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Applications of Non-Classical Logic

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ABSTRACT

This dissertation is composed of three projects applying non-classical logic to problems in history of philosophy and philosophy of logic.

The main component concerns Descartes' Creation Doctrine (CD) – the doctrine that while truths concerning the essences of objects (eternal truths) are necessary, God had voluntary control over their creation, and thus could have made them false. First, I show a flaw in a standard argument for two interpretations of CD. This argument, stated in terms of non-normal modal logics, involves a set of premises which lead to a conclusion which Descartes explicitly rejects. Following this, I develop a multimodal account of CD, according to which Descartes is committed to two kinds of modality, and that the apparent contradiction resulting from CD is the result of an ambiguity. Finally, I begin to develop two modal logics capturing the key ideas in the multi-modal interpretation, and provide some metatheoretic results concerning these logics which shore up some of my interpretive claims.

The second component is a project concerning the Channel Theoretic interpretation of the ternary relation semantics of relevant and substructural logics. Following Barwise, I develop a representation of Channel Composition, and prove that extending the implication-conjunction fragment of **B** by composite channels is conservative. Finally, I argue that

standard accounts of negation in the tradition of relevant logic are ill-suited to this interpretation, with the axiom form of contraposition the culprit.

The final component is a project concerning the implication fragment of Frege's *Grundgesetze*. In *Grundgesetze*, unlike *Begriffsschrift*, Frege uses a proof system consisting of few axioms and many rules – as Peter Schroeder-Heister noticed, this proof system bears a striking resemblance to Gentzen's sequent calculi. I note that while the system does bear this striking resemblance, one of the structural features of Gentzen style systems, the cut rule, has additional duties in Frege's system. Not only does it perform cut, but also performs structural manipulations on sets of premises which are more usually built in to the data type of the premise sequent. I discuss the features of Frege's system which result in the cut rule having this property.

Applications of Non-Classical Logic

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Contents

Chapter 1

Introduction

This dissertation is in the broad subject of philosophical logic – the application of logic to problems in philosophy, and the methodology throughout tends toward the model theoretic. So the approach to philosophical logic most in evidence in this dissertation is that of using model theoretic tools, in particular tools from relational semantics for modal and non-classical logics, to characterise and solve philosophical problems.

There are three projects undertaken in this dissertation, all of which share this approach to philosophical logic. In this introduction, I shall describe these projects, the structure of the dissertation, and discuss themes running through the work here.

The first, and most substantial, project in the dissertation concerns the problem of interpreting Descartes' *Creation Doctrine* – this is, in short, the view that God has voluntary control over the creation of truths concerning the essences of objects (called "eternal truths"). Descartes holds that this is the case, and so seems to hold that eternal truths are possibly false, while elsewhere seems to be committed to the necessity of these truths. Finding a consistent interpretation compatible with the text concerning the creation doctrine has been a major subject of work in Descartes scholarship for many years. There are a number of extant interpretations, but for my purposes in this introduction, two of them are particularly salient. The first, due to Harry Frankfurt [30], called *universal possibilism*, holds that Descartes's commitment to the creation doctrine just shows that Descartes is not committed to the claim that eternal truths are necessary – indeed, Frankfurt holds that Descartes is committed to the claim that *no* propositions are necessary, and that all are merely possible. The second, due to E.M. Curley [23], called *limited possibilism*, holds that while Descartes is not committed to the claim that all propositions are possible, he is committed to the claim that all propositions are *possibly* possible. Curley goes on to give a natural deduction argument in non-normal modal logic for limited possibilism, taking himself to have shown that Descartes is committed to the view. This same proof was altered by van Cleve [76] to be an argument for universal possibilism – from which conclusion van Cleve inferred that Descartes' modal metaphysics is internally inconsistent.

Three of the chapters of this dissertation concern the creation doctrine. Chapter 2 – The Creation Doctrine and Modal Arguments for Possibilism – shows that the premises of the natural deduction argument given by Curley (and those in the argument given by van Cleve) deliver a conclusion which Descartes explicitly rejects. This conclusion, stated informally, if that if God wills a proposition to be necessary, then God was necessitated to will that proposition true. Descartes seems to indicate that God's willing a proposition to be necessary does not entail God's being necessitated to will it, and so the premises invoked by Curley and van Cleve already miss their mark. The key upshot of this chapter is that Descartes may not be committed to universal or limited possibilism, and that other interpretations, perhaps incompatible with these, are available.

Chapter 3 – A Multimodal Interpretation of Descartes' Creation Doctrine – goes on to develop such an alternative interpretation. This is one according to which Descartes' writing hide an ambiguity between at least two distinct metaphysical modalities. One pair of which, the i-modalities, concern truths about the actual essences of created things, and the other of which, the o-modalities, concern truths about any essences God might have created, or assigned to created things. In this chapter, I first develop this view against the background of universal and limited possibilism, to show that the multimodal view is a natural combination of these, and then go on to consider in more detail Cartesian essences, God's relationship to these essences, and the relationship between i-modalities, o-modalities, and epistemic modality in the sense of conceivability. Along the way, I consider various details about the interpretation, compare it to other extant interpretations, and argue (informally) that the interpretation is consistent, and that according to the interpretation, Descartes' modal commitments are not especially bizarre.

Chapter 4 – Two Logics of Variable Essence – begins the process of developing a robust, rigorous model theory capturing the multimodal interpretation developed in Chapter 3. The core moves here are that there are two pairs of modal operators, one corresponding to the i-, the other to the o-modalities, and that objects have some of their properties essentially, with an object's essential properties at a world being a set of properties (in extension), and yet these can differ across the set of worlds in a model. Against this background, I develop two logics. The first sets the elements above against a simple background of a Kripke model with elements (worlds) which are consistent and closed under classical logic. This, the logic of *classical variable essences* (or **CVE**) is simple, and I provide a sound and complete proof system, as well as some further machinery to characterise it's models. The second logic allows for models with worlds at which the non-modal logical vocabulary can exhibit different behaviour – for instance, where negation doesn't obey the usual Boolean rules. This, the logic of *non-classical* variable essences (**NVE**) is introduced, and some results are proved characterising its models, and comparing it to **CVE**. Some results are proved to show that these logics capture key elements of the creation doctrine, and some further results are given which suggest, by way of proving some key lemmata, that the logics are consistent, and conservative extensions of classical **S5**. These two are formal analogues to the claims made in Chapter 3 – that the interpretation is consistent, and is not modally bizarre.

Chapter 5 – Channel Composition and Ternary Relation Semantics – is a paper concerning the project of modelling the flow of information using the ternary relation semantics. On this picture, the ternary relation itself $R\alpha\beta\gamma$ is to be interpreted as saying "α is channel permitting information flow between various states of information – in particular, it permits flow from the signal β to the target γ". The account of information flow over channels, originally proposed by Barwise [4], was fitted to the ternary relation semantics by Restall [59], and a number of others. The result is a reading of the ternary relation semantics wherein the points in a ternary relation frame are information states (closed under First Degree Entailment (or some sublogic of this, depending on the logical connectives under consideration)), the ternary relation is understood in these channel theoretic terms, and the various frame constraints placed on the ternary relation to get different logics have intuitive interpretations in terms of the operations of channels. Among the core requirements laid out by Barwise are that channels may be defined in terms of other channels – given two channels α, β, one can compose them either in parallel, or in series. Restall has proposed reading serial composition in terms of the relevant connective of fusion $\circ$. This provides a relatively natural reading of the semantics, but it has two major downsides. It collapses together the action of composing two information channels and the action of applying a channel to a signal (which $\circ$ always models in the ternary relation semantics) – these actions are clearly distinct. Second, and of more formal interest, Restall's approach winds up barring the channel theoretic interpretation of very weak relevant logics. For his account to work, $\circ$ must be associative – that is, R must obey frame constraints corresponding to the combinators B, B' – and some interesting logics fail this test – particularly the basic system of the ternary relation semantics **B**, and some of the depth relevant systems proposed by Sylvan and Brady over the years, like **DJ**, **DW**, and **MC**. These logics are of interest for a number of reasons, and this reading of the ternary relation semantics would be better for incorporating them.

In the paper, I propose a different definition of channel composition which looks much more like the usual function composition than Restall's account. I prove that the set of models of the conjunction-implication fragment of **B** – called $\mathbf{B}_\wedge$ – which have points behaving as composite channels (as I define them) enforces the same consequence relation as the full set of $\mathbf{B}_\wedge$. The upshot is that for $\mathbf{B}_\wedge$ my proposed interpretation fits nicely. However, the argument I give does not extend to **B** with the language including disjunction or negation. The most interesting connective is negation, because the reason seems to be that on standard ways of interpreting negation in the ternary relation semantics, the axiom form of contraposition seems to require an implausible frame constraint on the ternary relation – that is, implausible under the channel theoretic interpretation. The upshot is that some other way of interpreting negation is required, or that the channel theoretic interpretation runs into serious trouble in the realm of weak logics in the ternary relation family.

Chapter 6 – Structural Features of the Implication Fragment of Frege's *Grundgesetze* – concerns the interesting proof system developed by Frege in the *Grundgesetze*. In the *Begriffsschrift*, Frege's proof system looks very much like a Hilbert style axiomatic system. In the *Grundgesetze*, he changes this and adopts a system with a limited set of axioms and a large number of rules. Schroeder-Heister [70] noted that this system was more like a Gentzen-style sequent calculus than an axiomatic system, and proposed a sequent system to capture Frege's proof system. In this paper, I note that Frege's system has an interesting structural feature which Schroeder-Heister's presentation represses. The various 'structural' rules which Frege gives are stated in terms of left-nested conditional formulae, and the cut rule seems to allow for the rearranging of conditional formulae into this left-nested structure. This corresponds to the structural rule of associativity of the structural connective of sequent systems – which corresponds to the combinators B, B'. This structural rule is, usually, cooked into the data type – the association of some list of the members of a set, multiset, or sequence of formulae does not matter. In Frege's system, it's somewhat different, as this structural rule is not apparent from the way he writes his formulae, and its activity is built into his cut rule. I present a proof system which captures this feature of Frege's system, and discuss some salient details.

The projects here concern philosophical logic, all but Chapter 6 concern model theoretic techniques in particular, and all but Chapter 5 concern the application of philosophical logic in the study of historical figures in philosophy. While the dissertation consists in some different projects, these projects are unified by a methodology, and a focus on interpretive questions – I take it that interpreting the ternary relation semantics, and historical work in philosophy and logic are not radically different enterprises, though they may have different ends.

Chapter 2

Modal Arguments for Possibilism

2.1 The Creation Doctrine & Descartes' Modal Metaphysics

The status of necessity and possibility in Descartes' philosophy is a controversial and difficult topic, and there is a particular tension involving necessity and God's will which has troubled both his contemporaries and recent interpreters.

On on hand, most interpreters and commentators seem to agree that Descartes admits the existence of some necessary truths.[1] It certainly seems that Descartes holds that certain mathematical and physical truths are necessary. He even seems to use possible worlds as an intuition pump in favour of the claim that certain truths (laws) are such that 'we cannot but judge them infallible when we conceive them distinctly, nor doubt that, if God

[1]The claim that Descartes does admit some necessary truths has been questioned by ([30] *Descartes on the Creation...*). However, most accounts in this literature go the other way, and take it as read that Descartes does accept some necessary truths. Sources discussing Descartes on necessary truth are many, but some which are directly relevant to the issue of the creation doctrine include ([40] *Descartes' Creation Doctrine and ...*; [1] *Descartes, Conceivability, and Logical Modality*; [38] *The Status of Necessity and Impossibility...*; [54] *Divine Simplicity and the Eternal...*; [13] *Descartes's Method and the...*) and ([22] *Descartes' Modal Metaphysics*) gives a survey of the landscape.

had created many worlds, the laws would be as true in all of them as in this one.' (*Le Monde*, Ch. VII, AT 11:47, [19] *The Philosophical Writings of Descartes* Volume 1[2], p. 97)[3] In correspondence, a class of truths which Descartes seems to assume are necessary is referred to as the *eternal truths*, following a tradition in ancient and medieval philosophy to characterize necessity in temporal terms.[4]

On the other hand, there is Descartes' infamous doctrine that all truths are *freely* created by God. This is a form of *voluntarism* regarding truth, but I shall follow Curley and refer to this as the *creation doctrine*. This doctrine is expressed in a number of letters in Descartes' correspondence, and is briefly discussed in some responses to objections to the *Meditations*. It seems that Descartes held some version of the creation doctrine for much of the most philosophically productive period of his life, as it is in evidence in his correspondence from the Letter to Mersenne, 15 April 1630 (AT 1:135–147, CSMK,p. 23) to at least the Letter to Mesland, 2 May 1644 (AT 4:111–120, CSMK, p. 235).[5] So we are beholden to take it seriously as a part of his broader metaphysical commitments.

With this gloss, it is straightforward to how these two commitments – to the existence of necessary truths and to the creation doctrine – are in tension. Following Kaufman's presentation in ([40]) the difficulty can be laid out as between the following claims:

(i) Eternal truths are freely created by God.

(ii) Eternal truths are necessary.

[2]References to Descartes' writing will be to the three volumes of the Cottingham et. al. collections in English translation (along with page and volume numbers for the Adam-Tannery collections). Volume 1 ([19]) will be cited as CSM1, volume 2 ([18]) as CSM2, and volume 3 ([20].) will be cited as CSMK.

[3]Too much weight should not be placed on this point of similarity with possible worlds semantics, and it is at best a stretch to think of Descartes as any kind of possible worlds theorist.

[4]Some details on this tradition are available in Knuuttila's SEP article *Medieval Theories of Modality* ([42]).

[5]Some of the relevant passages will be discussed later in the paper.

(ii) simply notes that the eternal truths, including propositions from mathematics and physics and, in addition, certain theological claims, are necessary according to Descartes. These seem to be the kinds of truths which underwrite Descartes' broader rationalist project, as being those which we can clearly and distinctly perceive.

However, the creation doctrine is stated in universal generality – *all* truths are freely created by God. It is commonly assumed, and Curley takes it to be part of the concept of free action that when an agent does something freely, this means that they *could* have done otherwise. (i) states that God's creation of the eternal truths was free in just this way. A plausible, though contested, consequence of (i) is that God could have made any eternal truth false. From this it seems reasonable to draw the inference that for any eternal truth you choose, God could have made it false. Let us use lower case Greek letters as variables for propositions. It seems that 'it is possible that God made ϕ false' is a reasonable paraphrase of 'God could have made ϕ false.' Hence, by an inference from (i), we can obtain:

(iii) For any eternal truth ϕ, it is possible that God made ϕ false.

However, (ii) can be naturally parsed as:

(ii)' For any eternal truth ϕ, it is necessary that ϕ.

This makes the tension more clear. If (iii) is the case, it seems that any eternal truth could have been false, had God chosen to make it false.[6] It seems that Descartes is committed to the existence of a class of claims which, though necessarily true, are possibly false.

This issue makes interpreting Descartes' modal metaphysics a difficult project, and his few explicit claims on the subject provide a number of constraints which restrict potential interpretations. In addition to these textual restrictions, work by Curley and Van Cleve have

[6]Though this is a disputed conclusion, this is good enough to show the problem.

attempted to further restrict this space. Curley in [23] gives an argument in modal logic using premises which purport to express the core features of (i) and (ii), the conclusion of which is *limited possibilism*. According to limited possibilism, while some propositions are *necessary*, no propositions are *necessarily necessary*. Another way to put this point is to say that all propositions are *possibly possibly* false, even those which are in fact necessary. This interpretation is a response to the view presented in ([30]) that Descartes is committed to the claim that every proposition is possible, which has been called *universal possibilism*.

Van Cleve in ([76] *Destruction of the Eternal Truths*) uses Curley's argument structure, with minor changes, to argue that Descartes is committed not only to limited possibilism, but to universal possibilism, and that this commitment is inconsistent with Descartes' other commitments. It is Van Cleve's contention that Descartes modal metaphysics is incoherent, and that the creation doctrine seems the clear culprit.

Either of these consequences result in serious constraints on providing an interpretation of creation doctrine. If Van Cleve is correct, then no intelligible interpretation can be given, or at least no consistent interpretation. If Curley is correct, then any interpretation must be consistent with limited possibilism.

My purpose in this paper is to show that the premise set and inference rules given by Curley and Van Cleve do not express the commitments of Descartes' modal metaphysics. I argue that each author employs a premise and a principle of inference which allows one to infer a consequence which directly contradicts an explicit claim made by Descartes on the subject of the creation doctrine.

Namely, I shall argue that from Curley's and Van Cleve's premises and modal principles, we can infer the following *Principle of Divine Necessitation* (PDN):

(PDN) If God wills ϕ to be necessary, then it is necessary that God wills that ϕ.

This consequence contradicts a claim made by Descartes in the correspondence with Mersenne:

> And even if God has willed that some truths should be necessary, this does not mean that He willed them necessarily; for it is one thing to will that they be necessary, and quite another to will this necessarily, or to be necessitated to will it.
>
> (Letter to Mesland, 2 May 1644, AT 4:118–119, CSMK, p. 235)

This passage rules out the inference from 'God wills that ϕ is necessary' to 'it is necessary that God wills that ϕ' – a statement of which can be inferred from the premises used by Curley, and those used by Van Cleve in the course of their arguments. I argue that, as a result, neither of their conclusions are among Descartes' commitments. In section 2.2, I shall argue that the logical tools I shall employ are suited to the project, despite the potential objection that they are anachronistic or obscure. In section 2.3, I shall discuss Curley's argument for limited possibilism, giving some reasons why his premises should be taken seriously as expressing Descartes' metaphysics, and discuss a modal inference, Becker's Rule, which plays a substantial role in his argument. Then I'll give his argument and show how PDN can be inferred from Becker's Rule and one of his premises. In section 2.4, I'll discuss Van Cleve's argument and show how his assumptions also allow the derivation of PDN. In section 2.5, I briefly discuss a minor variation on PDN discussed by Curley, and argue that it is also derivable from Curley's assumptions.

2.2 Why Use Logical Tools?

This paper employs some tools from non-normal modal logics to study features of Descartes' modal metaphysics. Why would one want to do this?

The reasons to take this kind of approach seriously are many, and I'll start from the general reasons why one ought to use formal tools when studying this question at all to the more specific reasons why I shall use tools from *non-normal* modal logic in this paper.

2.2.1 Logical methods are a good fit for the problems raised by Creation Doctrine

The nature of the creation doctrine suggests logical tools because the questions which have been raised by commentators are largely questions either to do with (a) whether Descartes' modal metaphysics is coherent or, (b) whether Descartes' modal principles are especially bizarre. (a) has been raised by some of Descartes' contemporaries (including those in the correspondence) and more recently by ([76]) and ([32] *Logic Matters*, p. 179), both of whom claim that Descartes' total theory is incoherent (or "great nonsense" to use Geach's term). (b) has been raised by commentators, but perhaps most explicitly by ([39] *The Creation of the Eternal Truths*, p. vii). Both (a) and (b) are issues which are quite naturally addressed using logical tools. While it's somewhat unclear what makes a theory incoherent, a formal analogue is inconsistency, for which we do have a clear and rigorous definition.[7] Then we have the question of whether Descartes' theory is consistent, and this is a question which is best solved using formal tools. A logical method for showing that a theory is consistent is to build a model of the theory. Establishing that a theory is consistent in this way goes some way towards showing that the theory is coherent and can be made sense of. Certainly this doesn't go all the way to showing that the theory is plausible, but it certainly helps a theory's case to be provably consistent. In most cases, we can generally assume that a theory is consistent until we have some reason to suspect it isn't, but in the case of

[7] One might claim that a coherent theory might be inconsistent, as would a dialetheist such as ([56] *In Contradiction*), but I leave this point to the side.

giving a theory of Descartes' modal metaphysics, the question is on the table as to whether one can give a consistent theory at all. So some kind of formal methodology is called for by the nature of the criticisms raised against Descartes' modal metaphysics.

More specifically, since the creation doctrine is a doctrine about how God's will relates to modal notions, it seems that some kind of modal logic is a natural way to think about the problem, Descartes' claims, and what those claims commit him to. While Descartes himself did not and could not have known about modal logics in the modern sense, and the toolkit I propose to bring to bear in dealing with his modal metaphysics, these tools do allow us to provide a clearer picture of the consequences of Descartes' claims. While I don't provide a model of the creation doctrine here, the possible models of the theory are constrained by the claims to which Descartes is committed, so the argument of this paper is relevant to the project of modelling the creation doctrine in that it provides a clearer picture of the space of potential models.

2.2.2 Possibilisms and Non-Normal Modal Logics

More specifically, the use of *non-normal* modal logics is motivated by a couple of considerations. First, I shall clarify what non-normal modal logics are and briefly describe some of their salient features. The distinction between normal and non-normal modal logics is naturally stated using the Kripke semantics for modal logics ([43]; [44] *Semantical Analysis of Modal Logic* I and II). In Kripke semantics, modelling conditions result in picking out two principles which define the class of normal modal logics. The most important for our purposes is the rule of necessitation (RN), which allows one to infer from the fact that a proposition is *valid*, or true at all possible worlds in all models, that the necessitation of that proposition is also valid. Written using the $\Box$, this can be expressed:

$$\text{(RN)} \qquad \text{From } \phi \text{ infer that } \Box\phi.$$

This rule is valid in all normal modal logics. However, this inference principle is a bad fit with any kind of possibilism. Suppose that every proposition is possible. From this is follows that the *negation* of every proposition is possible. In addition, a feature of modal logics which is had in common between almost all seriously considered modal logics (normal or non-normal) is that necessity and possibility are related by negation in the following ways:

- $\neg\phi$ is possible ($\Diamond\neg\phi$) iff ϕ is not necessary ($\neg\Box\phi$)
- $\Box\neg\phi$ iff $\neg\Diamond\phi$

Given the rule of necessitation, if one admits that there are *any* logical validities, such as the law of excluded middle for instance, then one can show that the necessitation of that validity is also valid. So if ϕ is valid, then by necessitation, so is $\Box\phi$. However, from universal possibilism, one has that $\Diamond\neg\phi$ is also valid, and hence $\neg\Box\phi$ is valid as well. But then we have both $\Box\phi$ and $\neg\Box\phi$ as valid.

The problem remains for limited possibilism. There we have that every proposition is possibly possible. In symbols, for every ϕ, $\Diamond\Diamond\phi$. Just as before, a consequence of this is that the negation of every proposition is possibly possible: hence $\Diamond\Diamond\neg\phi$, which is equivalent to $\neg\Box\Box\phi$. Now if we suppose that ϕ is valid, we must simply apply the rule of necessitation twice to show that $\Box\Box\phi$ is valid.

So when dealing with possibilisms, we cannot have the rule of necessitation – it immediately undermines the possibilist's claim.

2.2.3 Curley, Van Cleve, and Non-Normal Modal Logics

A final reason for employing non-normal modal logics in particular is that these are the kinds of logic which Curley and Van Cleve employ. While neither provides a clear indication of exactly *which* modal logic they are using, it is quite clear that they logics they use are non-normal. First, because both present possibilist interpretations of Descartes and second, because in addition to using the notions of necessity and possibility, they also employ the notion of *entailment* by which they seem to mean *strict implication*. C.I. Lewis in two famous books ([45] *A Survey of Symbolic Logic*; [46] *Symbolic Logic*) introduced strict implication in order to develop a theory which avoided some issues with classical logic and material implication. Lewis' logics were among the first modern modal logics, and they were primarily focused on the modal features of the conditional connective, rather than on necessity and possibility by themselves.

These facts about Lewis are worth noting because the Curley style argument uses a strict implication, and the premises are stated in terms of strict implication. Even further, Van Cleve's argument uses both a strict implication and a material implication, and exploits the difference between these. While a kind of necessitated implication connective can be defined in all modal logics (with the appropriate vocabulary) – this kind of connective is just defined as $\Box(\phi \supset \psi)$, where $\supset$ is the material implication – using strict implication to model entailment usually takes place in a non-normal modal logics, and it was some of these logics which Lewis himself favoured.

These reasons motivate using some tools from non-normal modal logics in analysing the arguments given by Curley and Van Cleve, despite the fact that these logics are less well known than the usual stock of normal modal logics.

In order to distinguish between kinds of conditionals, I'll continue to use the horseshoe: $\supset$ for material implication, and I'll use Lewis' fish-hook symbol $\prec$ for the strict implication.

2.3 Curley's Argument for Limited Possibilism

Curley's argument proceeds from two premises by a collection of modal inferences. I'll give Curley's two premises and some justification for them. Then I'll discuss the most distinctively modal inference in Curley's argument, and show that PDN is a consequence of this logical background.

2.3.1 Curley's First Premise

The first premise is intended as an expression of the *freedom* of God's free creation. The the statement of the premise is:

(C1) If an agent wills some ϕ, then they could have not willed ϕ.

Curley justifies C1 as 'a general logical truth about acts of will' ([23],p. 580). It is worth noting that for the argument to go through that we don't need this premise in its full generality, but rather just the claim that when *God* wills ϕ, *God* could have not willed ϕ.

A reason to go with the form which refers only to God is that the universal formulation of C1 seems in tension with another of Descartes' famous views, on human free will. In the fourth meditation of the *Meditations on First Philosophy*, Descartes claims that when making a decision about which of two incompatible claims to accept 'In order to be free, there is no need for me to be inclined both ways; on the contrary, the more I incline in one direction – either because I clearly understand that reasons of truth and goodness point that way, or because of a divinely produced disposition of my inmost thoughts – the freer is my choice' (AT 7:57–58, CSM2, p. 40). He seems to hold that for human beings, a compulsion to believe some claim due to a recognition of its truth, or a disposition, is compatible with freedom of the will to believe it.

I mention this point only to say that this isn't clearly a problem if C1 is instead restated in a more particular form:

$$\text{(C1)}' \qquad \text{If God wills that } \phi\text{, then God could not have willed } \phi.$$

In order to express this claim in a formal language, I'll continue to use lower case Greek letters $(\phi, \psi, \chi, \ldots)$ standing in for propositions, and a collection of variables, lower case letters of the Latin alphabet $(a, b, c, \ldots)$, ranging over agents and a relation symbol $Wa\phi$, which should be read as 'agent a wills that ϕ.' I'll reserve g to refer to God. With this, the universal formulation of Curley's first premise can be stated in terms of strict implication as:

$$\text{(C1)} \qquad \forall a(Wa\phi \strictif \Diamond\neg Wa\phi)$$

and the restricted version formulated as:

$$\text{(C1)}' \qquad Wg\phi \strictif \Diamond\neg Wg\phi$$

To set the ground, I'll present some passages from Descartes showing that he may have accepted (C1)$'$.[8] Much of this evidence comes from the standard references regarding the

[8]The formalized version of (C1)$'$ should be read as an axiom scheme, where one may substitute any well-formed formula for ϕ. In Curley ([23]), the argument is stated using a kind of propositional quantification – where Curley also quantifies over the ϕ above. This involves some additional logical complexity that is, as far as I can tell, not needed to state the view. So I shall proceed with axiom schemata. One may worry about having formulas occur in relations alongside names for agents. This can be solved in any of the usual ways – Gödel numbering, Quine quotes, reading the relevant vocabulary as Priorean Connecticates. Nothing I say will distinguish between these various approaches, and all that matters is that we can distinguish names for agents from names for propositions. So any method will work, and I won't choose between them.

Creation Doctrine, taken largely from the correspondence. Consider the following two passages:

> It will be said that if God has established these truths He could change them as a king changes his laws. To this the answer is: Yes He can, if His will can change. 'But I understand them to be eternal and unchangeable.' – I make the same judgement about God. 'But His will is free.' – Yes, but His power is beyond our grasp. In general we can assert that God can do everything that is within our grasp but not that He cannot do what is beyond our grasp. It would be rash to think that our imagination reaches as far as His power.
>
> (To Mersenne, 15 April 1630, AT 1: 145–146, CSMK, p. 23)

> You ask also what necessitated God to create these [eternal] truths; and I reply that He was free to make it not true that all radii of the circle are equal – just as free as He was not to create the world. And it is certain that these truths are no more necessarily attached to His essence than are other created things.
>
> (To Mersenne, 27 May 1630, AT 1:151–152, CSMK, p. 25)

These passages make a compelling case for (C1)$'$. In particular, as regards the second, if Descartes is willing to claim that God was free to change the truths of geometry, then it is likely that he would claim that God was free to make any truths false. Indeed, the talk of these (eternal) truths being no more attached to His essence than any other truths seems to bear out the point that if God could have made these necessary truths false, then He could have made any truth false. The following passage, written some years later than the previous passages, expands on the point:

> I turn to the difficulty of conceiving how God would have been acting freely and indifferently if He had made it false that the three angles of a triangle were equal to two right angles, or in general that contradictories could not be true together. It is easy to dispel this difficulty by considering that the power of God cannot have any limits, and that our mind is finite and so created as to be able to conceive as possible the things which God has wished to be in fact

> possible, but not be able to conceive as possible things which God could have made possible, but which He has nevertheless wished to make impossible. The first consideration shows us that God cannot have been determined to make it true that contradictories cannot be true together, and therefore that He could have done the opposite. The second consideration assures us that even if this be true, we should not try to comprehend it since our nature is incapable of doing so.
>
> (To Mesland, 2 May 1644, AT 4:118, CSMK, p. 235)

Again, the text provides some evidence in favour of (C1)$'$. One may read the consequence Descartes draws from the second consideration as reason not to try to understand the creation doctrine, and thus as reason not to engage in this interpretative project. However, what the passage seems to suggest is not that the creation doctrine itself is incomprehensible, but rather that the impossible things God might have brought about will be incomprehensible to us, given that God has provided us with our conceptual capacities. There is much that has been, and can be, said about the relationship between conceivability and possibility in Descartes, and some of the extant literature on the creation doctrine delves into this relationship. For instance, in ([51] *Descartes on Modality*) and ([13]). This is an interesting literature, but one which extends beyond the focus of this paper.

In What Does God's Freedom Consist?

A natural question at this point is in what God's freedom consists. This is certainly an important question, and one about which there is controversy.[9] For the purposes of this paper, I take on the modal reading given by Curley and, apparently, adopted by Van Cleve. This is primarily for the reason that I want to show that even supposing that his premises

[9] For instance, ([40]) argues that the creation doctrine is not a modal principle, but rather an expression of divine indifference.

are acceptable as interpretation, Curley's argument still doesn't deliver the conclusion that Descartes is a limited possibilist. The argument I am giving is not that Descartes actually does take God's freedom to be, as Curley suggests, a matter of being able to create things in accordance with some other world, but rather that even assuming that this is the case, his comments do not bear out the conclusion that he is committed to limited or universal possibilism, at least on the basis of the argument structure we'll see shortly.

This is an involved and interesting question, and there have been interesting interpretations which read this freedom in other terms. However the question of how best to understand God's freedom is not the focus of this paper.

2.3.2 Curley's Second Premise

Curley's second premise reads:

A proposition ϕ is true if and only if God wills ϕ to be true.

Consider the following passage as evidence:

> As for the eternal truths, I say once more that they are true or possible only because God knows them as true or possible. They are not known as true by God in any way which would imply that they are true independently of Him. If men really understood the sense of their words they could never say without blasphemy that the truth of anything is prior to the knowledge which God has of it. . . . So, we must not say that if God did not exist nevertheless these truths would be true; for the existence of God is the first and most eternal of all possible truths and the one from which alone all others proceed.
>
> (To Mersenne, 6 May 1630, AT 1:149–150, CSMK, p. 24)

This passage provides some fairly strong evidence that Descartes would have accepted premise (2), that ϕ is true if and only if God wills that ϕ. At very least, the latter half provides compelling reason to think that Descartes would admit that if ϕ is true then God wills ϕ to be true, and the other direction of the biconditional is not particularly controversial.

In the formal language, I shall use the equality symbol $=$ as a shorthand for the strict biconditional. That is, $\phi = \psi$ should be read a $(\phi \strictif \psi) \wedge (\psi \strictif \phi)$. With this in mind, the formal statement of C2 is:

$$\text{(C2)} \qquad \phi = Wg\phi$$

2.3.3 Becker's Rule

The last piece of modal logic we need in place to put together Curley's argument is a rule governing interaction between necessity and strict implication. It allows you to infer from the fact that ϕ strictly implies ψ, that necessarily-ϕ strictly implies necessarily-ψ. In symbol form:

$$\text{(Becker's Rule)} \qquad \text{From } \phi \strictif \psi \text{ infer } \Box\phi \strictif \Box\psi.$$

Discovered by and named after Oskar Becker, this rule holds in Lewis' modal logic **S2**, and every non-normal modal logic stronger.[10] This includes most of the seriously studied systems of non-normal modal logic, with only few logics failing to validate this inference rule.

The way that Curley employs Becker's Rule also involves a somewhat hidden use of a rule of contraposition. This rule states that:

$$\text{(Contraposition)} \qquad \text{From } \phi \strictif \psi\text{, one may infer } \neg\psi \strictif \neg\phi.$$

[10]This fact, and other details about non-normal modal logics, are available in ([36] *New Introduction to Modal Logic*).

The inference which Curley employs is a mix of these two inference rules. Recall that $\Diamond\neg\phi$ is equivalent to $\neg\Box\phi$ – that a proposition is possibly false if and only if it isn't necessary. With this in mind, the inference principle which Curley employs is:

$$(*) \qquad \text{From } \phi \strictif \psi \text{ infer } \Diamond\neg\psi \strictif \Diamond\neg\phi.$$

One need only apply Becker's Rule to $\phi \strictif \psi$ to obtain $\Box\phi \strictif \Box\psi$. Then apply contraposition to obtain $\neg\Box\psi \strictif \neg\Box\phi$, and replace instances of $\neg\Box$ with $\Diamond\neg$ to obtain $(*)$. This inference is the most substantial modal inference Curley's argument employs. The rest of the argument is a matter of breaking down his premises.

Now with this in the background, I turn to the argument proper.

2.3.4 Curley's Argument

The argument, with some minor changes in notation, is as follows:

1.	$Wg\phi \strictif \Diamond\neg Wg\phi$	(C1)$'$
2.	$\phi = Wg\phi$	(C2)
3.	$\Box\phi$	Assumption for Conditional Proof
4.	$\Box\phi \strictif Wg\Box\phi$	2 Conjunction Elimination
5.	$Wg\Box\phi \strictif \Diamond\neg Wg\Box\phi$	1 Instantiation of $\Box\phi$ for ϕ
6.	$\Diamond\neg Wg\Box\phi$	3,4, and 5 Modus Ponens
7.	$\Diamond\neg Wg\Box\phi \strictif \Diamond\neg\Box\phi$	$(*)$ applied to 4
8.	$\Diamond\neg\Box\phi$	6,7 Modus Ponens
9.	$\Box\phi \strictif \Diamond\neg\Box\phi$	3–8 Conditional Proof
10.	$\Diamond\Diamond\phi$	Equivalent to 9

The step leading to line (7) is key, and this relies on ($*$), but as I'll now show, this principle along with (C2) leads directly to PDN. The last step involves some substantial modal reasoning, but I don't need to deal with the inferences used there to make my point.

2.3.5 PDN from Curley's Premises

The proof of PDN from Curley's premises is short, consisting of little more than (C2) and Becker's rule.

1. $\phi = Wg\phi$ (C2)
2. $\phi \prec Wg\phi$ 1 Conjunction Elimination
3. $\Box\phi \prec \Box Wg\phi$ 2 Becker's Rule

As mentioned at the end of §1, there is good textual reason to think that Descartes would have rejected this consequence.

One potential response to my claim that PDN is unacceptable might be that it is *vacuously true*. A universal possibilist, insofar as they claim that no propositions are necessary, may well say that it PDN is a non-problematic consequence of their view. Their view simply states that the antecedent can't be satisfied.

First, this line of response is not available to the proponent of *limited* possibilism, who may well admit that some propositions are *necessary*, while denying that any are *necessarily necessary*.

Second, this line of response seems unsatisfying when Descartes seems to explicitly deny *the conditional*. Descartes says 'even if God has willed that some truths should be necessary, this does not mean that He willed them necessarily.' (AT 4:118, CSMK, p. 235) This seems to be denying the *reasoning* from the antecedent to the consequent, and not just either the antecedent or the consequent itself. He appears to be denying that the *entailment* holds, which speaks against the truth of the conditional itself. So the vacuous satisfaction line doesn't provide much comfort to the possibilist giving this argument.

Since there are so few steps in the argument above, it seems that one has few options for altering Curley's argument to avoid this consequence. Either one has to reject Becker's Rule, or one has to replace (C2) with some other premise tying the truth of propositions to God's will. While not in response to this problem, Van Cleve does replace (C2) with a different premise we'll consider momentarily. First, there is a potential difficulty for my argument in Curley's paper which I'll address.

2.3.6 Curley's Second Formulation of Limited Possibilism

Curley, after giving the argument above, claims that $\Diamond\Diamond\phi$ is not the best statement of limited possibilism, and provides a different statement of the view. This later formulation is more complicated, and Curley notes that $\Diamond\Diamond\phi$ isn't much more palatable than $\Diamond\phi$ as a modal thesis, particularly as the distinctions between the model structures governing the various non-normal modal logics in question are quite subtle.

However, whether or not his preferred statement of limited possibilism is $\Diamond\Diamond\phi$ or something more complex, his argument still apparently provides us with the constraint that $\Diamond\Diamond\phi$. He doesn't, to my knowledge, anywhere reject either of the premises he assumed or the modal principles he uses. Indeed, Van Cleve, as we'll see, gives his argument in response to the argument for the $\Diamond\Diamond\phi$ formulation. The focus of the criticism in this paper is not on Curley's final interpretation, but rather on the modal argument he gives to justify $\Diamond\Diamond\phi$ in the first place. (C1)$'$ and (C2) look plausible as interpretative postulates, and my aim is to show that even if we accept these assumptions, we have good reason *not* to accept the conclusion of this argument. Curley's later formulation of limited possibilism, though interesting, is not on the table.

2.4 Van Cleve's Argument for the Incoherence Thesis

Van Cleve criticizes Curley on the grounds that Curley's own argument results in *universal possibilism* rather than just *limited possibilism* ([76], p. 60). His argument for this claim is a minor variation on Curley's, but the details of that argument are not necessary for my purposes here. What is of interest is that he proposes that one might attempt to avoid the claim that Descartes is committed to universal possibilism by altering Curley's (C2). He considers, and immediately dismisses, the premise:

$$\Box\phi = Wg\Box\phi$$

on the grounds that God's omnipotence can't apply *only* to necessary propositions, but must apply to propositions generally. 'If God is truly omnipotent, the truth of modal and non-modal propositions alike should be dependent on His will', Van Cleve claims ([76],p. 60). It is beyond my purposes here to criticize his characterization of omnipotence. What is important here is that Van Cleve proposes an additional alteration of (C2), which he claims is implied by a passage in the 6 May 1630 letter to Mersenne, particularly the comment about how eternal truths are true or possible *only* because God knows them to be so. The premise he takes this passage to imply is:

$$\text{(V2)} \qquad \Box\phi \supset (\phi \strictif Wg\phi)$$

This premise gives rise to an argument for universal possibilism using many of the same inference principles which are used in Curley's argument. Perhaps the only major difference in the principles used by these two arguments is that Van Cleve relies on an additional modal axiom – a version of axiom **T** ($\Box\phi \supset \phi$) stated with the strict rather than the material implication. Hence, I shall call this principle:

$$\text{(Strict-}\mathbf{T}\text{)} \qquad \Box\phi \strictif \phi$$

Whether Descartes was committed to Strict-**T** is not at issue, and it does not feature in my argument that Van Cleve's assumptions also deliver PDN, so I won't discuss it except to note that it is a difference in the modal principles these authors employ.

2.4.1 Van Cleve's Argument for Universal Possibilism

Van Cleve's argument is as follows. Once again, the structure is a conditional proof, though the conditional introduced is material, rather than strict. In addition, the key modal principles are Strict-**T**, and the contraposed Becker's rule, denoted with ($*$), used by Curley. [11]

1.	$\Box\phi$	Assumption
2.	$\Box\phi \supset (\phi \strictif Wg\phi)$	(V2)
3.	$\phi \strictif Wg\phi$	1,2 Material Modus Ponens
4.	$Wg\phi \strictif \Diamond\neg Wg\phi$	(C1)
5.	$\Box\phi \strictif \phi$	Strict-**T**
6.	ϕ	1,5 MP
7.	$Wg\phi$	3,6 MP
8.	$\Diamond\neg Wg\phi$	4,7 MP

[11] Van Cleve seems to use it as an *axiom*, referring to it as $((\phi \strictif \psi) \wedge \Diamond\neg\psi) \strictif \Diamond\neg\phi$. In this proof, I have used it as a *rule of inference* rather than an axiom for two reasons. (1) This reflects how Van Cleve writes out his argument and (2) the rule form of this principle is weaker than the axiom form, as the axiom allows one to substitute any formula into the places of ϕ, ψ, rather than having to first derive $\phi \strictif \phi$. This is a subtle difference, and one which makes no significant difference to my point here.

9. $\Diamond\neg Wg\phi \dashv\vdash \Diamond\neg\phi$ 3 (∗)

10. $\Diamond\neg\phi$ 8,9 MP

11. $\Box\phi \supset \Diamond\neg\phi$ 1–10 Conditional Proof

12. $\neg\Box\phi$ Equiv. to 11

So, Van Cleve has it that even with weakening (C2) to what he considers the weakest plausible version of the principle, Descartes still turns out to be committed to universal possibilism.[12] Van Cleve reasons further that this consequence commits Descartes to inconsistency because we have it that Descartes is committed to the claim that nothing is necessary, while at the same time he is committed to claims that some facts about God are necessary – 'that God *necessarily* exists and is *necessarily* omnipotent (i.e., is necessarily such that nothing happens save through His will)' ([76], p. 61).

One might attempt to respond to Van Cleve by arguing that Descartes is not committed to the necessity of these claims about God, but to my knowledge no one, not even ([30]) whose view is among the most radical in the literature, attempts to argue that. As I have done in response to Curley, I respond to Van Cleve by showing that PDN is a consequence of his collection of premises and the modal principles he accepts. On these grounds, I argue that this collection of assumptions cannot be jointly accepted in an interpretation of Descartes.

2.4.2 PDN from Van Cleve's Premises

The argument is slightly longer than that for Curley's (C2), but doesn't involve much more logical complexity.

[12]This version of the argument is slightly longer than that given by Van Cleve, as I have pulled out and displayed each of the applications of modus ponens on lines 6, 7, 8, and 10 which he collapses into one step – this makes no difference to whether the argument is valid.

1. $\Box\phi \supset (\phi \multimap Wg\phi)$ (V2)
2. $\Box\phi$ Assumption
3. $\phi \multimap Wg\phi$ 1,2 Material Modus Ponens
4. $\Box\phi \multimap \Box Wg\phi$ 3 Becker's Rule
5. $\Box Wg\phi$ 2,4 Modus Ponens
6. $\Box\phi \multimap \Box Wg\phi$ 2–6 Conditional Proof

This gives us our result. On Van Cleve's reading, Descartes is committed to a claim which he expressly denies. It's not that this argument shows that Descartes is committed to something inconsistent with what he has claimed elsewhere, but rather that it asserts a claim which Descartes expressly denies. Of course, one could reaffirm van Cleve's conclusion that Descartes's modal metaphysics is incoherent on the basis of my argument. So the options available on the basis of my argument regarding Van Cleve are (a) that Descartes is not committed to UP, and so at least one of principles or rules in van Cleve's proof is to be rejected, and (b) that Descartes has incoherent commitments related to PDN and UP. Option (b) has us retain Van Cleve's conclusion that Descartes' total system is inconsistent and simply note an additional set of claims where the inconsistency show itself, whereas (a) allows us to avoid this conclusion, and opens the way for consistent interpretations of Descartes' modal metaphysics. The principle of charity speaks for (a), as does the potential for learning positive lessons in contemporary modal metaphysics from Descartes. These considerations lead me to prefer (a) to (b), though they do not establish that it compulsory to accept (a).

2.5 PDN (Two Different Ones)

Before concluding there is one remaining issue to deal with. I have interpreted the passage in the 2 May 1644 letter to Mesland as ruling out the entailment:

(PDN) If God wills ϕ to be necessary, then it is necessary that God wills ϕ

Curley interprets this passage somewhat differently, though his reading and mine both motivate the arguments I have put forward here. He claims that this passage has 'Descartes invoking a scope distinction between $Wg\Box\phi$ and $\Box Wg\phi$' ([23], p. 582).

I think that this is a natural reading of the passage, but that it is also clear from what is written that Descartes is *rejecting* the entailment between $Wg\Box\phi$ and $\Box Wg\phi$. However, it involves only some minor variations on the arguments I have before to show that $Wg\Box\phi$ does entail $\Box Wg\phi$, given Curley's logical assumptions. To see this, note the following instance of (C2):

$$\text{(C2)} \qquad Wg\Box\phi = \Box\phi$$

An immediate consequence of this is $Wg\Box\phi \strictif \Box\phi$, and from (PDN) $\Box\phi \strictif \Box Wg\phi$, one immediately obtains the consequence $Wg\Box\phi \strictif \Box Wg\phi$. So one can even show Curley's preferred reading of PDN holds using his premises and modal inferences. In either case, this is an entailment which Descartes clearly rejects.

2.6 Conclusion

This paper has aimed to remove two constraints on giving a logical account of Descartes' creation doctrine and modal metaphysics more generally. First, that Descartes is committed to limited possibilism, and second that he is committed to universal possibilism and hence has an inconsistent modal theory. What is left?

One can still retain the premises Curley and Van Cleve admit, but this at the cost of rejecting Becker's rule, and the contraposed version thereof, which I've called $(*)$. Even most non-normal modal logics admit Becker's rule, and of the usual batch of non-normal modal logics, only systems strictly weaker than Lewis' **S2** are available.

Another option is to reject at least one premise of the pair (C1)$'$, (C2) or of the pair (C1)$'$, (V2). There are many ways to do this, either by attempting to reinterpret the creation doctrine as non-modal, which is Kaufman's preferred solution ([40]), or by reinterpreting omnipotence in some other way.

Yet another option is to reinterpret the modalities in some way as to distinguish some varieties of modality. One might, for instance, distinguish between metaphysical possibility and conceivability, as has been suggested by ([13]), or distinguish between different kinds of metaphysical possibility.

Adjudicating these options is also beyond the scope of this paper. The important upshot here is that the constraints on interpretation put forward by Curley and Van Cleve are not in force. Their arguments fail in that each commits Descartes to a claim which he explicitly denies. This suggests some avenues for formal interpretation which are worth consideration.

Chapter 3

A Multimodal Interpretation of the Creation Doctrine

3.1 The Problem with the Creation Doctrine

An especially difficult problem in the interpretation of Descartes, in particular his metaphysics, concerns understanding the status of *eternal truths*. Following some of his commentators (see citations below for examples), Descartes uses this term to refer to a class of necessary truths - in particular a class of necessary truths including truths about the *essences* or, in more Cartesian terminology, the *true and immutable natures* of objects.While natures and necessary truths seem to play a key role in Descartes' metaphysical project in his *Meditations*, within his total view, both notions involve serious problems. The substantial problem to which this paper is addressed is the doctrine of the creation of eternal truths, or as I shall call it, the *creation doctrine* (CD). Perhaps the simplest statement of this is as follows:

(CD) The eternal truths are *freely* created by God.

That is to say, God has full voluntary control over the eternal truths – including, it would seem, the power to make eternal truths false. Since God could make the eternal truths false, it follows that they could be false. Following Kaufman [40], the problem can be nicely stated as a tension between the following two claims (both of which Descartes seems to hold):

1. Eternal truths are necessary.

2. Eternal truths are possibly false.

According to the standard account of the interaction between necessity and possibility, to say that a proposition is possibly false is just to say that it is not necessary. So these two commitments of Descartes' seem to be obviously inconsistent.[1] This is the first interpretive problem raised by CD: can it be faithfully interpreted in a way to render Descartes' total view consistent? van Cleve [76] argues that it cannot be consistently interpreted, while others [30, 23, 38, 1, 40] have attempted to provide such interpretations.

Another problem, raised by certain putative solutions, is whether or not CD can be interpreted so that Descartes is not seen to be making any especially bizarre modal claims.[2] On the topic, Geach [32] has a polemical passage, cited in full by Curley [23][p. 590], in which he claims that an interpretation of Descartes which seeks to make CD consistent by employing a bizarre modal logic is suspect. So a desideratum for an interpretation of CD, along with consistency, is to avoid extreme bizarreness in the modal theory attributed to

[1] It has been questioned whether Descartes is actually committed to (1) (for instance, by [30]) – for my part, I take it for granted that he is so committed, but some reasons for this view will be evinced later.

[2] This was put explicitly in these terms by Kaufman [39][vii].

Descartes.[3]

In this paper I shall develop and defend an interpretation which satisfies both desiderata.[4] Before presenting my interpretation, I pause to consider some extant interpretations in detail. Against this background, it is easier to highlight the distinctive features of my account. After this, I'll present my view, go on to consider the roles of essences and of logic in Descartes' comments about CD, and finish by considering two disparate topics raised in the discussion.

3.2 Two Influential Extant Interpretations

Much of the debate surrounding CD has taken place against the background of two highly influential interpretations, due to Frankfurt and Curley. In this section, I'll discuss these interpretations, and set up my proposal in light of some problems with each.

3.2.1 Frankfurt: Universal Possibilism

Frankfurt's interpretation is one of the most influential of recent accounts, though its influence is most often felt in terms of commentators attempting to avoid his conclusion – that Descartes holds that there are no necessary truths after all, and that every proposition is possible. This view is a kind of *universal possibilism*.

Frankfurt puts due stress on the fact that CD concerns the essences of created things:

[3] Geach's point also seems to be that if Descartes can only be made sense of using a bizarre modal logic, then his position is indefensible – Geach also wrote that Descartes, in holding CD, was "clearly talking great nonsense."[32][p. 179]

[4] Though the modal logic suggested by my interpretation is unusual in some ways, I'll argue that the vast majority of Descartes' modal metaphysics is not bizarre, but that only when one considers God's role in making eternal truths does the bizarreness creep in.

> What Descartes calls "eternal truths" are truths about essences. The Pythagorean theorem, for example, is (or purports to be) an eternal truth about what is essential to right triangles. Asserting that the eternal truths are laid down by God is tantamount, then, to saying that God is the creator of essences.
>
> [30][p. 38]

Evidence for this reading of eternal truths as intimately tied to the essences of things is available in a letter to Mersenne in which Descartes responds to some questions:

> You ask me by what kind of causality God established the eternal truths. I reply: by the same kind of causality as He created all things, that is to say, as their efficient and total cause. For it is certain that He is the author of the essence of created things no less than of their existence; and their essence is nothing other than the eternal truths. You ask also what necessitated God to create these [eternal] truths; and I reply that He was free to make it not true that all radii of the circle are equal – just as free as He was not to create the world. And it is certain that these truths are no more necessarily attached to His essence than are other created things.
>
> (To Mersenne, 27 May, 1630 AT 1:152–53 [20][p. 25])

So CD arises from the fact that God has voluntary creative control over the essences of things, and those essences in some way determine the eternal truths.[5] Since God has voluntary control over the essences of the objects He creates, He has voluntary control over the eternal truths determined by those essences.

To be precise, Frankfurt notes that the eternal truths concern not just the essences of really existing things, but also the essences of things which God has not made to exist.

[5] One of the difficulties in understanding CD is to pin down the relation between essences and eternal truths. Descartes' comments on the topic leave open some interpretive options – some of which I canvas in §3.3.3 – but as it's not obvious that the interpretation I propose here provides compelling reason to choose between them. Hence, I'll continue writing loosely of a relation of "determination", and leave more detailed discussion to the mentioned section.

As Descartes indicates in the passage above, God is the author not just of the existence of things (as was a standard claim in the scholastic tradition), but also of their essences, and there seems to be no good reason to think that He didn't mean to include essences of non-existing things - these are subject to God's free act of creation just as much as the essence of any existing object.

Part of the freedom of God's will, on Frankfurt's reading, is that God's will and His intellect are identical. So, for God, knowing some proposition to be true and willing it to be true are one and the same activity. As a result, Frankfurt puts it, "there are no truths prior to God's creation of them, His creative will cannot be determined or even moved by any considerations of value or rationality whatever." [30][p. 41] This view is defended in the replies to the sixth objections to the *Meditations*, and I'll display at length a passage which defends this view. For now, it is enough to note that Descartes, on Frankfurt's reading, takes God's creation to be totally unconstrained by moral considerations, or considerations of truth. These considerations are dependent on God's creation, and hence the creation could not be determined, in any way, by them. So, for instance, moral truths, on Descartes' view, are contingent in the sense that God could have made them radically different – as we'll see, so are mathematical truths (these are Descartes' most commonly used examples), but Frankfurt makes a point of noting that logic also falls under this view. This is evident from the following passage:

> I turn to the difficulty of conceiving how God would have been acting freely and indifferently if He had made it false that the three angles of a triangle were equal to two right angles, or in general that contradictories could not be true together. It is easy to dispel this difficulty by considering that the power of God cannot have any limits, and that our mind is finite and so created as to be able to conceive as possible the things which God has wished to be in fact possible, but not be able to conceive as possible things which God could have

> made possible, but which He has nevertheless wished to make impossible. The first consideration shows us that God cannot have been determined to make it true that contradictories cannot be true together, and therefore that He could have done the opposite. The second consideration assures us that even if this be true, we should not try to comprehend it since our nature is incapable of doing so.
>
> (To Mesland, 2 May 1644, AT 4:118–119 [20][p. 235])

I'll have reason to come back and consider Descartes' second consideration, regarding conceivability and human comprehension, but for now the important part is the first consideration, that God cannot have been determined to make it true that contradictories cannot be true together, and therefore that He could have done the opposite. There is dispute as to how to read 'therefore that He could have done the opposite" (as we'll see in a later section regarding Ishiguro's interpretation), but on a flat-footed reading, the opposite of making it true that contradictories cannot be true together is to make it that (some) contradictories *can* be true together. From this reading of the passage, Frankfurt draws the following conclusion, quoted at length:

> God was free in creating the world to do anything, whether or not its description was logically coherent . . . Descartes evidently thinks that God could have omitted creating the essence "circularity" entirely. In that case there would be *no* eternal truths about circles: every proposition about a circle would have the status now enjoyed by the proposition that the diameter of the circle on a certain blackboard is one foot. Descartes also evidently thinks that God, while creating the essence "circularity", could have made it different from what we conceive it to be. In that case there would be eternal truths about circles, but they would differ from – and perhaps be the negations of – the propositions that are necessarily true of circularity as we now understand it.
>
> [30][p. 42–43]

So, to recap here, the interpretation offered by Frankfurt is as follows: God is the creator of essences, and essences determine the eternal truths. God could have failed to create some essences He did, hence making some eternal truths false, and furthermore, God could have made some essences to be different than we understand them to be, even in ways which are logically incoherent.

The issue arises: how are we to understand the "could" in the passages above? Clearly it can't pick out logical possibility, as we have understood Descartes to claim that God could bring about the logically impossible. Frankfurt's line, put briefly, is that this "could" expresses a genuine metaphysical possibility, whereas the kind of necessity had by the eternal truths is merely epistemic. Eternal truths appear to be necessary because God has created our minds in such a way that we can't conceive them as false. To return to the tension in §3.1, Frankfurt reads (1) "Eternal truths are necessary" as a fact about our conceptual faculties and (2) "Eternal truths are possibly false" as a fact about metaphysical possibility in the sense of what it is possible for God to bring about. On this line, the logical incoherence of the falsity of an eternal truth is no problem, as this is merely reflective of our conceptual faculties.

> The propositions we find to be necessary – like the Pythagorean theorem – need not be truths at all. The inconceivability of their falsity, which we demonstrate by the use of innate principles of reason, is not inherent in them. It is properly to be understood only as relative to the character of our minds. We cannot escape this character, of course, but we *can* realise that God might have made it different from what it is.. . . The necessities human reason discovers by analysis and demonstration are just necessities of its own contingent nature. In coming to know them, it is does not necessarily discover the nature of the world as it is in itself, or as it appears to God.
>
> [30][p. 45]

For Frankfurt's Descares, every truth is merely possible, and any truth which would appear necessary only does so because of the nature of our created minds. As a final point about

Frankfurt's interpretation, he considers a proposal, for which he cites Gueroult [33], that God has unlimited control only over essences which are not His own. So, Gueroult claims, there are some genuinely necessary truths: namely, truths about God's essence, such as that God is omnipotent and is not a deceiver, along with some mathematical truths. Frankfurt rejects this claim, pointing to some passages in which Descartes seems to claim that God's powers are fully limitless:

> For my part, I know that my intellect is finite and God's power is infinite, and so I set no bounds to it; I consider only what I can conceive and what I cannot conceive, and I take great pains that my judgement should accord with my understanding. And so I boldly assert that God can do everything which I conceive to be possible, but I am not so bold as to deny that He can do whatever conflicts with my understanding – I merely say that it involves a contradiction.
> (To More, 5 Feb. 1649, AT 5:272 [20][p. 363])

The view I seek to defend is a variation on Frankfurt's, but a key place in which we differ is with regard to the absolute impossibility of God's being a deceiver – a point to which I shall return. In the meantime, I now turn another influential interpretation before presenting my own view.

3.2.2 Curley: Limited Possibilism

Curley [23] develops his view, now called *limited possibilism*, in response to Frankfurt. According to limited possibilism, while the eternal truths are necessarily true, they are not *necessarily* necessary.

Perhaps the key motivation for Curley's view is that Frankfurt's interpretation fails to do justice to how eternal truths work in Descartes' philosophy separate of considerations regarding CD. When discussing truths dependent upon true and immutable natures elsewhere, such as in the *Meditations*, Descartes does seem to claim that they are *necessary*. Frankfurt weakens principle (1) of §3.1 to concern merely *conceptual necessity*, and Curley notes that this is inappropriate for understanding Descartes' broader philosophical project.

> Consider the ontological argument. As Descartes expounds this, it requires the assumption that I conceive of countless things which have true, immutable and eternal natures, even though they may never have existed or have been thought of (AT VII, p. 64). These eternal natures do not depend on my mind; my thought does not impose any necessity on things, rather the necessity of the things themselves determines me to think of them in the way that I do.... Moreover, not only do we perceive that the truths of mathematics are necessary, sometimes, at least, we perceive clearly and distinctly that they are necessary. If they aren't in fact necessary, then it looks as though Descartes will have to give up his criterion of truth. Not everything we perceive clearly and distinctly is true.
>
> [23][p. 572]

So, *pace* Frankfurt, Descartes is indeed committed to the claim that eternal truths are not just conceptually necessary, but, so Curley argues, in a robustly metaphysical sense as well. He points to Descartes' description of his project in *Le monde* as more evidence:

> Further, I showed what the laws of nature were, and without basing my arguments on any principle other than the infinite perfections of God, I tried to demonstrate all those laws about which we could have any doubt, and to show that they are such that, even if God created many worlds, there could not be any in which they failed to be observed.
>
> (*Discourse*, AT 7: 43, [19][p. 132])

Curley takes this as evidence that Descartes held that these laws of nature are necessary in the sense that they are true in all possible worlds.[6] Curley concludes that universal

[6] It's worth noting that while this use of something like possible worlds to characterise necessity is striking in Descartes, it would be hasty to conclude that Descartes held a fully contemporary view.

possibilism is not adequate to account for CD as against the background of Descartes' modal metaphysics.

He argues, further, that Descartes' claims regarding CD indicate not only that God could have willed eternal truths false, but in fact that He willed them to be necessarily true. Consider the following passage, in which Descartes responds to an objection from Gassendi regarding the immutability and eternality of the natures considered in the *Meditations*:

> You say that you think it is 'very hard' to propose that there is anything immutable and eternal apart from God. You would be right to think this if I was talking about existing things, or if I was proposing something as immutable in the sense that its immutability was independent of God. But just as the poets suppose that the Fates were originally established by Jupiter, but that after they were established he bound himself to abide by them, so I do not think that the essences of things, and the mathematical truths which we can know concerning them, are independent of God. Nevertheless I do think that they are immutable and eternal, since the will and decree of God willed and decreed that they should be so. Whether you think this is hard or easy to accept, it is enough for me that it is true.
>
> (Replies to the Fifth Objections, AT 7:380 [18][p. 261])

Taking this passage to extend from essences, which are eternal, to eternal truths, this passage seems to indicate that Descartes held them to be necessary, but it doesn't provide much reason to think that this necessity is conceptual, as Frankfurt's reading would have it.

These considerations lead Curley into a problem: how can some truth be both necessary and freely created, as the eternal truths are claimed to be? Limited possibilism provides the way out – God's freedom in creation is expressed by the fact that while eternal truths are necessary, God was not necessitated to create them. [23][p. 579–581] Evidence of this claim is available in the letter, quoted earlier, to Mesland in 1644:

> And even if God has willed that some truths should be necessary, this does not mean that He willed them necessarily; for it is one thing to will that they be necessary, and quite another to will this necessarily, or to be necessitated to will it.
>
> (Letter to Mesland, 2 May 1644, AT 4:119 [20][p. 235])

So while not every proposition is possible, they are all possibly possible, all this in a metaphysical sense. As Curley puts it:

> Descartes wants to allow that there are some propositions which are in fact impossible, but which might have been possible, and others that are in fact necessary, but might, nevertheless, not have been necessary. There is nothing epistemic about these "mights." We are not saying: "These things *seem* necessary, but, for all we know they might not *be* necessary." We are saying: "These things *are* necessary, but there is nothing necessary about *that*."
>
> [23][p. 583]

The most pressing problem with Curley's interpretation is that it seems to make God's power appear quite limited and weak, which is in tension with Descartes' apparent motivations for adopting CD. As Alanen puts the point, following some comments of Plantinga's [55]:

> *Modal* propositions (propositions ascribing modality to other propositions) would, according to this view, be subject to God's control. God could not have made 2+2=4 false, "He could only have made it the case that He could have made it false. He could have made it *possibly false*."[23][p. 581–583 and 589] This, however, is in conflict with Descartes' explicit claim that God, for instance, could make it untrue that all the lines from the center of a circle to its circumference are equal.
>
> [2][p. 169]

As Alanen points out, Plantinga suggests that Descartes is confused between universal and limited possibilism. This is an option. However, it seems to me more likely that these two views capture parts of Descartes' total account, while neither gets it quite right. One of the aims of my view is to make this other alternative more compelling – Descartes did not 'run together' limited and universal possibilism, but rather that both views are only part of the correct picture.

3.3 Inner and Outer Modalities

My account follows up on insights from Frankfurt and Curley, and is primarily aimed at preserving the obvious readings of both (1) and (2) from §3.1. Frankfurt, in stressing (2) paid undue attention to (1), and Curley, in stressing (1), wound up making (2) appear quite weak. Like Frankfurt, I take it that there are two distinct modal notions at work in (1) and (2), but unlike him, I hold that both modalities are genuinely *metaphysical*.

Roughly, one modality concerns what must or could be the case *given* that God created the world (including the essences of things) in the way He actually did. The other concerns how God must or could have chosen to create the world. Another way to think of the latter is that a proposition is necessary in that sense just when no matter how God creates a world, the proposition is true, whereas a proposition is necessary in the former sense if God chose to make it necessary (by setting up the essences in the correct way, for instance). I shall distinguish these two notions by calling the former *inner*-necessity (i-necessity), and the latter *outer*-necessity (o-necessity).

With this distinction, it is easy to see how to erase the tension raised in §3.1 – I simply replace (1) and (2) with the following:

(1') Eternal truths are i-necessary.

(2') It is o-possible that the eternal truths are false.

That is, intuitively speaking, the eternal truths are necessary in the sense that Curley emphasises. God *chose* to make them necessary, and they are genuinely necessary – however, they are necessary only in light of the kind of world God created, including, most importantly, the essences He actually created. On the other hand, the eternal truths are possible in the sense that Frankfurt calls for – God o-could have made the world in such a way (by not creating some essences or changing some essences) such that (almost) any eternal truth would be false. This disambiguation allows for the best of universal and limited possibilism, without falling prey to the central problems associated with either. Furthermore, it closely matches the key texts without providing obvious problems for the rest of Descartes' philosophical work. Finally, I claim that this interpretation is both consistent and not modally bizarre. At least, I'll argue, its modal bizarreness is quite restricted – the majority of Descartes' modal reasoning is fairly intuitive.

In this, and the following two sections, I'll discuss the details of my interpretation, including how Cartesian essences feature into the account, that God is free to both not create essences and to create essences to be different than they are. In so doing, I'll attempt to defend the claim that (1') and (2') capture the texts while avoiding the problems for both Frankfurt and Curley. Afterwards, I'll go on to consider the questions regarding consistency and modal strangeness.

3.3.1 Modalities in Light of Essences

One of the core theses I take over from Frankfurt, though it is standard in the literature, is that Descartes takes modal facts to be somehow determined by the essences God creates. On my interpretation, this holds of inner modality. Roughly put, the relationship between inner modalities and essences is as follows:

- ϕ is i-necessary just in case ϕ is made true by the essences of created things.

- ϕ is i-possible just in case ϕ is not made false by the essences of created things.

This is a totally standard approach to modality, for the time.[7] I-modalities are dependent on essences in the sense that to say that something is i-possible to say that it is not ruled out by the essences in question, and to say that something is i-necessary is to say that it is made true by the essences in question. Some evidence that Descartes stuck to this kind of view is available in the *Regulae*, in the discussion of Rule Twelve. Here Descartes is discussing the connection between simple things, some of which, called *simple natures*, are importantly related to his later work on *true and immutable* natures.

> [T]he conjunction between these simple things is either necessary or contingent. The conjunction is necessary when one of them is somehow implied (albeit confusedly) in the concept of the other so that we cannot conceive either of them distinctly if we judge them to be separate from each other.. . . if I say that 4 and 3 make 7, the composition is a necessary one, for we do not have a distinct conception of the number 7 unless in a confused sort of way we include 3 and 4 in it. . . . The union between such things, however, is contingent when the relation conjoining them is not an inseparable one. This is the case when we say that a body is animate, that a man is dressed, etc.
>
> (*Regulae*, AT 10: 421, [19][p. 46])[8]

An account like this is, as I say, reasonably intuitive, and in endorsing this kind of

[7] Only sweeping generalisations about medieval and ancient approaches to modality are compatible with the essence of this paper. Having said that, [74, 41] present the broad outlines of the development of medieval modal logic, and views matching this broad outline can be found there. For instance, Kilwardby's claim that "in order for a proposition to be necessary, it is not enough that it be true and be incapable of not being true; rather the proposition has to state an essential and insepearable cause of the predicate's inherence or non-inherence in the subject." [74][p. 361]

[8] That these comments are stated in terms of conceivability is something to which we'll return later – for now, the point is just that the broad structure of the account of necessary and contingent connections is in line with what I have claimed.

picture, Descartes is not radical.[9] What is radical about his view, on my reading, is that he goes farther and discusses not just modality relative to the essences of created things, but also the essences God might have assigned to created things. Hence, the rough reading for the outer modalities is as follows:

- ϕ is o-necessary just in case ϕ is made true by any essences God might have created.
- ϕ is o-possible just in case ϕ there are some essences God might have created which don't make ϕ false.

So the o-modalities consider not just the actually essential facts about the world, but other essential facts which may have held, had God created different essences. So the eternal truths are, indeed, necessary in that they are guaranteed by the essences God actually created, but they are not necessary in the sense that God might have created some other essences (or not have created some essences salient to the eternal truths in question). Eternal truths are contingent, in the sense that nothing determined God to create the world, along with its essential facts, in just the way that He did, but they are necessary in that they are guaranteed by the way in which He did create the world, along with its essential facts. A way of putting the distinction is in terms of absolute and relative modality – i-necessities are necessary relative to the collection of actual essences, whereas o-necessities are not relative to actual essences, and would be true regardless of which essences God had created.[10]

[9]There are, of course, many disputes that come into specifying how this picture is to be filled in to provide an *account*, but my point is just that this style of account does not set Descartes apart from his contemporaries and predecessors.

[10]Note, contemporary authors also drew the distinction between absolute and relative necessity, but did so in importantly different ways. I am not claiming that Descartes is taking over that account, but simply using the language to express his view. For an interesting discussion of this issue, see [2][§6].

Using talk of possible worlds, we can rephrase the view a bit.[11] Consider the collection of all the worlds God could have chosen to create – including ones where the essences differ in some way from the actual world (either because these fail to have some actual essence, or have some essence which is different from any actual essence). Call those worlds which have all the same essences as the actual world *normal* – the set of normal worlds is a proper subset of the total set of worlds. With these, we can restate the conditions above to better fit a more contemporary understanding of modality:

- ϕ is i-necessary iff ϕ is true at all normal worlds.
- ϕ is i-possible iff ϕ is true at some normal world.
- ϕ is o-necessary iff ϕ is true at all worlds (normal or otherwise).
- ϕ is o-possible iff ϕ is true at some world (normal or otherwise).

According to this interpretation we have, for instance, that any o-necessary truth is also i-necessary, and that any i-possible truth is o-possible, but otherwise we haven't specified much of the logic of these modal operators. What is important for now is just that we can have a proposition which is i-necessary, because it is guaranteed by some actual essences, and yet o-possibly false, because its negation is compatible with some essences God might have created.

This raises some questions, which I shall now consider.

[11] I'll use this possible worlds talk throughout the rest of the paper. Later on I shall argue that this anachronism is not a dangerous one.

3.3.2 What are Cartesian Essences?

I have been talking somewhat fast and loose about God creating a world with some essences or essential facts, but to get my account off the ground I need to clarify what I mean in light of Descartes' claims regarding true and immutable natures. My aim here is not to give a fully fleshed out account of Cartesian essences, but rather to point out some key features of these creatures which relate to CD.[12]

The discussion of simple natures in the *Regulae* is quite explicit and provides a good starting point. Rule Twelve of the *Regulae* [19][p. 39–51] concerns the order in which we should obtain knowledge – that we should start by working to obtain what Descartes would later call clear and distinct perceptions of propositions. Part of this project is to distinguish between simples and complexes composed of them – for instance:

> If, for example, we consider some body which has extension and shape, we shall indeed admit that, with respect to the thing itself, it is one single and simple entity. For, viewed in that way, it cannot be said to be a composite made up of corporeal nature, extension and shape, since these constituents have never existed in isolation from each other. Yet with respect to our intellect we call it a composite made up of these three natures, because we understood each of them separately before we were in a position to judge that the three of them are encountered in the same time in one and the same subject.
>
> (*Regulae* AT 10: 418, [19][p. 44])

So the simple natures are those which are simple with respect to the intellect, even though they may be combined in a body. It is by starting from these natures that one can obtain certain knowledge of the kind which Descartes' epistemic project is aimed at. It is in the *Meditations* that Descartes develops the mature version of the project laid out in the

[12]This picture is in line with Rozemond's [66] moderate platonism about Cartesian essences. Further details of this kind of account are available there.

Regulae, and while he doesn't continue to use the terminology "simple natures", the project is the same in outline, and to a large extent also in detail.

In particular, what is retained is something very much like the theory of ideas outlined in the *Regulae*. This is taken up in the Fifth Meditation, as a preliminary to the ontological argument. Here the notion of essences come up as objects of my ideas which do not seem to represent an existing thing in the external world. He goes to some pains to claim that essences, though we don't access them through the sense, are not mind-dependent:

> But I think the most important consideration at this point is that I find within me countless ideas of things which even though they may not exist anywhere outside me still cannot be called nothing; for although in a sense they can be thought of at will, they are not my invention but have their own true and immutable natures. When, for example, I imagine a triangle, even if perhaps no such figure exists, or has ever existed, anywhere outside my thought, there is a still a determinate nature, or essence, or form of the triangle which is immutable and eternal, and not invented by me or dependent on my mind. this is clear from the fact that various properties can be demonstrated of the triangle, for example that its three angles equal two right angles, that its greatest side subtends its greatest angle, and the like; and since these properties are ones which I now clearly recognise whether I want to or not, even if I never thought of them at all when I previously imagined the triangle, it follows that they cannot have been invented by me.
>
> (*Meditations*, AT 7:64, [18][p. 44-45]

To put it briefly, Cartesian essences are (human-)mind-independent[13], non-existent (or not necessarily existent[14]) things which my ideas can represent, and about which I can come to know, with certainty, eternal truths. In addition, as we have seen in the discussion regarding Frankfurt, God has voluntary control over them, and this is where CD gets its bite. The fact

[13]They are not, however, independent of God's mind – hence Rozemond's [66] label *moderate platonism*.

[14]For further comment on the existence of 'merely possible' essences, see the *Conversation with Burman* AT 5:160, [20][p. 343]

that they are mind-independent puts pressure on Frankfurt's contention that (1) in §3.1 should be understood purely epistemically – if essences are mind-independent entities then how can truths about them (eternal truths) be reliant on the makeup of the human mind? This consideration provide more evidence of the claim that, according to Descartes, the necessity of eternal truths is not just a matter of our inability to conceive them being false.[15] This is part of the grounds for my claiming that i-necessary truths are *genuine* metaphysical necessities for Descartes, *pace* Frankfurt.

3.3.3 Connection between Essences and Eternal Truths

Recall the passage, previously quoted, from the 27 May 1630 letter to Mersenne – particular the claim "For it is certain that He is the author of the essence of created things no less than of their existence; and their essence is nothing other than the eternal truths."[20][p. 25]

The standard line seems to be that this should be understood as claiming that the essences determine the eternal truths – it is the essence 'circularity' having certain features, such as having all equal radii, that determines that the proposition "all the radii of the circle are equal" is necessary. This, for instance, is the way Cottingham discusses it in the *Descartes Dictionary*. [17][p. 57–58] There he distinguishes two kinds of eternal truths – truths about essences and common notions (we'll come back to consider common notions in a later section). So on that reading, eternal truths are different in kind from essences in that the former are propositional whereas the latter are not. This is a natural distinction to make in light of contemporary distinctions between propositions and whatever it is that accounts for the truth values of propositions: facts, truthmakers, etc. However, the phras-

[15] The relationship between conceivability and possibility is important in Descartes' philosophical project, and it's one I'll come back to later in the paper. However, the point here is that they are at least distinct notions in the sense that a propositions' being necessary is not *merely* a matter of its falsity being inconceivable.

ing of this passage above also seems to allow that the eternal truths might just *be* essences themselves. There are two ways I can see a view like this going. One might claim that eternal truths are essences of a kind (and so are a kind of thing), and they bear important relations – some kind of inverse truthmaking relation, perhaps – to other essences. On this view the essence *circularity* stands in the truthmaking relation to another essence, namely, "All the radii of the circle are equal". So the essence *circularity* and this eternal truth about circles are distinct, but both are actually essences. Another, and perhaps more natural, way to go is with a stronger identity thesis. Alanen [2][p. 161] might suggest a line like this: "Descartes identifies the eternal truths with the essences of the creatures", as does Rozemond [66]. On a reading like this, it would seem, there is no difference at all between an essence (had by a creature) and an eternal truth ('about' that creature). So "Circles are trilateral" the essence 'circularity' are the same thing.[16]

There are a number of potentially interesting readings available here, and these seem well worth exploring, but for my purposes, it's not obvious that any one of these accounts tells for or against the reading of CD that I develop here. So I leave the matter at this, and continue developing the view without any particular attention to the distinctions between these various readings, and continue to talk loosely of a determination relation between eternal truths and the essences they concern.

[16]This identification of proposition and object may be in line with the kind of *self-induced confusion* Camp [14][p. 191–193] diagnoses in Descartes' identification of act of awareness and object of awareness. On this view Descartes would be intentionally blurring the, apparent, distinction between a proposition and its truthmaker. There is much more that needs to be said in order to state a Camp-style reading of the relation between essences and eternal truths, and I include it here merely to note that it's an option. Thanks to Dave Ripley for pointing out this connection.

3.3.4 Essences Dependent on God's Will

As I earlier suggested, the radical feature of CD as against Descartes' predecessors and contemporaries is that essences are fully dependent on God's will. While those in the Thomist tradition would hold that God's options in creating the world are limited by the fact that, for instance, trilaterality is contained in the essence of circularity, Descartes holds it is blasphemous to claim that circularity is independent of God's will.

> As for the eternal truths, I say once more that they are true or possible only because God knows them as true or possible. They are not known as true by God in any way which would imply that they are true independently of Him. If men really understood the sense of their words they could never say without blasphemy that the truth of anything is prior to the knowledge which God has of it. In God, knowing and willing are a single thing in such a way that by the very fact of willing something He knows it and it is only for this reason that such a thing is true. So, we must not say that if God did not exist nevertheless these truths would be true; for the existence of God is the first and most eternal of all possible truths and the one from which alone all others proceed.
>
> (To Mersenne, 6 May 1630, AT 1:146–150 [20][p. 24])

In addition to providing justification or the thesis that essences are dependent on God, this passage makes it eminently clear that both the truth and the possibility of the eternal truths is under God's direct voluntary control. This reinforces the problem, raised earlier, that limited possibilism doesn't capture the full scope of CD – it's not just modal features of eternal truths that are dependent on God's free will, but also their truth *simpliciter*.[17] Having said that, Descartes is fairly explicit here that God does have control over the modal properties of propositions – this provides part of the grist for my mill in that God does

[17] There is an interpretive difficulty with how the first occurrence of "or" in the passage should be understood – is it an a genuine disjunction, or is the second disjunct to be understood as a mere restatement of the first. In this case, it seems odd to take "the eternal truths are possible" as a restatement of "the eternal truths are true", so the former reading seems more natural.

make it that propositions are *necessary*. That is, He makes propositions i-necessary – the o-modalities track the limits of His options in performing this creation. That is to say, on the multimodal account, the i-modalities are those over which God has control. The o-modalities serve to express facts about God's powers.[18]

There is, I take it, plenty of evidence to read Descartes claims about the eternal truths as modal, but to defend this view, additional reason is needed to read Descartes' claims as invoking the second, o-modality. I do not suggest that he does so explicitly – that's why, to borrow a comment of Curley's [23][p. 570] my reading is an interpretation and not a restatement of Descartes' view. The best reason I know of is invited by the question of how one should understand uses of apparent modal talk to discuss the relationship of dependence between the eternal truths (and essences) and God. Consider the following claims, which have been, or will be, quoted in full context elsewhere in the paper:

- "our mind is finite and so created as to be able to conceive as possible things which God could have made possible, but which He nevertheless wished to make impossible." [20][p. 235]

- "if God has established these truths, He could change them as a king changes his laws. To this the answer is: Yes He can, if His will can change… [20][p. 23]

- "I boldly assert that God can do everything which I conceive to be possible, but I am not so bold as to deny that He can do whatever conflicts with my understanding…" [20][p. 363]

- "God cannot have been determined to make it true that contradictories cannot be true together, and therefore He could have done the opposite." [20][p. 235]

[18] So it does not make sense to ask whether God has control over the o- modalities – these just express facts about what God has control over.

These are some of the canonical statements of CD, and they involve fairly explicit modals "can" and "could", referring directly to God's scope of options in creating the world. That is, here, and elsewhere, we see talk of what options God has in creating the world, and the o-modalities I propose express this talk. To say that it is o-necessary that ϕ is to say that there was no option available to God, in creating the world, which failed to make ϕ true. The thrust of CD is to say that since the essences of created things are within the voluntary control of God's will, there are options available to God in which they are different, or fail to exist. To return to possible worlds talk, the non-normal worlds are available as options for God in creating the world, and at these worlds eternal truths can fail.

To recap here, the indirect textual evidence for multiple modalities is Descartes' use of apparent modal terminology to talk about God's options. In light of this, the claim of CD, that God's options include some situations where eternal truths are false, demand a distinction between the outer modality tracking God's options, and inner modality, in keeping with an essentialist picture.

3.4 The Freedom of God's Will

I have set out a multimodal interpretation of CD, and given some reasons in favour of it. In this section, I shall discuss further Descartes' account of God's nature, and in particular how this account makes sense of God's freedom in creation. The claim that God's will is free and unrestrained in the act of creation is central to CD, and so a clear picture of Descartes' views on God's will is important. In particular, some key issues I discuss are the relationship between God's intellect and God's will and the nature of indifference.

3.4.1 The Divine Understanding and the Divine Will

In the treatment of Frankfurt and elsewhere, we have seen glimpses of the peculiar kind of Divine Simplicity thesis Descartes adopts, according to which there is no distinction in God between knowing and willing. Perhaps the most clear statements are in the 6 May 1630 letter to Mersenne: " In God willing and knowing are a single thing in such a way that by the very fact of willing something He knows it and it is only for this reason that such a thing is true"(AT 1:149 [20][p. 24]) and the 27 May 1630 letter "In God, willing understanding and creating are all the same thing without one being prior to the other even conceptually." (AT 1:153 [20][p. 25–26])

Kaufman points out how this simplicity thesis distinguishes Descartes from Leibniz, who also holds the view that essences and eternal truths depend on God. As Kaufman puts it, "Leibniz believes that the eternal truths exist in and depend on God's *understanding* but not God's will."[40][p. 36] For Descartes, there is no such distinction. In God, seeing that it is necessary that triangles are trilateral, and willing the necessity of this proposition, are one and the same action. As another example, there is no difference, in Genesis, between God's creating light, and His seeing that it was good. In contrast to human beings – more on this in the following section – God's believing a proposition does not involve a movement in God's will following a light in God's intellect. Rather, they, being the same divine faculty, move in unison. So God's will is not determined by what God's intellect perceives – it's for this reason that God is not constrained by considerations of what is necessary. The response in God's intellect to what is necessary is the same as God's willing to create what is necessary in the first place.

3.4.2 Indifference: Divine and Human

This plays directly into Descartes' reliance on Divine Indifference when discussing CD. He famously holds that the human will is most free when responding to compelling reason to believe. This is most clearly stated in the fourth *Meditation*, in a discussion of the interaction of human will and intellect in the formation of ideas and judgements, and how we fall into error in forming beliefs. In this passage, Descartes propounds the view that human free will, unlike that of God, is not correctly characterised in terms of indifference:

> In order to be free, there is no need for me to be inclined in both ways [between a proposition and its negation]; on the contrary, the more I incline in one direction – either because I clearly understand the reasons of truth and goodness point that way, or because of a divinely produced disposition of my inmost thoughts – the freer is my choice. Neither divine grace nor natural knowledge ever diminishes freedom; on the contrary, they increase and strengthen it. But the indifference I feel when there is no reason pushing me in one direction rather than another is the lowest grade of freedom; it is evidence not of any kind of perfection of freedom, but rather of a defect in knowledge or a kind of negation. For if I always saw clearly what was true and good, I should never have to deliberate about the right judgement or choice; in that case, although I should be wholly free, it would be impossible for me ever to be in a state of indifference.
>
> (*Meditations*, AT 7: 57–57, [18][p. 40])

The human will is freest and most perfect when responding to what it is given by the intellect, responding to clear and distinct perceptions. God's will, however, cannot be so responsive to His intellect. This is a consequence of the Divine Simplicity thesis to which Descartes commits himself, and it's a consequence of which he is clearly aware, and which he embraces. The following passage sees a lengthy discussion of this point:

> As for the freedom of the will, the way in which it exists in God is quite different from the way in which it exists in us. It is self-contradictory to suppose

> that the will of God was not indifferent from eternity with respect to everything which has happened or will ever happen; for it is impossible to imagine that anything is thought of in the divine intellect as good or true, or worthy of belief or action or omission, prior to the decision of the divine will to make it so. I am not speaking here merely[19] of temporal priority: I mean that there is not even any priority of order, or nature, or of 'rationally determined reason' as they call it, such that God's idea of the good impelled Him to choose one thing rather than another. For example, God did not will the creation of the world in time because He saw it would be better this way than if He created it from eternity; nor did He will that the three angles of a triangle should be equal to two right angles because He recognised that it could not be otherwise, and so on. On the contrary, it is because He willed to create the world in time that it is better this way than if He had created it from eternity; and it is because He willed that the three right angles of a triangle should necessarily equal to two right angles that this is true and cannot be otherwise; and so on in other cases.
>
> (Sixth Replies, AT 7:431–432, [18][p. 291])

The characteristic fact of God's freedom is *divine indifference* – that is, following Kaufman's lead [40][p. 38], nothing independent of God impelled or determined God to will what He actually willed to be true. There couldn't be anything separate from God to which God's intellect could respond in guiding God's will, because God's will and intellect are identical. Thus, God's will is indifferent, and since the eternal truths are dependent on God's will, God is indifferent w.r.t. the eternal truths. Kaufman's view, to be further discussed in §3.8, has it that divine indifference is all there is to CD, which has no modal force. Here I diverge, and claim that it is because God's will is indifferent w.r.t. eternal truths that CD has modal force - i.e. that the eternal truths are o-possibly false.

According to this sketch, it is this divine indifference that explains why it's true that God o-could have made some i-necessary propositions false. This modal fact is explained

[19] This "merely" is omitted in the CSM translation, but is included in Haldane and Ross [34][p. 248] – I include it for emphasis.

by (1) divine indifference and (2) the fact that all of creation is entirely dependent on the act of God's creation. Roughly speaking, here are the inferential moves as I read them in Descartes:

(P1) Divine Simplicity: God's intellect and will are identical

(C1) Divine Indifference: God's will is not determined by any independent factor.

(P2) Dependence: Creation is entirely dependent upon God's will.

(P3) Essences, and therefore eternal truths, are created.

(C2) Given an eternal truth ϕ, God was not determined to make ϕ true.

(C3) Therefore, God *could* have done the opposite of making ϕ true.

(C4) Therefore, God could have made $\neg\phi$ true.

(C5) $\neg\phi$ is o-possible.

So, *pace* Kaufman, I don't take it that divine indifference exhausts the intended import of CD, but is rather part of a broader argument in favour of CD, which does have modal force. The move that most clearly distinguishes our views is that from C2 to C3 – I take it that God's will's being undetermined, as per C1, does not exhaust the meaning of the "could" in C3–C4, but rather entails that "could" claim, which is a modal claim.

This completes the presentation of my core account. In the rest of the paper, I'll discuss the way in which logic falls under the intended scope of CD, whether there are any outer necessary truths, how epistemic modalities fit into the proposed picture, and close with some sundry topics and potential objections.

3.5 God's Freedom to Create Logic

So far, I've focused on God's free creation of essences with little discussion for God's free creation of *logic*, but this involves some additional difficulties. Under the assumption that essences are objects, it is easy to understand how God's freedom with regard to the creation of those objects how God has free control over the eternal truths concerning them. It's harder to understand how to extend this modal to account for how God can create logic. We have seen a passage in which Descartes indicates that even logic is in the scope of CD, when he claims that God could have done the opposite of making it true that contradictories cannot be true together. [20][p. 235] Given the state of logic at the time, it's likely that this claim was intended to have a broader impact than simply to allow for some kind of paraconsistent possibilities. Rather, it was most likely intended to indicate that God has control over all of logic, and so could have made logic different in any number of ways – not just by making the law of non-contradiction (or, more aptly, *ex contradictione quodlibet*) false.

In what way does God's control over the essences of created things also grant Him control over logic? Logic concerns relations of consequence between propositions, and not objects directly (at least, it's not obvious that logic concerns particular objects). So the question is how changing the essences (of objects) could affect logic. The following passage from the *Regulae* gives the grist of an answer – logic is itself a matter determined by essences of a certain kind. In this passage, Descartes is once again considering *simple natures* and the role they play in our knowledge acquisition. Most interesting here is that he gives common notions as examples (the emphasis in the following quotation is mine):

> [T]hose things which are said to be simple with respect to our intellect are, on our view, either purely intellectual or purely material, or common to both. Those simple natures which the intellect recognises by means of a sort of innate

> light, without the aid of any corporeal image, are purely intellectual. . . . Those simple natures, on the other hand, which are recognised only in bodies – such as shape, extension and motion, etc. – are purely material. Lastly those simples are to be termed 'common' which are ascribed indifferently, now to corporeal things, now to spirits – for instance, existence, unity, duration, and the like. To this class we must also refer those common notions which are, as it were, links which connect other simple natures together, and *whose self-evidence is the basis for all the rational inferences we make.* Examples of these are: 'Things that are the same as a third thing are the same as each other'; 'Things that cannot be related in the same way to a third thing are different in some respect.' These common notions can be known either by the pure intellect or by the intellect as it intuits the images of material things.
>
> (*Regulae* AT 10: 419–420, [19][p. 44–45] – my emphasis)

This passage makes it seem that among the simple natures are logical rules. "Common notion" is the name given in the *Elements* for those most general principles which hold regardless of the figure under study. The content of these include some arithmetic axioms, but others, those mentioned by Descartes and Common Notion 5 "The whole is greater than its part", are more general than this and are, plausibly, basic logical principles. This passage extends that to include the basis for good inferences in general – this would seem to include logical inference rules like the law of non-contradiction, or *ex contradictione quodlibet*. Descartes gives some indication of the extended generality of this class of common notions in the *Principles*:

> But when we recognise that it is impossible for anything to come from nothing, the proposition *Nothing comes from nothing* is regarded not as a really existing thing, or even as a mode of a thing, but as an eternal truth which resides within our mind. Such truths are termed common notions or axioms. The following are examples of this class: *It is impossible for the same thing to be and not to be at the same time*; *What is done cannot be undone*; *He who thinks cannot but exist while he thinks*; and countless others.

(*Principles* Pt. 1, Art. 49, AT 8A: 23 – 24, [19][p. 209])

These passages provides a natural option – God's power over logic is just the same as His power over other eternal truths though essences. Logical inferences are themselves based on the links connecting essences (simple natures), which links are themselves just more essences. We can grasp them just as we grasp mathematical essences, and their truth guides our reasoning as we contemplate mathematical essences, or reason about extension, for instance. So, in creating a collection of essences, God is creating logic in just the same way that He is creating, for instance, numbers and geometrical figures, and so His power over logic is of a piece with His power over other essential truths.

There are many issues to be ironed out here. How do logical essences relate to non-logical essences, and how do these relations serve to guide logical reasoning? Further, if logical truths are wholly general and don't rely on particular names and predicates (for instance), then it seems that, whatever relations in which they stand to non-logical essences, they must stand in those relations to all non-logical essences. These are difficult problems, and I don't have fully satisfying answers to these questions, but the initial idea - that eternal truths about logic concern essences which stand in some relations to other essences - seems a promising way to proceed.

However, this view may provide one reason to prefer the strong identity thesis mentioned in §3.3.3. If essences are just eternal truths, then the relations between essences provided by logic are just those of consequence among eternal truths. Further, on the identity reading to say that logical truths are (determined by) essences is just to say that they are eternal truths, and that God creates these *ex nihilo*. This strikes me as less mysterious than saying that God creates some essences, which are not eternal truths, which determine logical truths by standing in various relations with other essences. I have tried to stay agnostic between various readings of the relation between eternal truths and essences, but this provides some reason to think that my account best suits an identity thesis. This is not to say that only the identity thesis makes sense, but the other theses raise problems which the identity thesis seems to dissipate.

Regardless of how one fills in the question regarding the relationship between essences and eternal truths, an interesting consequence of understanding logical truth as determined by essences is that logical necessity is not something separate from, and more broad than, metaphysical necessity. We can understand *metaphysical* necessity as being either i-necessity or o-necessity, and understanding logical necessity as determined by a class of essences has it that logical necessity is not broader than either of these. Logical necessity is stricter than o-necessity, as God could have made some other logical truths, and logical necessity is no broader than i-necessity, of which it is a species.

3.5.1 Descartes as a Dialethiest?

It should be clearly noted that I am *not* claiming that Descartes was a dialethiest - this would stretch credulity, and correctly so as it is clearly not the case. To make this clear, the dialethiest claims that some contradictions are true, whereas related weaker positions merely hold them to be possible. While I claim that there is a kind of possibility for which the latter claim is true, Descartes does not hold that there are actually true contradictions – indeed, they are i-impossible on his account. For instance, in the letter to Mesland 1944, he claims that God made the world such that contradictories *couldn't* be true together, and I read this *couldn't* as expressing this i-impossibility. That contradictions o-could be true does not entail that they i-could be true. There is a great deal of evidence that Descartes did hold the law of non-contradiction to be (i-) necessary – it's this presupposition that makes it noteworthy when claims that God could have *done the opposite*.

There aren't, to my knowledge, any contemporary positions which map on to the modal metaphysics which I attribute to Descartes – this is one of the reasons I take it to be interesting. In the broad sense, the view I am attributing to Descartes is that in considering some matters, namely God's will and His act of creation, impossible situations (worlds) are relevant to the discourse, because they are options for God, even if they are impossible. Unlike most recent authors who have defended a version of this view (see [62, 56, 49]) I don't take Descartes to be committed to a paraconsistent logic – his commitment to the law of non-contradiction, and more importantly to *ex contradictione quodlibet*, is obvious and clear. I don't have a fleshed out account of Descartes' logical commitments, and this a topic about which he writes so little that a detailed account is likely not forthcoming. All that he is committed to, on this reading, is that some i-necessary propositions (including some that are logical) are o-possibly false. This doesn't commit him to dialethiesm, nor to the position that contradictions are conceivable, nor to other related views developed in the literature on impossible worlds, or paraconsistent logic. Ideally, his view may provide some insight into positions available in that literature, and on the literature regarding counterpossible, countermathematical, and counterlogical reasoning, but drawing out these remarks is left for future work.

3.6 The Scope of CD

An important question which has been raised in some of the interpreters I've considered is that of delineating the intended scope of CD. Over just which necessary truths is God supposed to have free control? Frankfurt claimed, for instance, that God has free control even over eternal truths about Himself – and so, presumably, God could have made it that He doesn't exist – whereas Gueroult and others have claimed otherwise.

I hold that Descartes' view has it that certain eternal truths about God are o-necessary, against Frankfurt. The truths I consider are not exactly those listed by Gueroult, but only because I would exclude some he includes. I have two reasons for thinking that some eternal truths about God are o-necessary – the first serves to justify the existential claim, and the second serves to justify some particular instances.

A preliminary consideration is that in the most explicit passage connecting the eternal truths and essences, Descartes says that "God is the author of the essence of created things no less than their existence" [20][p. 25], not that God is the author of the essence of everything. If Descartes held that God was not a created thing, then presumably this passage provides some reason to think that Descartes did not mean to include in the scope of CD those eternal truths about the essence of God. That is, this omission is some evidence that Descartes held that God does not have free control over His own essence.[20]

A more serious consideration arises in some problems with fitting CD into Descartes' broader commitments. To bring these out, I'll first need to say something about how conceivability fits into the kinds of possibility I distinguish. Toward that, there seem to be good reasons to think that Descartes was committed to the claim that conceivability is co-extensive with some kind of possibility, and a result of this is that it is o-necessary that God is not a deceiver (and, per force, that He exists).

[20]Thanks are due to Donald Baxter for suggesting this point.

3.6.1 McFetridge and the Matching Argument

To start, I'll discuss an argument to the effect that i-possibility is co-extensive with conceivability. In view of this, I'll consider McFetridge's [51] treatment of an argument, what he calls the *matching argument*, of the Second Replies. Further, I'll argue that this co-extensivity is o-necessary on the grounds that this is what's needed for this argument to go through. The key upshot of this is that God is bound (by o-necessity) to make it that our faculty of conceivability, to be fleshed out more momentarily, matches the world, and its essential features. This, I take it, is tantamount to claiming that God is o-necessarily not a deceiver – God could have made the world in other ways, and with other essences, but whichever way He made the world, He o-cannot have made thinking things which are, in principle, incapable of obtaining knowledge about that world.

The co-extensivity thesis claims that ϕ is i-possible iff ϕ is conceivable. McFetridge [51] explicitly defends the direction: no claim is conceivable without being possible. This is the thesis which is most essential in Descartes' epistemic project – it seems to follow from the truth rule. Suppose I had a clear and distinct perception of ϕ, then I clearly and distinct perceive that it is true, and therefore that it is possible. Thus, by the truth rule, ϕ is possible.[21]

In addition to this, McFetridge points to a passage in the replies to the second set of objections to the *Meditations* which makes it clear that Descartes is committed to the claim that there can be nothing conceivable but not possible. The passage is worth quoting at length. Descartes is responding to Mersenne's criticism of Descartes' argument in the fifth meditation that God exists. Mersenne holds that what Descartes shows in that meditation is only that existence belongs to the nature of God, but not that God exists

[21] Thanks to Lionel Shapiro for helping me get clear on this point.

because Descartes has not successfully argued that God's nature is possible.[18][p. 91] Descartes takes Mersenne to open the argument with the major premise 'That which we clearly understand to belong to the nature of something can be truly asserted to belong to its nature."[18][p.106] In the following passage, he corrects this error (it should say "That which we clearly understand to belong to the nature of something can be truly asserted of it"), and goes on to argue that our clear and distinct perception of God's nature does provide us with knowledge that God's nature is possible. These comments give rise to the passage of most interest here, that possibility, understood metaphysically, must *match* conceivability.

> [To] deploy the objection which you go on to make, you should have denied the major premise and said instead 'What we clearly understand to belong to the nature of a thing cannot for that reason be affirmed of that thing unless its nature is possible, or non-contradictory.' But please notice how weak this qualification is. If by 'possible' you mean what everyone commonly means, namely, 'whatever does not conflict with our human concepts', then it is manifest that the nature of God, as I have described it, is possible in this sense, since I supposed it to contain only what, according to our clear and distinct perceptions, must belong to it; and hence it cannot conflict with our concepts. Alternately, you may well be imagining some other kind of possibility which relates to the object itself; *but unless this matches the first sort of possibility it can never be known by the human intellect*[22], and so it does not so much support a denial of God's nature and existence as serve to undermine every other item of human knowledge. . . . if we deny that the nature of God is possible, we may just as well deny that the angles of a triangle are equal to two right angles, or that he who is actually thinking exists; and if we do this it will be even more appropriate to deny that anything we acquire by means of the senses is true. The upshot will be that all human knowledge will be destroyed, though for no good reason. [My emphasis]

[22] Original ". . . nisi cum praecedente conveniat, nunquam ab humano intellectu cognisi potest. . . "

(AT 7:150-151 [18][p. 107])

The discussion is most important in that it states that the kind of possibility which relates to the object itself must *match* this common notion of possibility, which can be nicely characterised as Chalmer's *negative conceivability*. According to Chalmer's characterisation, ϕ is negatively conceivable iff ϕ is not ruled out a priori. [15][p. 149] This is the kind of conceivability I discuss throughout the paper. Descartes seems to explicitly endorse reading conceivability in these negative terms when, in the letter to Voetius, he remarks:

> "if he is asked how he knows that this is impossible, he must answer that he knows it from the fact that it implies a contradiction – that is, it cannot be conceived."
>
> (To Voetius May 1643, AT 8B:60 [20][p. 222])

The passage from the replies (AT VII 150–151) only directly defends the claim that conceivability entails possibility, which is some kind of 'matching'. Cottingham et al use "to match" to translate *convenire*, which has the alternate meaning "to fit" or "to agree with". This last reading may make the co-extensivity thesis more plausible than "to match", but there may be another way for conceivability and possibility to agree with one another while not being co-extensive.

McFetridge takes it for granted (saying that it's uncontroversial) that Descartes is committed to the claim that if ϕ is possible then ϕ is conceivable, but there is a better defence of this claim available. Suppose that ϕ were possible but not conceivable. Then we could rule out ϕ on purely a priori grounds, apparently coming to the clear and distinct perception that ϕ is necessarily false. By the truth rule, then, it must be true that ϕ is necessarily false, which itself is false.

To step back into my account, I take it that conceivability is co-extensive with i-possibility. In addition, there is good reason to suppose that Descartes is committed to the fact that this equivalence is not only true, but o-necessary. That is, God could not have created the essences of His creatures in such a way that modal facts determined by these essences were in-principle outside of the reach of our faculty of conceivability. However, this does not entail that outer possibility is co-extensive with conceivability. This last claim is evidenced by the 2 May 1644 letter to Mesland in which Descartes writes: "our mind is finite and so created as to be able to conceive as possible the things which God has wished to be in fact possible, but not be able to conceive as possible things which God could have made possible, which He has nevertheless wished to make impossible." [20][p. 235] I take it from this that Descartes not only doesn't think that our conceivability should match what God o-could have brought about, but that he explicitly holds that it cannot. Not only isn't it required that conceivability matches o-possibility, it is not i-possible for us to conceive of an i-impossibility which is o-possible.

So, a consequence of God's o-necessarily not being a deceiver is that it is also o-necessary that i-possibility and conceivability match. As an example: had God created the essence *triangularity* differently than He actually did, He would have been bound to alter our conceptual faculties in such a way that we could conceive this essence in just the same way we can conceive the actual essence of triangularity. Otherwise, we would be deceived about the natures of the triangles.

As a more general upshot for the interaction between my reading of CD and Descartes' epistemic project, for the purposes of vouchsafing certain knowledge, i-necessity is necessity enough. Put in slightly different terms, given that our knowledge is tied to the actual world, and the actual essences of things, o-modalities don't have a dramatic impact on our epistemic states.

3.7 Consistency and Bizarre Modal Theses

In §3.5.3 I pointed out that the multimodal reading does not commit Descartes to dialetheism. Indeed, the view is consistent in the sense that it does not commit Descartes to any pair of contradictory claims. To properly defend this, a logical argument is necessary, and that would involve developing a model theory capturing the view, which goes beyond the aims of this paper. Having said this, some reasons can be evinced for the claim that CD is consistent. For all I've said, while Descartes is committed to the o-possibility of propositions of the form ϕ and $\neg\phi$, this does not entail the i-possibility or truth of any such propositions. At worst, Descartes is committed to the truth of contradictions only under the scope of o-possibility, and this is just a commitment, which I take him to accept, to God's being able to create the world in such a way as to make contradictions true. He is not, however, committed to any inference form which would generate a contradiction from this set of entailments. For instance, it is not true that it is *not* o-possible for ϕ and $\neg\phi$ to be true, for in order for this to be the case it would mean that God is, while able to create the world so as to make ϕ and $\neg\phi$ true, also *unable* to create the world in this way. Nothing about the view gives reason to think that Descartes has this pair of commitments.

Further, CD does not commit Descartes to especially modally bizarre claims. Now, it is somewhat bizarre to hold that there are two different modalities, but, for the most part, the o-modalities don't enter into Descartes' reasoning. For my part, the only points in the text in which it seems that Descartes invokes the o-possibility are those in which he is explicitly reasoning about what it is in God's power to do (and the majority of such passages have been quoted, indicating the dearth of discussion on this topic in Descartes' philosophical output). Outside of these passages, and outside of reasoning about what God could have done, Descartes only concerns himself with i-modalities – these are what matters for the epistemic project of the *Meditations*, as I've argued, and these provide the actual constraints on physical and mathematical reasoning. The picture of i-modalities as determined by the actual essences of things is, for its part, quite standard. So, the modal bizarreness in the account is not widespread, but shows up only in certain kinds of reasoning, concerning what God could have brought about, and Descartes even casts doubt upon our abilities to reason fruitfully about those topics at all, since we can't conceive these o-possibilities.

I take these considerations to show that not only is the view consistent, but it is not modally bizarre in any way which should trouble us. Those bits of bizarreness are systematic, and explainable within the broader theory. Bizarreness of that sort looks less like confusion on Descartes', and more like bizarreness due to the unfamiliarity of his metaphysical picture.

3.8 Comparisons to Other Recent Interpretations

Kaufman: CD is Not a Modal Thesis

Kaufman [39, 40] claims that CD is not a modal thesis, but rather merely concerns divine indifference, which can be understood without invoking modality. While I take it that a reason in favour of my account is that one may take Descartes' use of modal terminology at face value, Kaufman argues that this surface appearance of modality is misleading. So I'll take a moment here to consider his reading of CD, and consider the extent to which his claims speak against my interpretation.

Kaufman's account [40] is given in response to attempts, like those given by some interpretations we discussed earlier, to understand the modal implications of the use of words like "can" and "could" in the various claims of and surrounding CD (like those cited in a previous section). His view is, where ϕ is an eternal truth, Descartes is committed to the claim that God could have willed $\neg\phi$, but that he is not committed to the claim that $\neg\phi$ is possible in virtue of this previous, or any other, commitments.

Kaufman argues that, since "a proposition is true only if God wills it to be true" is a consequence of CD, and that this holds for modal as well as non-modal propositions [40][p. 36] From this he concludes that in order for it to be true that the negation of some eternal truth ϕ is possible (or possibly possible), it must be that God willed the truth of the proposition "$\neg\phi$ is possible". But, Kaufman concludes, since God has willed that the eternal truths are necessary, He did not will that any eternal truth is possibly false. He takes this reason to motivate his view that CD is not a modal claim. How I take account of the various claims involving divine indifference has already been discussed, so I want to turn to some reasons to think, first, that a modal interpretation of CD has its benefits and, second, that my interpretation does not result in the problem Kaufman raises above.

The main reason to prefer a modal account of CD is that the terms Descartes uses are distinctly modal, and so an interpretation which attempts to avoid the modal talk apparent in the passages has a higher price to pay for its credibility. Furthermore, that the passage includes these apparent modals is borne out by the overwhelming tendency of interpreters of CD to give modal accounts. Of course, they could all be wrong, but this weight of previous interpretations and the apparent modalities used in expressing the doctrine lead me to think that, *ceteris paribus*, a modal account of CD is preferable to a non-modal account.

As a further consideration, it's not just bare modals like *could* and *can* which appear to be used in Descartes' discussion surrounding CD. He also seems to use modals embedded in counterfactual conditionals, for example (I emphasise the explicitly counterfactual reasoning):

> You ask whether there would be real space, as there is now, *if God had created nothing*. At first this question seems to be beyond the capacity of the human mind, like infinity, so that it would be unreasonable to discuss it; but in fact I think that it is merely beyond the capacity of the imagination, like the questions of the existence of God and of the human soul. I believe that our intellect can reach the truth of the matter, which is, in my opinion, that *not only would there not be any space, but even those truths which are called eternal, as that 'the whole is greater than its part' – would not be truths if God had not so established…*
>
> (To Mersenne, 27 May 1638, AT 2:138 [20][p. 102–103])

The passage does have a counterfactual conditional, though with a complex antecedent. For clarity, when I refer to *the* counterfactual conditional in the above passage, I take it to be the following (where I take it that saying that the eternal truths "would not be truths" is interchangeable with saying that they "would be false"):

If God had created nothing, then the eternal truths would be false.

Our best understanding of counterfactual conditionals is given by an account, due originally to Stalnaker [73] and Lewis [47], which employs modals, and it is unclear that there is a good account of counterfactuals available which does not employ modals. By working within a modal interpretation of CD, I have the tools necessary to interpret the above passage– indeed, I take it that my interpretation alone has the tools necessary to properly interpret this passage. On my reading, using the Lewis-Stalnaker approach to counterfactuals, the passage above should be read as saying that among the worlds where God created nothing, the closest have it that all the eternal truths are false – they are only i-necessary, after all.

The situation outlined in the antecedent, that God created nothing, is impossible in the sense that it is not compatible with the eternal truths since, as Descartes concludes, none of them would be true. Given that the eternal truths are necessary, it follows that the situation under consideration in this passage is impossible. So not only is the situation counterfactual, it is also *counterpossible*. My interpretation can account for this by having invoking worlds at which eternal truths are false, including some where God created nothing – these worlds aren't i-possible, they are o-possible. On my reading, Descartes is making a substantial claim to the effect that at any such world, the eternal truths would be false.

There are still difficulties and questions that need answering on my account – what does it mean to have a world in which God created nothing, for instance – but the rough-and-ready framework for counterfactuals provides a starting point for my interpretation. The opponent who takes CD not to be modal doesn't have this kind of analysis available – they have to explain away the apparent modality of the bare modals like *could* and *can*, as well as the apparent modality of this conuterfactual reasoning. This is not to say that Kaufman couldn't provide an account, but that this starts to look like a tall order.

Curley's Limited Possibilism and Geach's Suggestion

Curley's interpretation [23] follows a suggestion he attributes to Geach that the creation doctrine involves "not a denial that there are necessary truths, but a denial that those which are necessary are necessarily necessary." [2][p. 168] Geach seems to have intended this remark to indicate that the creation doctrine should be understood as involving a denial of the principle characterising **S4**: if ϕ is necessary then it is necessary that ϕ is necessary. On my interpretation, this same intuition can be more helpfully characterised by saying that it's not the case that if ϕ is i-necessary then ϕ is o-necessary – in other words, it is i-possible that ϕ is i-necessary while $\neg\phi$ is o-possible. This is built right into my interpretation, and in this sense, I am also following Geach's suggestion as against Frankfurt, while still holding, with Frankfurt, that o-necessity is largely unrestricted.

Another way of looking at this commitment of my interpretation is that most propositions, whether or not they are i-possible, are o-contingent. In this sense, the i-necessity and i-possibility of propositions is an o-contingent affair, but as against van Cleve [76], this does not lead to inconsistency. I think it's a virtue of this account that it makes the grounding of the modal in a contingent matter (so to speak) a natural result of a broader theory, rather than a bizarre and troublesome result in tension with the rest of Descartes' philosophical commitments.

3.8.1 Ishiguro on God's Possibly Doing The Opposite

Ishiguro's interpretation [38] has it that Descartes is not committed to the claim that God could have made contradictories true together. Her reading relies on a subtle reading of the passage in the 1644 letter to Mesland where Descartes claims "God cannot have been determined to make it true that contradictories cannot be true together, and therefore . . . He could have done the opposite."[20][p. 235] and related passages. She notes that in the various places where these examples are given, Descartes does not assert that God could bring about a positive, like "God could make contradictories true together", but only a double negative, as in the above passage. One of the conclusions she draws is that it is not appropriate to apply a double negation inference, and another is that, for Descartes, there is an asymmetry between the necessity of the eternal truths, and the impossibility of the falsehood of this logical truth. That is, she denies that Descartes holds the standard duality of the modalities – that ϕ is possible when it's not the case that $\neg\phi$ is necessary, and necessary when $\neg\phi$ is not possible.

> The reason why when we add two and three we cannot but get the sum

> of five is because eternal truths, according to Descartes, have their seat in the mind. (Principles I §49, AT 8A: 23 [19][p. 209]) They do not, it is true, depend on how particular individual minds are made – for example to be cleverer or to perceive better. We should therefore not call the Cartesian notion of modality "epistemic." Descartes' modality does not depend on historical states of our knowledge, nor on the state of knowledge either. In fact, what Descartes means by eternal truths having their seat in the mind seems closer to Kant's view on the a priori than it does to epistemic views like that of Hume. What is at issue is the universal validity of these eternal truths in our mental constitution – and this is something that Descartes discovers by the inspection of forms of our clear and distinct thoughts and not by empirical investigation of psychological dispositions.
>
> [38][p. 463]

On Ishiguro's reading, while the necessity of eternal truths is not epistemic in the sense of depending on particular minds, but is dependent on the nature of human minds broadly speaking. A way of understanding this in line with the discussion so far is that her view is that human minds, along with a faculty for conceiving, were created by God and that this creation determines the eternal truths. So they are tied down to conceivability, but this fact is bound by the human mind in general, and not by the particular powers of any thinking individual. In light of this, she goes on to claim that "what we take to be necessary truths are only necessary conditional on how our mind was created, it is easy to comprehend that all necessities be conditional." [38][p. 468]

In contrast with this, she argues that the impossibility of contradictions is absolute.

> [There] is an absolute nonepistemic modality even in Descartes: the impossibility of actualising something that falls under a contradictory concept is *absolute*. It is not true that Descartes' God *could have made* it true (let alone necessary) that two times four be seven or two plus two be five. . . . We do not ascribe anything at all to anything if we contradict ourselves. We cannot ascribe to God the power of creating that $2 + 3 = 6$ or that $2 > 3$, because we

> have not succeeded in describing a possible state of affairs that a creator could bring about.
>
> [38][p. 464–465]

The reasoning here is very similar to Thomist account of why God can't bring about contradictions. The reason is that there's nothing to bring about. Furthermore, to say that God cannot bring about a true contradiction is not to limit God's power, for there's nothing there which one says God cannot bring about.

There is a passage, troubling for my account, in the 5 Feb. 1649 letter to More in which we see Descartes engaging in this kind of reasoning. In particular, he claims:

> [We] do not take it as a mark of impotence when someone cannot do something which we do not understand to be possible, but only when he cannot do something which we distinctly perceive to be possible. Now we certainly perceive it to be possible for an atom to be divided, since we suppose it to be extended; and so, if we judge that it cannot be divided by God, we shall judge that God cannot do one of the things which we perceive to be possible. but we do not in the same way perceive it to be possible for what is done to be undone – on the contrary, we perceive it to be altogether impossible, and so it is no defect of power in God not to do it.
>
> (Letter to More, 5 Feb. 1649, AT 5:273 [20][p. 363])

On these grounds, Ishiguro claims that Descartes is not committed to the claim that God could bring about contradictory propositions, nor could God bring about the truth of claims like $2 + 2 = 5$ which contradict our clear and distinct understanding of mathematics. Rather, she holds: "When Descartes concluded that God freely ordained that two times four be necessarily eight, we saw that this was because the identity of *this truth* could not be given independently of the system of mathematics, which depends on the constitution of our mind which God created." [38][p. 469]

Ishiguro's denial that Descartes' modalities are epistemic seems to rely on understanding epistemic modality as connected to "degree of certitude or states of knowledge."[38][p. 469] However, the account she gives of the conditional necessity of eternal truths still seems to rely on epistemic modality in the sense I've discussed here – it's a matter of human conceivability, a faculty which God has created. However, a result of this seems to be that on her account the necessity of eternal truths is nothing more than our inability to conceive of their falsity. I have attempted to argue against this claim in my discussion of McFetridge on the matching argument.

Further, the evidence that Descartes does not assert that God could have made contradictories true together, but only that He is not determined *not* to do so, is perhaps less striking than it first appears. Alanen notes that Descartes' statement of CD in the letters to Mersenne appears "to have been formulated almost verbatim in opposition to the view defended by Suarez." [1][p. 68] Further sources and details are available in Cronin's [21], who also notes the connection between Descartes' statement and Suarez'. So the fact that Descartes uses this negative terminology has an available explanation on which he wasn't attempting to avoid stating the positive claims directly.

Finally there is the passage from the letter to More. Another passage which pushes this point is available in a letter to Regius, where Descartes claims: "God can surely bring about whatever we can clearly understand; the only things that are said to be impossible for God to do are those which involve a conceptual contradiction, that is, which are not intelligible." (Letter to Regius, June 1642 AT 3:565 [20][p. 214]) This is a problem for my account. I want to claim that, in general, Descartes discusses o-possibility only when discussing the limits of God's options in creating the world, and that all other discussions of modality are discussions of i-modality. My first move would be to try to claim that Descartes is here reasoning about inner modality when claiming that contradictions cannot be true, despite the fact that he is here considering what is impossible *for God*. I don't have a satisfactory account of these passages, but one is wanting. An option is that Descartes is here parroting the usual Thomist responses, rather than giving his real view. I don't endorse this reading, but hope that something along these lines might be made to work. In the meantime, I recognise this problem with my account, but hope that the weight of evidence in its favour outweighs this problem.

3.8.2 The Anachronism of Possible Worlds

Finally, I have used the terminology of possible worlds semantics to give an account of Descartes modal metaphysics, but this is quite anachronistic.

To this I have a few comments. First, it's not quite as anachronistic as it seems. As I've briefly suggested earlier, there were predecessors to Descartes who understood modality in terms of *simultaneous alternatives* (see the discussion in [42]) in the mind of God, and for my purposes the talk of *possible worlds* could be uniformly replaced with this talk. This is not to say, as might be claimed, that Descartes explicitly committed himself to understanding modality in terms of possible worlds (despite his rather famous claim in *Le Monde* about 'many worlds' [19]), but just to say that the resources I've used are, with the possible exception of my brief discussion of counterfactual conditionals, ones to which Descartes might have had access.

Second, and in a more methodological spirit, my project uses these tools to best capture what Descartes' claims committed him to. For this, it's not required that he thought about modality in terms of possible worlds or simultaneous alternative. Rather, we can use these tools to faithfully represent what he meant in making his claims, and what those claims commit him to. On this approach to the history of philosophy, I aim to characterise his views in such a way that we can more clearly express his philosophical commitments than he could, by being able to draw finer distinctions and use a more refined conceptual toolkit than was available to Descartes. A nice expression and defence of this methodological approach is given by Sellars[23]:

> The history of philosophy is appropriately rewritten by each generation, not because they have better historical methods, but because philosophy itself has made available not only finer distinctions but finer distinctions between distinctions. We can understand Plato better than Plato understood himself not primarily because we can see things that Plato did not see but because we see more complicated patterns of sameness and difference in the things he saw.
>
> ([71] *Berkeley and Desacrtes*, p. 377)

So the use of possible worlds talk in characterising Descartes' modal metaphysics does not involve taking Descartes to be committed to possible worlds, or even to understand modality in those terms, but merely to use this to better articulate the view he did espouse.

3.9 Conclusion

I have developed and defended a multimodal interpretation of Descartes' creation doctrine. While the multimodal account presented here requires more fleshing out, and while there are many related topics in Descartes' modal metaphysics about which my account must take a stand, I have attempted to argue that it is a viable picture, and one which puts the creation doctrine on firm ground.[24] My conviction here is that the creation doctrine, as I understand it, is a respectable theory in modal metaphysics, which fits in well with Descartes' broader commitments. It is genuinely modal, consistent, and doesn't result from some confusion

[23]Thanks to Lionel Shapiro for drawing my attention to this passage.

[24]This point is to say that I hope this account makes the creation doctrine the kind of view about which commentators will not be pushed to say unkind things – see Kaufman [40][p. 24]

on Descartes' part about the meanings of modal terms. In particular, Descartes is not speaking great nonsense (*pace* [32]), though his modal views interestingly differ from his contemporaries, and from recent accounts.

Chapter 4

Two Logics of Variable Essence

4.1 Motivations

Accounts of the relationship between essences and necessity are present throughout the history of logic, and has seen recent work by Fine [29], among many others. An interesting variation on this theme can be found in the work of Descartes on true and immutable natures, and on the controversy surrounding the doctrine of the creation of eternal truths. This provides the grist for some work in modal logic, on the study of logics of essence in which essences vary across a set of worlds. I'll give a brief sketch of the core elements of the creation doctrine, and an interpretation of the doctrine which motivates the logical work to follow.

Descartes holds that the set of essences determine a class of necessary truths (the eternal truths) – the most common examples used by Descartes for essences are mathematical objects, such as triangles, and for eternal truths, necessary truths regarding such objects, such as "the interior angles of a triangle sum to two right angles". In addition, he considers examples of logical propositions, such as "contradictories cannot be truth together". Beyond this broad essentialism, Descartes also holds that God has voluntary control over the essences which He creates (that is, all essences other than God's own). So, Descartes claims, God could have made it not true that the interior angles of a triangle sum to two right angles, by failing to create the essence "triangularity" or creating it in some other way. His other examples, concerning other mathematical objects, and essences which, somehow, underwrite logic, have the same pattern.

So, Descartes' modal metaphysics seems to commit him to the claim that some truths are necessary (because true of essences) but possibly false (because God could have made the essences in question differently). A natural way to respond is by distinguishing two different modalities – so that these apparently contradictory claims really just involve an ambiguity. This kind of reading motivates the logics I develop here.[1]

I'll distinguish two kinds of modality and, for the sake of simplicity, I'll talk about possibility (though each comes along with its dual). A proposition is possible in the first sense (call this "i-possibility", with "i" for "inner") just in case it is compatible with the essences God has actually created – for instance, on this line, it is i-possible for a triangle to have one right interior angle, but not to have two – and a proposition is possible in the second sense (call this "o-possibility" for "outer") just in case it is compatible with any essences God might have created. In this latter sense, then, there are many o-possibilities which are not i-possibilities: for instance, triangles o-possibly have three interior right angles. Both kinds of possibility are naturally analysed in possible worlds terms – using a standard piece of terminology, call those worlds at which God creates the same essences as the actual world "normal" and all others "non-normal" and interpret the i-modalities

[1]This interpretation is further developed in my "A Multimodal Interpretation of Descartes' Creation Doctrine."

as standard **S5** style modalities accessing all normal worlds, and the o-modalities by a universal accessibility relation.

With this picture in mind, I develop two logics which flesh out the idea of variable essences, with an aim of developing modal logics inspired by the creation doctrine.[2]

The first is presented as an extension of Kripke frame semantics for bimodal **S5**/U, and interprets essential predicates in a simple way which allows those essences to vary at non-normal worlds. Call this the logic of *Classical Variable Essences*, or **CVE**. An axiomatic system for **CVE** is presented, soundness and completeness is proved, as are some results relevant to characterising **CVE** frames and models.

The second logic extends this by allowing for the propositional connectives to exhibit non-classical behaviour at non-normal worlds, in accordance to something like Graham Priest's "open worlds" construction. [57] Call the resulting system *Non-Classical Variable Essences*, or **NVE**. A model theory for this system is given, and some results are proved.

Both **CVE** and **NVE** share a syntax and frame definition, so in the next two sections in which the formal work is relevant to both systems, I'll talk just of **VE**.

4.2 VE Syntax

The syntax of the language $\mathcal{L}$, which I'll employ throughout the paper, includes standard logical connectives, but with enough structure to atomic sentences to express that objects have some predicates essentially. $\mathcal{L}$ is constructed out of sets $\mathcal{N}, \mathcal{P}, \mathcal{C}$, defined as follows.

Definition 4.2.1. $\mathcal{N}$ is a denumerable set $\{a_1, a_2 \dots\}$ of name constants.

[2] In addition, some of the key results along the way are relevant to argumentative claims made in "A Multimodal Interpretation of Descares' Creation Doctrine".

Definition 4.2.2. $\mathcal{P}^*$ is a denumerable set $\{P_0^n, P_1^m, \ldots\}$ of predicate letters – subscripted letters denote an enumeration, superscripted letters denote arity.

Definition 4.2.3. $\mathcal{C} = \{\neg, \wedge, \Box, \Diamond, \Delta, \nabla\}$ is the set of logical connectives with arities 1,2,1,1,1,1 respectively.

Definition 4.2.4. $\mathcal{P}^{\Box}$ is a set of unary predicates $\{P_i^{\Box}, P_j^{\Box}, \ldots\}$ where for every P_i^1 in $\mathcal{P}^*$, there is a unique $P_i^{\Box} \in \mathcal{P}^{\Box}$. The use of subscripts indicates that the ordering of the $P^{\Box}$'s matches the ordering of P_i^1's. That is, if P_i^1 comes before P_j^1 in the ordering of $\mathcal{P}^*$, then $P_i^{\Box}$ comes before $P_j^{\Box}$ in the ordering of $\mathcal{P}^{\Box}$.

$P_i^{\Box}a$ is to be read "a is essentially P_i". Call these *essentialised* predicates.

Definition 4.2.5. $\mathcal{P} = \mathcal{P}^* \cup \mathcal{P}_{\Box}$

Definition 4.2.6. At is the set of sentences of the form $Pa_0, \ldots, a_n$ where $P \in \mathcal{P}$ is n-ary, and $a_0, \ldots, a_n \in \mathcal{N}$.

Definition 4.2.7. $\mathcal{L}$ is $At|A|A \wedge A|\neg A|\Box A|\Diamond A|\Delta A|\nabla A$

In what follows, I'll use $A \to B$ as shorthand for $\neg(A \wedge \neg B)$, as usual.

One may question the decision to express essential properties only in terms of unary predicates. The decision is partly one of simplicity – allowing for essential relations and complex properties significantly increases the complexity of the otherwise simple model theory developed here.[3] I work under the assumption that essential relational properties can be expressed by means of essential unary predications. For example, I assume that one can always introduce a unary predicate "is the child of Alice and Bob" to express the

[3]Fine's [29] logic of essence, for instance, is much more complex than what's developed here, at least in part due to the fact that he allows for a broader range of essential properties than I do.

essential relation a child has to its parents. With these considerations in mind, I continue with the simplifying assumption.[4]

4.3 VE Frames

Definition 4.3.1 (**VE** Frame)**.** A **VE** frame $\mathcal{F}$ is a tuple $\langle W, N, R, S, D, e, \{X_i; i \in \mathbb{N}\}\rangle$ where:

- $N \subseteq W$
- $R \subseteq W^2$
- $S \subseteq W^2$
- $D \neq \varnothing$
- $e : W \times D \longrightarrow \wp(\wp(D))$
- $X_i : W \longrightarrow \wp(D)$

Intuitively speaking, e assigns to each object, at each world, a collection of unary properties – i.e. those properties essential to that object at that world. R is the inner accessibility relation, S the outer, and N a set of normal worlds. Intuitively, the normal worlds are worlds where objects have the same essences as at the actual world. Further, where $\alpha \in W$, let $R\alpha = \{\beta; R\alpha\beta\}$.

Elements of $\mathcal{F}$ must obey the following conditions:

[4]Future work on this project calls for treating of properties in general, using the full expressive resources of lambda calculus. However, I take that as a further step after that of laying out the groundwork, as this paper aims to do.

(c1) R, S are equivalences.

(c2) $S = W^2$

(c3) $\forall \overline{a} \in D \forall \alpha \in W \forall i \in \mathbb{N}(X_i \alpha \in e(\alpha, \overline{a}) \Rightarrow \forall \beta \in R\alpha(X_i \beta \in e(\beta, \overline{a})))$

(c4) $\forall \overline{a} \in D \forall \alpha \in W \forall i \in \mathbb{N}(X_i \alpha \in e(\alpha, \overline{a}) \Rightarrow \forall \beta \in R\alpha(\overline{a} \in X_i \beta))$

(c1) and (c2) are self explanatory, but (c3) and (c4) are slightly more involved. The functions X_i interpret unary predicates, to to say that $X_i \alpha \in e(\alpha, \overline{a})$ is to say that the property (in extension) picked out by $X_i \alpha$ is in the essence of $\overline{a}$ at α. With this in mind, (c3) intuitively indicates that if a property (in intension) is essential to $\overline{a}$ at α, and β is R-accessible from α, then the property must be essential to $\overline{a}$ at β as well. In short: essential predications are necessary. (c4), then, indicates that essential properties are necessary properties.[5]

4.4 CVE: Model Theory

Definition 4.4.1 (**CVE** model)**.** M is a **CVE** model when $M = \langle \mathcal{F}; g, f \rangle$ where:

- $\mathcal{F}$ is a **VE** frame
- $g : \mathcal{N} \longrightarrow D$
- When P is an n-place predicate, $f(P^n_j)$ is a function of type $W \longrightarrow \wp(D^n)$. For unary predicates P^1_i, set $f(P^1_i) = X_i$.

Satisfaction, $\vDash$, is defined as follows, where A, B are metavariables over $\mathcal{L}$:

[5]These comments will be borne out as we come to some correspondence results on the way to proving completeness.

- $\alpha \vDash Pa_0, \ldots, a_n$ iff $\langle g(a_0), \ldots, g(a_n) \rangle \in f(P)\alpha$
- $\alpha \vDash P_i^{\Box} a$ iff $f(P_i)\alpha \in e(\alpha, g(a))$ iff $X_i \alpha \in e(\alpha, g(a))$
- $\alpha \vDash \neg A$ iff $\alpha \nvDash A$
- $\alpha \vDash A \wedge B$ iff $\alpha \vDash A$ & $\beta \vDash B$
- $\alpha \vDash \Box A$ iff $\forall \beta \in R\alpha(\beta \vDash A)$
- $\alpha \vDash \Diamond A$ iff $\exists \beta \in R\alpha(\beta \vDash A)$
- $\alpha \vDash \Delta A$ iff $\forall \beta \in S\alpha(\beta \vDash A)$ iff $\forall \beta \in W(\beta \vDash A)$
- $\alpha \vDash \nabla A$ iff $\exists \beta \in S\alpha(\beta \vDash A)$ iff $\exists \beta \in W(\beta \vDash A)$

Definition 4.4.2. Given a model M, $\Gamma \vDash_M A$ iff $\forall \alpha \in N^{\mathcal{M}}(\alpha \vDash \bigwedge \Gamma \Rightarrow \alpha \vDash A)$

Note that consequence on **CVE** models is a matter of truth preservation at normal worlds only.

Definition 4.4.3. $\Gamma \vDash_{\mathbf{CVE}} A$ iff for every model M, $\Gamma \vDash_M A$

4.5 CVE: Hilbert System

As before, A, B are metavariables over $\mathcal{L}$, while $P_{\Box}$ and a metavariables over $\mathcal{P}_{\Box}$, $\mathcal{N}$, respectively.

Axioms:

A1 All propositional tautologies

A2 $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$

A3 $\Box A \rightarrow A$

A4 $\Box A \rightarrow \Box\Box A$

A5 $\Diamond A \rightarrow \Box\Diamond A$

A6 $\Delta A \rightarrow A$

A7 $\Delta A \rightarrow \Delta\Delta A$

A8 $A \rightarrow \Delta\nabla A$

A9 $\Delta A \rightarrow \Box A$

A10 $P_i^{\Box} a \rightarrow \Box P_i^{\Box} a$

A11 $P_i^{\Box} a \rightarrow \Box P_i^{1} a$

Rules:

R1 $A \rightarrow B, A \Rightarrow B$

R2 $A \Rightarrow \Box A$

Definition 4.5.1 (**CVE** Proof)**.** A **CVE** proof is a finite series of formulae, each of which is either an instance of a **CVE** axiom or is obtained by one of the **CVE** rules from previous formulae in the proof.

Definition 4.5.2 (**CVE** Theorem)**.** $\vdash_{\mathbf{CVE}} A$ iff there is a **CVE** proof of which A is the last formula.

Intuitively speaking, A10 indicates that essential predications are necessary, and A11 that essential properties are necessary, corresponding, as we'll show, to (c3) and (c4).

4.6 CVE Adequacy

Theorem 4.6.1 (Soundness)**.** $\vdash_{\boldsymbol{CVE}} A$ *only if* $\vDash_{\boldsymbol{CVE}} A$

Proof. The proof is, as usual, by induction. I first show that each **CVE** axiom is true on all models, and then that the **CVE** rules preserve truth on all models. I'll omit the standard cases, namely (A1)–(A5). That leaves cases (A6)–(A14).

(A6) Suppose that $\alpha \in N$ and $\alpha \nvDash \Delta A \to A$. That is, $\alpha \vDash \Delta A$ and $\alpha \nvDash A$. Since $\alpha \vDash \Delta A$, it follows that $\forall \beta \in W(\beta \vDash A)$, and since $\alpha \in W$, it follows that $\alpha \vDash A$, contrary to the supposition.

(A7–8) Straightforward.

(A9) For this, note only that $\forall \alpha(R\alpha \subseteq W)$ – or, more to the point, $N \subseteq W$.

(A10) Suppose that $\alpha \in N$ and that $\alpha \vDash P_i^{\Box} a$. That is, $f(P_i^1)\alpha \in e(\alpha, g(a))$, and so $X_i\alpha \in e(\alpha, g(a))$. By (c3), $\forall \beta \in R\alpha(X_i\beta \in e(\beta, g(a)))$, and thus $f(P_i^1)\beta \in e(\beta, g(a))$. By the truth condition for $P_i^{\Box}$, $\forall \beta \in R\alpha(\beta \vDash P_i^{\Box} a)$, and by the condition for $\Box$, $\alpha \vDash \Box P_i^{\Box} a$.

(A11) Suppose that $\alpha \in N$ and that $\alpha \vDash P_i^{\Box} a$. It follows that $f(P_i^1)\alpha = X_i\alpha \in e(\alpha, g(a))$ and so, by (c4), $\forall \beta \in R\alpha(g(a) \in X_i\beta = f(P_i^1)\beta)$. Thus $\forall \beta \in R\alpha(\beta \vDash P_i^1 a)$, and so $\alpha \vDash \Box P_i^1 a$.

$\Box$

For the completeness proof, the method is the standard canonical model method with one twist due to [12][417–418]. We start by defining a pre-canonical model, which satisfies all conditions on **CVE** models except that $S = W^2$, and then we generate the genuine canonical model from this pre-canonical model which, in addition, satisfies this constraint.

Definition 4.6.2 (**CVE** Pre-Canonical Model)**.** The **CVE** pre-canonical model PM_c is a tuple

$\langle W_c, N_c, R_c, S_c, D_c, e_c, \{X_{i_c}; i \in \mathbb{N}\}, f_c, g_c\rangle$ where:

- $W_c = \{\alpha \subseteq \mathcal{L}; \alpha$ is maximally consistent and closed under **CVE**$\}$
- $N_c = W_c$
- $R_c = \{\langle \alpha, \beta\rangle \in W_c^2; \Box A \in x \Rightarrow A \in y\}$
- $S_c = \{\langle \alpha, \beta\rangle \in W_c^2; \Delta A \in x \Rightarrow A \in y\}$
- $D_c = \mathcal{N}$
- $X_{i_c}\alpha = \{a \in D_c; P_i^1 a \in \alpha\}$
- $e_c(\alpha, a) = \{X_{i_c}\alpha; P_i^{\Box} a \in \alpha\}$
- $f_c(P_i^n)\alpha = \{\langle a_0, \ldots, a_n\rangle \in D_c^n; P_i^n a_0, \ldots, a_n \in \alpha\}$
- g_c is the identity function
- $\alpha \vDash_c A \iff A \in \alpha$

Note that for unary predicates, $f_c(P_i^1)\alpha = X_{i_c}\alpha$ for every $\alpha \in W_c$.

Lemma 4.6.3. *PM_c satisfies (c1), (c3), and (c4) of the definition of* **VE** *frame, and, in addition, $R_c \subseteq S_c$.*

Proof. The proof proceeds by cases. (c1) is straightforward, and so is omitted.

(c3) Suppose that $\alpha \in W_c$, $a \in D_c$, $X_{i_c}\alpha \in e_c(\alpha, a)$, and $R_c\alpha\beta$. Since $X_{i_c}\alpha \in e_c(\alpha, a)$, $P_i^{\Box} a \in \alpha$. $\vdash_{\mathbf{CVE}} P_i^{\Box} a \rightarrow \Box P_i^{\Box} a$, so $\Box P_i^{\Box} a \in \alpha$, and since $R_c\alpha\beta$, $P_i^{\Box} a \in \beta$. Thus $X_{i_c}\beta \in e_c(\beta, a)$ as desired – this follows from the definition of e_c, X_{i_c}.

(c4) Suppose that $\alpha \in W_c, a \in D_c, X_{i_c}\alpha \in e_c(\alpha, a)$, and $R_c\alpha\beta$. As before, we have $P_i^{\Box}a \in \alpha$, and since $\vdash_{\mathbf{CVE}} P_i^{\Box}a \to \Box P_i^1 a$, $\Box P_i^1 a \in \alpha$, and since $R_c\alpha\beta$, $P_i^1 a \in \beta$. Thus, $a \in f_c(P_i^1)\beta$, and so $a \in X_{i_c}\beta$, since $f_c(P_i^1)\beta = X_{i_c}\beta$.

To show that $R_c \subseteq S_c$, it is enough to note that $\vdash_{\mathbf{CVE}} \Delta A \to \Box A$. This guarantees that if $R_c\alpha\beta$, and $\Delta A \in \alpha$, then $\Box A \in \alpha$, and thus $A \in \beta$.

$\Box$

Definition 4.6.4. Let $\alpha \in W_c$. $M_c^\alpha = \langle W_c', N_c', R_c', S_c, D_c', e_c', \{X_{i_c}'; i \in \mathbb{N}\}, f_c', g_c'\rangle$ where:

- $W_c' = S_c\alpha$
- $N_c' = W_c'$
- $R_c' = R_c$
- $D_c' = D_c$
- $e_c' = e_c$
- The X_{i_c}'s, f_c', g_c', and $\vDash_c'$ are defined similarly.

Lemma 4.6.5. *For all $\alpha \in W_c$, M_c^α is a **CVE** Model.*

Proof. Lemma 2 holds of M_c as well as of PM_c, since the latter is generated by the former. In M_c we have that $S_c = W_c'$, by the construction, so M_c satisfies (c2). All other conditions were satisfied by PM_c, and all are bounded universal generalisations, and so remain true in M_c.

All that remains are to check that the truth conditions hold in M_c for $\vDash_c'$. This proof proceeds by cases, which follow directly from the definitions of PM_c and M_c.

$\Box$

Theorem 4.6.6. $\Gamma \vDash_{CVE} A$ *only if* $\Gamma \vdash_{CVE} A$

Proof. Suppose that $\Gamma \nvdash_{\mathbf{CVE}} A$ – as a result, $\Gamma \cup \{\neg A\}$ is consistent. Lindenbaum's lemma ensures us that $\Gamma \cup \{\neg A\}$ can be extended to a maximally consistent set - call this α - and take M_c^α. By the previous lemma, M_c^α is a model of **CVE**, and we have $\Gamma \nvDash_{M_c^\alpha} A$. The result follows.

$\square$

4.7 Some CVE Consequences

Theorem 4.7.1. $P_i^\square a \wedge \nabla \neg P_i^1 a$ *is satisfiable. Hence* $P_i^\square a \to \Delta P_i^1 a$ *is not a theorem of* ***CVE***.

Proof. For simplicity, and without lack of generality, let the language signature consist of $\mathcal{P} = \{P_i^1, P_i^\square\}$ and $\mathcal{N} = \{a\}$. Let M be a **CVE** model with the following elements:

$$W = \{@, \alpha\} \qquad f(P_i^1)\alpha = X_i\alpha = \varnothing$$

$$N = \{@\} \qquad g(a) = \bar{a}$$

$$R = \{\langle @, @\rangle, \langle \alpha, \alpha\rangle\} \qquad e(@, \bar{a}) = \{X_i @\}$$

$$D = \{\bar{a}\} \qquad e(\alpha, \bar{a}) = \varnothing$$

$$f(P_i^1)@ = X_i @ = \{\bar{a}\}$$

On this model, $@ \vDash P_i^\square a$, $\alpha \vDash \neg P_i^1 a$. Hence $\vDash_M P_i^\square a \wedge \nabla \neg P_i^1 a$. Hence $\nvDash_M P_i^\square a \to \Delta P_i^1 a$.

$\square$

Corollary 4.7.2. $\square A \wedge \nabla \neg A$ *is satisfiable. Hence,* $\square A \to \Delta A$ *is not a theorem of* ***CVE***.

Proof. Let A be the statement $P_i^1 a$, and simply use M – given that $P_i^\square a \to \square P_i^1 a$, it follows that $@ \vDash \square P_i^1 a$, and thence the result.

$\square$

These results are desirable for a few reasons. Corollary 4.7.2 is nice in showing that Δ is strictly stronger than $\square$, but it and Theorem 4.7.1 have more direct textual importance.

On the intended reading of Descartes, Δ, ∇ characterise God's options in creating the world. So to say that ΔPa is true is to say that no matter how God created the world, a would have been a P. In other words, this is to say that God was necessitated to will that Pa. On the reading of the creation doctrine which I propose, Descartes is committed to the claim that (excluding God's own essence), just because He willed some property of a to be essential (and therefore $\square$-necessary), it does not follow that God was necessitated to will that a had that property essentially (or at all). With this reading in mind, the content of Theorem 4.7.1 captures Descartes' claim that God's willing a truth to be essential does not entail that God willed that fact necessarily, or was necessitated to will it, and Cor. 4.7.2 is a slight generalisation. [6]

4.8 Characterising Cells in CVE

Before going on to **NVE**, I pause to develop some methods for better characterising models of **CVE** – in particular, to flesh out the intuitive idea that worlds in these models come in collections of "co-normal" worlds, equivalence classes induced by R. The interaction between these equivalence classes and the universal modality can be characterised with a slight increase in expressive power.

Definition 4.8.1. Let $[\alpha] = \{\beta \in W; R\alpha\beta\}$. These are equivalence classes, given the definition of R, so let $\mathbb{C} = W/[\cdot]$.

[6] See the 2 May 1644 Letter to Mesland, AT 4:119 [20][p. 235].

Lemma 4.8.2. $|\mathbb{C}| \leq |W|$

Suppose that W, and hence $\mathbb{C}$, are countable.

Definition 4.8.3. Enumerate the members of $\mathbb{C}$ as c_i $(i \in \mathbb{N})$. Expand $\mathcal{L}$ by adding a countable set of propositional constants $T = \{t_i; i \in \mathbb{N}\}$ s.t.

$$\alpha \vDash t_i \Leftrightarrow \alpha \in c_i.$$

Furthermore, let $t_0 = t$ and $c_0 = N$.

The following simply restates the salient part of definition 4.8.1 as frame condition, in light of further notational conventions:

(c5) If $\alpha, \beta \in c_i$ then $R\alpha\beta$, and if $R\alpha\beta$ there exists a c_i s.t. $\alpha \in c_i$ and $\beta \in c_i$.

We can then add the following additional axiom schemata:

A12 t

A13 $\Delta(t_i \rightarrow \Box t_i)$

A14 $\Delta((\Box A \wedge t_i) \rightarrow \Delta(t_i \rightarrow A))$

Theorem 4.8.4 (Soundness)**.** *A12–A14 are true at all normal worlds in all models.*

Proof. I prove both cases.

A12 Obvious.

A13 Suppose $\alpha \in N$, and $\alpha \nvDash \Delta(t_i \rightarrow \Box t_i)$, thus $\exists \beta \in W(\beta \vDash t_i \;\&\; \beta \nvDash \Box t_i)$. Thus, there's a $\gamma \in R\beta$ s.t. $\gamma \nvDash t_i$. Thus $\gamma \notin c_i$ and $\beta \in c_i$ – however, given $R\beta\gamma$, this is impossible.

A14 Suppose $\alpha \in N$ and α does not satisfy A14. Then there must be a $\beta \in W$ s.t. $\beta \models \Box A \wedge t_i$ and yet $\beta \not\models \Delta(t_i \rightarrow A)$. In order for this last to be true, there must be a $\gamma \in W$ s.t. $\gamma \models t_i$ and $\gamma \not\models A$. However, we know that $\beta \models t_i$, so $\beta \in c_i$, and $\beta \models \Box A$. Since $\gamma \models t_i$, $\gamma \in c_i$, and thus $R\beta\gamma$. Hence, $\gamma \models A$ after all.

□

Definition 4.8.5. Extend the canonical model construction from the previous section to incorporate explicit cells talk by extending **CVE** to include A12–A14, and define the canonical cells c_i^c $(i \in \mathbb{N})$ as:

$$c_i^c = \{\alpha \in W_c; t_i \in \alpha\}$$

Define M_c as before, including the generated submodel construction, but alter the definition to say that $N_c = \{\alpha \in W_c; t \in \alpha\}$. I'll show that the additional conditions on cells hold in the generated submodel.

Lemma 4.8.6. *For all $\alpha, \beta \in W_c$, $i \in \mathbb{N}$, if $R'_c\alpha\beta$, then $\alpha \in c_i^c$ iff $\beta \in c_i^c$.*

Proof. Suppose that $R'_c\alpha\beta$. If $\alpha \in c_i^c$ then $t_i \in \alpha$, and since $\vdash \Delta(t_i \rightarrow \Box t_i)$, it follows that $\Delta(t_i \rightarrow \Box t_i) \in \alpha$, and since $S'_c\alpha\alpha$, $t_i \rightarrow \Box t_i \in \alpha$. Thus $\Box t_i \in \alpha$, and since $R_c\alpha\beta$, $t_i \in \beta$. The argument showing that if $\beta \in c_i^c$ then $\alpha \in c_i^c$ is similar.

□

Lemma 4.8.7. *For all $\alpha, \beta \in W_c$, $i \in \mathbb{N}$, if $\alpha, \beta \in c_i^c$ then $R'_c\alpha\beta$.*

Proof. If $\alpha, \beta \in c_i$ then $t_i \in \alpha \cap \beta$. To show that $R_c\alpha\beta$, suppose that $\Box A \in \alpha$. By A14, $(\Box A \wedge t_i) \rightarrow \Delta(t_i \rightarrow A) \in \alpha$, so $\Delta(t_i \rightarrow A) \in \alpha$. Thus $t_i \rightarrow A \in \beta$, and so $A \in \beta$.

□

Definition 4.8.8. Let M_c^* be the submodel of M_c (where M_c generated from PM_c by some max. consistent subset of $\mathcal{L} \cup \{t_i; i \in \mathbb{N}\}$) with $W_c^* = \{\alpha \in W_c; \exists i \in \mathbb{N}(t_i \in \alpha)\}$, and the other elements generated as usual.

Theorem 4.8.9. *M_c^* satisfies (c5).*

Proof. By lemmata 4.8.6, 4.8.7, we know that $R_c^*\alpha\beta \iff \forall i \in \mathbb{N}(\alpha \in c_i^c \iff \beta \in c_i^c)$. All that is required to obtain (c5) is that for each $\alpha \in W_c^*$, there exists a c_i^c s.t. $\alpha \in c_i^c$, but this is precisely what's delivered by the definition of W_c^*.

□

Corollary 4.8.10 (Completeness)**.** *The class of models on a **CVE** frames including $\mathbb{C}$ is completely characterised by the proof system comprising A1–A14 and R1–R2.*

Proof. The proof follows from Lemmata 4.8.6, 4.8.7 and Theorem 4.8.9 as usual.

□

I intend to develop this method further for application in **NVE** to come, but as of now, I leave the development here, and the rest for future work. In the next section, when I refer

4.9 NVE Model Theory

Now, to turn to the non-classical element of the construction. We also need worlds in the intended model at which logic is different (assuming, as we have, that classical logic is the default). There are, of course, many ways for logic to be different, and my aim is to rule out no consequence relation on the language.

A natural way to proceed is by the set of what Priest [57] calls "open worlds" – the set of all subsets of $\mathcal{L}$. However, there are some downsides to this proposal for my aims.

First, there are certain elements of the interpretation which I want to keep fixed, such as the truth conditions for the modals and essentialised predicates. These are the elements of Descartes' view which are being modelled, so they should be invariant across models. So just reading all formulas as getting arbitrary truth values is not ideal.

Second, the presence of open worlds makes the metatheory difficult – since they aren't closed under any uniform truth conditions, it's had to say anything substantial about them.[7]

Instead, the approach will involve using a different accessibility relation for each of $\wedge, \neg$ – allowing these relations to be arbitrary but for constraints ensuring that at normal worlds, $\wedge, \neg$ get their usual Boolean truth conditions.

Definition 4.9.1 (**NVE** Frame)**.** An **NVE** frame $\mathcal{F}$ is a tuple $\langle W, \leq, N, R^{\Box}, S, R^{\wedge}, R^{\neg}, D, e, \{X_i; i \in \mathbb{N}\}\rangle$ where $\langle W, N, R^{\Box}, S, D, e, \{X_i; i \in \mathbb{N}\}\rangle$ are as in **VE** frame (with $R^{\Box}$ in for R), $\langle W, \leq\rangle$ a poset, $R^{\wedge}$ a ternary relation on W, and $R^{\neg}$ a binary relation on W, satisfying the following conditions (along with c1–c4 above):

(c6.0) $\forall\alpha \in N \forall\beta, \gamma \in W(R^{\wedge}\alpha\beta\gamma \Rightarrow (\alpha \leq \beta \ \& \ \alpha \leq \gamma))$

(c6.1) $\forall\alpha, \beta, \gamma \in N(R^{\wedge}\alpha\beta\gamma \Leftrightarrow \alpha = \beta = \gamma)$

(c6.2) $\forall\alpha, \beta, \gamma, \alpha' \in W((\alpha' \leq \alpha \ \& \ R^{\wedge}\alpha\beta\gamma) \Rightarrow R^{\wedge}\alpha'\beta\gamma)$

(c7.0) $\forall\alpha \in N \forall\beta \in W(R^{\neg}\alpha\beta \Rightarrow \beta \leq \alpha)$

(c7.1) $\forall\alpha, \beta \in N(R^{\neg}\alpha\beta \Leftrightarrow \alpha = \beta)$

(c7.2) $\forall\alpha, \beta \in W((R^{\neg}\alpha\beta \ \& \ \alpha' \leq \alpha) \Rightarrow R^{\neg}\alpha'\beta)$

[7] This is, of course, part of the point of open worlds, but an approach which allows us *some* more power to make substantial claims would be helpful. An additional feature of this approach, for those motivated by this sort of thing, is that we can retain uniform truth conditions for the connectives at all worlds – something clearly lost in adopting Priest's approach.

(c8.0) $\forall \alpha \in N \forall \beta \in W(R^{\Box}\alpha\beta \Rightarrow \beta \in N)$

(c8.1) $\forall \alpha \in N \forall \beta \in W(\alpha \leq \beta \Rightarrow R^{\Box}\alpha\beta)$

Definition 4.9.2 (**NVE** Model)**.** An **NVE** model M is a tuple $\langle \mathcal{F}; g, f \rangle$ where g, f are as with **CVE** models, and so are the truth conditions excepting that those for $\wedge$ and $\neg$ are as follows:

- $\alpha \models A \wedge B$ iff $\forall \beta, \gamma \in W(R^{\wedge}\alpha\beta\gamma \Rightarrow (\beta \models A \ \& \ \gamma \models B))$
- $\alpha \models \neg A$ iff $\forall \beta \in R^{\neg}\alpha(\beta \not\models A)$

In addition, I need the following heredity condition on $\leq$:

$$\text{(c10)} \qquad \text{If } P \in \mathcal{P} \text{ and } \alpha \leq \beta, \text{ then } f(P)\alpha \subseteq f(P)\beta$$
$$\text{and for all } \overline{a} \in D,\ f(P)\alpha \in e(\alpha, \overline{a}) \text{ only if } f(P)\beta \in e(\beta, \overline{a}).$$

Lemma 4.9.3 (Heredity)**.** *For any formula $A \in \mathcal{L}$ and world $\alpha \in N$, if $\alpha \models A$ and $\alpha \leq \beta$, then $\beta \models A$.*

Proof. The proof is standard, by induction. For atomic sentences (with or without essentialised predicates, c10 does the work. For the induction cases, I'll cover $\wedge, \neg, \Box$, and Δ – $\Diamond$ and ∇ are similar to the latter two.

Suppose that A is $B \wedge C$. Since $\alpha \models A$, $\forall \gamma, \delta \in W(R^{\wedge}\alpha\gamma\delta \Rightarrow (\beta \models B \ \& \ \gamma \models C))$, and suppose that $R^{\wedge}\beta\gamma\delta$. Since $R^{\wedge}$ is antitone in its first place (c6.2), $R\alpha\gamma\delta$, and so $\gamma \models B$ and $\delta \models C$. Thus $\beta \models B \wedge C$.

Suppose that A is $\neg B$, and that $R^{\neg}\beta\gamma$. Since $\alpha \models \neg A$, if $R^{\neg}\alpha\delta$ then $\delta \not\models B$. By c8.2, since $R^{\neg}\beta\gamma$ and $\alpha \leq \gamma$, it follows that $R^{\neg}\alpha\gamma$, and so $\gamma \not\models B$. Thus, $\beta \models \neg B$, as desired.

Suppose A is $\Box B$. Since $R^{\Box}$ is transitive, $\alpha \models \Box\Box B$ when $\alpha \models \Box B$. Since $\alpha \leq \beta$, $R^{\Box}\alpha\beta$, by c9, and so $\beta \models \Box B$. If A is $\Diamond B$, then the key fact about $R^{\Box}$ is that it's symmetric, so that when $\alpha \models \Diamond B$, $\alpha \models \Box\Diamond B$.

If A is ΔB, $\alpha \models \Delta B$ iff $\forall\gamma \in W(\gamma \models B)$. This condition holds throughout the frame of the model, so if $\alpha \models \Delta B$ then for any $\beta \in W$, $\beta \models \Delta B$ – and this holds as well if $\alpha \leq \beta$. The case where A is ∇B is similar.

□

The heredity of $\leq$ makes it easy to show that $R^{\wedge}, R^{\neg}$ enforce the Boolean truth conditions in N.

Lemma 4.9.4. *If $\alpha \in N$, $\alpha \models A \wedge B$ iff $\alpha \models A$ & $\alpha \models B$.*

Proof. Suppose $\alpha \in N$.

($\Rightarrow$) Suppose $\alpha \models A \wedge B$. Since $\alpha \in N$, $R^{\wedge}\alpha\alpha\alpha$, so $\alpha \models A$ and $\alpha \models B$.

($\Leftarrow$) Suppose $\alpha \models A$ and $\alpha \models B$, and that $R^{\wedge}\alpha\beta\gamma$. By (c6.0), $\alpha \leq \beta$ and $\alpha \leq \gamma$, and so $\beta \models A$ and $\gamma \models B$. Thus $\alpha \models A \wedge B$.

□

Lemma 4.9.5. *If $\alpha \in N$, $\alpha \models \neg A$ iff $\alpha \not\models A$*

Proof. Suppose $\alpha \in N$.

($\Rightarrow$) Straightforward – since $\alpha \in N$, $R^{\neg}\alpha\alpha$.

($\Leftarrow$) Suppose $\alpha \not\models \neg A$ – then there is a $\beta \in R^{\neg}\alpha$ s.t. $\beta \models A$. By c7.0, it follows that $\alpha \models A$. The contrapositive is the desired result.

□

Definition 4.9.6. The **S5** *fragment* of $\mathcal{L}$ is that consisting of atomic sentences including the non-essentialised predicates, the connectives $\wedge, \neg$, and modalities $\Box, \Diamond$.

Theorem 4.9.7 (Conservative Extension)**.** ***NVE*** *is a conservative extension of* ***S5****.*

Proof. Suppose that Γ, A are in the **S5** fragment of $\mathcal{L}$, and that $\Gamma \not\models_{\mathbf{NVE}} A$. Then there is an $\alpha \in N$ of such a model where $\alpha \models \bigwedge \Gamma$ and $\alpha \not\models A$. Since Γ, A are in the **S5** fragment of $\mathcal{L}$, the only pieces of logical vocabulary which occur are $\Box, \Diamond, \wedge, \neg$. Let $N = \{\alpha, \alpha_0, \alpha_1, \dots\}$ – since $R^{\Box}$ is an equivalence relation on N, $\langle N, R \rangle$ is an **S5** frame. That $\wedge, \neg$ get their usual Boolean truth conditions in N guarantees that $\langle N, R^{\Box}, \models \rangle$ is an **S5** model, and thus $\Gamma \not\models_{\mathbf{S5}} A$.

Suppose, on the other hand that $\Gamma \not\models_{\mathbf{S5}} A$ – so it has a countermodel $\langle W, R, \models \rangle$. This model can be extended to at least one **NVE** model. The most boring way to do this is to set $\mathcal{M} = \langle W', N', R^{\Box}, S, R^{\wedge}, R^{\neg}, D, e, \{X_i; i \in N\}, f, g \rangle$ s.t.

- $W' = W$
- $R^{\Box} = S = R$
- $R^{\wedge} = \{\langle \alpha, \alpha, \alpha \rangle; \alpha \in W'\}$
- $R^{\neg} = \{\langle \alpha, \alpha \rangle; \alpha \in W'\}$
- $e = \varnothing$
- $D, f, g, \{X_i; i \in N\}$ as required for v to assign truth values to atomic sentences to fit the **S5** model in question.

That this model obeys the frame conditions for **NVE** is easily checked – note that the various $X_i's$ actually don't matter for the construction because the essentialised predicates are not in the language fragment under consideration.

Then $\Gamma \not\models_{\mathcal{M}} A$, and so $\Gamma \not\models_{\mathbf{NVE}} A$. Thus $\Gamma \models_{\mathbf{S5}} A$ iff $\Gamma \models_{\mathbf{NVE}} A$, when Γ, A in the **S5** fragment of $\mathcal{L}$.

$\square$

Theorem 4.9.8. *The proof system introduced for* **CVE** *is sound w.r.t. the class of* **NVE** *models.*

Proof. The proof proceeds by cases – that any instance of a classical tautology is valid on the set of **NVE** models is given by the fact that $\wedge, \neg$ have Boolean truth conditions at normal worlds, which fact also guarantees the validity of modus ponens. (c8.0) tells us that **NVE**-consequence will validate the **S5** $\Box, \Diamond$ axioms, as well as the rule of $\Box$ necessitation. The arguments for the remaining axioms are straightforward – for each, I'll list the key fact about **NVE** models which facilitates the proof.

(A6) S is a universal accessibility relation.

(A7) The interpretation of Δ is invariant across points in the model.

(A8) The interpretation of ∇ is invariant across points in the model.

(A9) $R^{\Box} \subseteq S$.

(A10) (c3) – the previous argument is unchanged except that the fact that the world of evaluation is assumed to be normal is now necessary to ensure that $\rightarrow$ has the appropriate truth conditions.

(A11) (c4) – with a similar comment to that above.

$\square$

This result guarantees that

4.10 Correspondence of $R^{\wedge}, R^{\neg}$ to the Boolean Truth Conditions

In this section, I show that the frame conditions $c6.0$–$c8$ all hold on a canonical model which gives the the skeleton of one for **NVE**. The upshot of this is that the frame conditions are well chosen as tightly characterising the behaviour of the connectives in question in **NVE** models.

$W_c = \wp(\mathcal{L})$

$N_c = \{\alpha \in W_c; \alpha \text{ maximally consistent and closed under } \mathbf{S5}\}$

$R_c^{\Box} = \{\langle \alpha, \beta \rangle \in W_c^2; \Box A \in \alpha \Rightarrow A \in \beta\}$

$R_c^{\wedge} = \{\langle \alpha, \beta, \gamma \rangle \in W_c^3; A \wedge B \in \alpha \Rightarrow (A \in \beta \;\&\; B \in \gamma)\}$

$R_c^{\neg} = \{\langle \alpha, \beta \rangle \in W_c^2; \neg A \in \alpha \Rightarrow A \notin \beta\}$

$\leq_c = \subseteq$

Lemma 4.10.1 (c6.0)**.** *If $\alpha \in N_c$ and $\beta, \gamma \in W_c$, $R_c^{\wedge}\alpha\beta\gamma$ only if $\alpha \subseteq \beta$ and $\alpha \subseteq \gamma$.*

Proof. Suppose $\alpha \in N_c$, $\beta \in W_c$, $R_c^{\wedge}\alpha\beta$, and $A \in \alpha$. Since $\alpha \in N_c$, $A \wedge A \in \alpha$, and thus, since $R_c^{\wedge}\alpha\beta$, $A \in \beta$ and $A \in \gamma$. Thus, $\alpha \subseteq \beta$ and $\alpha \subseteq \gamma$.

□

Lemma 4.10.2 (c6.1)**.** *If $\alpha, \beta, \gamma \in N_c$, $R_c^{\wedge}\alpha\beta\gamma \iff \alpha = \beta = \gamma$.*

Proof. Suppose that $\alpha, \beta, \gamma \in N_c$. First, suppose that $R_c^{\wedge}\alpha\beta\gamma$ – from this we already know that $\alpha \subseteq \beta$. Now suppose that $A \notin \alpha$. Since $\alpha \in N_c$, $\neg A \in \beta$, and since $\beta \in N_c$, $A \notin \beta$. Thus $\beta \subseteq \alpha$, so $\alpha = \beta$. To show that $\alpha = \gamma$ is similar.

Second, suppose that $\alpha = \beta = \gamma$, and that $A \wedge B \in \alpha$. Since $\alpha \in N_c$, $A \in \alpha$ and $B \in \alpha$, so $R_c^{\wedge}\alpha\beta\gamma$.

□

Lemma 4.10.3 (c6.2). *If $R_c^{\wedge}\alpha\beta\gamma$ and $\alpha' \subseteq \alpha$, then $R_c^{\wedge}\alpha'\beta\gamma$.*

Proof. Suppose that $A \wedge B \in \alpha'$. Since $\alpha' \subseteq \alpha$, $A \wedge B \in \alpha$, and since $R_c^{\wedge}\alpha\beta\gamma$, $A \in \beta$ and $B \in \gamma$ as desired.

□

Lemma 4.10.4 (c7.0). *If $\alpha \in N_c$, $\beta \in W_c$, $R^{\neg}\alpha\beta$, then $\beta \subseteq \alpha$.*

Proof. Make the suppositions of the lemma, and suppose that $A \in \beta$. By definition of $R_c^{\neg}$, $\neg A \notin \alpha$. Under the supposition that $\alpha \in N_c$, it follows that $A \in \alpha$, so $\beta \subseteq \alpha$,

□

Lemma 4.10.5 (c7.1). *If $\alpha, \beta \in N_c$, $R_c^{\neg}\alpha\beta \iff \alpha = \beta$.*

Proof. Suppose that $\alpha, \beta \in N_c$.

First, suppose that $\alpha = \beta$. If $\neg A \in \alpha$ and $\alpha \in N_c$, then $A \notin \alpha$, so $R_c^{\neg}\alpha\alpha$. So $R_c^{\neg}\alpha\beta$.

Second, suppose that $R_c^{\neg}\alpha\beta$. Now to show that $\alpha \subseteq \beta$, suppose that $A \in \alpha$. It follows that $\neg\neg A \in \alpha$, and so $\neg A \notin \beta$, which, since $\beta \in N_c$, implies that $A \in \beta$. Next, suppose that $A \in \beta$. It follows that since $R_c^{\neg}\alpha\beta$, $\neg A \notin \alpha$, which entails that $A \in \alpha$.

□

Lemma 4.10.6 (c7.2). *If $\alpha, \beta \in W_c$, $R_c^{\neg}\alpha\beta$ and $\alpha' \subseteq \alpha$, then $R_c^{\neg}\alpha'\beta$.*

Proof. Under the suppositions of the lemma, if $\neg A \in \alpha'$ then $\neg A \in \alpha$, and since $R_c^{\neg}\alpha\beta$, it follows that $A \notin \beta$. Thus $R_c^{\neg}\alpha'\beta$.

□

Lemma 4.10.7 (c8). *If $\alpha \in N_c$, $\beta \in W_c$, and $\alpha \subseteq \beta$ then $R_c^{\Box}\alpha\beta$.*

Proof. Suppose that all the constraints hold, and $\Box A \in \beta$. Since $\alpha \in N_c$, $A \in \alpha$, and since $\alpha \subseteq \beta$, $A \in \beta$. Thus $R_c^{\Box}\alpha\beta$.

□

Further work includes extending this model to a full canonical model for **NVE**, and using that to get a better grip on the class of **NVE** models. For now, these lemmata show that the frame conditions were well chosen, and will form the core of a further model theoretic work on **NVE**.

4.11 Conclusion

In this paper, I have begun to characterise two logics of variable essence – for **CVE** a proof theory and adequate model theory have been given. For **NVE**, a class of models has been defined, and I have begun the process of characterising these models. However, two important upshots available here are (1) that both logics are conservative extensions of **S5**, and (2) the groundwork for showing that both are consistent. (1) is important in providing the means to argue that Descartes' modal metaphysics is only bizarre when the essentialised predicates and the outer modality are in play. On my reading, the vast majority of Descartes' treatment of modality in, for instance, the *Meditations*, treats primarily of the inner modals. So, I hold, even if his view is bizarre, the bizarreness is part of his theory, and not due to a confusion regarding the modal language he employs. As for (2), I take it that Descartes' total view is consistent, and providing a proof of this fact will, hopefully, proceed by showing that normal worlds in **NVE** are closed under classical consequence and that we can build models at which some formula is false at some normal world. The latter is easy to show, and the former involves extending the proofs given in this paper to guarantee that classical consequence holds at normal worlds even when the o-modals are in play.

Chapter 5

Channel Composition in Ternary Relation Semantics

5.1 Ternary Relation Semantics and the Problem of Interpretation

The ternary relation semantic framework, as most famously developed in [64], though very powerful, has long presented difficulties in interpretation. [1] The proposal with which we are primarily concerned is that of *channel theory*, as developed by Barwise [4] and Restall [59], within the broader theoretical framework provided by *situation semantics* as developed in [5], [6], and elsewhere.[2]

The basic theoretical posit of situation semantics are *situations*. These can be understood, as following Barwise and Seligman [6], as *classification-systems* of a kind. A clas-

[1] Note the classic challenge in [16] that the semantics is merely a technical device with little of the intuitive grip had by truth-functional semantics or the Kripke semantic framework for intuitionist logic and modal logics.

[2] For a broad overview of the interaction between semantics for relevant logics and situation semantics/channel theory see [50].

sification system is a collection of *types* and *tokens* and an assignment, to each type, of a set of tokens. In some slightly different terms, a situation is a collection of *objects* and *properties*, and an assignment to each property of some (possibly empty) set of objects. Formally, a situation can simply be modelled as a set of sentences over some vocabulary containing predicate-symbols (for the types) and name constants (for the tokens).[3]

Barwise [4] develops an account of how information flows *between* situations by considering an additional kind of entity, namely, an *information channel*. A channel, for Barwise, supports information flow from a situation (the *signal* of the channel) to a situation (its *target*), and just which kind of information flow it supports determines what conditional propositions are made true at that channel. He employs a ternary relation, evocatively notated $\beta \overset{\alpha}{\mapsto} \gamma$, to indicate that the channel α supports information flow from the signal β to the target γ. It was not long in the waiting for relevant logicians to recognise the similarity between this semantics and that employed in the ternary relation (Routley-Meyer[4]) semantics for relevant logics. Indeed, the formal match makes the interpretation of relevant semantics by channels quite natural.

As another brief point of motivation, while situation semantics has fallen out of fashion, it is, by my estimation, a very natural setting for a theory of inference, and due for a reappraisal in the broader philosophical community. Situated inference is general enough that it can provide a (somewhat) neutral background for debates about logic, and those contested principles of logic. That is, since something like the situation semantics (particularly in the extended sense presaged by the *set-ups* of [65]) can be tweaked to provide semantics for

[3]There is a great deal more to be said of the situation semantics and, for instance, the interpretation of *negation* on such a framework. However, our interests are, for the most part, restricted to the conditional, so we leave these other considerations to the side.

[4]Though this semantic framework often goes by the name "the Routley-Meyer semantics", a better name, giving proper attention to the history, might be "the Maksimova-Urquhart-Routley(Sylvan)-Meyer-Fine semantics." What this name lacks in elegance, it makes up in correctness.

many logics with a variety of consequence relations, non-classical and otherwise, it could provide a fairly natural setting for debates between proponents of these various logics. In any case, despite the dearth of new work in situation semantics in the last decade or so, it is still a framework in which interesting new work is desirable (or so I hold, having missed situation semantics in its heyday).

Now let me come back to the work at hand. There are a number of ways to proceed in fleshing out a channel theory in the ternary relation semantic framework. First I shall focus on Greg Restall's approach as developed in [59], which is an interpretation of the ternary relation in terms quite similar to Barwise's theory. Of particular interest is Restall's treatment of *serial composition* (from here on just *composition*) and its relation to an operation of his related to the relevant connective *fusion*, which we shall come to define, as cashing out a notion of *application*. I shall provide some reasons to be dissatisfied with Restall's account, and shall instead go back to Barwise's approach, with a particular focus on his treatment of composition. The aim of this paper is to provide some more definition to the account by pulling Barwise's composition apart from Restall's application, and to set out an extension to the Ternary Relation semantics to provide a logic for channels and channel composition. I'll set out the basic semantic framework for the basic relevant logic **B** in the language of conjunction and implication, and show that this extension to the ternary relation semantics for this logic is conservative. Finally, I'll display some interesting differences between Restall's account and the account to be developed here as regards the idempotence and commutativity of composition, before closing with a problem about extending the approach to incorporate negation.

5.1.1 $\mathbf{B}_\wedge$ Frames and Models

Our formal language includes a set of propositional atoms [5] and the connectives $\wedge$ and $\rightarrow$ (both of arity 2). In §5.5 we shall also discuss $\leftarrow$, but shall set out its semantics as we come to it. $A, B, C, \ldots$ are metavariables ranging over propositions.

A ternary relation *model* for $\mathbf{B}_\wedge$ is defined as follows: [6]

Definition 5.1.1. A ternary relation model for $\mathbf{B}_\wedge$ is a pair $\langle \mathcal{F}, \vDash \rangle$ of a frame $\mathcal{F}$and valuation $\vDash$ meeting the following criteria:

$\mathcal{F} = \langle S, N, R, \sqsubseteq \rangle$

- $N \subseteq S$ and $N \neq \varnothing$
- $R \subseteq S^3$
- $\exists x \in N(Rx\alpha\beta) \Leftrightarrow \alpha \sqsubseteq \beta$
- $\sqsubseteq$ is a partial order
- $\alpha' \sqsubseteq \alpha$, $\beta' \sqsubseteq \beta$, $\gamma \sqsubseteq \gamma'$, and $R\alpha\beta\gamma$ imply that $R\alpha'\beta'\gamma'$. Tonicity Conditions
- For any proposition A, $\alpha \sqsubseteq \beta \Rightarrow (\alpha \vDash A \Rightarrow \beta \vDash A)$. Heredity

The following conventions will be useful for a short expression of some features of the ternary relation.

- $R^2\alpha\beta\gamma\delta$ iff $\exists\epsilon(R\alpha\beta\epsilon \;\&\; R\epsilon\gamma\delta)$

[5] The situation-semantic story we shall be interested in here is given in a first order language, but for our purposes it is sufficient to stay at the level of propositional logic.

[6] We shall occasionally employ $\Rightarrow$ and & as metalanguage connectives which shall behave in accordance with material implication and classical conjunction, respectively. In addition, we shall occasionally use metalanguage quantifiers $\forall, \exists$ for brevity, and these always range over situations.

- $R^2\alpha(\beta\gamma)\delta$ iff $\exists\epsilon(R\alpha\epsilon\delta \;\&\; R\beta\gamma\epsilon)$

- $R^3\alpha(\beta(\gamma\delta))\epsilon$ iff $\exists\zeta(R^2\alpha(\beta\zeta)\epsilon \;\&\; R\gamma\delta\zeta)$

Finally, to get a model $\mathcal{M} = \langle \mathcal{F}, \vDash \rangle$ of $\mathbf{B}_\wedge$, define the valuation $\vDash$ as follows:

- $\alpha \vDash A \wedge B$ iff $\alpha \vDash A$ and $\alpha \vDash B$

- $\alpha \vDash A \to B$ iff for all $\beta, \gamma \in S$ if both $R\alpha\beta\gamma$ and $\beta \vDash A$ then $\gamma \vDash B$

Given $\vDash$, we can define *theorem* and *model-validity* as usual:

Definition 5.1.2. A is valid on the $\mathbf{B}_\wedge$ model $\mathcal{M}$ $(\mathcal{M} \vDash A)$ iff $x \vDash A$, for all $x \in N$.

Definition 5.1.3. A is a theorem of $\mathbf{B}_\wedge$ $(\vdash_{\mathbf{B}_\wedge} A)$ iff for every model $\mathcal{M}$ of $\mathbf{B}_\wedge$, $\mathcal{M} \vDash A$.

5.1.2 $\mathbf{B}_\wedge$ – A Hilbert System

There are a handful of options [7] regarding how to axiomatise $\mathbf{B}_\wedge$ but we use the following axioms and rules: [8]

A1	$A \to A$	I
A2	$(A \wedge B) \to A$	Simplification$_1$
A3	$(A \wedge B) \to B$	Simplification$_2$
A4	$((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$	Lattice-$\wedge$

[7] See [65] for details.

[8] Upper case sans-serif letters are used as names for some axioms and these refer to the names of the combinators of which these formulae are the principal type schemata. See [27] for more information about this convention.

R1 $A, B \vdash A \wedge B$ Adjunction

R2 $A \to B, A \vdash B$ Modus Ponens

R3 $A \to B, C \to D \vdash (B \to C) \to (A \to D)$ Affixing

Note that these rules are *rules of proof*, in Smiley's sense (see Humberstone [37] for clarification). So, for instance, the statement of (R1) is intended to be understood as "when A and B are both theorems, then so is $A \wedge B$".

Axioms and frame conditions for some extensions of $\mathbf{B}_\wedge$

A11 $((A \to B) \wedge (B \to C)) \to (A \to C)$ Conjunctive Syllogism

A12 $(A \to B) \to ((C \to A) \to (C \to B))$ B

A13 $(A \to B) \to ((B \to C) \to (A \to C))$ B'

A14 $(A \to (A \to B)) \to (A \to B)$ W

The following conditions correspond to the above axioms.

S11 $R\alpha\beta\gamma \Rightarrow R^2\alpha(\alpha\beta)\gamma$

S12 $R^2\alpha\beta\gamma\delta \Rightarrow R^2\alpha(\beta\gamma)\delta$

S13 $R^2\alpha\beta\gamma\delta \Rightarrow R^2\beta(\alpha\gamma)\delta$

S14 $R\alpha\beta\gamma \Rightarrow R^2\alpha\beta\beta\gamma$

This correspondence is of the following sort: $\mathcal{M} \models$ A1i iff the frame of $\mathcal{M}$ obeys restriction S1i, for $1 \leq i \leq 4$. Proofs of these facts are sketched in [65][313] and [63][203–204]. We shall have reason to refer to other pairs of provabilities and corresponding ternary relation conditions from time to time, but for the most part, our concern shall be with those above. The following logics are notable for our interests:

$\mathbf{DJ}_\wedge$ is $\mathbf{B}_\wedge$ plus A11.

$\mathbf{TW}_\wedge$ is $\mathbf{B}_\wedge$ plus A12 and A13.

$\mathbf{T}_\wedge$ is $\mathbf{TW}_\wedge$ plus A14.

TW, **DJ**, and their neighbours, including **B** itself, have long been of interest as potential homes for theories of naïve truth and sets.[9] While this paper won't involve any substantial comment on the paradoxes, it's worth a passing note that we are in the neighbourhood and that the channel theoretic interpretation seems a good fit for these very weak logics.

5.2 Information Channels

To begin with, some comments are in order to set out our focal notions of *application* and *composition*.

Some notion of application plays an essential role in many interpretations of relevant logics and particularly the ternary relation. For instance, Restall interprets $R\alpha\beta\gamma$ as "the conditional information given in α applied to β results in no more than γ" [59] and elsewhere as "applying the information in α to β gives information which is already in γ." [58] One can find similar intuitions and terminology at work in [72], [49], and parts of [8]. A natural way to make these intuitions concrete, following the lead of algebraic semantics for relevant logics, is to introduce a collection of points $\alpha \circ \beta$ into the semantics which, speaking loosely, are the results of *applying* α to β. Well-known problems with an operational semantics for relevant logics [10] mean that we cannot, in general, assume that $\alpha \circ \beta$ is a

[9]The paradox to which we refer here is, of course, Curry's paradox and derivative paradoxes, like the validity curry [9]. For general information see [7].

[10]See [75] for details.

unique point in the frame. Thus, Restall [59] defines this as a set-forming operation:

Definition 5.2.1. $\alpha \circ \beta = \{\gamma \,|R\alpha\beta\gamma\}$.

In order to make sense of how points like this behave in the ternary relation semantics, we need to enforce at least the following condition: [11]

$$\alpha \circ \beta \vDash A \text{ only if for every } \gamma \in \alpha \circ \beta, \gamma \vDash A.$$

Even with this operation providing only a set of points, rather than a unique point, the application story is fairly natural, in abstract. The key fact here is that $(\alpha \vDash A \rightarrow B \,\&\, \beta \vDash A) \Rightarrow \alpha \circ \beta \vDash B$, as follows immediately from the definition of $\circ$. This fact provides the key intuition behind the ternary relation: when α is a channel from β to γ ($R\alpha\beta\gamma$), and $\beta \vDash A$ implies $\gamma \vDash B$, then $\alpha \vDash A \rightarrow B$.

On the other hand, we have *composition* $\alpha;\beta$ as channels. The key job we want composition to do is to enforce $(\alpha \vDash A \rightarrow C \,\&\, \beta \vDash C \rightarrow B) \Rightarrow \alpha;\beta \vDash A \rightarrow B$. Intuitively, when α is a channel supporting information flow from A-propositions to some propositions from which β supports information flow to B-propositions, there is a situation which is a channel cutting out the middle, as it were. As an example, suppose that a phone call allows for information to flow from my home in Connecticut to Edmonton, and an email allows for that (or some related) information to flow from Edmonton to my next-door neighbour in Connecticut. Then there is a channel resulting of the composition of the relevant bits of the phone-network connecting my house to Edmonton with the relevant bits of the internet and servers which support that email connection from Edmonton to my neighbour. It's the phone line *composed* with the email connection which allow for information to flow from me to my next door neighbour. Barwise [4] makes the following demands of composition,

[11] In addition, we need some posits governing how $\circ$ interacts with $\sqsubseteq$ but these details are not necessary for the comments in this paper.

where 0 is a *logic channel* behaving essentially as a member of N as set out in §1 [12] and $\beta \overset{\alpha}{\mapsto} \gamma$ is to be read as $R\alpha\beta\gamma$.

- For any α, β, there is a unique $\alpha;\beta$.
- $\gamma \overset{\alpha;\beta}{\mapsto} \delta \Leftrightarrow \exists\epsilon(\gamma \overset{\alpha}{\mapsto} \epsilon \,\&\, \epsilon \overset{\beta}{\mapsto} \delta)$
- $0;\alpha = \alpha = \alpha;0$ [13]
- $\alpha;(\beta;\gamma) = (\alpha;\beta);\gamma$

He proceeds to show, given his very abstract framework, that composition has these features, and leaves open the questions of whether $\alpha = \alpha;\alpha$ and $\alpha;\beta = \beta;\alpha$, taking these as substantial questions to be filled in by fuller channel theories.

5.2.1 Restall and Application

Restall's move [59] is to identify $\alpha;\beta$ and $\alpha \circ \beta$. An equivalent statement of $R\alpha\beta\gamma$, given his account of $\circ$, is $\alpha \circ \beta \sqsubseteq \gamma$. This has some interesting features. One nice feature is that we can carry over intuitions about composition in order to explain some features of application, and hence the ternary relation. Barwise's desiderata for composition, when read in terms of $\circ$, are either built into the ternary relation semantics, or underwrite what many take to be plausible axioms and arguments, understood in an information-theoretic terms. Consider Barwise's normality condition that $0;\alpha = \alpha = \alpha;0$. For Restall's reading to capture this it must at least demand the following condition, where $\alpha \circ (\beta \circ \gamma) \sqsubseteq \delta$ holds

[12] While, for the purposes of generality, we consider non-reduced models, that is, models with multiple normal points, in the remainder of §5.2 we follow Barwise and Restall in focusing on a distinguished normal point, 0.

[13] For this desideratum and some discussion, see [4][19].

just when there exists an ϵ s.t. $\alpha \circ \epsilon \sqsubseteq \delta$ and $\beta \circ \gamma \sqsubseteq \epsilon$, and $(\alpha \circ \beta) \circ \gamma \sqsubseteq \delta$ just in case some ϵ is s.t. $\alpha \circ \beta \sqsubseteq \epsilon$ and $\epsilon \circ \gamma \sqsubseteq \delta$.

$$\alpha \circ \beta \sqsubseteq \gamma \Leftrightarrow (0 \circ \alpha) \circ \beta \sqsubseteq \gamma \Leftrightarrow (\alpha \circ 0) \circ \beta \sqsubseteq \gamma$$

i.e.

$$R\alpha\beta\gamma \Leftrightarrow R^2 0\alpha\beta\gamma) \Leftrightarrow R^2\alpha 0\beta\gamma.$$

Note, that $R\alpha\beta\gamma \Leftrightarrow R^2 0\alpha\beta\gamma$ is immediate. For $R\alpha\beta\gamma \Rightarrow R^2\alpha 0\beta\gamma$, some more robust assumptions are required [14] but it is, perhaps, a plausible demand on this story.

For Restall, satisfying the desired associativity property, $\alpha; (\beta; \gamma) = (\alpha; \beta); \gamma$, involves at least admitting:

$$(\alpha \circ \beta) \circ \gamma \sqsubseteq \delta \Rightarrow \alpha \circ (\beta \circ \gamma) \sqsubseteq \delta \qquad \text{i.e. } R^2\alpha\beta\gamma\delta \Rightarrow R^2\alpha(\beta\gamma)\delta$$

which corresponds to (A12). In addition, in order to enforce the condition: $\alpha \models A \to B$ and $\beta \models B \to C$ imply $\alpha \circ \beta \models A \to C$, one needs the condition:

$$(\alpha \circ \beta) \circ \gamma \sqsubseteq \delta \Rightarrow \beta \circ (\alpha \circ \gamma) \sqsubseteq \delta \qquad \text{i.e } R^2\alpha\beta\gamma\delta \Rightarrow R^2\beta(\alpha\gamma)\delta$$

which corresponds to (A13). So, setting aside the concerns with our logic channels, the weakest logic which can be given a Restall-style channel account is around the strength of **TW**. [15]

There are a couple of potential problems here.

1. $\circ$ is not functional, so given some appropriate α, β, $\alpha \circ \beta$ is not unique.

[14] A far too strong one, for instance, being $R\alpha\beta\gamma \Rightarrow R\beta\alpha\gamma$, which collapses the distinction between $\to$ and $\leftarrow$ (see the semantics of this arrow below), and corresponds to the axiom $A \to ((A \to B) \to B)$. Perhaps a more natural answer is just $R0\alpha\beta \Rightarrow R\alpha 0\beta$, which is somewhat weaker.

[15] Adding disjunction and negation to $\mathbf{TW}_\wedge$ involves adding the axioms A5–A9 of [65][287] in addition to the contraposition axiom we consider in §6.

2. This version renders the associativity of composition as a fairly substantial property, corresponding to the provability of the B' axiom $(A \to B) \to ((B \to C) \to (A \to C))$. On Restall's scheme, one also gets the B axiom $(A \to B) \to ((C \to A) \to (C \to B))$, and so rules out some very weak logics as underwriting a channel theory. However, its unclear how the provability of the axiom corresponding to this property of composition is to be justified on channel-theoretic terms. The frame conditions corresponding to B and B' involve at least blurring the distinction between channels and situations operated upon by channels.[16]

3. Surely composition and application are distinct operations with distinct notions, and even if it makes sense to identify them, such an argument would need more detail about how they operate separately.

4. The requirement that $R\alpha\beta\gamma \Rightarrow R^2\alpha 0\beta\gamma$ seems potentially problematic on the intended reading, in that it seems to demand that we accept something like $R\alpha 0\alpha$ which is quite implausible, when understanding 0 as some kind of logic *channel*, in line with Barwise. Why should it be that any channel given a logic channel as an input produces itself?

In the rest of this paper, we shall be interested to develop an account more in line with Barwise's initial proposal. This proposal has quite broad applications, applying naturally to classical, intuitionist, and some other logics, as Barwise showed, and as we shall show, to an important fragment of the basic relevant logic **B**.

[16]While Restall takes this to be a feature rather than a bug, it is at least prima facie unclear whether this is the case.

5.3 Some Preliminaries

The ternary relation semantic framework set out in §5.1 by itself meets some of Barwise's desiderata for composition. In this section we shall show that important consequences of the identity and associativity contraints are met.

The best way to proceed would be to define and fully work out the details of a composition operation or function on the ternary relation semantics of the type $S^2 \longrightarrow S$. This would involve defining an operation which interacts with R and $\sqsubseteq$ in ways which produce the desired behaviour. The first demand is that for any pair of points in S there exists a point which behaves as their composite. That is, for any $\alpha \vDash A \to B$ and $\beta \vDash B \to C$, there is some $\alpha;\beta \vDash A \to C$.

First, however, we should set out just when a putative composite point is a channel between some points. The following condition fits the bill:

$$\exists\epsilon(R^2\beta(\alpha\gamma)\delta \Leftrightarrow R\epsilon\gamma\delta) \qquad \text{(Existence of Composites)}$$

We use $\alpha;\beta$ to name the ϵ for α,β in question. Then the above has the effect that $R^2\beta(\alpha\gamma)\delta \Leftrightarrow R(\alpha;\beta)\gamma\delta$, so long as $\alpha;\beta$ exists. On the intended reading, $R(\alpha;\beta)\gamma\delta$ tells us that there is some information one gets from applying α to γ which, when β is applied to it, results in δ. That is, there is a chain of channels along which we can reason where we take a signal γ for α, get its target, and then apply β to that target to get a situation which supports the target of β, namely δ.

That this fits the bill is easy to see. Suppose that $\alpha \vDash A \to B$ and $\beta \vDash B \to C$. Then since $\forall\gamma\forall\delta(R\alpha\gamma\delta \Rightarrow (\gamma \vDash A \Rightarrow \delta \vDash B))$ and $\forall\epsilon\forall\zeta(R\beta\epsilon\zeta \Rightarrow (\epsilon \vDash B \Rightarrow \zeta \vDash C))$, so there is an η s.t. $R\alpha\gamma\eta$ and $R\beta\eta\delta$ and $\gamma \vDash A$, then $\eta \vDash B$ and so $\delta \vDash C$ after all. So $R(\alpha;\beta)\gamma\delta \Rightarrow (\gamma \vDash A \Rightarrow \delta \vDash C)$ as desired.

We first present two preliminary results showing that if composite points are around in the frame, they behave in much the way that Barwise wanted. In lieu of $0;\alpha = \alpha = 0;\alpha$, we can show that something like $R\alpha\beta\gamma \Leftrightarrow R(0;\alpha)\beta\gamma \Leftrightarrow R(\alpha;0)\beta\gamma$, where we generalise to the non-reduced framework initially introduced. This shows that when α is a channel from β to γ, then so is the composition of α with some normal point, which captures at least part of the spirit of Barwise's desideratum.

Theorem 5.3.1. *If $\mathcal{M}$ satisfies the existence of composites condition, then it also satisfies $R\alpha\beta\gamma \Leftrightarrow \exists x \in NR(\alpha;x)\beta\gamma \Leftrightarrow \exists y \in NR(y;\alpha)\beta\gamma$.*

Proof. Suppose that $R\alpha\beta\gamma$. Then we have that $R\alpha\beta\gamma$ and $\exists x \in N(Rx\gamma\gamma)$, and thus $\exists y \exists x \in N(R\alpha\beta y \mathbin{\&} Rxy\gamma)$. That is, $\exists x \in NR^2x(\alpha\beta)\gamma$.

Suppose that $\exists x \in NR^2x(\alpha\beta)\gamma$. That is, $\exists x \in N\exists y(Rxy\gamma \mathbin{\&} R\alpha\beta y)$. Since $x \in N$ and $Rxy\gamma$, we have that $y \sqsubseteq \gamma$, and so since $R\alpha\beta y$, it is the case that $R\alpha\beta\gamma$, by the tonicity conditions on R.

So, we have that $R\alpha\beta\gamma \Leftrightarrow \exists x \in NR^2x(\alpha\beta)\gamma$, that is $R\alpha\beta\gamma \Leftrightarrow \exists x \in NR(\alpha;x)\beta\gamma$.

Suppose that $R\alpha\beta\gamma$. Note that $\exists x \in NRx\beta\beta$ and hence $\exists y \exists x \in N(R\alpha y\gamma \mathbin{\&} Rx\beta y)$, that is $\exists x \in NR^2\alpha(x\beta)\gamma$.

For the other direction, note that if $R^2\alpha(x\beta)\gamma$ then $\exists y$ s.t. $R\alpha y\gamma$ and $\beta \sqsubseteq y$, and so the tonicity conditions on R guarantee that $R\alpha\beta\gamma$. □

In a similar vein, we can show that the following important consequence of associativity for composition holds in any ternary relation model:

Theorem 5.3.2. *If $\mathcal{M}$ satisfies the existence of composites condition, then it also satisfies $R(\alpha;(\beta;\gamma))\delta\epsilon \Leftrightarrow R((\alpha;\beta);\gamma)\delta\epsilon$*

Proof. Suppose $R(\alpha;(\beta;\gamma))\delta\epsilon$, that is $R^2(\beta;\gamma)(\alpha\delta)\epsilon$. So $\exists\zeta(R(\beta;\gamma)\zeta\epsilon \mathbin{\&} R\alpha\delta\zeta)$. So $\exists\zeta(R^2\gamma(\beta\zeta)\epsilon \mathbin{\&} R\alpha\delta\zeta)$. Thus $\exists\zeta(\exists\lambda(R\beta\zeta\lambda \mathbin{\&} R\gamma\lambda\epsilon) \mathbin{\&} R\alpha\delta\zeta)$. Hence, since λ does not occur in $R\alpha\delta\zeta$, we have that $\exists\zeta\exists\lambda(R\beta\zeta\lambda$ & $R\gamma\lambda\epsilon$ & $R\alpha\delta\zeta)$. So $\exists\lambda(\exists\zeta(R\beta\zeta\lambda$ & $R\alpha\delta\zeta)$ & $R\gamma\lambda\epsilon)$. Thus $\exists\lambda(R^2\beta(\alpha\delta)\lambda$ & $R\gamma\lambda\epsilon)$. Hence $\exists\lambda(R(\alpha;\beta)\delta\lambda$ & $R\gamma\lambda\epsilon)$ and so $R^2\gamma((\alpha;\beta)\delta)\epsilon$ and so $R((\alpha;\beta);\gamma)\delta\epsilon$.

The other direction is similar (just do the above proof 'backwards', so to speak).

□

You can understand this proof as essentially giving us that both $R(\alpha;(\beta;\gamma))\delta\epsilon$ and $R((\alpha;\beta);\gamma)\delta\epsilon$ are equivalent to $R^3\gamma(\beta(\alpha\delta))\epsilon$ (from simply pulling the two existential quantifiers to the front as we did before). The series of equivalences can be nicely demonstrated linearly:

$$\begin{aligned} R^3\gamma(\beta(\alpha\delta))\epsilon \Leftrightarrow R^2(\beta;\gamma)(\alpha\delta)\epsilon \Leftrightarrow R(\alpha;(\beta;\gamma))\delta\epsilon \\ \Leftrightarrow R^2\gamma((\alpha;\beta)\delta)\epsilon \Leftrightarrow R((\alpha;\beta);\gamma)\delta\epsilon. \end{aligned}$$

Note that we only needed to appeal to the definition of R^2 and R^3, so nothing beyond the basic definition of ternary relation models is needed for the result. So, these facts hold even in **B**. So, our extension of the ternary relation framework by composite points will have at least these desired features.

5.4 Adequacy of $\mathbf{B}_\wedge$ for Channel Models

Call a $\mathbf{B}_\wedge$ ternary relation model a *channel model* just in case it includes composite points meeting our Existence of Composites condition from §5.3. The goal here is to prove that the axiom system for $\mathbf{B}_\wedge$ is adequate, i.e. sound and complete, for channel models. From this, one can obtain a conservative extension result that adding channel-composites to the $\mathbf{B}_\wedge$ ternary relation model structure does not alter the validities of that structure. First, make note of another salient adequacy fact.

Theorem 5.4.1. *The class of models in §5.1.1 is sound and complete with respect to $\mathbf{B}_\wedge$ as in §5.1.2.*

Proof. The proof can be found in Chapter 4 of [65], using essentially the canonical model construction as set out below. □

The composite points $\alpha;\beta$, for some α, β, will have to include all the arrow statements $A \rightarrow B$ s.t. α supports $A \rightarrow C$ and β supports $C \rightarrow B$. This feature is captured by simply incorporating the definition of $R(\alpha;\beta)\gamma\delta$ into something much like the usual valuation clause for $\rightarrow$:

$$\alpha;\beta \vDash A \rightarrow B \text{ iff } \forall\gamma\forall\delta(R^2\beta(\alpha\gamma)\delta \Rightarrow (\gamma \vDash A \Rightarrow \delta \vDash B)).$$

These are the points added to a ternary relation model for $\mathbf{B}_\wedge$ to result in a channel model. Essentially, all one needs to do to obtain a channel model from a $\mathbf{B}_\wedge$ model is to outfit the set of situations in that model with the appropriate composite points.

Theorem 5.4.2. *The class of channel models are sound with respect to $\mathbf{B}_\wedge$*

Proof. This is obvious, as every channel model is a model of $\mathbf{B}_\wedge$.

□

Completeness is a bit more involved. Our strategy is to show that the canonical model of $\mathbf{B}_\wedge$ is a channel model, and to do this, we need only show that each composite point is already in the canonical set S_c of $\mathrm{B}_\wedge$ situations, which we'll define shortly.

Definition 5.4.3. α is a $\mathbf{B}_\wedge$-theory just in case the following conditions hold:

- $A, B \in \alpha \Rightarrow A \wedge B \in \alpha$
- $(\vdash_{\mathbf{B}_\wedge} A \rightarrow B \;\&\; A \in \alpha) \Rightarrow B \in \alpha$

Furthermore, α is a *regular* $\mathbf{B}_\wedge$-theory just in case α is a $\mathbf{B}_\wedge$-theory and

- $\vdash_{\mathbf{B}_\wedge} A \Rightarrow A \in \alpha$

Definition 5.4.4. The canonical frame of $\mathbf{B}_\wedge$ is $\mathcal{F}_c = \langle S_c, N_c, R_c, \rangle$

- S_c is the set of $\mathbf{B}_\wedge$ theories.
- N_c is the set of regular $\mathbf{B}_\wedge$ theories.
- $R_c\alpha\beta\gamma \Leftrightarrow ((A \to B \in \alpha \;\&\; A \in \beta) \Rightarrow B \in \gamma)$.

To get the canonical model $\mathcal{M}_c = \langle \mathcal{F}_c, \vDash_c \rangle$, we need add only the canonical valuation as follows:

$$\alpha \vDash_c A \Leftrightarrow A \in \alpha.$$

In this setting, we are interested to find a point $\alpha;\beta$ in $\mathcal{F}_c$ which obeys the following condition:

$$A \to B \in \alpha;\beta \Leftrightarrow \forall\gamma\forall\delta((R^2\beta(\alpha\gamma)\delta \;\&\; A \in \gamma) \Rightarrow B \in \delta).$$

The question of finding a point like this can be recast as one of whether one can take a set of conditional formulae meeting this condition and build a $\mathbf{B}_\wedge$-theory out of it. Importantly, the process of building a theory out of this set of conditionals must not involve adding any conditionals beyond those added to satisfy the above condition. If this were not to be the case, then one of these new conditionals $A \to B$ would not be such that $\alpha \vDash A \to C$ and $\beta \vDash C \to B$. If we can build such a set without any additional conditionals, then we'll have shown that $\alpha;\beta$ is in the canonical model after all and behaves as we want.

5.4.1 Construction of $\alpha;\beta$

We employ a standard theory construction. Let us begin with the following:

$$\alpha;\beta_0 = \{A \to B \mid \forall\gamma\forall\delta(R^2\beta(\alpha\gamma)\delta \Rightarrow (A \in \gamma \Rightarrow B \in \delta))\}.$$

To make sure $\alpha;\beta$ is a theory in the language, we need only add conjunctions as follows:

$$A, B \in \alpha;\beta_n \Rightarrow A \wedge B \in \alpha;\beta_{n+1}$$

to get $\alpha;\beta$ as:

$$\alpha;\beta = \bigcup_{n<\omega} \alpha;\beta_n.$$

5.4.2 Verification of $\alpha;\beta$

Now, our concern is to verify that $\alpha;\beta$ is a $\mathbf{B}_\wedge$-theory after all. That it obeys the conjunction property clearly falls out of the construction (conjunctive formulae only get in when both conjuncts do). So, the remainder of the verification requires that we show that when $A \to B$ is a theorem of $\mathbf{B}_\wedge$, then $A \in \alpha;\beta \Rightarrow B \in \alpha;\beta$. First, we need some lemmata. The first is reported by Dezani-Ciancaglini et al. in [25][210]:

Lemma 5.4.5 (Bubbling)**.** *Suppose* $\vdash_{\mathbf{B}_\wedge} \bigwedge_{i\in I}(A_i \to B_i) \to (A \to B)$ *for some propositions indexed by* $I \subseteq \mathbb{N}$. *Then there is some non-empty finite* $J \subseteq I$ *s.t.* $\vdash_{\mathbf{B}_\wedge} A \to \bigwedge_{j\in J} A_j$ *and* $\vdash_{\mathbf{B}_\wedge} \bigwedge_{j\in J} B_j \to B$.

Proof. See Barendregt et al. [3][933] and note that as we don't have anything like $\top$ matching their ω, so the initial non-identity clause, which would amount to $B \neq \top$ as in [25], is unnecessary for our purposes. [17]

[17] It is worth noting here that the proof given here bears a substantial resemblance to work done by Dunn and Meyer to provide ternary relation semantics for Combinatory logic [28]. Indeed, Thanks are due to a referee for noting this point of resemblance.

$\square$

It is easy to extend this lemma to an equivalence:

Lemma 5.4.6 (Double Bubbling)**.** *Suppose that there exists a $J \subseteq I$ s.t. $\vdash_{\boldsymbol{B}_\wedge} A \to \bigwedge_{j\in J} A_j$ and $\vdash_{\boldsymbol{B}_\wedge} \bigwedge_{j\in J} B_j \to B$. Then $\vdash_{\boldsymbol{B}_\wedge} \bigwedge_{i\in I}(A_i \to B_i) \to (A \to B)$.*

Proof. The proof is completed in two stages. First is to show, in the ternary relation semantics for $\mathbf{B}_\wedge$, that if for every $x \in N$, $x \vDash A \to \bigwedge_{j\in J} A_j$ and $x \vDash \bigwedge_{j\in J} B_j \to B$ then for every $x \in N_c$ $x \vDash \bigwedge_{i\in J}(A_j \to B_j) \to (A \to B)$.

Suppose otherwise. That is, suppose that there exists α, β s.t. $\alpha \sqsubseteq \beta$ and $\alpha \vDash \bigwedge_{j\in J}(A_j \to B_j)$ and $\beta \nvDash A \to B$. Thus, there are γ, δ s.t. $R\beta\gamma\delta$ and $\gamma \vDash A$ and $\delta \nvDash B$. Since $\gamma \sqsubseteq \gamma$ it follows that $\gamma \vDash \bigwedge_{j\in J} A_j$, and so $\gamma \vDash A_j$ for each $j \in J$. The tonicity properties of R guarantee that $R\alpha\gamma\delta$, and so $\delta \vDash B_j$ for each $j \in J$. Hence, since $\delta \sqsubseteq \delta$, we have that $\delta \vDash B$ after all. So, each $x \in N$ must satisfy $\bigwedge_{j\in J}(A_j \to B_j) \to (A \to B)$.

So if $\vdash_{\mathbf{B}} A \to \bigwedge_{j\in J} A_j$ and $\vdash_{\mathbf{B}} \bigwedge_{j\in J} B_j \to B$ then $\vdash_{\mathbf{B}} \bigwedge_{j\in J}(A_j \to B_j) \to (A \to B)$. By (R3) $\vdash_{\mathbf{B}} (\bigwedge_{i\in I}(A_i \to B_i) \to \bigwedge_{j\in J}(A_j \to B_j)) \to (\bigwedge_{i\in I}(A_i \to B_i) \to (A \to B))$ follows. However, when $J \subseteq I$, $\bigwedge_{i\in I}(A_i \to B_i) \to \bigwedge_{j\in J}(A_j \to B_j)$ is provable with either (A2) or (A3). Hence, if there exists $J \subseteq I$ s.t. $\vdash_{\mathbf{B}} \bigwedge_{j\in J}(A_j \to B_j) \to (A \to B)$ then $\vdash_{\mathbf{B}} \bigwedge_{i\in I}(A_i \to B_i) \to (A \to B)$.

$\square$

Lemma 5.4.7. *If $\vdash_{\boldsymbol{B}_\wedge} C$ then there exists some $\bigwedge_{i\in I}(A_i \to B_i)$ s.t. $\vdash_{\boldsymbol{B}_\wedge} C \leftrightarrow \bigwedge_{i\in I}(A_i \to B_i)$, for formulae with indeces from some $I \subseteq \mathbb{N}$.*

Proof. Since $\vdash_{\mathbf{B}_\wedge} C$, we know that C must not be an atomic formula. Similarly, since a conjunction is provable in $\mathbf{B}_\wedge$ iff both conjuncts are provable, we can be sure that C does not have an atomic formula as a conjunct. So C must have some complex structure, and every non-atomic formulae in the language $\{\to, \wedge\}$ has the desired structure, hence every conjunction of non-atomic formulae in the language has the desired structure.

$\square$

Theorem 5.4.8. $(\vdash_{\boldsymbol{B}_\wedge} A \to B \;\&\; A \in \alpha;\beta) \Rightarrow B \in \alpha;\beta$

Proof. We proceed by structural induction on the proof of $A \to B$.

Base: $A \to B$ is an instance of a $\mathbf{B}_\wedge$ axiom.

(A1) That $A \in \alpha;\beta \Rightarrow A \in \alpha;\beta$ follows from the fact that $\alpha;\beta$ is a set.

(A2) Suppose that $A \wedge B \in \alpha;\beta$. Then at some stage n, $A \in \alpha;\beta_n$ and $B \in \alpha;\beta_n$. If $A \in \alpha;\beta_n$ then $A \in \alpha;\beta$. (A3) is similar.

(A4) Suppose that $(A \to B) \wedge (A \to C) \in \alpha;\beta$. Suppose that $A \to (B \wedge C) \notin \alpha;\beta$. So $\exists\gamma,\delta(R^2_c\beta(\alpha\gamma)\delta \;\&\; A \in \gamma \;\&\; B \wedge C \notin \delta)$. Since $R^2_c\beta(\alpha\gamma)\delta$ and $A \in \gamma$, it follows that $B \in \delta$ and $C \in \delta$. By supposition, δ is a $\mathbf{B}_\wedge$-theory, and so $B \wedge C \in \delta$ after all.

Induction Step: Consider the cases when $A \to B$ is a result of an application of a $\mathbf{B}_\wedge$ rule to some other $\mathbf{B}_\wedge$ theorems. In particular, suppose that each premise to the application of the rule is respected by $\alpha;\beta$, for induction, and we'll show that the consequence of the rule application is respected by $\alpha;\beta$.

(R2) Suppose that $\vdash_{\mathbf{B}_\wedge} C \to (A \to B)$ and $\vdash_{\mathbf{B}_\wedge} C$. From $\vdash_{\mathbf{B}_\wedge} C$, Lemma 5.4.7 guarantees that $\vdash_{\mathbf{B}_\wedge} C \leftrightarrow \bigwedge\limits_{i\in I}(A_i \to B_i)$ for some collection of arrow formulae. Thus, the replacement property guarantees that both $\vdash_{\mathbf{B}_\wedge} \bigwedge\limits_{i\in I}(A_i \to B_i) \to (A \to B)$ and $\vdash_{\mathbf{B}_\wedge} \bigwedge\limits_{i\in I}(A_i \to B_i)$. Suppose that $A \in \alpha;\beta$. Since $\vdash_{\mathbf{B}_\wedge} \bigwedge\limits_{i\in I}(A_i \to B_i) \to (A \to B)$, the bubbling lemma ensures that there exists a finite non-empty $J \subseteq I$ where $\vdash_{\mathbf{B}_\wedge} A \to \bigwedge\limits_{j\in J} A_j$ and $\vdash_{\mathbf{B}_\wedge} \bigwedge\limits_{j\in J} B_j \to B$. Under the supposition that both of these are respected by $\alpha;\beta$, given by the equivalence in Lemmata 5.4.5 and 5.4.6, we have that $\bigwedge_{j\in J} A_j \in \alpha;\beta$ and so each $A_j \in \alpha;\beta$. By IH we have that $A_i \in \alpha;\beta \Rightarrow B_i \in \alpha;\beta$ for all $i \in I$, so for each $j \in J$, $B_j \in \alpha;\beta$ and so $\bigwedge_{j\in J} B_j \in \alpha;\beta$. Thus $B \in \alpha;\beta$.

(R3) Suppose that $\vdash_{\mathbf{B}_\wedge} A \to B$ and $\vdash_{\mathbf{B}_\wedge} C \to D$ and, in addition, that $B \to C \in \alpha;\beta$ and $A \to D \notin \alpha;\beta$. By this last supposition, we have $\exists\gamma\exists\delta(R^2\beta(\alpha\gamma)\delta \;\&\; A \in \gamma \;\&\; D \notin \delta)$. It follows that $B \in \gamma$, since γ is a $\mathbf{B}_\wedge$-theory. Therefore, since $B \to C \in \alpha;\beta$, $C \in \delta$, and so $D \in \delta$ as δ is a $\mathbf{B}_\wedge$-theory as well.

□

On our way to the conservative extension result, we note an additional result. Namely, that whenever both $\alpha, \beta \in N_c$, then $\alpha;\beta \in N_c$. This is a straightforward result of the construction, but one which guarantees that these composite points are not only around in the frame, but are sensitive to the points of which they are composites.

Theorem 5.4.9. *When α, β are regular $\mathbf{B}_\wedge$-theories, then so is $\alpha;\beta$.*

Proof. By supposition, we have that $\vdash_{\mathbf{B}_\wedge} A \Rightarrow A \in \alpha \cap \beta$. Suppose that $\vdash_{\mathbf{B}_\wedge} A$, to show that $A \in \alpha;\beta$. We have already shown that $\alpha;\beta$ is closed under the rules of the system, so it is sufficient to our purposes to show that if α, β are normal, then $\alpha;\beta$ must contain each *axiom* of the system, as the fact that $\alpha;\beta$ is a $\mathbf{B}_\wedge$ theory will ensure that anything provable from the axioms is in $\alpha;\beta$. Note that every axiom of $\mathbf{B}_\wedge$ has $\to$ as its main connective. So, we need only consider the cases where the A in question is $B \to C$, and show that $B \to C \in \alpha;\beta$ when it a theorem.

Suppose that $\vdash_{\mathbf{B}_\wedge} B \to C$, $\exists x(R_c\beta x\delta \;\&\; R_c\alpha\gamma x)$, $\alpha, \beta \in N_c$, and $B \in \gamma$. Since $\alpha \in N_c$, we have that $B \to C \in \alpha$ as $\vdash_{\mathbf{B}_\wedge} B \to C$. Since $R_c\alpha\gamma x$ and $B \in \gamma$, we have that $C \in x$. Now, as $\beta \in N_c$, we have that $C \to C \in \beta$, and so since $R_c\beta x\delta$, we have that $C \in \delta$. Hence, $B \to C \in \alpha;\beta$ given the construction of $\alpha;\beta$.

□

Corollary 5.4.10. $\exists\epsilon(R_c^2\beta(\alpha\gamma)\delta \Leftrightarrow R_c\epsilon\gamma\delta)$

Proof. We start by showing that $R_c^2\beta(\alpha\gamma)\delta \Leftrightarrow R_c(\alpha;\beta)\gamma\delta$.

First, for the left to right direction suppose that $R_c^2\beta(\alpha\gamma)\delta$ and let $\alpha;\beta$ be in accordance with the construction in §5.4.1. Theorems 5 and 6 show that the defined set is, indeed, a $\mathbf{B}_\wedge$ theory (which is regular if α,β are) – since S_c is the set of all $\mathbf{B}_\wedge$ theories, then, $\alpha;\beta \in S_c$. The construction ensures that $R_c(\alpha;\beta)\gamma\delta$, and so $R_c^2\beta(\alpha\gamma)\delta \Rightarrow R_c(\alpha;\beta)\gamma\delta$.

Second, the right to left. Suppose that $R_c(\alpha;\beta)\gamma\delta$. We want to show that $\exists x(R_c\alpha\gamma x$ & $R_c\beta x\delta)$. Let $x' = \{B|\exists A(A \to B \in \alpha$ & $A \in \gamma)\}$ – we can construct a theory x from x' using a construction very much like that given in §5.4.1. Note that we have immediately that $R_c\alpha\gamma x$. Suppose, then, that $A \to B \in \beta$ & $A \in x$. Then, there is a C s.t. $C \to A \in \alpha$ & $C \in \gamma$. The construction of $\alpha;\beta$ guarantees that $C \to B \in \alpha;\beta$ when $C \to A \in \alpha$ and $A \to B \in \beta$, so, since $R_c(\alpha;\beta)\gamma\delta$ and $C \in \gamma$, $B \in \delta$. Hence $R_c\beta x\delta$ after all, and thus $R_c^2\beta(\alpha\gamma)\delta$. So $R_c(\alpha;\beta)\gamma\delta \Rightarrow R_c^2\beta(\alpha\gamma)\delta$.

Thus, $R_c^2\beta(\alpha\gamma)\delta \Leftrightarrow R_c(\alpha;\beta)\gamma\delta$, and so $\exists\epsilon(R_c^2\beta(\alpha\gamma)\delta \Leftrightarrow R_c\epsilon\gamma\delta)$. □

Note that $R_c^2\beta(\alpha\gamma)\delta \Rightarrow \exists\epsilon R_c\epsilon\gamma\delta$ follows from Corollary 5.4.10. This is, perhaps, the most natural statement that for any α,β standing in the correct relation, there is a composite point in the canonical model.

Corollary 5.4.11. *The logic characterised by the class of channel models conservatively extends that characterised by the class of* $\mathbf{B}_\wedge$ *models.*

Proof. Given Corollary 5.4.10, for any α,β in the $\mathbf{B}_\wedge$ canonical model, there exists a $\mathbf{B}_\wedge$ theory (which is regular whenever α,β are). That is, the canonical model satisfies the existence of composites property. So, the canonical model of $\mathbf{B}_\wedge$ is a channel model, and hence the class of channel models is complete for $\mathbf{B}_\wedge$. Hence, the class of channel models admits no validities not already admitted by the class of $\mathbf{B}_\wedge$ models, and so the extension of the $\mathbf{B}_\wedge$ model structures by additional composite points is conservative over $\mathbf{B}_\wedge$. □

The facts proven in §5.3 also provide an argument for the associativity and left/right normality of our composition 'operation'. Theorem 5.4.8 implies that Theorems 5.3.1 and 5.3.2 also hold in the canonical model (as they hold in any $\mathbf{B}_\wedge$-model). It is a key question whether or not this construction continues to work in logics extending $\mathbf{B}_\wedge$.

This proof strategy, at very least, does not extend any further, due to its reliance on the bubbling lemma. This lemma fails in $\mathbf{B}_{\wedge\vee}$, more commonly referred to as $\mathbf{B}^+$, because it includes all instances the following theorem for $\vee$:

$$\vdash_{\mathbf{B}^+} ((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$$

some instances of which are counterexamples to the bubbling lemma. [18] So, we cannot extend this proof to cover any of the extensions of $\mathbf{B}_\wedge$ including the usual disjunction. However we can say something interesting about other potential extensions of $\mathbf{B}_\wedge$ and how composition as we have defined it operates in those logics. First, we can say something about the question whether ; is commutative and idempotent under the intended interpretation, and how our answers stack up against Restall's. Second, there is a more serious problem into which this approach runs as soon as we consider logics with negations obeying a certain contraposition property.

[18] This can be seen by noting that, in general, $\nvdash_{B+} (A \vee B) \to (A \wedge B)$.

5.5 Idempotence and Commutativity of Composition

Whether $\alpha;\alpha = \alpha$ and $\alpha;\beta = \beta;\alpha$ are interesting questions, and ones which Barwise leaves open. Our concern here is to consider what results enforcing these conditions has on our approach, as opposed to Restall's. Unsurprisingly we get quite different answers, and answers which provide some indication of the split in the approaches. We'll display some of the key differences, and reflect on which approach is more natural for the channel theoretic interpretation.

$\alpha;\alpha = \alpha$ looks to be some kind of contraction principle. Understanding this condition as Restall does, the most salient consequence of this is, in Restall's notation, $\alpha \circ \alpha \sqsubseteq \alpha$, or more commonly $R\alpha\alpha\alpha$, which corresponds with the axiom $((A \to B) \wedge A) \to B$, which we call WI. As has been known since at least [53], this theorem is bad news for the usual naïve theories, as it provides for a straightforward Curry paradox. This is an interesting result, and, understood in Restall's terms, there is a nice story one can give for why this principle ought to fail along channel-theoretic terms. However, understood in our terms, the important related upshot of this idempotence principle is $R\alpha\beta\gamma \Leftrightarrow R(\alpha;\alpha)\beta\gamma \Leftrightarrow R^2\alpha(\alpha\beta)\gamma$. Under our interpretation, this is a somewhat different kind of 'reuse of resources' than one has in something like WI. With our interpretation, what this tells us is that exploiting a situation α *qua*-channel to get from β to γ is in no way different from exploiting α twice. This is to be contrasted with what contraction allows, namely, that one can exploit a situation *qua*-signal twice. In other words, that one can use the same proposition as a *premise* as many times as one likes, with no change in the validity of the argument.

Consider not $R\alpha\alpha\alpha$ and WI, which is contraction 'mixed' with identity, but rather the pure contraction axiom W: $(A \to (A \to B)) \to (A \to B)$. The frame condition corresponding to this axiom is $R\alpha\beta\gamma \Rightarrow R^2\alpha\beta\beta\gamma$. Using Restall's notation, this comes to

$\alpha \circ \beta \sqsubseteq \gamma \Rightarrow (\alpha \circ \beta) \circ \beta \sqsubseteq \gamma$, which can naturally be read in terms of allowing the exploitation of an antecedent-supporting situation multiple times when it can exploited once. Our way of cashing out composition provides for a kind of 'contraction' of the channel in use which is quite distinct from employing some premise information twice. Note that $R\alpha\beta\gamma \Rightarrow R^2\alpha(\alpha\beta)\gamma$ is a consequence of our precisification of the idempotence clause. This frame condition corresponds to (A11):

$$((A \to B) \wedge (B \to C)) \to (A \to C).$$

A natural reading of this formula, and its associated ternary relation condition, is that whenever some situation supports two constraints $A \to B$ and $B \to C$, then it must also 'act as its own composition' and support $A \to C$. This is, indeed, what $R\alpha\beta\gamma \Rightarrow R(\alpha;\alpha)\beta\gamma$ most naturally gives us. That is, that when α is a channel from β to γ, so is $\alpha;\alpha$. There may be reasons to accept this principle, perhaps resulting in a theory not too dissimilar from that of [48], but it is at least not obvious.

Consider a channel understood is purely physical terms, as a part of the world which connects some site of information to another site. Then employing a channel, as it were by applying it to some signal, is one instance of application of that channel. If the same channel supports $A \to B$ and $B \to C$, it is at least questionable that one can pass across both conditionals with only one application of the channel. [19]

We leave off the interpretation for now simply to point out two interesting features of this approach. These have to do with the interaction between $\to$ and the $\leftarrow$ which is available in this logical setting where we don't have $R\alpha\beta\gamma \Rightarrow R\beta\alpha\gamma$. [20] Briefly, the valuation clause for $\leftarrow$ is as follows:

[19] Thanks go to Dave Ripley for pushing me on this point.

[20] Details on this connective and its relation to $\to$ are available in many places, and [60] provides a nice overview.

$$\alpha \models B \leftarrow A \text{ iff } \forall\beta\forall\gamma(R\beta\alpha\gamma \Rightarrow (\beta \models A \Rightarrow \gamma \models B)).$$

When we enforce idempotence for our composition, $R\alpha\beta\gamma \Rightarrow R(\alpha;\alpha)\beta\gamma$, an immediate consequence is a contraction principle for $\leftarrow$, namely:

$$((B \leftarrow A) \leftarrow A) \rightarrow (B \leftarrow A).$$

This is noteworthy because of its connection to the commutativity of composition. An immediate consequence of the commutativity of ; in our sense is:

$$R(\alpha;\beta)\gamma\delta \Leftrightarrow R(\beta;\alpha)\gamma\delta \qquad \text{(i.e. } R^2\beta(\alpha\gamma)\delta \Leftrightarrow R^2\alpha(\beta\gamma)\delta)$$

which enforces:

$$((C \leftarrow B) \leftarrow A) \rightarrow ((C \leftarrow A) \leftarrow B).$$

A kind of permutation for $\leftarrow$.[21] However, there seems to be no obvious connection to the frame conditions which enforce prefixing or suffixing for $\leftarrow$, i.e. $R^2\alpha(\beta\gamma)\delta \Rightarrow R^2(\alpha\beta)\gamma\delta$ and $R^2\alpha(\beta\gamma)\delta \Rightarrow R^2(\alpha\gamma)\beta\delta$, respectively. This is an avenue for some future work.

5.6 The Axiom Form of Contraposition

Something difficult happens here when we consider a common contraposition axiom, namely:

$$\text{(A15) } (A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A).$$

[21]These formulae also correspond to the associated ternary relation restrictions.

Negation is generally interpreted in the ternary relation semantics for relevant logics by means of an operation * on worlds, the Sylvan-Plumwood star. [65] For the small point we make here, it is enough to note that the valuation condition on negation in this semantics is $\alpha \vDash \neg A$ iff $\alpha^* \nvDash A$, and that the above axiom corresponds to the following frame condition:

$$\text{(S15) } R\alpha\beta\gamma \Rightarrow R\alpha\gamma^*\beta^*$$

Note, given what we have up to this point, $\alpha;\beta$ does not provide what we want if we are to consider logics including this axiom. Suppose that $\alpha \vDash A \to B$ and $\beta \vDash B \to C$, so that $\alpha \vDash \neg B \to \neg A$ and $\beta \vDash \neg C \to \neg B$. We have $\alpha;\beta \vDash A \to C$, but not necessarily $\alpha;\beta \vDash \neg C \to \neg A$.

Fact 5.6.1. *If* $\alpha;\beta \vDash A \to B$ *then* $\beta;\alpha \vDash \neg B \to \neg A$

Proof. Suppose that $\alpha;\beta \vDash A \to B$ and $\beta;\alpha \nvDash \neg B \to \neg A$. Unpacking the latter, we have $\exists\gamma\exists\delta(R^2\alpha(\beta\gamma)\delta \ \& \ \gamma \vDash \neg B \ \& \ \delta \nvDash \neg A)$. Since $\gamma \vDash \neg B$, we get that $\gamma^* \nvDash B$ and since $\delta \nvDash \neg A$, we have $\delta^* \vDash A$. Since $R^2\alpha(\beta\gamma)\delta)$ we know there is an ϵ s.t. $R\alpha\epsilon\delta \ \& \ R\beta\gamma\epsilon$. Therefore, $R\alpha\delta^*\epsilon^* \ \& \ R\beta\epsilon^*\gamma^*$, and thus $R^2\beta(\alpha\delta^*)\gamma^*$. Since $\alpha;\beta \vDash A \to B$ and $\delta^* \vDash A$, we get that $\gamma^* \vDash B$, contrary to hypothesis.

□

If we impose merely $R\alpha\beta\gamma \Rightarrow R\alpha\gamma^*\beta^*$, what we get by assuming $R^2\beta(\alpha\gamma)\delta$ is not $R^2\beta(\alpha\delta^*)\gamma^*$, as we'd want, but rather $R^2\alpha(\beta\delta^*)\gamma^*$. So, we don't have:

$$R(\alpha;\beta)\gamma\delta \Rightarrow R(\alpha;\beta)\delta^*\gamma^*$$

but rather

$$R(\alpha;\beta)\gamma\delta \Rightarrow R(\beta;\alpha)\delta^*\gamma^*.$$

One way to enforce the result we want is to enforce that $R(\alpha;\beta)\gamma\delta \Rightarrow R(\beta;\alpha)\gamma\delta$, but this is a fairly hefty assumption, and hard to justify. At very least, I don't see any intuitive reason to think it holds.

For the general case, we need only to enforce:

$$R^2\alpha(\beta\gamma)\delta \Rightarrow R^2\alpha(\beta\delta^*)\gamma^*$$

Now, we needn't necessarily enforce the above commutativity principle to get this. The question is, what does this add to the frames of logics containing **DW**?

This raises a more general question, namely, what should negation look like in channel theory in the first place? While the ternary relation of the relevant semantics is fairly natural in the channel-setting, the Sylvan-Plumwood star is another matter entirely. [22]

5.7 Concluding Remarks

We have proven the main result, that one can supplement the ternary relation semantics for $\mathbf{B}_\wedge$ with points behaving as the composites of other points in the frame. This is a good first step to the larger project of fully laying out how Barwise's initial approach to channel theory helps to interpret the ternary relation semantics, but, as we have seen, there are some difficulties ahead. Some, like extending the result to full $\mathbf{B}^+$, seem solvable, but clearly require some other proof tactic. From there, one might hope that it is not a difficult manner to extend the picture upwards into other positive logics gotten by extending $\mathbf{B}^+$ by various additional axioms and rules. Incorporating negation into the picture, on the other hand, seems to require some more foundational work before the formalism can get off the ground.

[22] As are Dunn's Perp [26] and Restall's compatibility [61].

In addition to these forward looking comments, as mentioned at the start of §5.2, the work we have done here, though it is a step to giving us some insight into channel composition in relevant semantics, doesn't yet settle the behaviour of this operation in any other than purely extensional terms. Future work is certainly needed to expand the picture we've given to one which provides a more robust insight into just what kind of critter this composition operation really is, and not just that we can find composites when we want them, at least as far as $\mathbf{B}_\wedge$ is concerned.

Chapter 6

The Implication Fragment of Frege's *Grundgesetze*

The propositional logic of Frege's *Grundgesetze der Arithmetik* [31] does not seem to be the subject of much discussion in the Frege literature. Books and article which do discuss Frege's propositional logic tend to focus on *Begriffsschrift*, without expounding on the fact that this earlier system differs in some key ways from the *Grundgesetze* system. In *Begriffsschrift*, there are two conditional axioms and two negation axioms, roughly corresponding to Thinning (K), full Conditional Distribution (S), Contraposition, and Double Negation elimination, and one rule, Modus Ponens.[1] However, in *Grundgesetze*, there is only one Basic Law governing each connective, and an expanded set of rules designed to shorten and clarify inferences. One result is that while the system of *Begriffsschrift* is a standard looking axiomatic presentation of classical logic, that of *Grundgesetze* is unusual.

Schroeder-Heister has claimed that the logic of *Grundgesetze* is a precursor to Gentzen's

[1] The principles I've labelled K and S are displayed in §6.1, and since my focus is on the implication fragment of the *Grundgesetze* logic, we shall not discuss the negation axioms beyond these introductory remarks.

sequent calculus, and has produced an elegant Gentzen system for the *Grundgesetze* rules. [70] This way of understanding the *Grundgesetze logic* is englightening, but there are interesting structural features of the system which stand to be made more explicit. For this purpose, I shall employ a different kind of sequent calculus which permits a finer-grained study of the structural features of the conditional at work.

First, I set out a standard axiom system for the implication fragment of classical logic, pulling apart some key axioms which underwrite the kinds of structural rules employed in sequent calculi. Then, I shall go through the axiom and rules of *Grundgesetze*, setting out potential readings of the rules in modern terms, to settle on a collection of rules. Finally, I shall present a sequent calculus for this system using techniques from structurally free logics as developed in relevant and substructural logic.

6.1 $\mathbf{K}_{\rightarrow}$: A Hilbert System

The following is a more or less standard axiom system for the implication fragment of classical logic, namely, the system set out in [52].

Axioms

1. $A \rightarrow (B \rightarrow A)$ — K
2. $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$ — S
3. $((A \rightarrow B) \rightarrow A) \rightarrow A$ — Peirce's Law

Rule

$$\frac{\vdash A \qquad \vdash A \rightarrow B}{\vdash B}\text{ Modus Ponens}$$

This implication system is equivalent to Frege's system from *Begriffsschrift* except for the inclusion of Peirce's Law. Indeed, *Begriffsschrift* is essentially the same system of pure implication presented by Hilbert in [35]. That is, it includes K, in addition to axioms for prefixing and suffixing (B and B'), permutation (C) and contraction (W), as listed below.

As is well known, axioms K and S in the presence of MP present a structurally complete theory of implication. That is, all of the usual results of a full set of structural rules in a single-succedent sequent calculus are admissible. These rules roughly correspond to the following theorems, here expressed again with combinator names reflecting their status as the principle type schemata of their associated combinators:[2]

1. $(A \to B) \to ((C \to A) \to (C \to B))$ B
2. $(A \to B) \to ((B \to C) \to (A \to C))$ B'
3. $(A \to (B \to C)) \to (B \to (A \to C))$ C
4. $(A \to (A \to B)) \to (A \to B)$ W
5. $A \to A$ I

I shall follow Curry [24] in referring to the system including these formulae as axioms and MP as *Absolute Implication*; it is the implication fragment of Intuitionist logic. This system happens to have the same set of theorems as the implication fragment of *Grundgesetze*, although the particular construction of the theory is interestingly, importantly different. I shall set out Frege's system to compare with standard axiomatic treatments of absolute implication as compared to a standard sequent calculus treatment of the same theory. The result is that Frege's system much more closely corresponds to the latter, in particular due to his rule corresponding to B and B'.

[2] See [10] for a modern introduction to combinatory logic.

6.2 $G_{\rightarrow}$: Axiom and Rules Variants

I shall set out Frege's axiom and rules, distinguishing potential readings of the rules to draw out the peculiarly sequent-like features of his generalised rules. [3]

6.2.1 Basic Law I

$A \rightarrow (B \rightarrow A)$ $\qquad\qquad$ $A \rightarrow A$

Basic law I has two components, corresponding to K and I above. This second form of Basic law I is an immediate instance of the contraction rule, but it is so often used that Frege includes it as a basic form to shorten the proofs. For my purposes, it is more convenient to avoid using the full form of basic law I and focus only on K or I as necessary.

6.2.2 Permutation of subcomponents

This permutation rule is designed to recapture the key inferences permitted by the principal type scheme of C,

namely $(A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C))$. However, the statement of the rule is somewhat unclear. I'll distinguish instances of the permutation rule by C with number subscripts. One very natural, modern, reading of this rule is as follows:

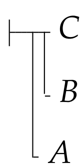
$$\frac{\vdash A \rightarrow (B \rightarrow C)}{\vdash B \rightarrow (A \rightarrow C)}\,\mathsf{C}_1$$

[3]It should be noted that modern informal interpretations of this system, and the system to be developed later in the paper, are not quite Frege's, as Frege had a very different conception of the logical vocabulary from that which is common now. I focus only on cases where complex formulae involved are those whose components are propositions, setting aside the additional formalism Frege put in place to ensure that *any* terms could be connected by the logical vocabulary.

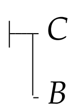
Simply, when given a theorem of the form shown in the premise of the rule, which is the antecedent of C, one may derive a theorem with the form of the conclusion. However, this does not appear to be Frege's rule. His statement, given in the summary of the rules, §48 of *Grundgesetze*, is the following:

> Subcomponents of the same proposition may be pe- rmuted with one another arbitrarily. [31][61]

A subcomponent for Frege is essentially an antecedent of a conditional. Consider the following conditional, in *Begriffsschrift* notation:



Frege tells us that this can be read so that C is the *supercomponent* while A and B are *subcomponents* or so that



is the supercomponent and A is the subcomponent. This provides flexibility, as one can instantiate formulae into the rule more or less without restriction. That is, one can apply the rule to a conditional formula with arbitrarily many antecendents, at least so long as it has a right-associated structure. As such, the standard reading of the rule, see [31][A-20–A-22] is the following:

$$\frac{\vdash A_0 \rightarrow (\ldots A_i \rightarrow (\ldots Aj \rightarrow (\ldots A_n \rightarrow B)\ldots)}{\vdash A_0 \rightarrow (\ldots A_j \rightarrow (\ldots A_i \rightarrow (\ldots A_n \rightarrow B)\ldots)}\ \mathsf{C}_2$$

However, there is a contextually sensitive variant of the rule in the area. Note that in the short statement of the rule Frege uses "proposition", which might allow the following rule, where $\Gamma[A]$ signifies that A is some a particular instance of a formula which is a component of Γ, a complex formula:

$$\frac{\vdash \Gamma[A \to (B \to C)]}{\vdash \Gamma[B \to (A \to C)]}\ \mathsf{C}_3$$

Now, clearly these rules are of differing strength, against a structurally free background. C_3 allows for the proofs of strictly more formulae than C_2, and that more than C_1. To see the difference between C_1 and C_2, consider a theorem of the form $A \to (B \to (C \to D)))$. Clearly either rule will license the inference to $B \to (A \to (C \to D)))$. However C_2 will also license the inference to

$A \to (C \to (B \to D)))$, while C_1 by itself will not. C_3 licences both of these, in addition to inferences such as $(A \to (B \to C)) \to D \vdash (B \to (A \to C)) \to D$ and similar sequents, which C_2 does not. In the presence of the provability of B or S, the distinction between these rules collapses, as instantiation into such forms provides the necessary manipulation of nested conditionals.

The most plausible of these rules as that of *Grundgesetze* is C_2.[4] Every example Frege provides is to a formula, preceded by $\vdash$, with the right-association form required by C_2. In addition, the use of "*Satz*", which is rendered "proposition" in [31], is generally used to refer to judged formulae, namely, theorems of the system. So, C_3 is rendered dubious by Frege's use of terminology. [5]

It is worth noting that from Basic Law I (BLI) and C_2 or C_3, one can easily prove the axiom form C given an appropriate instance of I.

[4] See, for instance, the treatment in Roy Cook's appendix to [31]

[5] To show fully that C_2 is Frege's rule, one would need to go through the proofs given throughout *Grundgesetze* to ensure that no instances of Permutation are licensed only by C_3. This is a daunting task as there might be inferences which resemble those allowed by C_3 which are made by appeal to other rules. Frege's notation for showing applications of rules allows for such applications to be chained together in a way which makes this far from an easy task, and for now we shall suppose that C_2 is the rule.

$$\frac{\vdash (A \to (B \to C)) \to (A \to (B \to C))}{\vdash (A \to (B \to C)) \to (B \to (A \to C))}\ \mathsf{C}_2$$

So, in this case, one can easily prove the axiom from the rule.

6.2.3 Fusion of equal subcomponents

The short statement of the rule given in §48 is as follows:

> A subcomponent that occurs repeatedly in the same proposition only needs to be written once.[31][61]

Again, the natural interpretation is that of a generalized contraction rule. In keeping with our previous numbering, I'll formally state this rule as:

$$\frac{\vdash A_0 \to (\cdots \to (A_i \to \ldots (A_i \to (\ldots A_n \to B) \ldots)}{\vdash A_0 \to (\cdots \to (A_i \to (\ldots A_n \to B) \ldots)}\ \mathsf{W}_2$$

An instance of this rule is the more familiar:

$$\frac{\vdash A \to (A \to B)}{\vdash A \to B}\ \mathsf{W}_1$$

Note that W_1 has all the same consequences as W_2 in the presence of C_2. Given a theorem of the form of the premise of W_2, one can simply permute antecedents to bring the two instances of the same antecedent to the far left of the formula, where one can then apply W_1.

In either case, using W_1 along with C_2 or just using W_2, one can derive the axiom form from an obvious instance of I.

6.2.4 Modus Ponens

Again, Frege states his MP rule in general terms.

$$\frac{\vdash A_i \qquad \vdash A_0 \rightarrow (\ldots A_i \rightarrow (\ldots A_n \rightarrow B)\ldots)}{\vdash A_0 \rightarrow (\ldots A_n \rightarrow B)\ldots)}\ \mathrm{MP}_2$$

The most familiar instance instance of this rule follows:

$$.\ \frac{\vdash A \qquad \vdash A \rightarrow B}{\vdash B}\ \mathrm{MP}_1$$

Similarly to the above, MP_1 has the same consequences as MP_2 in the presence of C_2.

6.2.5 Hypothetical Syllogism

As above, the HS rule is designed for generality, and hence becomes rather long in a our presentation. I shall instead state it linearly, rather than in a proof-tree format. Read the presentation below as follows – if (1) and (2) are theorems, then the formula following the $\therefore$ sign is a theorem.

(1) $C_1 \rightarrow (\cdots \rightarrow (C_n \rightarrow A_i)\ldots)$
(2) $A_1 \rightarrow (\ldots A_i \rightarrow (\ldots A_n \rightarrow B)\ldots)$
$\therefore A_1 \rightarrow (\ldots C_1 \rightarrow (\ldots C_n \rightarrow (\ldots A_n \rightarrow B)\ldots)$

This rule essentially allows one to replace an instance of some antecedent A_i in a conditional with $C_1, \ldots C_n$ when you have a proof of A_i from $C_1 \ldots C_n$. However, there are tricky details involved with how the replacement occurs. What is being done is that in formula (2), the occurrence of A_i is being deleted, and the symbol string "$C_1 \rightarrow (\cdots \rightarrow (C_n \rightarrow$" is inserted so that A_{i-1} (the immediate antecedent of A_i) is the immediate antecedent of C_1, and that A_{i+1} is the immediate consequent of C_n. With this string included, the other antecedents of formula (2) continue in the same order as before the application of the rule. It is precisely this feature which is important, and we shall provide some examples of applications of this rule in this and the following section.

A first-degree (with only on occurrence of $\to$) instantiation of this rule resembles B', as presented below:

$$\frac{\vdash A \to B \qquad \vdash B \to C}{\vdash A \to C}\ \mathrm{HS}_1$$

However, this more natural reading fails to capture the results of Frege's rule in one key regard. While HS_1 does something like a single cut rule in a sequent calculus, HS_2 does the same job while, at the same time preserving the structure of the antecedents allowing for the proof of the cut formula, A_i. Consider a relatively simple instance of this rule:

$$\frac{\vdash A \to (B \to C) \qquad \vdash D \to (C \to E)}{\vdash D \to (A \to (B \to E))}$$

Note that the nesting of the antecedent conditionals in the left premise are preserved in the conclusion. This is is key to providing the full system with the power necessary to produce absolute implication.

6.2.6 Full $G_\to$

So, as noted, the more generalized formulations of the rules other than Permutation and Hypothetical syllogism are derivable from their ungeneralized variations in the presence of C_2. That is, K plus C_2 W_2 MP_2 HS_2 has the same set of theorems as K plus C_2 W_1 MP_1 HS_2. Using the weaker variant rule for C or HS would result in a strictly weaker system, and one which is, strictly, substructural.

A result of $G_\to$ which is not a result of any proper subsystem is shown easily in the following, which is an instance of the inference shown in §6.2.5. I'll use $\dot{B}$ to pick out the ‘cut formula’, as it were. This is to say, the formula $\dot{B}$ is that in common between the antecedent of the formula on the rightmost ‘leaf’ of the proof tree and the consequent of the formula on the leftmost ‘leaf’.

$$\dfrac{\dfrac{\vdash (A \to B) \to (A \to \dot{B}) \qquad \vdash (A \to (B \to C)) \to (\dot{B} \to (A \to C))}{\vdash (A \to (B \to C)) \to ((A \to B) \to (A \to (A \to C)))}\,\mathsf{HS}_2}{\vdash (A \to (B \to C)) \to ((A \to B) \to (A \to C))}\,\mathsf{W}_2$$

This is a proof of the axiom S which nicely displays the way in which HS_2 is a powerful rule and which, as I’ll set out in a later section, can naturally be read as a cut rule in a sequent calculus. Before that, however, we should give some more indication of what $G_{\to}$ amounts to. We know that K and S are provable, so it is at least Absolute Implication.[6] To properly pin down precisely what system the implication of *Grundgesetze* amounts to, one would need to consider the various intermediate logics, which strays somewhat beyond the intended scope of this work. It is, however, an interesting open question.

6.3 *Grundgesetze* as Sequent Calculus and HS_2 as Cut

Schroeder-Heister, in [70] (and earlier noted these issues in an abstract [69]), has developed a reading of the propositional logic of *Grundgesetze* in terms of a sequent calculus with the following axioms and rules[7]:

Axioms

$A \vdash A$ $\qquad\qquad$ $A, B \vdash A$

[6]Note that from K, S, and the usual modus ponens rule all of the implication principles considered so far beside Peirce’s Law are provable.

[7]I ignore trivial differences in notation.

Rules

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B} \to \mathrm{I} \qquad \frac{\Gamma \vdash A \to B}{\Gamma, A \vdash B} \to \mathrm{R}$$

$$\frac{\Gamma \vdash A}{\Gamma' \vdash A}\ \text{Per.} \qquad \frac{\Gamma[A \ldots A] \vdash B}{\Gamma[A] \vdash B}\ \text{Contr.}$$

Where Γ' is an arbitrary permutation of Γ, and the conclusion of Permutation includes some fewer instances of A in the list Γ of premises than in the Γ occurring in the premise. Perhaps an even more straightforward way to state this is to simply treat the premise as a set, as opposed to a sequence or other structured entity, where arbitrary permutations and contractions fall out of the data type. His final rule is a standard form of Single Cut.

$$\frac{\Gamma \vdash A \qquad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}\ \text{Cut}$$

However, it seems that this proof system skates over the interestingly strong presentation of HS. As usual with Gentzen-style sequent calculi, the structural features of the conditional codified by B and B' are packed into the structural connective in a way which obscures their role. So, I develop a system which draws out the particular features of HS_2.

6.3.1 L$G_{\to}$

This calculus, L$G_{\to}$ draws on Structurally Free Logics developed by Dunn and Meyer [28].[8] The key feature of these calculi is that the structural connective ; is designed to build in no structural features beside those enforced by rules.[9] So, for instance, in a standard Gentzen-style sequent calculus B is provable without appeal to any of the structural rules.

[8] §5.2 in Bimbó [11] is an excellent source on these calculi.

[9] In Dunn and Meyer's work, one then adds explicit combinators the effects of which are the structural rules corresponding to them. The relationship between this system and their systems is one of inspiration only – I don't give a structurally free system here.

This is not the case in the calculus we develop. This shows the way in which $G_{\to}$ proofs rely on the particular statement of the rules, and showcases the way in which HS_2 features as cut.

The connective ; is dyadic, and we read a sequent $\Gamma;\Delta;\Theta \vdash A$ as left associated, that is as $(\Gamma;\Delta);\Theta \vdash A$, where Γ,Δ,Θ are (possibly empty) structures of varying complexity. We shall keep the brackets to explicitly show the structure of antecedents.

Axiom

$A;\Gamma \vdash A$

Rules

$$\frac{\Gamma \vdash A \qquad \Delta[B] \vdash C}{\Delta[A \to B;\Gamma] \vdash C}\to\vdash \qquad\qquad \frac{\Gamma;A \vdash B}{\Gamma \vdash A \to B}\vdash\to$$

$$\frac{(\Gamma;\Delta);\Theta \vdash A}{(\Gamma;\Theta);\Delta \vdash A}\,\mathsf{C}_2 \qquad\qquad \frac{(\Gamma;\Delta);\Delta \vdash A}{\Gamma;\Delta \vdash A}\,\mathsf{W}_2$$

$$\frac{\Gamma \vdash A \qquad \Delta[A] \vdash B}{\Delta[\Gamma] \vdash B}\,\mathsf{HS}_2$$

The string $\Delta[A]$ occurring in the presentation of HS_2 is intended to pick out the location of A in Δ, including its nesting in the parentheses. This is important in the intended reading of $\Delta[\Gamma]$. As with $\Delta[A]$, this is intended to indicate that the elements of Γ, in their order, are to be plugged into the previous location of A, where the resulting formula remains left-associated, with the elements of Γ appropriately distributed. Consider the following example:

$$\frac{A;(B;C) \vdash D \qquad E : D \vdash F}{(((E;A);B);C \vdash F}$$

The structure in the antecedent of the sequent which results from the application of the rule, now left associated, can be operated upon freely by the other rules.

This statement of the sequent rule is to make the rule accurately match Frege's HS_2, which results in formulae which are right-associated conditionals. The result of an application of Frege's HS_2 results in well-formed, right-associated conditionals, just as the result of the sequent HS_2 rule results in a left-associated structure in the antecedent which can be manipulated with rules which naturally match Frege's other rules. This is some indication of the way in which this rule is not standard cut rule but something stronger, which affects the structure in a way which forces the kinds of B-manipulations displayed above. The usual structurally free B rule is as follows:

$$\frac{\Gamma; (\Delta; \Theta) \vdash A}{(\Gamma; \Delta); \Theta \vdash A}$$

It's this rule which mimics prefixing, and our HS_2 rule, matching Frege's rule, builds this B behaviour into its construction. This is as opposed to building the B-features into the structural connective, as with a set-like structural connective is a standard Gentzen system. For instance, consider the following instance of HS_2 which, after preliminary applications of $\rightarrow\vdash$, precisely mimics the proof of S given above.

$$\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A \qquad B \vdash B}{A \to B; A \vdash B} \qquad \dfrac{\dfrac{\dfrac{\dfrac{B \vdash B \qquad C \vdash C}{B \to C; B \vdash C} \qquad A \vdash A}{(A \to (B \to C); A); B \vdash C}}{(A \to (B \to C); B); A \vdash C}\ \mathsf{C}_2}{((A \to (B \to C); A \to B); A); A \vdash C}\ \text{HS}_2}{(A \to (B \to C); A \to B); A \vdash C}\ \mathsf{W}_2}{\vdash (A \to (B \to C)) \to ((A \to B) \to (A \to C))}\ \vdash\!\to \text{x3}$$

It is not just that there is a translation between the axioms and conclusion formulae. This is true of Schroeder-Heister's sequent calculus, using a standard translation of $A_1, \ldots, A_n \vdash B$ as $A_1 \to (\cdots \to (A_n \to B) \ldots)$. $\text{LG}_{\to}$ clearly also proves $A \to (B \to A)$, and it seems clear that such a formula translation is also available here.

Simply read $(A_1; A_2); A_3 \vdash B$ as $A_1 \to (A_2 \to (A_3 \to B))$, and expand to more formula as expected. An advantage of $LG_{\to}$ over Schroeder-Heister's system is that it more faithfully reproduces the proofs available in Frege's system. There is an obvious translation available between *Grundgesetze*-proofs and proofs in $LG_{\to}$, where the latter mimic just the same moves that are permitted in G.

As set out, HS_2 clearly functions as a cut rule, though it has the extra feature of enforcing prefixing and suffixing behaviour. With HS_2, we have the following derivation:

$$\cfrac{\cfrac{\cfrac{A \vdash A \qquad B \vdash B}{A \to B; A \vdash B} \qquad \cfrac{B \vdash B \qquad C \vdash C}{B \to C; B \vdash C}}{(B \to C; A \to B); A \vdash B}}{\vdash (B \to C) \to ((A \to B) \to (A \to C))}$$

This corresponds to the Frege proof, where the notation $\dot{B}$ is as above. [10]

$$\frac{(A \to B) \to (A \to \dot{B}) \qquad (B \to C) \to (\dot{B} \to C)}{(B \to C) \to ((A \to B) \to (A \to C))}$$

The essential feature is just the B-like manipulation allowed. This is the most natural available proof in Frege's axiom system, and one which the sequent system naturally reproduces. This points out an interesting additional features of the hypothetical syllogism rule which utilising a polyadic structural connective, like Schroeder-Heister's comma, obscures.

6.3.2 Pulling out the T Rule

Our C_2 rule is stated to allow for one of the structures on the left of the turnstile to be possibly empty. To make this point more explicit, let us set out the following additional, redundant, rule T:[11]

[10] Note that the other instance of B occurring in the upper right formula cannot be the cut formula. This is clear from the statement of Frege's rule, as the first instance $B \to C$ in $(B \to C) \to (B \to C)$ is itself an antecedent.

[11] The implication formula associated with this rule and combinator is $A \to ((A \to B) \to B)$ which is clearly a kind of permutation principle.

$$\frac{\Gamma;\Delta \vdash A}{\Delta;\Gamma \vdash A}\,\mathsf{T}$$

T immediately allows the proof of B and B', and makes HS_2 redundant as one can then prove K and S without appeal to HS_2 (though with appeal to MP_2). Consider the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A \qquad B \vdash B}{A \rightarrow B; A \vdash B} \qquad C \vdash C}{B \rightarrow C; (A \rightarrow B; A) \vdash C}\,\mathsf{T}}{(A \rightarrow B; A); B \rightarrow C \vdash C}\,\mathsf{C}}{(A \rightarrow B; B \rightarrow C); A \vdash C}}{A \rightarrow B; B \rightarrow C \vdash A \rightarrow C}$$

From this end-sequent, one can, of course, prove either the B axiom or the B' axiom using the T rule. So, we have the axioms K, W, C, and B provable and modus ponens as a rule of proof. This means that S is indeed provable, and all these without use of HS_2.

So, the $\mathrm{L}G_{\rightarrow}$ calculus as presented has it that the HS_2 rule is redudant. This provides an immediate, interesting upshot. While Frege provided the extra rule for hypothetical syllogism, and, as we have suggested, prefigured the kind of cut principle later employed by Gentzen, this was, formally speaking, strictly unnecessary. He, like Gentzen, could have done without this rule, providing only the modus ponens rule, the K axiom, and his generalised C and W rules. Frege seems to have been lead by the desire to reproduce the inferential work done by the implication axioms in his *Begriffsschrift* by means of adding generalised rules, he wound up making a mistake we might have expected in producing a redundant rule set. However, the redundancy points to an interesting feature of his proof system, and one about which either he didn't know, or of which he chose to make no comment.

This result does not undermine the fact that this treatment of the hypothetical syllogism rule as enforcing prefixing and suffixing behaviour sheds some light on how the proof system fits together. It is very clear that Frege's proof system is, finally, an axiom system, which manipulates formulae, and for the most part implication formulae. This is a key formal difference between Frege's system and sequent-style systems, which operate on arguments. The insight provided by studying this logic in terms of the binary structural connective of structurally free logics is to, as closely as possible, mimic the behaviour of the binary conditional connective in which Frege carried out his proofs.

6.4 Conclusion

I have sought to clarify just how the structural features of Frege's conditional are produced by his rules, with a particular focus on his hypothetical syllogism rule and considered some ways in which does and does not resemble Gentzen's cut rule. As with all implication systems, structural features of Frege's conditional rules set out key features of how the proof system operates, and there is more to be said about how Frege's system is interestingly novel in building such features in. While his proof system is not a sequent calculus[12], one can capture key elements of his proof system using this formalism which sheds light on the interesting and unique system which Frege did develop. Frege's work prefigured the key role that cut plays in proofs, and seems to have noticed some subtle structural features of the conditional some decades before these same features were more explicitly specified by, for instance, Russell [67], Schönfinkel [68], or Curry [24]. While, perhaps not surprising, it is some indication that Frege, in the *Grundgesetze*, may have been aware of some subtle proof-theoretic issues not carefully studied for some decades after the publication of his work.

[12]Clearly the intended reading of his conditional is not as inferences or inference tickets, but as the name of a truth value, namely, The True in the case of theorems of the system.

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