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Numerical Computation of Traveling Wave and Standing Pulse Solutions to FitzHugh-Nagumo Equations in 2D

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Abdou Alzubaidi, Ph.D.
University of Connecticut, 2018

ABSTRACT

Algorithms are constructed to calculate standing pulse and traveling wave solutions for the FitzHugh-Nagumo equations in two dimensions. The algorithms are based on the application of a steepest descent method to some functionals. These algorithms are global in nature, in the sense that it does not require a good initial guess to guarantee convergence. Their numerical implementation involves construction of asymptotic boundary conditions on truncated domains; asymptotic boundary conditions make the computation less expensive.

Our focus is on two special types of solutions for the FitzHugh-Nagumo equations: radially symmetric standing pulses in the whole space with $\Omega = \mathbb{R}^2$ and traveling wave solutions in a strip $\Omega = \mathbb{R} \times [-L, L]$ for some $L > 0$. Using these algorithms we find multiple traveling pulse and front solutions for the same physical parameters. As an independent check, we test the traveling wave solutions from the steepest descent method using a parabolic solver, which reveals the stability of the solutions at the same time.
Numerical Computation of Traveling Wave and Standing Pulse Solutions to FitzHugh-Nagumo Equations in 2D

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M.S. University of Connecticut, 2013
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Numerical Computation of Traveling Wave and Standing Pulse Solutions to FitzHugh-Nagumo Equations in 2D

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University of Connecticut
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Chapter 1

Introduction

Reaction-diffusion systems are widely employed as models for studying complex patterns in various branches of science [1]. Besides such solutions, there are localized structures that are far away from an equilibrium state; the most prominent examples are standing and traveling waves, including fronts and pulses. In this work we will study standing pulses and traveling waves in two dimensional domains. Our focus will be on the FitzHugh-Nagumo equations due to the richness of their behavior in having many solutions.

Let $d > 0$ and $\gamma > 0$ be positive constants that need to be adjusted later, and let $0 < \beta < 1/2$ be a fixed constant. The FitzHugh-Nagumo equations are

$$
\begin{align*}
  u_t &= \Delta u + \frac{1}{2}(f(u) - v), \\
  v_t &= \Delta v + u - \gamma v
\end{align*}
$$

(1.0.1)

in a domain $\Omega \subseteq \mathbb{R}^2$. Here $f(u) = u(u - \beta)(1 - u)$. To look for standing pulses in
the whole space $\mathbb{R}^2$, we set $u_t = v_t = 0$ in (1.0.1). It is therefore natural to focus on radially symmetric solutions. As $r = |x| \to \infty$, there is no radial change of the solution; in other words $(u, v) \to (u_s, v_s)$ where $(u_s, v_s)$ satisfies

\[
\begin{cases}
  f(u_s) - v_s = 0, \\
  u_s - \gamma v_s = 0.
\end{cases}
\]

(1.0.2)

We study both $\gamma < \frac{4}{(1-\beta)^2}$ and $\gamma > \frac{4}{(1-\beta)^2}$. In the former case there is only one solution $(u_s, v_s) = (0, 0)$. In the latter case there are 3 constant equilibrium solutions.

Our second task is to study traveling waves in a strip domain $\Omega = \mathbb{R} \times [-L, L]$ for some $L > 0$. We impose zero Dirichlet boundary conditions at $|y| = L$ for both $u$ and $v$. Because of these boundary conditions, any non-trivial (constant) solutions of (1.0.2) can never be the asymptotic states that $(u, v)$ goes to as $|x| \to \infty$. In case we are dealing with a traveling front, $\lim_{x \to -\infty} u(x, y)$ has to depend on $y$. This is a major difference between 1D and 2D traveling waves.

Our goal is to construct a unified global algorithm to find the solutions in both cases; it is based on applying the steepest descent method to some functionals. This algorithm is global in nature, in the sense that it does not require a good initial guess to guarantee convergence, so long as this initial guess is in a suitable admissible set $\mathcal{A}$ that we will construct later [3]. The idea comes from [1,2] that a global minimizer of a certain functional, say $J$, in some constrained set $\mathcal{A}$ satisfies the FitzHugh-Nagumo equations. Asymptotic boundary conditions (see [5,6,9]) will be constructed on a truncated domain for both cases in order to speed up the computations.

In Chapter 2 we look for radially symmetric standing pulse solutions in the whole space with $\Omega = \mathbb{R}^2$. We construct an algorithm using the steepest descent method,
which numerically computes critical points of the functional $J$ without a good initial guess. Then we construct asymptotic boundary conditions for a suitable truncated domain. After that we describe a finite element discretization and other implementation details for our algorithm. Finally, we document some numerical results for the standing pulse profiles.

In Chapter 3 we study the behavior of the traveling wave equations of (1.0.1) as $x \to -\infty$. We expect any derivatives of $u$ with respect to $x$ are zero; however such asymptotic states can depend on $y$. We call these equations minimal energy equations. Then we follow similar procedures as in Chapter 2 to derive the algorithm. Finally we present some numerical results.

In Chapter 4 we study the traveling wave equations of (1.0.1) in 2D satisfying zero Dirichlet boundary condition on $\Omega = \mathbb{R} \times [-L, L]$; including both pulses and fronts. A pulse connects the same steady state at both ends, while a front connects distinct steady state solutions. The steady state solutions are those constructed in Chapter 3. The algorithm will be similar, except we need to have a way to determine the wave speed $c$ to be a root of a certain functional. Numerical results show that there co-exist traveling pulse and front solutions, as well as different kinds of fronts, for the same set of physical parameters. We test such solutions in a parabolic solver to determine their stability.
Chapter 2

Radially symmetric standing pulse

For standing pulses we study the time-independent solution of (1.0.1), i.e.

\[
\begin{align*}
\frac{d}{dr} \triangle u + f(u) - v &= 0, \\
\triangle v + u - \gamma v &= 0.
\end{align*}
\]

(2.0.1)

It is a bounded solution of (2.0.1) which satisfies \( \lim_{|x| \to \infty} (u(x), v(x)) = (0, 0) \). Observe that \( (u, v) = (0, 0) \) is a constant equilibrium solution of (2.0.1). When \( \Omega = \mathbb{R}^2 \), it is natural to seek radially symmetric solution with \( u = u(|x|) \) and \( v = v(|x|) \). Let \( r = |x| \) so that \( \triangle = \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} + \frac{1}{r} \frac{d}{dr} (r \frac{d}{dr}) \); the governing equations become

\[
\begin{align*}
&du'' + \frac{d}{r} u' + f(u) - v = 0, \quad (2.0.2a) \\
&v'' + \frac{1}{r} v' + u - \gamma v = 0. \quad (2.0.2b)
\end{align*}
\]

on the domain \([0, \infty)\). Moreover \( u'(0) = v'(0) = 0 \) and \( (u, v) \to (0, 0) \) as \( r \to \infty \).
In this chapter we find radially symmetric solutions of (2.0.1). By restricting ourselves to the space of radially symmetric functions, we give a variational formulation of the governing equations (2.0.2a)-(2.0.2b) in Section 2.1. The critical points of some functional $J$ correspond to standing pulses of (1.0.1). Next, in Section 2.2 we present the steepest descent method, which numerically computes critical points of the functional $J$ without a good initial guess. In order to speed up the calculations in Section 2.3 we construct asymptotic boundary conditions for a truncated computational domain. We observe that the asymptotic boundary conditions render a long domain unnecessary; this saves us computational time and yields more accurate results. In Section 2.4 we explain our finite element discretization schemes and other implementation details. Finally, in Section 2.5 we document some numerical results on the standing pulse profiles and perform an independent check on our algorithm.

## 2.1 Variational formulation

We introduce Hilbert spaces $H^1_r(0, \infty) = \{ w : \int_0^\infty r(w^2 + w'^2)dr < \infty \}$ and $L^2_r(0, \infty) = \{ w : \int_0^\infty rw^2 dr < \infty \}$. Since (2.0.2b) is linear in $v$, it can be shown that for a given $u \in L^2_r$ one can uniquely find a solution $v \in H^1_r$ with zero Neumann boundary conditions at $r = 0$. Designate this solution by $v = \mathcal{L}u$, where $\mathcal{L} : L^2_r \to H^1_r$ is a linear bounded operator. It can be checked that $\mathcal{L}$ is self-adjoint with respect to the weighted inner product for $L^2_r$. Let $F : \mathbb{R} \to \mathbb{R}$ be given by $F(\xi) = -\int_0^\xi f(\eta) d\eta = \xi^4 - \frac{(1+\beta)\xi^3}{3} + \frac{\beta\xi^2}{2}$. We now define a functional $J : H^1_r(0, \infty) \to \mathbb{R}$ such that for all $u$ in the domain

$$J(u) \equiv \int_0^\infty r\left\{ \frac{d}{2}u'^2 + \frac{1}{2}u\mathcal{L}(u) + F(u) \right\}dr. \quad (2.1.1)$$
Since \( \mathcal{L} \) is self-adjoint, we have for all \( \varphi \in H^1_r(0, \infty) \)

\[
J'(u)\varphi = \int_0^\infty r\{du'\varphi' + \mathcal{L}(u)\varphi - f(u)\varphi\}dr.
\]

(2.1.2)

Integrating by parts, we obtain

\[
J'(u)\varphi = \int_0^\infty \{-d(ru')' + rL(u) - rf(u)\}\varphi dr,
\]

(2.1.3)

provided \( u \) is smooth. Hence any minimizer \( u \) of \( J \) satisfies the Euler-Lagrange equation

\[
du'' + \frac{d}{r}u' + f(u) - \mathcal{L}u = 0
\]

in the weak sense. This is equivalent to (2.0.2a)-(2.0.2b). We will seek such a minimizer in some admissible set \( \mathcal{A} \).

Suppose \( \beta < \beta_1 < 1 < \hat{\beta}_2 \) satisfy \( F(\beta_1) = F(\hat{\beta}_2) = 0 \) and \( \beta_2 = \min\{1 + \frac{\beta}{2}, \hat{\beta}_2\} \).

Next take a constant \( M_1 = M_1(\gamma) \geq \beta_2 \) such that for all \( \xi \leq -M_1 \), we have \( f(\xi) \geq \frac{\beta_2}{\gamma} \).

**Definition 2.1.1.** A function \( u \in H^1_r(0, \infty) \) is in the class +/- if there exists \( r_1 \geq 0 \) s.t \( u \geq 0 \) on \([0, r_1]\) and \( u \leq 0 \) on \((r_1, \infty)\).

**Definition 2.1.2.** \( \mathcal{A} \equiv \{u \in H^1_r(0, \infty) : u \text{ is in class +/- and } u - \beta \text{ is in class +/-, } u \leq \beta_2 \text{ and } u \geq -(M_1 + 1)\} \).

Thus when \( u \in \mathcal{A} \), then there exist \( 0 \leq r_0 \leq r_1 \leq \infty \) such that \( \beta \leq u \leq \beta_2 \) on \((0, r_0)\), \( 0 \leq u \leq \beta \) on \((r_0, r_1)\) and \( -(M_1 + 1) \leq u \leq 0 \) on \((r_1, \infty)\).
In Chapter 2 we restrict our attention to \( J : \mathcal{A} \to \mathbb{R} \).

### 2.2 Steepest descent method

We propose to find a standing pulse solution \((u, v)\) numerically using a steepest descent algorithm which tracks a minimizer of \( J \) in the admissible set \( \mathcal{A} \). Let \( u \in \mathcal{A} \) be given. We need to calculate the steepest descent direction \( q \) at this point. To do so, we minimize \( J'(u)\varphi \) subject to \( \|\varphi\|_{H^1_r}^2 = 2 \). Let \( K(\varphi) = J'(u)\varphi + \lambda(\frac{\|\varphi\|_{H^1_r}^2}{2} - 1) \), where \( \lambda \) is a Lagrange multiplier which removes the equality constraints \( \|\varphi\|_{H^1_r}^2 = 2 \) (see [3]). For all \( p \in H^1_r(0, \infty) \),

\[
K'(q)p = J'(u)p + \lambda \int_0^\infty r(q'p' + qp)dr = 0 \quad (2.2.1)
\]

so that a smooth steepest descent direction \( q \) satisfies

\[
J'(u)p + \lambda \int_0^\infty [-rq' + rq]p dr = 0. \quad (2.2.2)
\]

Hence we have

\[
-d(ru')' + r\mathcal{L}(u) - rf(u) - \lambda(rq')' + \lambda rq = 0 \quad (2.2.3)
\]

which can be rewritten as

\[
-(\lambda q + du)'' - \frac{1}{r} (\lambda q + du)' + (\lambda q + du) = du - \mathcal{L}(u) + f(u). \quad (2.2.4)
\]
Here \( u \) is given and \( \lambda q \) is the only unknown in this equation. In order to reduce numerical errors, it is better to solve for \( du + \lambda q \) in (2.2.4) because otherwise we need to calculate numerically the second derivative of \( u \), which is less accurate. Consequently \( \lambda q \) can be computed, which is parallel to the steepest descent direction.

To be more precise, set \( u^* = (du + \lambda q) \) and \( G(u) = du - L(u) + f(u) \), then (2.2.4) gives

\[
-u^*_{rr} - \frac{1}{r} u^*_r + u^* = G(u)
\]

with \( u^*_r(0) = 0 \) and \( |u^*(r)| \to 0 \) as \( r \to \infty \). After solving \( u^* \), we have \( \lambda q = u^* - du \).

It can be checked that \( \lambda q \) points in the same direction as the steepest descent. In other words \( J(u + \alpha \lambda q) < J(u) \) for small \( \alpha > 0 \) unless \( u \) is the minimizer. In addition we have to make sure that \( u + \alpha \lambda q \) stays inside \( A \). In all our computations, this is found to be the case provided we make \( \alpha \) small enough.

We now tabulate the implementation details of our algorithm to find minimizers. The algorithm is global and works without a good initial guess. We will use a finite element method to solve (2.0.2b) and (2.2.5) numerically and the discretization details are explained in later sections. Here is our iterative algorithm to find a minimizer of \( J \) in the admissible set \( A \).

**Algorithm 2.2.1.** Given an initial guess \( u^0 \in A \) and initial descent step size \( 0 < \alpha^0 < 1 \). We iterate to generate updates \( u^n \) for \( n = 1, 2, \ldots \).

1- Solve \( v^n = Lu^n \).
2- Compute \( G(u^n) \), where \( G(u) = du - L(u) + f(u) \).
3- Compute \( u^{n+1} \) solving (2.2.5) with data \( G(u^n) \).
4- Set $\lambda q = (u^* n - du^n)$.  
5- Set $u^{n+1} = u^n + \alpha^n(\lambda q)$, where $u^{n+1}$ is the next update.  
6- Check $J(u^{n+1}) \leq J(u^n)$. If $J(u^{n+1}) > J(u^n)$ then set $\alpha^n = \frac{\alpha^n}{2}$ and recompute $u^{n+1}$ using step 5.  
7- Iterate using steps 1-6 until we have  

$$|J(u^{n+1}) - J(u^n)| \leq tol$$  

for a small prescribed tolerance $tol$. At this point we stop and the final $u$ that we have gotten from the last iteration represents a minimizer of $J$ numerically. Hence $u$ and $v = Lu$ solves (2.0.2a)-(2.0.2b). They represent a radially symmetric standing pulse solution of the FitzHugh-Nagumo equations. 

### 2.3 Asymptotic boundary conditions for $Lu$ and $u^*$

To actually compute numerical solutions we first consider a large truncated interval $[0, R]$ to approximate $(0, \infty)$. Instead of using zero Dirichlet boundary conditions at $r = R$, asymptotic boundary conditions will be constructed to speed up the calculation.

First we consider (2.0.2b). Its solution can be represented by $v = Lu$. In our algorithm, $u$ is known from the previous iteration or an initial guess. The equation

$$v'' + \frac{1}{r}v' - \gamma v = -u$$  

(2.3.1)
is equivalent to the system:

\[
\begin{pmatrix}
v' \\
v''
\end{pmatrix} = B_1 \begin{pmatrix} v \\ v'
\end{pmatrix} - \begin{pmatrix} 0 \\ u
\end{pmatrix}
\] (2.3.2)

where \( B_1 = \begin{pmatrix} 0 & 1 \\ \gamma & -\frac{1}{r} \end{pmatrix} \). The eigenvalues of \( B_1 \) are given by:

\[
\lambda_{1,2} = \frac{1}{2r}(-1 \pm \sqrt{1 + 4\gamma r^2})
\] (2.3.3)

with \( \lambda_1 < 0 < \lambda_2 \). For large \( r \), \( \lambda_1 \to \sqrt{\gamma} \) and \( \lambda_2 \to -\sqrt{\gamma} \). A direct computation gives \( \frac{d\lambda_1}{dr} = \frac{1+\sqrt{1+4\gamma r^2}}{2r^2\sqrt{1+4\gamma r^2}} = O\left(\frac{1}{r^2}\right) \) for large \( r \). Now we have \( L = \begin{pmatrix} -\lambda_1 \\ 1 \end{pmatrix} \) as a left eigenvector of \( B_1 \) for \( \lambda_2 \). Define

\[
\phi = L \cdot \begin{pmatrix} v \\ v'
\end{pmatrix} = -\lambda_1 v + v'
\] (2.3.4)

Then taking scalar product of \( L \) with (2.3.2) we get

\[
\phi' = \lambda_2 \phi - u + L' \cdot \begin{pmatrix} v \\ v'
\end{pmatrix}
= \lambda_2 \phi - u + O\left(\frac{1}{r^2}\right) \cdot v
\]
\[
\approx \lambda_2 \phi - u \text{ near } r = R \gg 1.
\] (2.3.5)

Ignoring the \( O\left(\frac{1}{r^2}\right) \) term, this is a first order differential equation that can be
written as

\[(e^{-\lambda_2 t} \phi)' = -ue^{-\lambda_2 t}.\]

Therefore, with \(\phi\) being bounded as \(r \to \infty\),

\[e^{-\lambda_2 r} \phi(r) = \int_r^\infty u(t)e^{-\lambda_2 t} dt,\]

which implies

\[\phi(r) = \int_r^\infty u(t)e^{\lambda_2 (r-t)} dt. \tag{2.3.6}\]

Simple calculations then yield

\[\lambda_2 \phi(R) - u(R) = -\lambda_2 \int_R^\infty e^{\lambda_2 (R-t)} (u(R) - u(t)) dt \]
\[= \int_R^\infty e^{\lambda_2 (R-t)} u'(t) dt \]

Therefore, assuming \(|u'|\) to be small for large \(r\),

\[|\lambda_2 \phi(R) - u(R)| \leq |o(1)| \int_R^\infty e^{\lambda_2 (R-t)} dt = \frac{|o(1)|}{\lambda_2}.\]

Hence, we can impose the asymptotic boundary condition \(\lambda_2 \phi = u\) at \(r = R\), which is equivalent to:

\[v' = \frac{u}{\lambda_2} + \lambda_1 v \quad at \quad r = R. \tag{2.3.7}\]

Second, we construct asymptotic boundary conditions for (2.2.5). With a known \(G(u) = du - \mathcal{L}(u) + f(u)\), we do the same steps as before for the equation

\[u^{**} + \frac{1}{r} u^{*'} - u = -G, \tag{2.3.8}\]
which is equivalent to the system:

\[
\begin{pmatrix}
    u^s \\
    u^{s'}
\end{pmatrix}' = B_2 \begin{pmatrix}
    u^s \\
    u^{s'}
\end{pmatrix} - \begin{pmatrix}
    0 \\
    G
\end{pmatrix}
\]

(2.3.9)

where \( B_2 = \begin{pmatrix}
    0 & 1 \\
    1 & -\frac{1}{r}
\end{pmatrix} \). The eigenvalues of \( B_2 \) are given by:

\[
\hat{\lambda}_{1,2} = \frac{1}{2r} (-1 \pm \sqrt{1 + 4r^2})
\]

(2.3.10)

where \( \hat{\lambda}_1 < 0 < \hat{\lambda}_2 \). The asymptotic boundary condition is given by

\[
\begin{align*}
     u^{s'} &= \frac{G}{\lambda_2} + \hat{\lambda}_1 u^s \\
     & \text{at } r = R.
\end{align*}
\]

(2.3.11)

\[2.4\text{ Numerical implementation}\]

In this section we will establish the well-posedness of our problems (2.0.2b) and (2.2.5) on the finite interval \([0, R]\) with the established asymptotic boundary conditions at \( r = R \). Next we describe our finite element discretization using piecewise Hermite cubic polynomials as the basis functions. For 1D problems it is easy to implement and produces fourth-order accurate results, with less memory requirement for a given computational grid compared to standard Lagrangian cubic element, see details in [4].
2.4.1 Well-posedness on $[0, R]$

First we derive the weak formulation for (2.0.2b). For $u \in L^2(0, R)$, then $v \in H^2(0, R) \subseteq C^1[0, R]$ by regularity estimate [11]. Take any $\phi \in H^1_r$, multiply the equation by $r\phi$ and integrate on $[0, R]$,

$$-\int_0^R ((rv')'\phi - r\gamma v\phi)dr = \int_0^R ru\phi dr.$$ 

Integrating by parts yields

$$-Rv'(R)\phi(R) + \int_0^R rv'\phi' dr + \int_0^R \gamma rv\phi dr = \int_0^R ru\phi dr.$$

By imposing the asymptotic boundary conditions (2.3.7) we obtain

$$-R\lambda_1(R)v(R)\phi(R) + \int_0^R rv'\phi' dr + \int_0^R \gamma rv\phi dr = \int_0^R ru\phi dr + Ru(R)\lambda_2(R)\phi(R).$$

Then let $\mathcal{A} : H^1_r \times H^1_r \to \mathbb{R}$ be a bilinear functional and $\mathcal{K} : H^1_r \to \mathbb{R}$ be a linear functional defined by

$$\mathcal{A}(v, \phi) = -R\lambda_1(R)v(R)\phi(R) + \int_0^R rv'\phi' dr + \int_0^R \gamma rv\phi dr,$$

$$\mathcal{K}(\phi) = \int_0^R ru\phi dr + Ru(R)\lambda_2(R)\phi(R).$$

The weak formulation is to find $v \in H^1_r(0, R)$ such that

$$\mathcal{A}(v, \phi) = \mathcal{K}(\phi) \quad (2.4.1)$$
for all $\phi \in H^1_r(0,R)$.

Next we derive the weak problem for the equation (2.2.5), which is also has a smooth solution. For any $\phi \in H^1_r$ multiply the equation by $r\phi$ and integrate on $[0,R]$ to get:

$$ - \int_0^R ((ru^*)'\phi - ru^*\phi)dr = \int_0^R rG\phi dr. $$

Integrating by parts gives

$$ -Ru^*(R)\phi(R) + \int_0^R ru^*\phi'dr + \int_0^R ru^*\phi dr = \int_0^R rG\phi dr. $$

By imposing the asymptotic boundary conditions (2.3.11) we obtain

$$ -R\lambda_1(R)u^*(R)\phi(R) + \int_0^R ru^*\phi'dr + \int_0^R ru^*\phi dr = \int_0^R rG\phi dr + R\frac{G(R)}{\lambda_2(R)}\phi(R). $$

Then let:

$$ \mathcal{A}(u^*, \phi) = -R\lambda_1(R)u^*(R)\phi(R) + \int_0^R ru^*\phi'dr + \int_0^R ru^*\phi dr $$

$$ \tilde{K}(\phi) = \int_0^R rG\phi dr + R\frac{G(R)}{\lambda_2(R)}\phi(R) $$

The weak formulation is to find $u^* \in H^1_r(0,R)$ such that

$$ \mathcal{A}(u^*, \phi) = \tilde{K}(\phi) \tag{2.4.2} $$

for all $\phi \in H^1_r(0,R)$. 
To establish the existence and uniqueness for solution of our weak problems, we use the Lax-Milgram Theorem [10]. First we establish a lemma.

**Lemma 2.4.1.** Let $R \geq 2$. Then for any $w, \phi \in H^1_r(0, R)$,

$$|Rw(R)\phi(R)| \leq 5 \|w\|_{H^1_r(0, R)} \|\phi\|_{H^1_r(0, R)}.$$

**Proof.** Let $w$ be smooth and $u(r) = \sqrt{r}w(r)$. Then

$$\sqrt{R}w(R) = |u(R)| \leq \|u\|_{L^\infty(R-1, R)} \leq \sqrt{2} \|u\|_{H^1_r(R-1, R)} \quad \text{(Sobolev imbedding)}$$

$$= \sqrt{2} \int_{R-1}^R [((\sqrt{r}w)^2 + ((\sqrt{r}w)'^2)dr$$

$$\leq \sqrt{2} \int_{R-1}^R [rw^2 + 2rw'w + \frac{2}{4r}w^2]dr$$

$$\leq 2 \int_{R-1}^R [rw^2 + rw'w + \frac{1}{4}rw^2]dr$$

$$\leq \sqrt{5} \int_{R-1}^R (rw^2 + rw'^2)dr$$

$$\leq \sqrt{5} \|w\|_{H^1_r(0, R)}.$$

Similarly

$$\sqrt{R}\phi(R) = \sqrt{5} \|\phi\|_{H^1_r(0, R)}.$$

Hence

$$|Rw(R)\phi(R)| \leq 5 \|w\|_{H^1_r(0, R)} \|\phi\|_{H^1_r(0, R)}.$$
As a consequence of Lemma 2.4.1, it is easily seen that there exist a positive constant $C$ such that

$$|\mathcal{A}(w, \phi)| \leq C \|w\|_{H^1_r} \|\phi\|_{H^1_r}$$

for all $w, \phi$ in $H^1_r$. Indeed, since $\lambda_1(R) \to -\sqrt{\gamma}$ as $R \to \infty$,

$$|\mathcal{A}(v, \phi)| \leq |\lambda_1| |Rv(R)\phi(R)| + \|v'\|_{L^2_r} \|\phi'\|_{L^2_r} + \gamma \|v\|_{H^1_r} \|\phi\|_{L^2_r}$$

$$\leq 5 |\lambda_1| \|v\|_{H^1_r} \|\phi\|_{H^1_r} + (1 + \gamma) \|v\|_{H^1_r} \|\phi\|_{H^1_r}$$

$$\leq C \|v\|_{H^1_r} \|\phi\|_{H^1_r}$$

for all $R \geq 2$, with $C$ being independent of $R$. It is now clear that $\mathcal{A}$ is a bilinear bounded functional.

Next, observe that for all $v \in H^1_r$,

$$\mathcal{A}(v, v) = -R\lambda_1 v^2(R) + \int_0^R r|v'|^2 + \gamma \int_0^R r|v|^2$$

$$\geq \int_0^R r(|v'|^2 + \gamma|v|^2)$$

$$\geq min(1, \gamma) \|v\|_{H^1_r}^2$$

since $\lambda_1 < 0$. This establishes coercivity for the bilinear functional $\mathcal{A}$. Therefore, by the Lax-Milgram theorem, there exists a unique solution $v \in H^1_r(0, R)$ satisfying (2.4.1). Proving the existence and uniqueness of solution for (2.2.5) will be similar.
2.4.2 Finite element method

Now we will study (2.0.2b) numerically using a finite element discretization on a truncated domain $[0, R]$. The treatment for (2.2.5) is similar and will be skipped in our discussion. Define $T_h(0, R) = \bigcup_{j=1}^{N} T_j$, $T_j = [r_j, r_{j+1}]$, where $r_1 = 0 < r_2 < r_3 < \ldots < r_{N+1} = R$ with uniform mesh size $h = \frac{r_{j+1} - r_j}{N}$. We use piecewise Hermite cubic polynomials as our basis functions [4]. Define piecewise cubic polynomials $\phi_j(r)$ and $\phi'_j(r)$ such that $\phi_j(r_i) = \phi'_j(r_i) = \delta_{ij}$ and $\phi'_j(r_i) = \phi_j(r_i) = 0$ for all $i, j$ where

$$
\delta_{ij} = \begin{cases} 
1, & \text{if } j = i \\
0, & \text{if } j \neq i 
\end{cases}
$$

(2.4.3)

With a known $u$ either from the previous step or an initial guess in our algorithm, we solve $v$ from (2.0.2b). Let $v_h$ be an approximation of $v$ in the vector space $\text{span}\{\phi_j, \phi'_j, j = 1, \ldots, N+1\}$ so that

$$
v_h(r) = \sum_{j=1}^{N+1} v_{j1} \phi_j + \sum_{j=1}^{N+1} v_{j2} \phi'_j
$$

(2.4.4)

for some coefficients $v_{ji}$, $j = 1, 2$, $i = 1, \ldots, N + 1$. By our choice of basis functions we have

$$
v_h(r_i) = v_{i1} \quad \text{for} \quad 1 \leq i \leq N + 1;
$$

$$
v'_h(r_i) = v_{i2} \quad \text{for} \quad 1 \leq i \leq N + 1.
$$
To satisfy the boundary condition $v'(0) = 0$, we require

$$v'_h(r_1) = v_{12} = 0.$$ 

Now we define a Hermite cubic finite element spaces as follows. Let

$$X_h = \{ w_h : [0, R] \to \mathbb{R} : w_h|_{[r_j, r_{j+1}]} \in \mathbb{P}_3([r_j, r_{j+1}]), 1 \leq j \leq N \} \cap C^1[0, R]$$

where $\mathbb{P}_3([a, b])$ is the set of polynomials with degree 3 or less on the interval $[a, b]$,

$$X^0_h = \{ w_h \in X_h : w_h'(0) = 0, w_h'(R) = \lambda_1 w_h(R) \},$$

and

$$Y_h = \{ w_h \in X_h : w_h'(0) = 0, w_h'(R) = \frac{u(R)}{\lambda_2} + \lambda_1 w_h(R) \}.$$

Our goal is to find $v_h \in Y_h$ such that

$$\mathcal{A}(v_h, \phi_h) = \mathcal{K}(\phi_h) \quad (2.4.5)$$

for all $\phi_h \in X^0_h$.

We need to incorporate the asymptotic boundary conditions (2.3.7) into (2.4.4); doing so leads to

$$v_h(r) = \sum_{j=1}^N v_{j1} \phi_{j1} + \sum_{j=2}^N v_{j2} \phi_{j2} + v_{N+1,1} \phi_{N+1,1} + \left( \frac{u(R)}{\lambda_2} + \lambda_1 v_{N+1,1} \right) \phi_{N+1,2}. $$
Note that $v_{N+1,2}$ has been eliminated. Set

$$g = \frac{u(R)}{\lambda_2} \phi_{N+1,2}$$

and

$$w_h(r) = \sum_{j=1}^{N} v_{j1} \phi_{j1} + \sum_{j=2}^{N} v_{j2} \phi_{j2} + v_{N+1,1} \psi \in X_h^0$$

where $\psi = \phi_{N+1,1} + \lambda_1 \phi_{N+1,2}$; then $v_h = w_h + g$. Therefore, we have

$$\mathcal{A}(w_h, \phi_h) = K(\phi_h) - \mathcal{A}(g, \phi_h).$$

Let $\mathcal{F}(\phi_h) = K(\phi_h) - \mathcal{A}(g, \phi_h)$. We want to find $w_h \in X_h^0$ such that

$$\mathcal{A}(w_h, \phi_h) = \mathcal{F}(\phi_h)$$

(2.4.6)

for all $\phi_h \in X_h^0$. Therefore, for any $\phi_h \in X_h^0$ we have:

$$\mathcal{A} \left( \sum_{j=1}^{N} v_{j1} \phi_{j1} + \sum_{j=2}^{N} v_{j2} \phi_{j2}, \phi_h \right) + v_{N+1,1} \mathcal{A}(\psi, \phi_h) = \mathcal{F}(\phi_h)$$

which is equivalent to:

$$\sum_{j=1}^{N} v_{j1} \mathcal{A}(\phi_{j1}, \phi_{ik}) + \sum_{j=2}^{N} v_{j2} \mathcal{A}(\phi_{j2}, \phi_{ik}) + v_{N+1,1} \mathcal{A}(\psi, \phi_{ik}) = \mathcal{F}(\phi_{ik})$$

for $\{(i, k) = (1, 1)\} \cup \{(i, k) : i = 2, 3, ..., N, \ k = 1, 2\}$ and

$$\sum_{j=1}^{N} v_{j1} \mathcal{A}(\phi_{j1}, \psi) + \sum_{j=2}^{N} v_{j2} \mathcal{A}(\phi_{j2}, \psi) + v_{N+1,1} \mathcal{A}(\psi, \psi) = \mathcal{F}(\psi)$$
Thus we have

\[ A\vec{w} = \vec{b} \]

where:

\[
\vec{w} = \begin{bmatrix}
v_{11} \\
v_{21} \\
v_{22} \\
. \\
v_{N1} \\
v_{N2} \\
v_{N+1,1}
\end{bmatrix},
\vec{b} = \begin{bmatrix}
\mathcal{F}(\phi_{11}) \\
\mathcal{F}(\phi_{21}) \\
\mathcal{F}(\phi_{22}) \\
. \\
\mathcal{F}(\phi_{N1}) \\
\mathcal{F}(\phi_{N2}) \\
\mathcal{F}(\psi)
\end{bmatrix}
\]

are vectors of size \(2N\), and \(A\) is a matrix with size \(2N \times 2N\). If we employ \(\vec{\psi}\) to denote \((\phi_{11}, \phi_{21}, \phi_{22}, \phi_{31}, \phi_{32}, ..., \phi_{N1}, \phi_{N2}, \psi)\), then \(A_{ij} = \mathcal{A}(\psi_i, \psi_j), \ 1 \leq i, j \leq 2N.\) It is clear that \(A\) is a symmetric positive definite matrix. We solve this linear system and the solutions \(w_h \in X_h^0\) satisfy \(\mathcal{A}(w_h, \phi_h) = \mathcal{F}(\phi_h)\) for all \(\phi_h \in X_h^0\). Finally \(v_h = w_h + g\).
2.5 Numerical results

2.5.1 Varying \(d\) while fixing \(\beta\) and \(\gamma\)

First, let \(\beta = \frac{1}{4}, \gamma = 0.1\) be fixed and vary \(d\). For such \(\beta\) and \(\gamma\), there is a unique constant equilibrium solution \((u, v) = (0, 0)\). We now give numerical results from the steepest descent algorithm. The domain \([0, \infty)\) is approximated by \([0, L]\) for some large \(L > 0\). As constructed in an earlier section, asymptotic boundary conditions for \(u\) and \(v\) are imposed at \(r = L\). Let \(M\) be the number of elements in the finite element discretization. We take \(M = 800\) and \(L = 20\) in our numerical experiments. We stop when \(J(u^n) \leq J(u^{n+1}) + \text{tolerance}\), where \(n\) is the iteration. In Table 2.5.1 when we decrease the tolerance, we see that the solution is converging. Figure 2.5.1 gives a plot of the standing pulse profiles for \(u\) and \(v\). As \(r\) increases from zero, we see \(u\) that increases first to its positive maximum, then decreases to a negative minimum, and finally decays to 0 as \(r \to \infty\). The corresponding \(v\) is always positive.

<table>
<thead>
<tr>
<th>tol</th>
<th>(u(r=0))</th>
<th>max((u))</th>
<th>min((u))</th>
<th>max((v))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1e-06</td>
<td>9.277152e-01</td>
<td>9.277152e-01</td>
<td>-8.244442e-02</td>
<td>6.366134e-02</td>
</tr>
<tr>
<td>1e-09</td>
<td>9.171316e-01</td>
<td>9.451418e-01</td>
<td>-8.144410e-02</td>
<td>4.873743e-02</td>
</tr>
<tr>
<td>1e-12</td>
<td>9.198347e-01</td>
<td>9.299525e-01</td>
<td>-7.396818e-02</td>
<td>4.957951e-02</td>
</tr>
<tr>
<td>1e-13</td>
<td>9.183126e-01</td>
<td>9.287271e-01</td>
<td>-7.494739e-02</td>
<td>5.032105e-02</td>
</tr>
<tr>
<td>1e-14</td>
<td>9.180782e-01</td>
<td>9.285277e-01</td>
<td>-7.508319e-02</td>
<td>5.043452e-02</td>
</tr>
<tr>
<td>1e-15</td>
<td>9.179054e-01</td>
<td>9.283844e-01</td>
<td>-7.518790e-02</td>
<td>5.051801e-02</td>
</tr>
</tbody>
</table>

Table 2.5.1: For \(d = 1e^{-4}\), \(u\) and \(v\) converge to steady profiles when we decrease the tolerance.
Figure 2.5.1: Radially symmetric standing pulse solution for $\beta = \frac{1}{4}$, $\gamma = 0.1$ and $d = 1e - 4$.

Figure 2.5.2: The radially symmetric standing pulse profiles of $u$ with different values of $d$. When $d = 7.4e - 4$, the steepest descent algorithm gives us the trivial solution $u = 0$. For our steepest descent algorithm it seems when $d \geq 7.4e - 4$ there is no non-trivial solution in the admissible set $\mathcal{A}$. 
2.5.2 Varying $\gamma$ while fixing $\beta$ and $d$

Second, let $\beta = \frac{1}{4}, d = 7e - 4$ be fixed and vary $\gamma$ in the range $7.11 \approx \frac{4}{(1-\beta)^2} < \gamma < \frac{9}{(1-2\beta)(2-\beta)} \approx 10.28$. For such $\beta$ and $\gamma$, there are three constant equilibrium solutions. Again we use $M = 800$ on the domain $[0, L]$ with $L = 20$. The resulting plots of $u$ are given in Figure 2.5.3. We noticed that when $\gamma$ gets closer to $\frac{9}{(1-2\beta)(2-\beta)} \approx 10.28$, the pulse width increases rapidly. As a consequence, it requires a much longer computational domain. It is likely that for $\gamma > 10.28$, the pulse disappears and becomes another kind of solution.

2.5.3 Validation of our algorithm

Let $\beta = \frac{1}{4}, \gamma = 0.1$ and $d = 1e - 4$. The number of elements $M$ is inversely proportional to the spatial mesh size $h$. As an independent check on our algorithm, we calculate the residuals of the numerical solutions $u_h$ and $v_h$. Recall that the numer-
ical solutions are represented in terms of the Hermite basis functions; we substitute them in the left hand sides of (2.0.2a)-(2.0.2b). Define the residuals

\[ R(u_h) \equiv d u''_h + \frac{d}{r} u'_h + f(u_h) - v_h, \quad \text{(2.5.1a)} \]

\[ R(v_h) \equiv v''_h + \frac{1}{r} v'_h + u_h - \gamma v_h. \quad \text{(2.5.1b)} \]

Because of numerical error due to discretization and premature stopping, neither \( R(u_h) \) or \( R(v_h) \) are zero. A measure of their magnitudes gives an indication of the accuracy. Also, convergence of the residuals to zero indicates convergence of \((u_h, v_h)\) to the true solution of (2.4.1)-(2.4.2). We focus on \( R(u_h) \) for simplicity. Suppose for some constant \( C \) and \( \alpha \), \( \|R(u_h)\|_{L^2} \sim Ch^\alpha \) as \( h \to 0 \). Then

\[ (\log 2)RU \equiv \log \left( \frac{\|R(u_{2h})\|_{L^2}}{\|R(u_h)\|_{L^2}} \right) \sim \log 2^\alpha = \alpha \log 2. \]

Thus \( RU \sim \alpha \) as \( h \to 0 \). Increasing \( M \) by a factor of 2 successively from \( M = 160 \) to 2560, we obtain Table 2.5.2. A similar investigation using \( v_h \) leads to \( RV \). Thus

<table>
<thead>
<tr>
<th>M</th>
<th>RU</th>
<th>RV</th>
</tr>
</thead>
<tbody>
<tr>
<td>320-640</td>
<td>2.1410395</td>
<td>2.0192919</td>
</tr>
<tr>
<td>640-1280</td>
<td>1.9716526</td>
<td>2.0226650</td>
</tr>
<tr>
<td>1280-2560</td>
<td>2.0384595</td>
<td>2.0205349</td>
</tr>
</tbody>
</table>

Table 2.5.2: Convergence rate validation.

we see that \( \alpha \approx 2 \) for our numerical experiments. This validates our algorithm.
Chapter 3

Minimal energy equations

In Chapter 4 we will study traveling wave solutions of the FitzHugh-Nagumo equations on the domain \( \mathbb{R} \times [-L, L] \) for some given \( L > 0 \). In order to do so, we first need to understand the asymptotic behavior of such solutions. This is the goal in this chapter. We recall the FitzHugh-Nagumo equations

\[
\begin{align*}
    u_t &= \Delta u + \frac{1}{2}(f(u) - v), \\
    v_t &= \Delta v + u - \gamma v
\end{align*}
\]  

(3.0.1)

and now derive the governing equations for their traveling waves in the domain \( \Omega = \mathbb{R} \times [-L, L] \) as follows. For some smooth functions \( \tilde{u} : \Omega \to \mathbb{R} \) and \( \tilde{v} : \Omega \to \mathbb{R} \), the traveling wave solution must be of the form \( u(x, y, t) = \tilde{u}(x - ct, y) \) and \( v(x, y, t) = \tilde{v}(x - ct, y) \) where \( c \) is the wave speed, yet to be determined. Let \( \xi = x - ct \), then
we have

\[
\begin{align*}
\frac{\partial^2 \tilde{u}}{\partial \xi^2} + \frac{\partial^2 \tilde{u}}{\partial y^2} + c \frac{\partial \tilde{u}}{\partial \xi} + \frac{1}{d} (f(\tilde{u}) - \tilde{v}) &= 0, \\
\frac{\partial^2 \tilde{v}}{\partial \xi^2} + \frac{\partial^2 \tilde{v}}{\partial y^2} + c \frac{\partial \tilde{v}}{\partial \xi} + \tilde{u} - \gamma \tilde{v} &= 0.
\end{align*}
\]

(3.0.2a)

Taking \((x, y) = (c\xi, y)\), \(u(x, y) = \tilde{u}(\frac{x}{c}, y)\) and \(v(x, y) = \tilde{v}(\frac{x}{c}, y)\), the traveling wave equations then become

\[
\begin{align*}
c^2 u_{xx} + u_{yy} + c^2 u_x + \frac{1}{d} (f(u) - v) &= 0, \\
c^2 v_{xx} + v_{yy} + c^2 v_x + u - \gamma v &= 0
\end{align*}
\]

(3.0.3a)

(3.0.3b)

on the domain \(\mathbb{R} \times [-L, L]\). Moreover we impose the boundary conditions \((u, v) = (0, 0)\) at \(|y| = L\). Our aim in this chapter is to study the behavior of the traveling wave solutions of (3.0.3) as \(x \to -\infty\). Since we are looking for a minimum energy traveling wave solution using a steepest descent algorithm, it is natural that as \(x \to -\infty\), the asymptotic limit corresponds to a minimal energy standing wave solution in a 1D domain. To be precise, we study

\[
\begin{align*}
U_{yy} + \frac{1}{d} (f(U) - V) &= 0, \\
V_{yy} + U - \gamma V &= 0
\end{align*}
\]

(3.0.4a)

(3.0.4b)

with boundary condition \((U, V) = (0, 0)\) at \(|y| = L\). These are obtained from (3.0.3) by setting \((u(x, y), v(x, y)) = (U(y), V(y))\) so that all the terms involving derivatives in \(x\) are zero. We restrict our attention to solutions symmetric about \(y = 0\), i.e. we study (3.0.4) on the domain \([0, L]\) with the boundary conditions \(U'(0) = V'(0) = 0\)
and \( U = V = 0 \) at \( y = L \). We let \( L = 1 \) throughout this and the next chapter.

### 3.1 Variational formulation

Let \( 0 < \beta < 1/2 \) be a fixed constant, while \( d > 0 \) and \( \gamma > 0 \) are constants whose magnitudes we need to adjust later. Making use of the fact that (3.0.4b) is linear in \( V \) with constant coefficients, we can find the Green’s function for the operator \((\gamma - \frac{d^2}{dy^2})\) and obtain

\[
V(y) = \mathcal{L}U(y) \equiv \int_0^1 G(y, s)U(s) \, ds
\]

where

\[
G(y, s) = \begin{cases} 
\frac{1}{\sqrt{\gamma}} e^{-\sqrt{\gamma} s} \cosh(\sqrt{\gamma} y) & \text{if } y < s, \\
\frac{1}{\sqrt{\gamma}} e^{-\sqrt{\gamma} s} \cosh(\sqrt{\gamma} y) & \text{if } y > s.
\end{cases}
\]

Thus \( G(y, s) = G(s, y) \) for all \( y, s \) in \([0, 1]\) and

\[
\mathcal{L}U(y) = \int_0^y \frac{1}{\sqrt{\gamma}} e^{-\sqrt{\gamma} y} \cosh(\sqrt{\gamma} s) U(s) \, ds + \int_y^1 \frac{1}{\sqrt{\gamma}} e^{-\sqrt{\gamma} y} \cosh(\sqrt{\gamma} y) U(s) \, ds.
\]

We introduce Hilbert spaces \( \mathcal{A}^* = \{ w : w(1) = 0 \text{ and } \int_0^1 (w^2 + w')dy < \infty \} \) and \( L^2(0, 1) = \{ w : \int_0^1 w^2dy < \infty \} \). Let \( F : \mathbb{R} \to \mathbb{R} \) is given by \( F(\xi) = -\int_0^\xi f(\eta) \, d\eta = \frac{\xi^4}{4} - \frac{(1+\beta)\xi^3}{3} + \frac{\beta \xi^2}{2} \). Define a functional \( J : \mathcal{A}^* \to \mathbb{R} \) such that

\[
J(U) = \int_0^1 \left\{ \frac{d}{2} U_y^2 + \frac{1}{2} U \mathcal{L}U + F(U) \right\} \, dy.
\]

Making use of the symmetry of the Green’s function \( G \), we can check that \( \mathcal{L} \) is self-
adjoint with respect to the $L^2$ inner product. Then for all $\varphi \in A^*$ we have

$$J'(U)\varphi = \int_0^1 \{dU'\varphi' + \mathcal{L}(U)\varphi - f(U)\varphi\} dy. \quad (3.1.1)$$

**Lemma 3.1.1.** If $\phi \in A^*$ is a smooth critical point of $J$, then $-d\phi'' + \mathcal{L}(\phi) - f(\phi) = 0$ and $\phi'(0) = 0$.

**Proof.** Let $\phi \in A^*$ be a smooth critical point of $J$, then for all $p \in A^*$ we have

$$0 = J'(\phi)p = \int_0^1 \{d\phi'p' + \mathcal{L}(\phi)p - f(\phi)p\} dy$$

$$= \int_0^1 \{-d\phi'' + \mathcal{L}(\phi) - f(\phi)\}p dy + \phi'p\bigg|_0^1$$

$$= \int_0^1 \{-d\phi'' + \mathcal{L}(\phi) - f(\phi)\}p dy - \phi'(0)p(0). \quad (3.1.2)$$

Step 1: take $p \in C_0^\infty(0, 1) \subseteq A^*$, then

$$0 = J'(\phi)p = \int_0^1 \{-d\phi'' + \mathcal{L}(\phi) - f(\phi)\}p dy \quad \text{for all } p \in C_0^\infty(0, 1)$$

which implies $-d\phi'' + \mathcal{L}(\phi) - f(\phi) = 0$.

Step 2: Putting back this new information into (3.1.2), we have for any $p \in A^*$

$$0 = J'(\phi)p = -\phi'(0)p(0).$$

Since $p(0)$ can be arbitrary, this implies $\phi'(0) = 0$. \qed

**Remark 3.1.2.** The requirement $\phi'(0) = 0$ in the above lemma is known as a natural boundary condition.
As a consequence of Lemma 3.1.1, any smooth local minimizer \( U \in \mathcal{A}^* \) of \( J \) satisfies the Euler-Lagrange equation

\[
dU'' + f(U) - \mathcal{L}U = 0 \quad (3.1.3)
\]

and \( U'(0) = 0 \). This is equivalent to the system (3.0.4a)-(3.0.4b) with the boundary conditions \( U'(0) = V'(0) = 0 \) and \( U(1) = V(1) = 0 \). We will seek minimizer of \( J \) in \( \mathcal{A}^* \) using a steepest descent algorithm.

### 3.2 Steepest descent method

We want to find a standing pulse solution \((U, V)\) numerically using a steepest descent algorithm in the admissible set \( \mathcal{A}^* \), therefore it is necessary to calculate the steepest descent direction \( \phi \). Let \( U \in \mathcal{A}^* \) be given, which is not a solution of (3.0.4). To evaluate the steepest descent direction at \( U \), we want to minimize \( J'(U)\phi \) subject to \( ||\phi||_{H^1}^2 = 2 \). Thus we define

\[
\mathcal{K}(\phi) = J'(U)\phi + \lambda \left( \frac{||\phi||_{H^1}^2}{2} - 1 \right),
\]

where \( \lambda \) is a Lagrange multiplier which removes the equality constraint \( ||\phi||_{H^1}^2 = 2 \) (see [3]). Its steepest descent direction \( \phi \) satisfies

\[
\mathcal{K}'(\phi)p = J'(U)p + \lambda \int_0^1 (\phi'p' + \phi p)dy = 0 \quad (3.2.1)
\]

for all \( p \in \mathcal{A}^* \). Hence a smooth \( \phi \) satisfies the natural boundary condition \( \phi'(0) = 0 \), and

\[
J'(U)p + \lambda \int_0^1 [-\phi'' + \phi]p dy = 0. \quad (3.2.2)
\]
Therefore we have

\[-dU'' + \mathcal{L}(U) - f(U) - \lambda \phi'' + \lambda \phi = 0,\]  

(3.2.3)

which can be recast as

\[-(dU + \lambda \phi)'' + (dU + \lambda \phi) = dU - \mathcal{L}(U) + f(U).\]  

(3.2.4)

Then \(\lambda \phi\) can be computed, which is parallel to the steepest descent direction. To be more precise, set \(U^* = (dU + \lambda \phi)\) and \(P(U) = dU - \mathcal{L}(U) + f(U)\), then (3.2.4) gives

\[-U^*_{yy} + U^* = P(U)\]  

(3.2.5)

with \(U^*_y(0) = 0\) and \(U^*(1) = 0\). After solving for \(U^*\), it follows that \(\lambda \phi = U^* - dU\). It can be checked that \(\lambda \phi\) points in the same direction as the steepest descent, so that \(\lambda > 0\). In other words, \(J(U + \alpha \lambda \phi) < J(U)\) for some small \(\alpha > 0\), unless \(U\) is a minimizer. The algorithm ensures that \(U + \alpha \lambda \phi\) stays inside \(\mathcal{A}^*\). In other words, \(U + \alpha \lambda \phi\) is the new \(U\) in \(\mathcal{A}^*\) with a lower energy of \(J\). We now repeat the above procedures with this new \(U\) until we arrive at a local minimizer.

The algorithm of this case will be analogous to the algorithm 2.2.1 of Chapter 2. We only substitute \(u\) by \(U\), \(v\) by \(V\) and \(G\) by \(P\).
3.3 Numerical implementation

Again we solve (3.0.4b) and (3.2.5) numerically with same steps as in Chapter 2 using a similar finite element method. In particular, we use piecewise Hermite cubic polynomials as the basis functions.

Deriving the weak formulation for (3.0.4b) and (3.2.5) is analogous to that in Chapter 2, except we need to impose a zero Dirichlet boundary condition at \( y = 1 \) here instead. The weak problem for (3.0.4b) is to find \( V \in \mathcal{A}^* \) such that

\[
a_\gamma(V, \phi) = \mathcal{K}_1(\phi)
\]

(3.3.1)

for all \( \phi \in \mathcal{A}^* \) where

\[
a_\gamma(V, \phi) = \int_0^1 (V' \phi' + \gamma V \phi) \, dy,
\]

\[
\mathcal{K}_u(\phi) = \int_0^1 U \phi \, dy.
\]

Also the weak problem for the updated equation (3.2.5) is to find \( U^* \in H^1(0,1) \) such that

\[
a_1(U^*, \phi) = \mathcal{K}_1^*(\phi)
\]

(3.3.2)

for all \( \phi \in \mathcal{A}^* \), where

\[
a_1(U^*, \phi) = \int_0^1 (U^{*'} \phi' + U^* \phi) \, dy,
\]

\[
\mathcal{K}^*_p(\phi) = \int_0^1 P \phi \, dy.
\]

By the Lax Milgram Theory there exist unique solutions for (3.3.1) and (3.3.2).
Finally, we discretize (3.3.1) and (3.3.2) in the finite element space that involves piecewise Hermite cubic polynomials to get linear algebraic systems, which we solve using MATLAB.

### 3.4 Numerical results

#### 3.4.1 Varying $\gamma$ while fixing $\beta$ and $d$

First, let $\beta = \frac{1}{4}, d = 7e - 4$ be fixed and vary $\gamma$ in the steepest descent algorithm. Zero Dirichlet boundary conditions for $U$ and $V$ are imposed at $|y| = 1$ and we restrict our attention to solutions that are symmetric about $y = 0$ on the domain $(-1, 1)$. We now present our numerical results from the steepest descent algorithm. Let $J = \inf_{w \in A} J(w)$ and $M$ be the number of elements in the finite element discretization. We take $M = 400$ in our numerical experiments. We stop when $J(U^n) \leq J(U^{n+1}) + \text{tolerance}$, where $n$ is the iterative count.

For various values of $\gamma$, we found different classes of local minimizers; they are classified according to the oscillation of $U$, described as -/+/- (see Figure 3.4.1A), +/+-/+ (see Figure 3.4.1B), + (see Figure 3.4.1C) and the trivial solution, respectively. In order to draw the above conclusions, we give results of our numerical experiments below.

Case 1: We employ the $U$ profile in Figure 3.4.1B, which is in the class -/+/-, as the initial guess of our algorithm and vary $\gamma$. The final local minimizer profiles are given in Figures 3.4.2, 3.4.3 and summarized in Table 3.4.1. In Figure 3.4.4 we compute the energy $J$ for these final states for comparison.
Figure 3.4.1: We fix $\beta = \frac{1}{4}$ and $d = 7e - 4$. Plot (A) shows the solution of class +/-/+ at $\gamma = 4$. Plot (B) shows the solution of class -/+/- at $\gamma = 5$. Plot (C) shows the solution of class + at $\gamma = 10$. Each solution is generated by distinct initial guesses in the algorithm; these initial guess are in the same class as their final solution profiles.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>final profiles of $U$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0 , 1.8]</td>
<td>Trivial solutions</td>
</tr>
<tr>
<td>[1.9 , 9.4]</td>
<td>-/+/-</td>
</tr>
<tr>
<td>[9.5 , $\infty$)</td>
<td>+</td>
</tr>
</tbody>
</table>

Table 3.4.1: Three types of profiles of $U$ for various $\gamma$ result from an initial guess in the class -/+/-.

Figure 3.4.2: Plots (A),(B) show the final profiles of $U,V$ respectively for both $\gamma = 1.8$ and $\gamma = 1.9$. In plot (A) we have the trivial solution when $\gamma = 1.8$, but a solution of class -/+/- when $\gamma = 1.9$. 
Figure 3.4.3: Plots (A),(B) show final profiles of $U,V$ respectively for both $\gamma = 9.4$ and $\gamma = 9.5$. In plot (A) the solution $U$ is of class -$+/-$ when $\gamma = 9.4$, but of class + when $\gamma = 9.5$.

Figure 3.4.4: The energy $J$ for standing pulse local minimizers obtained in Table 3.4.1. Two jumps occur as depicted in Figures 3.4.2 and 3.4.3.

Case 2: We employ the $U$ profile in Figure 3.4.1 plot (A), which is in the class $+/-/+$, as the initial guess in our algorithm and vary $\gamma > 0$, the final local minimizer profiles are given in Figure 3.4.5 and summarized in Table 3.4.2. In Figure 3.4.6 a plot of the energy $J$ of these final states versus $\gamma$ is given.
Table 3.4.2: Two types of profiles of $U$ for various $\gamma$ result from an initial guess in the class $+/\cdot+/\cdot$.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>final profiles of $U$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0, 13.9]$</td>
<td>$+/\cdot+/\cdot$</td>
</tr>
<tr>
<td>$[14, \infty)$</td>
<td>$+$</td>
</tr>
</tbody>
</table>

Figure 3.4.5: Plots (A),(B) show final profiles of $U, V$ respectively for both $\gamma = 13.9$ and $\gamma = 14$. In (A) the solution is of class $+/\cdot+/\cdot$ when $\gamma = 13.9$, but becomes of class $+$ when $\gamma = 14$.

Figure 3.4.6: The energy $J$ for standing pulse local minimizers obtained in Table 3.4.2. A jump occurs as depicted in Figure 3.4.5.
Case 3: We employ the $U$ profile in Figure 3.4.1 plot (C), which is in the class $+$, as the initial guess in our algorithm and vary $\gamma > 0$, the final local minimizer profiles are given in Figure 3.4.7 and summarized in Table 3.4.3. In Figure 3.4.8 a plot of the energy $\mathcal{J}$ of these final states versus $\gamma$ is given.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>final profiles of $U$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0, 6.5]$</td>
<td>$+/\cdot/+\cdot$</td>
</tr>
<tr>
<td>$(6.6, \infty)$</td>
<td>$+$</td>
</tr>
</tbody>
</table>

Table 3.4.3: Two types of profiles of $U$ for various $\gamma$ result from an initial guess in the class $+$.

Figure 3.4.7: Plots (A),(B) show final profiles of $U,V$ respectively for both $\gamma = 6.5$ and $\gamma = 6.7$. In (A) the solution is of class $+/\cdot/+\cdot$ when $\gamma = 6.5$, but becomes of class $+$ when $\gamma = 6.7$. 
Moreover, in addition with all cases above if we employ the initial guess $U_0(y) \cong 0$ then we will obtain the trivial solutions for all values of $\gamma$. Thus $(U, V) = (0, 0)$ is a local minimizer.

In Figures 3.4.4, 3.4.6, 3.4.8 we observe discontinuous jumps in $\mathcal{J}$ as we increase $\gamma$ gradually in our steepest descent algorithm. The solutions that we obtain are local minimizers. On the other hand we can follow these branches of solutions when we start with a large $\gamma = \gamma_0$ and decrease $\gamma$ gradually. Jumps are now observed at different values of $\gamma$.

As a consequence of combining all these information, we conclude that there can be more than one kind of local minimizers with the same physical parameters, see Table 3.4.4. In Figures 3.4.9 to 3.4.13 we show multiple profiles of $U$ for the same value of $\gamma$. 

**Figure 3.4.8:** The energy $\mathcal{J}$ for standing pulse local minimizers obtained in Table 3.4.3. A jump occurs as depicted in Figure 3.4.7.
### Table 3.4.4: A summary of all possible local minimizers for various $\gamma$.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>Trivial solutions</th>
<th>$+/ -$ class</th>
<th>$+/-/+$ class</th>
<th>$+$ class</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0, 1.8]$</td>
<td>✓</td>
<td>-</td>
<td>✓</td>
<td>-</td>
</tr>
<tr>
<td>$[1.9, 6.5]$</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>-</td>
</tr>
<tr>
<td>$[6.6, 9.4]$</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>$[9.5, 13.9]$</td>
<td>✓</td>
<td>-</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>$[14, \infty)$</td>
<td>✓</td>
<td>-</td>
<td>-</td>
<td>✓</td>
</tr>
</tbody>
</table>

**Figure 3.4.9:** A single non-trivial local minimizer $U$ for $\gamma=1$

**Figure 3.4.10:** Two non-trivial local minimizer $U$ for $\gamma=4$

**Figure 3.4.11:** Three non-trivial local minimizer $U$ for $\gamma=7$

**Figure 3.4.12:** Two non-trivial local minimizer $U$ for $\gamma=10$
Figure 3.4.13: A single non-trivial local minimizer $U$ for $\gamma=14$

Figure 3.4.14 shows the energy $J$ of various types of local minimizers versus $\gamma$ by combining Figures 3.4.4, 3.4.6, 3.4.8 and the corresponding information when we increase or decrease $\gamma$ along various branches of solutions. There may not be discontinuous jumps in $\inf_{w \in A^*} J(w)$ when we vary $\gamma$. It is likely that there are solutions in other bifurcation branches that connect to the discontinuous jump in Figure 3.4.14. We think these solutions are not local minimizers and hence will be missed by our steepest descent algorithm. A continuation method will be needed to investigate these bifurcation branches. To make sure that the results in Figure 3.4.14 are typical, we run the computation for the same $\beta = \frac{1}{4}$; however we now set $d = 1.4e - 3$. Upon varying $\gamma$ we obtain Figure 3.4.15. This is qualitatively similar to Figure 3.4.14.

We notice that the solutions of class $+/--$ always has lower energy than class $-/++$. In Chapter 4 we usually find traveling wave solution that has lower energy on the left boundary. In fact, we could not find traveling wave solutions whose asymptotic behavior as $x \to -\infty$ is of class $-/++$. A more details discussion will be deferred until Chapter 4.
Figure 3.4.14: The energy $\mathcal{J}$ versus $\gamma$ for all cases when $d = 7e - 4$ and $\beta = \frac{1}{4}$

Figure 3.4.15: The energy $\mathcal{J}$ versus $\gamma$ for all cases when $d = 1.4e - 3$ and $\beta = \frac{1}{4}$
3.4.2 Varying $d$ while fixing $\beta$ and $\gamma$

Let $\beta = \frac{1}{4}$ and $\gamma = 7$. We take each type of solution in Figure 3.4.1 as an initial guess in our algorithm; in turn, we run the code for various values of $d$. Figures 3.4.16 to 3.4.18 present the profiles of $U$ and $V$ for

**Figure 3.4.16:** Class $-/+/-$ solutions: Plots (A) and (B) present the profiles of $U$ and $V$, respectively, with various values of $d$. When $d \geq 3.1e-3$ we get trivial solutions.

**Figure 3.4.17:** Class $+/-/+$ solutions: Plots (A) and (B) present the profiles of $U$ and $V$, respectively, with various values of $d$. When $d \geq 3.9e-3$ we get trivial solutions.
each case. The $\mathcal{J}$ plots with respect to $d$ for the solution classes $+/-/+, -/+/-$ and $+$ are given in Figure 3.4.19.

**Figure 3.4.18:** Class $+$ solutions: Plots (A) and (B) present the profiles of $U$ and $V$, respectively, with various values of $d$. When $d \geq 3.2e-3$ we get trivial solutions.

**Figure 3.4.19:** The energy $\mathcal{J}$ versus $d$ for all cases when $\gamma = 7$ and $\beta = \frac{1}{4}$.
Chapter 4

Traveling wave in two dimensional domains

In this chapter we look for traveling wave solutions of (1.0.1) in two dimensional domains $\Omega = \mathbb{R} \times [-L, L]$. As derived in Chapter 3, the governing traveling wave equations are

\begin{align}
    c^2 u_{xx} + u_{yy} + c^2 u_x + \frac{1}{d} (f(u) - v) &= 0, \\
    c^2 v_{xx} + v_{yy} + c^2 v_x + u - \gamma v &= 0.
\end{align}

(4.0.1a) \hspace{1cm} (4.0.1b)

It is noted that the wave speed $c$ needs to be determined as well as $u$ and $v$. We restrict ourselves to solutions which are symmetric about $y = 0$ and impose zero Dirichlet boundary conditions $(u,v) = (0,0)$ at $|y| = L$.

For 1D case, there are multiple constant steady states when $\gamma > \frac{4}{(1-\beta)^2}$. As $|x| \to \infty$, the traveling wave converges to these constant states [8]. Coming back to
the 2D scenario, the Dirichlet boundary conditions prevent these constant solutions from being the steady state solutions. In fact, as $|x| \to \infty$, the 2D traveling wave will converge to the steady state solutions that we construct in Chapter 3.

We will investigate both traveling pulse and front solutions. As in the previous chapters, we employ a variational formulation to study (4.0.1) in Section 4.1. For each given $c > 0$, we look for its minimizer $u_c$ in an admissible set $A_c$, to be defined later. Thus $\mathcal{J}(c) \equiv J_c(u_c) = \min_{w \in A_c} J_c(w)$. As will be shown, the traveling speed $c_0$ is determined by $\mathcal{J}(c_0) = J_{c_0}(u_{c_0}) = 0$.

Next in section 4.2 we present the steepest descent method, which numerically computes local minimizers of the functional $J_c$ without a good initial guess; the wave speed $c_0$ will be chosen so that $\mathcal{J}(c_0) = 0$. In Section 4.3 we construct asymptotic boundary conditions for a truncated computational domain for pulses and fronts. Then in Section 4.4 we explain our finite element discretization schemes and other implementation details for the algorithm. Finally, in Section 4.5 we document some numerical results on the traveling wave profiles from the steepest descent algorithm and test them in a parabolic solver to determine their stability.

### 4.1 Variational formulation

Let $c > 0$. Define the Hilbert spaces $L^2_{e^x} \equiv \{ w : \int_\Omega e^x w^2 dx dy < \infty \}$ and $H^1_{e^x} \equiv \{ w : \int_\Omega e^x (w_x^2 + \frac{1}{c^2} w_y^2 + w^2) dx dy < \infty \}$ with inner products

$$\langle u, w \rangle_{L^2_{e^x}} \equiv \int_\Omega e^x uw \, dx dy,$$
\[ \langle u, w \rangle_{H^1_{ex}} \equiv \int_{\Omega} e^x (u_x w_x + \frac{1}{e^2} u_y w_y + uw) \, dx \, dy , \]

respectively. Although \( H^1_{ex} \) depends on \( c \), they are all equivalent norms for all \( c \neq 0 \). Let \( \mathcal{H} \equiv \{ w : \int_{\Omega} e^x (w_x^2 + \frac{1}{e^2} w_y^2 + w^2) \, dx \, dy < \infty , \, w(x, y) = 0 \text{ at } |y| = L \text{ and } w \text{ is symmetric about } y = 0 \} \). Define a functional \( J_c : \mathcal{H} \to \mathbb{R} \) by:

\[ J_c(u) \equiv \int_{\Omega} e^x \{ c^2 du_x^2 \frac{2}{2} + \frac{du_y^2}{2} + F(u) + \frac{1}{2} u \mathcal{L}_c u \} \, dx \, dy \]

where \( v = \mathcal{L}_c u \) by solving (4.0.1b) with \( v = 0 \) at \( |y| = L \). When \( u \) is symmetric about \( y = 0 \), \( v \) will automatically be symmetric as well. If \((u, v, c)\) is a traveling wave solution, any translation of \( u, v \) in the \( x \)-direction remains a solution. To restrict ourselves to a unique traveling wave, we let

\[ \mathcal{A}_c \equiv \{ w \in \mathcal{H} : \|w\|_{H^1_{ex}}^2 = 2 \} \]

be the admissible set and study \( J_c : \mathcal{A}_c \to \mathbb{R} \). As \( \mathcal{L}_c \) is self-adjoint with respect to the inner product in \( L^2_{ex} \),

\[ J'_c(u)\phi = \int_{\Omega} e^x \{ c^2 du_x \phi_x + du_y \phi_y - f(u)\phi + \mathcal{L}_c(u)\phi \} \, dx \, dy . \]

Suppose \( u \) is smooth, then

\[ J'_c(u)\phi = \int_{\Omega} [ -(c^2 e^x u_x)_x - e^x du_{yy} - e^x f(u) + e^x \mathcal{L}_c(u)] \phi \, dx \, dy . \]
Hence any minimizer \( u \) of \( J_c \) satisfies the Euler-Lagrange equation

\[
(e^x d^2 e^x u_x)_x + e^x d u_{yy} + e^x f(u) - e^x \mathcal{L}_c(u) = 0
\]

in the weak sense. This is equivalent to (4.0.1).

We seek minimizers of \( J_c \) in the admissible set \( \mathcal{A}_c \). Introduce a Lagrange multiplier \( \lambda \) to remove the equality constraint \( \|w\|_{H^1_{1\epsilon}}^2 = 2 \) in \( \mathcal{A}_c \). Suppose \( u_c \) is an unconstrained minimum point of \( \mathcal{I}_c \) where

\[
\mathcal{I}_c(u) := J_c(u) + \lambda \left( \frac{\|u\|_{H^1_{1\epsilon}}^2}{2} - 1 \right).
\]

Then for all \( \phi \in H \),

\[
0 = \mathcal{I}'_c(u_c)\phi = J'_c(u_c)\phi + \lambda \langle u_c, \phi \rangle_{H^1_{1\epsilon}}.
\]  \hfill (4.1.1)

If we set \( \phi = \frac{\partial u_c}{\partial x} \) in (4.1.1) we get

\[
0 = \int_\Omega e^x \left\{ dc^2 \frac{\partial}{\partial x} \left( \frac{u^2_{1\epsilon}}{2} \right) + \frac{\partial}{\partial x} \left( \frac{u^2_{2\epsilon}}{2} \right) + \frac{\partial}{\partial x} F(u_c) + \frac{\partial}{\partial x} \left( \frac{1}{2} u_c \mathcal{L}_c u_c \right) \right\} dxdy
\]

\[
+ \lambda \int_\Omega e^x \frac{\partial}{\partial x} \left( \frac{u^2_{1\epsilon}}{2} + \frac{1}{c^2} \frac{u^2_{2\epsilon}}{2} + \frac{u^2_{3\epsilon}}{2} \right) dxdy.
\]

Upon integration by parts,

\[
J_c(u_c) + \lambda = 0.
\]

If we pick \( c = c_0 \) so that \( J_{c_0}(u_{c_0}) = 0 \) then the Lagrange multiplier is \( \lambda = 0 \). Putting this information in (4.1.1), we have \( J'_c(u_{c_0})\phi = 0 \). Thus \( (u_{c_0}, v_{c_0}, c_0) \) will be a traveling wave solution.
Let $\Omega_M \equiv (M, M+1) \times (L, -L)$. Define $v_c = Lu_c$. As $u_c, v_c \in H^1_{c\epsilon}(\Omega)$, we have $u_c, v_c \in H^1(\Omega_M)$ with

$$\|u_c\|_{H^1(\Omega_M)} \to 0 \quad \text{and} \quad \|v_c\|_{H^1(\Omega_M)} \to 0 \quad \text{as} \quad M \to \infty. \quad (4.1.2)$$

By Sobolev estimates, $u_c, v_c \in L^p(\Omega_M)$ for any $p > 1$. With $f$ being a cubic polynomial, these imply $f(u_c) \in L^p(\Omega_M)$ for any $p > 1$. Typical regularity estimates on (4.0.1) ensure that $u_c, v_c \in W^{2,p}(\Omega_M) \subseteq C^1(\overline{\Omega_M})$. Higher norm bounds on $u_c$ and $v_c$ can now be obtained. Together with (4.1.2) we obtain $u \to 0$, $v \to 0$ uniformly as $x \to \infty$.

On the other hand, as $x \to -\infty$, the traveling wave solutions go to standing wave solutions $(U, V)$ of (3.0.4) in Chapter 3. As we focus on traveling wave solutions $(u, v)$ which are symmetric about $y = 0$, the corresponding $(U, V)$ will be symmetric about $y = 0$ as well.

### 4.2 Steepest descent method

We would like to find a traveling wave solution $(u, v, c)$ numerically by the steepest descent method, which tracks a minimizer $u_c$ of $J_c$ in the admissible set $A_c$ so that $J(c) = \inf_{w \in A_c} J_c(w) = J_c(u_c)$. We remark that the norm $H^1_{c\epsilon}$ depends on $c$. While this does not change the value of $J_c(u)$, it does affect the steepest descent direction we calculate below. Distinct equivalent norms give rise to distinct steepest descent directions.

Given $c > 0$ and an initial guess $u \in A_c$. Note that the initial guess of $u$ need
not be good. As there are multiple local minimizers, different initial guesses may lead to distinct minimizers. Thus it helps if we use an initial guess that resembles qualitatively the solution we are looking for. When we look for a pulse, we guess $u \in \mathcal{A}_c$ such that $|u(x,y)| \to 0$ as $|x| \to \infty$. However, for the case of a front, we expect the traveling wave solution goes to the minimal energy solution $U(y)$ for the same set of physical parameters in Chapter 3. We therefore employ an initial guess $u(x,y) = f^*(x)U(y)$ in the steepest descent method, with some smooth function $f^*$ that satisfies $f^* \to 0$ as $x \to \infty$ and $f^* \to 1$ as $x \to -\infty$.

At a given $c > 0$ and $u \in \mathcal{A}_c$, denote the steepest descent direction by $q = q(u,c)$ on the manifold $\mathcal{A}_c = \{u \in \mathcal{H} : \|u\|_{\mathcal{H}_1^{ex}}^2 = 2\}$, which is normalized so that $\|q\|_{\mathcal{H}_1^{ex}}^2 = 2$. We introduce two Lagrange multipliers $\rho$ and $\mu$ to remove the equality constraints $\|q\|_{\mathcal{H}_1^{ex}}^2 = 2$ and $\langle u, q \rangle_{\mathcal{H}_1^{ex}} = 0$ [7,3], respectively. Hence we can find $q$ as an unconstrained critical point of

$$K_c(\phi) = J_c'(u)\phi + \rho\left(\frac{1}{2}\|\phi\|_{\mathcal{H}_1^{ex}}^2 - 1\right) + \mu\langle u, \phi \rangle_{\mathcal{H}_1^{ex}} \quad \text{for all } \phi \in \mathcal{H}.$$ 

Therefore the steepest descent direction $q$ satisfies

$$K_c'(q)p = J_c'(u)p + \rho\langle q, p \rangle_{\mathcal{H}_1^{ex}} + \mu\langle u, p \rangle_{\mathcal{H}_1^{ex}} = 0 \quad \text{for all } p \in \mathcal{H}_1^{ex}. \quad (4.2.1)$$

Set $p = u$ in (4.2.1) we obtain

$$J_c'(u)u + \rho\langle q, u \rangle_{\mathcal{H}_1^{ex}} + \mu\|u\|_{\mathcal{H}_1^{ex}}^2 = 0.$$
Since $\|u\|^2_{H^1_{ex}} = 2$ and $(q, u)_{H^1_{ex}} = 0$ it follows that

$$\mu = -\frac{1}{2} J_c'(u) u,$$

which can be calculated. Suppose both $u$ and $q$ are smooth. From (4.2.1) we have

$$0 = \int_{\Omega} [-(c^2 e^x u_x)_x - e^x du_{yy} - e^x f(u) + e^x \mathcal{L}_c(u)] p \, dx \, dy$$
$$+ \rho \int_{\Omega} e^x (q_x p_x + \frac{1}{c^2} q_y p_y + qp) \, dx \, dy$$
$$+ \mu \int_{\Omega} e^x (u_x p_x + \frac{1}{c^2} u_y p_y + up) \, dx \, dy$$

for all $p \in H^1_{ex}$, which gives

$$-u_{xx}^* - \frac{1}{c^2} u_{yy}^* - u_x^* + u^* = Q$$

(4.2.2)

where $u^* = (dc^2 + \mu) u + \rho q$ and $Q = dc^2 u + f(u) - \mathcal{L}_c(u)$. With $u$ and $c$ being given and $\mu$ can be computed, $\rho q$ is the only unknown in this equation. Observe that (4.2.2) is a linear equation; we numerically solve $u^*$ using a finite element method. After solving $u^*$, we have $\rho q = u^* - (dc^2 + \mu) u$, which is parallel to the steepest descent direction. In fact, it can be checked that $\rho q$ points in the same direction as the steepest descent direction. In other words $J_c(u + \alpha \rho q) < J_c(u)$ for small $\alpha > 0$ unless $u$ is the minimizer. In addition, we have to make sure that $u + \alpha \rho q$ stays inside $\mathcal{A}_c$. In all our computations, this is found to be the case provided we make $\alpha$ small enough.

We now tabulate the implementation details of our algorithm to find the minimizer. The algorithm is global and works without a good initial guess. We will use
a finite element method to solve (4.0.1b) and (4.2.2) numerically and the discretization details are explained in later sections. Here is our iterative algorithm to find a minimizer of $J_c$ in the admissible set $\mathcal{A}_c$.

**Algorithm 4.2.1.** Given $c > 0$ and an initial guess $u^0 \in \mathcal{A}_c$ and initial descent step size $0 < \alpha^0 < 1$. We iterate to generate updates $u^n$ for $n = 1, 2, \ldots$

1- Solve $v^n = \mathcal{L}_c u^n$ using (3.0.1b).
2- Set $\mu^n = -\frac{1}{2}J'_c(u^n)u^n$
3- Compute $Q(u^n)$ where $Q(u) = dc^2u + f(u) - \mathcal{L}_c u$.
4- Solve $u^*\!\!_n$ using (4.2.2).
5- Set $\rho_q = (u^*\!\!_n - (\mu + dc^2)u^n)$.
6- Set $u^{n+1} = u^n + \alpha^n(\rho_q)$ where $u^{n+1}$ is the next update.
7- Check $J(u^{n+1}) \leq J(u^n)$. If $J(u^{n+1}) > J(u^n)$ then set $\alpha^n = \frac{\alpha^n}{2}$ and recompute $u^{n+1}$ using step 6.
8- Iterate using steps 1-7 until we have

$$|J(u^{n+1}) - J(u^n)| \leq tol.$$ 

for a small prescribed tolerance $tol$. The final $u_c$ that we have gotten from the last iteration represents a local minimizer of $J_c$ numerically. Hence $u_c$ is a minimizer of $J_c$ with $J_c(u_c) = \inf_{w \in \mathcal{A}_c} J_c(w) \equiv \mathcal{J}(c)$. Pick $c = c_0$ such that $\mathcal{J}(c_0) = J_{c_0}(u_{c_0}) = 0$ then $(u_{c_0}, v_{c_0}, c_0)$ is a traveling wave solution.
4.3  Asymptotic boundary conditions

4.3.1  Pulse case

To find numerical solutions we first consider a large truncated domain \( \Omega_t = [a, b] \times [0, L] \) to approximate the infinite strip \((-\infty, \infty) \times [0, L]\). We use zero Dirichlet boundary conditions for both \( u \) and \( v \) at \( y = L \) for all \( x \in \mathbb{R} \). Furthermore we impose \( \frac{\partial v}{\partial y}(\cdot, 0) = 0 \). As \( \frac{\partial u}{\partial y}(\cdot, 0) = 0 \) by the natural boundary condition associated with the variational formulation, our algorithm will automatically restrict to solutions that are symmetric about \( y = 0 \). We now construct asymptotic boundary conditions at \( x = a \) and \( x = b \) to speed up the calculation.

First we consider (4.0.1b). Its solution can be represented by \( v = \mathcal{L}_c u \). In our algorithm \( u \) is known from the previous iteration or as an initial guess. Fourier expansions for \( v = \mathcal{L}_c u \) and \( u \) are

\[
\begin{align*}
  u(x, y) &= \sum_{j=1}^{\infty} \hat{u}_j(x) \sin\left(\frac{(2j - 1)\pi(y + L)}{2L}\right), \\
  v(x, y) &= \sum_{j=1}^{\infty} \hat{v}_j(x) \sin\left(\frac{(2j - 1)\pi(y + L)}{2L}\right). \tag{4.3.1}
\end{align*}
\]

Insert the relation (4.3.1) into the equation \( v_{xx} + \frac{1}{c^2} v_{yy} + v_x - \frac{1}{c^2} \gamma v = -\frac{1}{c^2} u \) and obtain

\[
\begin{align*}
  \sum_{j=1}^{\infty} \left( \hat{v}_j''(x) + \hat{v}_j'(x) - \left( \frac{(2j - 1)^2 \pi^2}{4L^2 c^2} + \frac{\gamma}{c^2} \right) \hat{v}_j(x) \right) \sin\left(\frac{(2j - 1)\pi(y + L)}{2L}\right) &= \sum_{j=1}^{\infty} -\frac{1}{c^2} \hat{u}_j(x) \sin\left(\frac{(2j - 1)\pi(y + L)}{2L}\right).
\end{align*}
\]
Then the Fourier coefficients satisfy

\[
\hat{v}_j''(x) + \hat{v}_j'(x) - \left( \frac{(2j - 1)^2 \pi^2}{4L^2 c^2} + \frac{\gamma}{c^2} \right) \hat{v}_j(x) = -\frac{1}{c^2} \hat{u}_j(x).
\] (4.3.2)

In case \( x < 0 \) with \( |x| \gg 1 \), we assume that \( |\hat{u}_j(x)| \ll |\hat{u}_1(x)| \) and \( |\hat{v}_j(x)| \ll |\hat{v}_1(x)| \) for all \( j > 1 \), thus

\[
\begin{align*}
  u(x, y) &\approx \hat{u}_1(x) \sin \left( \frac{\pi(y + L)}{2L} \right), \\
  v(x, y) &\approx \hat{v}_1(x) \sin \left( \frac{\pi(y + L)}{2L} \right).
\end{align*}
\] (4.3.3)

In other words \( \hat{u}_1(x) \) and \( \hat{v}_1(x) \) are the dominant behavior in \( x \) for \( u \) and \( v \), respectively, as \( x \to -\infty \). We now focus on the equation

\[
\hat{v}_1''(x) + \hat{v}_1'(x) - \left( \frac{\pi^2}{4L^2 c^2} + \frac{\gamma}{c^2} \right) \hat{v}_1(x) = -\frac{1}{c^2} \hat{u}_1(x).
\] (4.3.4)

Since this equation depends only on \( x \), we do the same derivation as in Chapter 2.

The equation (4.3.4) is equivalent to the system

\[
\begin{pmatrix}
  \hat{v}_1 \\
  z
\end{pmatrix}' = A
\begin{pmatrix}
  \hat{v}_1 \\
  z
\end{pmatrix} - \begin{pmatrix}
  0 \\
  \hat{v}_1
\end{pmatrix}
\] (4.3.5)

where

\[
A = \begin{pmatrix}
  0 & 1 \\
  \frac{\pi^2}{4L^2 c^2} + \frac{\gamma}{c^2} & -1
\end{pmatrix}
\].

The eigenvalues of \( A \) are given by:

\[
\begin{align*}
  \nu_1 &= \frac{1}{2} \left( -1 - \sqrt{1 + \frac{\pi^2}{L^2 c^2} + \frac{4\gamma}{c^2}} \right), \\
  \nu_2 &= \frac{1}{2} \left( -1 + \sqrt{1 + \frac{\pi^2}{L^2 c^2} + \frac{4\gamma}{c^2}} \right)
\end{align*}
\] (4.3.6)
with $\nu_1 < 0 < \nu_2$. Now we have $L_1 = \begin{pmatrix} -\nu_2 \\ 1 \end{pmatrix}$ and $L_2 = \begin{pmatrix} -\nu_1 \\ 1 \end{pmatrix}$ as left eigenvectors of $A$ for $\nu_1$ and $\nu_2$, respectively. Taking scalar product of $L_1$ with (4.3.5) we get

$$\phi_1' = \nu_1 \phi_1 - \frac{\hat{u}_1}{c^2},$$

(4.3.7)

where

$$\phi_1 = L_1 \cdot \begin{pmatrix} \hat{v}_1 \\ z \end{pmatrix} = -\nu_2 \hat{v}_1 + z.$$  

(4.3.8)

This is a first order differential equation which can be written as

$$(e^{-\nu_1 t} \phi_1)' = -\frac{\hat{u}_1}{c^2} e^{-\nu_1 t}.$$  

Therefore with $\phi_1$ being bounded as $x \to -\infty$,

$$e^{-\nu_1 x} \phi_1(x) = -\int_{-\infty}^{x} \frac{\hat{u}_1}{c^2}(t)e^{-\nu_1 t} dt$$

which implies

$$\phi_1(x) = -\int_{-\infty}^{x} \frac{\hat{u}_1(t)}{c^2} e^{\nu_1(x-t)} dt.$$  

(4.3.9)

Further manipulation gives

$$\nu_1 \phi_1(a) - \frac{\hat{u}_1(a)}{c^2} = \frac{\nu_1}{c^2} \int_{-\infty}^{a} e^{\nu_1(a-t)}(\hat{u}_1(a) - \hat{u}_1(t)) dt$$

$$= -\frac{1}{c^2} \int_{-\infty}^{a} e^{\nu_1(a-t)} \hat{u}_1'(t) dt.$$
Therefore,

\[ |\nu_1 \phi(a) - \frac{\hat{u}_1(a)}{c^2}| \leq \frac{|o(1)|}{|\nu_1| c^2} \int_{-\infty}^{a} e^{\nu_1(a-t)} dt = \frac{|o(1)|}{|\nu_1| c^2}. \]

Hence, we can impose the asymptotic boundary conditions \( \nu_1 \phi_1 = \frac{\hat{u}_1}{c^2} \) at \( x = a \), which is equivalent to:

\[
\hat{v}'_1 = \frac{\hat{u}_1}{\nu_1 c^2} + \nu_2 \hat{v}_1 \quad \text{at} \quad x = a. \tag{4.3.10}
\]

Similarly

\[
\hat{v}'_1 = \frac{\hat{u}_1}{\nu_2 c^2} + \nu_1 \hat{v}_1 \quad \text{at} \quad x = b. \tag{4.3.11}
\]

Combining (4.3.10) and (4.3.11) with (4.3.3), the asymptotic boundary conditions for (4.0.1b) are

\[
v_x = \frac{u}{\nu_1 c^2} + \nu_2 v \quad \text{at} \quad x = a, \quad \text{for} \quad 0 < y < L, \tag{4.3.12}
\]

\[
v_x = \frac{u}{\nu_2 c^2} + \nu_1 v \quad \text{at} \quad x = b, \quad \text{for} \quad 0 < y < L. \tag{4.3.13}
\]

Next we construct asymptotic boundary conditions for (4.2.2). With a known \( Q(u) = dc^2 u - L_c(u) + f(u) \), the derivation of asymptotic boundary conditions of (4.2.2) will be analogous to (4.0.1b). We only substitute \( \frac{\gamma}{c^2} \) with 1 and \( \frac{u}{c^2} \) with \( Q \).

Then the new eigenvalues are

\[
\nu_1^* = \frac{1}{2} \left( 1 - \sqrt{5 + \frac{\pi^2}{L^2 c^2}} \right),
\]

\[
\nu_2^* = \frac{1}{2} \left( 1 + \sqrt{5 + \frac{\pi^2}{L^2 c^2}} \right). \tag{4.3.14}
\]
Therefore the asymptotic boundary conditions are given by

\begin{align}
  u_x^* &= \frac{Q}{\nu_1^*} + \nu_2^* u^* \quad \text{at} \quad x = a, \quad \text{for} \quad 0 < y < L, \\
  u_x^* &= \frac{Q}{\nu_2^*} + \nu_1^* u^* \quad \text{at} \quad x = b, \quad \text{for} \quad 0 < y < L.
\end{align}

\section*{4.3.2 Front case}

Asymptotic boundary conditions (4.3.12)-(4.3.13) and (4.3.15)-(4.3.16) were derived for the case of a traveling pulse. One is tempted to treat a traveling front without modification. Since \((u, v) \to 0\) as \(x \to \infty\) for all \(y\), the asymptotic boundary conditions (4.3.13) and (4.3.16) for a front at \(x = b\) will be the same as for a pulse. However we find that the asymptotic boundary condition at \(x = a\) is not accurate for the front case. To be more precise, by using the asymptotic boundary conditions (4.3.12) and (4.3.15), the numerical solution \((u_{c_0}, v_{c_0}, c_0)\) we find at \(x = a\) does not match the minimal energy solution \((U, V)\) with the same physical parameters. This is not a surprise as the assumption (4.3.3) is a poor one due to the shape of \(U\).

Therefore we need to derive a new asymptotic boundary condition at \(x = a\).

Fix \(y\) and regard \(v_{yy}\) as a known function. To construct the asymptotic boundary condition for

\begin{align}
v_{xx} + v_x - \frac{1}{c^2} \gamma v &= -\frac{1}{c^2} (u + v_{yy}),
\end{align}

(4.3.17)
we do the same steps as for equation (4.3.4). Then the new eigenvalues are

\[ \hat{\nu}_1 = \frac{1}{2} \left( -1 - \sqrt{1 + \frac{4\gamma}{c^2}} \right), \]
\[ \hat{\nu}_2 = \frac{1}{2} \left( -1 + \sqrt{1 + \frac{4\gamma}{c^2}} \right). \]  

(4.3.18)

Therefore the asymptotic boundary condition is given by

\[ v_x = \frac{1}{\hat{\nu}_1 c^2} (u + v_{yy}) + \hat{\nu}_2 v \quad \text{at} \quad x = a \quad \text{and} \quad 0 < y < L. \]  

(4.3.19)

The treatment for the term \( v_{yy} \) will become apparent in Section 4.4.2 below. Also we need to construct asymptotic boundary conditions for (4.2.2) at \( x = a \). Let \( u_{yy}^* \) be known and fix \( y \) then the derivation of the asymptotic boundary conditions for

\[ u_{xx}^* + u_x^* - u^* = -(Q + \frac{1}{c^2} u_{yy}^*). \]  

(4.3.20)

will be analogous to that for (4.3.17). We only substitute \( \frac{\gamma}{c^2} \) with 1 and \( \frac{u}{c^2} \) with \( Q \). Then the new eigenvalues are

\[ \hat{\nu}_1^* = \frac{1}{2} \left( -1 - \sqrt{5} \right), \]
\[ \hat{\nu}_2^* = \frac{1}{2} \left( -1 + \sqrt{5} \right). \]  

(4.3.21)

The corresponding asymptotic boundary conditions are given by

\[ u_x^* = \frac{1}{\hat{\nu}_1 c^2} (c^2 Q + u_{yy}^*) + \hat{\nu}_2 u^* \quad \text{at} \quad x = a, \quad \text{for} \quad 0 < y < L, \]  

(4.3.22)

The new asymptotic boundary conditions at \( x = a \) give us a good behavior of the
solutions \((u_{c_0}, v_{c_0}, c_0)\) at \(x = a\) which are converging to the minimal energy solutions \((U, V)\).

4.4 Numerical implementation

In this subsection we will establish the well-posedness of our problems (4.0.1b) and (4.2.2) on the finite strip \(\Omega = [a, b] \times [0, L]\) with the established asymptotic boundary conditions at \(x = a\) and \(x = b\), zero Dirichlet boundary conditions at \(y = L\) and zero Neumann boundary condition at \(y = 0\). Next we describe our finite element discretization using Lagrange polynomials [4] as the basis functions.

4.4.1 Well-posedness on truncated domain for the pulse case

First we derive the weak formulation for (4.0.1b). Observe that \(v\) is a smooth function by known regularity estimates. For any \(\phi \in H_{e_x}^1\), multiply the equation by \(\phi\) and integrate over \(\Omega_t\),

\[
\int_a^b \int_0^L \left( c^2 v_{xx} \phi + v_{yy} \phi + c^2 v_x \phi - \gamma v \phi \right) dy dx = - \int_a^b \int_0^L u \phi dy dx.
\]

Integrating by parts yields

\[
\int_a^b \int_0^L \left( c^2 v_{xx} \phi_x + v_y \phi_y - c^2 v_x \phi + \gamma v \phi \right) dy dx - \int_0^L c^2 v_x \bigg|_{x=a}^{x=b} \phi dy = \int_a^b \int_0^L u \phi dy dx. \tag{4.4.1}
\]
By inserting the asymptotic boundary conditions (4.3.12) and (4.3.13) we obtain
\[
\int_a^b \int_0^L \left( c^2 v_x \phi_x + v_y \phi_y - c^2 v_x \phi + \gamma v \phi \right) dy dx - \int_0^L c^2 \phi (\nu_1 v|_{x=b} - \nu_2 v|_{x=a}) dy
\]
\[= \int_a^b \int_0^L u \phi dy dx + \int_0^L \phi \left( \frac{1}{\nu_2} u|_{x=b} - \frac{1}{\nu_1} u|_{x=a} \right) dy.
\]
Let $\mathcal{A}_p : H^1_{e^x} \times H^1_{e^x} \to \mathbb{R}$ be a bilinear functional and $\mathcal{K}_p : H^1_{e^x} \to \mathbb{R}$ be a linear functional defined by
\[
\mathcal{A}_p(v, \phi) = \int_a^b \int_0^L \left( c^2 v_x \phi_x + v_y \phi_y - c^2 v_x \phi + \gamma v \phi \right) dy dx
\]
\[- \int_0^L c^2 \phi (\nu_1 v|_{x=b} - \nu_2 v|_{x=a}) dy
\]
and
\[
\mathcal{K}_p(\phi) = \int_a^b \int_0^L u \phi dy dx + \int_0^L \phi \left( \frac{1}{\nu_2} u|_{x=b} - \frac{1}{\nu_1} u|_{x=a} \right) dy.
\]
The weak formulation is to find $v \in \mathcal{H}$ such that
\[
\mathcal{A}_p(v, \phi) = \mathcal{K}_p(\phi)
\]
for all $\phi \in \mathcal{H}$.

We perform a similarly analysis for (4.2.2). Let $\mathcal{A}_p^* : H^1_{e^x} \times H^1_{e^x} \to \mathbb{R}$ be a bilinear
functional and $K_p^* : H^1_{e^x} \to \mathbb{R}$ be a linear functional defined by

$$A^*_p(u^*, \phi) = \int_a^b \int_0^L (c^2 u^*_x \phi_x + u^*_y \phi_y - c^2 u^*_x \phi + c^2 u^* \phi) \, dy \, dx$$

$$- \int_0^L c^2 \phi (\nu^*_1 u^*_x |_{x=b} - \nu^*_2 u^*_x |_{x=a}) \, dy$$

and

$$K_p^*(\phi) = \int_a^b \int_0^L c^2 Q \phi \, dy \, dx + \int_0^L c^2 \phi (\frac{1}{\nu^*_2} Q |_{x=b} - \frac{1}{\nu^*_1} Q |_{x=a}) \, dy.$$ 

The weak formulation is to find $u^* \in \mathcal{H}$ such that

$$A^*_p(u^*, \phi) = K_p^*(\phi)$$

(4.4.3)

for all $\phi \in \mathcal{H}$.

### 4.4.2 Well-posedness on truncated domain for the front case

First we derive the weak formulation for (4.0.1b). This is the same as in the pulse case except that we insert the asymptotic boundary conditions (4.3.13) and (4.3.19) in (4.4.1) to obtain

$$\int_a^b \int_0^L (c^2 v_x \phi_x + v_y \phi_y - c^2 v_x \phi + \gamma v \phi) \, dy \, dx - \int_0^L (c^2 \nu_1 v |_{x=b} \phi + \frac{1}{\nu_1} v_y |_{x=a} \phi_y - c^2 \nu_2 v |_{x=a} \phi) \, dy.$$
\[
\int_a^b \int_0^L u \phi \, dy \, dx + \int_0^L \phi \left( \frac{1}{\nu_2} u|_{x=b} - \frac{1}{\nu_1} u|_{x=a} \right) \, dy.
\]

Then let \( \mathcal{A}_f : H^1_{e_x} \times H^1_{e_x} \to \mathbb{R} \) be a bilinear functional and \( \mathcal{K}_f : H^1_{e_x} \to \mathbb{R} \) be a linear functional defined by

\[
\mathcal{A}_f(v, \phi) = \int_a^b \int_0^L \left( c^2 v_x \phi_x + v_y \phi_y - c^2 v_x \phi + \gamma v \phi \right) \, dy \, dx \\
- \int_0^L \left( c^2 \nu_1 v|_{x=b} + \frac{1}{\nu_1} v_y|_{x=a} \phi_y - c^2 \nu_2 v|_{x=a} \phi \right) \, dy
\]

and

\[
\mathcal{K}_f(\phi) = \int_a^b \int_0^L u \phi \, dy \, dx + \int_0^L \phi \left( \frac{1}{\nu_2} u|_{x=b} - \frac{1}{\nu_1} u|_{x=a} \right) \, dy.
\]

The weak formulation is to find \( v \in \mathcal{H} \) such that

\[
\mathcal{A}_f(v, \phi) = \mathcal{K}_f(\phi) \tag{4.4.4}
\]

for all \( \phi \in \mathcal{H} \).

Similarly for (4.2.2), let \( \mathcal{A}_f^* : H^1_{e_x} \times H^1_{e_x} \to \mathbb{R} \) be a bilinear functional and \( \mathcal{K}_f^* : H^1_{e_x} \to \mathbb{R} \) be a linear functional defined by

\[
\mathcal{A}_f^*(u^*, \phi) = \int_a^b \int_0^L \left( c^2 u^*_x \phi_x + u^*_y \phi_y - c^2 u^*_x \phi + c^2 u^* \phi \right) \, dy \, dx \\
- \int_0^L \left( c^2 \nu_1^* u^*_x|_{x=b} \phi_x + \frac{1}{\nu_1^*} u^*_y|_{x=a} \phi_y - c^2 \nu_2^* u^*|_{x=a} \phi \right) \, dy
\]
and
\[ K_f^*(\phi) = \int_a^b \int_0^L c^2 Q \phi \, dy \, dx + \int_0^L c^2 \phi \left( \frac{1}{\nu_1^2} Q \big|_{x=b} - \frac{1}{\nu_1^2} Q \big|_{x=a} \right) \, dy. \]

The weak formulation is to find \( u^* \in \mathcal{H} \) such that
\[ \mathcal{A}_f^*(u^*, \phi) = K_f^*(\phi) \] (4.4.5)
for all \( \phi \in \mathcal{H} \).

By the Lax Milgram theory there exist unique solutions for (4.4.2)-(4.4.3) and (4.4.4)-(4.4.5).

### 4.4.3 Finite element method

Now we solve (4.4.2) numerically using finite element discretization on the finite strip \([a, b] \times [0, L]\). The treatment for (4.4.3), (4.4.5) and (4.4.6) is similar and will be skipped in our discussion. Given \( p \leq 1 \), let \( \mathbb{P}_p \) be the set of all polynomials with degrees at most \( p \), and \( N_x, N_y \in \mathbb{N} \). Define Lagrangian finite element spaces as follows

\[
X_{\Delta x} = \{ g \in C^0[a, b] : g|_{[x_i, x_{i+1}]} \in \mathbb{P}_p[x_i, x_{i+1}] , \ i = 0, 1, \ldots, N_x \},
\]

\[
Y_{\Delta y} = \{ g \in C^0[0, L] : g|_{[y_i, y_{i+1}]} \in \mathbb{P}_p[y_i, y_{i+1}] , \ i = 0, 1, \ldots, N_y \},
\]

where \( x_i = a + i \Delta x \) with a uniform spacing \( \Delta x = \frac{b-a}{N_x} \) in the \( x \)-direction and \( y_i = i \Delta y \) with a uniform spacing \( \Delta y = \frac{L}{N_y} \) in the \( y \)-direction.

Based on the order \( p \) of the polynomial that we choose, there will be \( pN_x + 1 \) nodes in the \( x \)-direction and \( pN_y + 1 \) nodes in the \( y \)-direction. In the \( x \)-direction,
let $z_k = a + k \Delta x$, $k = 0, 1, \ldots, p N_x + 1$, denote the locations of the nodes. In other words all $x_i$, $i = 0, 1, \ldots, N_x + 1$, are nodes, moreover there are $(p - 1)$ interior nodes inside $(x_i, x_{i+1})$, $i = 0, 1, \ldots, N_x - 1$. Thus $I_x = \{0, \ldots, p N_x + 1\}$ forms an index set for the nodes in the $x$-direction. Similarly we have $I_y = \{0, \ldots, p N_y + 1\}$ as the index set in the $y$-direction, and $w_r = r \Delta y$ with $r \in I_y$ are the nodes in the $y$-direction.

On the interval $[x_i, x_{i+1}] = [z_{i*p}, z_{(i+1)*p}]$, there is a unique $p^{th}$ degree polynomial $Q_j^{(i)}$, $j = i*p, i*p + 1, \ldots, (i+1)*p$, such that $Q_j^{(i)}(z_k) = \delta_{jk}$ whenever $i*p \leq j, k \leq (i+1)*p$. At the interior nodes when $j = i*p+1, i*p+2, \ldots, (i+1)*p-1$, we define $\phi_j = Q_j^{(i)}$ and extend $\phi_j$ to be zero outside $[x_i, x_{i+1}]$. For $j = i*p$, we let

$$\phi_{i*p} = \begin{cases} 
Q_{i*p}^{(i)}, & x_i \leq x \leq x_{i+1} \\
Q_{i*p}^{(i-1)}, & x_{i-1} \leq x \leq x_i \\
0, & \text{otherwise.}
\end{cases}$$

A picture of the shape of $\phi_j$ for $p = 2$ is given in Figure 4.4.1. Thus $\{\phi_j\}_{j=0}^{p N_x + 1}$ are piecewise degree polynomials of order $p$ and they form a basis for $X_{\Delta x}$. Similarly we construct a basis $\{\phi_k\}_{k=0}^{p N_y + 1}$ for $Y_{\Delta y}$ consisting of piecewise polynomials of order $p$ in the $y$-direction. Define

$$\psi_l(x, y) = \phi_k(x) \phi_r(y) \quad \text{for} \quad k \in I_x, r \in I_y$$

where $l = k + r(1 + p N_x)$. Recall that $v$ represents the solution in (4.4.2). Let $v_h$ be its approximation in the vector space $\text{span}\{\psi_l : l = 0, 1, \ldots, (p N_x + 1)(p N_y + 1) - 1\}$
Figure 4.4.1: Let $p = 2$. Plot (A) is the basis function we use at a boundary node of $[x_i, x_{i+1}]$. Plot (B) is that at an interior node.

so that

$$v_h(x, y) = \sum_l v_l \psi_l(x, y) \quad (4.4.6)$$

for some coefficients $v_l$. By our choice of basis functions we have

$$v_h(z_k, w_r) = \begin{cases} 
  v_l, & \text{if } l = k + r(1 + pN_x) \\
  0, & \text{otherwise}.
\end{cases} \quad (4.4.7)$$

By inserting (4.4.6) into (4.4.2) and choosing the test functions $\phi$ to be $\psi_l$, $l = 0, 1, \ldots, (pN_x + 1)(pN_y + 1) - 1$, we get a linear system which can be solved by MATLAB.
4.5 Numerical results

For all numerical computations we fix $\beta = \frac{1}{4}$ and find traveling wave profiles for both pulses and fronts as well as their speeds, when we vary $\gamma$ and $d$ on the finite strip $[a, b] \times [0, L]$. In Section 4.5.1 we fix $d = 7e - 4$ and vary $\gamma$; while in Section 4.5.2 we fix $\gamma = 2.5$ and $\gamma = 7$, and vary $d$.

4.5.1 Varying $\gamma$ while fixing $\beta$ and $d$

Let $\beta = \frac{1}{4}$, $d = 7e - 4$ and vary $\gamma$ in the steepest descent algorithm. Both traveling fronts and pulses are found in different regimes of physical parameters. Sometimes they even co-exist for the same set of physical parameters. When a front is found, the $(u, v)$ profiles at $x = a$ always match those of the standing pulse $(U, V)$ in Chapter 3. We take $b - a = 200$ with 4800 intervals in the $x$-direction; at the same time we employ 60 intervals on the interval $[0, 1]$ in the $y$-direction. The wave speed $c$ is determined so that $\mathcal{J}(c) = 0$. Our experience tells us that a larger $b - a$ is needed for bigger $\gamma$ in order for the traveling wave to converge to a steady profile at $x = a$.

Here are the numerical results that we obtain.

(a) Pulse

When $\gamma \in (0, 3.6]$ we find traveling pulses. Figure 4.5.1 shows the pulse profiles of $u$ and $v$ when $\gamma = 2.5$; the corresponding speed is found to be $c \approx 10.7937$. For $\gamma \geq 3.7$ we cannot find any pulse solutions. As our algorithm tracks only local minimizers, there is a possibility of existence of traveling pulses which are not local minimizer.

This result is consistent with the 1D case. The same phenomenon has been ob-
Figure 4.5.1: Contour plots of $u$ and $v$ when $\beta = \frac{1}{4}$, $\gamma = 2.5$ and $d = 7e - 4$. Plots (a) and (b) represent the traveling pulse solutions $u$ and $v$, respectively, with a wave speed $c \approx 10.7937$.

served: traveling pulses exist for small $\gamma$, but are absent for large $\gamma$.

(b) Front

(i) For $\gamma \leq 1.9$ we cannot find any fronts.

(ii) When $\gamma \geq 2$ we begin to observe traveling fronts; the $u$ profile is of the type $+/-/+ at x = a$. Figure 4.5.2 shows the front profiles $u$ and $v$ when $\gamma = 2.5$; the wave speed is $c \approx 10.790$. Note that this $\gamma$ value (as well as both $\beta$ and $d$) is the same as for the pulse represented in Figure 4.5.1. Hence traveling fronts and pulses can co-exist for the same set of physical parameters. In our case the co-existence happens in the range $\gamma \in [2, 3.6]$ with $\beta = \frac{1}{4}$ and $d = 7e - 4$. The same set of physical parameters can give rise to waves with completely different tail behaviors at $x = a$. Though the wave speeds for both the pulse and front are roughly the same, the pulse does travel slightly faster; a fact predicted in the 1D case [8].

(iii) When $\gamma \geq 6.7$ a different kind of front solution emerges: the $u$ profile is of the class $+$ at $x = a$. Figure 4.5.3 shows the front profiles of $u$ and $v$ when $\gamma = 7$;
Figure 4.5.2: Contour plots of $u$ and $v$ when $\beta = \frac{1}{4}$, $\gamma = 2.5$ and $d = 7e - 4$. Plots (a) and (b) represent the traveling front solutions $u$ and $v$, respectively, with a wave speed $c \approx 10.79$. At $x = a$, $u$ is of class $+/-/+$. The wave speed is $c \approx 11.095$. We will investigate if these two kinds of traveling fronts can co-exist for the same set of physical parameters. When we try to find solution $u$ of the class $+/-/+$ at $x = a$ for $\gamma = 7$, this requires a very long domain in the $x$-direction, i.e. $b - a$ has to be much larger. The numerical results are given in Figure 4.5.4. While the traveling wave settles down to a steady profile $+/-/+$ at $x = a = -500$ with $\gamma = 5$, the situation is less clear when $\gamma = 7$. The observed level curves of $u$ in Figure 4.5.4(a) stop oscillating for large, negative $x$; however this is not the case in Figure 4.5.4(b).

Remark 4.5.1. In the next subsection we keep $\beta = \frac{1}{4}$, $\gamma = 7$ and vary $d$. The issue of co-existence of two different kinds of fronts will be easier to investigate there when $d$ takes on a different value; the required computational effort is less intense.

We note that tail behaviors of $+/-/+$ and $+$ for $u$ at $x = a$ for different $\gamma$ are found. For easy contrast we put the solutions in Figure 4.5.5. Plots (B1) and (B2) clearly illustrate the differences. These tails at or near $x = a$ are the same as the
Figure 4.5.3: Contour plots of $u$ and $v$ when $\beta = \frac{1}{4}$, $\gamma = 7$ and $d = 7e - 4$. Plots (a) and (b) represent the traveling front solutions $u$ and $v$, respectively, with a wave speed $c \approx 11.095$. At $x = a$, $u$ is of class $\oplus$.

Figure 4.5.4: Contour plot of $u$. Plot (a) shows the traveling front solution $u$ with a wave speed $c \approx 10.9775$ when $\beta = \frac{1}{4}$, $\gamma = 5$ and $d = 7e - 4$. Plot (b) shows the minimizer profile $u$ with a wave speed $c \approx 11.095$ when $\beta = \frac{1}{4}$, $\gamma = 7$ and $d = 7e - 4$. 
solutions $U$ found in Chapter 3 for the same set of physical parameters.

![Figure 4.5.5](image)

**Figure 4.5.5**: Plots (A1) with $\gamma = 2.5$ and (A2) with $\gamma = 7$ are the same as Figure (4.5.2a) and (4.5.3a). Plots (B1) and (B2) are the solutions of (A1) and (A2) at $x = a$, respectively; they coincide with the minimal energy solutions $U$ in Chapter 3 for the same physical parameters.

Traveling front with a tail $-+/-$ has never been found. Likely this is because their energy is quite high so that these 1D standing pulses may be unstable with respect to 2D perturbations. To be more precise, we set an initial guess with a tail of class $-+/-$ then we run the code. The profile changing while iteration to minimizer with a tail of class $+/-/+$.  

The solution $u$ for both $\gamma = 2.5$ and $\gamma = 7$ is stable. Let the characteristic length of a pulse be the size of the length of domain in the $x$-direction. We put solutions
from the steepest descent algorithm in a parabolic solver as initial conditions,

\[
\begin{align*}
  u_t &= c^2 u_{xx} + u_{yy} + c^2 u_x + \frac{1}{\delta} (f(u) - v), \\
  v_t &= c^2 v_{xx} + v_{yy} + c^2 v_x + u - \gamma v,
\end{align*}
\]

then run the code for one characteristic time \( T_c \) (the time required for the wave to traverse its characteristic length). The resulting solutions are then translated back by a distance of \( c \cdot T_c \) in the \( x \)-direction for comparison with the initial profiles. Figure 4.5.6a shows the pulse profile \( u \) obtained by the steepest descent algorithm. Figure 4.5.6b shows the translated profile from the parabolic solver. The two profiles are essentially identical. This demonstrates that the solution is stable.

**Figure 4.5.6:** Contour plots of \( u \). Plot (a) shows the pulse profile \( u \) of Figure 4.5.1a obtained from the steepest descent algorithm. We use this solution as an initial condition in the parabolic solver, along with \( v \). Plot (b) shows the profile of \( u \) from the parabolic solver after translation.

Similar investigations are performed for the fronts when \( \gamma = 2.5 \) and \( \gamma = 7 \) in Figure 4.5.7 and 4.5.8, respectively. They show that the traveling fronts are stable in both cases.
Figure 4.5.7: Contour plots of $u$. Plot (a) shows the front solution $u$ of Figure 4.5.2a obtained from the steepest descent algorithm. We use this solution as an initial condition in the parabolic solver, along with $v$. Plot (b) shows the profile of $u$ from the parabolic solver after translation.

Figure 4.5.8: Contour plots of $u$. Plot (a) shows the front solution $u$ of Figure 4.5.3a obtained from the steepest descent algorithm. We use this solution as an initial condition in the parabolic solver, along with $v$. Plot (b) shows the profile of $u$ from the parabolic solver after translation.
4.5.2 Varying $d$ while fixing $\beta$ and $\gamma$

In this set of numerical experiments we fix $\beta$ and $\gamma$, and allow $d$ to vary. We observe that for the same physical parameters, there can be multiple traveling pulses with distinct speeds. The same is true for traveling fronts. Upon combining with the results from the last subsection, we have the co-existence of fronts with pulses, as well as multiple fronts and multiple pulses. Our theory indicates that a traveling wave exists with a speed $c$ whenever $J(c) = 0$. In case there are multiple roots for $J$, there will be multiple traveling waves for the same physical parameters, each with a distinct wave speed.

Let $\beta = \frac{1}{4}$. We will fix $\gamma = 2.5$ and $\gamma = 7$. The former case gives rise to pulses while the latter corresponds to fronts. When we compute with a smaller wave speed $c$, minimizers go to steady profiles at $x = a$ with a faster rate. So for $c \leq 7$ we can shorten the domain, compared to Section 4.5.1, to $b - a = 60$ with 2400 intervals in the $x$-direction and 60 intervals from $y = 0$ to $y = 1$.

(i) The pulse case when we fix $\beta = \frac{1}{4}$, $\gamma = 2.5$ and vary $d$.

Figure 4.5.9 gives the graph of $J$ versus the wave speed $c$ for three values of $d$. For $d = 7e - 4$ we find a unique traveling pulse solution that corresponds to a single root of $J$ at $c \approx 10.7937$. When $d = 1e - 3$ there are three traveling pulses with $c_0^p \approx 6.774$, $c_1^p \approx 5.357$ and $c_2^p \approx 4.696$. Finally when $d = 1.5e - 3$, $J$ has no root, that means there is no traveling pulse solution.
Figure 4.5.9: The function $J$ versus the wave speed $c$ for $\beta = \frac{1}{4}$, $\gamma = 2.5$ and three values of $d$.

Figure 4.5.10: The graph of $J$ when $d = 1e-3$, $\beta = \frac{1}{4}$ and $\gamma = 2.5$. We find three pulse solutions corresponding to its roots $c^p_0$, $c^p_1$ and $c^p_2$. Two are stable and the other is unstable.
Each of the three graphs of $J$ in Figure 4.5.9 is composed of two smooth curves cutting one another at a sharp angle. To illustrate this point, we blow up the graph for $d = 1e - 3$ in Figure 4.5.10. On the branch of large $c$, the minimizer $u$ for both $c = c_0^p$ and $c = c_1^p$ has a single bump (see Figure 4.5.11 (a) and (b)). On the branch of small $c$, $u$ has two bumps (see Figure 4.5.11 (c)). Starting with an initial guess that has a single bump, we calculate the minimizer using the steepest descent algorithm, while decreasing $c$ gradually from 10. When $c$ gets below 5, the minimizer $u$ of one bump starts splitting into two bumps; this is because the latter has a lower energy. The reverse happens when we use an initial guess with two bumps and calculate the minimizer while increasing $c$ gradually from 1.

We put the pulse solutions corresponding to the wave speeds $c_0^p, c_1^p, c_2^p$ as initial conditions in a parabolic solver. The pulses corresponding to $c_0^p$ and $c_2^p$ are stable while that for $c_1^p$ is unstable.

**Figure 4.5.11:** The traveling pulse solutions $u$. Plots (a), (b) and (c) show the profiles of $u$ with the corresponding wave speeds $c_0^p, c_1^p$ and $c_2^p$, respectively.
(ii) The front cases when we fix $\beta = \frac{1}{4}$, $\gamma = 7$ and vary $d$.

Case 1: in our steepest descent algorithm we set the initial guess to look like Figure 4.5.8; in particular its tail behavior is of class + at $x = a$. Figure 4.5.12 shows the function $J$ versus the wave speed $c$ for three values of $d$. For $d = 9e - 4$ we find a single traveling front solution, which satisfies $J(c) \approx 0$ with $c \approx 9.0688$. When $d = 1.4e - 3$ we find three traveling front solutions with speeds $c_0^f \approx 5.071$, $c_1^f \approx 2.940$ and $c_2^f \approx 2.519$. Finally when $d = 2e - 3$, $J$ has no root so that there is no traveling front solution.

![Figure 4.5.12: The function $J$ versus the wave speed $c$ for $\beta = \frac{1}{4}$, $\gamma = 7$ and three values of $d$.](image)

Again we observe each of the three graphs of $J$ in Figure 4.5.12 is composed of two smooth curves intersecting one another at a sharp angle. This is clearly seen when we blow up the graph for $d = 1.4e - 3$ in Figure 4.5.13. On the branch of large $c$, the minimizer $u$ has a single bump supported on a long strip (see Figure 4.5.14 (a) and (b)), while on the branch of small $c$, $u$ has two bumps supported on two
Figure 4.5.13: the graph of $J(c)$ when $d = 1.4e - 3$, $\beta = \frac{1}{4}$ and $\gamma = 7$. We find three front solutions corresponding to its roots $c_f^0$, $c_f^1$ and $c_f^2$. Two are stable and the other is unstable.

long, parallel strips (see Figure 4.5.14 (c)). We start with an initial guess that has a single strip and compute a minimizer using the steepest descent algorithm, decreasing $c$ gradually from 10. When $c$ gets below 2.75, the profile of one strip starts splitting into two strips with a lower energy. The reverse happens when we use an initial guess with two strips and increase $c$ gradually from 1.

Next we put the front solutions corresponding to these wave speeds as initial conditions in the parabolic solver. The fronts corresponding to speeds $c_f^0$ and $c_f^1$ are stable while that for $c_f^2$ is unstable.
Figure 4.5.14: The traveling front solutions $u$. Plots (a), (b) and (c) show the profiles of $u$ with corresponding wave speeds $c_0^f$, $c_1^f$ and $c_2^f$, respectively.

Case 2: In our steepest descent algorithm we set the initial guess to look like Figure 4.5.7; in particular, its tail behavior is of class +/-/+ at $x = a$. We then calculate the minimize by decreasing $c$ gradually from 10. Figure 4.5.15 shows the graph of $J$ versus the wave speed $c$ for three values of $d$. When $d = 9e - 4$ we find a single traveling front solution that satisfies $J(c) \approx 0$ with $c \approx 9.0688$. The situation is similar for $d = 1.4e - 3$; a single traveling front solution exists with $c \approx 5.0788$. Finally, when $d = 2e - 3$ $J(c)$ has no root; that means there is no traveling front solution. All the front solutions in this Case have their tail behavior in the class +/-/+ at $x = a$. 
Figure 4.5.15: The function $J$ versus the wave speed $c$ for $\beta = \frac{1}{4}$, $\gamma = 7$ and three values of $d$.

Figure 4.5.16: Three curves of the function $J(c)$ corresponding to three different minimizer profiles for $d = 1.4e^{-3}$, $\beta = \frac{1}{4}$ and $\gamma = 7$. Also, this figure shows four traveling front solutions, with the corresponding wave speeds $c_{f0}^f$, $c_{f1}^f$, $c_{f2}^f$ and $c_{f3}^f \approx 5.0788$. 
We note that when $d = 1.4e - 3$, two different types of traveling fronts coexist. For this value of $d$ we superimpose the graphs of $J$ in Figure 4.5.13 and 4.5.15 to obtain Figure 4.5.16. There are four front solutions with the same physical parameters, each traveling with distinct speeds. Three of them are stable and the other is unstable. The (unique) front with tail behavior $+/-/+\$ and the fastest of the fronts with tail behavior $+$ have almost identical wave speeds, with the former one being slightly faster. They are both stable. Figure 4.5.17 shows these two stable front solutions for $u$.

![Contour plot of $u$ for $d = 1.4e - 3$, $\beta = \frac{1}{4}$ and $\gamma = 7$. Plot (a) and (b) show the different front solutions with corresponding wave speeds $c \approx 5.0710$ and $c \approx 5.0788$, respectively.](image)

**Figure 4.5.17:** Contour plot of $u$ for $d = 1.4e - 3$, $\beta = \frac{1}{4}$ and $\gamma = 7$. Plot (a) and (b) show the different front solutions with corresponding wave speeds $c \approx 5.0710$ and $c \approx 5.0788$, respectively.
Bibliography


