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New Directions in Justification Logic

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ABSTRACT

Justification logics are constructive analogues of modal logics. As such, they provide perspicuous models of those modalities that have inherently constructive character, such as intuitionistic mathematical provability or the knowledge operator of evidentialist epistemology. In this dissertation, I examine a variety of positions in epistemology, along with their associated ontological commitments, and develop various classical and non-classical justification logics that are suitable for use as models of these positions.

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Any errors that may remain in this dissertation, as well as the fatal flaws that lead to the removal of the section to which Greg contributed, are entirely the fault of the author, and not due to any of the contributors acknowledged above.

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Contents

Chapter 1

History and Formal Presentation of Artemov's JT4 and Other Normal Justification Logics

Convention 1.0.1 (Originality of Results)**.** In this chapter and in the appendices, all of the formal results that are presented were previously published by some other author. In all other chapters, any result which is formally presented as a numbered theorem is original. Results that were previously published by some other author will either be stated in the text without an assigned theorem number, or else will be incorporated into a (possibly numbered) formal definition.

1.1 Logic of Intuitionistic Provability.

Intuitionism is a philosophy of mathematics founded by L. E. J. Brouwer in the 1920s.[1] Its metaphysical basis is the claim that mathematical objects have no independent existence, but rather are constructed (in a neo-Kantian sense) by mathematicians through proofs. In particular, mathematical existence claims are true only if there is a (possibly indeterministic)[2] algorithm which generates an object with the desired property. Because mathematical entities have no existence independent of mathematical proof, no proposition is true or false of them unless it has been so proven. This metaphysical position invalidates many of the classical rules of inference in mathematical contexts, and has led to a new branch of mathematical inquiry that operates without such inferences.

The pure logic required for work in intuitionistic mathematics, known as intuitionistic logic, was formalized by Brouwer's student, Arend Heyting. Proof theoretically, intuitionistic logic is just classical logic with one of the axioms or rules governing negation removed.[3] As there are countless proof systems equivalent to classical logic, the missing axiom or rule will vary. The most common choices are double negation elimination, the excluded middle, and positive conclusion *reductio ad absurdum*—all of which are equivalent to each other, given suitable axioms and rules governing disjunction and conditionals.[4]

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[1]The following interpretation of Brouwer's ideas is essentially my own; it is supported by evidence in sources such as the first chapter of [25].

[2]The possibility of indeterministic constructions, or free choice sequences, is required to reconcile the facts that Cantor's diagonal argument proceeds in an intuitionistically acceptable manner and that there is only a countable infinity of deterministic algorithms for constructing numerical quantities.

[3]This is a slight oversimplification; many of the common axiomatizations of classical logic are unsuitable for this procedure, as in intuitionistic logic, separate axioms or rules are needed for conjunction, disjunction, and the conditional; they are not interdefinable in the usual manner.

[4]Obviously, given a base system which is equivalent to intutitionistic logic, all possibilities for the

However, in mathematics, there are many other concepts to be formalized in addition to those which are contained in a logical system proper. One important example is the concept of provability. Usually this is defined metalinguistically, but in mathematics it is sometimes needed as an object-language notion. There are two major strategies for accomplishing this, both developed by Kurt Gödel. One strategy is to map the sentences of the language onto objects in the language's first-order domain using a metalinguistic function known as a "Gödel coding," and then to find the first-order predicate which is true of exactly those objects with which the Gödel coding represents provable sentences. This technique leads to some powerful results (notably, Gödel's incompleteness theorems), but also imposes significant limitations. Obviously, it can only be used in a language containing at least first-order predication. Moreover, aside from degenerate cases, the requirement that sentences be mapped onto objects forces the first-order domain to be infinite, and in order to be sure of the existence of a provability predicate, the language must allow complete first-order comprehension. These restrictions make this version of object-language provability unusable in simple cases (such as that of propositional languages, which will be employed for simplicity throughout this dissertation). Moreover, the resulting provability predicate is only interpretable as such by reference to the metalinguistic Gödel coding, making it unsuitable as a model for the communication of provability data within the object language.

The other strategy is to replace the object language with one that contains the desired provability operator. Naïvely, this might be accomplished by simply adding such an operator to the original object language. However, that solution is not rec-

additional axiom or rule that gives classical logic must be equivalent to each other, but it would be circular to employ this fact in a demonstration of what proof systems constitute intuitionistic logic itself.

ommended, as it leads to systems which are difficult to characterize in both proof theory and semantics. Instead, one typically searches for a translation of the original object language into a well-understood language which already contains an equivalent to the desired proof operator. For the propositional logics in which we are presently interested, the normal modal logics provide suitable targets for this translation.

Definition 1.1.1. The normal modal logics K, T, S4, S5, and GL are the systems that can be constructed proof-theoretically from the following axiom schemas and inference rules:

PC Any set of axiom schemas whose closure under modus ponens is sound and complete with respect to classical propositional logic

K $\vdash \Box(\psi \supset \varphi) \supset (\Box\psi \supset \Box\varphi)$

T $\vdash \Box\varphi \supset \varphi$

4 $\vdash \Box\varphi \supset \Box\Box\varphi$

5 $\vdash \neg\Box\varphi \supset \Box\neg\Box\varphi$[5]

W $\vdash \Box(\Box\varphi \supset \varphi) \supset \Box\varphi$

Modus Ponens If $\vdash \psi \supset \varphi$ and $\vdash \psi$ then $\vdash \varphi$.

Necessitation If $\vdash \varphi$ then $\vdash \Box\varphi$.

- System K is the closure of the PC and K schemas under modus ponens and necessitation.

[5]The 5 axiom is usually presented in terms of the $\Diamond$ operator rather than the negated $\Box$, but for reasons that will become clear in the next section, the equivalent form given here is more convenient.

- System T is the closure of the PC, K, and T schemas under modus ponens and necessitation.
- System S4 is the closure of the PC, K, T, and 4 schemas under modus ponens and necessitation.
- System S5 is the closure of the PC, K, T, and 5 schemas under modus ponens and necessitation. All instances of the 4 schema are included as theorems in S5.
- System GL is the closure of the PC, K, and W schemas under modus ponens and necessitation. As in S5, instances of the 4 schema are theorems of GL.
- In general, a normal modal logic is any consistent system that includes the PC and K schemas, and is closed under modus ponens and necessitation.

Definition 1.1.2. Let the translation function † from intuitionistic propositional logic into S4 be defined recursively as follows:[6]

- For any atomic formula p, $p^\dagger = p$.
- $(\neg\varphi)^\dagger = \Box\neg\varphi^\dagger$
- $(\varphi \supset \psi)^\dagger = \Box\varphi^\dagger \supset \Box\psi^\dagger$
- $(\varphi \vee \psi)^\dagger = \Box\varphi^\dagger \vee \Box\psi^\dagger$
- $(\varphi \wedge \psi)^\dagger = \varphi^\dagger \wedge \psi^\dagger$.

[6]This is the form of the translation as initially conjectured in [49]. [72] and [98] provide proofs for translations nearly identical to this one as well as for alternative translations with different forms. For present purposes, it suffices that we can identify at least one sound translation.

Theorem 1.1.3. *Let $\vdash_{int}$ be the usual metatheoretic provability operator in intuitionistic logic, $\vdash_{S4}$ be provability in S4, and † the translation function defined above. Then $\vdash_{int} \varphi$ iff $\vdash_{S4} \Box\varphi^{\dagger}$.*

In essence, Theorem 1.1.3 provides us with a manner in which the S4 $\Box$ operator can be used as an intuitionistic provability operator. The translation procedure employed therein was originally proposed by Gödel in [49], without any proof of its correctness. The first published proof of the theorem comes from McKinsey and Tarski [72]; I will not reproduce their proof here. It is, however, worth noting that the proof uses no metatheory beyond the definition of provability.[7] Moreover, one who has access to S4 as an object language also has access to intuitionistic logic, as the former is a proof-theoretic extension of the latter.[8] Therefore, the procedure of Theorem 1.1.3 can be taken as a genuine object-language account of intuitionistic provability, suitable as a model for object-language communication of proof.

A similar theorem is available relating the S5 $\Box$ operator to classical provability. One important feature that the classical-S5 and intuitionistic-S4 provability translations have in common is that they both operate only at the level of pure logic. If mathematical axioms were added to either base logic and run through the provability translation, the result is an inconsistent system. The inconsistency is not in the

[7]The McKinsey and Tarski proof of the theorem does utilize semantic methods (specifically, algebraic semantics, that being the only semantics for modal and intuitionistic logic available at the time). Technically, one could therefore insist that the proof requires metatheory in the form of a completeness theorem. However, this does not seem to be a significant objection, as proof theory and semantics are usually considered equally acceptable as representations of the resources that an object language speaker can utilize—indeed, it is more common to treat the semantics as fundamental rather than the proof-theoretic approach that I take here.

[8]I mean this in the sense that I, as a classical logician, can write a theorem such as $(p \supset q) \supset (\neg q \supset \neg p)$ on the chalkboard, prove it, and correctly assert of this very theorem that is also provable in intuitionistic logic (perhaps by the same proof, depending on what classical proof of the theorem I chose); and moreover that the same is true of every intuitionistic theorem.

addition of something like the Peano axioms to S4 or S5; S5 + PA is a consistent system. Rather, the inconsistency results from the interpretation of the S4 or S5 $\Box$ operator as a provability operator for PA or HA, as is illustrated by Löb's Theorem:

Theorem 1.1.4 (Löb's Theorem)**.** *Let Bew be an object-language provability operator for Peano arithmetic—either a provability predicate of the type that can be developed using Gödel coding within PA, or an operator such as* $\Box$ *in a modal translation. For any formula* φ*, if* $PA \vdash Bew\ulcorner\varphi\urcorner \supset \varphi$*, then* $PA \vdash \varphi$*.*

Proof. For the case involving Bew predicates constructed by Gödel coding, see Löb [67]. For the case involving modal Bew operators, see Boolos [22]. ■

Modal systems S4 and S5 include axiom schema T, and so $\Box\varphi \supset \varphi$ holds for all formulae φ. But if the $\Box$ operator is interperable as provability, then Löb's Theorem entails that PA $\vdash \varphi$ for all φ, which is equivalent to the claim that PA is inconsistent. If we wish to hold onto the assumption that mathematics is consistent, we need a weaker logic to model mathematical provability. This is the focus of George Boolos' famous work on provability logic [22, 23], based on Solovay's proof [91] that the $\Box$ operator of GL is a suitable provability operator for classical mathematics. To the best of my knowledge, no analogous result has been discovered for intuitionistic mathematics.[9] Presumably, there is some logic slightly weaker than GL that will do the job, in direct analogy to the results that have been proven for the pure logics. A precise characterization of the requisite logic is an interesting task, but I shall not attempt it in this dissertation.

For the present, I wish to direct the reader to another problem, which applies

[9]The task of finding a provability logic for Heyting Arithmetic is discussed in Beklemishev and Visser's 2006 article [19] on unsolved problems in provability theory.

specifically to the use of S4 to model intuitionistic logical provability (or to the hypothetical GL-analogue, in its use to model intuitionistic mathematical provability). Recall the philosophical basis for intuitionism: that mathematical objects are constructed in mathematical practice, and that such objects have no existence (and thus, no truth or falsity to claims about them) until they have been appropriately constructed. This applies as much to mathematical proofs as to any other mathematical objects. A claim of provability is simply a claim of the existence of a proof. Such a claim cannot properly be said to be true unless one has constructed an actual proof (or a non-trivial algorithm[10] for generating one). If one uses Theorem 1.1.3 along with reasoning in S4 to determine that a statement is intuitionistically provable, one has not thereby devised a procedure for proving it in intuitionistic logic, and so one has violated the fundamental principle of intuitionism.

The problem, then, is to devise a provability translation for intuitionistic logic that preserves the benefits of Theorem 1.1.3 while also containing within it an algorithm for devising a proof in intuitionistic logic of every formula which the translation theorem deems to be provable. To meet this demand, Sergei Artemov [4] devised the Logic of Proofs, the first of the formal systems which are now known collectively as justification logics.

[10]There is a trivial algorithm that will produce a proof of any provable sentence: enumerate all of the objects that satisfy the recursive definition of being a proof in the logical system under consideration, and then check whether each one is a proof of the target sentence. This algorithm is correct, but like the previous account fails to satisfy the constructive intuition.

1.2 Justification Logic

1.2.1 Axiomatic Systems

Unlike modal logics, which add to propositional logic only a single sentential operator $\Box$, justification logics use a more complicated syntax that could be regarded as a highly restricted first-order logic. In addition to the propositional constants and connectives, justification systems include proof constants (which will be denoted throughout this dissertation with letters a, b, c, etc.), proof variables (denoted x, y, z, etc.), functions ranging over proof polynomials (some examples of which will be denoted !, $\cdot$, +, and ?), and a special operator which will be denoted with a colon (:), which takes a proof polynomial as its left input, a proposition as its right input, and outputs a proposition. By proof polynomials, I mean all of the grammatical terms which can be formed using the proof constants and variables—formally, this can be defined as follows:

Definition 1.2.1 (Proof Polynomials)**.** The set of proof polynomials (PP) is defined by the following:

- All proof variables x, y are members of PP.
- All proof constants a, b are members of PP.
- If $s, t \in PP$, then $s \cdot t \in PP$, $s + t \in PP$, $!s \in PP$, and $?s \in PP$.
- Nothing else is a member of PP.

The fundamental nature of these objects will depend on the field of inquiry in which the logic is being employed. In the mathematical provability setting discussed earlier, the proof polynomials might be Gödel codes for mathematical proofs. In an epistemic application, the proof constants might be names for pieces of evidence, and higher proof polynomials the new evidence that can be produced by combining pieces of existing evidence in different ways. In a computer science application, the proof constants might be pointers to entries in a database, and the functions used to generate higher proof polynomials would represent functions that could be implemented within the database software. The logic itself is sufficiently abstract as to not distinguish among these or other possible usage cases. The name "proof polynomial" does suggest the mathematical provability interpretation, but this is merely an artifact of history; the first justification logics were developed by Artemov [4] specifically as responses to the problem of intuitionistic provability discussed earlier. In other portions of the justification logic literature (and possibly in this dissertation!), one will find the same entities referred to by such labels as "proof terms," "justification terms," etc. I do not expect that such usage will cause any great confusion.

Convention 1.2.2. Throughout this dissertation, unbound proof variables shall not be understood as meaning their universal closures—in most contexts, the language won't have quantification at all, so that common reading wouldn't make sense. Unbound proof variables are interpreted as constants. In most cases, I will use true constants for semantically significant naming, and variables for ad hoc naming.

Definition 1.2.3 (Formulae)**.** The formulae of justification logics are as follows:

- All propositional constants p, q are formulae.
- If φ and ψ are formulae, then so are $\neg\varphi$, $\varphi \vee \psi$, $\varphi \wedge \psi$, and $\varphi \supset \psi$.
- If φ is a formula and t is a proof polynomial, then $t : \varphi$ is a formula.

- Nothing else is a formula.

Formula construction is essentially what any reader with training in logic would expect it to be: there are atomic formulae, and the usual Boolean connectives. The only new element is the formula $t:\varphi$. This formula is usually read aloud as "t proves φ," mainly because this reading is conveniently short. When a justification system is used as a provability logic, "t proves φ" is indeed the correct interpretation of $t:\varphi$. In epistemic contexts, a variety of interpretations are possible. $t:\varphi$ might be read as "t is evidence for φ," "t is an epistemic justification of φ," "[Some agent] knows φ on the basis of t," etc.

Convention 1.2.4. Throughout this dissertation, formulae of justification logics will be written with the : operator taking the narrowest possible scope, and the $\supset$ operator taking the widest possible scope. Parentheses will be used only where needed for clarity.

Having defined the syntax of a formal language, one would normally be in a position to formally present the logic. However, in the case of justification logic, we need to introduce one more item first. This is the notion of a constant specification.

Definition 1.2.5 (Constant Specifications)**.**

1. A constant specification is a relation between the set of proof constants and the set of formulae of the logic.

2. A multi-constant specification is a relation between the set of ω-sequences of proof constants and the set of formulae of the logic.

3. A constant or multi-constant specification $\mathcal{C}$ is said to be axiomatically appropriate if Range$(\mathcal{C}) = \{\varphi | \varphi$ is a substitution instance of one of the axioms of the logical system.$\}$.

4. The total constant (or multi-constant) specification $\mathcal{C}_t$ is the [multi-]constant specification such that for every c in the set of proof constants (or $\vec{c}$ in the set of ω-sequences of proof constants), $\mathcal{C}_t(c, \varphi)$ (or $\mathcal{C}_t(\vec{c}, \varphi)$) holds iff φ is an axiom instance. Equivalently, $\mathcal{C}_t$ is the union of all axiomatically appropriate constant or multi-constant specifications for the logical system.

Constant and multi-constant specifications are not exactly part of the formal language, but rather are metatheoretic bookkeeping tools. Their function is to coordinate the interpretations of the various proof constants, which is required in order to rigorously establish connections—such as soundness and completeness theorems—between different presentations of the same logic. Some authors prefer to define constant specifications as functions rather than relations. I have chosen to avoid this, because it is not helpful to require either that each constant justifies only a single formula or that each formula be justified only by a single constant. Indeed, the noted special case of the total [multi]-constant specification entails that neither of these restrictions hold. A functional form can still be achieved by replacing the codomain with its power set, but I think it is more natural to use the relational form of the constant specification in this case.

Definition 1.2.6. The normal justification logics J, JT, JT4, and JT5 are the systems that can be constructed proof-theoretically from the following axiom schemas and inference rules, in relation to a fixed constant or multi-constant specification $\mathcal{C}$:

PC Any set of axiom schemas whose closure under modus ponens is sound and complete with respect to classical propositional logic

$\mathbf{K^j}$ $\vdash t : (\varphi \supset \psi) \supset (s : \varphi \supset (t \cdot s) : \psi)$. This axiom is more commonly referred to as the axiom of application.

$\mathbf{T^j}$ $\vdash t : \varphi \supset \varphi$. This is called the axiom of reflection.

$\mathbf{4^j}$ $\vdash t : \varphi \supset\ !t : t : \varphi$. This is sometimes called the axiom of positive introspection, or the proof checker axiom.

$\mathbf{5^j}$ $\vdash \neg t : \varphi \supset\ ?t : \neg t : \varphi$. This is called the axiom of negative introspection.

Sum1 $\vdash s : \varphi \supset (s + t) : \varphi$

Sum2 $\vdash t : \varphi \supset (s + t) : \varphi$

Modus Ponens If $\vdash \psi \supset \varphi$ and $\vdash \psi$ then $\vdash \varphi$.

Iterated Axiom Justification If φ is an instance of one of the axiom schemas included in a particular system, and if $c_0, \ldots, c_n$ are proof constants that comprise an initial fragment of a sequence $\vec{c}$ such that $\mathcal{C}(\vec{c}, \varphi)$, then $\vdash c_n : c_{n-1} : \cdots : c_0 : \varphi$.

Simple Axiom Justification If φ is an instance of one of the axiom schemas included in a particular system, then for any proof constant c such that $\mathcal{C}(c, \varphi)$, $\vdash c : \varphi$.

- System J is the closure of the PC and $\mathrm{K^j}$ schemas under modus ponens and iterated axiom justification.

- System JT is the closure of the PC, $\mathrm{K^j}$, and $\mathrm{T^j}$ schemas under modus ponens and iterated axiom justification.

- System JT4 (which is Artemov's original Logic of Proofs)[11] is the closure of the PC, $\mathrm{K^j}$, $\mathrm{T^j}$, and $\mathrm{4^j}$ schemas under modus ponens and simple axiom justification.

[11] Artemov originally denoted his logical system with the abbreviation LP, and most authors working on justification logics retain this nomenclature. I have abandoned it in favor of the more systematic JT4 in order to avoid confusion during the discussion of paraconsistent systems in Chapters 3–4 of this dissertation.

- System JT5 is the closure of the PC, K^j, T^j, 4^j, and 5^j schemas under modus ponens and simple axiom justification. Technically, 4^j could be eliminated by defining the ! function in a manner analogous to the proof of the 4 axiom in S5. However, it is more convenient to include it in the axiomatization.

- The two sum axioms can be added conservatively to any of the above systems. Throughout this dissertation, I will not bother to note the inclusion or exclusion of these axioms unless it makes some crucial difference in the formal question at hand.

Note that iterated axiom justification requires the use of a multi-constant specification, whereas simple axiom justification requires the use of a standard constant specification. For this reason, only one of these rules may be utilized in any given system. However, it is possible to formulate any of the above systems using either of the axiom justification rules. To use iterated axiom justification in place of simple axiom justification, no further changes are required. To use simple axiom justification in place of iterated axiom justification, add schema 4^j to the system, subject to the restriction that the formula substituted for φ must be an axiom instance (allowing recursive instances of this restricted 4^j schema).

Another noteworthy feature of the axiom justification rules is that each of them specifies two antecedent conditions, which are partially redundant. In the intended application of justification logic, the constant or multi-constant specification $\mathcal{C}$ is axiomatically appropriate. In this case the requirement that φ be an axiom instance is entirely redundant, as $\mathcal{C}(c, \varphi)$ does not obtain unless φ is an axiom instance. In a case where $\mathcal{C}$ is not axiomatically appropriate, we get deviant behavior regardless of whether the requirement that φ be an axiom instance is included. However, omitting this restriction magnifies the deviance: with the restriction in place, the only nonstandard behavior is that the logic might fail to prove a justification of some axiom instance; whereas if the restriction is omitted, the logic might additionally prove justifications of formulae that are not axiom instances, or indeed not even theorems. As the intended applications of the logic involve axiomatically appropriate constant specifications, there is no definitive basis for establishing which behavior is preferable when $\mathcal{C}$ is not axiomatically appropriate.

If we choose to retain the restriction that the axiom justification rules only apply to formulae φ that are axiom instances, we can instead eliminate the restriction in the simple axiom justification rule that $\mathcal{C}(c, \varphi)$ (or the analogous restriction in the iterated axiom justification rule). In the absence of such restrictions, we would allow arbitrary constants to be used in the axiom justification rules. This is licensed by the fact that we can choose $\mathcal{C}$ to be the total [multi-]constant specification $\mathcal{C}_t$, which would then satisfy $\mathcal{C}_t(c, \varphi)$ for any constant c. I personally consider the presentation of axiom justification that omits reference to constant specifications to be the most perspicuous option, because in actual practice, an agent who is performing a mathematical proof (or other epistemic reasoning) does not proceed by consulting a giant lookup table of all the substitution instances of all the axioms whenever she comes across a new axiom instance. Instead, she examines the formula, determines that it is an axiom instance, and then immediately moves on, deeming the formula to be justified without any further work. Additionally, eliminating constant specifications entirely has the advantage that it rules out the sort of deviant cases which are generated by inappropriate constant specifications. Strictly speaking, the logics that would be generated using such inappropriate specifications are not legitimate instances of the normal justification logics, as they fail to satisfy the normality criterion of full axiom justification, and so these logics ought to be excluded from consideration.

1.2.2 On the Use of Justification Logic in Intuitionism

The system JT4 provides a satisfactory solution to the problem raised at the end of Section 1.1 in the case where we choose the propositional axiom schemas to be one of the axiomatizations that results by adding an axiom such as double negation elimination to an axiomatization of intuitionistic logic. We can then interpret the proof constants as names for proofs in that logic. The $\cdot$ function, as defined by the application axiom, represents the process of creating a new proof by combining conclusions of two existing proofs under modus ponens. The ! function represents the algorithm for verifying that a given sequence of formulae is actually a proof. The + function, if included, represents the result of simply concatenating two existing proofs without attempting to derive any new conclusions.[12] Ultimately, when we apply the JT4-analogue of Theorem 1.1.3, the proof polynomial input into the outermost : operator will correspond structurally to a proof of the equivalent formula in intuitionistic logic.

First, one must show that there is a formula relating JT4 to intuitionistic logic in the same manner in which S4 is related to intuitionistic logic. Artemov, in [4], does this by first demonstrating various properties of JT4 and its relationship with S4, and then combining these results with the existing proof of Theorem 1.1.3. Note that all of the theorems that will be proven below have analogues which hold for the

[12] Obviously, for this interpretation to work, proofs must be viewed as "proving" all of the formulae contained within them. If one insists on each proof having a unique conclusion, one should utilize the version of JT4 without sum axioms here.

other normal justification logics and their corresponding modal logics.[13]

Convention 1.2.7. Vector notation used in place of a name of a proof polynomial (e.g., $\vec{c}$ or $\vec{s}$) indicates a sequence of proof polynomials (in this context, a finite sequence, as opposed to the ω-sequences that are employed in a multi-constant specification), each of which is to be applied to some formula. The exact nature of this application depends on the syntactic form of the context in which it is written. In a single well-formed formula such as $\vec{c}:\varphi$, the proof polynomials in $\vec{c}$ are to be applied iteratively to φ, in the manner of the iterated axiom justification rule. When a vector proof polynomial is applied to a syntax representing a sequence of formulae, as in $\vec{s}:\Gamma$, then the proof polynomials are to be applied individually to the formulae, producing in this example the new formulae $s_0 : \gamma_0$, $s_1 : \gamma_1$, and so forth. In this dissertation, I will be using the latter case of this convention almost exclusively, but it is nonetheless convenient to recognize both cases.

Theorem 1.2.8 (Lifting Lemma). *If $\vec{s}:\Gamma, \Delta \vdash_{JT4} \varphi$, then there is a proof polynomial t such that $\vec{s}:\Gamma, \vec{y}:\Delta \vdash_{JT4} t:\varphi$.*

Proof. By induction on the derivation of φ. If φ is an axiom, let t be the proof constant specified for that axiom by the simple axiom justification rule. If φ was obtained by identity from Δ, let t be the matching y_i, and the conclusion follows by identity. If φ was obtained by identity from $\vec{s}:\Gamma$, let $t = !s_i$ corresponding to the relevant element of $\vec{s}:\Gamma$, and the conclusion follows by axiom 4^{j}. If φ was obtained by the simple axiom justification rule, then φ has the form $c:\psi$. Let $t = !c$, and the conclusion follows by axiom 4^{j}. Finally, if φ was obtained by modus ponens from ψ

[13] Also note that all of the theorems in this and the next two sections are due to Artemov [4, 5].

and $\psi \supset \varphi$, apply the induction hypothesis to get $u : \psi$ and $v : (\psi \supset \varphi)$, let $t = v \cdot u$, and the conclusion follows by axiom K^{j}. ■

The lifting lemma tells us that any reasoning that can be performed using the propositional aspects of the logic can also be reproduced within the justification mechanism. Obviously, this is essential if the justification terms are to represent proofs in propositional logic. The lifting lemma also allows us to rectify the glaring disanalogy between the normal modal and justification logics, as we get the following analogue of the modal necessitation rule as an obvious corollary of the lifting lemma:

Corollary 1.2.9 (Justification)**.** *If* $\vdash_{JT4} \varphi$*, then there is a proof polynomial* t *such that* $\vdash_{JT4} t : \varphi$*.*

If we examine the justification axioms K^{j}, T^{j}, and 4^{j}, and the general justification principle that we have derived from the lifting lemma, we can see that they have exactly the same logical form as the modal axioms K, T, 4, and the modal necessitation rule, with the only difference being the use of proof polynomials and the $:$ operator in place of the $\Box$ operator. Logical provability only depends on form, so we should intuitively expect that the justification logics prove the same things as the modal logics, and thus that Theorem 1.1.3 can be extended to them. But to do that, we need to prove the formal relationship that appears to be obvious on visual inspection. One direction of this relationship is straightforward:

Definition 1.2.10. For any JT4-sentence φ, let φ° (the forgetful projection of φ) be the S4-sentence produced by recursively replacing every subformula of φ that is a substitution instance of $t{:}\psi$ with $\Box\psi$ (such that nested : operators project to multiple boxes).

Theorem 1.2.11 (Projection Theorem)**.** *If* $\vdash_{JT4} \varphi$*, then* $\vdash_{S4} \varphi^{\circ}$*.*

The proof of Theorem 1.2.11 proceeds by induction on the axioms and inference rules of JT4, all of which are clearly S4-valid under the $\circ$ transformation.

The converse relationship between the logical systems is known as realization. Unlike projection, the realization theorem cannot be proven by direct inspection of the axiomatic systems. Two procedures for proving realization are known. In this dissertation, I will present realization proofs in the style of Artemov [4], who carries out the proof using sequent calculi equivalent to the modal and justification logics in question. Fitting [41] presents an alternative proof using model theories of these two logics. Fitting's proof, however, is not only more difficult than Artemov's, but it does not extend well to justification systems other than the normal justification logics that we are examining in this chapter. It therefore would not advance the purposes of this dissertation to examine the Fitting realization proof, and so I shall only present the Artemov proof. To start, we must develop the sequent calculus.

1.3 Sequent Calculus for JT4

Theorem 1.3.1. *A sequent calculus equivalent to JT4 can be produced by taking a sequent calculus for classical propositional logic*[14] *(including cut and contraction rules) and adding the following inference rules:*

$$\frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, t : \varphi \Rightarrow \Delta} \; {:}L$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, c : \varphi} \; {:}R$$

[14]For a formal presentation of such a calculus, see Appendix B.

Restriction: φ must be an instance of the PC, K^j, T^j, or 4^j schemas, and c must be chosen so that $\mathcal{C}(c,\varphi)$ holds for some constant specification $\mathcal{C}$. Note that for soundness and completeness to hold, the same $\mathcal{C}$ must be used consistently in both the axiomatic and sequent systems.

$$\frac{\Gamma \Rightarrow \Delta, s:\varphi}{\Gamma \Rightarrow \Delta, (s+t):\varphi}\ +R1$$

$$\frac{\Gamma \Rightarrow \Delta, t:\varphi}{\Gamma \Rightarrow \Delta, (s+t):\varphi}\ +R2$$

$$\frac{\Gamma \Rightarrow \Delta, s:(\varphi \supset \psi) \qquad \Gamma \Rightarrow \Delta, t:\varphi}{\Gamma \Rightarrow \Delta, (s\cdot t):\psi}\ \cdot R$$

$$\frac{\Rightarrow t:\varphi}{\Rightarrow !t:t:\varphi}\ !R$$

In presenting this theorem, Artemov omits the proof of equivalence, dismissing it as "a straightforward induction both ways." In the interest of conserving space, I shall follow his example and likewise omit this proof. Artemov does formally prove a cut-elimination theorem for the sequent calculus. However, this version of cut-elimination is not especially useful—notice that the subformula property fails for cut-free proofs in the given JT4 sequent calculus, as the ·R rule has cut-like behavior built into it.

1.4 Realization of S4 in JT4

To prove realization, we need, in addition to the sequent calculus for JT4, a sequent calculus for the modal system S4. The derivation of such a system is well-known, and can be found in standard proof theory texts such as [98]. One way of producing a sequent calculus for S4 is to add the following two rules to a classical propositional sequent calculus:

$$\frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \Box\varphi \Rightarrow \Delta}\,\Box\text{L}$$

$$\frac{\Box\Gamma \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi}\,\Box\text{R}$$

Theorem 1.4.1 (Realization Theorem). *If $\vdash_{S4} \theta$, then, assuming that the JT4 proof theory is developed using an axiomatically appropriate constant specification, there is some JT4-sentence φ such that $\varphi^\circ = \theta$ and $\vdash_{JT4} \varphi$.*

Proof. The proof proceeds by induction on a cut-free S4 sequent proof of θ.[15] At each step where $\Box$ is introduced via identity, weakening, or $\Box$L, simply replace $\Box$ with an unused proof variable (as per Convention 4).

At each step where $\Box$ is introduced via $\Box$R, the lifting lemma guarantees the existence of a suitable proof polynomial to replace the box. We construct a proof-step to replace the S4 sequent $\Box$R by a procedure analogous to the proof of the lifting lemma. First, we examine the structure and derivation of the premise sequent $\Box\Gamma \Rightarrow \varphi$. If φ is an axiom instance, we can simply replace $\Box$R in S4 with :R in JT4. If φ is a single modal formula, then we get a JT4 sequent derivation of its analogue by the inductive hypothesis, and then replace the $\Box$R step with a !R step. If φ is a formula such that $\varphi \in \Gamma$, then we can rewrite the original S4 derivation such that $\Box$R is not used at all, the sequent $\Box\varphi$ instead being introduced on both sides as an identity step, which is realized in JT4 by an arbitrary proof variable, as per the above. Minor propositional variants of any of the above cases (for example, cases in which a $\vee$R step is applied on a modal which would otherwise be suited

[15] Formally, a sequent calculus derivation is defined by a recursive algorithm—all identity sequents are valid derivations, and then the result of applying a sequent rule to an existing derivation produces a new valid derivation. Any such recursive definition yields an induction principle. In this proof, we are applying the induction procedure based on the sequent calculus definition. This may be contrasted with cases such as the standard proof of cut-elimination, which proceeds by a mathematical induction on a numerically-defined property of the calculus (the cut-rank).

for the !R translation) can be handled in a fairly obvious manner, though the exact details will depend on the structure of the propositional component of the axiomatic systems S4 and JT4. If none of the cases discussed above apply, then the sequent $\Box\Gamma \Rightarrow \varphi$ must have been derived by using $\Box$L to add boxes to some valid sequent $\Gamma \Rightarrow \varphi$. That sequent will correspond to a modus ponens step in axiomatic S4, and we can construct a $\cdot$R step using the premises of that modus ponens and suitable proof polynomials (which will probably be arbitrary choices from the translation of the $\Box$L steps in the original S4 sequent derivation, though this may be complicated somewhat if the sequent derivation involved the rules for propositional connectives other than $\supset$).

All other proof rules are common to both systems, so the end result of this procedure is a JT4 sequent proof of a sentence φ that satisifies the criterion $\varphi^\circ = \theta$. ∎

1.5 Model Theory for Normal Justification Logics

The basic semantic technique for justification systems, first expounded in [75], is to model the : operator by an arbitrary relation between the set of proof polynomials and the set of sentences, which is then restricted in accordance with the properties of the particular logic, as follows:

Definition 1.5.1. An evidence relation for the system J (or JT, JT4, or JT5) is a relation E between the set of proof polynomials and the set of sentences which satisfies the following:

- If φ is an axiom of the system's axiomatic presentation, and the axiomatic system in question employs iterated axiom justification, then there is an ω-

sequence of proof constants $\vec{c}$ such that $E(c_0, \varphi)$ and $E(c_n, c_{n-1} : \cdots : c_0 : \varphi)$ hold for all $n \in \mathbb{N}$. For systems employing simple axiom justification, replace the infinite sequence $\vec{c}$ with a single proof constant that plays the role of c_0 in this condition.

- For all proof polynomials s, t and sentences φ, ψ, if $E(s, \varphi \supset \psi)$ and $E(t, \varphi)$, then $E(s \cdot t, \psi)$.
- For all proof polynomials s, t and sentences φ, if $E(s, \varphi)$ or $E(t, \varphi)$, then $E(s + t, \varphi)$.
- For systems JT4 and JT5, if $E(t, \varphi)$, then $E(!t, t : \varphi)$ for all proof polynomials t and sentences φ. For simple axiom justification formulations of other justification logics, the condition that if $E(t, \varphi)$, then $E(!t, t : \varphi)$ is required only for formulae φ that are axiom instances.
- For system JT5, if it is not the case that $E(t, \varphi)$, then $E(?t, \neg t : \varphi)$ for all proof polynomials t and sentences φ.

Definition 1.5.2. A Mkrtychev model of the system J (or JT, JT4, or JT5) consists of an evidence relation E for that system (as defined above) plus a valuation function v from the set of sentences to $\{0, 1\}$ that satisfies the following:[16]

- $v(\neg\varphi) = 1$ iff $v(\varphi) = 0$.
- $v(\psi \supset \varphi) = 1$ iff $v(\psi) = 0$ or $v(\varphi) = 1$.

[16] Note that the propositional components of all of these systems are classical rather than intuitionistic. It is thus acceptable to give semantic clauses for $\neg$ and $\supset$ only, and to let other connectives be defined from them in the usual manner. For the sake of brevity, I will do so whenever it is appropriate.

- $v(t : \varphi) = 1$ iff $E(t, \varphi)$.
- For systems JT, JT4, and JT5, if there is any t such that $E(t, \varphi)$, then $v(\varphi) = 1$.

Definition 1.5.3 (Consequence for Mkrtychev models)**.** For each justification system $L \in \{J, JT, JT4, JT5\}$, $\Gamma \vDash_L \varphi$ iff for every Mkrytchev model of L such that $v(\gamma) = 1$ for every $\gamma \in \Gamma$, it is also the case that $v(\varphi) = 1$.

The given definition of an evidence relation builds in a form of axiomatic appropriateness, but does so without reference to any particular constant specification. This is in keeping with the view that axiomatically inappropriate cases are not normal justification logics at all, and ought to be ruled out entirely, possibly by the method of formulating the system without reference to constant specifications. If it is desirable to model the deviant cases that result from allowing axiomatically inappropriate constant specifications, this can be done by omitting the clause in the evidence relation definition that mimics the effect of axiom justification.

However, whether we incorporate axiom justification into the definition of an evidence relation or not, in order to prove soundness and completeness with respect to the axiomatic system, we need constant specifications to coordinate the systems, and so we must still provide a mechanism to incorporate constant or multi-constant specifications into the model theory. For logics with simple axiom justification, we say that a model of the logic meets a given constant specification $\mathcal{C}$ iff $\mathcal{C} \subseteq E$,[17] and define a consequence relation $\vDash_{L,\mathcal{C}}$ by restricting the class of Mkrytchev models to only those that meet $\mathcal{C}$. For logics with iterated axiom justification, we can define the consequence relation similarly, but the notion of meeting a multi-constant

[17]The ability to specify this condition as mere subsethood was one of the primary motivations behind the particular relational form used in Definition 1.2.5.

specification must be spelled out more tediously:

Definition 1.5.4. A Mkrytchev model $M = \langle E, v\rangle$ of J or JT is said to meet the multi-constant specification $\mathcal{C}$ if for every $\langle \vec{c}, \varphi\rangle \in \mathcal{C}$, we have it that $E(c_0, \varphi)$ and for every $n > 0$, $E(c_n, c_{n-1} : \cdots : c_0 : \varphi)$, where each c_i is the ith term of the ω-sequence $\vec{c}$.

Theorem 1.5.5. *For each justification system $L \in \{J, JT, JT4, JT5\}$, $\Gamma \vdash_{L,\mathcal{C}} \varphi$ iff $\Gamma \models_{L,\mathcal{C}} \varphi$, where $\vdash_{L,\mathcal{C}}$ is the axiomatic provability relation in which the axiom justification rule uses the axiomatically appropriate constant or multi-constant specification $\mathcal{C}$.*

Proof. The forward direction (soundness) is proven directly from the formal definition of axiomatic provability in the usual manner. The soundness of the classical axioms and modus ponens with respect to the model-theoretic clauses describing $\neg$ and $\supset$ is well-known. The first clause of the evidence relation definition validates the (simple or iterated) axiom justification rule in general,[18] and the use of a common constant [multi-]specification ensures that the results of each axiom justification instance are valid for the models which meet the [multi-]constant specification. The second clause validates axiom K$^{\mathrm{j}}$. The third validates the sum axioms. If relevant, the remaining two clauses validate axiom 4$^{\mathrm{j}}$ and axiom 5$^{\mathrm{j}}$ respectively. Finally, the clause in the valuation restrictions that pertains to systems JT, JT4, and JT5 validates axiom T$^{\mathrm{j}}$. The formulae provable in the axiomatic system are simply the inductive closure of these axioms and rules, so soundness is established universally.

For the reverse direction (completeness) we proceed according to the standard Lindenbaum-Henkin technique. We start by defining a maximal consistent set of

[18]The defined notions of meeting a constant or multi-constant specification also suffice for the soundness of either form of axiom justification rule even if the corresponding clause is omitted from the general definition of an evidence relation.

sentences M as a set such that $M \nvdash_{L,\mathcal{C}} \psi \wedge \neg\psi$ but for every $\varphi \notin M$, $M, \varphi \vdash_{L,\mathcal{C}} \psi \wedge \neg\psi$. Every proof-theoretically consistent set of sentences can be extended to a maximal consistent set by the algorithm of running through an enumeration of the sentences and at each step adding the next sentence to the set if doing so would not create inconsistency (Lindenbaum's Lemma). We also note that maximal consistent sets are closed under modus ponens and have the property that for every maximal consistent set M and sentence φ, either $\varphi \in M$ or $\neg\varphi \in M$.

Next, we show that for every maximal consistent set M, there is a Mkrytchev model such that $v(\varphi) = 1$ iff $\varphi \in M$. We start constructing a suitable model by letting $E(t, \psi)$ hold of all and only those proof terms t and sentences ψ such that $t : \psi \in M$. It is simple to verify that this E is an evidence relation. We know that for each axiom φ there are constants $\vec{c}$ such that $E(c_0, \varphi)$ and $E(c_n, c_{n-1} : \cdots : c_0 : \varphi)$ because if not, M would contain sentences $\neg\vec{c}{:}\varphi$ for every sequence of proof constants $\vec{c}$, which would make M an inconsistent set as per the axiom justification rule. Likewise, M contains every axiom instance (for if not, it would contain the negation and be inconsistent). This combined with the closure of M under modus ponens forces the satisfaction of the clauses that if $E(s, \varphi \supset \psi)$ and $E(t, \varphi)$, then $E(s \cdot t, \psi)$ and that if $E(s, \varphi)$ or $E(t, \varphi)$, then $E(s + t, \varphi)$, and likewise the similar clauses that apply to JT4 and JT5. Thus, E is an evidence relation. We can verify that the specified valuation meets the remaining conditions for a model in similar fashion.

Finally, we prove completeness by *reductio*. Assume that $\Gamma \vDash_{L,\mathcal{C}} \varphi$ but $\Gamma \nvdash_{L,\mathcal{C}} \varphi$. Then we have it that the set of sentences $\Gamma \cup \{\neg\varphi\}$ is proof-theoretically consistent, and so it can be extended to a maximal consistent set M. But then by the lemma proven in the last paragraph, we have a model satisfying Γ such that $v(\neg\varphi) = 1$, and thus $v(\varphi) = 0$. But this contradicts the assumption that $\Gamma \vDash_{L,\mathcal{C}} \varphi$, so completeness must hold. ■

It is sometimes philosophically useful to embed the evidence relation that models justification logic into the sort of frame structure that is used to model modal logics. Such a procedure might be used, for example, to model the relationship between epistemic certainty and ontological necessity which is stipulated under some epistemic theories. This sort of embedding is known as a Fitting model, based on its development in [41]. It turns out that such an embedding causes little trouble in the model theory—though it greatly enlarges the set of admissible models,[19] there is no change in the consequence relation.

Definition 1.5.6. A Fitting model of the system J (or JT, JT4, or JT5) is a quadruple $\langle \mathcal{W}, R, \mathcal{E}, V \rangle$ such that $\mathcal{W}$ is a non-empty set, R is a binary relation on $\mathcal{W}$, $\mathcal{E}$ is a binary function from $\mathcal{W}$ and the set of proof polynomials to the power set of the set of sentences, V is a binary function from $\mathcal{W}$ and the set of sentences to $\{0, 1\}$, and all of the following restrictions hold:

- If φ is an axiom of the system's axiomatic presentation, then there is an infinite sequence of proof constants $\vec{c}$ such that $\varphi \in \mathcal{E}(w, c_0)$ and $c_{n-1}{:}\cdots{:}c_0{:}\varphi \in \mathcal{E}(w, c_n)$ for all $w \in \mathcal{W}$ and $n \in \mathbb{N}$. (For systems JT4 and JT5, the infinite sequence $\vec{c}$ may be replaced with a single proof constant that plays the role of c_0 in this condition.)

- For all proof polynomials s, t, sentences φ, ψ, and $w \in \mathcal{W}$, if $(\varphi \supset \psi) \in \mathcal{E}(w, s)$ and $\varphi \in \mathcal{E}(w, t)$, then $\psi \in \mathcal{E}(w, s \cdot t)$.

[19] "Enlarges" is a misnomer here, given that we are dealing with infinite sets. However, the method creates great redundancy in the set of models, in that there are some severely restricted subsets that yield a consequence relation that is sound and complete with respect to the whole (or equivalently, to the proof theory or to the class of Mkrytchev models of the logic).

- For all proof polynomials s, t, sentences φ, and $w \in \mathcal{W}$, if $\varphi \in \mathcal{E}(w, s)$ or $\varphi \in \mathcal{E}(w, t)$, then $\varphi \in \mathcal{E}(w, s + t)$.

- For systems JT4 and JT5, if $\varphi \in \mathcal{E}(w, s)$, then $(s : \varphi) \in \mathcal{E}(w, !s)$ for all proof polynomials s, sentences φ, and $w \in \mathcal{W}$.

- For system JT5, if it is not the case that $\varphi \in \mathcal{E}(w, s)$, then $\neg(s : \varphi) \in \mathcal{E}(w, ?s)$ for all proof polynomials s, sentences φ, and $w \in \mathcal{W}$.

- $V_w(\neg\varphi) = 1$ iff $V_w(\varphi) = 0$.

- $V_w(\psi \supset \varphi) = 1$ iff $V_w(\psi) = 0$ or $V_w(\varphi) = 1$.

- $V_w(t{:}\varphi) = 1$ iff $\varphi \in \mathcal{E}(w, t)$ AND for all $w' \in \mathcal{W}$ such that $R(w, w')$, $V_{w'}(\varphi) = 1$.

- For systems JT, JT4, and JT5, and for every $w \in \mathcal{W}$, $R(w, w)$.

- For systems JT4 and JT5, and for every $w, w', w'' \in \mathcal{W}$, if $R(w, w')$ and $R(w', w'')$, then $R(w, w'')$.

- For system JT5, and for every $w, w' \in \mathcal{W}$, if $R(w, w')$ then $R(w', w)$.

- Also note that the JT4 and JT5 R restrictions may be imposed even in systems where they are not required. This works because the countermodels to undesired modal properties that would otherwise be generated from the unrestricted R can be reproduced using the $\mathcal{E}$ element of the semantic clause governing the : operator.

- Another restriction that may be optionally imposed upon Fitting models of any of the normal justification logics is that if $V_{w'}(\varphi) = 1$ for all $w' \in \mathcal{W}$ such that $R(w, w')$, then there is some proof polynomial t such that $\varphi \in \mathcal{E}(w, t)$. Obviously, this does not work if unnecessary R restrictions are employed.

Definition 1.5.7.

- A Fitting model $\langle \mathcal{W}, R, \mathcal{E}, V \rangle$ is said to meet a constant specification $\mathcal{C}$ if, for every $\langle c, \varphi \rangle \in \mathcal{C}$ and $w \in \mathcal{W}$, $\varphi \in \mathcal{E}(w, c)$.

- A Fitting model $\langle \mathcal{W}, R, \mathcal{E}, V \rangle$ is said to meet a multi-constant specification $\mathcal{C}$ if, for every $\langle \vec{c}, \varphi \rangle \in \mathcal{C}$ and $w \in \mathcal{W}$, $\varphi \in \mathcal{E}(w, c_0)$ and for every $n > 0$, $c_{n-1} : \cdots : c_0 : \varphi \in \mathcal{E}(w, c_n)$, where each c_i is the ith term of the ω-sequence $\vec{c}$.

- For each justification system $L \in \{J, JT, JT4, JT5\}$ and constant or multi-constant specification $\mathcal{C}$, a consequence relation $\vDash_{L,\mathcal{C}}$ is defined such that $\Gamma \vDash_{L,\mathcal{C}} \varphi$ iff, for every Fitting model $\langle \mathcal{W}, R, \mathcal{E}, V \rangle$ which meets $\mathcal{C}$ and for which $V_w(\gamma) = 1$ for all $\gamma \in \Gamma$ and $w \in \mathcal{W}$, it is also the case that $V_w(\varphi) = 1$ for all $w \in \mathcal{W}$.

Notice that each Mkrytchev model is structurally equivalent to a Fitting model in which W is a singleton and R is reflexive. This equivalence and the optional restrictions that are mentioned in the Fitting model definition illustrate the redundancy of this model theory that was alluded to above. Nevertheless, the various restricted and unrestricted classes of Fitting models are sound and complete with respect to each other and to the proof theory. These various completeness properties can be proven by procedures similar to the completeness proof given above for the Mkrytchev models, or by other procedures illustrated in [41]; I will not reproduce those proofs here.

1.6 Motivation as Epistemic Logic

The classical analysis of knowledge is that it is justified true belief. This analysis has been challenged by Gettier's famous counterexample [46], and by other related examples such as Goldman's fake barns [51]. The common element of all these counterexamples is that an agent seems to have justified belief in a proposition, but seems to lack knowledge because the putative justification does not have the proper sort of connection with the proposition's truth. For this reason, the general consensus is that the justified true belief analysis is mostly correct, and simply needs to be refined to rule out any unacceptable Gettier-like cases.[20] There is, however, no consensus as to how such refinement should proceed. For the present purposes, I shall ignore such details and treat knowledge as simply justified true belief.

Now, given that knowledge is (at least) justified true belief, an adequate epistemology must clearly include some understanding of what truth is and of what justification is. At present, we shall not concern ourselves too much with the nature of truth, though we shall return to such considerations in Chapters 3–5. The immediately pressing question is the nature of justification. There are three important theories of justification: evidentialism, reliabilism, and coherentism. Some proponents of the latter two take justification to be evidentialist and treat their proposals as alternatives to the justified true belief analysis of knowledge, but this is merely a terminological distinction.[21]

[20] The most noteworthy genuine dissent is the view of Williamson [102], who argues that knowledge is best treated as an unanalyzable primitive. Most other dissents from the JTB analysis in the literature are merely verbal.

[21] Another interesting take on the terminology of epistemology can be found in Bergmann [21]. Bergmann uses the term "warrant," which he defines explicitly as "that which makes the difference between knowledge and mere true belief." Bergmann also notes that Plantinga [81] uses "warrant" similarly, and suggests that Chisholm [27] makes analogous use of the term "justification." It would be perfectly fair to read my own usage of "justification" in a similar fashion, though I am

Evidentialism is the thesis that an agent is justified in believing a proposition just when she has access to the right sort of evidence to support that belief. It is the most traditional account of justification. Evidentialist justification is inherently a constructive notion; to have knowledge of a proposition, one must have access to some specific piece of evidence that supports it. Thus, given evidentialism, the most perspicuous model of knowledge would be a justification logic, with the proof polynomials representing individual pieces of evidence.

If justification is connected to evidence, one must consider the ontological status of evidence. What kind of thing is it? Intuitively, in at least some cases, the evidence that justifies a belief consists of one or more additional beliefs (e.g., when a belief is justified using a deductive argument from antecedently believed premises). But then it is equally intuitive that for this to count as justification, the beliefs that serve as evidence must themselves be justified. Ultimately, this leads to a trilemma: either some beliefs are justified using some sort of evidence that isn't another belief, or some beliefs are justified without any evidence outside of themselves (analogously to axioms in mathematical proof), or we have an infinite regress of belief. The last option is obviously unacceptable. The second option is a paradigmatic example of what Sellars calls "the myth of the given." (Sellars' characterization of the "myth" also encompasses some versions of the non-doxastic evidence account, but that subtlety isn't critical here.) Sellars wrote many papers attacking views of this sort,[22] and the philosophical community has found his arguments persuasive. Most evidentialists instead adopt the remaining option, accepting some form of non-doxastic evidence,

unconvinced by Bergmann's argument that this should be considered distinct from our ordinary usage of "justification."

[22]For an example of a well-developed theory of evidence along these lines, see Chisholm's account in [26] and Sellars' response to it in [87].

usually sense-data or the like. This position is much more plausible, but it does have its detractors. Notably, Davidson [31] argues that the connection between sense-data and belief is fundamentally a matter of causation, and that it is not clear that this provides true epistemic justification. This line of objection is legitimate, but not decisive; Davidson does not show that sense-data cannot provide epistemic justification, but merely expresses doubt as to whether it actually does so.

Davidson's recommended solution to the trilemma of beliefs-as-evidence is to adopt a coherentist account of justification. The coherentist thesis is that instead of beliefs serving as evidence to justify each other individually, the elements of an agent's system of beliefs justify each other collectively. The epistemic status of any particular belief depends primarily on two factors: on how plausible that belief is, given the agent's other beliefs; and on how important that belief is to the mutual plausibility of the other beliefs. Some coherentists (e.g., Shogenji [89] and Fitelson [40]) have attempted to formalize these factors into mathematical calculations called coherence measures. A perspicuous model for knowledge under coherentist justification would probably be a non-classical logic built around one of these coherence measures. Working out the details of such a system is the project of my colleague, Michael Hughes,[23] and I shall leave it to him and not discuss it further here.

The other major account of justification, reliabilism, is the thesis that a belief is justified iff it is formed by a reliable process, which is defined in terms of the objective probability that beliefs formed by the process are true. Goldman [52] argues for this view over evidentialism on the basis that it more genuinely explains knowledge by analyzing it into non-epistemic concepts. Reliabilist accounts might also be helpful

[23]Michael presented work on exactly this topic at a research session in 2012 [57]. However, none of the material from this presentation was incorporated into his final dissertation [58], nor has any of it been published elsewhere.

in avoiding Gettier problems, but only because in reducing justification to accuracy, they have already rejected the intuition on which the Gettier argument is based. Finally, because reliabilist accounts of justification are concerned only with objective probabilities, they are non-constructive, and are perspicuously modeled using modal logics.

Justification logics are important in epistemic contexts for several reasons. In my opinion, the most persuasive argument for their use is the one that I presented above—that justification systems provide the most perspicuous model for evidentialist justification. However, there are two additional cases for justification logic that must be considered. The first, from van Bentham [99], is a perspicuity argument similar to the one presented above. However, whereas the above analysis worked with the standard justified true belief account of knowledge as presented within the framework of a standard truth-conditional semantics, van Bentham works within a class of alternative semantic systems that he calls "information processing semantics." These semantic systems are inherently constructive, and thus require a constructive knowledge operator. To this end, van Bentham sketches out an epistemic logic which is essentially a forerunner of the Artemov justification logics presented above in Section 1.2.

Perspicuity arguments, such as van Bentham's argument from information processing, my argument from evidentialism, or for that matter the argument for JT4 over S4 in the context of intutionistic mathematics, suffer from certain glaring weaknesses. An obvious weakness is that all such arguments depend on the adoption of some particular background theory. If one does not accept information processing semantics, evidentialist justification, or mathematical intuitionism, then none of the aforementioned arguments gets off the ground. However, this is not a fatal weakness, given that there are many people who do accept these background theses. A more fundamental concern is that perspicuity arguments are essentially aesthetic objections rather than practical ones. For example, in the intuitionist case, if one is merely concerned with finding out what propositions are intuitionistically provable, then there is no reason to prefer JT4 over S4; on the contrary, the latter system is better suited to this task because it is simpler. JT4 only becomes essential when one is concerned with matters of deeper principle—intuitionistic propriety as per Brouwer's philosophy. Similar distinctions can be drawn in the epistemic cases. All things considered, then, the perspicuity arguments should convince the reader that the justification logics are a topic of theoretic interest, worthy of further study; but these arguments will probably fail to convince anyone to actually adopt justification logic who is not already concerned with the deeper philosophical questions associated with logical perspicuity. This result suffices to motivate the study of justification logics that I will undertake in the remainder of this dissertation, but it is far from ideal.

Perhaps recognizing this concern, Artemov presents in his 2011 article [7] an entirely practical argument for the use of justification systems as epistemic logics. Artemov begins by observing the large variety of thought experiments that have been constructed by various epistemologists to test our epistemic intuitions—for example, the Gettier and fake-barn scenarios that I mentioned earlier. Subtle changes in the components of these scenarios can often produce significant changes in our intuitive judgment. Artemov argues that if we are to accept all of these intuitions as genuine, a modal logic would not be able to express them without falling into inconsistency, as its notation is too crude to model the subtleties of the scenarios. As an example, Artemov analyzes the Red Barn scenario of Kripke [64], which has the following salient features:

- Henry is looking at a genuine barn, which happens to be painted red.
- Henry and the barn are located in a region which contains many fake barn facades, none of which are painted red.
- For the present discussion, let us assume that Henry is aware of the region's fake barns, and of their color uniformity.[24]
- Intuitively, in this case, Henry cannot know that a barn is present by observing a barn simpliciter, because the barn could be fake. However, when he observes a red barn, he knows that there is a red barn, because an observed red barn cannot be a fake.
- Finally, let us stipulate that the proposition that x is a red barn is simply to be analyzed as the conjunction "x is red $\wedge$ x is a barn," and likewise for the corresponding observations.

Under a modal logic of knowledge, any reasonable formalization of this scenario leads to a contradiction. Henry does not know (based on his observation) that the object in front of him (call it x) is a barn: $\neg\Box Bx$. He does know that it is a red barn: $\Box(Rx \wedge Bx)$. However, the $\Box$ operator is closed under propositional logic—and most epistemic theories would have knowledge closed under at least known-to-the-agent logical entailment—so $\Box(Rx \wedge Bx)$ entails $\Box Bx$, creating a contradiction. It should be emphasized that no epistemologist, upon analyzing this scenario, would actually

[24]In Kripke's original deployment of this example in an objection to Nozick's epistemic theory, part of the case's objectionability stems from the fact that the argument goes through even if Henry does not know that there are fake barns, or that all of the fakes are non-red. The question of whether these features are acceptable or problematic is a matter of pure epistemology. For the logical concern in which Artemov and I are interested, it is better to simply assume that Henry has been informed of all the salient facts.

assert the contradictory propositions. One would instead say something like "Henry does not know, in virtue of seeing a barn, that there is a barn, but he does know, in virtue of seeing a red barn, that there is a red barn (and thus, a barn simpliciter)." However, the notation of modal logic does not include the expressive resources needed to represent this distinction, and so the contradiction comes about as an artifact of the formalization.

Artemov points out that this contradiction is easily avoided by modeling knowledge with a justification logic: the denial of knowledge and the assertion of knowledge occur under different justification terms, and thus these assertions are no more contradictory than the assertions that one set of statements is a proof of the Pythagorean theorem and that another is not. Indeed, we can describe the case more precisely. I stipulated above that "red barn" is simply the conjunction of being red and being a barn, and that the observations should be treated likewise.[25] So let us do so, by writing the observation of a red barn as a proof polynomial $r + b$. We then have it that $\neg b : B$, but also $(r + b) : (B \wedge R)$, and thus $(r + b) : B$. (Technically, the last of those should be $(c \cdot (r + b)) : B$, where c is the proof constant specified for the axiom of conjunction simplification, but that distinction isn't important here.) Now look carefully at the axiom (or model-theoretic clause) defining the + function. Notice that it is a mere conditional: $\vdash t : \varphi \supset (s + t) : \varphi$. This was stipulated quite deliberately—combining two pieces of evidence (even in a merely concatenation-like manner) at least proves everything that the originals do, but it is allowable that the combination proves more, and this particular scenario is a case which is best modeled in just such a manner. Therefore, we conclude that justification logic provides a satisfactory formalization of this particular case.

[25]The latter is my own addition to Artemov's argument.

Artemov conjectures that similar analyses will be helpful in most of the complicated scenarios that occur in the epistemology literature, and presents examples of several Gettier-type cases in [6]. The technique that Artemov proposes for those cases involves modifying the logic by taking the minimal normal justification logic J and then adding a restricted version of the T^j schema that can be instantiated only by a proper subset of the set of proof polynomials. This allows a Gettiered justification to be represented by a proof polynomial that falls outside of the domain of application for T^j, whereas a genuine justification would be represented by a proof polynomial to which T^j does apply.[26]

Let us grant for the moment that one might be persuaded by one of the above arguments to adopt a justification logic to model the knowledge operator. Another pressing question arises: which justification logic should one use? To answer this, we must examine the ontology of knowledge. In particular, we are concerned with the following question: What is it which must be true for it to be the case that a subject S knows a proposition p? Obviously, given that knowledge is justified true belief, we have it that p must be true, and that S must have justification for it. Moreover, we have discussed already several accounts of justification, any of which would suffice for the present inquiry. Let us now adopt the evidentialist account, simply to provide a concrete example with which to work. So we have it that p is true and that S possesses evidence for p. But what is it for S to possess evidence? We noted earlier that S's other justified beliefs can constitute evidence, but that to avoid regress, some of S's beliefs must be justified by some other sort of evidence, and we must now consider

[26] This proposal is merely the simplest of the several Gettier-type scenarios that Artemov discusses. Artemov provides different analyses of some other cases. However, all of these analyses share the features that the base logic is something intermediate between J and JT, and that Gettiered justifications are represented by proof polynomials for which T^j fails.

possible explanations of what it is for S to possess evidence of this sort.

The ontological question with which we are presently concerned cuts close to a distinction that is typically drawn in epistemology, the divide between internalist and externalist theories of knowledge. However, there is an important difference. Not all epistemologists who use the internal/external terminology provide clear definitions of what is meant, but for those who do, the distinction is not fundamentally ontological, but rather wholly epistemic. A condition is said to be internal if an agent can determine whether or not the condition obtains by means of reflection or introspection—a property which is also commonly known as epistemic accessibility. An external condition is such that one cannot determine whether or not it obtains in such a manner. Epistemic internalism, on most accounts, is the thesis that epistemic justification is internal in this sense, and likewise for epistemic externalism.[27] As I see it, this wholly epistemic characterization of the internalism/externalism distinction is useful only for sorting epistemologists into groups. The truly fundamental question is ontological: whether justification (or warrant, or the possession of evidence) is a property of the mind, or whether it is a property of the world.

An internalist theory of epistemology, under the "epistemic access" definition of internalism, entails a KK principle, which is the epistemologist's term for an analogue of the 4 axiom. The access definition tells us that the determination that we possess some particular knowledge is a process of reflection, and then iterated knowledge is nothing more than the reflective process of tracking that lower-level reflection. The ontological definition of internalism does not directly guarantee either epistemic access in general or the KK principle in particular. However, it is universally agreed that in

[27] For arguments that this is indeed the predominant usage, see the literature analyses of Bergmann [21] and Harper [54].

the case of a purely mental state which does not satisfy the requirement of epistemic access, there is no plausible argument that the state should be classified as knowledge rather than as a mere belief, emotion, etc.. In practice, then, every ontological internalism is also an access internalism, and has the same logical properties.

Besides the KK principle, the other important logical property to consider is whether knowledge should be modeled by a *normal* modal or justification logic. Recall that normality consists of two principles: a K axiom, which provides closure under logical inference, and a necessitation rule, which makes the modality hold of all logical truths. In regard to these principles, the epistemic internalist can coherently either accept or reject them. The internalist who accepts normality (and whose justification logic is thus JT4) emphasizes that there is nothing of any external character in the process of logical reasoning; logic is entirely *a priori.* The internalist who rejects normality points out that no agent's belief system or other mental states contains all of the theorems of logic, and that this lack is indeed accessible (as a non-constructive existential thesis) by reflection. Moreover, from an ontological perspective, it may be impossible for the external world to contain logical contradictions, but the internal content of an agent's mind is subject to no such restriction. It is thus defensible for an internalist to reject either or both aspects of normality. Justification logics that are non-normal with respect to necessitation will be examined in the next chapter and in Chapter 5. Logics that are non-normal with respect to the K axiom will be examined in Chapter 4.

What about externalism? We still have two different ways to define externalism, one based on ontology and the other based on the rejection of access internalism. Contraposing the relationship between the internalist views gives us that every anti-access externalism is also an ontological externalism. As was hinted earlier, it seems to hold uncontroversially that the nature of the universe must be logically consistent.[28] The consistency of the universe inevitably leads to normality in the epistemic logic, though the exact mechanism depends on the externalist epistemology in question. Take, for example, reliabilism, which is a paradigm case of epistemic externalism. A belief is justified under reliabilism if it is formed by an objectively reliable process. Sound logical argument is necessarily such a process, as the objective reality cannot violate the laws of logic. Therefore, the justification logic for reliabilism must be a normal logic, specifically, JT.[29]

For anti-access externalists, the KK principle is a fundamental part of the very notion of epistemic accessibility that they are rejecting. Their epistemic logic must therefore be free of any version of the 4 axiom, leaving modal system T and justification system JT. However, it does not necessarily follow that an ontological externalist must adopt this logic.

Consider the view that S possesses evidence for p insofar as S bears an appropriate physical relationship to those objects in the universe that are relevant to the truth of p. Moreover, let us suppose that it is true in a given case that S knows that p. What features of the universe are relevant to this truth? The only possible candidates are S himself, the objects relevant to the truth of p, and that it is the case that these objects are all related in a manner conducive to knowledge. What, then, is required for S to know that S knows that p? Obviously, it is that the relevant objects are related in the appropriate way. What are these objects? The relationship between

[28] Even a dialetheist would not deny this, but rather would deny that the logic to which the universe conforms is classical.

[29] We earlier concluded that reliabilist epistemology does not actually require a justification logic, and so can be perspicuously modeled by the modal system T. However, the present concern is the question of which modal/justification axioms are needed to model the epistemology, so the example remains legitimate.

objects is not itself an object,[30] so the relevant objects cannot differ between the simple and iterated knowledge cases. Perhaps there might be some difference in the role that the objects play in the simple and iterated knowledge relations that would allow the truth values of simple and iterated knowledge to come apart—given that we haven't actually specified the relevant relationship, we cannot determine whether or not this could be the case. But there are certainly examples in which it turns out that this is impossible, and indeed that the truth-conditions for simple and iterated knowledge are identical. This, in turn, makes the justification logic of knowledge of such a theory JT4 rather than the usual externalist logic, JT.

[30] An anonymous reviewer of an earlier draft of this chapter pointed out that this claim is controversial, but let's take it for granted, as the present aim is only to set forth a plausible epistemic theory, not to argue that this theory should be adopted.

Chapter 2

Non-Normal Justification Logic

In Chapter 1, I formally introduced the class of logical systems known as justification logics—more specifically, the normal justification logics. I also presented various philosophical arguments as to why justification logics ought to be used in place of modal logics as representations of knowledge. In addition, I briefly analyzed the logical properties of various positions within epistemology, concluding that most externalist positions and some internalist positions could be perspicuously represented by the normal modal logics that were discussed in that chapter, but that other positions require different logics.

In this instance, the need for alternatives to the normal logics goes beyond mere perspicuity. Indeed, the feature of normal modal (and justification) logics that the $\Box$ operator holds of all logical truths is often cited as being implausible by opponents of those epistemologists who have adopted theories based on normal logics. Here's a simple version of the argument: if knowledge is defined as something like justified true belief, then an agent's knowledge must be a subset of the agent's beliefs. In the real world, all epistemic agents are finite beings with finite brain capacities, and so their set of beliefs (and, *a fortiori,* of knowledge) must also be finite. So any epistemic theory that entails that agents possess infinite knowledge (such as the knowledge of all logical truths) must be false. This objection is known as the problem of logical omniscience.

There are two standard ways in which an epistemologist who is charged with the problem of logical omniscience might respond. One option is to insist that what the epistemic logic is describing is not an agent's actual knowledge, but rather potential knowledge—that is, the sum total of information that an idealized epistemic agent with infinite capacity would be able to claim knowledge of under the same epistemic circumstances which apply to the actual agent under consideration. This move is a reasonable defense of the philosophical use of the normal modal or justification logic, but does not provide the expressive resources to describe the knowledge possessed by the actual, finite agent.

The other standard response is to change the logic. There are many possible changes that could be made, but the most common choice[1] is to introduce into the semantic frame of epistemic possibilities an "impossible possibility," that is, a situation in which certain logical truths may fail to hold, but which the epistemic agent, owing to her finite limitations, has failed to rule out. In the model theory, this is represented by the transition from the simple frame semantics of the normal modal logics (and of the Fitting models of the normal justification logics) to a partitioned frame semantics. In this chapter, we will develop justification analogues of the non-normal modal logics that result from this transition.

[1]A notable example of a philosopher advocating this move is Hintikka [56].

2.1 The Justification System JS0.5

2.1.1 Axiomatic Development

Our starting point will be the modal system S0.5, first described by E. J. Lemmon in [65]. This system has the advantage of simplicity in both its axiomatic and model-theoretic characterizations. In its axiomatic formulation, S0.5 consists of the following axiom schemata and proof rules:

PC Any set of axiom schemata whose closure under modus ponens is sound and complete with respect to classical propositional logic

K $\vdash_{S0.5} \Box(\psi \supset \varphi) \supset (\Box\psi \supset \Box\varphi)$

T $\vdash_{S0.5} \Box\varphi \supset \varphi$

Modus Ponens If $\vdash_{S0.5} \psi \supset \varphi$ and $\vdash_{S0.5} \psi$, then $\vdash_{S0.5} \varphi$.

Necessitation If φ is a tautology,[2] then $\vdash_{S0.5} \Box\varphi$.

A justification system JS0.5 can be produced by replacing the modal operators in these axioms with justification functions, just as in Artemov's original development of the Logic of Proofs from S4 [4]:

PC Any set of axiom schemas whose closure under modus ponens is sound and complete with respect to classical propositional logic

$\mathbf{K^j}$ $\vdash_{JS0.5} t : (\psi \supset \varphi) \supset (s : \psi \supset (t \cdot s) : \varphi)$.

[2]Here taken to denote an arbitrary substitution instance of a classically valid scheme; in other words, any formula that can be proven using only the PC axioms and modus ponens, regardless of whether the formula contains any instance of the $\Box$ operator.

$\mathbf{T^j}$ $\vdash_{JS0.5} t : \varphi \supset \varphi$.

Sum1 $\vdash_{JS0.5} s : \varphi \supset (s + t) : \varphi$

Sum2 $\vdash_{JS0.5} t : \varphi \supset (s + t) : \varphi$

Modus Ponens If $\vdash_{JS0.5} \psi \supset \varphi$ and $\vdash_{JS0.5} \psi$, then $\vdash_{JS0.5} \varphi$.

Classical Axiom Necessitation If φ is an instance of an axiom scheme in PC, and c is a proof constant such that $\mathcal{C}(c, \varphi)$ holds for a chosen constant specification $\mathcal{C}$, $\vdash_{JS0.5} c : \varphi$.

As in the normal justification logics of Chapter 1, the sum axioms are not needed to capture any feature of the modal system, but they are conservative extensions of the logic, and their inclusion can be useful for modeling certain philosophical applications. I will thus continue my policy from Chapter 1 of not noting the inclusion or exclusion of these axioms except where it makes a significant difference in the application at hand.

The following theorems describe fundamental logical properties of JS0.5:

Theorem 2.1.1 (Classical Lifting Lemma)**.** *If* $\Gamma \vdash_{PC} \varphi$*,*[3] *then for any axiomatically appropriate constant specification* $\mathcal{C}$ *and any sequence of proof polynomials* $\vec{s}$ *with length equal to the cardinality of* Γ*, there exists a proof polynomial* t *such that* $\vec{s} : \Gamma \vdash_{JS0.5,\mathcal{C}} t : \varphi$*.*

Proof. The proof proceeds via the inductive procedure that defines the classical provability function $\vdash_{PC}$. In the base case where φ is a PC axiom, we apply the JS0.5 classical axiom necessitation rule, to get $\vdash_{JS0.5} c : \varphi$. Let t be the proof constant c

[3] As in the S0.5 axiomatization, the notion of classical provability used here should be taken to include modal substitution instances of classical proofs.

specified by that rule, and use the structural weakening property of $\vdash_{JS0.5}$ to show that $\vec{s} : \Gamma \vdash_{JS0.5} t : \varphi$.

In the base case where φ is a member of Γ, we simply let t be the element of $\vec{s}$ that is applied to φ in the premise set $\vec{s} : \Gamma$. We then have it that $\vec{s} : \Gamma \vdash_{JS0.5} t : \varphi$. by the identity property of $\vdash_{JS0.5}$.

Finally, we have the inductive case, in which φ is deduced by modus ponens from provable formulae ψ and $\psi \supset \varphi$. By inductive hypothesis, the theorem holds for $\Gamma \vdash_{PC} \psi$ and $\Gamma \vdash_{PC} \psi \supset \varphi$, yielding proof polynomials t_1 and t_2 such that $\vec{s} : \Gamma \vdash_{JS0.5} t_1 : \psi$ and $\vec{s} : \Gamma \vdash_{JS0.5} t_2 : (\psi \supset \varphi)$. We also have $\vec{s} : \Gamma \vdash_{JS0.5} t_2 : (\psi \supset \varphi) \supset (t_1 : \psi \supset (t_2 \cdot t_1) : \varphi)$, as the right-hand formula is an instance of the application axiom. Combining these three results via modus ponens yields $\vec{s} : \Gamma \vdash_{JS0.5} (t_2 \cdot t_1) : \varphi$. Let $t = t_2 \cdot t_1$, and we get $\vec{s} : \Gamma \vdash_{JS0.5} t : \varphi$. ■

Corollary 2.1.2 (S0.5-like Justification)**.** *For any formula φ and axiomatically appropriate constant specification $\mathcal{C}$, if $\vdash_{PC} \varphi$, then there exists a proof polynomial t such that $\vdash_{JS0.5,\mathcal{C}} t : \varphi$.*

Theorem 2.1.3 (Projection of S0.5 into JS0.5)**.** *If $\Gamma \vdash_{JS0.5} \varphi$, then $\Gamma^\circ \vdash_{S0.5} \varphi^\circ$, where $\circ$ is the forgetful projection operator, as defined in Chapter 1.*

I will omit the full proof of this theorem, as it is a completely obvious application of the inductive definition of $\vdash_{JS0.5}$. All of the axioms become S0.5 axioms under the $\circ$ transformation, and the inference rules also remain valid.

2.1.2 The Realization Theorem

The converse property of projection is standardly denoted by "realization." Unlike projection, there is no convenient proof of realization in any axiomatic justification logic. There are two known methods for proving realization theorems: a procedure based on sequent calculus, due to Artemov [4], and an alternative based on model theory, due to Fitting [41]. Of the two, the Artemov realization proof is both simpler and better suited for adaptation to non-normal and non-classical logics, and as such will be adopted for all of the logics presented in this dissertation.

As a precondition for the development of the realization proof, we need a sequent calculus for the base modal system S0.5. Unlike all of the other modal systems discussed in this dissertation, the sequent calculus for S0.5 is an original development, and so I will take the time to formally prove its correctness.

Theorem 2.1.4. *The system that results from adding the following rules to a sequent calculus for classical logic (including the contraction, weakening, and cut rules) is sound and complete with respect to the axiomatic system S0.5:*

$$\frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \Box\varphi \Rightarrow \Delta}\ \Box L$$

$$\frac{\Gamma, \Box(\psi \supset \varphi), \Box\psi \Rightarrow \Delta, \varphi}{\Gamma, \Box(\psi \supset \varphi), \Box\psi \Rightarrow \Delta, \Box\varphi}\ \Box R$$

$$\frac{\Rightarrow \varphi}{\Rightarrow \Box\varphi}\ Nec$$

Restriction: The sequent $\Rightarrow \varphi$ *must have been derived without using the rules* $\Box L$*,* $\Box R$*, or Nec.*

Proof. Every proof theory textbook contains a theorem relating classical axiomatic propositional logic with its sequent calculus in the same manner as this theorem does for S0.5. In particular, Girard [48] provides a fairly good discussion of how the classical portion of the proof can be adapted to the different (logically equivalent) variants of the classical axiomatic and sequent systems. (For the reader's convenience, a less comprehensive discussion of this matter can be found in Appendix B of this disseration.) For the sake of brevity, I shall omit most of the details concerning the classical portions of the modal systems, and focus instead on the purely modal rules and axioms.

The left-to-right direction of the biconditional can be proven by a strong induction on the depth of the sequent derivation, where depth is a numerical index defined as follows:

- Axiomatic sequents have depth 1.
- Sequents derived by weakening or contraction have depth equal to the depth of the preceding sequent in the derivation.
- Sequents derived by single-premise rules (such as $\Box$L or $\Box$R) have depth 1 greater than the depth of the preceding sequent.
- Sequents derived by two-premise rules (such as $\supset$L or cut) have depth 1 greater than the larger of the depths of the preceding sequents.

Depth 1 sequents contain the same formula in both Γ and Δ; that formula can thus be proven as a hypothesis. Suppose for strong induction that we have a proof corresponding to each derivable sequent of depth less than n. The last step of any derivation of depth n will either be a premised rule applied to a sequent derivation of depth $n-1$ (and possibly an additional sequent derivation of depth $n-1$ or less), or it will be an application of weakening or contraction to another depth n sequent. We then devise a procedure to construct a suitable proof in the axiomatic system depending on the last rule applied in the sequent derivation.

If the last rule is a $\Box$L instance $\dfrac{\varphi, \Gamma \Rightarrow \Delta}{\Box\varphi, \Gamma \Rightarrow \Delta}$ we apply the inductive hypothesis to the depth $n-1$ derivation of the sequent $\varphi, \Gamma \Rightarrow \Delta$ to get a proof of one of the formulas in Δ from hypotheses φ, Γ. If the hypothesis φ was not used in the proof, then the same proof is legitimate for the desired hypothesis set $\Box\varphi, \Gamma$. Otherwise, we derive the formula φ in the desired hypothesis set by using modus ponens on the premise $\Box\varphi$ and the T-axiom instance $\Box\varphi \supset \varphi$, and plug these steps into the appropriate place in the proof.

If the last rule of the derivation is a $\Box$R instance $\dfrac{\Gamma, \Box(\psi \supset \varphi), \Box\psi \Rightarrow \Delta, \varphi}{\Gamma, \Box(\psi \supset \varphi), \Box\psi \Rightarrow \Delta, \Box\varphi}$ we apply the inductive hypothesis to the derivation of the sequent $\Gamma, \Box(\psi \supset \varphi), \Box\psi \Rightarrow \Delta, \varphi$ and examine the resulting proof. If it is a proof of a formula in Δ, then we can use that proof. Otherwise, we discard it and produce a proof of $\Box\varphi$ by taking the K-axiom instance $\Box(\psi \supset \varphi) \supset (\Box\psi \supset \Box\varphi)$ and applying modus ponens with the hypotheses $\Box(\psi \supset \varphi)$ and $\Box\psi$.

If the last rule of the derivation is a Nec instance $\dfrac{\Rightarrow \varphi}{\Rightarrow \Box\varphi}$, then according to the restriction on the Nec rule, the sequent $\Rightarrow \varphi$ was obtained by a derivation that does not use $\Box$L, $\Box$R, or Nec. An S0.5 sequent derivation without those three rules is simply a classical sequent derivation. Therefore, we can apply the soundness and completeness theorem for the sequent calculus of classical logic to show that φ is a tautology in the sense required for the necessitation rule of axiomatic S0.5. Thus, we can construct an axiomatic proof of $\Box\varphi$ using necessitation.

If the last rule of the derivation is an instance of Cut $\dfrac{\Gamma \Rightarrow \varphi, \Delta \qquad \Gamma', \varphi \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$ we apply the inductive hypothesis to the upper-left premise and examine the resulting proof. If it is a proof of a formula in Δ, then we can use the same proof. If it is a proof of φ, we combine that proof with the proof that results from applying the inductive hypothesis to the other premise.

If the last rule of the derivation is an instance of one of the (left or right) introduction rules for the classical connectives, we can produce the desired proof using a procedure similar to the cases discussed above. The exact technique depends on the choice of axioms used to define the axiomatic system; cf. [48].

If the last rule of the derivation is an instance of weakening or contraction, note that in any application of these rules, a proof that satisfies the theorem for the preceding step will also do so for the weakening or contraction result. So we examine the last previous step that was not a weakening or contraction instance, which will have depth n, as weakening and contraction do not alter depth. We then find a proof according to the appropriate procedure specified above.

This completes all of the possible cases, so by strong induction, all derivable sequents correspond to hypothetical proofs in the axiomatic system as per the left-to-right direction of the theorem statement.

For the right-to-left direction, we need to show that for every formula θ provable in the axiomatic system from hypotheses Γ, the sequent $\Gamma \Rightarrow \theta$ is derivable. We proceed using the inductive definition of provability in the S0.5 axiomatic system. The base cases are when θ is an axiom instance, when $\theta \in \Gamma$, and when θ is derived by the necessitation rule, and the inductive case is when θ is derived by modus ponens.

For the axiom case, we derive sequents corresponding to arbitrary instances of the various axioms. In all these derivations, the left-hand side of the sequent is left empty; the members of Γ can be added in by weakening as needed. For the classical axioms, the derivation depends on the choice of axiom schema, and in any event is fairly trivial; cf. [48]. For a T-axiom instance $\Box\varphi \supset \varphi$, we give the following sequent derivation:

$$\cfrac{\cfrac{\varphi \Rightarrow \varphi}{\Box\varphi \Rightarrow \varphi}\,\Box\text{L}}{\Rightarrow \Box\varphi \supset \varphi}\,\supset\text{R}$$

Finally, for a K-axiom instance $\Box(\psi \supset \varphi) \supset (\Box\psi \supset \Box\varphi)$, the following sequent derivation suffices:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\psi \Rightarrow \psi \qquad \varphi \Rightarrow \varphi}{\psi, \psi \supset \varphi \Rightarrow \varphi}\,\supset\text{L}}{\psi, \Box(\psi \supset \varphi) \Rightarrow \varphi}\,\Box\text{L}}{\Box\psi, \Box(\psi \supset \varphi) \Rightarrow \varphi}\,\Box\text{L}}{\Box\psi, \Box(\psi \supset \varphi) \Rightarrow \Box\varphi}\,\Box\text{R}}{\Box(\psi \supset \varphi) \Rightarrow \Box\psi \supset \Box\varphi}\,\supset\text{R}}{\Rightarrow \Box(\psi \supset \varphi) \supset (\Box\psi \supset \Box\varphi)}\,\supset\text{R}$$

For the $\theta \in \Gamma$ case, we simply take the axiom instance $\theta \Rightarrow \theta$ and apply weakening to get $\Gamma \Rightarrow \theta$.

For the necessitation case, $\theta = \Box\varphi$, and we know that φ is a tautology, and thus is provable in the axiomatic system of classical propositional logic from an empty hypothesis set. We can apply the already-proven classical analogue of this theorem to get a derivation $\Rightarrow \varphi$ in classical sequent calculus. This result is also a valid derivation in the sequent calculus discussed here, and it meets the conditions for an application of **Nec** to derive the sequent $\Rightarrow \Box\varphi$, which can then be weakened to $\Gamma \Rightarrow \Box\varphi$ as needed.

For the modus ponens case, assume for induction that we have derivations $\Gamma \Rightarrow \psi \supset \varphi$ and $\Gamma \Rightarrow \psi$, where $\theta = \varphi$. We derive $\Gamma \Rightarrow \varphi$ as follows:

$$\cfrac{\cfrac{}{\Gamma \Rightarrow \psi \supset \varphi}\,\text{Ind hyp} \qquad \cfrac{\cfrac{}{\Gamma \Rightarrow \psi}\,\text{Ind hyp} \qquad \cfrac{}{\varphi, \Gamma \Rightarrow \varphi}\,\text{Axiomatic}}{\Gamma, \psi \supset \varphi \Rightarrow \varphi}\,\supset\text{L}}{\Gamma \Rightarrow \varphi}\,\textbf{Cut}$$

These cases describe all of the formulas that can possibly appear in proofs with hypotheses Γ. The sequent derivations given thus show that every proof of a formula

θ from hypotheses Γ corresponds to a derivable sequent $\Gamma \Rightarrow \theta$, as stated by the right-to-left direction of the theorem.

■

For JS0.5, we simply transform the rules that were added to the classical propositional sequent calculus from modal properties into justification properties, as follows:[4]

$$\frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, t : \varphi \Rightarrow \Delta} \text{:L}$$

$$\frac{\Gamma, t : (\psi \supset \varphi), s : \psi \Rightarrow \Delta, \varphi}{\Gamma, t : (\psi \supset \varphi), s : \psi \Rightarrow \Delta, (t \cdot s) : \varphi} \text{:R}$$

$$\frac{\Rightarrow \varphi}{\Rightarrow x : \varphi} \text{Nec}$$

Restriction: The sequent $\Rightarrow \varphi$ must have been derived without using the rules :L, :R, or Nec.

Proposition 2.1.5. *The cut rule is* not *eliminable from the sequent calculi for S0.5 and JS0.5 as presented above.*

Proof. Ciabattoni and Terui [28] show that for a certain general class of sequent calculi, a pair of conditions that they call reductivity and weak substitutivity are jointly necessary and sufficient for a property that they call modular cut eliminability. The sequent calculi for S0.5 and JS0.5 satisfy the conditions needed for the Ciabattoni and Terui result to hold, but the □R and :R rules generate counterexamples to weak

[4] Note the use of schematic letters t and s in :L and :R versus the use of a free variable x in Nec. This represents the fact that the rules :L and :R are properties which hold true for any proof polynomials that may be substituted for the schematic letters, whereas Nec applies to a specific proof polynomial, whose identity simply isn't determined by the proof theory. The motivating interpretation of this rule is analogous to that of presentations of justification logic proof theories that omit constant specifications; for ordinary logical purposes, it suffices to know that there is some instance which justifies the conclusion.

substitutivity. This does not quite suffice to prove the desired result, as modular cut eliminability is strictly stronger than cut eliminability proper. However, Ciabattoni and Terui do prove a theorem that exhaustively characterizes the cases in which weak substitutivity or reductivity fails to hold but cut is nonetheless eliminable, and the counterexamples generated by □R and :R do not appear to meet the conditions of that theorem. ■

In other contexts, this cut ineliminability result would be devastating. However, in the present setting, it is unimportant. As in Chapter 1, we are only using these sequent calculi to establish realization, the proof of which does not use cut elimination. Interestingly, if cut elimination did hold, the sorts of counterexamples to the subformula property that were available in Chapter 1 would not hold of JS0.5. In this light, we might think of a cut-elimination result for JS0.5 as being literally "too good to be true." If cut elimination did hold for systems presented in the manner of S0.5 and JS0.5, it would be advantageous to develop sequent systems of that form for all of the justification logics, rather than relying on the Artemov-style systems that were presented in Chapter 1. But, as that turns out to not be the case, let us move on.

Theorem 2.1.6 (Approximate Soundness and Completeness for the JS0.5 Sequent Calculus)**.** *A sequent $\Gamma \Rightarrow \Delta$ is derivable in the above sequent calculus iff there is a proof in JS0.5 of one of the members of Δ (or a substitution instance of a proof variable in Δ) from hypotheses Γ, with the same constant specification $\mathcal{C}$ being employed in both the axiomatic and sequent systems.*

Because of the close structural analogy that has been maintained within both the axiomatic and sequent formulations of S0.5 and JS0.5, the proof of this theorem is directly analogous to the above proof of the corresponding theorem for the S0.5 sequent and axiomatic systems—the one noteworthy change is that instead of appealing to the axiomatic system's necessitation rule to validate the sequent rule Nec, one must apply the S0.5-like justification corollary of the classical lifting lemma.

Theorem 2.1.7 (Realization Theorem for JS0.5)**.** *If $\Gamma \vdash_{S0.5} \theta$, then for any axiomatically appropriate constant specification $\mathcal{C}$, there is some JS0.5-sentence φ such that $\varphi^\circ = \theta$ and $\Gamma^\circ \vdash_{JS0.5,\mathcal{C}} \varphi$.*

Proof. This theorem is proven by taking an arbitrary S0.5 sequent calculus derivation of $\Rightarrow \theta$, and transforming it into a JS0.5 sequent calculus derivation of a suitable $\Rightarrow \varphi$. At each step in the initial derivation where a subformula $\Box\psi$ is introduced through an axiomatic sequent, a weakening step, or a $\Box$L step, simply replace $\Box\psi$ with some arbitrary $t : \psi$ (an unused proof constant or free variable would be a good choice); this will, of course, transform $\Box$L steps into valid :L steps. At each step in the derivation where a subformula ψ is introduced by a $\Box$R step, replace the left-side formulae with justification functions as above, and then build a proof polynomial to use on the right ride using the $\cdot$ operator, as specified in the :R rule. Finally, at steps where a subformula $\Box\psi$ is introduced using Nec, replace it with a suitable free variable $x : \psi$. All other proof rules are identical in both systems, so the result will be a valid sequent derivation of $\Rightarrow \varphi$ for a formula φ satisfying the condition $\varphi^\circ = \theta$, which in turn proves $\vdash_{JS0.5} \varphi$, as per the preceding theorem. ■

2.1.3 Model Theory for JS0.5

Although there is a complete structural analogy between S0.5 and JS0.5 in both types of proof-theoretic presentations that we have examined, this analogy disappears when we turn to model theory. The model theory for S0.5, as developed by M. J. Cresswell [30], is equivalent to a non-normal Kripke frame structure with S5-like universal access between worlds. This is, of course, the simplest genuinely non-normal model theory, which is a major factor motivating the study of S0.5.

JS0.5 can also be modeled in a simple fashion, but its model theory bears little resemblance to the S0.5 models.

Definition 2.1.8. A Mkrtychev model of JS0.5 is a pair $\langle E, v\rangle$ such that E is a relation between the set of proof polynomials and the set of JS0.5 sentences, v is a function from the set of JS0.5 sentences to $\{0, 1\}$, and the following conditions are satisfied:

- For every PC axiom instance φ, there is at least one proof constant c such that $E(c, \varphi)$.
- For all proof polynomials s, t and sentences φ, ψ, if $E(s, \varphi \supset \psi)$ and $E(t, \varphi)$, then $E(s \cdot t, \psi)$.
- For all proof polynomials s, t and sentences φ, if $E(s, \varphi)$ or $E(t, \varphi)$, then $E(s + t, \varphi)$.
- For all formulae φ, if $v(\varphi) = 0$, then for every proof polynomial t, it is not the case that $E(t, \varphi)$.
- For all formulae φ, $v(\neg\varphi) = (1 - v(\varphi))$.
- For all formulae φ and ψ, $v(\psi \supset \varphi) = 0$ iff $v(\psi) = 1$ and $v(\varphi) = 0$.
- For all formulae φ and proof polynomials t, $v(t : \varphi) = 1$ iff $E(t, \varphi)$.

Because JS0.5 uses plain constant specifications, we get the simple implementation of meeting a constant specification as $\mathcal{C} \subseteq E$, and can define a consequence relation $\vDash_{JS0.5,\mathcal{C}}$ over the class of models that meet $\mathcal{C}$.

Theorem 2.1.9 (Soundness and Completeness)**.** *For all formulae* Γ, φ*,* $\Gamma \vdash_{JS0.5,\mathcal{C}} \varphi$ *iff* $\Gamma \vDash_{JS0.5,\mathcal{C}} \varphi$*.*

Proof. The forward direction (soundness) is proven directly from the formal definition of axiomatic provability in the usual manner. The soundness of the classical axioms and modus ponens with respect to the model-theoretic clauses describing $\neg$ and $\supset$ is well-known. The requirement in the model that there be proof constants in E for every PC axiom validates the classical axiom necessitation rule. The clauses restricting values of $E(s \cdot t)$ and $E(s + t)$ validate the application and sum axioms respectively. Reflection is validated by the clause requiring that if $v(\varphi) = 0$, then for every proof polynomial t, $\varphi \notin E(t)$. Thus, all of the axioms and rules of JS0.5 are sound with respect to the Mkrtychev model theory.

For the reverse direction (completeness) we proceed according to the standard Lindenbaum-Henkin technique. We start by defining a maximal consistent set of sentences M as a set such that $M \nvdash_{JS0.5} \psi \wedge \neg\psi$ but for every $\varphi \notin M$, $M, \varphi \vdash_{JS0.5} \psi \wedge \neg\psi$. Every proof-theoretically consistent set of sentences can be extended to a maximal consistent set by the algorithm of running through an enumeration of the sentences and at each step adding the next sentence to the set if doing so would not create inconsistency (Lindenbaum's Lemma). We also note that maximal consistent sets are closed under modus ponens and have the property that for every maximal consistent set M and sentence φ, either $\varphi \in M$ or $\neg\varphi \in M$.

Next, we show that for every maximal consistent set M, there is a Mkrtychev model such that $v(\varphi) = 1$ iff $\varphi \in M$. We start constructing a suitable model by letting $E(t, \psi)$ hold of all and only those proof terms t and sentences ψ such that $t : \psi \in M$. It is simple to verify that the combination of this E and the valuation v given by M is a Mkrytchev model. We know that for each PC axiom φ there is a constant c such that $E(c, \varphi)$ because if not, M would contain sentences $\neg c_i : \varphi$ for every proof constant c_i, which would make M an inconsistent set as per the classical axiom necessitation rule. Likewise, M contains every JS0.5 axiom instance (for if not, it would contain the negation and be inconsistent). This combined with the closure of M under modus ponens forces the satisfaction of the clauses governing $E(s \cdot t)$ and $E(s + t)$. Finally, the properties of M clearly suffice for all of the remaining restrictions on v in the model definition.

With this result in hand, we can prove completeness by reductio. Assume that $\Gamma \vDash \varphi$ but $\Gamma \nvdash_{JS0.5} \varphi$. Then we have it that the set of sentences $\{\Gamma, \neg\varphi\}$ is proof-theoretically consistent, and so it can be extended to a maximal consistent set M. But then by the lemma proven in the last paragraph, we have a model satisfying Γ such that $v(\neg\varphi) = 1$, and thus $v(\varphi) = 0$. But this contradicts the assumption that $\Gamma \vDash \varphi$, so completeness must hold. ■

As we did with the normal justification logics of Chapter 1, we can embed the evidence structure of the Mkrtychev models into a Kripke-style frame semantics to create Fitting models.

Definition 2.1.10. A Fitting model for JS0.5 is a structure $\langle W, N, R, \mathcal{E}, v \rangle$ such that W and N are non-empty sets such that $N \subseteq W$, R is a reflexive relation on W, $\mathcal{E}$ is a binary function from W and the set of proof polynomials to the power set of the set of JS0.5 sentences, v is a binary function from W and the set of JS0.5 sentences to $\{0,1\}$, and the following conditions are satisfied:

- For every PC axiom instance φ, there is at least one proof constant c such that $\varphi \in \mathcal{E}(w,c)$ for all $w \in W$.
- For all $w \in W$, formulae φ and ψ and proof polynomials s and t, if $(\psi \supset \varphi) \in \mathcal{E}(w,s)$ and $\psi \in \mathcal{E}(w,t)$, then $\varphi \in \mathcal{E}(w, s \cdot t)$.
- For all $w \in W$ and proof polynomials s and t, $(\mathcal{E}(w,s) \cup \mathcal{E}(w,t)) \subseteq \mathcal{E}(w, s+t)$.
- For all formulae φ, $v(w, \neg\varphi) = (1 - v(w,\varphi))$.
- For all formulae φ and ψ, $v(w, \psi \supset \varphi) = 0$ iff $v(w,\psi) = 1$ and $v(w,\varphi) = 0$.
- For all formulae φ and proof polynomials t, $v(w, t : \varphi) = 0$ iff either $\varphi \notin \mathcal{E}(w,t)$ or there exists a point $w' \in W$ such that $R(w,w')$ and $v(w',\varphi) = 0$.

As in Chapter 1, we define the notion of meeting a constant specification such that a Fitting model $\langle W, N, R, \mathcal{E}, V \rangle$ is said to meet a constant specification $\mathcal{C}$ if, for every $\langle c, \varphi \rangle \in \mathcal{C}$ and $w \in N$, $\varphi \in \mathcal{E}(w,c)$, and then define a consequence relation $\vDash_{JS0.5,\mathcal{C}}$ over the class of Fitting models that meet $\mathcal{C}$ such that $\Gamma \vDash_{L,\mathcal{C}} \varphi$ iff, for every Fitting model $\langle W, N, R, \mathcal{E}, V \rangle$ which meets $\mathcal{C}$ and for which $V_w(\gamma) = 1$ for all $\gamma \in \Gamma$ and $w \in N$, it is also the case that $V_w(\varphi) = 1$ for all $w \in N$.

Why is the Fitting model theory interesting, when Mkrtychev models represent justifications such as JS0.5 more perspicuously? Fitting gives several arguments in the paper where he presents this type of model theory [4], but the most important factor in this case is one which doesn't apply in the context of his paper. You may have noted that I did not specify as many restrictions on the frame structure in the Fitting model as one might expect given the usual model theory for S0.5. In fact, no such restrictions are required: the frame structure in question need not correspond to the modal system on which the logic is based. It must correspond to an extension of that base logic,[5] but in the case of S0.5, nearly all of the commonly studied modal logics meet that criterion. For the same reason, though I did allow for non-normal worlds above, we can still get a sound and complete set of models of JS0.5 if we impose a restriction that $W = N$.

This property of the Fitting models can be employed to do useful work in epistemology. Suppose, for example, that one holds to the epistemic principle that in order to be known, something must be necessarily true—a typical example of an extreme skeptical position. This can be captured by letting the frame structure corresponding to one's position on alethic necessity (which is usually taken to be S5, though some philosophers use S4 instead), and then adding in addition to the justification formulae, a $\Box$ operator defined over the same frame structure. Reading justification functions as knowledge and $\Box$ as necessity, we have it that whatever is known must be necessary, but that the converse doesn't hold, because the minimum standard for knowledge is given by the weaker base system (in this case, S0.5, though the same interpretation is also available for the Fitting models of the normal justification logics of Chapter 1). This result seems to be an accurate representation of the epistemological view of the skeptic who asserts the principle in question.

[5]This holds because having an extension of the base logic ensures that the $\Box$-like property will hold for all of the desired formulae (and perhaps more), and then the $\mathcal{E}$-function eliminates the excess formulae.

2.1.4 Epistemic Application of JS0.5

Although our motivations for investigating non-normal justification logics were primarily internalist, JS0.5 is actually most useful as a model for a strongly externalist epistemology. JS0.5 differs from JT, which was our preferred logic for externalist knowledge in Chapter 1, only by virtue of having a weaker axiom justification rule. Epistemically, this can be interpreted as a more thorough rejection of the epistemic access principle. In epistemically interpreted JT, although the KK principle is not generally valid, it does hold in the case of logical theorems, due to the presence of iterated axiom justification. The axiom justification rule of JS0.5 does not include iteration, so this logic allows for the rejection of those KK instances. Moreover, the axiom justification rule of JT applies also to the justification axioms; in epistemic interpretation, this amounts to a claim that although an agent does not necessarily know that she knows any particular proposition, she does know that her knowledge conforms to certain logical principles. Because its justification rule applies only to propositional axioms, JS0.5 does not entail that an agent possess any non-tautological knowledge of her own epistemic status.

2.2 Justification Analogues for Other Non-Normal Modal Logics

Because JS0.5 does not satisfy the desire for a non-normal system that adequately models the features of those internalist epistemologies that cannot be modeled by normal justification logics, we must look for justification analogues of other non-normal modal logics. Treated as a formal logic exercise, developing justification analogues of various modal logics is simple. Given an arbitrary modal axiom scheme, choose an arbitrary function on the proof polynomials of appropriate arity, and let the proof polynomial used in the consequent of the axiom be determined by applying that function to the proof polynomials used in the antecedent (as in the application axioms of JS0.5). Modal rules can either be handled with functions, or by simply allowing the introduction of suitable proof polynomials under the required conditions (as in the classical axiom necessitation rule). At this level, the details of justification logics for particular modal systems can be appropriately assigned as exercises in a graduate-level logic course.

Unfortunately, there are some complications, both formal and philosophical. On the formal side, the biggest problem is that there is no suitable translation for the $\Diamond$ operator. One might be tempted to try rewriting $\Diamond\varphi$ as $\neg\Box\neg\varphi$ and then translating this as $\neg t : \neg\varphi$. An easy way to see why this fails is to attempt to read the various formulae epistemically. $\Diamond\varphi$ would be read as something like "For all we know, it could be the case that φ," and $\neg\Box\neg\varphi$ would be read as "We don't know that it isn't the case that φ." These are indeed equivalent. But $\neg t : \neg\varphi$ would become "Evidence t does not provide sufficient reason to disbelieve φ," which is not at all the same thing—we might possess some other evidence that does refute φ. (There are, of course, contexts in which the proposition that we can express is useful, and justification analogues of systems containing $\Diamond$ can fill that purpose.) To properly translate $\Diamond$ in its more general use, it would be necessary to employ quantification over proof polynomials. However, doing that would endanger the philosophical advantages that we receive from the constructive nature of justification formulae. Moreover, there seems to be a great potential for paradox if we additionally attempt to replace the propositional atoms with first-order predication.

Another philosophical problem is that when we introduce a function to deal with a given axiom, this function does not convey anything useful in a philosophical application (such as epistemology) unless we can interpret it. For the syntax of JS0.5, we have such interpretations readily available: proof polynomials correspond to pieces of evidence, the $+$ operator to arbitrary combinations of evidence, and the $\cdot$ operator to the process of combining evidence in the structure of a deductive argument. However, functions introduced to represent arbitrary axioms may not have any such intuitive meaning; in such cases, it's not clear how we could make any sense of the resulting formal system.

It should be noted that these problems do not spell doom for the entire program of non-normal justification logic. There are many interesting systems that can be axiomatized without any problematic instances of $\Diamond$—this can be done, for example, for all three of the original C. I. Lewis non-normal modal systems—and it seems likely that suitable interpretations can be found for at least some of these logics.

Indeed, there is one obvious candidate for the desired internalist epistemic logic that is subject to neither of the aforementioned problems. This is the modal system E4, set forth by Lemmon [65]. E4 consists of the same axioms as S4 (and so the axioms are all readily interpretable) and remains closed under modus ponens. However, instead of a necessitation rule, it features the rule that from a deduction $\vdash \varphi \supset \psi$, we derive the deduction $\vdash \Box\varphi \supset \Box\psi$. In this case, there does seem to be a plausible interpretation of the new rule as a feature of a justification system: if we have just proven that φ materially implies ψ, then all we need to add to a proof of φ to create a proof of ψ is an encoding of what we have just done. However, I have yet to discover any adequate way to formalize this intuitive interpretation of the modal rule as a justification operator. Neither does the modal system E4 have any extant model theory or sequent calculus; there doesn't seem to be any fundamental reason why these systems cannot be developed, but there is no time to solve those problems and the formalization problem all within the scope of the present project, so such investigations must be postponed.

Chapter 3

Paraconsistent Justification Logic

3.1 Motivation

Throughout this dissertation, I have been constructing justification logics with the intent that they should be used as formal models for various philosophical theories of knowledge, specifically those in which knowledge is analyzed as some variant of "justified true belief." Given this analysis, every account of knowledge must bear an appropriate relationship to some theory of truth, and ideally we should analyze this relationship. However, I have been persuaded that it would not be fruitful to perform such analysis here for two reasons. First, the question of the proper relationship between epistemology and the metaphysics of truth would be a dissertation topic in itself. It is not feasible to summarize the relevant material in a few paragraphs without resorting to extremely vague generalization, which would be an injustice to the reader. Second, *this* dissertation is not really a work of epistemology, but rather of logic. For the present purpose, it suffices to consider only the logic of truth, setting aside metaphysics entirely.

From a purely logical perspective, truth is standardly characterized by the schema $T\ulcorner\varphi\urcorner \leftrightarrow \varphi$, proposed by Tarski [97].[1] Using this schema in an unrestricted form quickly leads to the problematic case of the sentence λ defined by $\lambda = \neg T\ulcorner\lambda\urcorner$ (better known as the liar sentence), from which we can deduce the contradiction $\lambda \leftrightarrow \neg\lambda$. One might be tempted to conclude from this that either the λ construction or the truth schema is incorrect, but that is too hasty. The λ construction can be performed within any natural language; for example, let $\lambda =$ "This sentence is not true." As for the correctness of Tarski's truth schema, that is a question to be settled by the theoretical inquiry into the nature of truth, which we cannot go into here. In the absence of argumentation to the contrary, we must treat the liar as a genuine paradox. Given that knowledge is defined in terms of truth, our logic of knowledge must therefore accommodate this and other related paradoxes of truth.

One possible response to this problem (other than promulgating a non-Tarskian analysis of the truth predicate) is to note that knowledge is usually modeled as a sentential operator, not a predicate. While it may be obvious that the "justified true belief" account of knowledge requires our epistemic logic to contain a model of truth, there's no reason why it must contain a truth *predicate* such as is characterized by the Tarski schema. All that is clearly required is a truth *operator*, which does not lead to any paradox at all—it is merely an identity operator on the set of sentences. This requirement being absolutely trivial, the "true" part of the justified true belief definition can be understood as merely requiring that any epistemic logic validate the

[1]Note that we make no commitment as to whether Tarski's schema is to be treated as the definition of truth, or as merely a logical consequence of some other truth concept. It should also be noted that Tarski himself did not consider this schema to be the definition of truth, but rather as a condition that must be entailed by any adequate definition of truth.

T axiom (or its justification analogue $\mathrm{T^j}$).

Given this response, I think it must be conceded that the word “true” as used in the “justified true belief” definition of knowledge can be treated as a non-paradoxical operator truth. That conclusion is inevitable for anyone who is inclined to treat knowledge itself as an operator, as we do throughout this dissertation. However, the usage of truth in natural language in general cannot be treated this way. The reason is that many of the contexts in which truth is employed involve quantification into the scope of the truth predicate,[2] as in the sentence “Everything that is written in the Bible is true,” which can only be formalized as $\forall x(Bx \to Tx)$. If truth were an operator, this example sentence would be ungrammatical, as it would require the quantifier to bind a variable that ranges over propositions rather than objects, which are the only allowable range of interpretation for first-order variables. The only plausible way to fix this problem under an operator interpretation of truth would be to allow quantification over sentences. That would resolve the expressibility problem, but unfortunately it would also destroy operator truth’s immunity to paradox. Whichever way we interpret things, our language must contain a paradox-vulnerable truth notion. Sentences containing that notion would be just as good candidates to appear within the scope of the knowledge operator as any other sentences, so the account of knowledge would still need to be able to cope with paradox.

One may, perhaps, wish to continue ignoring the question of truth entirely, as we have done in previous chapters of this dissertation. Such a move is acceptable; however, there are other sources of paradox that must be considered. In particular, there are paradoxes that are generated within epistemology. The most famous of

[2] Beall argues in [15] and elsewhere that such contexts are fundamental in the sense that if not for them, we could get by perfectly well without including truth in our language at all.

these is the skeptical paradox. The paradox begins by considering any of the class of skeptical propositions, which include such examples as Descartes' "I am being constantly deceived by an evil demon;" "I am a character in a dream (or a computer simulation, etc.);" and "I am a disembodied human brain, with electrical wires attached to certain neurons to provide artificial sensory perceptions." Under an evidentialist account of justification, we are not justified in believing the negations of any of these propositions, as there is no possible evidence that we could obtain that would be incompatible with their truth. For concreteness, let us use the disembodied brain example sentence in the following discussion, and denote it by v. Based on the evidentialist considerations above, we should hold that $\neg K \neg v$.

To get a paradox from v, we need two other premises. First is the principle that knowledge is closed under (known) logical consequence.[3] This seems to be an obvious truth; surely, a sound deductive argument is an acceptable means of gaining epistemic justification and thus knowledge. If closure did not hold, it would be especially problematic for mathematical knowledge, which is only obtained by deductive argumentation in the form of mathematical proof.[4] Second, we need the premise that we do possess some knowledge of the world. In particular, given that I am not an amputee, I have overwhelming evidence for the proposition h: I have hands. I have visual evidence that there are hands attached to my body. I also receive certain sensory impressions whenever my hands come into contact with other physical objects. Finally, I have reflective access to my ability to causally influence my hands in such

[3]If closure is applied unrestrictedly, this leads to a very blatant form of the problem known as logical omniscience, which is discussed in Chapters 2 and 5. However, the restriction that the entailment be known to the agent in question does not interfere with the derivation of the skeptical paradox, so in this chapter, let us ignore it.

[4]This claim is a bit too strong; obviously, many people obtain mathematical knowledge based on testimonial evidence from their textbooks and math teachers. But all such knowledge is ultimately grounded in deductive proof.

a way as to accomplish various actions, such as typing out this very sentence. All of these evidences firmly establish that Kh.

Now, the paradox. Proposition h entails $\neg v$, as disembodied brains do not possess hands. By closure, we can deduce $K\neg v$ from Kh. But we already established from fundamental principles that $\neg K\neg v$. Thus, we have a clear contradiction.

Another case of interest is the epistemic liar, "I know that this sentence is false." Given the factivity of knowledge, if we assume the sentence to be true, we have a contradiction as in the standard liar λ. It is logically possible for the sentence to be false, as I could lack knowledge of its truth value. Since it cannot be true and can be false, the sentence must be false. However, if we take the initial assertion to have been made by *me*, then there is a problem. I have just proven the sentence to be false, and logical proofs are a source of epistemic justification. Therefore, I do know that the sentence is false, which makes it a paradox.

These paradoxes can be resolved within epistemology. One option is to reject closure, as is famously advocated by Dretske [35] and Nozick [77]. Another possibility is that, instead of rejecting closure generally, one can reject the paradoxical arguments by asserting that there is a sufficient shift in the semantic value of the word "know" between premises that the conclusion is not a true contradiction, but merely an equivocation. An example of a response along these lines is the epistemic contextualism of DeRose [33].

However, it seems that there is enough similarity among the standard liar paradox, the epistemic liar, and the skeptical paradox that one should seek a uniform solution for all three. Such a uniform solution would have to be located at some level more fundamental than the definitions of truth and of knowledge, given that the standard liar and the skeptical paradox each involve only one of those definitions. The only place to look for a solution is in the underlying logic, and the most obvious candidate is to adopt a paraconsistent logic. Paraconsistent logics have the property that explicit contradictions do not entail anything unacceptable (such as the truth of every sentence of the language, as in classical logic). Therefore, by using a paraconsistent justification logic as our epistemic logic, we can simply accept that the contradictory conclusions of all three types of paradox are correct.

Although the ability to present a unified account of paradox is perhaps the strongest motivation for paraconsistent epistemic logic, it is by no means essential. If you accept a paraconsistent approach to semantic paradoxes such as the liar, but reject such a treatment of the skeptical paradox, you will probably need a paraconsistent epistemic logic in order to deal with the epistemic liar, as it is fundamentally more similar to the standard liar than to the skeptical paradox. In principle, one could also hold the reverse position—that the epistemic paradoxes are to be resolved paraconsistently, but the semantic paradoxes are to be solved in some other manner—but this is much less plausible. Furthermore, there are other applications of paraconsistent logics besides the types of semantic and epistemic paradoxes presented above,[5] and it may be needful to do epistemology in a setting where such paraconsistent phenomena may occur within the scope of the knowledge operator. Any of these scenarios will suffice as motivation for the following logical developments.

[5]For example, Weber [101] uses a paraconsistent logic to address the phenomenon of vagueness.

3.2 The Paraconsistent System LP

The simplest paraconsistent logic[6] is the system LP, developed independently by Asenjo [9] and Priest [82]. A modal extension of LP is presented by Priest in [83]. To avoid redundancy in material, I shall formally present that system rather than the propositional base logic, which can be recovered by merely taking the clauses for the truth-functional connectives. (I will also be using a different formulation of the model theory than in the latter Priest citation, but these are well-known equivalents.)[7]

Definition 3.2.1. Models of the modal system LPK consist of a non-empty set of worlds $\mathcal{W}$, a relation R on $\mathcal{W}$, and a valuation function v from pairs of worlds and sentences to the set of truth values $\{0, \frac{1}{2}, 1\}$ which satisfies the following principles:

- $v_w(\varphi \vee \psi) = \max(v_w(\varphi), v_w(\psi))$.
- $v_w(\varphi \wedge \psi) = \min(v_w(\varphi), v_w(\psi))$.
- $v_w(\neg\varphi) = \begin{cases} 0, \text{if } v_w(\varphi) = 1; \\ \frac{1}{2}, \text{if } v_w(\varphi) = \frac{1}{2}; \\ 1, \text{if } v_w(\varphi) = 0. \end{cases}$

[6] Here, I mean "simplest" in the sense of requiring the least added semantic machinery as compared to classical logic; LP only adds one new truth value, and no new logical vocabulary. As Jc Beall pointed out to me, there are other natural notions of simplicity that would favor other candidate logics instead of LP.

[7] Priest uses the relational semantics for LP, whereas I am using the three-valued semantics. As far as I can tell, the equivalence of these semantics was initially proven for the four-valued logic FDE, of which LP is a sublogic. The four-valued semantics for FDE comes from Dunn [36], whereas the relational semantics is from Dunn [37]. It should also be noted that Asenjo [9] uses the same three-valued semantics for LP that I present below, though of course in propositional form rather than the modal extension.

- $$v_w(\Box\varphi) = \begin{cases} 0, \text{if } \exists w' \in \mathcal{W} \text{ such that } R(w,w') \text{ and } v_{w'}(\varphi) = 0; \\ 1, \text{if } \forall w' \in \mathcal{W} \text{ such that } R(w,w'),\ v_{w'}(\varphi) = 1; \\ \frac{1}{2}, \text{otherwise.} \end{cases}$$

Definition 3.2.2. An LPK model $\mathcal{M} = \langle \mathcal{W}, R, v \rangle$ satisfies a proposition φ (written $\mathcal{M} \vDash \varphi$) iff $\forall w \in \mathcal{W}$, $v_w(\varphi) \neq 0$. A proposition φ is valid (written $\models \varphi$) iff it is satisfied by all LPK models.

We can, of course, extend LPK to any of the other normal modal logics by imposing the familiar restrictions on R: reflexivity to get LPT, reflexivity and transitivity for LPS4, etc.

Take careful note of the semantic clause for the $\Box$ operator. The clause presented here seems to be the most natural reading of modality for a person who adopts paraconsistent logic on account of the motivations presented in the previous section of this chapter. However, there is an important alternative semantics for modality, which replaces that clause with the following:

$$v_w(\Box\varphi) = \begin{cases} 1, \text{if } \forall w' \in \mathcal{W} \text{ such that } R(w,w'),\ v_{w'}(\varphi) \neq 0; \\ 0, \text{otherwise.} \end{cases}$$

This version of modality will be satisfied by every model that satisfies the modal clause of the original definition. The difference is that modal propositions are limited to values 0 and 1, which rules out the possibility of modal dialetheia. There are technical advantages to imposing such a limitation, and it may be perspicuous for some applications of paraconsistent logic, but is less so for the present application. If we want to say that the liar sentence λ is both true and false, then we would probably wish to say the same for $\Box\lambda$.[8]

Of course, the real focus of this dissertation is justification logic, not modal logic. Nothing in the above-discussed motivation for paraconsistency affects the arguments made in Chapter 1 for using justification logic rather than modal logic to model knowledge. Our goal is thus to develop a justification system corresponding to the modal system that we have just presented. One might expect that we could continue to apply the methodology of the previous chapters, and create an analogous modal system by replacing the $\Box$ operator with the $t : \varphi$ syntax. Unfortunately, that procedure only works on axiomatic modal logics; recall that all of the model theories and sequent calculi given for justification logics in previous chapters contained ineliminable references to the axiomatic system, usually in the form of a clause affirming $c : \varphi$ iff φ is an axiom of the system. In the cases of LP and LPK (and extensions thereof), however, the only possible axiomatic system[9] is the trivial axiomatization—the axiomatic system in which all valid sentences of the logic are axioms, and there are no inference rules. A justification logic based on an axiomatic system of this sort would admit as proof polynomials only constants and perhaps the trivial sum

[8]On our target epistemic reading, $\Box\lambda$ is interpreted as "[Some agent] knows that the standard liar, λ, is true," and as noted in the above discussion of closure, we assume that the agent in question possesses all relevant logical competencies. In particular, the agent knows that λ is both true and false, and thus is true—but the agent equally knows that λ is not true, and so $\frac{1}{2}$ is the proper semantic value for the knowledge ascription. It might be objected that there are many logically competent agents who do not subscribe to a paraconsistent theory of truth. Such an agent would not believe (and thus would not know) that λ is both true and false, and so might not know or believe that λ is true. Although this objection is reasonable, it seems best for now to set such cases aside. Assuming that the paraconsistent approach is actually the "correct" account of the logic of truth, the analysis of the epistemic state of such an agent will probably be comparable to that of an agent who lacks the logical competence to deal with λ at all.

[9]At least under the standard understanding of an axiomatic system, as the closure of a class of axiom schemas under a smaller class of rules. The natural deduction formulation of LP that we will examine in Chapter 4 could be viewed as an axiomatic system with a single axiom schema but many rules, but as we will see, that interpretation will require significant changes to the formulation of the justification logic.

operation; there would not be enough expressive resources to provide any genuine advantage over modal LPK.

The ultimate source of LP's unaxiomatizability is its lack of a genuine conditional.[10] A material conditional can be defined in the usual manner, but modus ponens is invalid for such a conditional. In the next chapter of this dissertation, we shall develop an LP-based justification logic in a manner that does not depend on the axiomatic system. For the remainder of this chapter, we will instead work with paraconsistent logics that can be axiomatized.

3.3 Simple Axiomatizable Paraconsistent Systems

The simplest way to produce a non-trivially axiomatizable paraconsistent system is to extend LP or LPK with a truth-functional conditional for which modus ponens is valid. There are actually two candidate conditionals, which produce distinct but expressively equivalent logics. The system $LP^{\rightarrow}$ is produced by adding the semantic

[10]Throughout this dissertation, I shall use the term "axiomatizable" to refer to the possibility of constructing a Hilbert-style axiomatic system (with *modus ponens* as the primary inference rule) for the logic. This usage is admittedly non-standard, but I cannot find any other apt term for the concept. The usual definition of an axiomatizable theory is that the theory has a [finite] subset which entails the whole under some consequence relation. LP and LPK both clearly satisfy this version of axiomatizability, as we can simply take the empty set under the model-theoretic LP or LPK consequence relation. Moreover, LP and LPK are both decidable.

clause:[11,12]

$$v_w(\varphi \to_{\mathrm{LP}^{\to}} \psi) = \begin{cases} 1, \text{if } v_w(\varphi) = 0; \\ v_w(\psi), \text{otherwise.} \end{cases}$$

The system RM_3 is produced by adding the following instead:

$$v_w(\varphi \to_{\mathrm{RM}_3} \psi) = \begin{cases} \frac{1}{2}, \text{if } v_w(\varphi) = v_w(\psi) = \frac{1}{2}; \\ 0, \text{if } v_w(\varphi) > v_w(\psi); \\ 1, \text{otherwise.} \end{cases}$$

These systems are expressively equivalent in that each of them can define the conditional of the other. The $\mathrm{LP}^{\to}$ conditional can be defined in RM_3 by $(\varphi \to_{\mathrm{RM}_3} \psi) \vee \psi$, and the RM_3 conditional can be defined in $\mathrm{LP}^{\to}$ by $(\varphi \to_{\mathrm{LP}^{\to}} \psi) \wedge (\neg\psi \to_{\mathrm{LP}^{\to}} \neg\varphi)$. By contrast, neither of these conditionals can be defined within plain LP.

Convention 3.3.1. The unsubscripted $\to$ symbol shall be taken to refer to whichever of $\to_{\mathrm{LP}^{\to}}$ and $\to_{\mathrm{RM}_3}$ is a semantic primitive in the language in which the formula containing $\to$ is written, i.e., $\to_{\mathrm{LP}^{\to}}$ in $\mathrm{LP}^{\to}$ extensions and $\to_{\mathrm{RM}_3}$ in RM_3 extensions. Subscripts will only be used for those $\to$ connectives that are defined from the primitive $\to$ as noted above.[13]

[11] The names $\mathrm{LP}^{\to}$ and RM_3 are the most commonly used names for these logics in recent literature. $\mathrm{LP}^{\to}$ has been independently discovered at least three times: by Asenjo and Tamburino [10], by Priest [82], and by Avron [11]. RM_3 (strictly speaking, its $\to\neg$ fragment) was first put forth by Sobociński [90]; the name RM_3 comes from a connection with the system RM of Anderson and Belnap [2], which was observed by Parks [79].

[12] I have left the world-references in the semantic clauses because our ultimate interest is in modal and justification extensions of these logics. These should properly be omitted from the semantics for the propositional base languages $\mathrm{LP}^{\to}$ and RM_3.

[13] Such defined conditionals will not actually be used at all in any of the following material, but they are nonetheless genuine sentences of their respective languages, and so the notation should be preserved.

The modal extensions of $LP^{\to}$ and RM_3 can be constructed in the same manner as LPK above. Since these logics are axiomatizable, we can also produce justification extensions. This work has been anticipated by Che-Ping Su in [94, 95], where he presents a system that he calls PJ_b.

Definition 3.3.2. A Fitting model of the logic PJ_b consists of a non-empty set of worlds $\mathcal{W}$, a relation R on $\mathcal{W}$, an evidence function $\mathcal{E}$, which maps pairs of worlds and proof polynomials to sets of sentences, and a valuation function v from pairs of worlds and sentences to the set of truth values $\{0, \frac{1}{2}, 1\}$, all such that the following principles are satisfied:

- If φ is an axiom of the axiomatic presentation of PJ_b, then there is a sequence of proof constants c_i such that $\varphi \in \mathcal{E}(w, c_0)$, $c_0 : \varphi \in \mathcal{E}(w, c_1)$, $c_1 : c_0 : \varphi \in \mathcal{E}(w, c_2)$, and so forth.[14]
- If $(\varphi \to \psi) \in \mathcal{E}(w, s)$ and $\varphi \in \mathcal{E}(w, t)$, then $\psi \in \mathcal{E}(w, s \cdot t)$.
- $\mathcal{E}(w, s) \cup \mathcal{E}(w, t) \subseteq \mathcal{E}(w, s + t)$.
- $v_w(\varphi \vee \psi) = \max(v_w(\varphi), v_w(\psi))$.
- $v_w(\varphi \wedge \psi) = \min(v_w(\varphi), v_w(\psi))$.
- $v_w(\neg\varphi) = \begin{cases} 0, \text{if } v_w(\varphi) = 1; \\ \frac{1}{2}, \text{if } v_w(\varphi) = \frac{1}{2}; \\ 1, \text{if } v_w(\varphi) = 0. \end{cases}$

[14] As in Chapter 1, the infinite sequences of conditions in this clause should be simplified to just the c_0 case when using constant rather than multi-constant specifications, and likewise one should add a restricted ! clause if constant specifications are used in systems such as PJ_b proper that don't contain the unrestricted version of that clause.

- $v_w(\varphi \to \psi) = \begin{cases} 1, \text{if } v_w(\varphi) = 0; \\ v_w(\psi), \text{otherwise.} \end{cases}$

- $v_w(t{:}\varphi) = \begin{cases} 1, \text{if } \varphi \in \mathcal{E}(w,t) \text{ and for every } w' \text{ such that } R(w,w'),\ v_{w'}(\varphi) \neq 0; \\ 0, \text{otherwise.} \end{cases}$

We will get to the axiomatic presentation in a moment. First, note from the model structure that $\mathrm{PJ_b}$ is an extension of $\mathrm{LP}^{\to}$ with the justification analogue of the bivalent semantics for modality. In particular, it is the analogue of bivalent-modal LPK. We can extend it to the other normal modal logics by adding additional restrictions on $\mathcal{E}$ and R, as discussed in Chapter 1; let us refer to such extensions as $\mathrm{PJ_bT}$, $\mathrm{PJ_bT4}$, and so forth. We can also change the system to incorporate the more genuinely paraconsistent modality found in our initial presentation of LPK by replacing the semantic clause for the operation $t:\varphi$ by:

$$v_w(t:\varphi) = \begin{cases} 1, \text{if } \varphi \in \mathcal{E}(w,t) \text{ and for every } w' \text{ such that } R(w,w'),\ v_{w'}(\varphi) = 1; \\ 0, \text{if } \varphi \notin \mathcal{E}(w,t) \text{ or there exists } w' \text{ such that } R(w,w') \text{ and } v_{w'}(\varphi) = 0; \\ \frac{1}{2}, \text{otherwise.} \end{cases}$$

Let us call the system with this semantic clause $\mathrm{LP}^{\to}\mathrm{J}$, and its extensions $\mathrm{LP}^{\to}\mathrm{JT}$, $\mathrm{LP}^{\to}\mathrm{JT4}$, etc. We could also replace the $\mathrm{LP}^{\to}$ conditional with the $\mathrm{RM_3}$ conditional.[15] This leads to two different families of justification systems, corresponding to the two versions of the semantics for the justification (or modal) operator. I recommend reserving the $\mathrm{RM_3J}$ notation for the fully paraconsistent version, and inventing some

[15] For reasons that will be explained in Appendix A, the $\mathrm{RM_3}$-based justification logics also require an additional semantic clause specifying that if $\varphi \in \mathcal{E}(w,t)$ and $\psi \in \mathcal{E}(w,t)$ then $(\varphi \wedge \psi) \in \mathcal{E}(w,t)$.

other nomenclature for justification systems employing the RM_3 conditional along with the bivalent justification semantics used in PJ_b.

Exactly as in Chapter 1, we can use either constant or multi-constant specifications for any of these logics, though it's more convenient to use constant specifications for the 4- or 5-extensions and multi-constant specifications for the weaker systems. The definitions of meeting a constant [multi-]specification and of consequence are exactly as in Definition 1.5.7, except that the value $\frac{1}{2}$ is treated the same as the value 1 in the consequence definition.

Axiomatic systems for the propositional base systems $LP^{\rightarrow}$ and RM_3 are presented in [73], and have been reproduced in Appendix A for the reader's convenience. Intuitively, axiomatizations of their justification extensions, $LP^{\rightarrow}J$ and RM_3J should be obtainable simply by adding axiom K^j and the iterated axiom justification rule to the axiomatization of the propositional system,[16] and further extensions such as $LP^{\rightarrow}JT$ should be obtainable by additionally adding the requisite axioms for those systems.

This intuition is mostly sound. However, there are two complications, both involving negated justification operators. First, in the $LP^{\rightarrow}$ cases, contraposition is not valid for general conditionals, and so the contrapositives of the justification axioms must be explicitly specified in order to axiomatize the negative cases.[17] The second complication is that one might expect to also require a negative analogue of the [iterated] axiom justification rule. In fact this is not required for the particular modalities

[16]As noted in Appendix A, RM_3J also requires an additional justification logic axiom to deal with the conjunction rule.

[17]One might question why we want those contrapositions to hold, given the choice of a conditional that does not validate contraposition as a logical rule. The answer is that they are required for completeness with the model theory, as the fully paraconsistent treatment of modality makes it impossible to produce modal or justification logic models of the $LP^{\rightarrow}$ counterexamples to contraposition. A similar problem will arise in the development of an LP-based justification logic in Chapter 4.

that we are considering. In the cases of $\mathrm{LP}^{\rightarrow}\mathrm{J}$ and $\mathrm{RM}_3\mathrm{J}$, such a rule would be invalid on the grounds that the relation R may be empty. In such a case, the semantics of justification reduces to the question of whether a sentence is present in $\mathcal{E}(w, t)$, and there are no constraints preventing any sentence from being so included. For all of the extensions of these systems that we are considering here, the negative axiom justification rule is redundant, as it can be replaced in any proof by modus ponens on an instance of the contrapositive of axiom T^{j}. However, if we were to develop a paraconsistent justification analogue of a modal system lacking axiom T, we would need to include such a rule.

For PJ_{b}, the situation is complicated in a different fashion. The bivalent justification operator eliminates all of the negative justification theorems that proved important in the non-bivalent systems,[18] but it also enables us to get an explosive proposition $\bot$ by taking any dialethia of the form $t\!:\!\varphi \wedge \neg t\!:\!\varphi$. Given the existence of such $\bot$, we can define an external negation by $\sim\!\varphi = \varphi \rightarrow \bot$. In [95], Su proves the soundness and completeness of the PJ_{b} Fitting models with respect to the following lengthy axiomatic system that incorporates both negations:

Definition 3.3.3. The axiomatic system for PJ_{b} consists of the closure of the following axiom schemas under modus ponens and iterated axiom justification:

1. $\varphi \rightarrow (\psi \rightarrow \varphi)$

2. $(\theta \rightarrow (\varphi \rightarrow \psi)) \rightarrow ((\theta \rightarrow \varphi) \rightarrow (\theta \rightarrow \psi))$

3. $(\varphi \vee \psi) \rightarrow (\psi \vee \varphi)$

4. $(\varphi \vee \psi) \rightarrow ((\varphi \rightarrow \theta) \rightarrow (\theta \vee \psi))$

[18] For a counterexample, take a model that includes a world where all atomic propositions receive value $\frac{1}{2}$.

5. $\varphi \vee \neg\varphi$

6. $\varphi \vee {\sim}\varphi$

7. $\varphi \to ({\sim}\varphi \to \psi)$

8. $\neg(\varphi \to \psi) \leftrightarrow (\varphi \wedge \neg\psi)$

9. ${\sim}(\varphi \to \psi) \leftrightarrow (\varphi \wedge {\sim}\psi)$

10. $\varphi \to (\psi \to (\neg\psi \to \neg(\varphi \to \psi)))$

11. ${\sim}\varphi \to {\sim}\neg(\varphi \to \psi)$

12. ${\sim}\neg\psi \to {\sim}\neg(\varphi \to \psi)$

13. $\varphi \to (\psi \to (\varphi \wedge \psi))$

14. $(\varphi \wedge \psi) \to \varphi$

15. $(\varphi \wedge \psi) \to (\psi \wedge \varphi)$

16. $\neg\varphi \to \neg(\varphi \wedge \psi)$

17. ${\sim}\varphi \to {\sim}(\varphi \wedge \psi)$

18. $({\sim}\neg\varphi \wedge {\sim}\neg\psi) \leftrightarrow {\sim}\neg(\varphi \wedge \psi)$

19. $\varphi \to (\varphi \vee \psi)$

20. $(\neg\varphi \wedge \neg\psi) \leftrightarrow \neg(\varphi \vee \psi)$

21. $({\sim}\varphi \wedge {\sim}\psi) \leftrightarrow {\sim}(\varphi \vee \psi)$

22. ${\sim}\neg\varphi \to ({\sim}\neg(\varphi \vee \psi))$

23. $\sim\neg\psi \to (\sim\neg(\varphi \vee \psi))$

24. $\neg\neg\varphi \leftrightarrow \varphi$

25. $\sim\neg\neg\varphi \leftrightarrow \sim\varphi$

26. $s : (\varphi \to \psi) \to (t : \varphi \to (s \cdot t) : \psi)$

27. $s : \varphi \to (s + t) : \varphi$

28. $t : \varphi \to (s + t) : \varphi$

29. $t : \varphi \to ((\neg t : \varphi) \to \psi)$

3.4 Sequent Calculi and Realization for the Axiomatizable Paraconsistent Justification Logics

As in previous chapters, our preferred methodology is to develop sequent calculi for the justification logics under consideration, and for their modal analogues, and then to use those calculi to prove a realization result in the style of Artemov [4]. (By contrast, Su [94, 95] attempts to prove realization using the semantic techniques of Fitting [41], and only succeeds in proving a weaker result known as quasi-realization.)

A sequent calculus for $\mathrm{LP}^{\to}$ has been presented by Avron [12]. Extending this calculus to $\mathrm{PJ_b}$ and its kin is a fairly routine task.

Theorem 3.4.1. *The following sequent calculus is sound and complete with respect to the axiomatic presentation (and thus also to the model theory) of the logic PJ_b:*

Begin with unrestricted forms of the structural rules of identity, weakening, contraction, and cut. Additionally include the following rules:

$$\frac{}{\Gamma \Rightarrow \Delta, p, \neg p}\ \textit{Exhaustion}$$

$$\frac{}{\bot \Rightarrow \Delta}\ \bot L$$

$$\frac{}{\Rightarrow \neg\bot}\ \neg\bot R$$

$$\frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta}\ \wedge L$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}\ \wedge R$$

$$\frac{\Gamma, \neg\varphi \vee \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \wedge \psi) \Rightarrow \Delta}\ \neg\wedge L$$

$$\frac{\Gamma \Rightarrow \neg\varphi \vee \neg\psi, \Delta}{\Gamma \Rightarrow \neg(\varphi \wedge \psi), \Delta}\ \neg\wedge R$$

$$\frac{\Gamma, \varphi \Rightarrow \Delta \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta}\ \vee L$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}\ \vee R$$

$$\frac{\Gamma, \neg\varphi \wedge \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \vee \psi) \Rightarrow \Delta}\ \neg\vee L$$

$$\frac{\Gamma \Rightarrow \neg\varphi \wedge \neg\psi, \Delta}{\Gamma \Rightarrow \neg(\varphi \vee \psi), \Delta}\ \neg\vee R$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \rightarrow \psi \Rightarrow \Delta}\ {\rightarrow}L$$

$$\frac{\Gamma, \varphi \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \rightarrow \psi}\ {\rightarrow}R$$

$$\frac{\Gamma, \varphi, \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \to \psi) \Rightarrow \Delta} \neg \to L$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma \Rightarrow \Delta, \neg\psi}{\Gamma \Rightarrow \Delta, \neg(\varphi \to \psi)} \neg \to R$$

$$\frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \neg\neg\varphi \Rightarrow \Delta} \neg\neg L$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \neg\neg\varphi} \neg\neg R$$

$$\frac{}{t : \varphi, \neg t : \varphi \Rightarrow} \textit{:Biv}$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, c_n : \cdots : c_0 : \varphi} \textit{::R}$$

Restriction: The formula φ used in any application of ::R must be an instance of one of the axiom schemas of the axiomatic presentation of PJ_b given above, and the proof constants $c_0, \ldots, c_n$ an initial fragment of a $\vec{c}$ such that $\mathcal{C}(\vec{c}, \varphi)$.

$$\frac{\Gamma \Rightarrow \Delta, s : \varphi}{\Gamma \Rightarrow \Delta, (s + t) : \varphi} \textit{+R1}$$

$$\frac{\Gamma \Rightarrow \Delta, t : \varphi}{\Gamma \Rightarrow \Delta, (s + t) : \varphi} \textit{+R2}$$

$$\frac{\Gamma \Rightarrow \Delta, s : (\varphi \to \psi) \qquad \Gamma \Rightarrow \Delta, t : \varphi}{\Gamma \Rightarrow \Delta, (s \cdot t) : \psi} \cdot R$$

For PJ_bT, add the additional rule:

$$\frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, t : \varphi \Rightarrow \Delta} \textit{:L}$$

For PJ_bT4, add the above rule and the following:

$$\frac{\Gamma \Rightarrow \Delta, t : \varphi}{\Gamma \Rightarrow \Delta, !t : t : \varphi} \textit{!R}$$

Additionally, if constant specifications are to be used instead of multi-constant specifications, replace ::R with the simpler :R rule presented in Chapter 1 in the sequent calculus for PJ_bT4.

Proof. This proof proceeds in the usual manner for proofs of theorems of this sort. The full proof consists of sequent derivations of all axiom schemes, axiomatic proofs of all the axiomatic sequents, and algorithms that transform axiomatic proofs of the premises of each sequent rule to axiomatic proofs of the conclusion. Proofs of this sort are generally simple but tedious, and as such are usually omitted from published papers. This proof is especially tedious due to the unusually large number of axioms in the axiomatic system. Thus, I will comment only on selected cases.

To derive the axioms involving $\sim$ in the sequent calculus, one must restate the axioms using the definition of $\sim\varphi$ as $\varphi \to \bot$, and create sequent derivations of the corresponding conditionals. For example, axiom 6, $\varphi \vee (\sim\varphi)$, can be derived as:

$$\dfrac{\dfrac{\dfrac{}{\varphi \Rightarrow \varphi, \bot}\text{Identity}}{\Rightarrow \varphi, \varphi \to \bot}\to\text{R}}{\Rightarrow \varphi \vee \varphi \to \bot}\vee\text{R}$$

To justify the $\bot$ rules in the axiomatic system, we must unpack the definitions further, replacing $\bot$ with an arbitrarily chosen dialethia $t : \varphi \wedge \neg\, t : \varphi$. Thus, $\bot$L is actually just a special case of :Biv. To justify :Biv (and thus $\bot$L), we need a proof from premise $t{:}\varphi \wedge \neg t{:}\varphi$ to a conclusion of an arbitrary-and-unrelated proposition ψ. We begin by using modus ponens on our premise and suitable instances of axioms 14 and 15 to derive each conjunct individually. We then apply modus ponens on axiom 29 with each of these results to complete the proof of ψ.

Similar methods can be applied to $\neg\bot$R, but far more tediously. Alternatively, note that this sequent is an obvious theorem of the model theory, and as such can be justified in the axiomatic system via Su's soundness and completeness result for the model theory.

Nearly all of the remaining subproofs are either shown or intentionally omitted by Artemov [4] and Avron [12] in their sequent calculi for normal justification logic and $LP^{\rightarrow}$ respectively. Axiom 29 is unique to this particular justification logic, but its derivation in the sequent calculus is a very simple application of :Biv and $\rightarrow$R. ■

This result gives us a usable sequent calculus for PJ_b and its extensions. To get a sequent calculus for $LP^{\rightarrow}J$ and extensions thereof, we must omit the two $\bot$ rules and the :Biv rule, and also add negative justification rules as follows:

$$\frac{\varphi, \neg\psi \Rightarrow \neg(\varphi \rightarrow \psi)}{t : \varphi, \neg(s \cdot t) : \psi \Rightarrow \neg s : (\varphi \rightarrow \psi)} \neg{\cdot}\mathrm{L}$$

$$\frac{\Gamma, \neg s : \varphi \Rightarrow \Delta}{\Gamma, \neg(s + t) : \varphi \Rightarrow \Delta} \neg{+}\mathrm{L1}$$

$$\frac{\Gamma, \neg t : \varphi \Rightarrow \Delta}{\Gamma, \neg(s + t) : \varphi \Rightarrow \Delta} \neg{+}\mathrm{L2}$$

For systems containing T^j:

$$\frac{\Gamma \Rightarrow \Delta, \neg\varphi}{\Gamma \Rightarrow \Delta, \neg t : \varphi} \neg : \mathrm{R}$$

For systems containing 4^j:

$$\frac{\Gamma, \neg t : \varphi \Rightarrow \Delta}{\Gamma, \neg !t : t : \varphi \Rightarrow \Delta} \neg!\mathrm{L}$$

To get a calculus for RM_3J and its extensions, make all of the above changes. Additionally replace the two positive $\rightarrow$ rules with the forms below, devised by Avron [13], and add the $: \wedge$ rules, which are also given below.[19] Note that the negated $\rightarrow$ rules for the RM_3 extensions are the same as those for the PJ_b and $LP^{\rightarrow}$ systems.

[19] These latter additions are required because conjunction is handled by an inference rule rather than an axiom in propositional RM_3, and thus propositional inferences involving conjunction cannot be justified via ::R as they are in PJ_b and $LP^{\rightarrow}J$.

$$\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma \Rightarrow \Delta, \neg\psi \qquad \Gamma, \neg\varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \rightarrow \psi \Rightarrow \Delta} \rightarrow\text{L}$$

$$\frac{\Gamma, \varphi \Rightarrow \Delta, \psi \qquad \Gamma, \neg\psi \Rightarrow \Delta, \neg\varphi}{\Gamma \Rightarrow \Delta, \varphi \rightarrow \psi} \rightarrow\text{R}$$

$$\frac{\Gamma \Rightarrow \Delta, t : \varphi \qquad \Gamma \Rightarrow \Delta, t : \psi}{\Gamma \Rightarrow \Delta, t : (\varphi \wedge \psi)} : \wedge\text{R}$$

$$\frac{\Gamma, \neg(t : \varphi) \vee \neg(t : \psi) \Rightarrow \Delta}{\Gamma, \neg t : (\varphi \wedge \psi) \Rightarrow \Delta} \neg : \wedge\text{L}$$

With respect to cut-elimination, all of these sequent calculi follow the pattern of the classical justification logic sequent calculi discussed in Chapter 1. A cut-elimination theorem is readily proven for all of the calculi in question; the proof does not differ in any interesting way from the cut-elimination proof mentioned there and presented by Artemov [5], so I will not repeat it here. All of these logics also share the property that although cut-elimination holds, the subformula property does not hold of the resulting cut-free derivations. As in Chapter 1, this unfortunate state of affairs is due to the cut-like behavior built into the ·R rule.

With these sequent calculi in hand, we can turn to the topic of realization. To state the projection and realization results, we first must identify the modal logics corresponding to each of the justification logics under consideration. This is easily done within the model theories; from the Fitting models of PJ_b, $\text{LP}^{\rightarrow}\text{J}$, and RM_3J, simply remove the evidence function $\mathcal{E}$ from the structure, and likewise eliminate the conjunct or disjunct mentioning $\mathcal{E}$ from each of the semantic clauses for $t : \varphi$ to get the corresponding semantic clause for $\Box\varphi$. As nomenclature, Su uses PE_b to refer to the modal logic corresponding to PJ_b, and I will retain this name. For the $\text{LP}^{\rightarrow}\text{J}$ and RM_3J analogues, I will use $\text{LP}^{\rightarrow}\text{K}$ and RM_3K. Extensions to other normal modal logics will be designated by replacing K with the traditional name of the modal logic in the $LP^{\to}$ and RM_3 cases (e.g., $LP^{\to}T$ or RM_3S4), or by appending axiom-designations to PE_b in the same manner as was done for PJ_b(e.g., PE_bT or PE_bT4).

Convention 3.4.2. For the remainder of this section, the following notation will be used to express theorems that hold for all of the logics which we have thus far discussed:

Let $\models_J$ denote the consequence relation (using an axiomatically appropriate [multi-] constant specification) of any of the justification systems PJ_b $LP^{\to}J$, RM_3J, or any of the extensions thereof, and $\models_M$ denote the consequence relation of the corresponding modal logic. Either of these labels may be used on its own to refer to any of the logics within its range, but whenever both are used in the same theorem, the two logics represented must "match" in the manner described just before the introduction of this convention.

Theorem 3.4.3 (Projection Theorem)**.** *If $\Gamma \models_J \varphi$, then $\Gamma^\circ \models_M \varphi^\circ$, where $^\circ$ is the forgetful projection operator, as defined in Chapter 1.*

In earlier chapters, projection theorems were proven by noting that the axioms of the modal system are all forgetful projections of the axioms of the axiomatic system. In this case, we have not specified axiomatic systems for the modal logics, but a similar proof-by-inspection can be performed on the model theories. To do this, one must take note of two properties: first, that every Fitting model of the justification system is also a model of the modal system, and second, that there are no models of the modal system that cannot be extended to Fitting models of the justification system by adding suitable evidence functions. For all of the logics under consideration, both of these properties clearly hold, and thus the projection theorem also holds.

The projection theorem is always the simpler of the two results. To get to the realization theorem, we'll need to develop some more machinery. The proof of that theorem requires sequent calculi that are sound and complete with respect to the various consequence relations $\models_M$. It also requires a result known as the "Lifting Lemma." We've already seen lifting lemma proofs for all of the logics discussed in previous chapters; the statement and proof don't depend much on the exact details of the logic, so we can continue to treat all of the logics under consideration together.

Theorem 3.4.4 (Lifting Lemma). *If $\Gamma \models_J \varphi$, then for any sequence of proof polynomials $\vec{s}$, there is another proof polynomial t such that $\vec{s} : \Gamma \models_J t : \varphi$.*

Proof. Although the theorem is stated here in terms of the model theory, to prove it we will pass to the axiomatic system via Su's soundness and completeness proof (or the similar results for the other logics that are easily obtained as corollaries of the soundness and completeness proofs for propositional $\mathrm{LP}^{\rightarrow}$ and RM_3).

Recall the formal definition of an axiomatic proof, which has been stated in previous chapters. A proof of $\Gamma \vdash \varphi$ is a finite sequence of formulae, of which the last member is φ, such that all formulae in the sequence are either members of Γ, axioms of the system, or results of applying inference rules (in this case, modus ponens or iterated axiom justification) to earlier members of the sequence.

Assume the antecedent of the theorem statement, $\Gamma \models_J \varphi$. By the soundness and completeness result, there must be an axiomatic proof of this as defined above. To prove the consequent, we need only transform this proof into an axiomatic proof of $\vec{s} : \Gamma \vdash t : \varphi$. To do this, we build a transformation procedure for each component of the inductive definition of formulae that may appear in a proof.

For formulae that appear in the initial proof as members of Γ, we take the corresponding member of $\vec{s}:\Gamma$. For formulae that appear in the initial proof as axioms, we take the $n = 0$ case of iterated axiom justification as applied to that axiom. For formulae that are derived by an $n = k$ case of iterated axiom justification in the initial proof, we take the $n = k + 1$ case in the new proof. Finally, for formulae θ which are derived in the initial proof by using modus ponens on earlier formulae ψ and $\psi \rightarrow \theta$, there are proof polynomials s and t such that we have already included $s : \psi$ and $t : \psi \rightarrow \theta$ among the formulae of the new proof. Thus, we can add the following instance of the application axiom (Axiom 26 in the $\mathrm{PJ_b}$ axiomatization): $t : \psi \rightarrow \theta \rightarrow (s : \psi \rightarrow (t \cdot s) : \theta)$, and then derive $(t \cdot s) : \theta)$ using modus ponens on this axiom instance and the earlier formulae $s : \psi$ and $t : \psi \rightarrow \theta$. ■

Stronger versions of the lifting lemma can be proven for some of the systems under discussion. For example, systems including the 4^{j} axiom allow for the theorem statement to be strengthened to a form analogous to that given for JT4 in Chapter 1. However, for the present application, such strengthening is unnecessary.

Readers who only care about validity of results rather than constructiveness of proofs will be able to simplify the forthcoming realization proof using the following machinery:

Corollary 3.4.5 (Cheating Lemma)**.** *The following rule is approximately admissible (in the sense of Theorem 2.1.6; i.e., there is a choice of proof polynomials that may be substituted for the proof variables such that the resulting sequent is derivable) in the sequent calculus corresponding to any system $\models_J$, subject to the restrictions that y and the proof variables contained in $\vec{x}$ are all distinct, and that none of those variables occur free in the sequent $\Gamma \Rightarrow \varphi$:*

$$\frac{\Gamma \Rightarrow \varphi}{\vec{x} : \Gamma \Rightarrow y : \varphi}\ \mathit{Cheat}$$

Proof. This follows from the Lifting Lemma and the soundness and completeness result for the sequent calculus with respect to the axiomatic system. In this instance, we are using proof variables to represent the arbitrary proof polynomials $\vec{s}$ and t of the lifting lemma statement. This gives a rule with structure analogous to the Nec sequent rule that was incorporated into the JS0.5 sequent calculus of Chapter 2. ■

The various methods for creating a modal sequent calculus by extending classical propositional sequent calculi are well-known; Negri [76] provides an excellent survey of all the possibilities. As usual, we can simply replace the justification rules (both the rules governing the : operator and those governing functions on proof polynomials such as ·R) in the earlier-developed sequent calculi with suitable modal rules. For $\mathrm{PE_b}$ the following will suffice:

$$\frac{}{\Box\varphi, \neg\Box\varphi \Rightarrow \Delta}\ \Box\mathrm{Biv}$$

$$\frac{\Gamma \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi}\ \Box\mathrm{LR}$$

For $\mathrm{PE_bT}$, add

$$\frac{\Gamma \Rightarrow \varphi}{\Box\Gamma \Rightarrow \varphi}\ \Box\mathrm{L}$$

For $\mathrm{PE_bT4}$, include □L and replace □LR with the following:

$$\frac{\Box\Gamma \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi}\ \Box\mathrm{R}$$

For $\mathrm{LP^{\rightarrow}K}$, $\mathrm{RM_3K}$, and their extensions, omit □Biv (and, as in the justification cases, the ⊥ rules), and add rules pertaining to negated □ formulae:[20]

[20] As with the justification systems, these rules are unneeded in $\mathrm{PE_b}$ systems, as that logic has no negated-modal theorems.

$$\frac{\Box\Gamma, \neg\varphi \Rightarrow \neg\Delta}{\Box\Gamma, \neg\Box\varphi \Rightarrow \neg\Box\Delta}\ \neg\Box\text{LR}$$

For T-containing systems:

$$\frac{\Gamma \Rightarrow \Delta, \neg\varphi}{\Gamma \Rightarrow \Delta, \neg\Box\varphi}\ \neg\Box\text{R}$$

For 4-containing systems:

$$\frac{\Box\Gamma, \neg\varphi \Rightarrow \neg\Box\Delta}{\Box\Gamma, \neg\Box\varphi \Rightarrow \neg\Box\Delta}\ \neg\Box\text{L}$$

Theorem 3.4.6 (Realization)**.** *If* $\Gamma \models_M \varphi$*, then there exist* Δ, ψ *such that* $\Delta \models_J \psi$*,* $\Delta^\circ = \Gamma$*, and* $\psi^\circ = \varphi$*.*

Proof. As in previous theorems of this section, we begin by applying soundness and completeness results to translate the theorem statement from a statement regarding model theories to one regarding sequent calculi. So what we need to create is an algorithm translating a modal sequent derivation of $\Gamma \Rightarrow \varphi$ to a justification sequent derivation of $\Delta \Rightarrow \psi$. Sequent derivations are defined inductively, with axiomatic sequents as base cases and the other rules as inductive cases. So we can set up our translation following the same pattern.

Most of the sequent rules are identical between the corresponding justification and modal systems, and so the translation can leave those steps in place. Steps using $\Box$Biv, $\Box$L, or $\neg\Box$R rules can simply be translated to instances of :Biv, :L, or $\neg$:R with arbitrary proof polynomials. $\Box$LR steps can be translated directly by the approximately admissible cheat rule,[21] or more tediously by employing reasoning steps in the sequent calculus analogous to the proof of the lifting lemma in the axiomatic system. Nearly identical reasoning works for the $\Box$R rule of the 4-containing systems.

[21] Since we are only trying to prove the existence of a realization rather than an actual algorithm for constructing a realization, it does not matter that cheat is only approximately admissible.

The above reasoning completes the proof for the PJ_b and PE_b systems. For the $LP^{\rightarrow}$ and RM_3 extensions, we must also consider cases involving the negated modal rules. I will address the $\neg\Box$LR case directly; as with the positive rules, the $\neg\Box$L case is substantially similar.

The essential element of a $\neg\Box$LR (or $\neg\Box$L) step is a negated formula on the left side of the premise sequent. To reach the conclusion sequent, we must consider how that negation was introduced. If it was introduced by a weakening step (or equivalently, as a side formula in an axiomatic step), we can reach the conclusion sequent by simply introducing a negated justification formula instead. If it was introduced as the main formula of an identity axiom, then there may be an identical formula on the right side. In this case, we replace the initial identity step by one introducing the desired negated justification formula to get the conclusion sequent.

Many of the remaining $\neg\Box$LR cases can be translated directly by $\neg\cdot$L. For those which cannot, first note that the relevant cases all involve compound formulae, as the pure atomic cases were handled in the previous paragraph. Our first step will be to pass from the sequent calculus back to the axiomatic system, and find a formula equivalent to the $\neg\Box$LR premise that has the correct form to be a $\neg\cdot$L premise. Because we're working in axiomatic logic, this method only alters the right side of the sequent; if the form-fitting problem with the original was that the left premise was too complex, we may end with a very complex right side, but there will be a workable solution. Moreover, the steps used to derive that solution are all axioms (or necessitations thereof) and modus ponens. We use the final result of this derivation as the premise in one $\neg\cdot$L step, and the intermediate steps of the derivation are all "modus ponens on an axiom" procedures, which can themselves be translated into separate derivations ending in ¬·L steps.[22] So we collect all of these derivations, and perform cuts to produce a derivation of the originally desired sequent. ■

3.5 Other Axiomatizable Paraconsistent Systems

The logics discussed in the previous sections are valuable because they are reasonably simple and well-behaved in all three presentations: the axiomatic system, the sequent calculus, and the model theory. However, they are less than ideal for the philosophical purposes that were discussed in the introduction of this chapter. In particular, while these logics can treat the liar sentence as an unproblematic dialethia, they cannot adequately account for the Curry sentence $\kappa = T\ulcorner\kappa\urcorner \to \bot$.[23] Thus, on pain of triviality, we cannot simply include a truth predicate in a first-order extension of $LP^{\to}$, RM_3, or PJ_b/PE_b. In order to handle truth in this way, we need a logic that can render both the liar and Curry sentences non-paradoxical.

In [15], Beall proposes the paraconsistent system BX as a suitable logic for this purpose. Axiomatic and model-theoretic presentations of this system are known (see Fine [39]). We can produce an axiomatic justification extension simply by adding justification axioms, as we did to produce RM_3J (BX allows contraposition, so the $LP^{\to}J$ negative axioms are not needed). To my knowledge, nobody has developed a sequent calculus for BX, but if such a calculus were available, we could again proceed as above in adding justification and modal rules to get sequent calculi that could be

[22] For readers who are not convinced that such derivations must exist, here's a slightly more thorough explanation. Modify the axiomatic derivation by replacing axioms with their justifications and modus ponens steps with application axiom instances. Even in $LP^{\to}$, application instances are all contraposable, and these contrapositions are exactly the sequents we need to derive here. So by the soundness and completeness results, we know that such derivations exist.

[23] Depending on the logic, we can either use the $\bot$ operator defined in Section 3.3, or take $\bot = \forall x Tx$.

used to prove a realization theorem.

As noted, BX does have a known model theory, and there are also known procedures for extending such models to the various normal modal systems. The best reference on the latter is probably Fuhrmann [43]. There is no reason in principle why such models cannot be further extended into axiomatic justification logics. The resulting class of models would be extraordinarily complicated, posing both a practical problem of being difficult to use, and a philosophical problem of being difficult to provide the formal system with a genuine semantic interpretation[24].

So far we have focused on BX, motivated by Beall's proposal of that particular logic as a solution to the philosophical problems with which we began this chapter. However, it turns out that most of the other commonly-discussed paraconsistent logics have formal structures (especially model theories) similar to either the structure of BX or to that of the simple axiomatizable systems discussed earlier. Such logics can thus be extended into justification systems using the same methodology presented here.

[24]Much ink has been spilled over the question of whether the type of model theory used for logics like BX has an adequate interpretation even in the propositional case. The best attempt at providing such an interpretation is probably the many-authored [18]. Even if one accepts such an account, however, the modal and justification extensions will have to account for how the frame structure manages to do double-duty in explaining both conditionals and modals without any undesirable consequences; this question is unproblematic for the weaker modal systems, but potentially devastating for modalities like S5.

Chapter 4

Non-Applicative Justification Logic

4.1 Introduction

Up to this point, we have regarded justification logics as being fundamentally axiomatic systems, with the sequent and model-theoretic presentations being derivative forms whose foundation rests in the axiomatic system. This view of justification logic is essential to the project of intuitionistic provability, as the axiomatization is what allows us to trace each provability statement in JT4 to a corresponding proof in intuitionistic logic. Although such strong foundationalism is not strictly necessary in an epistemic logic application, it does seem to be desirable. After all, there does seem to be an epistemic difference between a belief generated by a simple deduction and one generated by a long formal proof; the latter is less certain due to the greater likelihood of the agent committing an error in deduction. It would thus be useful to be able to track what deductive steps are performed by the agent in justifying a belief, and working from an axiomatic justification system allows us to do so.

However, there may be other considerations weighing against the use of axiomatic justification logic. One major concern is the status of modus ponens. Axiomatic propositional logic has always been grounded in the use of modus ponens with a material conditional. Moreover, normal justification logics (whether presented axiomatically or not) are based on a principle of application, which is a modalization of modus ponens. If modus ponens is to be abandoned, we must abandon normality along with axiomatization.[1]

Must we then abandon modus ponens? In Chapter 3, we discussed motivations for paraconsistent logic, then presented paraconsistent systems that incorporated some form of modus ponens, noting that these systems did not fully satisfy the motivating desiderata. Beall [17] argues that the goals of paraconsistent logic can be more simply and completely achieved by rejecting modus ponens. In particular, he recommends a form of LP as the best paraconsistent working logic. If we are to take his argument seriously (as we ought), then we must be prepared to at least consider a justification logic that lacks modus ponens, and thus also normality and axiomatization.

Given that this motivation has pushed us to abandon axiomatic justification logic, how shall we construct our justification system? As in the last chapter, the solution to the problem has been partially anticipated by Che-Ping Su. The key factor is his choice to regard the model theory as fundamental instead of the axiomatic system in developing his paraconsistent justification logics.

[1] Strictly speaking, this does not quite follow. Return to the modalization of LP that was presented briefly in Chapter 3. Is the application principle (K axiom) valid? Under the bivalent formulation of modality it is invalid. Under the fully paraconsistent formulation it is valid, but inferentially useless in the absence of modus ponens. For the remainder of this chapter, we will always assume that modality is treated bivalently, as it would not be practical to build a justification system incorporating an inferentially-useless "justification."

4.2 Che-Ping Su's Non-Applicative System $\mathrm{PJ_F}$

Working on the problem of belief revision for inconsistent beliefs, Su [93] proposes a justification system based on something that he calls "fusion models" which are related to the semantics for impossible worlds developed in Restall [84].

Definition 4.2.1. A fusion model of the system $\mathrm{PJ_F}$ is a quadruple $\langle W, R, \mathcal{E}, v\rangle$, where W is a non-empty set of worlds; R is a relation between W and $(\wp(W) \setminus \{\emptyset\})$; $\mathcal{E}$ is a function from W and the set of proof polynomials to the power set of the set of sentences; v is a classical valuation function, i.e., a function from W and the set of sentences to $\{0, 1\}$; and the following restrictions hold:

- $\mathcal{E}(w, s) \cup \mathcal{E}(w, t) \subseteq \mathcal{E}(w, s + t)$.
- If $\varphi \in \mathcal{E}(w, t)$, then $(t : \varphi) \in \mathcal{E}(w, !t)$ for all proof polynomials t, sentences φ, and $w \in W$.
- $v_w(\neg\varphi) = 1$ iff $v_w(\varphi) = 0$.
- $v_w(\varphi \supset \psi) = 0$ iff $v_w(\varphi) = 1$ and $v_w(\psi) = 0$.
- $v_w(t : \varphi) = 1$ iff $\varphi \in \mathcal{E}(w, t)$ and for every non-empty set of worlds $U \subseteq W$ such that $R(w, U)$, there is some $w' \in U$ such that $v_{w'}(\varphi) = 1$.

The fusion model defined here is a variant of a Fitting model for justification logic; it has a modal structure built into itself—in this case, the modal structure is exactly that of the corresponding justification logic (by which I mean modal correspondence in the sense of the realization theorem—contrast this claim with the results from Chapters 1 and 2, where the modality built into the Fitting models could be an extension of the modality that corresponds to the justification system being modeled).

Su has proven for his system many of the properties that are shown for Fitting models by Fitting [41], including compactness and some of the lemmas pertaining to special classes of models, but he has not as yet proven the realization theorem.

We must note an important difference between the modality built into the system PJ_F and that which Restall employs in [84]. Both systems allow sets of worlds to be treated as if they were a single world in the semantics. However, in Restall's semantics, this combination distributes over propositional connectives. For example, in Restall's semantics, a conjunction could be satisified by a set of worlds if one conjunct is satisfied by one world in the set, and the other by another world, even though neither world would satisfy both conjuncts. In Su's system, this is not permitted; a conjunction is only satisfied if there is a world which itself satisfies both conjuncts. Because of this discrepancy, where Restall's system is equivalent to a modalization of the simple paraconsistent logic LP, Su's is equivalent to a weaker logic.

The axiomatic system for PJ_F is simply the normal justification logic J with the application axiom removed. That is, PJ_F is the closure of the PC and sum axiom schemas under modus ponens and iterated axiom justification. The soundness of this axiomatization is distinctly less obvious than it is for all the model theories that have been presented up to now. One reason for this is because of a presentational difference between my development of Fitting models in Chapter 1 and Fitting's original [41]. In my preferred presentation of the model theories for justification logic, the requirement of axiomatic appropriateness is built into the conditions that must be satisfied for a structure to count as a model of the logic at all. This makes the notion of a constant specification a mostly redundant feature, which is only needed for the metatheoretical work of coordinating models and axiomatic proofs in order to allow a rigorous soundness and completeness theorem. Fitting, by contrast, omits this, allowing structures that don't meet any axiomatically appropriate constant specification to still count as genuine models of the logic. In [93], Su chooses to present the model theory in Fitting's style, and I have preserved that here.

If we define the requirements of meeting a multi-constant specification and of consequence relative to a multi-constant specification in the usual manner applied to Fitting models (i.e., following Definition 1.5.7), we can prove soundness and completeness.[2] However, there are other related concerns in the non-applicative system. In particular, the fusion models with axiomatically appropriate consequence yield a counterexample to realization. In the underlying modal system, we have $\Box p \vDash \Box(p \vee q)$, because disjunctions behave classically at all worlds. In the justification system, however, being given $s : p$ is not enough to guarantee the existence of a proof polynomial t such that $t : (p \vee q)$. In a normal justification logic, we could build such a proof polynomial by taking a proof constant specified for the axiom of addition and then applying application with s, but without application, such a procedure is unavailable. One possible workaround is to add additional restrictions to $\mathcal{C}$ (or directly to the model's evidence function $\mathcal{E}$) which will provide the missing proof polynomials for valid inferences such as disjunction introduction.

For $\mathrm{PJ_F}$, Su proposes a very clever version of this approach. He extends the syntax of the language with an application operator $\cdot$, which is then defined in the evidence function exactly as in a normal justification logic. Even with this addition, the general application principle, $t : (\varphi \supset \psi) \supset (s : \varphi \supset (t \cdot s) : \psi)$, remains invalid, as it has counterexamples within the modal framework of the fusion model. However, the principle does hold for the special case in which either φ or $\varphi \supset \psi$ is a valid formula, which gives us the application instances that we need to avoid counterexamples like

[2]For the actual proofs, see Su [93].

the one presented above.

4.3 LP-Based Non-Applicative Systems

As was discussed in Chapter 3, LP is difficult to extend into a justification logic because it cannot be axiomatized. However, Su's model-theoretic development of the system PJ_F suggests two possible approaches. Implementing the Restall semantics of satisfaction at a set of worlds in a fusion model would produce a partially paraconsistent justification logic—a system where connectives occurring within justification formulae receive an LP-like interpretation while the base logic remains classical. An alternative approach is to add justification operators directly to a model of LP, producing a Fitting model with structure directly analogous to the modal LP system presented by Priest in [83]. In the following subsections, I shall develop LP-based justification systems of both types, which I shall call pLPJT4 and tLPJT4 respectively.[3]

Before we start modeling the specific LP-based systems, we must address a general problem concerning all of them. In the normal systems and in PJ_F, soundness, completeness, and realization were secured through axiomatic appropriateness and the application principle (restricted application for PJ_F). In LP, that solution is unavailable; LP has no axiomatic system, and necessity of premises is not sufficient to allow application. We must find an alternative, and we should start our search by

[3]For the present, these acronyms are to be expanded as "partially/totally LP-based JT4," though a phrasing in which "p" stands for something like "possible-worlds" might be more perspicuous. My classification of these non-applicative systems as variants of the normal justification logic JT4 might be objectionable. As I see it, however, these systems are non-normal only in the failure of application, which is an inevitable result of the switch from a classical base logic to LP, and so they can properly be viewed as forms of the normal justification logic.

asking what it is about axiomatic appropriateness and application that allows them to perform the desired function in normal systems and PJ_{F}. The answer to that question is, of course, that these elements build into the set of proof polynomials the entire structure of the base logic (via the latter's axiomatic presentation). So what we need is a way to "polynomialize" the structure of LP.

The simplest form of the LP structure to incorporate into the proof polynomials is the natural deduction formulation given by Kooi and Tamminga [63]. This system consists of the axiom schema of the excluded middle, the classical introduction and elimination rules for $\wedge$ and $\vee$, and bidirectional inference rules corresponding to the double negation and DeMorgan equivalences. There are several ways in which these rules can be polynomialized, but for the present, let us employ the following: Instead of relativizing models to constant specifications, fix a single proof constant e that will prove all instances of the excluded middle. For conjunctions, let us keep everything transparent, holding a proof of a conjunction to be identical to that proof polynomial which proves both conjuncts independently. The latter can be obtained when needed by using the sum operator that is already familiar from normal justification logics. For the equivalence rules, let us define a unary operator $\mathfrak{E}$ that transforms a proof of a given formula to a proof of its equivalent. Likewise, for disjunction introduction, we can define a unary operator ∂ to transform a proof of a disjunct to a proof of the whole disjunction. The problem is disjunction elimination. For this to work, we need to have proof polynomials which prove the metalinguistic assertion $\varphi \vdash \psi$ rather than any object-language sentence. For the intended epistemic application of the logic, it makes sense that such proof polynomials would exist. There is, after all, an articulable difference between having evidence of a logical entailment and evidence of a material conditional. However, if we formally codify this distinction, we run the risk of introducing self-referential paradoxes that are not solved by the paraconsistent base logic. Instead, I propose the following compromise: the metalinguistic assertion $\varphi \vdash \psi$ is to be treated as a mere atomic sentence, with no logical structure.[4] Disjunction elimination will then be modeled by a ternary operator $\blacktriangledown$, which will take proof polynomials for atomic propositions of the form $\varphi \vdash \psi$ in two of its input slots.

4.3.1 The System pLPJT4

Definition 4.3.1. A fusion model of the system pLPJT4 is a quadruple $\langle W, R, \mathcal{E}, V \rangle$, where W is a non-empty set of worlds; R is a relation between W and $(\wp(W) \setminus \{\emptyset\})$; $\mathcal{E}$ is a function from W and the set of proof polynomials to the power set of the set of sentences; V is a function from $(\wp(W) \setminus \{\emptyset\})$ and the set of sentences to $\{0, 1\}$; and the following restrictions hold:[5]

- $\mathcal{E}(w, s) \cup \mathcal{E}(w, t) \subseteq \mathcal{E}(w, s + t)$.
- For all sentences φ and $w \in W$, $(\varphi \vee \neg\varphi) \in \mathcal{E}(w, e)$.
- For all sentences φ, ψ, proof polynomials t, and $w \in W$, $(\varphi \wedge \psi) \in \mathcal{E}(w, t)$ iff $\varphi \in \mathcal{E}(w, t)$ and $\psi \in \mathcal{E}(w, t)$.
- For all sentences φ, ψ, proof polynomials t, and $w \in W$, if ψ is a double-negation or DeMorgan equivalent of φ and $\varphi \in \mathcal{E}(w, t)$, then $\psi \in \mathcal{E}(w, \mathfrak{E}t)$.
- For all sentences φ, ψ, proof polynomials t, and $w \in W$, if either $\varphi \in \mathcal{E}(w, t)$ or $\psi \in \mathcal{E}(w, t)$, then $(\varphi \vee \psi) \in \mathcal{E}(w, \partial t)$.

[4]Importantly, what is conceded here is that $\varphi \vdash \psi$ is to have no logical structure in the object language. We will continue to use its ordinary structure in metalinguistic argument.

[5]In the following statements, let U be a schematic letter ranging over $(\wp(W) \setminus \{\emptyset\})$.

- For all sentences φ, ψ, θ, proof polynomials r, s, t, and $w \in W$, if $(\varphi \vee \psi) \in \mathcal{E}(w, r)$, $(\varphi \vdash \theta) \in \mathcal{E}(w, s)$, and $(\psi \vdash \theta) \in \mathcal{E}(w, t)$, then $\theta \in \mathcal{E}(w, \blacktriangledown(r, s, t))$.

- If $\varphi \in \mathcal{E}(w, t)$, then $(t : \varphi) \in \mathcal{E}(w, !t)$ for all proof polynomials t, sentences φ, and $w \in W$.

- For every $w \in W$, $R(w, \{w\})$.

- For every $w \in W$ and $U, U' \in (\wp(W) \setminus \{\emptyset\})$, if $R(w, U)$ and for every $w' \in U$, $R(w', U')$, then $R(w, U')$.

- For every atomic proposition p, $V_U(p) = 0$ iff for every $w \in U$, $V_{\{w\}} = 0$.

- $V_U(\neg\varphi) = 1$ iff there is some $U' \subseteq U$ such that $V_{U'}(\varphi) = 0$.

- $V_U(\varphi \vee \psi) = 1$ iff there is some $U' \subseteq U$ such that $V_{U'}(\varphi) = 1$ or $v_{U'}(\psi) = 1$.

- $V_U(\varphi \wedge \psi) = 1$ iff there are some $U', U'' \subseteq U$ such that $V_{U'}(\varphi) = 1$ and $V_{U''}(\psi) = 1$.

- $V_U(t : \varphi) = 1$ iff there is some $w \in U$ such that $\varphi \in \mathcal{E}(w, t)$ and for every non-empty set of worlds $U' \subseteq W$ such that $R(w, U')$, $V_{U'}(\varphi) = 1$.

Note that this model theory calls for the semantic value of some normal object-language propositions to depend truth-functionally on the propositions of the form $\varphi \vdash \psi$, which have been declared to be semantically inert. This is acceptable, as modeling a language is fundamentally a metalinguistic enterprise. However, it still requires us to have a formal mechanism available to pick out the target propositions in the object language. One possibility is to add $\vdash$ to the object-language syntax as a propositional connective, with the understanding that this usage is syntactic only—the semantics of $\vdash$ are to be understood as entirely non-compositional, making the formulae built from it effectively equivalent to atoms.

There is also an alternative method of handling $\vdash$ that is more complicated, but which better fits the characterization of this operator as being an entirely metalinguistic item that is to be represented in the object language as a mere propositional atom. We begin by noting that the total collection of propositional atoms is taken to be countably infinite, as is standard in propositional logics. Let the propositional atoms be partitioned into two disjoint ω-sequences, p_i and q_i. Moreover, choose a Gödel coding to represent all the metalinguistic $\vdash$ formulae by natural numbers. We can then declare the convention that the ps all represent "ordinary" atomic propostions (for example, p_1 might be "Snow is white," p_2 "Grass is green," and so forth), whereas the qs represent $\vdash$ formulae according to the Gödel coding. Let all the $\vdash$ formulae in the model theory be replaced by q atoms as per this convention, and we have a model-theoretic representation of the language containing entirely object-language vocabulary, with the metalinguistic $\vdash$ only being needed for the properly metalinguistic function of explaining the intended interpretation of the various sentences of the object language. This account is my intended reading of all the languages in this chapter, but nothing in the discussion to follow will be harmed if a reader chooses to adopt the alternative from the last paragraph instead.

Given either treatment of $\vdash$, however, there remains one potential problem to be addressed: what happens if $\vdash$ operators are nested, producing formulae such as $\varphi \vdash (\psi \vdash \theta)$? In the normal metalinguistic role of $\vdash$, such constructions are ungrammatical, but if $\vdash$ is to be incorporated into the object language in any form, then they may become permissible. However, let us nonetheless rule out such constructions as ill-formed in the object language. Doing so avoids the philosophical problem of how to interpret such nested constructions—which would remain uninterpretable if object-language $\vdash$ were interpreted as equivalent to a metalinguistic $\vdash$. It also eliminates the hazard of producing self-referential paradoxes via diagonalization when the Gödel coding formulation of object-language $\vdash$ is used.

Definition 4.3.2 (Consequence for pLPJT4 Fusion Models)**.** A fusion model $\mathcal{M} = \langle W, R, \mathcal{E}, V \rangle$ satisfies a formula φ (written $\mathcal{M} \models_{\text{pLPJT4(F)}} \varphi$) iff for every $w \in W$, $V_{\{w\}}(\varphi) = 1$.[6] A formula φ is a consequence of the set of formulas Γ (written $\Gamma \models_{\text{pLPJT4(F)}} \varphi$) iff every fusion model that satisfies all of the members of Γ also satisfies φ. (Note the extra (F) in the subscripts of the turnstyles; this denotes that the consequence relation is derived from fusion models. An alternative model theory will be presented later.)

As discussed above, the fusion models of pLPJT4 differ from those of PJ_{F} (properly, from its reflexive and transitive extension) primarily in that the treatment of sets of worlds from Restall [84] has been restored. There are only two other differences between pLPJT4 and PJ_{F}: the use of sets of worlds instead of single worlds in the valuation function, and the introduction of operators on proof polynomials corresponding to the LP natural deduction rules. The former is simply a notational convenience; the consequence definition uses only singleton sets, which are clearly equivalent to single worlds in the PJ_{F} presentation. The latter is a substitute for the notion of axiomatic appropriateness, and should have the same metatheoretic behavior as the original. These differences are minor enough that all of the metatheorems proven for PJ_{F} in [93] should be easily recoverable for pLPJT4.

[6] Note that, because there are no axioms in pLPJT4, there is no version of axiom necessitation, and so there is no need to relativize this consequence relation to any constant specification.

Restall [84] proved that his account of consequence on sets of worlds is equivalent to the LP consequence relation. Obviously, this proof gives us confidence that the label pLPJT4 is apt, in that the system can genuinely be viewed as LP-based. In addition, however, it suggests that we can offer a simpler model theory for the logic, based on the standard trivalent models of LP.

Definition 4.3.3. A trivalent Fitting model of the system pLPJT4 is a structure $\langle W, R, \boldsymbol{@}, \mathcal{E}, v\rangle$, where W is a non-empty set of worlds; R is a reflexive and transitive relation on W; $\boldsymbol{@} \in W$; $\mathcal{E}$ is a function from W and the set of proof polynomials to the power set of the set of sentences; v is a function from W and the set of sentences to $\{0, \frac{1}{2}, 1\}$; and the following restrictions hold:

- $\mathcal{E}(w, s) \cup \mathcal{E}(w, t) \subseteq \mathcal{E}(w, s + t)$.
- For all sentences φ and $w \in W$, $(\varphi \vee \neg\varphi) \in \mathcal{E}(w, e)$.
- For all sentences φ, ψ, proof polynomials t, and $w \in W$, $(\varphi \wedge \psi) \in \mathcal{E}(w, t)$ iff $\varphi \in \mathcal{E}(w, t)$ and $\psi \in \mathcal{E}(w, t)$.
- For all sentences φ, ψ, proof polynomials t, and $w \in W$, if ψ is a double-negation or DeMorgan equivalent of φ and $\varphi \in \mathcal{E}(w, t)$, then $\psi \in \mathcal{E}(w, \mathfrak{E}t)$.
- For all sentences φ, ψ, proof polynomials t, and $w \in W$, if either $\varphi \in \mathcal{E}(w, t)$ or $\psi \in \mathcal{E}(w, t)$, then $(\varphi \vee \psi) \in \mathcal{E}(w, \partial t)$.
- For all sentences φ, ψ, θ, proof polynomials r, s, t, and $w \in W$, if $(\varphi \vee \psi) \in \mathcal{E}(w, r)$, $(\varphi \vdash \theta) \in \mathcal{E}(w, s)$, and $(\psi \vdash \theta) \in \mathcal{E}(w, t)$, then $\theta \in \mathcal{E}(w, \blacktriangledown(r, s, t))$.
- If $\varphi \in \mathcal{E}(w, t)$, then $(t : \varphi) \in \mathcal{E}(w, !t)$ for all proof polynomials t, sentences φ, and $w \in W$.

- $v_{@}(\varphi) \neq \frac{1}{2}$.
- $v_w(\neg\varphi) = \begin{cases} 1, \text{if } v_w(\varphi) = 0; \\ 0, \text{if } v_w(\varphi) = 1; \\ \frac{1}{2}, \text{if } v_w(\varphi) = \frac{1}{2}. \end{cases}$
- $v_w(\varphi \vee \psi) = \max(v_w(\varphi), v_w(\psi))$.
- $v_w(\varphi \wedge \psi) = \min(v_w(\varphi), v_w(\psi))$.
- $v_w(t : \varphi) = 1$ if $\varphi \in \mathcal{E}(w, t)$ and for every $w' \in W$ such that $R(w, w')$, $v_{w'}(\varphi) \neq 0$. Otherwise, $v_w(t : \varphi) = 0$.

Definition 4.3.4 (Consequence for pLPJT4 Trivalent Models)**.** A trivalent Fitting model $\mathcal{M} = \langle W, R, @, \mathcal{E}, v \rangle$ satisfies a formula φ (written $\mathcal{M} \models_{\text{pLPJT4(T)}} \varphi$) iff $v_{@}(\varphi) \neq 0$. A formula φ is a consequence of the set of formulas Γ (written $\Gamma \models_{\text{pLPJT4(T)}} \varphi$) iff every trivalent Fitting model that satisfies all of the members of Γ also satisfies φ.

In developing these model theories, we have so far treated sentences of the form $\varphi \vdash \psi$ as mere atoms, and left it entirely to chance whether there is any proof polynomial justifying such sentences. This choice guaranteed that the models could be coherently constructed, but is not really satisfactory. An improvement can be made by reintroducing the notions of constant specifications and $\mathcal{C}$-validity. The general idea will be to build a constant specification whose range includes all and only the desirable $\varphi \vdash \psi$ instances, and then to utilize $\mathcal{C}$-validity instead of absolute validity as the intended semantic interpretation of the language. However, to enable such a constant specification to do the desired work, we must also refine the notion of what it takes for a model to meet a specification, and thus to be included as part of the $\mathcal{C}$-validity condition:

Definition 4.3.5. A model $\mathcal{M}$ meets a constant specification $\mathcal{C}$ factively iff $\mathcal{M}$ meets $\mathcal{C}$ (as in Definition 1.5.7) and, for every $w \in W$ and $\varphi \in \text{Range}(\mathcal{C})$, $v_w(\varphi) \neq 0$. A formula is factively $\mathcal{C}$-valid if it is satisfied by all models which meet $\mathcal{C}$ factively.

This notion of factive $\mathcal{C}$-validity was not needed in the normal justification logics or in PJ_F because the constant specifications of interest there ranged over axioms—formulae already known to be true on all worlds of all models—where the question of satisfying $c : \varphi$ really only concerned the evidence function. In this application, however, we are using the constant specification to create a semantics for the otherwise atomic formulae $\Gamma \vdash \varphi$, so factivity must be explicitly specified.

With this notion in hand, we can build our desired constant specification as follows:

Lemma 4.3.6. *Construct a sequence of constant specifications as follows: let $\mathcal{C}_0$ be the empty constant specification. Let each successor $\mathcal{C}_{i+1}$ contain all and only those pairs $\langle b, \Gamma \vdash \varphi \rangle$ such that $\Gamma \vDash_{\mathcal{C}_i} \varphi$, where $\vDash_{\mathcal{C}_i}$ is the consequence relation associated with factive $\mathcal{C}_i$-validity, Γ and φ are free of $\vdash$ instances, and b is an arbitrary proof constant employed uniformly throughout this construction. At limit ordinals, let $\mathcal{C}_i$ be the union of all preceding constant specifications. Given that consequence relations are monotonic in the sense that restricting the class of models cannot render an established validity invalid, we have it that this sequence is monotonic in the sense that for ordinals $i < j$, $\mathcal{C}_i \subseteq \mathcal{C}_j$. This latter sense of monotonicity suffices to show that the sequence of constant specifications converges under transfinite recursion to a fixed point $\mathcal{C}^*$.*[7]

[7]For the proof of convergence, consult any elementary treatise on inductive definability. For

Definition 4.3.7. A constant specification $\mathcal{C}$ is said to be provability-appropriate just in case $\text{Range}(\mathcal{C}) = \text{Range}(\mathcal{C}^*)$, where $\mathcal{C}^*$ is the constant specification produced according to the above lemma.

Recalling that the structure of the proof polynomials is intended to match the structure of LP natural deduction, we can interpret the sequence of constant specifications $\mathcal{C}_i$ as a sequence of proof constructions. At the initial specification, we have all of the proofs that we can construct without using any $\vee$-elimination steps. Then the next specification gives us all of the proofs that we can construct using a single $\vee$-elimination step, and the following specification gives all that can be constructed with two $\vee$-eliminations, and so forth. Given that proofs are of finite length, once we reach step ω, we are assured of having all possible proofs. The formalization above uses a technically unnecessary transfinite recursion to get the result as a consequence of a more general theorem about recursive definitions, in which the convergence to a fixed point sometimes occurs at steps beyond ω; there seems to be little advantage in formally proving our intuitive judgment that the convergence should occur at ω. Finally, the generalization to provability-appropriateness allows us to dispense with the artificial assumption that all provability statements will be modeled by a single proof constant, yielding a perspicuous model of a provability logic for the natural deduction proof system.

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example, Chapter VI of [14] provides a clear account. That text deals with relations on a single set, but the relations are not required to be either surjective or injective, so we can simply work in the union of the sets of proof constants and sentences.

4.3.2 The System tLPJT4

Having laid out a trivalent model theory for pLPJT4, the only change that must be made to get tLPJT4 is the removal of the classical base world. This could be accomplished by simply deleting the clause that mandates $v_{@}(\varphi) \neq \frac{1}{2}$, but after doing so, the designation **@** becomes completely redundant. We can produce a trivalent Fitting model that omits the **@** designation and instead defines consequence over all worlds. Furthermore, because no particular world has any semantic significance, we can eliminate worlds from the model theory entirely, yielding a trivalent Mkrtychev model of tLPJT4. It turns out that this latter change will simplify the proof of Theorem 4.3.11 below, so we shall present the model theory in that manner.

Definition 4.3.8. A trivalent Mkrtychev model of the system tLPJT4 is a pair $\langle E, v\rangle$, where E is a relation between the set of proof polynomials and the set of sentences of tLPJT4; v is a function from the set of sentences to $\{0, \frac{1}{2}, 1\}$; and the following restrictions hold:

- There is a proof constant e such that for all sentences φ, $E(e, \varphi \vee \neg\varphi)$.
- For all proof polynomials s, t and sentences φ, if $E(s, \varphi)$ or $E(t, \varphi)$, then $E(s + t, \varphi)$.
- For all sentences φ, ψ, and proof polynomials t, $E(t, \varphi \wedge \psi)$ iff $E(t, \varphi)$ and $E(t, \psi)$.
- For all sentences φ, ψ, and proof polynomials t, if ψ is a double-negation or DeMorgan equivalent of φ and $E(t, \varphi)$, then $E(\mathfrak{E}t, \psi)$.
- For all sentences φ, ψ, and proof polynomials t, if either $E(t, \varphi)$ or $E(t, \psi)$, then $E(\partial t, \varphi \vee \psi)$.

- For all sentences φ, ψ, θ, and proof polynomials s, t, u, if $E(u, \varphi \vee \psi)$, $E(s, \varphi \vdash \theta)$, and $E(t, \psi \vdash \theta)$, then $E(\blacktriangledown(u, s, t), \theta)$.

- For all sentences φ and proof polynomials t, if $E(t, \varphi)$, then $E(!t, t : \varphi)$.

- For all sentences φ, if there exists any proof polynomial t such that $E(t, \varphi)$, then $v(\varphi) \neq 0$.

- $v(\neg\varphi) = \begin{cases} 1, \text{if } v(\varphi) = 0; \\ 0, \text{if } v(\varphi) = 1; \\ \frac{1}{2}, \text{if } v(\varphi) = \frac{1}{2}. \end{cases}$

- $v(\varphi \vee \psi) = \max(v(\varphi), v(\psi))$.

- $v(\varphi \wedge \psi) = \min(v(\varphi), v(\psi))$.

- $v(t : \varphi) = 1$ if $E(t, \varphi)$. Otherwise, $v(t : \varphi) = 0$.

Definition 4.3.9 (Consequence for tLPJT4 Trivalent Models)**.** A trivalent Mkrtychev model $\mathcal{M} = \langle E, v \rangle$ satisfies a formula φ (written $\mathcal{M} \models_{\text{tLPJT4}} \varphi$) iff $v(\varphi) \neq 0$. A formula φ is a consequence of the set of formulas Γ (written $\Gamma \vDash_{\text{LPJT4}} \varphi$) iff every trivalent Mkrtychev model that satisfies all of the members of Γ also satisfies φ.

We can define the notion of provability-appropriateness using the same lemma construction as in the pLPJT4 case. However, because we are working with Mkrtychev models, we can use the simpler definition of meeting a constant specification from Chapter 1, that $\mathcal{M} = \langle E, v \rangle$ meets $\mathcal{C}$ iff $\mathcal{C} \subseteq E$. Moreover, because this is a JT system, the Mkrtychev model theory entails that any model that meets a constant specification $\mathcal{C}$ does so factively, so we can omit that condition from any of our theorems involving provability-appropriateness if we so choose.

A sequent calculus that is sound and complete with respect to LP has been developed independently by Beall [16] and by Palau and Oller [78].[8] In principle, we should be able to obtain a sequent calculus for tLPJT4 simply by adding justification-operator rules to their sequent calculus, thusly:

Definition 4.3.10. The sequent calculus for tLPJT4 consists of unrestricted forms of the structural rules of identity, weakening, contraction, and cut, along with the following rules:

$$\frac{}{\Gamma \Rightarrow \Delta, p, \neg p}\text{ Exhaustion}$$

$$\frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta}\wedge\text{L}$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}\wedge\text{R}$$

$$\frac{\Gamma, \neg\varphi \vee \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \wedge \psi) \Rightarrow \Delta}\neg\wedge\text{L}$$

$$\frac{\Gamma \Rightarrow \neg\varphi \vee \neg\psi, \Delta}{\Gamma \Rightarrow \neg(\varphi \wedge \psi), \Delta}\neg\wedge\text{R}$$

$$\frac{\Gamma, \varphi \Rightarrow \Delta \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta}\vee\text{L}$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}\vee\text{R}$$

$$\frac{\Gamma, \neg\varphi \wedge \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \vee \psi) \Rightarrow \Delta}\neg\vee\text{L}$$

[8] Arguably, priority should be granted to Avron [12], as the LP calculus is nothing more than the result of omitting the $\to$ rules from the $\text{LP}^{\to}$ calculus. However, Avron does not mention the possibility of getting LP simpliciter (or even the existence of that logic) at any point in his body of work.

$$\frac{\Gamma \Rightarrow \neg\varphi \wedge \neg\psi, \Delta}{\Gamma \Rightarrow \neg(\varphi \vee \psi), \Delta}\ \neg\vee\mathrm{R}$$

$$\frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \neg\neg\varphi \Rightarrow \Delta}\ \neg\neg\mathrm{L}$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \neg\neg\varphi}\ \neg\neg\mathrm{R}$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi \vee \neg\varphi}{\Gamma \Rightarrow \Delta, e : (\varphi \vee \neg\varphi)}\ \mathrm{e{:}R}$$

$$\frac{\Gamma \Rightarrow \Delta, t : \varphi}{\Gamma \Rightarrow \Delta, !t : t : \varphi}\ !\mathrm{R}$$

$$\frac{\Gamma \Rightarrow \Delta, s : \varphi}{\Gamma \Rightarrow \Delta, (s + t) : \varphi}\ +\mathrm{R1}$$

$$\frac{\Gamma \Rightarrow \Delta, t : \varphi}{\Gamma \Rightarrow \Delta, (s + t) : \varphi}\ +\mathrm{R2}$$

$$\frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, t : \varphi \Rightarrow \Delta}\ {:}\mathrm{L}$$

$$\frac{\Gamma, t : \varphi, t : \psi \Rightarrow \Delta}{\Gamma, t : (\varphi \wedge \psi) \Rightarrow \Delta}\ : \wedge\mathrm{L}$$

$$\frac{\Gamma \Rightarrow \Delta, t : \varphi \qquad \Gamma \Rightarrow \Delta, t : \psi}{\Gamma \Rightarrow \Delta, t : (\varphi \wedge \psi)}\ : \wedge\mathrm{R}$$

$$\frac{\Gamma \Rightarrow \Delta, t : \varphi, \psi}{\Gamma \Rightarrow \Delta, \partial t : (\varphi \vee \psi)}\ \partial\mathrm{R1}$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi, t : \psi}{\Gamma \Rightarrow \Delta, \partial t : (\varphi \vee \psi)}\ \partial\mathrm{R2}$$

$$\frac{\Gamma, s : (\varphi \vdash \theta), t : (\psi \vdash \theta) \Rightarrow \Delta, u : (\varphi \vee \psi)}{\Gamma, s : (\varphi \vdash \theta), t : (\psi \vdash \theta) \Rightarrow \blacktriangledown(u, s, t) : \theta}\ \blacktriangledown\mathrm{R}$$

$$\frac{\Gamma, t : \varphi \Rightarrow \Delta}{\Gamma, \mathfrak{E}t : \psi \Rightarrow \Delta}\ \mathfrak{E}\mathrm{L}$$

$$\frac{\Gamma \Rightarrow \Delta, t:\varphi}{\Gamma \Rightarrow \Delta, \mathfrak{E}t:\psi}\ \mathfrak{E}\mathrm{R}$$

Restriction: φ and ψ must be double negation or DeMorgan equivalents in any application of the $\mathfrak{E}$ rules.

$$\frac{\Gamma \Rightarrow \Delta}{\Rightarrow c:(\Gamma \vdash \Delta)}\ {:}{\vdash}\mathrm{R}$$

Restriction: Γ and Δ must not contain any instances of $\vdash$, and the constant c must be such that $\mathcal{C}(c, \Gamma \vdash \Delta)$ for a provability-appropriate constant specification $\mathcal{C}$.

Theorem 4.3.11 (Soundness and Completeness). *A sequent $\Gamma \Rightarrow \varphi$ is derivable in the above sequent calculus iff the consequence $\Gamma \vDash \varphi$ holds in the sense of [factive] $\mathcal{C}$-validity in tLPJT4 for a provability-appropriate constant specification $\mathcal{C}$.*

Soundness and completeness for the LP calculus are already known from Beall [16] and Palau and Oller [78]. The modal rules are structurally quite similar to those used in the normal justification logic sequent calculi of Artemov [4, 5]; Artemov proves soundness and completeness there with respect to an axiomatic system rather than a model theory, but it is not difficult to fit his techniques into the model-theoretic completeness proof. Note that a rule of the form $\frac{\Gamma \Rightarrow \Delta}{\Rightarrow \Gamma \vdash \Delta}$, which corresponds to the model's factively meeting the provability-appropriate constant specification, is easily shown to be admissible in the sequent calculus via identity, :L, :⊢R, and cut.

At first glance, this sequent calculus appears fairly useless. It has an inconveniently large collection of rules. Worse, it shares the property of the Artemov sequent calculi for normal justification logics that even if cut were eliminable,[9] the subformula property does not hold for cut-free sequent derivations, as the justification rules

[9] In fact, cut is not eliminable for this sequent calculus. As in Proposition 2.1.5, the present version of the ▼R rule generates a counterexample to the weak substitutivity criterion of Ciabattoni and Terui [28]. An earlier draft of this chapter used a three-premise rule

(specifically ▼R in this calculus, or ·R in the normal calculi) have cut-like behavior built into them. However, the sequent calculus does inherit one useful property from Artemov's normal justification calculi; both may be used to provide simple proofs of the realization theorem.

Lemma 4.3.12. *A sequent calculus for the modal logic corresponding to tLPJT4 can be produced by replacing all of the justification rules in the tLPJT4 sequent calculus with the following modal rules:*

$$\frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \Box\varphi \Rightarrow \Delta}\,\Box L$$

$$\frac{\Box\Gamma \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi}\,\Box R$$

As stated, this "lemma" is really more of a definition. The thing-to-prove aspect which makes it a lemma is that the specified calculus is sound and complete with respect to the desired modal logic. That proof is a simple corollary of the soundness and completeness proof for the LP sequent calculus, and so shall be omitted.

Theorem 4.3.13 (Realization)**.** *If a sequent* $\Gamma \Rightarrow \Delta$ *is derivable in the sequent calculus of the above lemma, then there is a sequent* $\Gamma' \Rightarrow \Delta'$ *derivable in the sequent calculus of tLPJT4 such that the forgetful projections of* Γ' *and* Δ' *are* Γ *and* Δ *respectively.*

Proof. To prove this theorem, we build an algorithm for converting modal derivations into tLPJT4 derivations. Start by going through the derivation from top to bottom.

$\dfrac{\Gamma \Rightarrow \Delta, t:(\varphi \vee \psi) \qquad \varphi \Rightarrow \theta \qquad \psi \Rightarrow \theta}{\Gamma \Rightarrow \blacktriangledown(t, c_1, c_2):\theta}$ instead of the current ▼R rule. That alternative would have allowed cut-elimination, but at the cost of producing a calculus that was only weakly complete with respect to the model theory.

LP rules can be left alone, as they are common to both systems. $\Box$L rules can be translated as arbitrary :L instances, and likewise arbitrary proof polynomials can be used to turn any $\Box$ formulae introduced by identity or weakening into justification formulae. For $\Box$R rules, examine the form of the premise sequent. There will be six possibilities for the nature of the formula φ:

1. $\varphi \in \Gamma$.[10]

2. φ is a modal formula.

3. φ is an LP theorem (and thus Γ is either empty or consists of irrelevancies such as formulae which were introduced by weakening).

4. φ is a conjunction of modal formulae.

5. φ is a mixed conjunction of a modal formula with either an LP theorem or a formula contained in Γ.

6. φ is a disjunction of a modal formula with an arbitrary formula

In case 1, the use of $\Box$R in the original derivation was entirely redundant, as the conclusion is an identity axiom. To translate this case, discard the entire derivation above the $\Box$R rule, introduce the $\Box$R conclusion as an identity instance, and then translate all $\Box$ formulae into justification formulae using arbitrary proof polynomials.

Case 2 can simply be translated as a !R rule.

For case 4 conjunctions, note that these must have been formed earlier in the proof by applying the $\wedge$R rule to two modal formulae. For the translation, remove that

[10] A variant of this case where $\varphi \notin \Gamma$ but $\Gamma \vDash_{LP} \varphi$ is also possible. Treat that version as if it were an instance of the LP theorem case, by producing and translating a natural deduction proof. The only change is that arbitrary justifications of formulae from Γ that are used as premises in the natural deduction proof should be introduced as identity sequents.

initial $\wedge$R step. Then apply !R rules to both premises, and use +R rules as needed to make the outermost proof polynomials match. Once this match is obtained, the premises can be recombined using the :$\wedge$R rule to get the desired derivation.

For case 6, note as in case 4 that the disjunction must have been created using a right-introduction rule; in this case, $\vee$R. Remove the $\vee$R step, translate $\Box$R by a !R rule, and then restore the disjunction by following the !R with a ∂R rule.

This leaves the LP theorem cases. For the simple theorem of case 3, begin by obtaining a natural deduction proof of φ. That proof can then be turned directly into a sequent calculus derivation. First, identify any subproofs that are used as minor premises in $\vee$-elimination steps. For these subproofs, write LP sequent derivations of the conclusions, apply $\vdash$R, and cut the result with the result of the $\blacktriangledown$R rule when that step occurs in the main proof. For the remaining portions of the natural deduction proof, start with sequent derivations of all the excluded middle instances that are used as axioms, and apply the e:R rule to each. For $\wedge$-introduction steps, apply +R and :$\wedge$R as in the modal conjunction case above. For $\vee$-introduction steps, apply ∂R. For $\wedge$-elimination steps, create a suitable second sequent premise using duplicates of earlier steps along with structural rules and :$\wedge$L, then apply cut to eliminate the conjunction. As mentioned above, $\vee$-elimination steps are handled by simply applying the $\blacktriangledown$R rule, and then cutting with the subproofs that were prepared earlier. Double-negation and DeMorgan steps (in either direction) are translated by $\mathfrak{C}$R. Finally, if Γ was nonempty in the $\Box$R step of the original derivation, translate all of the formulae in $\Box\Gamma$ using arbitrary proof polynomials, and then add the resulting formulae to the new derivation by weakening.

The mixed conjunction cases will, like the purely modal conjunctions, be produced by $\wedge$R applications earlier in the proof. Remove that step to produce separate derivations of a modal theorem and a formula that is either an LP theorem or a member of Γ. Apply !R to the former and the relevant algorithm from the preceding paragraphs to the latter. Then use +R to make their proof polynomials match, and get the desired derivation by :∧R. As □L and □R are the only rules requiring translation, the procedures noted will successfully translate every modal sequent derivation into a tLPJT4 derivation with the property indicated in the theorem statement, Q.E.D. ■

The converse of realization is often referred to as the projection theorem. It holds for all of the logics described in this chapter. This fact is most easily proven by simply noting that if you remove the evidence function from each of the fusion or trivalent Fitting model theories presented, the result is a model of the corresponding modal logic.

4.3.3 Philosophical Applications of the LP-based Systems

The initial task that motivated our investigation into non-applicative justification logics was to devise a suitable logic of knowledge (i.e., of justified true belief) for those like Beall who hold that the logic of truth is LP. Assuming an epistemology whose logic of knowledge would be JT4 if the logic of truth were classical, the system tLPJT4 ideally suits this purpose. The total LP-basis accords with the view that the non-modal property of truth is dialethic. The ability to prove both projection and realization is also important, as it confirms that a theory of knowledge employing tLPJT4 agrees with theories that employ a simpler modal account of knowledge, while still gaining the benefits of justification logic that were discussed in Chapter 1.

Su's philosophical project that motivated him to develop PJ_F was to develop an account of sound reasoning from inconsistent beliefs under the assumption that the logic of truth is classical. For this purpose, a partially LP-based system (probably not pLPJT4, but rather the J-analogue) might be just as suitable as PJ_F. Indeed, Su indicated to me in correspondence that he has considered employing such a system, but has not formally developed one as I have done in this chapter. The partially LP-based justification logics can likewise be employed to model any other phenomena of interest for which the application property of a normal justification (or modal) logic is undesirable, though I am not presently aware of any such cases in the literature. If there is a need to model a phenomenon which requires a logic that is non-normal in virtue of lacking justification as well as application, one approach is to take the trivalent Fitting models, and replace the normal modal frame in the model with the partitioned frame structure of some target non-necessitative modal logic, as was discussed in Chapter 2. Another way to obtain both types of non-normality is to change the base logic from LP to FDE, which will be discussed briefly in the next chapter.

The systems pLPJT4 and tLPJT4 were developed as JT4 analogues for two reasons. First, JT4 is one of the normal justification logics that is most often applied as a logic of knowledge (the other, of course, is JT). Second, some of the formal results are easier to rigorously prove for that system than for a system lacking a 4-analogue property. Nonetheless, it is feasible to obtain analogues of the less-restrictive normal justification logics. To eliminate the T-analogue property, yielding p/tLPJ4, simply remove the reflexivity requirement from the (trivalent and fusion) Fitting models, the requirement that when $E(t, \varphi)$, $v(\varphi) \neq 0$ from the Mkrtychev models, and the :L rule from the sequent calculus. However, in the sequent calculus, the rule $\dfrac{\Gamma \Rightarrow \Delta}{\Rightarrow \Gamma \vdash \Delta}$ is still required to represent factive $\mathcal{C}$-validity, and so it must be added back into the calculus as a new basic rule, as it is no longer admissible in the absence of :L.

Eliminating the 4-analogue property from the logics, yielding p/tLPJT, is somewhat harder. In a normal justification logic, doing this would require a switch from simple to iterated axiom necessitation. In p/tLPJT, there is no axiom necessitation rule, but it is desirable to retain a similar iterative property for justifications of valid formulae in general. In the (trivalent and fusion) Fitting models, this can be accomplished by eliminating the transitivity requirement on the relation R, while retaining the clause defining the ! operator in $\mathcal{E}$. Likewise, in the sequent calculus, we can simply restrict the !R rule by banning side-formulae (i.e., requiring that Γ and Δ be empty). However, there is no way to incorporate an analogous change into a Mkrtychev model of tLPJT.[11] We would thus be required to use Fitting models of tLPJT instead of Mkrtychev models, which in turn would make the proof of soundness and completeness for the sequent calculus much more complicated. Finally, the systems pLPJ and tLPJ can be produced by adopting all of the changes that were indicated for the p/tLPJ4 and p/tLPJT cases.

[11] In Chapter 1, we could use ! instead of iterated axiom necessitation in Mkrtychev models of J and JT by restricting the ! clause to apply to axioms only, but that move is not available in this context, as there are no axioms.

Chapter 5

Paracomplete Justification Logic

5.1 Introduction

In the previous two chapters, we have examined several variants of paraconsistent justification logic. Our motivations for doing so consisted of the arguments from Chapter 1 that knowledge (qua justified belief) is perspicuously modeled by a justification logic, combined with arguments for paraconsistency based on the existence of semantic paradoxes (in both alethic and directly epistemic forms). However, these motivations do not exclusively indicate the use of paraconsistent justification logic.

A possible alternative to paraconsistency is the adoption of paracomplete logic. A paracomplete logic is one in which it is permissible that a sentence lack a truth value (or equivalently, a third value is provided with the interpretation being its complete distinctness from truth and falsity, rather than including both truth and falsity as in the paraconsistent case). Having this option available completely solves paradoxical cases such as "This sentence is false," as the sentence's lack of either of the standard truth values blocks our ability to infer that it must paradoxically have both of them.

The standard objection to this proposal is the revenge problem: when the liar is phrased as "This sentence is not true," and we claim that it is neither true nor false, this is a way of not being true, so intuitively the liar should then be true, leading back into paradox. The only response to this is to step back from logic and rely on whatever philosophical interpretation one gives to the situation of being neither true nor false. The reasoning would be something along the lines of paradoxical sentences being so defective that it never makes sense to consider them as candidates for having a classical truth value, and so it is inappropriate to reason from the sentence being assigned the non-classical value to its not being true and then to its being true.

This defense appears to be adequate for liar-like paradoxes. However, it is not appropriate for skeptical paradoxes. Recall that skeptical paradoxes are contradictions that arise between our everyday knowledge claims and the logical consequences of our inability to refute skeptical scenarios. If one were to judge these contradictions to be neither true or false, this would mean that everyday knowledge claims likewise lack classical truth values. This would effectively be a concession that the skeptics are correct, given that ordinary knowledge claims would not be true.

One option that remains available to the non-skeptical epistemologist is to adopt a paracomplete solution only to liar-like paradoxes, and to resolve skeptical paradoxes in a classical manner, such as by denying closure. This technically provides a motivation for paracomplete epistemic logic, as liar-like sentences are as much a potential object of knowledge as any others (as evinced by the epistemic liar). However, it's a weak motivation. If knowledge itself is treated classically, there is no good reason not to simply set alethic paradoxes aside and work in classical epistemic logics.

A better motivation for paracomplete justification logic comes from one of the standard objections to the project of epistemic logic as a whole, the problem of logical omniscience. Any normal modal or justification logic has the property that $\Box\varphi$ holds for all tautologies φ. However, intuitively, there are logical truths that are not known, because nobody has ever performed the calculations needed to discover them. If we are modeling the knowledge of a single agent, the situation is even worse, as the agent might not have any knowledge of logic at all. Clearly, then, the normal modal and justification logics are not felicitous models of knowledge.

One possible response is to adopt non-normal systems, as discussed in Chapter 2. Those who wish to continue working in normal modal or justification logics typically argue that any agent is in a position to know all logical truths, but might simply not have done the work needed to make that knowledge accessible. How should this sort of potential knowledge be analyzed? It is not really felicitous to say the the agent has knowledge, nor that the agent lacks knowledge. The agent seems to be in an epistemic state analogous to the third truth value of a paracomplete logic. In particular, the logic K_3, which we are about to introduce, represents this situation perfectly by behaving in a maximally classical fashion while still permitting that any proposition whatsoever (including classical tautologies) might receive the third value rather than either of the classical values.

5.2 Paracomplete Base Systems K_3 and $Ł_3$, With Their Modal Extensions

Kleene [59] devised truth tables corresponding to two different paracomplete logics. We will attend directly to only one of them, the “strong” system, which produces a logic that most current literature denotes as K_3. (Kleene’s weak system is also of philosophical interest, but its extensions can easily be recovered by altering the model theories of the K_3 extensions that will be presented in this chapter.) Viewed as a truth-table, K_3 is identical to the paraconsistent system LP that was discussed in previous chapters; the difference is that the third value is regarded as excluding (rather than including) both truth and falsity, and so is not incorporated in the definition of satisfaction. As was the case with LP, a modal extension of K_3 has been provided by Priest [83],[1] and we shall proceed directly to that system rather than presenting the propositional base by itself.

Definition 5.2.1. Models of the modal system K_3K consist of a non-empty set of worlds $\mathcal{W}$, a relation R on $\mathcal{W}$, and a valuation function v from pairs of worlds and sentences to the set of truth values $\{0, \frac{1}{2}, 1\}$ which satisfies the following principles:

- $v_w(\varphi \vee \psi) = \max(v_w(\varphi), v_w(\psi))$.
- $v_w(\varphi \wedge \psi) = \min(v_w(\varphi), v_w(\psi))$.
- $v_w(\neg\varphi) = \begin{cases} 0, \text{if } v_w(\varphi) = 1; \\ \frac{1}{2}, \text{if } v_w(\varphi) = \frac{1}{2}; \\ 1, \text{if } v_w(\varphi) = 0. \end{cases}$
- $v_w(\Box\varphi) = \begin{cases} 0, \text{if } \exists w' \in \mathcal{W} \text{ such that } R(w, w') \text{ and } v_{w'}(\varphi) = 0; \\ 1, \text{if } \forall w' \in \mathcal{W} \text{ such that } R(w, w'),\ v_{w'}(\varphi) = 1; \\ \frac{1}{2}, \text{otherwise.} \end{cases}$

Definition 5.2.2. An K_3K model $\mathcal{M} = \langle \mathcal{W}, R, v \rangle$ satisfies a proposition φ (written $\mathcal{M} \vDash \varphi$) iff $\forall w \in \mathcal{W}$, $v_w(\varphi) = 1$. A proposition φ is valid (written $\models \varphi$) iff it is

[1] Though unlike with the LP case, Priest cannot claim any credit for the system.

satisfied by all LPK models.

As usual, we extend K_3K to the other normal modal logics by imposing appropriate restrictions on R. Another similarity to the paraconsistent modal logics that we discussed in previous chapters is the alternative bivalent semantics for modality, produced by replacing the clause for the modal operator with

$$v_w(\Box\varphi) = \begin{cases} 1, \text{if } \forall w' \in \mathcal{W} \text{ such that } R(w,w'),\ v_{w'}(\varphi) = 1; \\ 0, \text{otherwise.} \end{cases}$$

An important property of K_3 (and its modal extensions) is that it has no theorems at all, and thus cannot be presented as an axiomatic system. (A counterexample to any putative theorem is a model consisting of a single world in which all sentences are assigned the value $\frac{1}{2}$.) Without an axiomatic system, we cannot produce a standard axiomatic justification logic. Later we will handle this in the manner of Chapter 4, but for now, let us consider an axiomatizable paracomplete system.

The simplest axiomatizable paracomplete logic is the three-valued system of Łukasiewicz [69]. This system can be produced by extending K_3 with a primitive conditional, in exactly the way that $LP^{\rightarrow}$ and RM_3 are produced from LP. Specifically, the semantic clause for the $Ł_3$ conditional is

$$v_w(\varphi \rightarrow \psi) = \begin{cases} 0, \text{if } v_w(\varphi) = 1 \text{ and } v_w(\psi) = 0; \\ \frac{1}{2}, \text{if } v_w(\varphi) = 1 \text{ and } v_w(\psi) = \frac{1}{2}; \\ \frac{1}{2}, \text{if } v_w(\varphi) = \frac{1}{2} \text{ and } v_w(\psi) = 0; \\ 1, \text{otherwise.} \end{cases}$$

Adding this conditional leads to some surprising results in modal logic. In classical modal logic, and indeed in all of the other nonclassical systems presented in this dissertation, the modal operator is obviously independent of all the truth-functional connectives. In Ł$_3$, this is not so; Łukasiewicz [69] argued that a modal $\Diamond$ operator can be defined by $\neg\varphi \to \varphi$, which maps values 1 and $\frac{1}{2}$ to 1, the concept being that the third value is to be interpreted as an unactualized possibility, while value 0 is reserved for necessary falsehoods. But although this procedure allows an adequate definition of a modal operator (for at least some purposes), there is no way to define an analogous justification operator. Therefore, for our purposes we must rule out this sort of modality, instead following the example of Schotch *et al.* [86] in modalizing Ł$_3$ by adding primitive modal operators exactly the way as was done for K$_3$ and for all of the other logics that we have discussed.

The relationship between the Łukasiewicz conditional and disjunction is also somewhat unusual. When classical logic is presented using only the negation and conditional, it is customary to define disjunction by $\varphi \vee_c \psi = \neg\varphi \to \psi$. Łukasiewicz does not discuss disjunction at all in [69] (the paper which provides the otherwise most comprehensive account of Łukasiewicz's three-valued logic), but in an earlier paper, [68], Łukasiewicz defines a disjunction by $\varphi \vee_{\text{Ł}} \psi = (\varphi \to \psi) \to \psi$. In the context of classical logic, these two definitions of disjunction are equivalent, but when the $\to$ connective is the Ł$_3$ conditional, they are not. In writing out the truth-tables for these two defined connectives, one will find that they differ in a single case. Specifically, when the inputs φ and ψ both have value $\frac{1}{2}$, the formula $\varphi \vee_c \psi$ gets value 1, whereas $\varphi \vee_{\text{Ł}} \psi$ gets value $\frac{1}{2}$. In the model-theoretic presentation of Ł$_3$ given above, $\varphi \vee \psi$ is assigned value $\frac{1}{2}$ in this case, so in axiomatizing the Ł$_3$ model-theory, we must use $\vee = \vee_{\text{Ł}}$.

Another important consequence of the $Ł_3$ conditional is that it allows an external negation $\sim$ to be defined by $\sim\varphi = \varphi \rightarrow \neg\varphi$. We saw in Chapter 3 how the presence of such a negation complicated the axiomatic system for PJ_b. In the present case, the external negation causes little trouble in the propositional case, but more in the modal extension. We begin with the propositional axiomatic system, due to Wajsberg [100]:

Definition 5.2.3. The logic $Ł_3$ consists of the closure of the following axiom schemas under modus ponens:

1. $\varphi \rightarrow (\psi \rightarrow \varphi)$
2. $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \theta) \rightarrow (\varphi \rightarrow \theta))$
3. $(\sim\varphi \rightarrow \varphi) \rightarrow \varphi$
4. $(\neg\varphi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \varphi)$

Wajsberg's axiomatization presents the system $Ł_3$ using only the negation and conditional. To apply this axiomatization for the full system $Ł_3$, we must define disjunction by the $\vee_{Ł}$ definition given above. We can then define conjunction by $\varphi \wedge \psi = \neg(\neg\varphi \vee \neg\psi)$ as usual, and likewise define the biconditional by $\varphi \leftrightarrow \psi = (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$.

Next, we must extend this $Ł_3$ axiomatization to axiomatize modal and justification systems. One would normally expect to get an axiomatization for $Ł_3$K from the corrected propositional system by adding the K axiom and the necessitation rule. However, Schotch *et al.* [86] show that this is insufficient. The exact requirements for axiomatic $Ł_3$K depend on whether one intends to employ the bivalent semantics for modality or the fully paracomplete semantics. In the bivalent case, the only additional axiom needed is $\neg(\Box\varphi \leftrightarrow \neg\Box\varphi)$, which is an unproblematic statement of bivalence. The fully paracomplete semantics, however, requires two additional axioms with less intuitive justification. The axioms in question are $\neg\Box\varphi \rightarrow (\Box(\neg\varphi \rightarrow \varphi) \rightarrow \Box\varphi)$ and $\sim\neg\Box\varphi \rightarrow \Box\sim\neg\varphi$.

5.3 Justification Extensions of Ł$_3$

As in the cases of other logics that we have discussed previously, we could adopt a justification analogue of fully paracomplete Ł$_3$K, but the additional axioms found in that system will require us to define new operations on the proof polynomials, and to give adequate interpretations of them. These challenges are not insurmountable, but neither do they seem to be worthwhile to pursue. Instead, let us work from the bivalent semantics, which requires no such mechanism.

For the sake of clarity, let us begin by setting out the resulting justification logic precisely, though the details should by now be familiar to the reader.

Definition 5.3.1. The logic Ł$_3$J consists of the closure of the following axiom schemas under modus ponens and iterated axiom justification:

1. $\varphi \rightarrow (\psi \rightarrow \varphi)$

2. $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \theta) \rightarrow (\varphi \rightarrow \theta))$

3. $(\sim\varphi \rightarrow \varphi) \rightarrow \varphi$

4. $(\neg\varphi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \varphi)$

5. $s : (\varphi \to \psi) \to (t : \varphi \to (s \cdot t) : \psi)$

6. $s : \varphi \to (s + t) : \varphi$

7. $t : \varphi \to (s + t) : \varphi$

8. $\neg(t : \varphi \leftrightarrow \neg t : \varphi)$

Definition 5.3.2. A Fitting model of the logic Ł$_3$J consists of a non-empty set of worlds $\mathcal{W}$, a relation R on $\mathcal{W}$, an evidence function $\mathcal{E}$, which maps pairs of worlds and proof polynomials to sets of sentences, and a valuation function v from pairs of worlds and sentences to the set of truth values $\{0, \frac{1}{2}, 1\}$, all such that the following principles are satisfied:

- If φ is an axiom of the axiomatic presentation of Ł$_3$J, then there is a sequence of proof constants c_i such that $\varphi \in \mathcal{E}(w, c_0)$, $c_0 : \varphi \in \mathcal{E}(w, c_1)$, $c_1 : c_0 : \varphi \in \mathcal{E}(w, c_2)$, and so forth.

- If $(\varphi \to \psi) \in \mathcal{E}(w, s)$ and $\varphi \in \mathcal{E}(w, t)$, then $\psi \in \mathcal{E}(w, s \cdot t)$.

- $\mathcal{E}(w, s) \cup \mathcal{E}(w, t) \subseteq \mathcal{E}(w, s + t)$.

- $v_w(\varphi \vee \psi) = \max(v_w(\varphi), v_w(\psi))$.

- $v_w(\varphi \wedge \psi) = \min(v_w(\varphi), v_w(\psi))$.

- $v_w(\neg\varphi) = \begin{cases} 0, \text{if } v_w(\varphi) = 1; \\ \frac{1}{2}, \text{if } v_w(\varphi) = \frac{1}{2}; \\ 1, \text{if } v_w(\varphi) = 0. \end{cases}$

- $v_w(\varphi \to \psi) = \begin{cases} 0, \text{if } v_w(\varphi) = 1 \text{ and } v_w(\psi) = 0; \\ \frac{1}{2}, \text{if } v_w(\varphi) = 1 \text{ and } v_w(\psi) = \frac{1}{2}; \\ \frac{1}{2}, \text{if } v_w(\varphi) = \frac{1}{2} \text{ and } v_w(\psi) = 0; \\ 1, \text{otherwise.} \end{cases}$
- $v_w(t : \varphi) = \begin{cases} 1, \text{if } \varphi \in \mathcal{E}(w, t) \text{ and for every } w' \text{ such that } R(w, w'),\ v_{w'}(\varphi) = 1; \\ 0, \text{otherwise.} \end{cases}$

As is usual for a Fitting model, we employ the notions of meeting a constant specification and of consequence from Definition 1.5.7. The soundness and completeness result for the Ł$_3$J axiomatic system and Fitting model theory is a routine adaptation of the one given for the corresponding modal systems in [86]. Likewise, extending to the other normal justification logics is simply a matter of adding the usual axioms to the axiomatic system, adding the usual restrictions on R and $\mathcal{E}$ to the model theory, and possibly switching from using multi-constant specifications and iterated axiom justification to constant specifications and simple axiom justification.

The next step is to construct a sequent calculus. As with the paraconsistent systems, the procedure will be to extend the Ł$_3$ sequent calculus of Avron [13] with justification rules.

Theorem 5.3.3. *The following sequent calculus is sound and complete with respect to the logic Ł$_3$J:*

Begin with unrestricted forms of the structural rules of identity, weakening, contraction, and cut. Additionally include the following rules:

$$\frac{}{\Gamma, p, \neg p \Rightarrow \Delta}\ \textit{Explosion}$$

$$\frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta} \wedge L$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi} \wedge R$$

$$\frac{\Gamma, \neg\varphi \vee \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \wedge \psi) \Rightarrow \Delta} \neg\wedge L$$

$$\frac{\Gamma \Rightarrow \neg\varphi \vee \neg\psi, \Delta}{\Gamma \Rightarrow \neg(\varphi \wedge \psi), \Delta} \neg\wedge R$$

$$\frac{\Gamma, \varphi \Rightarrow \Delta \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta} \vee L$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi} \vee R$$

$$\frac{\Gamma, \neg\varphi \wedge \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \vee \psi) \Rightarrow \Delta} \neg\vee L$$

$$\frac{\Gamma \Rightarrow \neg\varphi \wedge \neg\psi, \Delta}{\Gamma \Rightarrow \neg(\varphi \vee \psi), \Delta} \neg\vee R$$

$$\frac{\Gamma, \neg\varphi \Rightarrow \Delta \qquad \Gamma, \psi \Rightarrow \Delta \qquad \Gamma \Rightarrow \Delta, \varphi, \neg\psi}{\Gamma, \varphi \rightarrow \psi \Rightarrow \Delta} \rightarrow L$$

$$\frac{\Gamma, \varphi \Rightarrow \Delta, \psi \qquad \Gamma, \neg\psi \Rightarrow \Delta, \neg\varphi}{\Gamma \Rightarrow \Delta, \varphi \rightarrow \psi} \rightarrow R$$

$$\frac{\Gamma, \varphi, \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \rightarrow \psi) \Rightarrow \Delta} \neg \rightarrow L$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma \Rightarrow \Delta, \neg\psi}{\Gamma \Rightarrow \Delta, \neg(\varphi \rightarrow \psi)} \neg \rightarrow R$$

$$\frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \neg\neg\varphi \Rightarrow \Delta} \neg\neg L$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \neg\neg\varphi} \neg\neg R$$

$$\frac{}{\Gamma \Rightarrow \Delta, t : \varphi, \neg t : \varphi} \textit{:Biv}$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, c_n : \cdots : c_0 : \varphi} \textit{::R}$$

Restriction: The formula φ used in any application of ::R must be an instance of one of the axiom schemas of the axiomatic presentation of Ł$_3$J given above, and $c_0, \ldots, c_n$ an initial fragment of a sequence $\vec{c}$ such that $\mathcal{C}(\vec{c}, \varphi)$.

$$\frac{\Gamma \Rightarrow \Delta, s : \varphi}{\Gamma \Rightarrow \Delta, (s + t) : \varphi} \textit{+R1}$$

$$\frac{\Gamma \Rightarrow \Delta, t : \varphi}{\Gamma \Rightarrow \Delta, (s + t) : \varphi} \textit{+R2}$$

$$\frac{\Gamma \Rightarrow \Delta, s : (\varphi \to \psi) \qquad \Gamma \Rightarrow \Delta, t : \varphi}{\Gamma \Rightarrow \Delta, (s \cdot t) : \psi} \cdot R$$

$$\frac{\varphi, \neg\psi \Rightarrow \neg(\varphi \to \psi)}{t : \varphi, \neg(s \cdot t) : \psi \Rightarrow \neg s : (\varphi \to \psi)} \neg\cdot L$$

$$\frac{\Gamma, \neg s : \varphi \Rightarrow \Delta}{\Gamma, \neg(s + t) : \varphi \Rightarrow \Delta} \neg\textit{+L1}$$

$$\frac{\Gamma, \neg t : \varphi \Rightarrow \Delta}{\Gamma, \neg(s + t) : \varphi \Rightarrow \Delta} \neg\textit{+L2}$$

For Ł$_3$JT, add the additional rules:

$$\frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, t : \varphi \Rightarrow \Delta} \textit{:L}$$

$$\frac{\Gamma \Rightarrow \Delta, \neg\varphi}{\Gamma \Rightarrow \Delta, \neg t : \varphi} \neg : R$$

For Ł$_3$JT4, add the above rules and the following:

$$\frac{\Gamma \Rightarrow \Delta, t : \varphi}{\Gamma \Rightarrow \Delta, !t : t : \varphi} \textit{!R}$$

$$\frac{\Gamma, \neg t : \varphi \Rightarrow \Delta}{\Gamma, \neg !t : t : \varphi \Rightarrow \Delta} \neg ! L$$

Additionally, if simple axiom justification is employed in the axiomatic system, replace ::R with the simpler :R rule presented in Chapter 1 in the sequent calculus for Ł$_3$*JT4.*

Proof. Most of the rules in this calculus are identical to those used in the PJ_b and RM_3J calculi in Chapter 3, so there is little new material to discuss. Because the current system is paracomplete rather than paraconsistent, exhaustion is replaced by explosion, and the :Biv rule is likewise flipped. As in RM_3J, the conditional contraposes, but that contraposition is presupposed in the introduction rules of the calculus, so it is necessary to include the negative forms of the justification sequent rules in order to be able to prove the conditionals of the justification axiom schemas.

The sequent derivations of the propositional base axioms, we can take as given, considering Avron's result. For the justification axioms, the derivations are only slightly harder than in the classical and PJ_b cases; instead of simply invoking the corresponding sequent rule and following it with ⊃R, we are now using the positive and negative justification sequent rules as premises for →R (formalizing what we did implicitly in Chapter 3 for the justification extensions of RM_3). The bivalence axiom $\neg(t : \varphi \leftrightarrow \neg t : \varphi)$ requires a slightly more complicated proof:

$$\dfrac{\dfrac{\dfrac{}{\Rightarrow t : \varphi, \neg t : \varphi}\text{:Biv} \quad \dfrac{}{\Rightarrow t : \varphi, \neg t : \varphi}\text{:Biv}}{\Rightarrow t : \varphi, \neg(t : \varphi \to \neg t : \varphi)} \neg \to R \quad \dfrac{\text{Symmetric reasoning and } \neg\neg\text{R}}{\Rightarrow \neg\neg t : \varphi, \neg(t : \varphi \to \neg t : \varphi)}}{\dfrac{\dfrac{\dfrac{\Rightarrow \neg(t : \varphi \to \neg t : \varphi), \neg(\neg t : \varphi \to t : \varphi)}{\Rightarrow \neg(t : \varphi \to \neg t : \varphi) \vee \neg(\neg t : \varphi \to t : \varphi)} \vee\text{R}}{\Rightarrow \neg((t : \varphi \to \neg t : \varphi) \wedge (\neg t : \varphi \to t : \varphi))} \neg\wedge\text{R}}{\neg(t : \varphi \leftrightarrow \neg t : \varphi)} \text{Definition of} \leftrightarrow} \neg \to R$$

In the other direction, verifying the validity of the sequent rules, we have the propositional base from Avron, and the justification rules are utterly routine to verify, especially as we can pass to the model theory to deal with :Biv. ■

The similarity of this system to RM_3J also means that the proofs of projection and realization are essentially identical to those given in Chapter 3, so they needn't be repeated here.

5.4 Justification Extensions of K_3

Just as we arrived at justification extensions of $Ł_3$ by techniques analogous to those of Chapter 3, we can obtain extensions of K_3 by techniques analogous to those of Chapter 4. In the absence of an axiomatic system, the starting point is to incorporate a natural deduction system into the structure of the proof polynomials. Tamminga [96] presents a natural deduction formulation of K_3, consisting of introduction and elimination rules for $\vee$ and $\wedge$, double negation and DeMorgan equivalences, and *ex falso quodlibet* (henceforth, EFQ). Note that this is identical to the natural deduction formulation of LP, except that LP's excluded middle is here replaced with EFQ. Thus, we retain the conventions of Chapter 4 regarding the form of proof polynomials: the conjunction rules are represented transparently, using the $+$ operator when needed; the equivalence rules are represented by a unary operator $\mathfrak{E}$; and disjunction introduction and elimination are represented respectively by unary ∂ and ternary $\blacktriangledown$, the latter of which requires as inputs polynomials proving the special atomic propositions $\varphi \vdash \psi$. The only new element is the EFQ rule, which will be represented with a binary operator $\mathfrak{F}$.

With these operators in place, we can establish model theories. In Chapter 4, we employed two different model theories for the LP extensions: fusion models and trivalent models. For simplicity, we will work exclusively in trivalent model theory here.

Definition 5.4.1. A trivalent Fitting model of the system pK_3JT or tK_3JT is a structure $\langle W, R, @, \mathcal{E}, V\rangle$, where W is a non-empty set of worlds; R is a reflexive relation on W; $@ \in W$; $\mathcal{E}$ is a function from W and the set of proof polynomials to the power set of the set of sentences; and V is a function from W and the set of sentences to $\{0, \frac{1}{2}, 1\}$, such that the following restrictions hold:

- $(\mathcal{E}(w, s) \cup \mathcal{E}(w, t)) \subseteq \mathcal{E}(w, s + t)$.
- For all sentences φ, ψ, proof polynomials s, t, and $w \in W$, if $\varphi \in \mathcal{E}(w, s)$ and $\neg\varphi \in \mathcal{E}(w, t)$, then $\psi \in \mathcal{E}(w, \mathfrak{F}(s, t))$.
- For all sentences φ, ψ, proof polynomials t, and $w \in W$, $(\varphi \wedge \psi) \in \mathcal{E}(w, t)$ iff $\varphi \in \mathcal{E}(w, t)$ and $\psi \in \mathcal{E}(w, t)$.
- For all sentences φ, ψ, proof polynomials t, and $w \in W$, if ψ is a double negation or DeMorgan equivalent of φ and $\varphi \in \mathcal{E}(w, t)$, then $\psi \in \mathcal{E}(w, \mathfrak{E}t)$.
- For all sentences φ, ψ, proof polynomials t, and $w \in W$, if $\varphi \in \mathcal{E}(w, t)$, then $(\varphi \vee \psi) \in \mathcal{E}(w, \partial t)$.
- For all sentences φ, ψ, proof polynomials t, and $w \in W$, if $\psi \in \mathcal{E}(w, t)$, then $(\varphi \vee \psi) \in \mathcal{E}(w, \partial t)$.
- For all sentences φ, ψ, θ, proof polynomials r, s, t, and $w \in W$, if $(\varphi \vee \psi) \in \mathcal{E}(w, r)$, $(\varphi \vdash \theta) \in \mathcal{E}(w, s)$, and $(\psi \vdash \theta) \in \mathcal{E}(w, t)$, then $\theta \in \mathcal{E}(w, \blacktriangledown(r, s, t))$.
- For all sentences φ, proof polynomials t, and $w \in W$, $\varphi \in \mathcal{E}(w, t)$ iff $(t : \varphi) \in \mathcal{E}(w, !t)$.

- For the system pK_3JT, $V_@(\varphi) \neq \frac{1}{2}$.
- $V_w(\varphi \vee \psi) = \max(v_w(\varphi), v_w(\psi))$.
- $V_w(\varphi \wedge \psi) = \min(v_w(\varphi), v_w(\psi))$.
- $V_w(\neg\varphi) = \begin{cases} 0, \text{if } v_w(\varphi) = 1; \\ \frac{1}{2}, \text{if } v_w(\varphi) = \frac{1}{2}; \\ 1, \text{if } v_w(\varphi) = 0. \end{cases}$
- $V_w(t : \varphi) = 1$ if $\varphi \in \mathcal{E}(w, t)$ and for every $w' \in W$ such that $R(w, w')$, $V_{w'}(\varphi) = 1$. Otherwise, $V_w(t : \varphi) = 0$.

From here, we can proceed in the usual manner: a formula φ is satisfied by a model if $V_@(\varphi) = 1$, it is valid simpliciter if it is satisfied by all models, and [factively] $\mathcal{C}$-valid if it is satisfied by all models which [factively] meet the specification $\mathcal{C}$. With these definitions in hand, we can then perform the same recursive construction as in Chapter 4 to define the class of provability-appropriate constant specifications.

Note the presence of the clause for the ! operator. In previous chapters, we have encountered two uses for this operator. It is used to implement the 4^{j} axiom schema in systems that require it, and it is used in some presentations of Fitting (or Mkrtychev) models as an alternative to the iterated version of axiom justification. In this case, the former function will be needed in the 4-extension, but the latter is not needed at all, as the system lacks any form of axiom justification. However, what does hold in the corresponding modal system is that for any consequence $\Box\Gamma \vDash \Box\varphi$, the iterated form $\Box\Box\Gamma \vDash \Box\Box\varphi$ is also a logical consequence. In order to capture these theorems, the justification system must include some sort of iterative property, and ! seems to be the simplest solution. This particular usage does require the ! clause to be specified as a biconditional rather than the usual conditional, but that does no harm; in all of the other systems that we have examined, these clauses could be strengthened to biconditionals without changing any significant properties of the resulting logics.[2]

For tK_3JT4, as in Chapter 4, it is more convenient to use a trivalent Mkrtychev model in place of a Fitting model:

Definition 5.4.2. A trivalent Mkrtychev model of the system tK_3JT4 is a pair $\langle E, v\rangle$, where E is a relation between the set of proof polynomials and the set of sentences of tK_3JT; v is a function from the set of sentences to $\{0, \frac{1}{2}, 1\}$; and the following restrictions hold:

- For all sentences φ, ψ, and proof polynomials s, t, if $E(s, \varphi)$ and $E(t, \neg\varphi)$, then $E(\mathfrak{F}(s, t), \psi)$.
- For all proof polynomials s, t and sentences φ, if $E(s, \varphi)$ or $E(t, \varphi)$, then $E(s + t, \varphi)$.
- For all sentences φ, ψ, and proof polynomials t, $E(t, \varphi \wedge \psi)$ iff $E(t, \varphi)$ and $E(t, \psi)$.
- For all sentences φ, ψ, and proof polynomials t, if ψ is a double-negation or DeMorgan equivalent of φ and $E(t, \varphi)$, then $E(\mathfrak{C}t, \psi)$.
- For all sentences φ, ψ, and proof polynomials t, if either $E(t\varphi)$ or $E(t, \psi)$, then $E(\partial t, \varphi \vee \psi)$.

[2]The only difficulty with this procedure is that it may result in some theorems concerning propositions not justified by !t in which there may be other proof polynomials that justify the proposition, particularly in the systems that admit irreflexive models. This in turn would require the statement of the projection theorem to be weakened for such systems. However, I would regard this problem as a mere technicality, of no philosophical concern.

- For all sentences φ, ψ, θ, and proof polynomials s, t, u, if $E(u, \varphi \vee \psi)$, $E(s, \varphi \vdash \theta)$, and $E(t, \psi \vdash \theta)$, then $E(\blacktriangledown(u, s, t), \theta)$.

- For all sentences φ, and proof polynomials t, $E(t, \varphi)$ iff $E(!t, t : \varphi)$.

- For all sentences φ, if there exists any proof polynomial t such that $E(t, \varphi)$, then $v(\varphi) = 1$.

- $v(\neg\varphi) = \begin{cases} 1, \text{if } v(\varphi) = 0; \\ 0, \text{if } v(\varphi) = 1; \\ \frac{1}{2}, \text{if } v(\varphi) = \frac{1}{2}. \end{cases}$

- $v(\varphi \vee \psi) = \max(v(\varphi), v(\psi))$.

- $v(\varphi \wedge \psi) = \min(v(\varphi), v(\psi))$.

- $v(t : \varphi) = 1$ if $E(t, \varphi)$. Otherwise, $v(t : \varphi) = 0$.

As with tLPJT4, we can construct a sequent calculus for $\text{tK}_3\text{JT4}$ that is sound and complete with respect to [factive] $\mathcal{C}$-validity for provability-appropriate $\mathcal{C}$.

Definition 5.4.3. The sequent calculus for $\text{tK}_3\text{JT4}$ consists of unrestricted structural rules of identity, weakening, contraction, and cut, plus the following additional rules:

$$\frac{}{\Gamma, p, \neg p \Rightarrow \Delta}\ \text{Explosion}$$

$$\frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta}\ \wedge\text{L}$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}\ \wedge\text{R}$$

$$\frac{\Gamma, \neg\varphi \vee \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \wedge \psi) \Rightarrow \Delta}\ \neg\wedge\text{L}$$

$$\frac{\Gamma \Rightarrow \neg\varphi \vee \neg\psi, \Delta}{\Gamma \Rightarrow \neg(\varphi \wedge \psi), \Delta}\ \neg\wedge\mathrm{R}$$

$$\frac{\Gamma, \varphi \Rightarrow \Delta \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta}\ \vee\mathrm{L}$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}\ \vee\mathrm{R}$$

$$\frac{\Gamma, \neg\varphi \wedge \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \vee \psi) \Rightarrow \Delta}\ \neg\vee\mathrm{L}$$

$$\frac{\Gamma \Rightarrow \neg\varphi \wedge \neg\psi, \Delta}{\Gamma \Rightarrow \neg(\varphi \vee \psi), \Delta}\ \neg\vee\mathrm{R}$$

$$\frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \neg\neg\varphi \Rightarrow \Delta}\ \neg\neg\mathrm{L}$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \neg\neg\varphi}\ \neg\neg\mathrm{R}$$

$$\frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, t : \varphi \Rightarrow \Delta}\ :\mathrm{L}$$

$$\frac{\Gamma \Rightarrow \Delta, s : \varphi}{\Gamma \Rightarrow \Delta, (s+t) : \varphi}\ +\mathrm{R1}$$

$$\frac{\Gamma \Rightarrow \Delta, t : \varphi}{\Gamma \Rightarrow \Delta, (s+t) : \varphi}\ +\mathrm{R2}$$

$$\frac{\Gamma, \neg s : \varphi \Rightarrow \Delta}{\Gamma, \neg(s+t) : \varphi \Rightarrow \Delta}\ \neg{+}\mathrm{L1}$$

$$\frac{\Gamma, \neg t : \varphi \Rightarrow \Delta}{\Gamma, \neg(s+t) : \varphi \Rightarrow \Delta}\ \neg{+}\mathrm{L2}$$

$$\frac{\Gamma \Rightarrow \Delta, t : \varphi}{\Gamma \Rightarrow \Delta, !t : t : \varphi}\ !\mathrm{R}$$

$$\frac{\Gamma, t : \varphi, t : \psi \Rightarrow \Delta}{\Gamma, t : (\varphi \wedge \psi) \Rightarrow \Delta}\ : \wedge\mathrm{L}$$

$$\frac{\Gamma \Rightarrow \Delta, t : \varphi \qquad \Gamma \Rightarrow \Delta, t : \psi}{\Gamma \Rightarrow \Delta, t : (\varphi \wedge \psi)} : \wedge\text{R}$$

$$\frac{\Gamma \Rightarrow \Delta, t : \varphi, \psi}{\Gamma \Rightarrow \Delta, \partial t : (\varphi \vee \psi)} \partial\text{R1}$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi, t : \psi}{\Gamma \Rightarrow \Delta, \partial t : (\varphi \vee \psi)} \partial\text{R2}$$

$$\frac{\Gamma, s : (\varphi \vdash \theta), t : (\psi \vdash \theta) \Rightarrow \Delta, u : (\varphi \vee \psi)}{\Gamma, s : (\varphi \vdash \theta), t : (\psi \vdash \theta) \Rightarrow \blacktriangledown(u, s, t) : \theta} \blacktriangledown\text{R}$$

$$\frac{\Gamma \Rightarrow \Delta, s : \varphi \qquad \Gamma \Rightarrow \Delta, t : \neg\varphi}{\Gamma \Rightarrow \Delta, \mathfrak{F}(s, t) : \psi} \mathfrak{F}\text{R}$$

$$\frac{\Gamma, t : \varphi \Rightarrow \Delta}{\Gamma, \mathfrak{E}t : \psi \Rightarrow \Delta} \mathfrak{E}\text{L}$$

$$\frac{\Gamma \Rightarrow \Delta, t : \varphi}{\Gamma \Rightarrow \Delta, \mathfrak{E}t : \psi} \mathfrak{E}\text{R}$$

Restriction: φ and ψ must be double negation or DeMorgan equivalents in any application of the $\mathfrak{E}$ rules.

$$\frac{\Gamma \Rightarrow \Delta}{\Rightarrow c : (\Gamma \vdash \Delta)} :\vdash\text{R}$$

Restriction: Γ and Δ must not contain any instances of $\vdash$, and the constant c must be such that $\mathcal{C}(c, \Gamma \vdash \Delta)$ for a provability-appropriate constant specification $\mathcal{C}$.

To get the corresponding modal system, replace all of the justification operator rules in the above sequent calculus with the customary modal rules:

$$\frac{\Box\Gamma \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi} \Box\text{R}$$

and

$$\frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \Box\varphi \Rightarrow \Delta}\,\Box\mathrm{L}$$

With these sequent calculi, we can prove realization in the usual manner.

Theorem 5.4.4 (Realization)**.** *For any sequent $\Gamma \Rightarrow \Delta$ that is derivable in tK_3KT (i.e., in the modal analogue of tK_3JT), there is a sequent $\Theta \Rightarrow \Lambda$ derivable in tK_3JT such that $\Theta^\circ = \Gamma$ and $\Lambda^\circ = \Delta$, where $\circ$ is the forgetful projection operator defined in Chapter 1.*

Proof. As usual, we proceed by developing an inductive algorithm for transforming $\mathrm{tK_3KT}$ derivations into $\mathrm{tK_3JT}$ derivations. The base cases of axiomatic sequents and the inductive cases corresponding to non-modal rules are valid in both systems, and are left unaltered, except that any modal formulae introduced by axioms or side-formulae are replaced with justification formulae featuring arbitrarily chosen proof polynomials. Similarly, cases of $\Box$L are replaced with cases of :L featuring arbitrary proof polynomials.

For the $\Box$R inductive cases, we must provide procedures to address all the possible subcases, specifically the case where φ is a modal formula, the case where φ is a non-modal formula that is either a member of Γ or a consequence of formulae in Γ, and the case where φ is a conjunction or disjunction of formulae of the preceeding types. In the modal φ case, we have a translation of the premise sequent by inductive hypothesis, and simply apply the !R rule to that to get the desired conclusion. In the case where φ is a conjunction of modal formulae, note that this conjunction must have been formed earlier in the proof by applying the $\wedge$R rule to two modal formulae. For the translation, remove that initial $\wedge$R step. Then apply !R rules to both premises, and use +R rules as needed to make the outermost proof polynomials match. Once this match is obtained, the premises can be recombined using the :$\wedge$R rule to get the desired derivation. Similarly, a disjunction of a modal formula with an arbitrary formula must have been produced by applying $\vee$R earlier in the proof. Remove the $\vee$R step, translate $\Box$R by a !R rule, and then restore the disjunction by following the !R with a ∂R rule.

In the fully non-modal φ cases, we discard the steps of the original derivation above the $\Box$R step, and simply create a new derivation. In cases where where $\varphi \in \Gamma$, then the conclusion of $\Box$R is simply an identity axiom, so we can get the conclusion as an identity axiom in the justification logic by simply replacing all of the $\Box$ instances with justifications by arbitrary proof polynomials. In cases where φ is merely a consequence of Γ, we begin by obtaining a natural deduction proof of φ from hypotheses Γ. That proof can then be turned directly into a sequent calculus derivation. First, identify any subproofs that are used as minor premises in $\vee$-elimination steps. For these subproofs, write K_3 sequent derivations of the conclusions, apply $\vdash$R, and cut the result with the result of the $\blacktriangledown$R rule when that step occurs in the main proof. For the remaining portions of the natural deduction proof, start with identity sequents corresponding to justifications of the formulae from Γ that are used as premises in the natural deduction proof, and then combine or transform these sequents following the natural deduction proof. For $\wedge$-introduction steps, apply +R and :$\wedge$R as in the modal conjunction case above. For $\vee$-introduction steps, apply ∂R. For $\wedge$-elimination steps, create a suitable second sequent premise using duplicates of earlier steps along with structural rules and :$\wedge$L, then apply cut to eliminate the conjunction. As mentioned above, $\vee$-elimination steps are handled by simply applying the $\blacktriangledown$R rule, and then cutting with the subproofs that were prepared earlier. Double-negation and DeMorgan steps (in either direction) are translated by $\mathfrak{E}$R, and EFQ steps are translated by applying the $\mathfrak{F}$R rule. Finally, use weakening as needed to add in justifications of any formulae in Γ that were not used in the natural deduction proof, and the result will be a derivation of a realization of the conclusion of the original □R step.

Mixed conjunction cases will, like the purely modal conjunctions, be produced by ∧R applications earlier in the proof. Remove that step to produce separate derivations of a modal formula and a non-modal formula. Apply !R to the former and the algorithm from the preceding paragraph to the latter. Then use +R to make their proof polynomials match, and get the desired derivation by :∧R. ■

As in Chapter 4, the sequent calculus is given for tK_3JT4. The intended function of the ! operator in systems without the 4-analogue is characterized by the rule:

$$\frac{\vec{s} : \Gamma \Rightarrow t : \varphi}{\vec{!s} : \vec{s} : \Gamma \Rightarrow !t : t : \varphi} \; !\text{LR}$$

This rule is derivable using :L and !R in tK_3JT4. To get a sequent calculus for tK_3J4, :L is removed; to get a sequent calculus for tK_3JT, !R is removed. In either case, !LR must be added back to the system as a basic sequent rule. (As in Chapter 4, the T-free systems must also include the rule $\frac{\Gamma \Rightarrow \Delta}{\Rightarrow \Gamma \vdash \Delta}$.) The model-theoretic changes in moving between the various justification extensions of K_3 are just as in the LP extensions developed in Chapter 4.

Another thing to note is that neither the model theory nor the sequent calculus, as stated here, include rules governing the application (·) operator. These rules are, however, admissible, given that application is just a special case of ▼ in which the two ⊢ premises are instances of EFQ and identity provability-claims, both of which obviously hold for all provability-appropriate constant specifications.

5.5 Justification Extensions of FDE

FDE, developed in Anderson and Belnap [2], is a four-valued logic, where the two non-classical values exhibit the same logical properties as the non-classical values of LP and K_3 respectively, producing a logic that is both paraconsistent and paracomplete. This logic is thus useful when the philosophical interpretations of paraconsistency and paracompleteness are both desirable. In justification extensions, FDE also allows for the combination of the two forms of non-normality that occur as side-effects of the non-classical behaviors of LP and K_3, namely the failure of application in the LP-based systems and the failure of justification in the K_3-based systems.

Given that we have developed justification extensions of both LP and K_3, it is a routine exercise to combine these systems into extensions of FDE. Start with the trivalent (Fitting or Mkrtychev) models for both the p-systems and the t-systems, and extend the valuation criteria of the propositional connectives to reflect the four-valued FDE truth table. The clause for the : operator becomes $v_w(t : \varphi) = 1$ if $\varphi \in \mathcal{E}(w, t)$ and for every $w' \in W$ such that $R(w, w')$, $v_{w'}(\varphi)$ is either 1 or the LP-like non-classical value; and $v_w(t : \varphi) = 0$ otherwise (omitting the accessible-worlds condition when working in Mkrtychev models). Eliminate the clauses defining $\mathcal{E}$ for the $\mathfrak{F}$ operator and for the special constant e, as neither the excluded middle nor *ex falso quodlibet* are valid in FDE. Finally, because FDE is theorem-free like K_3, the biconditional ! clause from the K_3JT models is required.

The tFDEJT4 sequent calculus is produced similarly, starting from the sequent calculi for tLPJT4 and tK_3JT4. The e:R and $\mathfrak{F}$R rules are omitted, as are the explosion and exhaustion rules. All of the rules that are common to both logics are included; for systems not containing both the T-analogue and the 4-analogue, also include the !LR rule from the corresponding K_3-based logics. The proofs of soundness, completeness, and realization are also basically the same as in the tLPJT4 and tK_3JT4 cases.[3]

[3] A sequent calculus and a natural deduction system for FDE, equivalent to the ones which are developed here (and whose soundness and completeness are used as lemmas in the soundness, completeness, and realization proofs for tFDEJT4) are developed in Anderson and Belnap [2]. A more comprehensive survey of sequent calculus methods for FDE can be found in Shapiro [88].

Chapter 6

Probabilistic Justification Logic

6.1 Introduction

Thus far in this dissertation, we have presented various arguments for the use of justification systems as epistemic logics. We have further examined a wide range of philosophical positions, and constructed appropriate justification logics to model these positions. In particular, we have twice partitioned the space of the relevant philosophies: once based on the question of whether the theory of truth (and thus, the propositional base logic) is classical or non-classical, and once based on the internalism/externalism divide within epistemology. Logics suitable for the non-classical truth-theories were set forth in Chapters 3–5. For the classical theories, we found that externalist epistemologies were best modeled by the normal justification logic JT, or in unusual cases by JT4 or by the non-normal system JS0.5. Internalist epistemologies proved to vary more widely based on the details of the particular epistemology, with some being modeled by JT4, and others requiring either non-normal systems of the type developed in Chapter 2 or restrictedly-paracomplete systems like the system pK_3JT4 of Chapter 5.

However, if the purpose of employing justification logic is to create a perspicuous model of real epistemic situations, then there is a sense in which the models given thus far miss the mark. Evidence rarely guarantees the truth of a proposition, but rather provides a certain probability that the proposition is true. One is generally considered justified in believing a proposition if the probability given by the available evidence exceeds a certain epistemic threshold. This understanding of evidence is widely accepted, but the question of how this threshold is determined has spawned a major debate in epistemology, with at least three major competing views: contextualism (the view of DeRose [32]), which claims that the threshold is set by the conversational context of an utterance of "I know that p"; subject-sensitive invariantism (the view of Stanley [92]), which claims that conversational contexts don't affect the epistemic threshold, but that the threshold does vary depending on the real-world situation of an epistemic agent; and pure invariantism (the view of Williamson [103]), which fixes a single epistemic threshold for all agents in all epistemic circumstances. In addition to the major three, there are other epistemic theories that seek to accomplish the same goal using a different sort of framework, such as the relativist theory of MacFarlane [71]. In this chapter, I do not advocate any of these views, but rather I seek to devise a logical model of the proposition-evidence relation that can be applied to any of them.[1]

The appropriate mathematical tools for use in this project are fuzzy logic and

[1]There is also a significant class of theories addressing the same general epistemic question which cannot be modeled by the justification logic presented in this chapter. Notable examples include the contrastivist theory of Schaffer [85] and the older relevant alternatives theories of Dretske [34, 35] and Goldman [51].

probability theory. For the sake of simplicity, I will apply these tools to a single justification system, the normal justification logic JT4. The resulting logic can easily be converted into a system based on JT by removing all references to the ! operator and 4^{j} axiom and changing the axiom justification clauses to an iterated form, or to a non-normal system by restricting axiom justification in the manner of Chapter 2. At present, it is not feasible to produce analogues of the non-classical justification systems of Chapters 3–5, as the non-classical models of these systems are incompatible with the techniques of fuzzy logic.

6.2 Probability Theory and Fuzzy Logic

Probability theory is a mathematical model of uncertainty. Fuzzy logic is a mathematical model of vagueness. Giangiacomo Gerla, the fuzzy logician who has done the most work investigating probabilistic logics, emphasizes the distinction between these topics in [45] using the example sentence, "The rose on the table is red." In a case where you can see the rose on the table and its color is something intermediate between red and pink, the truth of the sentence is vague but not uncertain. In a case where you are shown two roses, one determinately red and one determinately not-red, and then one of the roses is placed on a table out of your sight, the truth of the sentence is uncertain but not vague. However, when epistemic operators are added to the language, this sharp distinction begins to collapse. If I have a small degree of uncertainty concerning the truth of proposition p, this creates vagueness in the truth of the proposition that I know that p. If the proposition p were vague as well as uncertain, this would presumably give an even weaker degree of truth to the knowledge claim, but it is unclear how those two sources of vagueness should interact. For the purposes of this chapter, let us assume that the only vagueness present is that which is caused by uncertainty in the propositions to which epistemic operators are applied. The logic developed under this assumption could be used directly by those who advocate a reductive account of vagueness, or it could perhaps be modified to account for the separate phenomena of vagueness and uncertainty. I shall take no stand here as to which application is to be preferred, but the latter must be postponed to another work.

In Section 6.4, I will set forth a justification logic based on this assumption—a probabilistic fuzzy logic similar to Gerla's. But first, let us examine competing proposals for logically modeling the sorts of epistemic applications discussed in the introduction.

6.3 Previous Justification Logic Approaches to the Vagueness of Epistemic Justification

6.3.1 Milnikel's Logic of Uncertain Justifications

Definition 6.3.1. The Logic of Uncertain Justifications, J^{U}, is formed by augmenting the axiomatic system of classical propositional logic with a family of additional binary operators $:_r$, where r is a rational number in the interval $(0, 1]$. All of these operators shall take proof polynomials as left inputs and formulae as right inputs, and shall be governed by the following axioms and inference rules:

Application $\vdash s :_p (\varphi \supset \psi) \supset (t :_q \varphi \supset (s \cdot t) :_{pq} \psi)$

Monotonicity$_1$ $\vdash s :_r \varphi \supset (s + t) :_r \varphi$

Monotonicity$_2$ $\vdash t :_r \varphi \supset (s + t) :_r \varphi$

Confidence Weakening If $p \leq q$, then $\vdash t :_q \varphi \supset t :_p \varphi$

Iterated Axiom Justification If φ is a substitution instance of any schema listed above, or of any axiom schema of the chosen axiomatization of classical propositional logic, then we may select an arbitrary sequence of proof constants $c_1, \ldots, c_n$ and infer $\vdash c_n :_1 c_{n-1} :_1 \ldots c_1 :_1 \varphi$

The intended interpretation of a formula $t :_r \varphi$ in J^{U} is something along the lines of "Evidence t provides justification for believing φ with at least confidence level r." With this interpretation, J^{U} has the resources to at least express the sort of epistemic uncertainty that was set forth in the introduction to this chapter. Indeed, J^{U} appears to be the simplest possible logic possessing those expressive resources. Additionally, it has the advantage of being fully expressible as an axiomatic system; Milnikel also provides a model theory and the associated strong completeness proof. By contrast, most forms of fuzzy logic (including the probabilistic fuzzy logic that will be the basis of my probabilistic justification logic in Section 6.4) are only axiomatizable with weak completeness (or some intermediate completeness property; compare Pavelka [80]). Having weak but not strong completeness means that although an axiomatization can be used for certain metatheoretic tasks, it does not contain all of the information that is found in the model theory; in particular, axiomatizations of fuzzy logics do not have any feature corresponding to the "fuzziness" of the truth values, but rather differ from axiomatic classical logic only in omitting those theorems that are invalidated by fuzzy counterexamples. In an Artemov-style justification logic, the axiomatic system is taken to be the canonical presentation of the logic, and alternative presentations such as model theories or sequent calculi are derived from the axiomatic system. For example, the definition of the Mkrytchev model theory in Chapter 1 includes a clause that directly refers to the axiomatic system. This methodology requires strong completeness, as we cannot take the axiomatic system to be semantically fundamental when it lacks core semantic information.

The advantages of Milnikel's approach are offset by a major weakness: J^{U} does not provide a genuine model of the phenomenon of epistemic uncertainty. The knowledge[2] operator of natural language is semantically vague, and the most important source of this vagueness is the probabilistic uncertainty of the truth of the embedded proposition. A true model of knowledge thus requires both a fuzzy logic and a probabilistic system. J^{U} is neither. It does not provide any formula that expresses the vague knowledge operator, but rather an infinite collection of deterministic uncertain-justification formulae. Also, the only specifications made in J^{U} as to which proof polynomials are to justify which propositions at which level are that axioms must be justified to degree 1 by proof constants, and that justification must be transmissible by modus ponens. Nothing more is specified; in particular, nothing is specified to ensure that the justifications obey the laws of probability. Even such extreme violations such as a single proof constant c satisfying both $c :_{0.7} A$ and $c :_{0.7} \neg A$ are not ruled out by J^{U}.

[2]Or epistemic justification—I phrase the point in terms of knowledge because that is the notion that is generally employed in natural language, but what really matters here is the justification aspect of the "justified true belief" analysis of knowledge, and not either the truth or belief aspects.

6.3.2 Kokkinis' Probabilistic Justification Logic

Kokkinis *et al.* [60, 61] develop a class of genuinely probabilistic justification logics using methods analogous to Milnikel's. Instead of Milnikel's "uncertain justification" operator, Kokkinis' logics employ two separate operators: a probability operator $P_{\geq r}$ and a standard justification operator :. The semantics is straightforward: the justification operator is handled in the style of the Mkrytchev evidence function, while the probability operator is handled by incorporating an algebraic probability theory.

The outstanding feature of the Kokkinis logic is that it is a simple justification logic that also contains a probability operator. As a logic of *probabilistic justification*, however, it is decidedly lacking. For one thing, there is no syntax in the language that can be identified as a good representation of a particular case of probabilistic justification. In presenting a formalization of the lottery paradox in [61], Kokkinis *et al.* are forced to translate extra-logically by stipulating that for every proof polynomial t, there exists another proof polynomial $pb(t)$ such that $t : (P_{\geq 0.99}\, \varphi) \supset pb(t) : \varphi$. Admittedly, that case involves belief rather than justification, but there doesn't seem to be any adequate approach to probabilistic justification proper aside from the same strategy of defining a new operation on proof polynomials; the P operators are not justifications, and the : operator does not distinguish between probabilistic and non-probabilistic cases.

The Kokkinis logic also suffers from the converse problem that it cannot adequately represent certain justification. A typical epistemic agent possesses some knowledge that is derived from completely certain sources (e.g., mathematical proof) and other knowledge that is subject to probabilistic uncertainty. It is therefore important that an epistemic logic be able to represent both types of knowledge. The fundamental logical property of certain knowledge is factivity. The : operator of Kokkinis logic is not factive. Normally, this would be a minor issue; factivity can be provided to a justification logic by adding T^j to the axiomatic system, making the Fitting model reflexive, and so forth. However, the proposed approach to probabilistic justification requires that the : operator *not* be factive, as certain instances of : are defined as representing merely probabilistic justification. Perhaps the problem might be solved by defining certain justification as the special case of probabilistic justification where the probability is 1, and then reworking the account of probabilistic justification so that the principal operator is $P_{\geq r}$ rather than :. This would require $P_{\geq 1}$ to be factive; as specified in Kokkinis *et al.* [60, 61], it is not, but that feature can probably be changed without difficulty.

6.3.3 Ghari's Hájek-Pavelka-Style Justification Logics

Ghari [47] presents a class of fuzzy justification logics whose semantics is substantially similar to that of the probabilistic justification logic that I present below. Ghari's logics are all justification versions of well-known non-probabilistic fuzzy logics such as Łukasiewicz continuum-valued logic and Pavelka logic. These logics do have an important epistemic application, which Ghari demonstrates with the following example:

> Suppose that you are invited to the fourth birthday party of your nephew Mark. When you meet Mark, based on your observation, you are justifying [sic] to believe that 'Mark is a child.' One second after, your first observation in the birthday party is still an evidence to believe that he is a child, and one second after that, you believe that he is still a child for

> the same evidence, and so on. Hence, you believe that Mark is a child for the same evidence after any number of seconds have elapsed. But after an appropriate number of seconds have elapsed, e.g. when Mark is aged thirty-five, your first observation in the birthday party is not an evidence to believe that he is a child. [47, p.771]

In this example, your evidence (and thus knowledge) that Mark is a child is completely certain at the time when it is acquired (Mark's fourth birthday party). Moreover, the quality of the evidence itself does not vary over time; discounting cases where you forget information or Mark dies prematurely, you are absolutely certain of *Mark's age* at the initial observation, 13 years after the initial observation (when Mark is 17), and indeed 21 years after the initial observation (when Mark is 35). What does change is your certainty regarding the proposition that *Mark is a child*: you are certain of this proposition's truth at the first time reference, uncertain at the second, and certain of its falsity at the third. The judgements of certainty and uncertainty here are not genuinely epistemic phenomena at all; they are entirely due to the semantic vagueness of the word "child."

The form of uncertainty resulting from epistemic judgements involving vague predicates is an important topic of study. Indeed, it is an unavoidable topic, given that nearly all natural language predicates are semantically vague. However, as explained in Section 6.2, they are not the intended target of the present inquiry. In this chapter, we are seeking to build a logical model of situations where the evidence *itself* is uncertain, and that task requires a probabilistic fuzzy logic rather than any of Ghari's non-probabilistic systems.

6.4 Probabilistic Justification Logic

Given our arguments for the inadequacy of all of the aforementioned logics, how can we give an adequate logical model of epistemic vagueness? As indicated above, we must develop a system of justification logic that functions as both a fuzzy logic and a probabilistic logic.

We begin setting forth our new logic, pr-JT4, as a variant of the Mkrtychev model theory for JT4 given in Chapter 1. Because this will be a fuzzy logic, the assignment function v will now map sentences to an interval of truth values $[0,1]$, and the evidence function E will likewise map ordered pairs of proof polynomials and sentences to $[0,1]$. The model must also include an additional component, a constant $\kappa \in [0,1]$, representing the epistemic threshold discussed in section 6.1. It seems felicitous to use this same threshold κ as the cut-off for proper assertability (which is known as designation in the multi-valued logic literature); given the assumption that there is no vagueness outside of epistemic contexts, the assertability criterion will only be needed for epistemic operators and their truth-functional compounds.

Definition 6.4.1. A Mkrtychev premodel of pr-JT4 is a triple $\mathcal{M} = \langle v, E, \kappa \rangle$, where v is a function from the set of sentences to $[0,1]$, E is a binary function from the sets of proof polynomials and sentences to $[0,1]$, $\kappa \in [0,1]$, and the following conditions are satisfied:

- $v(\bot) = 0$.
- $v(\top) = 1$.
- $v(\neg\varphi) = 1 - v(\varphi)$.
- $v(\varphi \vee \psi) \geq \max(v(\varphi), v(\psi))$.

- The connective $\supset$ can be defined from $\neg$ and $\vee$ in the usual manner and evaluated per the above.[3]

- $v(\varphi \wedge \psi) \geq v(\varphi) + v(\psi) - 1$.

- The conventional relationship between probabilities of conjunctions and disjunctions must also hold: $v(\varphi \vee \psi) = v(\varphi) + v(\psi) - v(\varphi \wedge \psi)$.[4]

- $v(t : \varphi) = E(t, \varphi)$.

- If $E(t, \varphi) = 1$ for any proof polynomial t, then for the corresponding φ, $v(\varphi) = 1$.

- $E(s \cdot t, \psi) \geq E(s, \varphi \supset \psi) + E(t, \varphi) - 1$.

- If $E(t, \varphi) \geq \kappa$, then $E(!t, t : \varphi) \geq \kappa$[5]

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[3]Reed Solomon suggested to me that instead of using the material conditional, we might understand $\varphi \to \psi$ as a conditional probability $P(\psi|\varphi)$. Computationally, this would give the valuation

$$v(\varphi \to \psi) = \begin{cases} \pm 1, \text{if } v(\varphi) \leq 0; \\ \frac{v(\varphi \wedge \psi)}{v(\varphi)}, \text{otherwise.} \end{cases}$$

where the ± 1 option represents an interpretational choice between having false antecedent cases interpreted as true-by-default (as in the material conditional) or as undefined/error conditions (as in the definition of conditional probability). This valuation represents an interesting alternative to the material conditional, and it might, in principle, be a more perspicuous translation of some natural-language conditionals. Unfortunately, the proposal has one fatal flaw: it is ultimately grounded in an abuse of notation. Conditionals, like all logical connectives, can be embedded in other sentential contexts. To perform an analogous embedding with conditional probability, however, would require us to treat $P(A|B)$ as if it were merely a substitution instance of $P(A)$, which it is not; notations like $P(A \vee (B|C))$ or $P((A|B)|C)$ are meaningless. If, instead of conditional probability, we take the specified valuation function as fundamental, then nested conditionals will at least be grammatical, but the semantic clauses will not provide any interpretation of such a sentence beyond the general condition $v(\varphi) \in [0, 1]$. This follows from the triviality result proved by Lewis [66].

[4]Note that the three restrictions given for conjunction and disjunction are not redundant. For example, algebraically combining the $\wedge$ inequality with the relating equation results only in a statement that $v(\varphi \vee \psi) \leq 1$, which tells us nothing useful about the behavior of $\vee$.

[5]In a previous draft, I used the condition "If $E(t, \varphi) = x$, then $E(!t, t : \varphi) = x$" instead of the present form. This clause was originally designed to model the particular JT4-based externalist epistemology that is set forth at the end of Chapter 1; when applied to other epistemologies, it was proposed to serve as a simplifying assumption. This assumption's benefit is that it provides a more concrete valuation for the ! operator than the current clause does. However, in most cases

- If $E(s, \varphi) = x$ and $E(t, \psi) = y$, where φ and ψ have no common subformula, then $E(s + t, \varphi) \geq x$, $E(s + t, \psi) \geq y$, and $E(s + t, \varphi \wedge \psi) \geq x + y - 1$.

- If φ is an instance of one of the axioms of the axiomatic presentation of JT4 given in Chapter 1, except for axiom T^{j}, then there must exist some proof constant c such that $E(c, \varphi) = 1$.

- If ψ is a sentence of the form $t : \varphi \supset \varphi$ (that is, an axiom T^{j} instance) and $E(t, \varphi) = 1$, then there must exist some proof constant c such that $E(c, \psi) = 1$.

Definition 6.4.2. A model of pr-JT4 is a Mkrtychev premodel $\langle v, E, \kappa \rangle$ that satisfies the following additional properties:

Classicality For any atomic proposition p, $v(p) = 0$ or $v(p) = 1$.

Projective Consistency For every proof polynomial t, there is an evidence function F such that the triple $\langle \lambda x.E(t, x), F, \kappa \rangle$ is also a Mkrtychev premodel of pr-JT4.

Definition 6.4.3 (Satisfaction)**.** A model of pr-JT4, $\mathcal{M} = \langle v, E, \kappa \rangle$, satisfies a formula φ (written $\mathcal{M} \vDash \varphi$) iff $v(\varphi) \geq \kappa$.

Definition 6.4.4 (Consequence)**.** $\Gamma \models \varphi$ iff every model of pr-JT4 that satisfies all of the formulae within Γ also satisfies φ.

It is also helpful to supplement the language with an additional sentential operator

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there is no particular philosophical justification for such a restriction, and I have been convinced by a reviewer's suggestion that mere simplification is not enough of a motive to prefer the concrete valuation over the more general form. If one prefers the concrete form, making this change will not have any significant effect on the resulting logic.

K representing knowledge, with semantics given by

$$v(K\varphi) = \begin{cases} 1, \text{ if there is a proof polynomial } t \text{ such that } E(t,\varphi) \geq \kappa; \\ 0, \text{ otherwise.} \end{cases}$$

This additional operator becomes unnecessary if we allow quantification over proof polynomials, as the quantified formula $\exists x(x : \varphi)$ would have truth value $\geq \kappa$ (and thus be assertible) iff $K\varphi$ would have truth value 1. Quantification is usually omitted in justification logics for historic reasons;[6] JT4 in particular was created to serve as a constructive account of provability for intuitionistic logic, and allowing quantification would risk allowing non-constructive demonstrations of provability. However, in the present context, I see no strong reason to prefer the primitive K operator over quantification. Nor do I see any strong reason to prefer quantification over a primitive K operator. Given some particular epistemic theory, one of these methods will likely be a more perspicuous model than the other, but for now, there is no need to make a decision on this matter.

The semantic clauses for the truth-functional connectives are based on the account of probabilistic logic given in Adams [1]; Gerla builds his probabilistic fuzzy logics using an algebraic semantics, though he notes in [45] an equivalence between one of his systems and a system similar to that given here. The given truth clauses will seem strange to a reader who is familiar with probability theory from a high school or undergraduate math course; in such settings, for example, there is a theorem giving

[6]Quantified justification logic is investigated in Fitting [42]. Even after the publication of that paper, almost all work in justification logic has been conducted in purely propositional systems. This constitutes a striking divergence from the majority of other fields of logic, where first-order systems are standard.

a precise probability for a conjunction by $P(A \wedge B) = P(A) \times P(B)$.[7] However, the truth of that theorem depends crucially on the assumption that the conjuncts are probabilistically independent. This assumption holds for the examples normally discussed in math classes: coin tosses, dice rolls, unbiased statistical samples from a sufficiently large population, etc. However, the assumption of independence fails in epistemic contexts. For example, if you go to a reputable zoo, you can conclude that there is a very strong probability that the animal in the zebra cage is a zebra, and likewise that the animal in the tapir cage is a tapir. These probabilities are clearly not independent; if the "zebra" should turn out to be a disguised mule, this would greatly increase the probability that the "tapir" is actually a pig. In this example, the cause of the dependence is quite obvious; a major part of our evidence that the animals are what they are claimed to be is that the zoo is reputable, and displaying fake animals would undermine this reputability. In more general cases, there may not be such blatant dependence, but the possibility of it must always be accounted for. Probabilistic dependence may either increase or decrease the probability of a conjunction (or disjunction, etc.) as compared with the independent case; this is why most of the probability values are given as lower bounds rather than as exact equations. It can be shown by simple algebraic computation that the values that conjunctions and disjunctions take in the independent case fall within the range of possible values that is given for the general case presented here. It is also routine to demonstrate that in the cases where all subformulae receive the values 0 and 1, the probabilistic conjunction, disjunction, and negation are equivalent to the standard

[7] This theorem is the motivation underlying the multiplication of indices in the application schema of $\mathrm{J^U}$. Milnikel [74] addresses the challenge that independence may fail by suggesting that the product of indices be replaced with the minimum. This solution coheres with the general probability theory presented here, but the logic $\mathrm{J^U}$ still is not genuinely probabilistic for the reasons discussed above.

Boolean connectives.

The preferred general theory of probability in the mathematics literature is known as Kolmogorov probability, after its development in [62]. The probabilistic logic of pr-JT4 does in fact encode a version of Kolmogorov probability, as we will now show:

Theorem 6.4.5. *Every Mkrtychev premodel of pr-JT4 is also a model of [finitely additive] Kolmogorov probability.*

Proof. Let the atomic sentences of pr-JT4 serve as events of a probability theory, with $\bot$ as the impossible event and $\top$ as the certain event. We define the complement of an event as the negation of the corresponding sentence, the intersection of events as the conjunction of sentences, and the union of events as the disjunction of sentences. This gives us a field of events with the required structure for at least a finitely additive formulation of probability theory. Note, however, that the atomic sentences are not all elementary events in Kolmogorov's sense, as some of them have non-empty intersections. We then define the probability of an event as the semantic value given to the corresponding sentence by the v function. The union of all events is the disjunction of all atomic propositions, including $\top$, which has value 1. Given that $v(\varphi \vee \psi) \geq \max(v(\varphi), v(\psi))$, the value of this universal disjunction is also 1, as specified by one of the Kolmogorov axioms. The semantic rules for pr-JT4 also include a version of the addition rule, $v(\varphi \vee \psi) = v(\varphi) + v(\psi) - v(\varphi \wedge \psi)$, which is a well-known theorem of Kolmogorov probability, and which suffices to entail the remaining Kolmogorov axiom, that the probability of a union of disjoint events is the sum of the individual probabilities. ∎

Thus, we have it that the atomic sentences of a premodel encode a probability theory. What about the evidential sentences $t : \varphi$? After all, these sentences are the aspect of the logic that is intended to be probabilistic. That problem is solved by the projective consistency criterion of the full pr-JT4 model. Projective consistency mandates that the interpretation of each proof polynomial, as used in the : operation, must itself be a Mkrtychev premodel, and thus also a probability model. And thus we have it, in contrast to $\mathrm{J^U}$, that pr-JT4 rules out justification instances that contradict the laws of probability, such as $v(c : A) = 0.7$ and $v(c : \neg A) = 0.7$, as there can be no Mkrtychev premodel whose valuation assigns the value 0.7 to both A and $\neg A$, given that $v(\neg A)$ is required to be $1 - v(A)$.

Philosophically, the interpretation of this projective consistency criterion is similar to that of the modular semantics of justification logic proposed by Artemov [8]. The underlying idea of modular semantics is that the semantic interpretation of a proof polynomial is the set of propositions for which it serves as evidence. By interpreting these propositions as sets of worlds in a Fitting model,[8] we have it the interpretation of a proof polynomial in a modular semantics is an epistemic situation—the set of worlds which are compatible with the given evidence. Because pr-JT4 is developed exclusively in Mkrtychev semantics, we don't have this sort of world-talk, but what we do have is that the interpretation of a proof polynomial is a Mkrtychev premodel. As with any propositional model theory, this Mkrtychev premodel can be viewed as a set of propositions that are interpreted as true—in this case, with the truth value being interpretable as the degree to which the evidence represented by the proof polynomial justifies the proposition.

The classicality condition on pr-JT4 models is philosophically licensed by the assumption declared at the end of Section 6.2, that there is no genuine non-epistemic

[8]Treating propositions as sets of worlds is ubiquitous in philosophical interpretation of modal logic, so this is an uncontroversial move.

vagueness. One might assume that we can dispense with this assumption simply by eliminating the classicality constraint, but there a two major problems with that proposal. First, pr-JT4 does not have the resources to account for how the non-epistemic vagueness of an atomic sentence affects the semantic value of a knowledge claim about that atom. Intuitively, knowledge should be weaker given the same evidence in a vague case than in a non-vague case, but perhaps we could accept pr-JT4 as it stands by pushing back against this intuition. The more fundamental problem is that non-epistemic vagueness may not be probabilistic in nature. All of the arguments made above for the use of probabilistic logic proceed under the assumption that the vagueness being analyzed derives from epistemic uncertainty. Indeed, most logicians who apply fuzzy logics to model non-epistemic vagueness find the most perspicuous model of the phenomenon to be a non-probabilistic fuzzy logic; for example, see Goguen [50] or Hájek [53]. The fuzzy justification logics of Ghari [47] provide a good model for epistemic reasoning involving such non-epistemic vagueness; unfortunately, there is no way to unify the approach to that phenomenon with the approach presented here to epistemic vagueness.

The assumption that non-epistemic propositions which are not subject to epistemic vagueness behave classically also justifies the decision to base pr-JT4's axiom justification principle on the axioms of normal JT4 rather than an axiomatization of pr-JT4 itself. All of the axioms of JT4 are true for analytic reasons—either as pure logic or as conceptual analyses of epistemic concepts—and as such do not depend on evidence that is subject to probabilistic uncertainty. This design decision also makes the model theory of pr-JT4 much easier to employ, as its underlying axiomatic system is a well-studied normal justification logic rather than an obscure probabilistic logic. Unfortunately, it makes it more difficult to devise an axiomatization of pr-JT4 itself;

at present, no such axiomatization is known.

The restrictions placed on the E function are fairly straightforward. The factivity clause and its corresponding justification case are restricted to apply only to value 1 evidence for the obvious reason that if we have only uncertain evidence (however strong) for the truth of a proposition, then it might actually turn out to be false. The clause governing proof polynomials of the form $s \cdot t$ has a conjunction-like formula because these proof polynomials represent the result of modus ponens arguments, and such arguments provide no information about the conclusion unless both premises are true.

As was the case for the normal justification logics of Chapter 1, the $+$ operator and its semantic clauses may be omitted, and are only included for the convenience of the reader who may wish to employ it; unlike in the systems of Chapters 4–5, the intended interpretation of pr-JT4 does not utilize $+$ for any important function. The function of the $+$ operator is to monotonically concatenate two pieces of evidence, without explicitly drawing any new inferences. Thus, the first two requirements of the semantic clause, which state that $s+t$ constitutes evidence for everything evinced by s and t, with strength not lower than the strength of the original evidence. This only makes sense in cases where φ and ψ are unrelated formulae, as otherwise one could apply summation to a case where $v(b : A) = 0.7$ and $v(c : \neg A) = 0.7$ to get $v((b + c) : A) = 0.7$ and $v((b + c) : \neg A) = 0.7$, violating projective consistency. The conjunction part of the semantic clause for $+$ is purely a notational convenience. In the absence of this assumption, we have that $((c \cdot (s + t)) \cdot (s + t)) : (\varphi \wedge \psi)$, where $s : \varphi$, $t : \psi$, and c is the proof constant corresponding to the propositional axiom $\varphi \supset (\psi \supset (\varphi \wedge \psi))$. And to be completely pedantic, this only holds if the chosen axiomatization of propositional logic includes that tautology as an axiom; if not, c

must be replaced by some complex proof polynomial. If this justification system were intended to be applied as a logical metatheory (as was the case in the original development of JT4 by Artemov [4]), it would be necessary to track such polynomials, but in the epistemic application, it is pointless. Hence the proposed simplification. The value assigned to the conjunction as justified by the sum coheres with the result of the repeated application principle in the longer form, given that all of the constituents except $s + t$ itself are axiomatic, and their evidence functions thus output value 1.

The ! operator, whose semantics is given by the clause that if $E(t, \varphi) \geq \kappa$, then $E(!t, t : \varphi) \geq \kappa$ (or the footnoted alternative), corresponds in the epistemic interpretation to the KK principle. The KK principle lies at the heart of one of the major divides in epistemology: practically all epistemic internalists are committed to it as a fundamental law of epistemology, whereas most externalists reject it. We can certainly model an epistemology that lacks the KK principle using the methods developed in this chapter. However, it is not quite as simple as merely removing the ! semantic clause from the model theory and the 4^{j} clause from the underlying JT4 axiomatization. As is noted in Artemov [6] and other justification logic survey texts (such as Chapter 1 of this dissertation), it is also necessary to strengthen the axiom justification rules to their iterated forms. In the model theory, instead of requiring a single proof constant c such that $E(c, \varphi) = 1$, these clauses now require an infinite sequence of proof constants c_i such that $E(c_0, \varphi) = 1$ and $E(c_i, c_{i-1} : \cdots : c_0 : \varphi) = 1$ for all $i > 0$. Note that this change must be mode for both the clause pertaining to generic axiom instances and the special clause for T^{j} instances.

Another technical consideration relating to the behavior of pr-JT4 as a justification logic is the feasibility of proving projection and realization theorems. These results, which relate each justification logic to a corresponding modal logic, are considered to be essential to the intended function of the entire justification logic research program. Before we can address whether these results are provable, we must decide how they ought to be stated. One option is to formulate a modal extension of the probabilistic propositional logic that is employed in the Mkrtychev premodels. However, this seems to be the wrong track to take. The intended interpretation of the "forgetful projection" operator is precisely to discard the distinctions between the sources of justification represented by the various proof polynomials, and simply to output a modal formula that indicates, for every formula that is justified within the justification logic, that it is justified. Given this interpretation, it ought to be that part of the information that is "forgotten" in the pr-JT4 case is whether the particular justification that is given for a formula is certain or uncertain. Because the intended modal logic is to ignore the distinction between certain and uncertain justification, it need only be a normal modal logic rather than a probabilistic modal logic. Which modal logic is it? One might be tempted to assume S4, but this is incorrect. In pr-JT4, the T^{j} axiom and its axiom justification result are restricted to only hold for certain justifications; they have countermodels involving uncertain justification. The only formulae that are required to possess certain justifications are logical truths—the set of classical consequences of the propositional, K^{j}, and 4^{j} axioms. Moreover, instances of $\Box\varphi \supset \varphi$ where φ is a logical truth are theorems of the T-free system K4, simply by virtue of these formulae being material conditionals with logically true consequents. Thus, it appears that K4 is the suitable logic to be related to pr-JT4 via projection theorem.[9] Indeed, the actual proof of projection of pr-JT4 into K4 is

[9] We can also formulate another probabilistic justification logic that forgetfully projects to K4: the system pr-J4. This logic will have the same set of theorems as pr-JT4, and thus it is fitting that they have the same forgetful projection. The difference between the two logics is that, in the case of a particular model that contains a certain justification of a formula that is not logically true, pr-J4 will permit that the formula be false on that model, whereas pr-JT4 forbids this.

a routine induction on the semantic clauses of the pr-JT4 model theory.

We have it that, with the right formulation of the theorem, projection is established easily. But that is true for most justification logics; realization is generally the more difficult of the two results. There are two routes that one might take. Considering that pr-JT4 is formulated in an entirely model-theoretic manner, one might attempt to adapt the model-theoretic realization proof of Fitting [41]. Perhaps this procedure will work, but in previous research (e.g., Su [94]) it has been found that details of Fitting's proof fail when applied to non-classical justification logics, and so I am not certain whether the proof is feasible in the probabilistic setting. A more promising route is to adapt the realization proof that is used in Artemov [5] and Brezhnev [24], which proceeds by induction on a cut-free sequent formulation of the modal logic. Because the sequent calculus that is doing the essential work is that of the modal logic K4 rather than the justification logic pr-JT4, it should not matter that we do not presently have a sequent formulation of the justification logic. The construction of proof polynomials to validate a realization of each modal sequent step can be accomplished using the pr-JT4 model theory in the absence of a sequent calculus.

In a typical epistemic application of pr-JT4, the proof constants are used to represent particular pieces of evidence.[10] The corresponding outputs of the E function

[10]Some authors (e.g., Antonakos [3]) approach the semantics of justification logic in such a way that proof constants can only be interpreted as justifications of logical axioms; if justifications of any other information are wanted, proof variables must be used. This usage, however, is not good practice. Treating constants and variables in such a manner makes it difficult to add quantification to the language without engendering confusion. It also does not cohere with the use of constants and variables in the majority of logical and mathematical practice, where constants are used to denote any object that is explicitly specified and variables are reserved for objects that are unknown, or whose identity is genuinely variable in the sense of not being fixed over all situations. Individual pieces of evidence are constants according to this standard usage, and so ought to be represented by proof constants in justification logic. If one desires to make a formal separation between justifications of axioms and extra-logical epistemic justifications, it is better to make this separation by partitioning

are populated with whatever information that evidence grants about various features of the world (as represented by probabilities of the truth of propositions—an appropriate representation given that evidence is probabilistic in nature). The values (v) of atomic propositions are filled in by the actual facts about how the world really is. The value of κ is filled in by the appropriate (physical or linguistic) context. The result is a fairly simple and usable logical system with a K operator that models the physical and epistemic world about as accurately and perspicuously as is possible.[11]

Although the formal system is committed to the notion that the semantic values involved are probabilities, it makes no commitment as to exactly what these probabilities represent. The answer to this question seems to depend on the ontological foundation of one's chosen epistemology. For an externalist, the probabilities in question will be Bayesian conditional probabilities, as this is the only coherent account of the objective probability of an event depending on a condition. For an internalist, the probabilities might also be Bayesian, or they might be something more abstract, such as the subjective credences of a rational agent who is attending to specific pieces of evidence.

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the set of proof constants rather than by involving proof variables.

[11] An anonymous reviewer suggested that I support this claim with a concrete example. This suggestion turns out to be surprisingly unhelpful. Simple toy examples will show that the model coheres with some intuitive principles of uncertain reasoning, for example, that reasoning from two certain premises preserves certainty, whereas reasoning from two uncertain premises magnifies uncertainty. However, I have not been able to devise a more complex example that yields interesting conclusions. I did search the literature on Bayesian epistemology in the hopes of finding usage examples of probabilistic epistemic reasoning that I could adapt, and to my surprise, found no such examples in the literature. The closest thing to a useful concrete example would be an epistemic Dutch book argument, but this is just as contrived as any of the simple toy examples, and really doesn't show anything other than that an agent's total body of evidence, when collected together by something like the justification logic + operator, must still conform to the laws of probability on pain of incoherence. This criterion is satisfied by pr-JT4, given that the projective consistency requirement holds for all proof polynomials and not merely for individual proof constants.

Appendix A: Axiomatic Systems for Classical and Paraconsistent Logics

A.1 Introduction

Throughout this dissertation, axiomatic systems for modal and justification logics are given as extensions of an unspecified axiom scheme that is sound and complete with respect to classical propositional logic. There are many different axiomatizations that could be used to fulfill this role, and for most of the purposes of the dissertation text, it is best that we do not make any commitment as to which is the most "fundamental" axiomatization.[12] Indeed, the choice of classical axioms turns out not to affect the modal systems at all. It does, however, have a minor (but possibly significant) effect in the justification logic; while every valid modal formula $\Box\varphi$ will be realized by some justification formula $t : \varphi$, the exact proof polynomial t will vary depending on the choice of axiomatization in the underlying logic.

Because of this behavior, anyone who intends to use a justification logic must first

[12]Another advantage of not specifying the axiomatization is that it avoids the waste of space inherent in repeating the axiomatization of classical logic for every new logical system to be presented.

select an axiomatization, even if they do not intend to use the logic in axiomatic form. In this appendix, we will examine various candidate axiomatizations, with an eye toward determining which are most suitable to the various uses of justification logic that have been discussed in the dissertation proper.

In Chapter 3, we additionally presented the justification logics $LP^{\to}J$ and RM_3J as extensions of the propositional logics $LP^{\to}$ and RM_3. As with the classically-based justification logics, the axiomatic systems were described as the results of adding certain justification schemas to the axiomatization of the base logic. That presentational choice provided the same general costs and benefits as the analogous choice did in the classical case. To mitigate the costs of the lack of axiom specification, this appendix will also provide axiomatizations of $LP^{\to}$ and RM_3.

A.2 Classical Logic

The most popular axiomatization of classical logic is a set of three schemas that are traditionally attributed to Łukasiewicz[13]. The main benefit of this axiomatization is concision; it strikes an excellent balance between the conflicting goals of minimizing the number of axiom schemas and minimizing the complexity of each schema.

Proposition A.2.1 (Łukasiewicz Axiomatization)**.** *The closure of the following axiom schemas under modus ponens is sound and complete with respect to classical propositional logic:*

⊃-Irrelevance $\vdash \varphi \supset (\psi \supset \varphi)$

[13] This attribution, however, is always given without citation, and the axiomatization in question does not appear in the collection of Łukasiewicz's papers to which I have access, so the attribution may be apocryphal.

⊃-Distribution $\vdash (\varphi \supset (\psi \supset \theta)) \supset ((\varphi \supset \psi) \supset (\varphi \supset \theta))$

C-Contraposition $\vdash (\neg\psi \supset \neg\varphi) \supset (\varphi \supset \psi)$

A note on axiom naming: Many authors refer to the first two axioms of this system as K and S respectively, a nomenclature derived from combinatory logic via the Curry-Howard correspondence. However, those names are not appropriate in contexts involving modal logic (such as this dissertation), as in modal logic, the name K is reserved for the axiom schema $\vdash \Box(\psi \supset \varphi) \supset (\Box\psi \supset \Box\varphi)$. Troelstra and Schwichtenberg [98] solve this clash by using bold lower-case **k** and **s** for the propositional axioms, leaving capital K for the modal axiom. However, I find this resolution unsatisfactory for several reasons, the principal one being that the solution does not provide a way to distinguish between the modal and propositional K axioms in speech. Therefore, I have instead chosen to use descriptive names for all of the propositional axioms that are to be discussed in this appendix, reserving the conventional letter-names solely for modal axioms. The name ⊃-Irrelevance for the first axiom originates with certain remarks made by Nuel Belnap in his logic course that I took as an undergraduate.[14] I am also adopting the convention that the familiar names of traditional logical equivalences refer to the full equivalences. Hence, the modifier C-contraposition on the third axiom, to indicate that only one direction of the equivalence is included as an axiom. The "C" nomenclature indicates that this is the direction of the equivalence that is classically but not intuitionistically valid; the converse would be denoted by I-contraposition.

As a basis for justification logic, the type of concision embodied by the Łukasiewicz axiomatization is not necessarily desirable. The cost of minimizing the number of

[14] As I recall, he informally designated that axiom as the "Principle of Complete Irrelevance." His course text [20] simply numbers the axioms rather than providing official names.

axiom schemas is that proofs of theorems increase in complexity, as they often must derive intermediate steps that other axiomatizations might include as axioms. In a justification logic, the structure of proof polynomials mirrors that of proofs, so these too will increase in complexity. This complexity increase would be acceptable if it corresponded to a phenomenon of philosophical import, but it does not. As an illustration, consider the following proof of $p \supset p$ using axiomatic proof theory based on the Łukasiewicz axiomatization:

1. $p \supset ((p \supset p) \supset p)$—$\supset$-irrelevance instance
2. $(p \supset ((p \supset p) \supset p)) \supset ((p \supset (p \supset p)) \supset (p \supset p))$—$\supset$-distribution instance
3. $(p \supset (p \supset p)) \supset (p \supset p)$—by modus ponens from lines 1 and 2
4. $p \supset (p \supset p)$—$\supset$-irrelevance instance
5. $p \supset p$—by modus ponens from lines 3 and 4

Let us emphasize that the displayed proof is not one of those "joke" proofs where one intentionally employs sub-optimal means to reach an obvious conclusion. It is in fact the most efficient proof of $p \supset p$ possible in the Łukasiewicz axiomatization. Thus, if we use a Łukasiewicz-based justification logic to assert the provability of $p \supset p$, we are forced to write it as something like $((a_1 \cdot b) \cdot a_2) : (p \supset p)$. This result is ideal if our intended interpretation of the : operator is something like provability in classical propositional logic (where the justification logic is JT5). But if the intended interpretation of : is something more like knowledge, then the result is ludicrous. The proof presented above is not the basis of our knowledge that $p \supset p$. Rather, the basis of that knowledge is the meaning of the $\supset$ connective (and our familiarity with it).

What would be a better alternative to the Łukasiewicz axiomatization, given that the intended interpretation of the : operator is something like knowledge? Obviously, we want to trade off features in the other direction, accepting more axioms in exchange for simplifying the justification portion of the language. How far we want to go depends on what epistemic purpose that logic is intended for. When the main purpose is to model the interactions among justification terms that represent non-logical evidence for various propositions, it is often useful to proceed all the way to the extreme case, the total axiomatization.

Definition A.2.2. The total axiomatization of classical propositional logic is the set $\mathcal{T} = \{\varphi \mid \models \varphi\}$, where $\models$ is the model-theoretic consequence relation corresponding to classical propositional logic.

What the total axiomatization does is that it simply declares all of the theorems of classical propositional logic to be axioms. The closure of $\mathcal{T}$ under modus ponens is just $\mathcal{T}$ itself, which is trivially sound and complete with respect to classical propositional logic, so it is suitable to fill the role of the PC schema in any of the axiomatic modal or justification logics.

The benefit of using the total axiomatization is that it renders purely logical inference unobtrusive in situations where justification logic is used to model processes where knowledge is derived by a mixture of logical and extra-logical reasoning. Given any justification premise $t : \varphi$, a formula ψ that is logically equivalent to φ can be justified in a single application step $(c \cdot t) : \psi$. Likewise, any tautology can be justified by a single constant, and we can easily select our constant specification such as to draw a clear distinction between the justification constants representing logical inferences and those that represent extra-logical evidence, thus making the justification logic a highly perspicuous formal tool for tracking the basis of our knowledge of propositions of all types.

However, despite its excellent suitability for many of the philosophical purposes advocated in this dissertation, the total axiomatization has significant drawbacks. Obviously, it is not an *effective* axiomatization, and thus is unsuitable for some of the metatheoretic applications to which axiomatic logic is typically put. Moreover, there are some philosophical settings for which the collapse of logical inferences to a single step is not desirable. One important case is Artemov's original application, where JT4 was used to model provability in intuitionistic logic. But there are also examples in epistemology. As we argued in the introduction to Chapter 4, there is a significant epistemic difference between formulae that are obviously tautologous (such as $p \supset p$) and those that are so complicated as to require substantial logical reasoning to verify their status. In the former case, a typical agent has no doubt whatsoever as to the proposition's status, whereas in the latter case, there is some room for doubt insofar as the agent could have made a mistake in the logical reasoning to ascertain that the proposition is a tautology. Because these cases have different statuses, it is helpful to have a formalism that represents this difference.

A case can be made that the best solution to this problem is to abandon the axiomatic-based formalization of justification logic, and instead employ the natural deduction methodology of Chapter 4 in the classical systems as well as in the LP-analogues; natural deduction has the advantage that it limits the ability to introduce complex formulae as substitution instances of axioms,[15] and so the complexity of the

[15] Arbitrary formulae are permitted in disjunction introduction, and in special circumstances such as the excluded middle axiom that is used in natural deduction LP, but these cases are still easier for an average person to recognize as instances of the basic rules than arbitrary instances of axioms like $\supset$-distribution would be.

proof polynomial better represents the "obviousness" of a tautology in justification logic based on natural deduction than in justification logic based on axiomatization. However, as we found in Chapter 4, natural deduction-based justification logic is complicated in other ways that may not be desirable, such as the proliferation of operations required on proof terms, and the difficulties that may be encountered in incorporating certain rules (notably disjunction elimination).

Let us then set aside the natural deduction proposal, and look for a solution within the realm of axiomatic systems. What we want is a compromise system, one that abandons the concision of the Łukasiewicz axiomatization in favor of providing a richer set of axiom schemas, yet does not go far as to render every tautology whatsoever into an axiom (as in practice, some tautologies are only available by being derived from more basic principles).

One obvious starting point in the hunt for such a system is in the axiomatization of intuitionistic logic. Proof-theoretically, classical and intuitionistic logics are very closely related. However, intuitionistic logic does not allow conjunction and disjunction to be defined from negation and the conditional, and so its axiomatizations must all describe the properties of each of these connectives separately. As is the case with classical logic, there are many possible axiomatizations of intuitionistic logic. For the present purpose, let us employ the axiomatization of Heyting [55], in which intuitionistic logic was first introduced. Given that the justification logics in this dissertation are based on classical rather than intuitionistic logic, I will recast the system as an axiomatization of classical logic by adding the additional schema of $\neg\neg$-Elimination; simply omit this schema to recover the axiomatization of intuitionistic logic itself.

Proposition A.2.3 (Heyting Axiomatization)**.** *The closure of the following axiom* *schemas under modus ponens is sound and complete with respect to classical propositional logic:*

Duplication $\vdash \varphi \supset (\varphi \wedge \varphi)$

$\wedge$**-Commutativity** $\vdash (\varphi \wedge \psi) \supset (\psi \wedge \varphi)$

$\supset$**-Irrelevance** $\vdash \varphi \supset (\psi \supset \varphi)$

$\supset$**-**$\wedge$**-Weakening** $\vdash (\varphi \supset \psi) \supset ((\varphi \wedge \theta) \supset (\psi \wedge \theta))$[16]

$\supset$**-Transitivity** $\vdash ((\varphi \supset \psi) \wedge (\psi \supset \theta)) \supset (\varphi \supset \theta)$

Pseudo-Modus Ponens $\vdash (\varphi \wedge (\varphi \supset \psi)) \supset \psi$

Addition $\vdash \varphi \supset (\varphi \vee \psi)$

$\vee$**-Commutativity** $\vdash (\varphi \vee \psi) \supset (\psi \vee \varphi)$

$\wedge \supset$ **Antecedent Merge** $\vdash ((\varphi \supset \theta) \wedge (\psi \supset \theta)) \supset ((\varphi \vee \psi) \supset \theta)$

$\neg$**-**$\supset$**-Irrelevance** $\vdash \neg\varphi \supset (\varphi \supset \psi)$

I-*Reductio* $\vdash ((\varphi \supset \psi) \wedge (\varphi \supset \neg\psi)) \supset \neg\varphi$

$\neg\neg$**-Elimination** $\vdash \neg\neg\varphi \supset \varphi$

This Heyting axiomatization comes about as close as any axiomatic system can to capturing how human agents actually perform logical reasoning. All of the axioms are principles that can be easily accepted as "obvious" logical truths, as an axiom should be. Because it was originally intended as an axiomatization of intuitionistic

[16]Read the axiom name as right-associative; this axiom specifies that $\supset$ admits the process of weakening-by-$\wedge$.

logic, where the propositional connectives are not interdefinable, separate axioms are provided to describe behaviors of $\supset$, $\wedge$, $\vee$, and of the interactions among these, rather than simply axiomatizing $\supset$ as the Łukasiewicz axiomatization does. This is a tremendous improvement in perspicuousness, as human beings genuinely don't think of "and" as shorthand for "it is not the case that if not this, then that." Similarly, Heyting's direct axiomatization of $\neg$ is preferable to more modern axiomatizations of intuitionistic logic that let $\neg\varphi$ be defined as $\varphi \supset \bot$, as ordinary human reasoning takes negation as a fundamental logical operation, and does not consider anything analogous to $\bot$ at all.

A.3 Paraconsistent Logics

As in the classical logic case, for each of the paraconsistent logics $\mathrm{LP}^{\rightarrow}$ and RM_3, there is a significant choice to be made between adopting a total axiomatization and a concise axiomatization. The philosophical merits of either choice are exactly as above: the total axiomatization allows logical reasoning steps to be compressed in the proof polynomial in order to highlight those proof constants that represent extra-logical justification, whereas the concise axiomatization requires the proof polynomial to represent logical reasoning steps explicitly, providing a good model for cases where the logical reasoning is epistemically salient.

An important difference between these paraconsistent logics and classical propositional logic is that the paraconsistent logics do not have the variety of concise axiomatizations that are found in the classical case. There are two reasons for this. One is simply that there are fewer people working with paraconsistent logics in general, and none who are specifically engaged in the project of looking for alternative axiomatizations (as Łukasiewicz was for classical logic). The other is that the paraconsistent semantics imposes more restrictions on the logic. In particular, neither the $LP^{\rightarrow}$ nor the RM_3 conditional can be interdefined with conjunction or disjunction as the material conditional is. Moreover, because the logics are paraconsistent, we lose the interdefinability of $\neg$ and $\bot$. There does remain some room for alternative axiomatizations, but most of these can be regarded as mere presentational alternatives (for example, the choice of whether to give commutativity axioms or just to give left- and right-handed versions of the basic axioms defining $\wedge$ and $\vee$). The following axiomatizations are adapted from Middelburg [73], but I make different decisions in regards to some of these presentational choices, in order to keep the forms of these axioms as close as possible to those presented earlier in the discussion of classical logic axiomatizations.

Proposition A.3.1 ($LP^{\rightarrow}$ Axiomatization)**.** *The closure of the following axiom schemas under modus ponens is sound and complete with respect to the logic* $LP^{\rightarrow}$*:*

$\rightarrow$-Irrelevance $\vdash \varphi \rightarrow (\psi \rightarrow \varphi)$

$\rightarrow$-Distribution $\vdash (\varphi \rightarrow (\psi \rightarrow \theta)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \theta))$

Peirce's Law $\vdash ((\varphi \rightarrow \psi) \rightarrow \varphi) \rightarrow \varphi$

Conjunction $\vdash \varphi \rightarrow (\psi \rightarrow (\varphi \wedge \psi))$

Simplification $\vdash (\varphi \wedge \psi) \rightarrow \varphi$

$\wedge$-Commutativity $\vdash (\varphi \wedge \psi) \rightarrow (\psi \wedge \varphi)$

Addition $\vdash \varphi \rightarrow (\varphi \vee \psi)$

$\vee$-Commutativity $\vdash (\varphi \vee \psi) \rightarrow (\psi \vee \varphi)$

$\wedge \rightarrow$ Antecedent Merge $\vdash ((\varphi \rightarrow \theta) \wedge (\psi \rightarrow \theta)) \rightarrow ((\varphi \vee \psi) \rightarrow \theta)$

Excluded Middle $\vdash \varphi \vee \neg\varphi$

$\neg\neg$-Intelim $\vdash \neg\neg\varphi \leftrightarrow \varphi$

Negated Conditional $\vdash \neg(\varphi \rightarrow \psi) \leftrightarrow (\varphi \wedge \neg\psi)$

DeMorgan$_1$ $\vdash \neg(\varphi \wedge \psi) \leftrightarrow (\neg\varphi \vee \neg\psi)$

DeMorgan$_2$ $\vdash \neg(\varphi \vee \psi) \leftrightarrow (\neg\varphi \wedge \neg\psi)$

Comparing the $\text{LP}^{\rightarrow}$ and Heyting axiomatizations, one finds that most of the axioms are either identical between the two, or else have obviously analogous functions (as is the case, for example, with the conditional distribution and transitivity axioms). There are several formulae that appear in $\text{LP}^{\rightarrow}$ but not in the Heyting axiomatization: the Excluded Middle, Peirce's Law, and the various biconditional axioms. Under intuitionistic logic, all of these are equivalent to $\neg\neg$-elimination, which is the principle that was added to the Heyting axiomatization in order to axiomatize classical logic rather than intuitionistic logic. Similarly, one formula is notably present in the Heyting axiomatization but absent from $\text{LP}^{\rightarrow}$: the axiom of $\neg$-$\supset$-Irrelevance. Middelburg reports that adding the $\rightarrow$ version of this axiom to $\text{LP}^{\rightarrow}$ produces an axiomatization of classical logic, exactly as $\neg\neg$-elimination does in the Heyting axiomatization.

Proposition A.3.2 (RM_3 Axiomatization)**.** *The closure of the following axiom schemas under modus ponens and the rule form of conjunction introduction ($\varphi, \psi \vdash \varphi \wedge \psi$) is sound and complete with respect to the logic RM_3:*

$\rightarrow$**-Reflexivity** $\vdash \varphi \rightarrow \varphi$

$\rightarrow$**-Transitivity** $\vdash ((\varphi \rightarrow \psi) \wedge (\psi \rightarrow \theta)) \rightarrow (\varphi \rightarrow \theta)$

Pseudo-Modus Ponens $\vdash (\varphi \wedge (\varphi \rightarrow \psi)) \rightarrow \psi$

$\rightarrow$**-Contraction** $\vdash (\varphi \rightarrow (\varphi \rightarrow \psi)) \rightarrow (\varphi \rightarrow \psi)$

Simplification $\vdash (\varphi \wedge \psi) \rightarrow \varphi$

$\wedge$**-Commutativity** $\vdash (\varphi \wedge \psi) \rightarrow (\psi \wedge \varphi)$

$\wedge \rightarrow$ **Consequent Merge** $\vdash ((\varphi \rightarrow \psi) \wedge (\varphi \rightarrow \theta)) \rightarrow (\varphi \rightarrow (\psi \wedge \theta))$

Addition $\vdash \varphi \rightarrow (\varphi \vee \psi)$

$\vee$**-Commutativity** $\vdash (\varphi \vee \psi) \rightarrow (\psi \vee \varphi)$

$\wedge \rightarrow$ **Antecedent Merge** $\vdash ((\varphi \rightarrow \theta) \wedge (\psi \rightarrow \theta)) \rightarrow ((\varphi \vee \psi) \rightarrow \theta)$

$\wedge$**-**$\vee$**-Distribution** $(\varphi \wedge (\psi \vee \theta)) \rightarrow ((\varphi \wedge \psi) \vee (\varphi \wedge \theta))$

$\neg\neg$**-Elimination** $\vdash \neg\neg\varphi \rightarrow \varphi$

C-Contraposition $\vdash (\neg\psi \rightarrow \neg\varphi) \rightarrow (\varphi \rightarrow \psi)$

Mingle $\vdash \varphi \rightarrow (\varphi \rightarrow \varphi)$

$\vee$**-** $\rightarrow$**-Irrelevance** $\vdash \varphi \vee (\varphi \rightarrow \psi)$

Once again, most of the axioms of the RM_3 axiomatization are familiar from the previous axiomatic systems that we have examined. All of the new axioms are fairly obvious theorems of classical (or, in most cases, intuitionistic) logic.

RM_3 is classified as a semi-relevant logic by virtue of the fact that neither of the blatant irrelevance principles, $\rightarrow$-Irrelevance and $\neg$-$\rightarrow$-Irrelevance, is provable in RM_3. The weaker irrelevance principle, $\vee$-$\rightarrow$-Irrelevance, is the result of the restriction of the RM_3 model theory to a three-valued truth-functional semantics.[17] Removing this axiom produces an axiomatization of the more general semi-relevant logic RM. Additionally removing the mingle axiom yields the fully relevant logic R. By contrast, adding $\rightarrow$-Irrelevance to the RM_3 axiomatization (or to the R axiomatization, as $\rightarrow$-Irrelevance entails mingle) produces an axiomatization of classical logic.

The reason why the RM_3 axiomatization incorporates the rule form of conjunction introduction rather than the axiom $\varphi \rightarrow (\psi \rightarrow (\varphi \wedge \psi))$ is that a single application of modus ponens to the axiom form results in the assertion of the conditional $\psi \rightarrow (\varphi \wedge \psi)$, which is blatantly irrelevant because there is no formal connection between the antecedent ψ and the occurrence of φ in the consequent. Unfortunately, this feature complicates the formulation of the justification logic based on RM_3, the system RM_3J developed in Chapter 3. In general, justification logics are built under the assumption that the propositional axiomatization features only modus ponens as an inference rule; the axiom justification rule incorporates all of the propositional axioms into the system of proof polynomials, and the K^j axiom does the same for modus ponens. Because of the additional inference rule of conjunction introduction in RM_3, we need to add another justification logic axiom to RM_3J in addition to the standard justification axioms that are used in all the justification logics of Chapters 1–3. Following the convention of Chapter 4, we refrain from adding a new operator on

[17]By contrast, full RM requires either an infinite-valued Sugihara matrix semantics or a modalized semantics; cf. Dunn [38]

proof polynomials to represent conjunction, and instead use the axiom $(t : \varphi \wedge t : \psi) \rightarrow t : (\varphi \wedge \psi)$. This also requires a corresponding clause to be added to the model theory of $\mathrm{RM}_3\mathrm{J}$, presented in a footnote in Chapter 3.

Appendix B: Sequent Calculus for Classical and Intuitionist Logics

B.1 Introduction

Sequent calculus is a presentation of logic developed by Gentzen [44] to combine the best features of natural deduction systems (which were also introduced by Gentzen in the same paper) and axiomatic systems. Like a natural deduction system, a sequent calculus interprets each logical operator by providing a pair of rules to govern the use of the operator as a premise or as a conclusion, which is a better model of actual logical reasoning than any axiomatization. Like an axiomatic system, a sequent calculus is "logistic" (Gentzen's word) in the sense that there is a precise syntactic characterization of all the operations that are permissible at any stage of a derivation; by contrast, natural deduction systems allow for arbitrary hypotheses to be introduced at any point, and then provide rules governing when such hypotheses may be discharged. This logistic character makes axiomatic and sequent systems useful for the proof of certain metatheoretic results that would be harder to prove in a natural deduction system or in a model theory (or other semantic presentation). In the context of this dissertation, sequent calculi are introduced because they enable the most efficient proof of realization theorems for justification logics.

B.2 Formal Characterization of the Sequent Calculus

Definition B.2.1. A sequent is a syntactic object of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are (possibly empty) collections of formulae, and $\Rightarrow$ is simply a punctuation symbol identifying the object as a sequent (and distinguishing Γ from Δ, when this distinction is unclear).

"Collection" as used in the above definition is not a technical term, but rather a deliberately ambiguous informality. In modern presentations of sequent calculus, these collections are formalized as either sets or multisets. Strictly classical logicians favor set formalizations for the sake of simplicity; non-classical logicians usually favor multiset formalizations. Multiset formalizations of sequent calculus require additional structural rules; this turns out to be an advantage because it allows for the possibility of substructural logics—that is, logics that differ from classical logic in that these structural rules are omitted. This is a non-classical logic dissertation, so the text implicitly assumes the multiset formalization throughout. However, no substructural logics are presented; all of the logics in this dissertation can be presented using set-based sequent calculi.

However, in this appendix, we will not use either the set or multiset formalizations of the sequent definition. Instead, we will revert to Gentzen's original formulation, in which Γ and Δ are finite sequences of propositions rather than sets. The reason for this choice is that the vector notation of Convention 1.2.7 utilizes sequences of proof polynomials and formulae. By presenting sequent calculus in Gentzen's style, we can employ this convention directly without sacrificing rigor. If we instead used sets or multisets, we would have to map these objects to sequences via some choice of canonical ordering in order to rigorously interpret the notation $\vec{t}{:}\Gamma$, and then collapse the resulting sequence of formulae back into a set or multiset if it is used within a sequent. Interpreting Convention 1.2.7 in such a manner along with a set or multiset formalization of sequent calculus would be completely legitimate, but it is simpler to retain sequences throughout.

Definition B.2.2. The class of derivable sequents is defined inductively as follows:

- Any identity sequent $\Gamma, \varphi \Rightarrow \Delta, \varphi$ is derivable.
- If all of the sequents "above the line" in any of the following inference figures are derivable, then so is the sequent "below the line."

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta}\ \text{Weakening-L}$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}\ \text{Weakening-R}$$

$$\frac{\Gamma, \varphi, \varphi \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta}\ \text{Contraction-L}$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi}\ \text{Contraction-R}$$

$$\frac{\Gamma, \varphi, \psi, \Gamma' \Rightarrow \Delta}{\Gamma, \psi, \varphi, \Gamma' \Rightarrow \Delta}\ \text{Exchange-L}$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi, \psi, \Delta'}{\Gamma \Rightarrow \Delta, \psi, \varphi, \Delta'}\ \text{Exchange-R}$$

$$\frac{\Gamma, \varphi \Rightarrow \Delta \qquad \Gamma' \Rightarrow \Delta', \varphi}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \text{Cut}$$

$$\frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta}\ \wedge\text{L}$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}\ \wedge\text{R}$$

$$\frac{\Gamma, \varphi \Rightarrow \Delta \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta}\ \vee\text{L}$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}\ \vee\text{R}^{18}$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma, \neg\varphi \Rightarrow \Delta}\ \neg\text{L}$$

$$\frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi}\ \neg\text{R}$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \Gamma', \psi \Rightarrow \Delta'}{\Gamma, \Gamma', \varphi \supset \psi \Rightarrow \Delta, \Delta'}\ \supset\text{L}$$

$$\frac{\Gamma, \varphi \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \supset \psi}\ \supset\text{R}$$

- No other sequent is derivable.

Note that the contraction, weakening, and exchange rules (and sometimes also the identity base case and the cut rule) are referred to collectively as structural rules. This is because these rules are not determined by the semantics of any logical operator, but rather by the structure of the sequent calculus itself.

[18] Alternatively, the $\vee$R rule may be replaced by two rules, one with the premise sequent $\Gamma \Rightarrow \Delta, \varphi$, and the other with the premise sequent $\Gamma \Rightarrow \Delta, \psi$. In classical sequent calculus, these are obviously equivalent to the single rule given here via contraction and weakening. However, the two-rule presentation is required for intuitionistic sequent calculus.

Theorem B.2.3 (Soundness and completeness for the sequent calculus)**.** *A sequent* $\Gamma \Rightarrow \Delta$ *is derivable iff there is a proof of some* $\delta \in \Delta$ *in axiomatic classical propositional logic using hypotheses* Γ*.*

Proof. First, we need to make the theorem statement precise. Fix an axiomatization of classical propositional logic (as discussed in Appendix A). A proof using hypotheses is a finite sequence of formulae such that the last formula is δ and each formula in the sequence is either an axiom instance, a member of the hypotheses Γ, a repetition of a previous formula in the sequence, or a result of applying modus ponens to two previous formulae in the sequence.

The left-to-right direction is proven by the inductive procedure of Definition B.2.2. In the base case of an identity sequent, the desired conclusion δ is among the hypotheses Γ, so the singleton sequence $\langle \delta \rangle$ will suffice as a proof.

In the cases of the sequents derived by any of the rules presented among the inference figures in Definition B.2.2, assume for induction that there is a suitable proof corresponding to each sequent above the line, and use this to construct a proof of the sequent below the line. In all of the weakening, contraction, and exchange cases, the proof from the inductive hypothesis can be retained unaltered. In the cut case, examine the proof corresponding to the premise sequent $\Gamma' \Rightarrow \Delta', \varphi$. If this is a proof of a formula in Δ', it can stand unaltered. Otherwise, it is a proof of φ. Append to this proof the proof corresponding to the other premise sequent $\Gamma, \varphi \Rightarrow \Delta$. Because φ was derived in the first part of the proof, it does not need to be justified as a hypothesis, and so the result will be a proof corresponding to the conclusion sequent $\Gamma, \Gamma' \Rightarrow \Delta, \Delta'$.

For any of the other sequent rules, the induction procedure will behave similarly to that for cut; it will divide into a trivial case and a case where we can construct a proof using axioms. The details will depend on the exact choice of axiomatization, and in many cases will be tedious. To illustrate the method, let's examine a simple case: $\vee$R under the Heyting axiomatization.[19] Start by assuming a proof corresponding to $\Gamma \Rightarrow \Delta, \varphi, \psi$ as inductive hypothesis. If this is a proof of $\delta \in \Delta$, it can stand unaltered. Otherwise, it is a proof of either φ or ψ. If it's a proof of φ, add to it the addition axiom instance $\varphi \supset (\varphi \vee \psi)$, and then use modus ponens to get the desired $\varphi \vee \psi$. If it's a proof of ψ, add the addition axiom instance $\psi \supset (\psi \vee \varphi)$ and the $\vee$-commutativity instance $(\psi \vee \varphi) \supset (\varphi \vee \psi)$, and then use modus ponens twice to get $\varphi \vee \psi$.

For the right-to-left direction of the theorem, use the inductive definition of proofs from hypotheses to show that there is a derivable sequent corresponding to every formula that can appear as the conclusion of a proof. Any such formula must be an axiom, a hypothesis, a repetition, or a modus ponens instance. We can ignore the repetition case, as any formula that is introduced that way must have been previously introduced by one of the other clauses. In the hypothesis case, $\delta \in \Gamma$, so $\Gamma \Rightarrow \delta$ is an identity sequent. In the modus ponens case, we have derivable sequents $\Gamma \Rightarrow \varphi$ and $\Gamma \Rightarrow \varphi \supset \psi$ by inductive hypothesis. We derive the desired sequent $\Gamma \Rightarrow \psi$ as follows:

$$\dfrac{\dfrac{\Gamma \Rightarrow \varphi \qquad \psi \Rightarrow \psi}{\Gamma, \varphi \supset \psi \Rightarrow \psi}\supset\text{L} \qquad \Gamma \Rightarrow \varphi \supset \psi}{\Gamma \Rightarrow \psi}\text{Cut}$$

The remaining cases are the axioms, which again depend on the chosen axiomatization, and are likely to be tedious rather than useful. The procedure is simply to create a sequent derivation of $\Gamma \Rightarrow \varphi$ for each axiom schema φ. For example, the

[19] Here we are using the single-rule form of $\vee$R; each of the rules in the two-rule version can be justified by omitting the subcase that does not apply.

duplication axiom (of the Heyting axiomatization) can be derived as follows:

$$\dfrac{\dfrac{\Gamma, \varphi \Rightarrow \varphi \qquad \Gamma, \varphi \Rightarrow \varphi}{\Gamma, \varphi \Rightarrow \varphi \wedge \varphi} \wedge R}{\Gamma \Rightarrow \varphi \supset (\varphi \wedge \varphi)} \supset R$$

■

Definition B.2.4. A sequent $\Gamma \Rightarrow \Delta$ is called an intuitionistic sequent if Δ contains at most one formula.

A sequent $\Gamma \Rightarrow \Delta$ is said to be intuitionistically derivable if it can be derived using the recursive algorithm of Definition B.2.2 with only intuitionistic sequents.

Corollary B.2.5. *A sequent* $\Gamma \Rightarrow \Delta$ *is intuitionistically derivable iff there is a proof of some* $\delta \in \Delta$ *in axiomatic intuitionistic propositional logic using hypotheses* Γ*.*

Proof. After completing all of the cases of the Theorem B.2.3 proof for the Heyting axiomatization (with the $\vee$R sequent rule split into two rules as noted above), examine the results. In the right-to-left direction, the only case that requires the presence of multiple formulae on the right side of a sequent is the sequent derivation of the $\neg\neg$-elimination axiom, which looks like this:

$$\dfrac{\dfrac{\dfrac{\Gamma, \varphi \Rightarrow \varphi}{\Gamma \Rightarrow \varphi, \neg\varphi} \neg R}{\Gamma, \neg\neg\varphi \Rightarrow \varphi} \neg L}{\Gamma \Rightarrow \neg\neg\varphi \supset \varphi} \supset R$$

Removing $\neg\neg$-elimination from the Heyting axiomatization of classical logic yields the Heyting axiomatization of intuitionistic logic. Thus, given that we have it that all of the classical Heyting axioms are intuitionistically derivable except for $\neg\neg$-elimination, it follows that intuitionistic logic is intuitionistically derivable, Q.E.D.

In the left-to-right direction of the proof, the $\neg\neg$-elimination axiom is only needed to create proofs corresponding to the $\neg$ rules. The only instances of these rules that can appear in an intuitionistic sequent derivation are those where Δ is empty. These special cases can be proven without $\neg\neg$-elimination. The proofs in question are simple in axiomatizations of intuitionistic logic based on $\bot$; in the Heyting axiomatization, they are tedious proofs involving the I-*reductio* axiom. ■

Notice that the soundness and completeness proofs presented here are proofs for the sequent calculus with respect to the axiomatic system. Proofs of soundness and completeness of these proof theories with respect to the model theory can be conducted using either an axiomatic or sequent system, given that the proof theories are sound and complete with respect to each other. The details of the model-theoretic completeness proof are somewhat less tedious for an axiomatic system than for a sequent calculus, though it is certainly feasible to give the proof in either form.

B.3 Cut Elimination

In the sequent calculus for classical (or intuitionistic) logic, sequent derivations that are performed without the cut rule have a useful characteristic called the subformula property.

Theorem B.3.1 (Subformula property)**.** *Every formula that appears in a cut-free derivation of the sequent $\Gamma \Rightarrow \Delta$ is a subformula of at least one of the formulae contained in Γ or Δ.*

Proof. Suppose that the theorem statement is false. Then there is some formula ξ that does not appear as a subformula in the final sequent of the derivation, but does appear in some earlier sequent. Let $\Gamma' \Rightarrow \Delta'$ be the lowest sequent in the derivation where ξ does appear as a subformula. This sequent must be used as a premise in some sequent calculus step where the conclusion does not contain ξ. What step could this be? Definition B.2.2 exhausts the possibilities. The only rules appearing in that definition that remove formulae from a sequent rather than incorporating them as subformulae of a newly introduced formula are the contraction and cut rules. A contraction step operating on ξ would have the form $\dfrac{\Gamma'', \xi, \xi \Rightarrow \Delta''}{\Gamma'', \xi \Rightarrow \Delta''}$ or $\dfrac{\Gamma'' \Rightarrow \Delta'', \xi, \xi}{\Gamma'' \Rightarrow \Delta'', \xi}$; both of these retain the formula ξ in the conclusion. Cut is also ruled out, as the theorem statement specifies cut-free derivations. Thus, by Definition B.2.2, there is no sequent derivation compatible with the assumption that the theorem is false. ■

Most of the practical applications of sequent calculus are simplified by invoking the subformula property. The applications discussed in this dissertation are counterexamples to this claim, as the subformula property does not hold for the sequent calculus presentations of most justification logics. But for most other applications, the subformula property is valuable. In order to make use of it, one must invoke the cut elimination result.

Definition B.3.2. The rank of a formula is defined as follows:

- Propositional atoms have rank 1.
- $\neg\varphi$ has rank $1 + \text{rank}(\varphi)$.
- $\psi \supset \varphi$, $\psi \wedge \varphi$, and $\psi \vee \varphi$ have rank $1 + \text{Max}(\text{rank}(\psi), \text{rank}(\varphi))$.

Theorem B.3.3 (Cut elimination)**.** *If the sequent $\Gamma \Rightarrow \Delta$ is derivable (or intuitionistically derivable) then it has a cut-free derivation.*

Proof. Assume that we have a derivation of $\Gamma \Rightarrow \Delta$ that contains cuts; we will construct a cut-free derivation. First, we replace all occurrences of the cut rule with the equivalent multicut, presented below, in which Φ and Φ' are nonempty sequences consisting solely of any number of occurrences of a formula φ, and Γ and Δ' do not contain φ except as a subformula of a larger formula.

$$\frac{\Gamma, \Phi \Rightarrow \Delta \qquad \Gamma' \Rightarrow \Delta', \Phi'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'} \text{Multicut}$$

Multicut is clearly equivalent to standard cut. A multicut may be turned into a standard cut by using contraction to reduce each of Φ and Φ' to a single occurrence of φ. A standard cut may be turned into a multicut by using exchange to group all occurrences of φ together, and then using weakening to add φ occurrences back to the conclusion sequent if required.

After converting cuts to multicuts, scan through the derivation to identify the uppermost multicut step; if there is more than one multicut with no multicut above it, pick one arbitrarily. We will create a new derivation that produces the conclusion of that multicut without using any cut or multicut steps. We can then substitute the resulting derivation back into the original, and repeat this algorithm of scanning for multicuts, selecting one, and replacing it with a cut-free derivation until no multicuts remain. At that point, we will have a cut-free derivation of the same sequent $\Gamma \Rightarrow \Delta$ that we started with, Q.E.D.

It remains only to produce the algorithm for eliminating an uppermost multicut. This proceeds by strong induction on the rank of the cut-formula φ. In the rank 1 base case, φ is atomic. This tells us two important things about the derivations that occur above the multicut. First, if any sequent rules other than the structural rules were used in the derivation above the multicut, the resulting formula is part of the side-formula-sequences Γ, Γ', Δ, and Δ' of the multicut rather than the cut-formula-sequences Φ and Φ', because formulae introduced by rules other than the structural rules cannot be atomic. Second, all instances of the cut-formula in the derivations above the multicut must have been introduced as the principal formula of an identity sequent, or else as either side-formulae or weakening formulae. In a case where all of the instances of the cut-formula φ in one of the premise sequents (for example, the premise $\Gamma, \Phi \Rightarrow \Delta$; the result is the same if it is the other premise that has this property) are side-formulae or weakening formulae, then the same derivation that produced $\Gamma, \Phi \Rightarrow \Delta$ can also serve as a derivation of $\Gamma \Rightarrow \Delta$ simply by omitting the side-formula instances of φ and the weakening steps introducing φ (and any contractions or exchanges that are rendered irrelevant by these omissions). Moreover, that derivation can be transformed into a derivation of $\Gamma, \Gamma' \Rightarrow \Delta, \Delta'$ by using weakening and exchange only, thus eliminating the multicut.

Suppose then that each premise sequent derivation of the multicut contains a cut-formula instance that was initially derived as the principal formula of an identity sequent. If one of the premises is such that the only steps occurring above the multicut are the identity itself and possibly some weakenings, contractions, and exchanges, then we have a direct elimination of the multicut similar to that which we found in the previous case. Assuming without loss of generality that it's the right premise which is an identity instance, the derivation looks like this (though with possibly a different ordering of formulae, or with extra stuctural steps):

$$\dfrac{\begin{array}{c}\mathcal{D}_0\\ \Gamma, \Phi \Rightarrow \Delta\end{array} \qquad \Gamma', \varphi \Rightarrow \Delta', \Phi'}{\Gamma, \Gamma', \varphi \Rightarrow \Delta, \Delta'}\ \text{Multicut}$$

We can derive the conclusion of this derivation from the left premise $\Gamma, \Phi \Rightarrow \Delta$ without using multicut. First, use contraction to reduce Φ to the single instance φ (which must appear in both the right premise and the conclusion of the multicut by the assumption that the right premise is an identity sequent). Then use weakening to add the formulae in Γ' and Δ' to the sequent, and exchange to correct the ordering.

This takes care of two subcases of the rank 1 base case. The remaining subcase is that both premises of the multicut contain cut-formulae derived from identity rather than weakening and side-formulae, but non-structural rules are used above the multicut in both premises. For this, we refer back to our earlier observation that the resulting formulae from such usages are part of the side-formula-sequences Γ, Γ', Δ, and Δ' of the multicut rather than the cut-formula-sequences Φ and Φ'. As such, they are completely inert in the derivation step based on multicut, which offers us the opportunity to rework the derivation to postpone such inferences. For example, consider the following derivation, in which $\wedge$L occurs above a multicut:

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_0}{\Gamma, \psi, \theta \Rightarrow \Delta, \Phi'}}{\Gamma, \psi \wedge \theta \Rightarrow \Delta, \Phi'}\wedge\text{L} \qquad \dfrac{\mathcal{D}_1}{\Gamma', \varphi \Rightarrow \Delta'}}{\Gamma, \Gamma', \Box\psi \Rightarrow \Delta, \Delta'}\text{Multicut}$$

We can replace this with an equivalent derivation in which the $\wedge$L step occurs after the multicut, thusly:

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_0}{\Gamma', \psi, \theta \Rightarrow \Delta', \Phi'} \qquad \dfrac{\mathcal{D}_1}{\Gamma, \Phi \Rightarrow \Delta}}{\Gamma, \Gamma', \psi, \theta \Rightarrow \Delta, \Delta'}\text{Multicut}}{\Gamma, \Gamma', \psi \wedge \theta \Rightarrow \Delta, \Delta'}\wedge\text{L}$$

There is nothing special about $\wedge$L; the same reasoning applies to all single-premise non-structural rules. We then thus repeat this postponement reasoning on whatever non-structural rules are present until there are no non-structural rules above one premise of the cut, in which case we can eliminate the cut as was demonstrated earlier.

For derivations involving two-premise derivations above the multicut, we can also apply a postponement method, but the situation is more complicated. It may be that only one of the premises contains the cut-formula of the multicut, in which case the postponement can occur just as in the single-premise case. But it may also be that cut-formulae occur in both premises, as in this example:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1 \\ \Gamma, \Phi \Rightarrow \Delta, \psi_1\end{matrix} \qquad \begin{matrix}\mathcal{D}_2 \\ \Gamma, \Phi \Rightarrow \Delta, \psi_2\end{matrix}}{\Gamma, \Phi \Rightarrow \Delta, \psi_1 \wedge \psi_2} \wedge\text{R} \qquad \begin{matrix}\mathcal{D}_0 \\ \Gamma', \Rightarrow \Delta', \Phi'\end{matrix}}{\Gamma, \Gamma' \Rightarrow \Delta, \psi_1 \wedge \psi_2, \Delta'} \text{Multicut}$$

In this case, we can still rewrite the derivation to postpone the two-premise ∧R step, but two multicuts are required, producing this derivation:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1 \\ \Gamma, \Phi \Rightarrow \Delta, \psi_1\end{matrix} \qquad \begin{matrix}\mathcal{D}_0 \\ \Gamma', \Rightarrow \Delta', \Phi'\end{matrix}}{\Gamma, \Gamma' \Rightarrow \Delta, \psi_1, \Delta'} \text{Multicut}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', \psi_1} \text{Exchange-R} \qquad \dfrac{\dfrac{\begin{matrix}\mathcal{D}_2 \\ \Gamma, \Phi \Rightarrow \Delta, \psi_2\end{matrix} \qquad \begin{matrix}\mathcal{D}_0 \\ \Gamma', \Rightarrow \Delta', \Phi'\end{matrix}}{\Gamma, \Gamma' \Rightarrow \Delta, \psi_2, \Delta'} \text{Multicut}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', \psi_2} \text{Exchange-R}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', \psi_1 \wedge \psi_2} \wedge\text{R}$$

This may seem to be a step backward—the goal is cut-elimination, not cut-proliferation! However, both resulting multicuts do occur above the ∧R step, so the non-structural postponement is successful. We can continue to apply non-structural postponement to both of the new multicuts until they are eliminable by weakening via one of the subcases presented earlier.

That finally completes the base case. Suppose for the inductive case that we have a derivation where the topmost multicut uses a cut-formula φ of rank n, and that we already have access to algorithms for eliminating cuts where the cut-formula is of any lower rank. Depending on the form of the derivation, we will either eliminate this cut directly, or rewrite the derivation using only cut-formulae of lower rank, which are eliminable by inductive hypothesis. In particular, we examine the derivations of the two premise sequents of the multicut to see how the cut-formula φ was introduced. If one of the premise sequents is such that all instances of φ are introduced structurally (i.e., as identity sequents, weakenings, or side-formulae), then we eliminate the cut directly, by operating on the premise sequent in question as in the base case, treating φ as if it were an atom.

Suppose then that both premises of the multicut contain instances of φ that are derived by non-structural rules. We can assume that the non-structural rules in question occur immediately above the multicut step, as otherwise we could make it so using postponement techniques as in the base case. We also assume the cut-formula occurs only once in one of the premises, and moreover that we can choose which premise is free of repeated cut-formulae; if necessary, we can adjust the derivation by using multiple copies of the second premise derivation to cut away each occurrence of the cut formula individually in the derivation of the premise where we want the cut-formula to occur only once. This process may introduce extra side-formulae, but these can be contracted away after the final multicut.

By the definition of rank, we know that φ has the form $\neg\psi$, $\psi_1 \wedge \psi_2$, $\psi_1 \vee \psi_2$, or $\psi_1 \supset \psi_2$, where in each case at least one of the ψs has rank exactly $n-1$, and the other (when applicable) has rank at most $n-1$. The plan is to replace the derivation with one where these lower-rank formulae are used as the cut-formulae. We need to consider each case separately.

The $\neg\psi$ case proves to be really simple, as we have already assumed that the cutformulae are introduced by $\neg$L and $\neg$R steps. Simply omit those steps from the derivation, leaving ψ on the opposite side of each derivation. Then use multicut on ψ in the new derivations, and eliminate that multicut by inductive hypothesis.

In the $\psi_1 \wedge \psi_2$, $\psi_1 \vee \psi_2$, and $\psi_1 \supset \psi_2$ cases, we will have one premise featuring a two-premise introduction rule ($\wedge$R, $\vee$L, or $\supset$L), and the other featuring a one-premise introduction rule ($\wedge$L, $\vee$R, or $\supset$R). Let the side with the two-premise introduction rule be the one which is forced to have only one cut-formula occurrence by repeated multicuts; the other side can be reduced to single occurrences by contraction. Then simply omit the introduction rules, leaving us with a derivation containing ψ_1, a derivation containing ψ_2, and a derivation containing both. Then use two multicuts to combine these into a single derivation of the desired conclusion, and use the inductive hypothesis to eliminate the multicuts. ■

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