laue condition (30), and all densities comply with the Q-ball specific relation (96). For the global Q-ball properties we obtain

$$\begin{aligned}
Q &= N_0\,\omega\,,\\
M &= N_0\,(B+\omega^2)\,,\\
D &= -N_0^2\,(B+\omega^2)\,, \qquad (106)
\end{aligned}$$

where we defined the constant

$$N_0 \equiv \phi_0^2\left(\frac{\pi}{2B}\right)^{3/2}.$$

It is important to stress that the same result for D follows in 3 different ways, from Eqs. (31a, 31b) and (97). The expressions for the mean square radii for the energy density and charge density are given by

$$\begin{aligned}
\langle r_E^2\rangle &= \frac{3}{4B}\,\frac{3B+\omega^2}{B+\omega^2}\,,\\
\langle r_Q^2\rangle &= \frac{3}{4B}\,. \qquad (107)
\end{aligned}$$

The parameters A and B cannot be chosen arbitrarily. Especially the parameter B has to comply with certain requirements. The conditions (32b, 90, 32a) imply (in this order):

$$\frac{2}{3}s(r) + p(r) = 2B\phi(r)^2 \geq 0 \quad \Leftrightarrow \quad B \geq 0\,, \tag{108}$$

$$\frac{\mathrm{d}}{\mathrm{d}\omega}\left(\frac{M}{Q}\right) = \frac{\mathrm{d}}{\mathrm{d}\omega}\left(\omega + \frac{B}{\omega}\right) \geq 0 \quad \Leftrightarrow \quad \omega^2 \geq B\,, \tag{109}$$

$$T_{00}(r) = (\omega^2 - 2B + 4B^2 r^2)\,\phi(r)^2 \geq 0 \quad \Leftrightarrow \quad \omega^2 \geq 2B\,. \tag{110}$$

All conditions are satisfied and the solutions classically stable if $2\omega^2 \geq B > 0$. We exclude $B = 0$ in (108) as it reproduces free theory.

For the limiting value $\omega^2 = 2B$ the energy density vanishes in the center, which is a feature not observed so far in the Q-ball literature to the best of our knowledge. For $2B < \omega^2 < 4B$ the energy density exhibits a dip in the center. Such dips occur when the “surface tension” of the Q-matter is strong enough to produce a peak in $T_{00}(r)$ at the “edge” of the Q-ball [61]. Finally, for $\omega^2 \geq 4B$ we have a $T_{00}(r)$ which has no dip and is monotonically decreasing for all r.

Notice that the parameter A is completely unconstrained. We can choose $\sqrt{B}$ to serve as the unit of mass in our theory, and $1/\sqrt{B}$ as the length unit. Then the role of A is to provide an overall rescaling of the fields by the factor $\exp(\frac{1}{2}AB^{-1})$, as can be seen from (101). This implies a corresponding rescaling of the properties in (106) via $N_0 \propto \exp(AB^{-1})$. Thus, at this point we may assume for A any value. However, in Sec. 5.6 we will see that certain restrictions exist for A.

5.4 Logarithmic Q-balls with $D = -1$

Now we discuss how the parameters can be fixed such that $D = -1$. First let us notice that in general we have for our logarithmic Q-balls the relation

$$\frac{(-D)}{Q^2} = 1 + \frac{B}{\omega^2} > 1\,, \tag{111}$$

where the inequality arises from $2\omega^2 \geq B$ and $B > 0$. Clearly, the parameters can be chosen such that either $D = -1$ or $Q = 1$ but not both simultaneously (unless one considered a limit like $\omega \to \infty$ for fixed B). Notice that Q is a conserved but not a topological quantum number and not required to be an integer. Q also does not need to correspond in general to the electric charge. Thus, there is no principle obstacle to have $D = -1$ and $Q \neq 1$ (in our conventions). Notice that, if we wished to do it, we could simply redefine the unit in which the charges are measured to have integer-valued charges.

Notice that similarly $M^2 = (-D)\,(\omega^2 + B)$ holds, implying the nice result $M = \sqrt{\omega^2 + B}$ if we manage to adjust the parameters such that $D = -1$ holds.

To obtain the desired value for the D-term $D = -1$ we may fix A and ω as follows. We define a positive parameter α which is arbitrary at this stage and which will be constrained shortly,

$$\omega^2 = \alpha B\,, \quad A = \frac{B}{2}\left[\alpha - 4 - \log\left[\frac{\pi^3}{8}(1+\alpha)\right]\right]. \tag{112}$$

In this way we obtain

$$D = -1\,,\ M = \sqrt{B}\sqrt{1+\alpha}\,,\ Q = \sqrt{\frac{\alpha}{1+\alpha}}\,,\ \langle r_E^2 \rangle = \frac{3}{4B}\frac{3+\alpha}{1+\alpha}\,,\ \langle r_Q^2 \rangle = \frac{3}{4B}\,, \tag{113}$$

For all values of α we have $D = -1$. Stability considerations (108–110) require

$$\alpha \geq 2 \tag{114}$$

leaving this parameter otherwise unconstrained. The criterion (79) for the smallness of relativistic corrections $\delta_{\rm rel}$ with $R^2 \to \langle r_Q^2 \rangle$ (which is always smaller than $\langle r_E^2 \rangle$ and yields a more restrictive criterion) we obtain

$$\delta_{\rm rel} = \frac{2}{3}\frac{1}{1+\alpha}\,. \tag{115}$$

At this point the parameter α is still not fixed, and we are free to choose its value to make relativistic corrections as small as we wish, e.g. if we choose $\alpha \gtrsim 66$ we have $\delta_{\rm rel} \lesssim 1\,\%$.

5.5 Logarithmic Q-balls and smeared out particles

In order to close the loop and make contact with the heuristic discussion in Secs. 4.6 and 4.2 we remark that the densities can be rewritten in terms of the Gaussian (77) introduced to smear out the δ-functions as

$$T_{00}(r) = M\left(\frac{\alpha-2}{\alpha+1}+\frac{2}{1+\alpha}\frac{r^2}{R^2}\right)f(r) \tag{116a}$$

$$p(r) = -\frac{1}{6M}\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}\right)f(r), \tag{116b}$$

$$s(r) = \frac{1}{4M}\left(\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}\right)f(r), \tag{116c}$$

$$\rho_{\rm ch}(r) = Q\, f(r), \tag{116d}$$

where we defined

$$f(r) \equiv \frac{1}{\pi^{3/2}R^3}\exp\left(-\frac{r^2}{R^2}\right) \quad \text{with} \quad R=\frac{1}{\sqrt{2B}}. \tag{116e}$$

The "smeared out δ-function representation" for the energy density differs from that of the other densities (which in general can be expected, see Sec. 4.6). Notice that $f(r) \equiv M\,\phi(r)^2$.

We can consider several limits.

- Limit $\alpha \to \infty$ with B kept fixed: $T_{00}(r)$ and $p(r)$ and $s(r)$ are expressed in terms of the same smeared-out function $f(r)$ which defines $p(r)$ and $s(r)$. In this interesting limit $D=-1$, $Q\to 1$, $\langle r_Q^2\rangle$ and $\langle r_E^2\rangle \to 3/(4B)$ are fixed, while the mass grows as $M\to\sqrt{\alpha B}$ justifying the applicability of the 3D-density description with $\delta_{\rm rel}\to 0$. This limit corresponds to a point-like particle smeared out by the Gaussian with a finite radius $R=1/\sqrt{2B}$.

- Limit $B\to\infty$ and $\alpha\to\infty$: We recover a heavy particle which becomes point-like as all mean square radii $\langle r_i^2\rangle \to 0$. In this limit $f(r)\to\delta^{(3)}(r)$ and we literally recover the description of a heavy point-like particle with $D=-1$, which we wrote down heuristically in Eq. (75) in Sec. 4.2.

- Limit $\alpha \to \infty$ and $B \to 0$ such that $M \to \sqrt{\alpha B}$ remains fixed: in this limit M could be even light (but must be non-zero) because the size of our (potentially light) particle grows as $\langle r_i^2 \rangle \to 3\alpha/M$ which guarantees the smallness of $\delta_{\rm rel}$ in (115) and applicability of the 3D-density description. We are not aware of systems of this kind in particle physics, but Rydberg atoms (moderate mass, extremely large size) provide an example from atomic physics.

It is gratifying to notice that there is no way to take a limit in which one could recover a light *and* small (point-like) particle, even if one were willing to pay the price of large relativistic corrections in Eq. (115). This is not surprising: our very starting point was the assumption that the 3D-density description *is* applicable, so our theory does not permit to have us take such a limit.

5.6 Boundary conditions for logarithmic Q-ball theory

This section is devoted to important technical details regarding the logarithmic potential in Eq. (99). This potential is not bound from below and does not constitute an "acceptable" Q-ball potential in the sense of Ref. [89]. Here we present a potential which is acceptable, bound from below, and contains our log-potential as limiting case.

Let us denote for simplicity $V = V(\phi)$ where $\phi = \phi(r)$ is the radial field. V is an "acceptable" Q-ball potential if (i) V is two times continuously differentiable with $V(0) = 0$, $V'(0) = 0$, $V''(0) = \omega_{\rm max}^2 \equiv m_\Phi^2 > 0$, $V(\phi) > 0$ for $\phi \neq 0$, (ii) $V(\phi)/\phi^2$ has a minimum at some $\phi_{\rm min} \neq 0$ which defines the lower limit $\omega_{\rm min}^2 = 2V(\phi_{\rm min})/\phi_{\rm min}^2$ for frequencies, (iii) positive numbers a, b, c exist with $c > 2$ such that $\frac{1}{2}m_\Phi^2\Phi^2 - V(\phi) \leq \min[a, b|\phi|^c]$ [89].

To construct a potential complying with the above criteria and containing (99) as a limiting case, we introduce the dimensionless parameters $0 < \varepsilon_i \ll 1$ with $i = 1, 2$. One acceptable regular potential V_{reg} is defined by

$$V_{\text{reg}} = A\,\phi^2 + \varepsilon_1\phi^4 - B\,\phi^2\,\log\left(\varepsilon_2 + \frac{\phi^2}{B}\right). \tag{117}$$

The role of the term with $\varepsilon_1\phi^4$ is to make sure the potential is bound from below for $\varepsilon_1 > 0$. The effect of ε_2 is to ensure a regular small field expansion of the potential exists, $V_{\text{reg}} = (A - B\log\varepsilon_2)\phi^2 + \mathscr{O}(\phi^4)$, which generates a finite mass term for the fundamental field. In the limit that the ε_i are negligible we recover the log-potential (99). Below we will see how this limit is understood. We begin by considering the limiting frequencies (89) and their difference,

$$\omega_{\text{max}}^2 = [V''(\phi)]_{\phi=0} = 2A - 2B\log\varepsilon_2 \equiv m^2\,, \tag{118}$$

$$\omega_{\text{min}}^2 = \min_\phi\left[\frac{2V(\phi)}{\phi^2}\right] = 2A + 2B(1 + \log\varepsilon_1 - \varepsilon_1\varepsilon_2)\,, \tag{119}$$

$$\Delta\omega^2 = \omega_{\text{max}}^2 - \omega_{\text{min}}^2 = 2B\,f(\varepsilon_1\varepsilon_2)\,,\;\; f(z) = z - \log z - 1\,. \tag{120}$$

We first show $\Delta\omega^2 > 0$, i.e. that there is a finite ω-range for solitons to exist. This is the case because $B > 0$ holds due to (108) (which remains valid if we have small $\varepsilon_i \neq 0$) and $f(z) > 0$ for $0 < z < 1$.

Next we will show that $\omega_{\text{min}}^2 > 0$ which means that $V(\phi)/\phi^2 > 0$ at its minimum. This will also show that $V(\phi) > 0$ for $\phi \neq 0$ confirming that $\phi = 0$ is the correct vacuum of the theory. Notice that in the general situation the expression for ω_{min}^2 in (119) does not need to be positive: for given A and B one cannot have arbitrarily small ε_1. This imposes a constraint on the parameters. The general condition is

$$\omega_{\min}^2 > 0 \quad \Leftrightarrow \quad \varepsilon_1 \exp(1-\varepsilon_1\varepsilon_2) < \exp(-A/B)\,. \tag{121}$$

Here we are interested in the specific situation with $D=-1$ where A, B are related to each other by Eq. (112) modulo negligible $\mathscr{O}(\varepsilon_i)$ corrections. This implies

$$\omega_{\min}^2 > 0 \quad \Leftrightarrow \quad \varepsilon_1 > c_0\sqrt{\frac{\alpha+1}{e^\alpha}} + \mathscr{O}(\varepsilon_i^2)\,, \quad c_0 = e\sqrt{\frac{\pi^3}{8}}\,, \tag{122}$$

i.e. ε_1 cannot be arbitrarily small. In practice, however, this is a loose bound as α must be large enough to ensure small relativistic corrections $\delta_{\rm rel}$, Eq. (115). For instance, if we demand $\delta_{\rm rel} \lesssim 1\,\%$ then $\alpha \gtrsim 66$ and $\varepsilon_1 \gtrsim 2.1\times 10^{-13}$. Thus ε_1 can be chosen so small that it can be neglected for all practical purposes. Even the limit $\varepsilon_1 \to 0$ can be realized for $\alpha \to \infty$ in which case we deal with the heavy mass limit of a fixed-size particle, see Sec. 5.3.

Obviously also $\omega_{\max}^2 > 0$ since $\omega_{\max}^2 = \omega_{\min}^2 + \Delta\omega^2$ and we have already proven that $\omega_{\min}^2$ and $\Delta\omega^2$ are both positive. This is also clear from (118) where (for $\varepsilon_2 \ll 1$) we see that $\omega_{\max}^2$ is evidently positive and defines the mass of the Φ_i-quanta. This completes the demonstration that $V_{\rm reg}$ satisfies the criteria (i) and (ii) of an acceptable potential.

Finally we turn to the criterion (iii), and introduce the notation

$$\begin{aligned} U_{\rm eff}(\phi) &\equiv \frac{1}{2}m^2\phi^2 - V(\phi) = \varepsilon_2 B^2 h(z)\,, \\ h(z) &= z\log(1+z) - \varepsilon z^2\,, \quad z = \frac{\phi^2}{\varepsilon_2 B}\,, \quad \varepsilon = \varepsilon_1\varepsilon_2\,. \end{aligned} \tag{123}$$

The function $h(z)$ satisfies

$$h(z) \leq z\log(1+z) \leq z^2 \quad \Leftrightarrow \quad U_{\text{eff}}(\phi) \leq b\,|\phi|^c\,, \quad b = \varepsilon_2 B^2\,, \quad c = 4\,. \tag{124}$$

This bound is useful for $\phi < \phi_{\text{eff, max}}$ where $U_{\text{eff}}(\phi)$ exhibits a maximum. For $\phi \geq \phi_{\text{eff, max}}$ a stronger bound is provided by $U_{\text{eff}}(\phi) \leq U_{\text{eff}}(\phi_{\text{eff,max}})$. To determine the extrema of $U_{\text{eff}}(\phi)$ we consider

$$h'(z) = \log(1+z) + \frac{z}{1-z} - 2\varepsilon z \stackrel{!}{=} 0 \tag{125}$$

which has a solution at $z = 0$ corresponding to a local minimum and a solution describing the global maximum at large $z \gg 1$ where $h'(z) = \log(z) + 1 - 2\varepsilon z + \mathcal{O}(1/z^2) \stackrel{!}{=} 0$ which is solved by

$$z = -\frac{1}{2\varepsilon} W_{-1}\left(-\frac{2\varepsilon}{e}\right) = \frac{1}{2\varepsilon}\log\left(\frac{e}{2\varepsilon}\right) + \frac{1}{2\varepsilon}\log\left(\log\left(\frac{e}{2\varepsilon}\right)\right) + \dots\,. \tag{126}$$

$W_{-1}(x)$ denotes the inverse function of $y = x\exp(x)$ known as Lambert W-function which is defined for $x \geq -1/e$ and multivalued at negative x. More precisely, $W_{-1}(x)$ denotes the branch with $W_{-1}(x) \leq -1$. In the second step in (126) we explored the asymptotic expansion of $W_{-1}(x)$ for small $(-x) \to 0$ [93] with the dots indicating subsubleading terms. Retaining only the leading terms we obtain

$$\phi^2_{\text{eff,max}} = \frac{B}{2\varepsilon_1}\log\left(\frac{e}{2\varepsilon_1\varepsilon_2}\right) + \dots\;, \quad U(\phi_{\text{eff,max}}) = \frac{B^2}{4\varepsilon_1}\log^2\left(\frac{e}{2\varepsilon_1\varepsilon_2}\right) + \dots \tag{127}$$

which shows that a maximum exists for $\varepsilon_i > 0$. Thus $U_{\text{eff}}(\phi) \leq \min[a, b\,|\phi|^c]$ where we can choose $a = U(\phi_{\text{eff,max}})$ and b, c as shown in Eq. (124). This completes the demonstration that also the criterion (iii) is satisfied.

To end this section we briefly report the results of a numerical check we made with the parameters $B = 2.5$, $\alpha = 65$ and a common value $\varepsilon \equiv \varepsilon_1 = \varepsilon_2 = 10^{-5}$ for sake of easier comparison. Recall that other Q-ball parameters are fixed by Eq. (112) which ensures $D = -1$ for $\varepsilon_i \to 0$. The scope of this exercise it to investigate the size of the deviations for D and other quantities for $\varepsilon_i = \varepsilon \neq 0$. Let us in the following denote the additional dependence on ε of the quantities as $\phi(r,\varepsilon)$, $M(\varepsilon)$, etc. with $\phi(r,0)$, $M(0)$, etc. corresponding to $\phi(r)$, M, etc. in Sec. 5.3 where the ε_i were strictly zero. To measure the deviations we introduce $\delta\phi(r) = \phi(r,\varepsilon) - \phi(r,0)$, $\delta M = M(\varepsilon) - M(0)$, etc. For the radial field we obtain

$$-0.6 \times 10^{-3} < \frac{\delta\phi(r,\varepsilon)}{\phi(r)} < 0.3 \times 10^{-3} \tag{128}$$

with the largest negative deviation at small r and the largest positive deviation around $r =$(1–2). For the integrated quantities we obtain

$$\frac{\delta Q}{Q} = -0.5 \times 10^{-3}\,, \quad \frac{\delta M}{M} = -0.6 \times 10^{-3}\,, \quad \frac{\delta D}{D} = 4 \times 10^{-3}\,,$$
$$\frac{\delta\langle r_E^2\rangle}{\langle r_E^2\rangle} = 3 \times 10^{-3}\,, \quad \frac{\delta\langle r_Q^2\rangle}{\langle r_E^2\rangle} = 3 \times 10^{-3}\,. \tag{129}$$

6 The D-term of spin-$\frac{1}{2}$ particles

Having discussed the D-term for spin-0 bosons we now turn our attention to spin-$\frac{1}{2}$ fermions. We start with the free theory, and find the surprising result that for a free fermion the D-term is zero. We discuss two heuristic arguments why this result is plausible in free theories. In the next section we will discuss dynamical models which will show how the D-terms of fermions can arise from interactions.

6.1 EMT form factors for a free Dirac particle

The simplest fermionic theory is a free Dirac fermion described by the Lagrangian

$$\mathscr{L} = \bar{\Psi}(i\,\partial\!\!\!/ - m)\Psi \tag{130}$$

For a free Dirac particle Eq. (10) yields the EMT operator given by

$$\hat{T}_{\mu\nu}(x) = \frac{1}{4}\,\bar{\Psi}(x)\left(i\gamma^\mu\overrightarrow{\partial}^\nu + i\gamma^\nu\overrightarrow{\partial}^\mu - i\gamma^\mu\overleftarrow{\partial}^\nu - i\gamma^\nu\overleftarrow{\partial}^\mu\right)\Psi(x)\,, \tag{131}$$

where the arrows indicate on which fields the derivatives act. For completeness let us recall that in equal time quantization the fields are given by

$$\Psi(x) = \int\frac{d^3k}{(2\pi)^3}\,\frac{m}{k_0}\sum_{a=1,2}\left[b_{(a)}(k)\,u_{(a)}(k)\,e^{-ikx} + d^\dagger_{(a)}(k)\,v_{(a)}(k)\,e^{ikx}\right] \tag{132}$$

with $\{b_{(a)}(k),\,b^\dagger_{(b)}(k)\} = (2\pi)^3\,\frac{k_0}{m}\,\delta^{(3)}(\vec{k}-\vec{k}')\,\delta_{ab}$ and analogously for $d_{(a)}(k)$, while all other anti-commutators vanish.

Evaluating the matrix elements yields

$$\langle p'|\hat{T}_{\mu\nu}(x)|p\rangle = \frac{1}{4}\,\bar{u}(p')\Big[\gamma_\mu p_\nu + p_\mu\gamma_\nu + \gamma_\mu p'_\nu + p'_\mu\gamma_\nu\Big]u(p)\,e^{i(p'-p)x}\,. \tag{133}$$

Exploring the Gordon identity

$$2m\bar{u}(p')\gamma^\alpha u(p) = \bar{u}(p')(2P^\alpha + i\sigma^{\alpha\kappa}\Delta_\kappa)u(p) \tag{134}$$

we can rewrite the result in Eq. (133) as

$$\langle p'|\hat{T}_{\mu\nu}(x)|p\rangle = \bar{u}(p')\Big[\frac{P_\mu P_\nu}{m} + \frac{1}{2}\,\frac{i(P_\mu\sigma_{\nu\rho} + P_\nu\sigma_{\mu\rho})\Delta^\rho}{2m}\Big]u(p)\,e^{i(p'-p)x}\,, \tag{135}$$

from which we read off the predictions of the free Dirac theory for the EMT form factors, namely

$$A(t) = 1\,,\quad J(t) = \frac{1}{2}\,,\quad D(t) = 0\,. \tag{136}$$

Several comments are in order. The form factors are constant functions of t as expected for a free point-like particle. The values of $A(t)$ and $J(t)$ are therefore unambiguously fixed for all t in terms of the constraints at $t = 0$ in (27), and our result in (136) reflects this fact. The only non-trivial result from this exercise is therefore the form factor $D(t)$. It is remarkable that the free Dirac theory predicts the D-term of a free point-like fermion to be zero.

This prediction is on the same footing as the predictions $g = 2$ of the free Dirac theory for the gyromagnetic ratio of the electron. In that case $g = 2$ is obtained by minimally coupling the free theory (130) to a classical background electromagnetic field. The value gets modified by radiative QED corrections starting with the famous Schwinger contribution $g = 2(1 + \frac{\alpha}{2\pi} + \ldots)$. Nevertheless the free theory calculation provides a benchmark and a very useful reference point. Analogously, the Dirac theory minimally coupled to a gravitational background unambiguously predicts that the D-term of a free spin-$\frac{1}{2}$ fermion is zero.

This is remarkable: it means that the D-term of a fermion is entirely of dynamical origin. In the following sections we shall provide heuristic plausibility arguments why $D = 0$ should be expected for a free spin-$\frac{1}{2}$ fermion. The next section will illustrate in dynamical models how interactions can generate a non-zero D-term.

6.2 Heuristic argument I: Why can't the Dirac equation predict a non-zero D-term?

The vanishing of the D-term of a free point-like fermion can be made plausible on the basis of two independent arguments. For the first argument we recall that for a spin-zero particle already the *free* Klein-Gordon equation yields a non-vanishing $D_{\text{boson}} = -1$. It is instructive to review how this happens. The D-term appears in the decomposition of the matrix elements of the EMT operator (3.1) with the structure $\Delta_\mu \Delta_\nu - g_{\mu\nu}\Delta^2$. In the spin-zero case such a structure emerges already from the kinetic term in the Lagrangian (33) which contains two derivatives of the fields and generates the contribution $\partial^\mu \Phi \partial^\nu \Phi$ to the EMT operator. This is sufficient to generate the needed structure $\Delta_\mu \Delta_\nu - g_{\mu\nu}\Delta^2$ in the EMT matrix elements even in the absence of interactions (which may affect the D-term but preserve its negative sign). In the case of free Dirac fields the Lagrangian contains only one derivative, and consequently no D-term can be generated. Let us notice that *if* interactions are present they of course may generate a D-term in the Dirac case, see next section for some illustrations.

6.3 Heuristic argument II: consistency in 3D density framework

For the second argument we assume the fermion to be heavy which is necessary to justify the exploration of the 3D-density framework. Our argumentation is analogous to that in Sec. 4.7 and based on two assumptions: (i) form factors of a free point-like particle are constants, and (ii) energy density of a heavy point-like particle is formally given by $T_{00}(\vec{r}\,) = m\,\delta^{(3)}(\vec{r}\,)$, cf. Sec. 4.3. We recall that the assumption that the fermion mass is large is of formal character. In order to define the heavy mass limit one strictly speaking should introduce an additional scale in the theory, such that it can be specified with respect to what the mass m of the fermion is supposed to be large. As no such additional scale is available in the free theory, one could proceed as in Sec. 4.4 and introduce e.g. a finite size of the fermion. We shall not follow this path here, and assume simply that $T_{00}(\vec{r}\,) = m\,\delta^{(3)}(\vec{r}\,)$ holds formally. The following arguments are not invalidated by this assumption.

Due to assumption (i) we can replace the form factors in Eq. (26) by their values at zero-momentum transfer. Next, we notice that the result in the square brackets in the following equation must be zero to comply with assumption (ii),

$$\frac{1}{m}\int d^3\vec{r}\,e^{-i\vec{r}\vec{\Delta}}\,T_{00}(\vec{r},\vec{s}) = A(0) - \frac{t}{4m^2}\underbrace{\big[A(0) - 2J(0) + D(0)\big]}_{\overset{!!}{=}\,0} \overset{!?}{=} 1\,. \tag{137}$$

With the constraints in Eq. (27) it then immediately follows that the D-term must vanish for a point-like particle for consistency reasons. Again we have to stress that interactions invalidate this argument as in that case form factors in general are not constants, and $D(t)$ in general does not need to be zero.

7 How interactions generate D-terms of fermions

In Sec. 6 we have seen that a free fermion has a vanishing D-term. This is a rather interesting result and in stark contrast to the spin-0 case, where the free theory predicts $D = -1$. The purpose of this section is to show how interactions in fermionic systems can give rise to $D \neq 0$. For that we first conduct a study in the bag model which will show how the D-term "emerges" when interactions are "introduced." We will also briefly review the case of the chiral quark soliton model, where it can be shown how the D-term "vanishes" if one "removes" the interactions in that model.

7.1 The bag model

The bag model describes one or several non-interacting fermions confined inside a "bag." In its rest frame, the bag is a spherical region of radius R carrying the energy density $B > 0$. If $N_c = 3$ quarks or a $\bar{q}q$-pair are placed inside the bag in a color-singlet state, this yields the popular model of hadrons with confinement simulated by the bag boundary condition [94]. Despite its simplicity the model is very popular [95, 96] and, despite its long history, continues giving good services [97, 98, 99] for getting some first insights into the nucleon structure. In particular, generalized parton distribution functions and EMT form factors of the nucleon were studied in that model in [34]. Here we will use the model to investigate EMT densities especially in the context of the D-term. More details will be reported elsewhere [68].

We note that while the fermions do not interact with each other, the boundary condition provides a simple type of interaction. Thanks to its simplicity, the bag model is an ideal model to shed some light on how interactions can generate the D-term of a fermion. It should be stressed that the "many body character" is irrelevant: a non-zero D-term emerges independently of how many fermions are placed inside the bag, and one finds $D \neq 0$ even for a single fermion inside the bag (of course, to model baryons one has to place N_c quarks inside the bag).

The Lagrangian of the bag model can be written as [95]

$$\mathscr{L} = \Big(\bar{\psi}\,(i\,\partial\!\!\!/ - m)\,\psi - B\Big)\,\Theta_V + \frac{1}{2}\,\bar{\psi}\,\psi\,\eta^{\mu}\partial_{\mu}\Theta_V\ , \tag{138}$$

In the rest frame of the bag, Θ_V and δ_S (whose indices V and S denote respectively the volume and the surface of the bag) and η_μ are given by

$$\Theta_V = \Theta(R-r), \quad \delta_S = \delta(R-r), \quad \eta^{\mu} = (0,\vec{e}_r), \quad r = |\vec{x}\,|, \quad \vec{e}_r = \vec{x}/r. \tag{139}$$

The equations of motion of the theory (138) are given by

$$\begin{aligned} (i\,\partial\!\!\!/ - m)\,\psi = 0 \quad & \text{for } r < R \text{ (free fermions)} \\ i\,\eta\!\!\!/\,\psi = \psi \quad & \text{for } \vec{x} \in S \text{ (linear boundary condition)} \\ -\frac{1}{2}\,\eta_\mu \partial^{\mu}(\bar{\psi}\psi) = B \quad & \text{for } \vec{x} \in S \text{ (nonlinear boundary condition)} \end{aligned} \tag{140}$$

The boundary conditions are designed such that there is no energy-momentum flow out of the bag, i.e.

$$T^{\mu\nu}\eta_\mu = 0. \tag{141}$$

If the radius R is finite, one finds discrete energy states. The lowest energy solution in the bag has positive parity and is given by the ground state wave-function

[94]

$$\psi_0(t,\vec{x}) = e^{-i\omega_0 t/R}\,\phi_0(\vec{x})\,, \quad \phi_0(\vec{x}) = \frac{A}{\sqrt{4\pi}} \begin{pmatrix} \alpha_+\, j_0(\omega_0\, r/R)\,\chi_s \\ \alpha_-\, j_1(\omega_0\, r/R)\, i\vec{\sigma}\vec{e}_r\chi_s \end{pmatrix} \tag{142}$$

where σ^i denote Pauli matrices and χ_s are two-component Pauli spinors. The normalization constant is

$$A = \left(\frac{\Omega(\Omega - mR)}{R^3\, j_0^2(\omega_0)(2\Omega(\Omega-1)+mR)}\right)^{1/2}$$

such that

$$\int d^3x\, \phi_0^\dagger(\vec{x})\,\phi_0(\vec{x}) = 1. \tag{143}$$

The $\alpha_\pm$ and Ω are defined as

$$\alpha_\pm = \sqrt{1 \pm mR/\Omega} \quad \Omega = \sqrt{\omega_0^2 + m^2R^2}. \tag{144}$$

The dimensionless quantity ω_0 denotes the lowest solution of the transcendental equation

$$\omega = (1 - mR - \Omega)\tan\omega. \tag{145}$$

If the quarks are massless, i.e. $m = 0$, the lowest solution is given by $\omega_0 \approx 2.04$.

7.2 EMT in the bag model

The starting point is as follows. If no bag boundary condition is present, i.e. in the limit $R \to \infty$ in Eq. (138), we deal with the free Lagrangian (130) with an additive constant B which is irrelevant and can be discarded. In such a free theory the

D-term is zero, as we have shown in Sec. 6.1.

If we assume that the radius R is finite, we obtain the bound state solution quoted in Eq. (142). We will use this solution to evaluate the EMT densities. In the bag model the total EMT has 3 contributions: the contribution $T_D^{\mu\nu}$ of the Dirac fermions, the contribution $T_S^{\mu\nu}$ from the bag surface, and the contribution $T_{\mathscr{L}}^{\mu\nu}$ which is basically due to the bag constant $B \neq 0$. These contributions are given by

$$\begin{aligned}
T^{\mu\nu} &= T_D^{\mu\nu} + T_S^{\mu\nu} + T_{\mathscr{L}}^{\mu\nu}\,, \\
T_D^{\mu\nu} &= \frac{i}{4}\Big(\bar\psi\gamma^\mu(\partial^\nu)\psi + \bar\psi\gamma^\nu(\partial^\mu)\psi - (\partial^\mu\bar\psi)\gamma^\nu\psi - (\partial^\nu\bar\psi)\gamma^\mu\psi\Big)\,, \\
T_S^{\mu\nu} &= \frac{1}{4}\,\bar\psi\psi\,(\eta^\mu\partial^\nu\Theta_V + \eta^\nu\partial^\mu\Theta_V)\,, \\
T_{\mathscr{L}}^{\mu\nu} &= -g^{\mu\nu}\mathscr{L}\,. \qquad (146)
\end{aligned}$$

We evaluate the densities of the EMT in the nucleon rest frame, in nucleon states with the non-relativistic normalization $\langle N|N\rangle = 1$. Evaluating the operators $T_D^{\mu\nu}$, $T_S^{\mu\nu}$, $T_{\mathscr{L}}^{\mu\nu}$ in the quark bag eigenfunctions (142) yields

$$\langle N|T_D^{00}(\vec{x})|N\rangle = \frac{N_c A^2}{4\pi}\,\frac{\omega}{R}\left(\alpha_+^2 j_0^2 + \alpha_-^2 j_1^2\right)\Theta(R-r) \qquad (147)$$

$$\langle N|T_D^{0k}(\vec{x})|N\rangle = -\frac{1}{2}\,\frac{A^2}{4\pi}\alpha_+\alpha_-\left(\frac{2\omega}{R}\,j_0\,j_1 + \frac{j_1^2}{r}\right)\varepsilon^{klr}\,e_r^l S^r \qquad (148)$$

$$\begin{aligned}
\langle N|T_D^{ik}(\vec{x})|N\rangle &= \frac{N_c A^2}{4\pi}\,\alpha_+\alpha_-\left[\left(j_0\,j_1' - j_0'\,j_1 - \frac{j_0\,j_1}{r}\right)e_r^i e_r^k \right. \\
&\qquad \left. + \frac{j_0\,j_1}{r}\,\delta^{ik}\right]\Theta(R-r) \qquad (149)
\end{aligned}$$

$$\langle N|T_S^{\mu\nu}(\vec{x})|N\rangle = 0 \qquad (150)$$

$$\langle N|T_{\mathscr{L}}^{\mu\nu}(\vec{x})|N\rangle = g^{\mu\nu}\,B\,\Theta(R-r) \qquad (151)$$

The arguments of the Bessel functions are $\omega r/R$ in all cases and omitted for brevity, and the primes denote here differentiation with respect to r. In the following we will discuss the energy density $T^{00}(r)$ and the stress tensor $T^{ik}(r)$. For a discussion of the density $T^{0k}(\vec{r},\vec{c})$ we refer to [68].

7.3 The energy density $T_{00}(r)$ in bag model

From Eqs. (147, 151) we obtain for the energy density the result

$$T_{00}(r) = \left[\frac{N_c A^2}{4\pi}\,\frac{\omega}{R}\left(\alpha_+^2 j_0^2 + \alpha_-^2 j_1^2\right) + B\right]\Theta(R-r). \tag{152}$$

Integrating this energy density over r yields the nucleon mass

$$M = \int d^3x\, T_{00}(r) = N_c\,\frac{\Omega}{R} + \frac{4}{3}\pi B R^3\,. \tag{153}$$

The constant B and the radius R are not independent of each other but related by the nonlinear boundary condition in Eq. (140) [94]. Interestingly this relation can also be obtained from minimizing the nucleon mass in Eq. (153) with respect to R:

$$M'(R) = 0 \quad \Leftrightarrow \quad N_c\,\omega = 4\pi B R^4 \quad \text{for } m = 0. \tag{154}$$

If $m \neq 0$ the relation is more complicated, because $\omega_0 = \omega_0(mR)$ due to Eq. (145). But in any case the nonlinear boundary condition in Eq. (140) is equivalent to the fact that the mass of a hadron in the bag model corresponds to the minimum with respect to R which reflects how the equations of motion mimize the action in this model.

The numerical result for $T_{00}(r)$ is shown in Fig. 4a. The energy density is always positive. It assumes its largest value in the center of the nucleon and monotically decreases with r until it suddenly drops to zero at the distance $R \approx 1.71\,\text{fm}$ which marks the radius of the bag. It should be noted that the bag radius should not be confused with the "size" of the nucleon.

7.4 The stress tensor $T^{ik}(r)$ in bag model

From the stress tensor in Eqs. (149, 151) we find the following expressions for the pressure and the shear forces in the bag model

$$\begin{aligned} p(r) &= \left[\frac{N_c A^2}{4\pi}\,\alpha_+\alpha_-\left(j_0 j_1' - j_0' j_1 + \frac{2}{r} j_0 j_1\right) - B\right]\Theta(R-r), \\ s(r) &= \left[\frac{N_c A^2}{4\pi}\,\alpha_+\alpha_-\left(j_0 j_1' - j_0' j_1 - \frac{1}{r} j_0 j_1\right)\right]\Theta(R-r). \end{aligned} \tag{155}$$

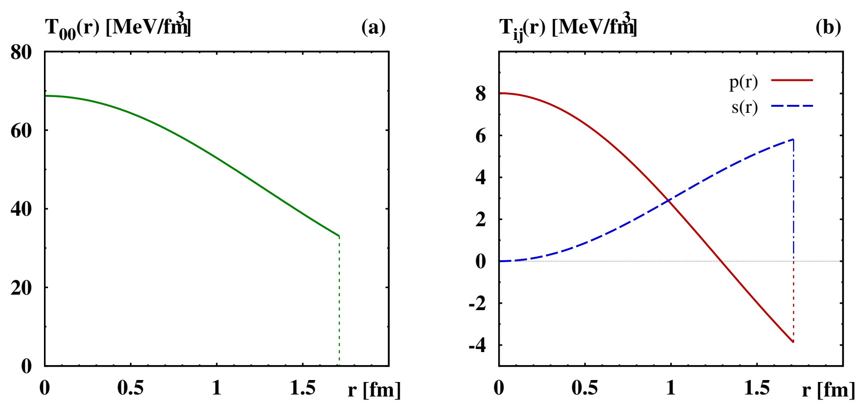


Figure 4: (a) The energy density $T_{00}(r)$ and (b) the densities of the stress tensor $T_{ij}(r)$, $s(r)$ and $p(r)$, in units of MeV/fm^3 as functions of r in units of fm in the bag model for massless quarks. The vertical lines mark the position of the bag boundary which is at $R \approx 1.71\,\text{fm}$ in our case (for $m = 0$).

The numerical results for massless quarks are shown in Fig. 4b. While the results do not look realistic due to the peculiar way that the bag stabilizes the nucleon, the results are nevertheless theoretically consistent. The pressure distribution and the shear forces satisfy the differential equation (29) and $p(r)$ obeys the von Laue condition (30), which can be proven analytically and is illustrated in Fig. 5. Since both equations are consequences of the conservation of the EMT, $\partial_\mu T^{\mu\nu} = 0$, this shows that in the bag model the EMT is conserved and the description of baryons is internally consistent [68].

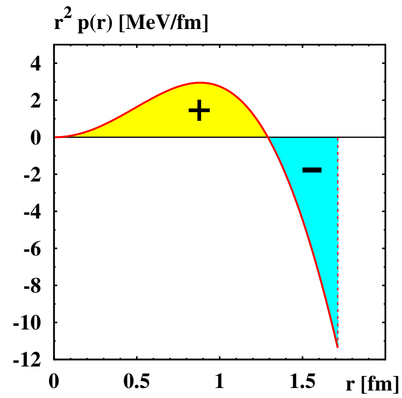


Figure 5: The $r^2 p(r)$ in units of MeV/fm as function of r in units of fm in the bag model for massless quarks. The vertical line at $R \approx 1.71$ fm indicates the position of the bag boundary. The figure illustrates how the von Laue condition, a necessary condition for stability, is realized: the areas above and below the r-axis are equal and compensate each other in the integral $\int_0^R dr\, r^2 p(r) = 0$ according to Eq. (30).

7.5 The D-term in the bag model

From the expressions for $p(r)$ and $s(r)$ in Eq. (155) we find by exploring Eq. (31) for the D-term the result

$$D = \frac{1}{3} M N_c \frac{A^2 R^4}{\omega^4} \alpha_+ \alpha_- \left(-\frac{\omega^3}{3} + \frac{5}{4} (\omega - \sin\omega \cos\omega) - \frac{\omega}{2} \sin^2\omega - \frac{5}{4} \sin\omega \cos\omega \right). \tag{156}$$

where M is given by Eq. (153). For $N_c = 3$ colors and assuming the fermions to be massless quarks (in which case $\omega \approx 2.04$) one obtains

$$D = -1.145 \tag{157}$$

in agreement with the numerical calculation in the bag model of nucleon GPDs and EMT form factors from Ref. [34]. Figure 5 illustrates how the negative sign of D emerges: the integral of $r^2 p(r)$ must vanish (and does so) to comply with the von Laue condition Eq. (30). The D-term is proportional to $r^4 p(r)$ and evidently one obtains a negative result if one provides an additional factor of r^2 in Fig. 5. For massive quarks the numerical result in Eq. (157) is alterted, but one always has $D < 0$ in this model [68].

7.6 The limit $mR \to \infty$

As an application of Eq. (156) it is insightful to consider the limit $mR \to \infty$. In this limit the ground state solution of the transcentendal equation (145) goes to $\omega \to \pi$ and for the D-term we obtain the result

$$D = N_c^2 \frac{(-4\pi^2 + 15)}{45} \approx -4.896 \tag{158}$$

for $N_c = 3$ colors. This result can be interpreted in two ways.

For the first interpretation we may assume that m is fixed and R becomes much larger than the Compton wave length of the particle, $R \gg 1/m$. When the region occupied by the bag grows with R, the "interaction" acting on the fermions effectively decreases: the confined particle(s) can occupy an increasingly larger and larger volume as the boundary is being "moved" further and further away. However, no matter how far away we move the boundary *some* interaction remains, and generates a non-zero D-term. (We recall that at this point R can be arbitrarily large, but must be finite since the boundary conditions have been implemented for some *finite* value of R. The strict limit $R \to \infty$ reproducing a free fermion with a vanishing D-term must be taken on the level of the Lagrangian (138) before starting the bound state calculation, see Sec. 7.2.)

For the second interpretation we may assume a fixed R and $m \to \infty$. This is known as the non-relativistic limit, in which $\alpha_- \to 0$ and the lower component of the spinor in (142) vanishes. The D-term in Eq. (156) is proportional to α_- which vanishes, and to the mass of the system which behaves as $M \propto N_c m$ for $m \to \infty$. The product $M\alpha_-$ is finite in the limit $m \to \infty$. As a result the D-term assumes a finite value as quoted in Eq. (158). This result demonstrates that also non-relativistic systems have a D-term, i.e. this property is not a relativistic effect.

7.7 The D-term in the large N_c limit

As a by-product of our study we comment on the large-N_c behavior of the D-term. The limit of a large number of colors N_c is a useful theoretical guideline. In nature $N_c = 3$ does not seem to be a large number, but in theoretical calculations it may be convenient to assume that $1/N_c$ is a small parameter and expand in it. In the large-N_c limit baryons are heavy with masses scaling like $M \sim N_c$ with the number of colors. However, the size of a baryon R_B remains fixed and scales as $R_B \sim N_c^0$. (It is interesting to compare this to the descriptions of large-A nuclei discussed in Sec. 2.10: the masses of nuclei grow as A and their radii grow as $A^{1/3}$ which reflects the saturation property of nuclear forces. Large-N_c baryons exhibit a much different behavior.) For the D-term of the nucleon it was proven from general principles that it scales like

$$D \sim N_c^2 \tag{159}$$

in the large-N_c limit [16].

Even though the bag model is a very simplistic model of the nucleon, it nevertheless correctly describes the large-N_c behavior of the D-term. For that we recall that the quarks are non interacting in the bag model. In order to describe a nucleon made from N_c quarks one would simply occupy the ground state of the bag with N_c quarks in a color-antisymmetric state. From Eq. (153) we see that in the bag model the nucleon mass scales as $M \sim N_c$. Notice that from Eq. (154) it is clear that the bag constant must scale as $B \sim N_c$, while the bag radius $R \sim N_c^0$ remains constant as do all other bag model parameters such as e.g. ω_0. After these preparations it is clear that from Eq. (156) we conclude that $D \sim N_c^2$ in agreement with the general result in Eq. (159). This conclusion is of course also valid in the limit $mR \to \infty$ discussed in Sec. 7.6, as can be explicitly seen in Eq. (158).

7.8 The D-term of nucleon in a chiral model

One virtue of the bag model is its transparency, which we explored to show how $D \neq 0$ emerges if one introduces a simple interaction in a fermionic theory. But the bag model does not comply with chiral symmetry which is violated by the bag boundary condition [95]. A realistic theory of the nucleon should obey chiral symmetry whose importance was discussed in Sec. 3.5. In this section we briefly review the results from a model which is more realistic in this respect: the chiral quark soliton model [100]. In this model it is possible to demonstrate how the D-term disappears if one removes interactions. Here we will briefly review the arguments. More details can be found in Ref. [67].

The chiral quark soliton model is based on an effective chiral theory describing the interaction of massive constituent quarks with chiral pion fields $U = \exp(i\tau^a \pi^a)$ [101, 102] which was derived from the instanton model of the QCD vacuum [103, 104, 105, 106, 107]. It is a nonlinear and strongly coupled theory, which must be solved non-perturbatively in the large-N_c limit. The D-term in the chiral quark soliton model was studied in [35, 36, 37, 38, 25, 39]. An approximate expression useful for our purposes was derived in [36] and is given by

$$D = -F_\pi^2 M \int \mathrm{d}^3x\, P_2(\cos\vartheta)\, \vec{x}^2\, \mathrm{tr}_F[\nabla^3 U][\nabla^3 U^\dagger] + \ldots \tag{160}$$

where tr_F is the trace over flavor indices, M denotes the nucleon mass, and the dots indicate higher order derivatives. In order to obtain the exact model prediction one, of course, must sum all (infinitely many) derivatives of the U-fields which was done (numerically) in [25].

The result in Eq. (160) is useful because it shows what happens if one removes the chiral interaction in this model. By taking the formal limit $U \to 1$, one can recover the free theory in the chiral quark soliton model. In such a limit, the leading expression and higher derivatives indicated by the parenthesis in Eq. (160) vanish.

This illustrates how the D-term vanishes in the chiral quark soliton model in the formal limit when one "switches off" the chiral interactions in this model. For a more detailed discussion see Ref. [67].

8 Conclusions

In this section the results are summarized which were obtained in this thesis. In addition, we also present a brief update on some recent works which have appeared after this thesis was basically completed.

8.1 Summary

In this thesis work studies of form factors of the EMT were performed in spin-0 and spin-$\frac{1}{2}$ systems with particular focus on the stress tensor and the D-term, a particle property as fundamental as mass and spin which is not known experimentally for any particle.

As a starting point we studied the D-term in free spin-0 theory, and showed that the free Klein-Gordon theory makes the unambiguous prediction $D = -1$. This result, obtained first by Pagels in 1965 and largely overlooked in recent literature, is analogous to the prediction $g = 2$ for the anomalous magnetic moment from Dirac theory.

We showed the particular sensitivity of the D-term to the dynamics and interactions by exploring Φ^4 theory. The value of the D-term is changed from its free theory value $D = -1$ to $-\frac{1}{3}$ no matter how infinitesimally weak the interaction is. This is due to renormalization (assuming that mass is renormalized such that it coincides with its counterpart in the classical Lagrangian). This is a very interesting observation which illustrates how strongly sensitive the D-term is to the dynamics of a system. It was observed in the literature that when model parameters are varied the D-term is the particle property which exhibits the largest variations. Our inspection of the D-term in Φ^4 theory confirms and supports these observations.

In order to investigate the D-term in a strongly interacting system, we reviewed the description of the D-term of the Goldstone bosons of the spontaneous breaking of chiral symmetry which is a key feature of the strong interactions. In the chiral limit the Goldstone bosons, pions, kaons and η, are massless and their D-terms is $D = -1$, like in the free field theory. This is a non-trivial consequence of chiral symmetry breaking. We have used results from the literature to predict the D-terms of real (i.e. massive) pions, kaons and η-mesons in one-loop-order chiral perturbation theory. Not unexpectedly, we found the D-term of lighter pions to be closer to the chiral limit predictions. The deviations from this limit are larger for kaons and η but the D-terms are always found to be negative. This is in line with results from the literature where the D-terms of various particles were always found negative in all theoretical studies so far.

We then studied the interpretation of the D-term and EMT form factors in terms of 3D-densities in the spin-0 systems. This interpretation gives insights on the stress tensor and the internal mechanical forces inside composed particles and is of particular interest for hadrons. We applied the 3D density formalism to the description of a point-like particle, and showed that a consistent 3D-description is obtained in the heavy mass limit. If the mass is not heavy, there are corrections which are a manifestation of the relativistic corrections which are known also from the 3D interpretation of the electric form factor in terms of the electric charge distribution in the nucleon and in nuclei. The presence of such corrections was known in the literature, but to the best of our knowledge, it has not yet been addressed from the point of view taken in this thesis.

In order to define a heavy mass limit in which the relativistic corrections are small, it is necessary to introduce an additional intrinsic scale such as the size of a particle. We quantified the corrections to this picture and found that they are reasonably small for a particle with the mass and size of the nucleon and completely negligible for larger and heavier particles like nuclei. This is of importance because it allows us to estimate to which a 3D interpretation is applicable to the nucleon and nuclei. It is important to stress that the corrections concern only the interpretation, but the theoretical description remains rigorous and consistent.

We showed that the free theory result $D = -1$ persists even when the spin-0 boson is not point-like but given "some internal structure." The EMT densities of a point-like particle are given in terms of delta-functions or their derivatives. We proceeded heuritically and "smeared out" the delta-functions and their derivatives by hand. We showed that such smeared out densities still comply with all general properties of the EMT densities. In particular, the D-term of a particle smeared out in such a way is still $D = -1$ and the EMT densities obey all relations derived from the conservation of the EMT. In particular, we have shown that if the particle size is much larger than the Compton wave length of the particle, then the relativistc corrections are small.

We also showed that one can formally derive the result $D = -1$ for a free point-like spin-0 boson from consistency arguments based on the notion of a (heavy) point-like particle by exploring the 3D interpretation. One naturally expects that the energy density is formally given by $T_{00}(r) = m\,\delta^{(3)}(\vec{r}\,)$ for a point-like particle. Combing this with the expectation that form factors of point-like particles (in non-interacting theories) are constants, yields $D = -1$ for a free point-like boson. This is not a rigourous derivation, but it is a helpful plausibility argument and shows the internal consistency of the 3D description.

We constructed an explicit microscopic theory where the notion of "giving" an internal structure to a particle is implemented dynamically in a consistent way. For this we were able to use the framework of Q-balls. From the logarithmic Q-ball we were able to recover a point-like particle in a certain parametric limit. The bosonic self-interaction which allows to interpolate in this way between point-like and extended particles contains a logarithmic potential. Interactions corresponding to the class of such potentials have been discussed in the context of beyond standard model Higgs physics and in cosmology. As an interesting by-product, we also found an exactly solvable soliton model in 3 + 1 space dimensions, where it is non-trivial to find exact solutions.

We then studied the D-term in spin $\frac{1}{2}$ theories, and found that the D-term of a free non-interacting fermion vanishes. This is a prediction of the free Dirac equation which is, in principle, also analogous to the prediction $g = 2$ for the anomalous magnetic moment of a charged point-like fermion. This result is remarkable for several reasons and has interesting implications.

The prediction of a vanishing D-term from the free Dirac equation should be contrasted with the bosonic case. The free Klein-Gordon equation predicts an intrinsic non-zero D-term already for free and non-interacting bosons. When interactions are introduced in bosonic theories, the value of D is in general affected and the effects can be sizable. However, in the fermionic case interactions do not modify the D-term. Rather in the fermionic case interactions *generate* the D-term. In other words, the D-term of a spin-$\frac{1}{2}$ particle is entirely of dynamical origin.

We have provided heuristic consistency arguments which make it plausible why the D-term of a free point-like spin $\frac{1}{2}$ particle should vanish. Similar arguments were used to explain why a free point-like boson must have $D = -1$. While not a rigorous derivation, it is again very helpful to see the internal consistency of the 3D interpretation in terms of EMT densities.

We have explored the bag model to illustrate how the *D*-term is generated in interacting system. In the bag model, a non-zero *D*-term emerges when we "switch on" interactions. The interactions are formulated in terms of boundary conditions which confine the otherwise free fermions. As a by product we have shown that the description of the nucleon in the bag model is internally consistent and the results for the *D*-term from the bag model comply, for instance, with the general scaling of the *D*-term of the nucleon in the large-N_c limit. We also briefly reviewed the chiral quark soliton model where the *D*-term vanishes when the chiral interactions of that model are "switched off." These are simple models of the nucleon, but these results solidify our conclusions: in a fermionic system the *D*-term is generated by dynamics, it arises entirely from interactions.

With its relation to the internal forces and the stress tensor [12] the *D*-term emerges therefore as a valuable window to gain new insights on the structure of composite particles, and especially the QCD dynamics inside the nucleon.

This thesis work contributes to a better understanding of this interesting property, and it will be interesting to learn about the *D*-terms of nucleons and nuclei in experiments running or planned at Jefferson Lab, COMPASS, and the envisioned future Electron-Ion-Collider.

8.2 Publications on which this thesis is based.

This thesis is based on the works [65, 66, 67, 68]. Some of the new results presented in this thesis were mentioned in the proceeding [65]. The new material presented in this thesis in Secs. 3–5 was published in Ref. [66]. The new results presented in Secs. 6–7 were published in [67]. Parts of the material presented in

Sec. 7 are not yet published and will appear in the preprint [68].

8.3 Updates

After this work was completed the following work appeared. This update helps to illustrate how timely this thesis work is.

The first phenomenological information on the D-term of a particle was reported for the neutral pion in [110] where the Belle data on the process $\gamma\gamma^*$ to $\pi^0\pi^0$ were studied. The obtained value for the quark contribution to the D-term of π^0 is $D^Q \approx -0.7$ at a scale of $Q^2 = 8.92\,\text{GeV}^2$. This value has an unestimated uncertainty due to the statistical accuracy of the data and the model dependence in this first exploratory study.

The first phenomenological information on the D-term of the proton was reported in [113] where data on the Deeply Virtual Compton Scattering (DVCS) process from Jefferson Lab were analyzed. The following result for the quark contribution to the proton D-term was reported: $D^Q \approx -2.04 \pm 0.14$ at a scale of about $Q^2 = 1.5\,\text{GeV}^2$. This result has a significant unestimated systematic error.

The first complete study of the quark and gluon contributions to the EMT form factors of the nucleon and pion was reported in [112]. In this work the mass of the pion was 450 MeV and disconnected diagrams were not included. The quark and gluon contributions to the D-terms of the nucleon and pion were all found to be negative.

An attempt to develop an interpretation of the EMT form factors in terms of 2D-densities was recently proposed in [111].

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A Notation

In Ref. [32] the notation for the EMT form factors was used

$$\text{Ref. [3], Eq. (8):} \quad \frac{G_1(q^2)}{2m^2} = A(t), \quad \frac{G_2(q^2)}{2m^2} = -D(t), \quad q^2 = t\,, \tag{161}$$

$$\text{Ref. [2], Eq. (3.152):} \quad \theta_2(\Delta^2) = A(t), \quad \theta_1(\Delta^2) = -D(t), \quad \Delta^2 = t\,, \tag{162}$$

$$\text{Ref. [32], Eq. (25):} \quad \theta_2(q^2) = A(t), \quad \theta_1(q^2) = -D(t), \quad q^2 = t\,, \tag{163}$$

$$\text{Ref. [27], Eq. (2):} \quad \frac{1}{2}M_A(t) = A(t), \quad \frac{2}{5}d_A(t) = -D(t), \tag{164}$$

We also remark that in many works discussing GPDs including e.g. [11, 13] the notion of D-term is often defined in a wider sense than here. In our work the D-term is defined more narrowly as the form factor associated with the Lorentz structure $(\Delta^\mu\Delta^\nu - g^{\mu\nu}\Delta^2)$ in the Lorentz decomposition of the matrix elements of the EMT operator. In contrast to this, our $D = \frac{4}{5}d_1$ in the notation of [11] where d_1 is the leading coefficient in an expansion of the Polyakov-Weiss-term in Gegenbauer polynomials.

B Alternative definition of form factors (spin $\frac{1}{2}$)

The following alternative definition of form factors of the EMT is commonly used in the literature, see e.g. [8],

$$\begin{aligned} \langle p'|\hat{T}^{Q,G}_{\mu\nu}(0)|p\rangle &= \bar{u}(p')\Bigg[A^{Q,G}(t)\,\frac{\gamma_\mu P_\nu + \gamma_\nu P_\mu}{2} + B^{Q,G}(t)\,\frac{i(P_\mu\sigma_{\nu\rho} + P_\nu\sigma_{\mu\rho})\Delta^\rho}{4m} \\ &+ C^{Q,G}(t)\,\frac{\Delta_\mu\Delta_\nu - g_{\mu\nu}\Delta^2}{m} \pm \bar{c}(t)g_{\mu\nu}\Bigg]u(p)\,. \end{aligned} \tag{165}$$

By means of the Gordon identity $2m\bar{u}'\gamma^\alpha u = \bar{u}'(i\sigma^{\alpha\kappa}\Delta_\kappa + 2P^\alpha)u$ Eq. (165) can be rewritten as Eq. (15) with

$$\begin{aligned} A^{Q,G}(t) &= A^{Q,G}(t)\,, \\ A^{Q,G}(t) + B^{Q,G}(t) &= 2J^{Q,G}(t)\,, \\ C^{Q,G}(t) &= \frac{1}{4}D^{Q,G}(t)\,. \end{aligned} \tag{166}$$

The constraints (27) translate in this language into $A^Q(0)+A^G(0) = 1$ and $B^Q(0)+B^G(0) = 0$. The latter constraint means that the total anomalous nucleon "gravitomagnetic moment" vanishes.

In models, in which the only dynamical degrees of freedom are effective quark degrees of freedom, the constraint $B^Q(0) = 0$ must hold. Such is the situation in the CQSM where consequently this constraint is satisfied [37].

Interestingly, it was argued [114] that also in QCD the quark and gluon gravitomagnetic moments of the nucleon could vanish separately, i.e. $B^Q(0) = 0$ and $B^G(0) = 0$. That would imply that $A^Q(0) = 2J^Q(0)$ and $A^G(0) = 2J^G(0)$ at any scale, and not only in the asymptotic limit of a large renormalization scale [8], see [114] for details.

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