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Continued Fractions in Cluster Algebras, Lattice Paths and Markov Numbers

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ABSTRACT

In this work we present results from three different, albeit related, areas. First, we construct an explicit formula for the F-polynomial of a cluster variable in a surface type cluster algebra. Second, we define lattice paths and order them by the number of perfect matchings of their associated snake graphs. Lastly, we prove two conjectures from Martin Aigner's book, *Markov's theorem and 100 years of the uniqueness conjecture* that determine an ordering on subsets of the Markov numbers based on their corresponding rational.

The common thread throughout this work is the interplay between cluster algebras, lattice paths, snake graphs, Markov numbers and their connections to continued fractions. In the first section we give the necessary background on finite continued fractions and then in each of the following three sections, we introduce a topic and follow it with our related results.

Continued Fractions in Cluster Algebras, Lattice Paths and Markov Numbers

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Continued Fractions in Cluster Algebras, Lattice Paths and Markov Numbers

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"The whole is more than the sum of its parts." - Aristotle

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Last but never least, Dan Martin. You are my favorite.

Contents

Chapter 1

Continued Fractions

Continued fractions can be seen in the literature as early as 300BC in Euclid's elements. However, continued fractions weren't studied in their own right until the 17th century when John Wallis published many of the basic properties in his 1655 book *Arithmetica Infinitorum*. Since that time, continued fractions have been used in number theory, specifically in approximation theory and the study of Diophantine equations, as well as other areas.

For our purposes continued fractions are a valuable tool. We use continued fractions and their many properties to better understand objects in the field of cluster algebras and number theory. In this chapter we give a brief overview of the basic background knowledge needed to use continued fraction throughout the other chapters. A list of additional necessary properties of continued fractions is given in Appendix A.

Definition 1.0.1. A finite continued fraction is a function

$$[a_1, \ldots, a_n] := a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}$$

of n variables $a_1, \ldots, a_n$.

Typically we restrict the variables $a_1 \in \mathbb{Z}$ and $a_i \in \mathbb{Z}^+$ for $i > 1$. In this case, we call the continued fraction *simple*. Even more specifically, we say that a continued fraction is *positive* if each $a_i \in \mathbb{Z}^+$.

Example 1.0.2. Here we compute the continued fraction $[3, 8, 2, 5]$.

$$\begin{aligned} [3, 8, 2, 5] &= 3 + \cfrac{1}{8 + \cfrac{1}{2 + \cfrac{1}{5}}} \\ &= \frac{290}{93} \end{aligned}$$

In general, a simple continued fraction is a representation of a simplified rational number.

Lemma 1.0.3. *[CS4]*

a.) There is a bijection between $\mathbb{Q}$ and the set of finite simple continued fractions whose last entry is at least 2.

b.) There is a bijection between $\mathbb{Q}_{>1}$ and the set of finite positive continued fractions whose last entry is at least 2.

The condition that the last entry must be at least two removes the ambiguity in the way we list the entries of a continued fraction. In Equation (A.0.4), we see that the continued fraction $[a_1, \ldots, a_n, 1] = [a_1, \ldots, a_n + 1]$. The proof is obvious[1] by the definition of a continued fraction. Thus if we remove the possibility of a continued fraction ending in one, then each rational number corresponds to a unique simple continued fraction. This unique continued fraction can be determined from the rational number using the Euclidean algorithm.

Example 1.0.4. Suppose we would like to determine the positive continued fraction associated to the rational $\frac{38}{7}$. Then the coefficient in each step of the Euclidean algorithm is an entry in the continued fraction. i.e. $[5, 2, 3] = \frac{38}{7}$.

$$\begin{aligned} 38 &= (5)7 + 3 \\ 7 &= (2)3 + 1 \\ 3 &= (3)1 + 0 \end{aligned}$$

Since each continued fraction $[a_1, \ldots, a_n]$ is rational, it has a numerator denoted by $N[a_1, \ldots, a_n]$ and a denominator, $D[a_1, \ldots, a_n]$ such that

$$[a_1, \ldots, a_n] = \frac{N[a_1, \ldots, a_n]}{D[a_1, \ldots, a_n]}.$$

While the classical computation of a continued fraction yields both the numerator and denominator of the rational number it represents, sometimes when the numerator

[1] "Obvious is the most dangerous word in mathematics." - E. T. Bell

alone is of interest, another type of computation is performed instead. Euler observed that the numerator of a continued fraction can be determined by a sum of products obtained by deleting consecutive pairs of entries in the continued fraction. In [GKP], the authors describe the "Morse code" method of describing this process. We consider all possible sequences of dots and dashes where a dot represents a single entry and a dash represents a consecutive pair, a_i, a_{i+1}. Then we construct a product from each sequence. We include the entry a_i if there is a dot in its place, and leave out the entries corresponding to dashes. If a product is empty (all dashes), we let the product be one. The sum of these products yield the numerator of the continued fraction.

Example 1.0.5. First, consider a continued fraction with four entries. We would like to use the Morse code method to obtain the numerator of the continued fraction $[a_1, a_2, a_3, a_4]$.

We take every combination of dots and dashes that fit in the four entries of the continued fraction, then take the product of the entries corresponding to dots.

$$\begin{array}{ccccc} \cdots\cdot & -\cdot\cdot & \cdot - \cdot & \cdot\cdot - & -- \\ a_1a_2a_3a_4 & a_3a_4 & a_1a_4 & a_1a_2 & 1 \end{array}$$

Thus we obtain $N[a_1, a_2, a_3, a_4] = a_1a_2a_3a_4 + a_3a_4 + a_1a_4 + a_1a_2 + 1$. Next, suppose we have a continued fraction with three entries. We would like to use the Morse code method to obtain $N[a_1, a_2, a_3] = a_1a_2a_3 + a_3 + a_1$.

$$\begin{array}{ccc} \cdots & -\cdot & \cdot - \\ a_1a_2a_3 & a_3 & a_1 \end{array}$$

Euler also noted Equation (A.0.6) is true from this method. The sum will remain the same even if the entries of the continued fraction are in reverse order, i.e. $N[a_1, \ldots, a_n] = N[a_n, \ldots, a_1]$.

Another well known technique for constructing the numerator of a continued fraction is to use the following recursion equations.

Lemma 1.0.6. *Let* $N[\ ] = 1$.

for $n > 2$

$$a.)\ N[a_1, a_2, a_3, \ldots, a_n] = a_1 N[a_2, a_3 \ldots, a_n] + N[a_3, \ldots, a_n] \tag{A.0.1}$$

$$b.)\ N[a_1, \ldots, a_{n-2}, a_{n-1}, a_n] = a_n N[a_1 \ldots, a_{n-2}, a_{n-1}] + N[a_1, \ldots, a_{n-2}] \tag{A.0.2}$$

for $n = 2$

$$c.)\ N[a_1, a_2] = a_1 N[a_2] + N[\ ] = a_1 a_2 + 1. \tag{A.0.3}$$

We give the proof of this well known result because we will make use of these recursion equations often throughout this work.

Proof. First we prove part a.).

$$\begin{aligned}
\frac{N[a_1, \ldots, a_n]}{D[a_1, \ldots, a_n]} &= [a_1, \ldots, a_n] \\
&= a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}} \\
&= a_1 + \frac{1}{[a_2, \ldots, a_n]}
\end{aligned}$$

$$\begin{aligned}\frac{N[a_1,\ldots,a_n]}{D[a_1,\ldots,a_n]} &= a_1 + \frac{D[a_2,\ldots,a_n]}{N[a_2,\ldots,a_n]} \\ &= \frac{a_1 N[a_2,\ldots,a_n] + D[a_2,\ldots,a_n]}{N[a_2,\ldots,a_n]}\end{aligned}$$

From this computation, and the simplified nature of a continued fraction, we have that $N[a_1,\ldots,a_n] = a_1 N[a_2,\ldots,a_n] + D[a_2,\ldots,a_n]$ and $D[a_1,\ldots,a_n] = N[a_2,\ldots,a_n]$. Thus $D[a_2,\ldots,a_n] = N[a_3,\ldots,a_n]$ and part a.) holds.

From the Morse code method of obtaining the numerator of a continued fraction, we observed Equation (A.0.6) holds. Thus $N[a_1,\ldots,a_n] = N[a_n,\ldots,a_1]$ and hence part b.) holds due to part a.).

Part c.) is clear by a quick computation. $[a_1,a_2] = a_1 + \dfrac{1}{a_2} = \dfrac{a_1 a_2 + 1}{a_2}$. $\square$

In the proof above we showed that $D[a_1,\ldots,a_n] = N[a_2,\ldots,a_n]$, therefore we can always write the denominator of a continued fraction as the numerator of some other continued fraction. This should illuminate the emphasis on computing the numerator of a continued fraction.

More properties of finite continued fractions are listed in Appendix A and will be utilized in other sections.

Chapter 2

Cluster Algebras

Cluster algebras were introduced in 2002 by Fomin and Zelevinsky [FZ1]. The theory has since been developed and the connections to fields such as quiver representations, dynamical systems, algebraic geometry and string theory have been studied. It is important to note that we concentrate on a specific class of cluster algebras, that is cluster algebras from surfaces.

2.1 Background

Cluster algebras are commutative rings, $\mathcal{A}(\mathbf{x}, \mathbf{y}, Q)$, that depend on the initial cluster variables $\mathbf{x} = (x_1, \ldots, x_s)$, cluster coefficients $\mathbf{y} = (y_1, \ldots, y_s)$ and a quiver Q. The elements of the cluster algebra are determined by a choice of an initial seed $(\mathbf{x}, \mathbf{y}, Q)$. A recursive operation called mutation is applied first to the initial seed and each iteration of the mutation yields a new cluster variable in the form of a Laurent polynomial in terms of the initial cluster variables $x_1, \ldots, x_s$. The mutation operation gives a combinatorial structure to the cluster algebra.

To better understand these commutative rings, we must first understand the ground ring of the cluster algebra. First we let $(\mathbb{P}, \cdot)$ be a free abelian group on the cluster coefficients, $y_1, \ldots, y_s$. Then we define the operation $\oplus$ in $\mathbb{P}$ such that

$$\prod_j y_j^{a_j} \oplus \prod_j y_j^{b_j} = \prod_j y_j^{\min\{a_j, b_j\}}$$

and call $(\mathbb{P}, \oplus, \cdot)$ a tropical semifield, where a semifield is a torsion-free multiplicative abelian group endowed with an additional operation $\oplus$, which is commutative, associative and distributive with respect to the multiplication. The group ring, $\mathbb{ZP}$, of $\mathbb{P}$ is the ring of Laurent polynomials in $y_1, \ldots, y_s$. In the case when $y_1 = \cdots = y_s = 1$, we have that $\mathbb{P} = 1$, $\mathbb{ZP} = \mathbb{Z}$ and we call the coefficients of the cluster algebra trivial coefficients. When we keep the coefficients arbitrary, we call $y_1, \ldots, y_s$ principal coefficients. The principal coefficients are most important because a cluster variable in an arbitrary cluster algebra can be computed from a cluster variable in a corresponding cluster algebra with principal coefficients [FZ4].

Next we construct a field of rational functions over the initial cluster variables. Every cluster variable in the cluster algebra is a Laurent polynomial in $\mathbb{QP}(x_1, \ldots, x_s)$ with coefficients in $\mathbb{QP}$, the field of fractions of $\mathbb{ZP}$.

A quiver, Q is a directed graph with a set Q_0 of vertices and Q_1 of arrows. The number of vertices in the quiver is the same as the number of initial cluster variables, s. We only consider quivers that do not have loops or 2-cycles.

loop 2-cycle

Cluster algebras are constructed by performing a recursive operation called mutation on the initial cluster variables in order to obtain all possible cluster variables. Then these cluster variables generate the cluster algebra. Starting with the initial seed $(\mathbf{x}, \mathbf{y}, Q)$, we mutate at the vertex k and obtain a new seed $\mu_k(\mathbf{x}, \mathbf{y}, Q) = (\mathbf{x}', \mathbf{y}', Q')$.

Definition 2.1.1. The mutation $\mu_k(\mathbf{x}, \mathbf{y}, Q) = (\mathbf{x}', \mathbf{y}', Q')$ is given by the following.

- $\mathbf{x}' = (x_1, \dots x_{k-1}, x'_k, x_{k+1}, \dots x_n)$ where

$$x'_k = \frac{1}{x_k} \frac{1}{y_k \oplus 1} \left(y_k \prod_{i \to k} x_i + \prod_{k \to i} x_i \right)$$

  The first product is taken over all arrows from a vertex i to the vertex k in Q and the second is taken over arrows from k to i in Q. We let an empty product be equal to one. The operation $\oplus$ is the operation from our tropical semifield $(\mathbb{P}, \oplus, \cdot)$.

- $\mathbf{y}' = (y'_1, \dots, y'_n)$ where

$$y'_j = \begin{cases} y_k^{-1} & \text{if } j = k, \\ y_j \prod_{k \to j} y_k (y_k \oplus 1)^{-1} \prod_{j \to k} (y_k \oplus 1) & \text{if } j \neq k \end{cases}$$

- Q' is obtained by applying these steps to the quiver Q.

  1. for every path $i \to k \to j$ add an arrow $i \to j$
  2. reverse all arrows at the vertex k
  3. delete any 2-cycles

It is important to note that mutations are involutions, so applying the same mutation twice in a row yields the same seed i.e. $\mu_k \mu_k(\mathbf{x}, \mathbf{y}, Q) = (\mathbf{x}, \mathbf{y}, Q)$.

Definition 2.1.2. The *exchange graph* of a cluster algebra is a graph whose vertices are seeds in the cluster algebra and edges are the mutations between those seeds.

To give a simple computation of a mutation, we consider a cluster algebra in which the cluster coefficients are all equal to one, i.e. are trivial. This is the case in Example 2.1.3.

Example 2.1.3. Let $\mathbf{x} = (x_1, x_2, x_3)$, let the cluster coefficients be trivial and let Q be the quiver $Q:\ 1 \longrightarrow 2 \longrightarrow 3$. We first perform a mutation μ_2 to obtain $(\mathbf{x}', \mathbf{y}', Q')$ and then mutate the new seed by μ_3 to obtain $(\mathbf{x}'', \mathbf{y}'', Q'')$.

$$Q':\ \begin{tikzcd} 1 \arrow[rr, bend left] & 2 \arrow[l] & 3 \arrow[l] \end{tikzcd} \qquad \mathbf{x}' = (x_1, x_2', x_3) = \left(x_1, \frac{x_1 + x_3}{x_2}, x_3 \right)$$

$$Q'':\ \begin{tikzcd} 1 & 2 \arrow[r] & 3 \arrow[ll, bend right] \end{tikzcd} \qquad \mathbf{x}'' = (x_1, x_2', x_3') = \left(x_1, \frac{x_1 + x_3}{x_2}, \frac{x_1 + x_2'}{x_3} \right)$$

$$= \left(x_1, \frac{x_1 + x_3}{x_2}, \frac{x_1 x_2 + x_1 + x_3}{x_2 x_3} \right)$$

Next, we consider a case when the coefficients are principal.

Example 2.1.4. Consider the initial seed with $\mathbf{x} = (x_1, x_2, x_3)$ and quiver Q as in Example 2.1.3. Now, let $\mathbf{y} = (y_1, y_2, y_3)$. Again we mutate first in the vertex 2 and then in the vertex 3. Q' and Q'' will remain the same as in Example 2.1.3, but now

$\mathbf{x}'$ and $\mathbf{x}''$ will change as well as $\mathbf{y}'$ and $\mathbf{y}''$.

$$\mathbf{x}' = (x_1, x_2', x_3) = \left(x_1, \frac{y_2x_1 + x_3}{x_2}, x_3\right) \qquad \mathbf{y}' = (y_1', y_2', y_3') = \left(y_1, \frac{1}{y_2}, y_2y_3\right)$$

$$\mathbf{x}'' = \left(x_1, \frac{y_2x_1 + x_3}{x_2}, \frac{y_3'x_1 + x_2'}{x_3}\right) \qquad \mathbf{y}'' = \left(y_1'(y_3' \oplus 1), y_2'y_3'(y_3' \oplus 1)^{-1}, \frac{1}{y_3'}\right)$$

$$= \left(x_1, \frac{y_2x_1 + x_3}{x_2}, \frac{y_2y_3x_1x_2 + y_2x_1 + x_3}{x_2x_3}\right) \qquad = \left(y_1, y_3, \frac{1}{y_2y_3}\right)$$

In Example 2.1.3 we let the cluster coefficents be trivial (equal to one) and gave some cluster variables. It will become important later to have the analogous concept of letting the initial cluster variables be trivial and have the cluster variables written only in terms of the principal coefficients. The resulting cluster variable is a polynomial in the principal coefficients, which we call the F-polynomial.

Definition 2.1.5. The *F-polynomial* of a cluster variable is the polynomial in the principal coefficients $y_1, \ldots, y_s$ obtained from setting the initial cluster variables equal to one.

For instance, the cluster variable $\dfrac{y_2y_3x_1x_2 + y_2x_1 + x_3}{x_2x_3}$ from Example 2.1.4 yields the F-polynomial $y_2y_3 + y_2 + 1$ when we set $x_1 = x_2 = x_3 = 1$.

In general, the number of cluster variables in a cluster algebra is not always finite. In fact, a cluster algebra is said to be of finite type if this is the case. In Example 2.1.3 and Example 2.1.4 the quiver Q is of Dynkin type[1] $\mathbb{A}$ and thus the cluster algebra is of finite type. Any quiver of Dynkin type $\mathbb{A}$, $\mathbb{D}$ or $\mathbb{E}$ yields a cluster algebra of finite type. Moreover any quiver that can be mutated to a Dynkin type $\mathbb{A}$, $\mathbb{D}$ or $\mathbb{E}$ quiver

[1] See Appendix B

after a finite sequence of mutations yields a cluster algebra of finite type.

Cluster algebras of surface type are defined for a surface (S, M) where S is a Riemann surface with a boundary ∂S (possibly empty) and M is a finite set of marked points with at least one marked point on each connected component of ∂S. We define an arc γ as in Definition 2.1.6.

Definition 2.1.6. For a surface (S, M), we define an *arc* γ to be a curve in S up to isotopy such that

a.) the endpoints of γ are in M;

b.) γ has no self intersections except possibly at the endpoints;

c.) γ is disjoint from ∂S except possibly at the endpoints;

d.) γ does not cut out an unpunctured monogon or an unpunctured bigon.

Considering a triangulation of the surface, the arcs in the triangulation correspond to vertices in the quiver and the initial cluster variables. Moreover the triangulation itself determines the arrows in the quiver.

In [FST] the authors proved that there is a bijection between the tagged arcs of (S, M) and the cluster variables in the cluster algebra. From an arc γ in a triangulated surface, one can construct a labeled snake graph [MSW]. From this point on, we will use this connection between snake graphs and cluster algebras from surfaces to study the cluster variables.

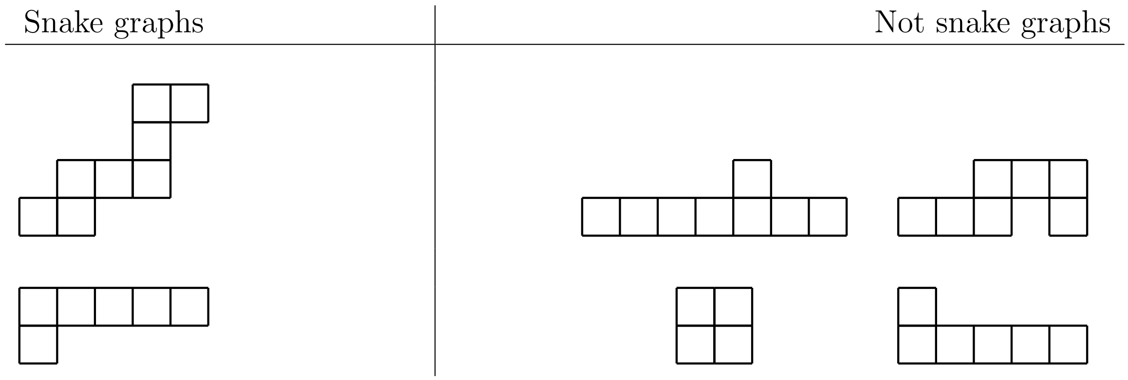


FIGURE 2.2.1: Some examples and non-examples of snake graphs.

2.2 Snake Graphs Background

Before we can use snake graphs to study cluster variables in a meaningful way, we must first define them and study their properties.

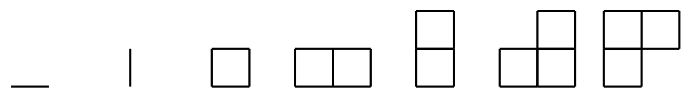
Definition 2.2.1. A *snake graph*, $\mathcal{G}$, is a connected planar graph consisting of a finite sequence of square tiles $G_1, G_2, \ldots, G_d$ such that each tile has edges parallel or orthogonal to the standard orthonormal basis in the plane and consecutive tiles G_i, G_{i+1} share exactly one edge. The shared edge is either the north edge of G_i with the south edge of G_{i+1} or the east edge of G_i with the west edge of G_{i+1}.

In Figure 2.2.1 we give some examples and non-examples of snake graphs. Any edge of a snake graph $\mathcal{G}$ that is a shared edge between two consecutive tiles is called an *interior edge* and all other edges are called *boundary edges*. By construction, a snake graph with d tiles will have $d-1$ interior edges.

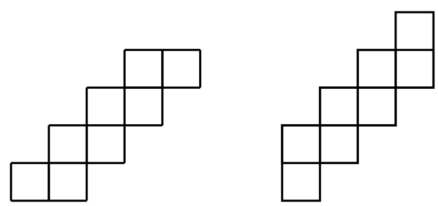
Remark 2.2.2. It is possible to have a snake graph with no tiles. Such a snake graph would consist of a single edge.

One special shape of a snake graph is a *zigzag* snake graph. A zigzag snake graph is a snake graph such that the interior edges alternate between being vertical and horizontal.

Example 2.2.3. Here are some examples of zigzag snake graphs when the number of tiles in the snake graph is d.



$d=0$ $d=0$ $d=1$ $d=2$ $d=2$ $d=3$ $d=3$



$d=8$ $d=8$

From [CS4] we know that to each snake graph we can associate a continued fraction and vice versa, so that the continued fraction describes the shape of the snake graph. To do this we construct a sign sequence and then from the sign sequence we obtain a continued fraction $[a_1, \ldots, a_n]$ and then denote the snake graph by $\mathcal{G}[a_1, \ldots, a_n]$. See Example 2.2.4 for a depiction of the process described below.

Constructing a sign sequence to formulate the continued fraction associated to a snake graph:

1. Label the south edge of the first tile with a negative sign if the east edge of the first tile is an interior edge. Otherwise, label the west edge of the first tile with a negative sign i.e if the north edge is an interior edge.

2. Sequentially label the interior edges of the snake graph according to the following rules:

   - If the edge is perpendicular to the previous edge, label with the same sign as the previous edge.

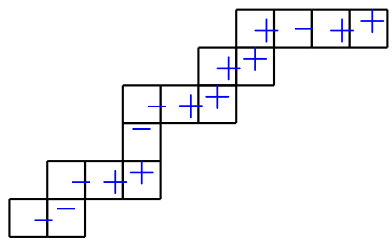
   - If the edge is parallel to the previous edge, label with the opposite sign as the previous edge.

3. In the last tile, label either the eastern or northern edge according to the rules in step 2 to obtain the same sign as the previous edge.

4. Count the length of each maximal subsequence of the same sign. Each length gives an entry in the continued fraction.

Notice that Step 1 and Step 3 force the continued fraction to begin and end in a positive integer greater than one. This is to avoid ambiguity in the entries of the continued fraction which will become useful later. However, this convention equates snake graphs that are 180° rotations of each other. For example, both snake graphs in Example 2.2.3 with three tiles ($d = 3$) are denoted by $\mathcal{G}[4]$. For our purposes, it is better to have ambiguity in the orientation of the snake graph than the entries in its associated continued fraction.

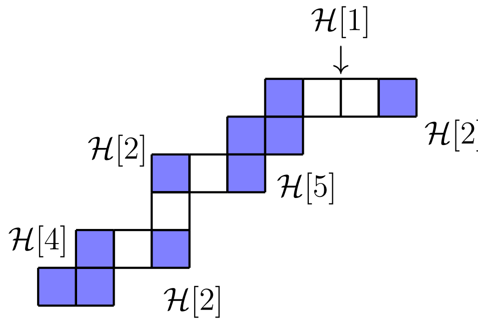
Example 2.2.4. Given the snake graph, we construct the sign function to determine the associated continued fraction. Thus we denote the snake graph as $\mathcal{G}[4, 2, 2, 5, 1, 2]$.



$$\underbrace{-,-,-,-}_{4},\underbrace{+,+}_{2},\underbrace{-,-}_{2},\underbrace{+,+,+,+,+}_{5},\underbrace{-}_{1},\underbrace{+,+}_{2} \qquad \rightarrow \qquad [4, 2, 2, 5, 1, 2]$$

It can be helpful to think of zigzag snake graphs as the building blocks of other snake graphs. The continued fraction associated to a zigzag snake graph can always be written as a single entry. If the zigzag snake graph has d tiles, then it can be denoted by $\mathcal{G}[d + 1]$. In Example 2.2.3, the snake graphs are denoted by $\mathcal{G}[1], \mathcal{G}[1], \mathcal{G}[2], \mathcal{G}[3], \mathcal{G}[3], \mathcal{G}[4], \mathcal{G}[4]$, $\mathcal{G}[9]$ and $\mathcal{G}[9]$ respectively. Moreover, notice that any other snake graph is a concatenation of zigzag snake graphs separated by single tiles.

Example 2.2.5. The snake graph from Example 2.2.4 can be split into 6 zigzag subgraphs separated by single tiles. If we let $\mathcal{H}$ denote a subgraph, then the shaded subgraphs are as follows $\mathcal{H}[4], \mathcal{H}[2], \mathcal{H}[2], \mathcal{H}[5], \mathcal{H}[1], \mathcal{H}[2]$. Notice that the entries in the continued fractions of the zigzag subgraphs make up the entries in the continued fraction of the entire snake graph $\mathcal{G}[4, 2, 2, 5, 1, 2]$.

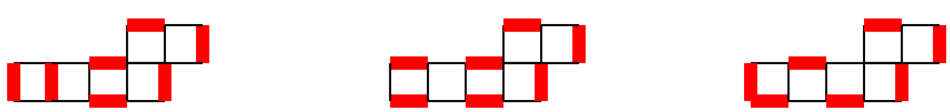


2.2.1 Perfect Matchings

Definition 2.2.6. A *perfect matching* P of a graph $\mathcal{G}$ is a collection of edges of the graph such that each vertex of $\mathcal{G}$ is incident to exactly one edge in P. We denote the set of perfect matchings of a snake graph $\mathcal{G}$ by $\text{Match}(\mathcal{G})$.

Each snake graph with at least one tile has exactly two perfect matchings that consist entirely of boundary edges. We call these matchings the minimal P_- and maximal P_+ matchings. Our convention is to let the perfect matching containing only boundary edges and the south edge of G_1 to be the minimal matching. The other perfect matching containing all boundary edges is therefore the maximal matching by this convention.

Example 2.2.7. Below is a snake graph with six tiles. The edges included in the matching are marked.



perfect matching P | minimal perfect matching P_- | not a perfect matching

We use the terminology minimal and maximal because we can construct a poset of all perfect matchings for a single snake graph. This can be done by the following systematic approach of turning tiles. One can turn a tile if two edges of the tile are in the given perfect matching. By turning the tile we replace the two edges in the perfect matching with the other two edges of the tile, thus forming a new perfect matching.

Determining the perfect matchings of a snake graph by turning tiles:

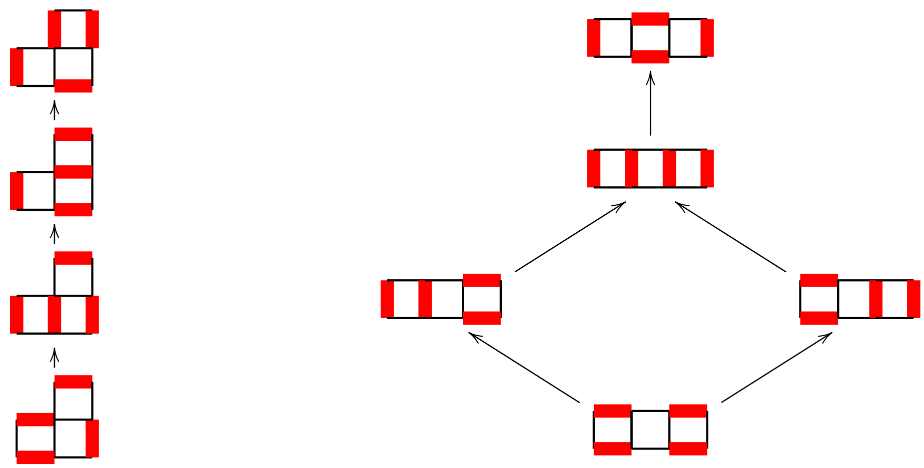
1. Start with the minimal matching. By convention, this is the perfect matching of boundary edges containing the southern edge of the first tile.

2. Any tile with two of its edges in the previous perfect matching can be turned independently, yielding a new matching.

3. For each new matching, repeat step 2 turning only tiles that have not already been turned. Continued this process until the maximal matching is obtained.

Notice in Figure 2.2.2 the linear matching poset is constructed from a zigzag snake graph. This is not coincidental, zigzag snake graphs always have linear matching posets because we can only turn tiles in a linear order.

While this process always works to determine the perfect matchings of a snake graph and the number of perfect matchings, it is not an efficient method for large or complicated snake graphs. Thus, we introduce a new method for computing the



This matching poset is linear. This matching poset is non-linear.

FIGURE 2.2.2: Here we have two examples for which we found all of the perfect matchings of the given snake graph by the process of turning tiles. The minimal matching is at the bottom and the maximal matching is at the top.

number of perfect matchings of a snake graph.

Theorem 2.2.8. *[CS4] The number of perfect matchings, $\#Match(\mathcal{G})$ of a snake graph $\mathcal{G}$ is equal to the numerator of the continued fraction associated to the snake graph by its sign sequence.*

Implementing this result for the snake graph $\mathcal{G}[4,2,2,5,1,2]$ from Example 2.2.4 we find that $\mathcal{G}[4,2,2,5,1,2]$ has $N[4,2,2,5,1,2] = 401$ perfect matchings.

In [CS4], the authors use the method of grafting snake graphs to obtain Proposition 2.2.9, an identity on numerators of continued fractions.

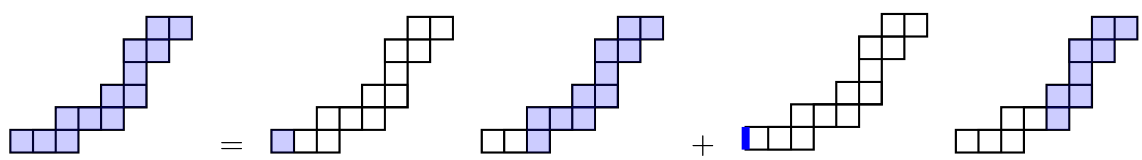
Proposition 2.2.9. *Let $a_i \in \mathbb{Z}_{\geq 0}$.*

$$N[a_1,\ldots,a_i,a_{i+1},\ldots,a_n] = N[a_1,\ldots,a_i]N[a_{i+1},\ldots,a_n]+N[a_1,\ldots,a_{i-1}]N[a_{i+2},\ldots,a_n]$$

Example 2.2.10. To illustrate Proposition 2.2.9, consider the continued fraction,

$[2,3,4,5]$. We first compute $N[2,3,4,5]$ and then the equivalent sum for each possible place we could graft.

$$\begin{array}{rcccl} N[2,3,4,5] & = & & & 157 \\ N[2]N[3,4,5]+N[\ ]N[4,5] & = & 2(68)+21 & = & 157 \\ N[2,3]N[4,5]+N[2]N[5] & = & 7(21)+2(5) & = & 157 \\ N[2,3,4]N[5]+N[2,3]N[\ ] & = & 30(5)+7 & = & 157 \end{array}$$



$$N[2,3,4,5] \quad = \quad N[2] \quad N[3,4,5] \quad + \quad N[\ ] \quad N[4,5]$$

2.3 F-polynomial Result

Returning to the subject of cluster algebras, the labeled snake graph $\mathcal{G}_\gamma$ associated to an arc γ on a surface (S, M) can be used to better understand the cluster variable x_γ associated to that arc. This is done by considering the expansion formula that gives the cluster variable x_γ as a sum over the perfect matchings of the snake graph as seen in [MSW].

In [CS4] the authors give a formula for the cluster variable of a labeled snake graph in terms of a continued fraction of Laurent polynomials in $x_1, \ldots, x_s$. However, this formula only works for cluster algebras with trivial coefficients. Here we extend this result to cluster algebras with principal coefficients by giving an explicit formula for the F-polynomial as a continued fraction of Laurent polynomials in $y_1, \ldots, y_s$. In order to accomplish this, we do not need the labeling of the snake graph $\mathcal{G}_\gamma$, but instead just the snake graph itself.

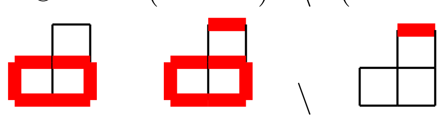
Then the expansion formula from [MSW] in the specific case of F-polynomials is given by the Equation 2.3.1 where $F(\mathcal{G}_\gamma)$ denotes the F-polynomial of the cluster variable x_γ associated to the snake graph $\mathcal{G}_\gamma$.

$$F(\mathcal{G}_\gamma) = \sum_{P \in \mathrm{Match}(\mathcal{G}_\gamma)} \prod_{G_i \in G_p} y_i \tag{2.3.1}$$

where G_p is the union of tiles in the subgraph of $\mathcal{G}$ whose boundary edges are the set $P_- \ominus P = (P_- \cup P) \setminus (P_- \cap P)$.

Example 2.3.1. Consider the linear matching poset of the zigzag snake graph from Figure 2.2.2. Let P be the matching one step below the maximal matching. In this example we will compute $P_- \ominus P$ where P_- is the minimal matching shown in the matching poset.

$$P_- \ominus P = \quad (P_- \cup P) \quad \setminus \quad (P_- \cap P)$$



Therefore in this example, G_P is $G_1 \cup G_2$, so $\prod_{G_i \in G_P} y_i = y_1 y_2$. If we were to do this for every perfect matching of the snake graph, we would get the F-polynomial $1 + y_1 + y_1y_2 + y_1y_2y_3$. The F-polynomial associated to the snake graph from the non-linear matching poset in Figure 2.2.2 would be $1 + y_1 + y_3 + y_1y_3 + y_1y_2y_3$.

From another perspective, the height $y(P)$ of P is defined recursively by $y(P_-) = 1$ and if P' is above P and obtained by turning the tile G_i then $y(P') = y_i y(P)$. The F-polynomial of $\mathcal{G}$ is defined as $F(\mathcal{G}) = \sum_P y(P)$, where the sum is over all perfect matchings of $\mathcal{G}$.

Notice that the F-polynomial for the zigzag snake graph in Example 2.3.1 is quite nice. We start with 1 and then for each consecutive term, we multiply the previous term by the next cluster coefficient until our final term is the product of all the cluster coefficients. In general, the F-polynomials for zigzag snake graphs look this way. Therefore since any snake graph is a concatenation of zigzag snake graphs separated by single tiles, we can somewhat break down the structure of the F-polynomial associated to a snake graph by the F-polynomials associated to the zigzag subgraphs of that snake graph. The question then becomes, how do we graft these F-polynomials together?

In [CS, CS2, CS3], identities in the cluster algebra have been expressed in terms of snake graphs. Here we look at the F-polynomial of a snake graph as a grafting together of the F-polynomials of two subgraphs. Let $\mathcal{G}[a_1, \dots, a_n]$ be a snake graph and define $\ell_i = \sum_{s=1}^{i} a_s$ for $i > 1$ and $\ell_0 = 0$. We also label each tile in the snake graph with its corresponding cluster coefficient, i.e. label the first tile y_1 and so on. Under the correspondence between continued fractions and snake graphs each a_i in the continued fraction $[a_1, \dots, a_n]$ corresponds to a zigzag subgraph, $\mathcal{H}_i$, consisting of $a_i - 1$ tiles of $\mathcal{G}[a_1, \dots, a_n]$. The subgraph $\mathcal{H}_i$ is isomorphic to $\mathcal{G}[a_i]$, but inherits its tile labels from $\mathcal{G}[a_1, \dots, a_n]$, thus is the zigzag snake graph consisting of tiles $G_{(\ell_{i-1})+1}, \dots, G_{(\ell_i)-1}$. Equation (2.3.2) below follows from the grafting with a single edge formula from Theorem 7.3 of [CS2], where $n \geq 2$ and the grafting takes place at tile $G_{\ell_{n-1}}$.

$$F(\mathcal{G}[a_1, \dots, a_n]) = y_{34} F(\mathcal{G}[a_1, \dots, a_{n-1}]) F(\mathcal{H}_n) + y_{56} F(\mathcal{G}[a_1, \dots, a_{n-2}]) \tag{2.3.2}$$

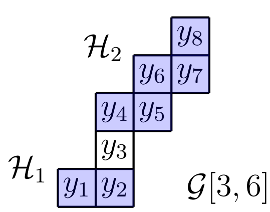
where the coefficients y_{34} and y_{56} are defined as follows, where $y_0 = 1$.

$$y_{34} = \begin{cases} y_{\ell_{n-1}} & \text{if } n \text{ is odd,} \\ 1 & \text{if } n \text{ is even,} \end{cases} \qquad y_{56} = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ \displaystyle\prod_{j=\ell_{n-2}}^{(\ell_n)-1} y_j & \text{if } n \text{ is even.} \end{cases} \tag{2.3.3}$$

For the case when $n = 2$, we define the snake graph of an empty continued fraction to be a single edge. There is only one perfect matching of a single edge, which corresponds to an F-polynomial equal to one. Therefore Equation (2.3.2) for the case when $n = 2$ becomes

$$F(\mathcal{G}[a_1, a_2]) = F(\mathcal{G}[a_1])F(\mathcal{H}_2) + \prod_{j=0}^{(\ell_2)-1} y_j. \tag{2.3.4}$$

Example 2.3.2. Consider the snake graph $\mathcal{G}[3,6]$ shown below. The first subgraph, $\mathcal{H}_1 = \mathcal{G}[3]$ has F-polynomial $F(\mathcal{G}[3]) = 1 + y_1 + y_1y_2$. The second subgraph $\mathcal{H}_2$ is isomorphic to $\mathcal{G}[6]$, but not equal due to the coefficient labelling. The F-polynomial is $F(\mathcal{H}_2) = 1 + y_4 + y_4y_5 + y_4y_5y_6 + y_4y_5y_6y_7 + y_4y_5y_6y_7y_8$.



The union of any perfect matching of $\mathcal{H}_1$ and any perfect matching of $\mathcal{H}_2$ is a perfect matching of $\mathcal{G}[3,6]$. This gives us the first product in the equation for $F(\mathcal{G}[3,6])$. In addition to this, we would also have a perfect matching that includes the boundary edges of the tile between these two subgraphs. This perfect matching would be the maximal matching of $\mathcal{G}[3,6]$ and would give us the term $y_1y_2y_3y_4y_5y_6y_7y_8$ in $F(\mathcal{G}[3,6])$.

Therefore by Equation (2.3.4) we have $F(\mathcal{G}[3,6]) = y_{34}F(\mathcal{G}[3])F(\mathcal{H}_2) + y_{56}F(\mathcal{G}[\ ])$, so

$$F(\mathcal{G}[3,6]) = (1 + y_1 + y_1y_2)(1 + y_4 + y_4y_5 + y_4y_5y_6 + y_4y_5y_6y_7 + y_4y_5y_6y_7y_8) + \prod_{j=0}^{8} y_j.$$

The equation for the F-polynomial of a snake graph $\mathcal{G}[a_1, \ldots, a_n]$ gets more complicated as n increases. For instance, Equation (2.3.4) gives a formula for grafting two zigzag snake graphs together with a single tile between them. However, the larger n gets, the more zigzag snake graphs we are grafting together. Our goal is to give an explicit formula for the F-polynomial written as a continued fraction of Laurent polynomials. Therefore, we define these Laurent polynomials in Definition 2.3.3 in terms of the continued fraction associated to the snake graph.

Definition 2.3.3. [R] For any continued fraction $[a_1, \ldots, a_n]$ with $a_1 > 1$, we define an associated continued fraction of Laurent polynomials $[\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_n]$, where each $\mathcal{L}_i = \varphi_i C_i$ and

$$C_i = \begin{cases} \displaystyle\prod_{j=1}^{\ell_{i-1}} y_j & \text{if } i \text{ is odd,} \\ \displaystyle\prod_{j=1}^{(\ell_i)-1} y_j^{-1} & \text{if } i \text{ is even,} \end{cases} \qquad \varphi_i = \begin{cases} \displaystyle\sum_{k=\ell_{i-1}}^{(\ell_i)-1} \prod_{j=(\ell_{i-1})+1}^{k} y_j & \text{if } i \text{ is odd,} \\ \displaystyle\sum_{k=(\ell_{i-1})+1}^{\ell_i} \prod_{j=k}^{(\ell_i)-1} y_j & \text{if } i \text{ is even.} \end{cases}$$

In the definition the C_i's are what we call correction factors because the purpose of them is to adjust the formula in order to take into account the single tiles between the zigzag subgraphs of $\mathcal{G}[a_1, \ldots, a_n]$. The φ_i's are the F-polynomials of the zigzag subgraphs themselves. First, we will prove that the F-polynomial of each zigzag subgraph $\mathcal{H}_i$ is φ_i.

Lemma 2.3.4. *[R]* $F(\mathcal{H}_i) = \varphi_i, \quad$ *for all* $1 \leq i \leq n.$

Proof. Let i be odd. The tiles $G_{(\ell_{i-1})+1}, \ldots, G_{(\ell_i)-1}$ make up $\mathcal{H}_i$. The subgraph $\mathcal{H}_i$ is the zigzag snake graph $\mathcal{G}[a_i]$ with the inherited labeling and completed minimal perfect matching P_-^i discussed previously. In this completion, the first tile of the subgraph $G_{(\ell_{i-1})+1}$ can be turned immediately and is the only such tile. If we turn tile $G_{(\ell_{i-1})+1}$, we can then turn the next tile $G_{(\ell_{i-1})+2}$ and so on. Therefore

$$\begin{aligned} F(\mathcal{H}_n) &= 1 + y_{(\ell_{i-1})+1} + y_{(\ell_{i-1})+1}y_{(\ell_{i-1})+2} + \cdots + y_{(\ell_{i-1})+1} \cdots y_{(\ell_i)-1} \\ &= \sum_{k=\ell_{i-1}}^{(\ell_i)-1} \prod_{j=(\ell_{i-1})+1}^{k} y_j = \varphi_i. \end{aligned}$$

Let i be even. In this case, the very last tile of $\mathcal{H}_i$, $G_{(\ell_i)-1}$ has two edges in the minimal matching of the completion, P_-^i, and can be turned. Therefore in order to determine the F-polynomial of $\mathcal{H}_i$, we must first turn the last tile and work our way down the snake graph.

$$\begin{aligned} F(\mathcal{H}_i) &= 1 + y_{(\ell_i)-1} + y_{(\ell_i)-1}y_{(\ell_i)-2} + \cdots + y_{(\ell_i)-1} \cdots y_{(\ell_{i-1})+1} \\ &= \sum_{k=(\ell_{i-1})+1}^{\ell_i} \prod_{j=k}^{(\ell_i)-1} y_j = \varphi_i. \qquad \square \end{aligned}$$

Next we will use Definition 2.3.3 and Lemma 2.3.4 to construct and prove a formula for the F-polynomial associated to the snake graph $\mathcal{G}[a_1, \ldots, a_n]$.

Theorem 2.3.5. *[R] The F-polynomial associated to the snake graph of the continued fraction* $[a_1, \ldots, a_n]$ *denoted by* $F(\mathcal{G}[a_1, \ldots, a_n])$ *is given by the equation:*

$$F(\mathcal{G}[a_1,\ldots,a_n]) = \begin{cases} N[\mathcal{L}_1,\mathcal{L}_2,\ldots,\mathcal{L}_n] & \textit{if } n \textit{ is odd,} \\ \\ C_n^{-1}N[\mathcal{L}_1,\mathcal{L}_2,\ldots,\mathcal{L}_n] & \textit{if } n \textit{ is even,} \end{cases}$$

where $N[\mathcal{L}_1,\mathcal{L}_2,\ldots,\mathcal{L}_n]$ *is defined by the recursion*

$$N[\mathcal{L}_1,\mathcal{L}_2,\ldots,\mathcal{L}_n] = \mathcal{L}_n N[\mathcal{L}_1,\mathcal{L}_2,\ldots,\mathcal{L}_{n-1}] + N[\mathcal{L}_1,\mathcal{L}_2,\ldots,\mathcal{L}_{n-2}]$$

where $N[\mathcal{L}_1] = \mathcal{L}_1$ *and* $N[\mathcal{L}_1,\mathcal{L}_2] = \mathcal{L}_1\mathcal{L}_2 + 1$.

Proof. Proof by induction. Let $n = 1$. In the case where n is odd, by Equation (2.3.3), we have $y_{34} = y_{\ell_{n-1}}$ and $y_{56} = 1$. It is clear that $F(\mathcal{G}[a_1]) = F(\mathcal{H}_1)$ simply because $\mathcal{G}[a_1]$ and $\mathcal{H}_1$ are the same snake graph. Then, by Lemma 2.3.4 we have $F(\mathcal{H}_1) = \varphi_1$. Note that $C_1 = \prod_{j=1}^{0} y_j = 1$ because it is an empty product. Therefore $\varphi_1 = C_1\varphi_1 = \mathcal{L}_1 = N[\mathcal{L}_1]$. Thus we have shown that $F(\mathcal{G}[a_1]) = N[\mathcal{L}_1]$.

Let $n = 2$. In this case we use Equation (2.3.4) and note that $\prod_{j=0}^{(\ell_2)-1} y_j = C_2^{-1}$.

$$\begin{aligned} F(\mathcal{G}[a_1,a_2]) &= F(\mathcal{G}[a_1])F(\mathcal{H}_2) + \prod_{j=0}^{(\ell_2)-1} y_j \\ &= F(\mathcal{G}[a_1])F(\mathcal{H}_2) + C_2^{-1} \end{aligned}$$

Using Lemma 2.3.4 and the case $n = 1$, we see that the right hand side is equal to $\varphi_2 N[\mathcal{L}_1] + C_2^{-1}$ and this is equal to the following, where the second equation holds by Definition 2.3.3 and the last equation holds by the definition of $N[\mathcal{L}_1, \mathcal{L}_2]$.

$$\begin{aligned} F(\mathcal{G}[a_1, a_2]) &= C_2^{-1}\left(C_2\varphi_2 N[\mathcal{L}_1] + 1\right) \\ &= C_2^{-1}\left(\mathcal{L}_2 N[\mathcal{L}_1] + 1\right) \\ &= C_2^{-1} N[\mathcal{L}_1, \mathcal{L}_2] \end{aligned}$$

Now let $n > 2$ be odd. Assume that for all $m < n$ our statement holds. In this situation by Equation (2.3.3), $y_{34} = y_{\ell_{n-1}}$ and $y_{56} = 1$. Additionally, we know from Equation (2.3.2) that the F-polynomial of $\mathcal{G}[a_1, \ldots, a_n]$ is given by the following.

$$F(\mathcal{G}[a_1, \ldots, a_n]) = y_{\ell_{n-1}} F(\mathcal{G}[a_1, \ldots, a_{n-1}]) F(\mathcal{H}_n) + F(\mathcal{G}[a_1, \ldots, a_{n-2}])$$

Applying our inductive step we obtain:

$$F(\mathcal{G}[a_1, \ldots, a_n]) = y_{\ell_{n-1}} C_{n-1}^{-1} N[\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_{n-1}] F(\mathcal{H}_n) + N[\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_{n-2}].$$

From here we can apply Lemma 2.3.4.

$$F(\mathcal{G}[a_1, \ldots, a_n]) = y_{\ell_{n-1}} C_{n-1}^{-1} N[\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_{n-1}] \varphi_n + N[\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_{n-2}]$$

Using the fact that $C_n = y_{\ell_{n-1}} C_{n-1}^{-1}$ and $\mathcal{L}_n = C_n \varphi_n$ we obtain our desired result as follows.

$$
\begin{aligned}
F(\mathcal{G}[a_1,\ldots,a_n]) &= C_n\varphi_n N[\mathcal{L}_1,\mathcal{L}_2,\ldots,\mathcal{L}_{n-1}] + N[\mathcal{L}_1,\mathcal{L}_2,\ldots,\mathcal{L}_{n-2}] \\
&= \mathcal{L}_n N[\mathcal{L}_1,\mathcal{L}_2,\ldots,\mathcal{L}_{n-1}] + N[\mathcal{L}_1,\mathcal{L}_2,\ldots,\mathcal{L}_{n-2}] \\
&= N[\mathcal{L}_1,\mathcal{L}_2,\ldots,\mathcal{L}_n]
\end{aligned}
$$

In the case where $n > 2$ is even, our argument is very similar. Assume that for all $m < n$ our statement holds. In this case, $y_{34} = 1$ and $y_{56} = \prod_{j=\ell_{n-2}}^{(\ell_n)-1} y_j$. Again, we make the corresponding replacements based on our induction hypothesis.

$$
F(\mathcal{G}[a_1,\ldots,a_n]) = N[\mathcal{L}_1,\ldots,\mathcal{L}_{n-1}]F(\mathcal{H}_n) + \left(\prod_{j=\ell_{n-2}}^{(\ell_n)-1} y_j\right) C_{n-2}^{-1}N[\mathcal{L}_1,\ldots,\mathcal{L}_{n-2}])
$$

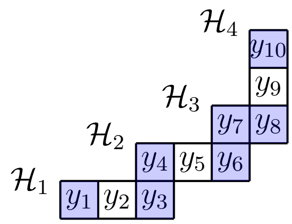
Then we apply Lemma 2.3.4 and the rest follows similarly to the previous case.

$$
\begin{aligned}
F(\mathcal{G}[a_1,\ldots,a_n]) &= N[\mathcal{L}_1,\mathcal{L}_2,\ldots,\mathcal{L}_{n-1}]\varphi_n + \left(\prod_{j=\ell_{n-2}}^{(\ell_n)-1} y_j\right) C_{n-2}^{-1}N[\mathcal{L}_1,\mathcal{L}_2,\ldots,\mathcal{L}_{n-2}] \\
&= N[\mathcal{L}_1,\mathcal{L}_2,\ldots,\mathcal{L}_{n-1}]\varphi_n + C_n^{-1}N[\mathcal{L}_1,\mathcal{L}_2,\ldots,\mathcal{L}_{n-2}] \\
&= C_n^{-1}\left(\varphi_n C_n N[\mathcal{L}_1,\mathcal{L}_2,\ldots,\mathcal{L}_{n-1}] + N[\mathcal{L}_1,\mathcal{L}_2,\ldots,\mathcal{L}_{n-2}]\right) \\
&= C_n^{-1}\left(\mathcal{L}_n N[\mathcal{L}_1,\mathcal{L}_2,\ldots,\mathcal{L}_{n-1}] + N[\mathcal{L}_1,\mathcal{L}_2,\ldots,\mathcal{L}_{n-2}]\right) \\
&= C_n^{-1}N[\mathcal{L}_1,\mathcal{L}_2,\ldots,\mathcal{L}_n]. \qquad \square
\end{aligned}
$$

Next, we apply this result to some examples.

Example 2.3.6. Consider the continued fraction $[2,3,4,2] = \frac{67}{29}$. Because the numerator of the continued fraction is 67, the F-polynomial has 67 terms. According to Theorem 2.3.5, since the continued fraction has an even number of entries, the F-polynomial of the snake graph $\mathcal{G}[2,3,4,2]$ is given by $C_n^{-1}N[\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3,\mathcal{L}_4]$ where

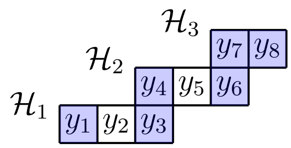
$$\begin{aligned} \mathcal{L}_1 &= 1+y_1 & \mathcal{L}_3 &= (1+y_6+y_6y_7+y_6y_7y_8)(y_1y_2y_3y_4y_5) \\ \mathcal{L}_2 &= \frac{1+y_4+y_3y_4}{y_1y_2y_3y_4} & \mathcal{L}_4 &= \frac{1+y_{10}}{y_1y_2y_3y_4y_5y_6y_7y_8y_9y_{10}} \end{aligned}$$



$\mathcal{G}[2,3,4,2]$

$$\begin{aligned} F(\mathcal{G}[2,3,4,2]) &= C_n^{-1}N[\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3,\mathcal{L}_4] \\ &= \left(\prod_{j=1}^{(\ell_4)-1} y_j\right)(\mathcal{L}_1\mathcal{L}_2\mathcal{L}_3\mathcal{L}_4+\mathcal{L}_1\mathcal{L}_2+\mathcal{L}_1\mathcal{L}_4+\mathcal{L}_3\mathcal{L}_4+1) \\ &= (1+y_1)(1+y_4+y_3y_4)(1+y_6+y_6y_7+y_6y_7y_8)(1+y_{10})y_5 \\ &\quad + (1+y_1)(1+y_4+y_3y_4)y_5y_6y_7y_8y_9y_{10} + (1+y_1)(1+y_{10}) \\ &\quad + (1+y_6+y_6y_7+y_6y_7y_8)(1+y_{10})y_1y_2y_3y_4y_5 \\ &\quad + y_1y_2y_3y_4y_5y_6y_7y_8y_9y_{10} \end{aligned}$$

Example 2.3.7. Consider the continued fraction $[2,3,4]$. Since $[2,3,4]=\frac{30}{13}$, the F-polynomial has 30 terms. According to Theorem 2.3.5, the F-polynomial of the snake graph $\mathcal{G}[2,3,4]$ is given by $N[\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3]$ where $\mathcal{L}_1,\mathcal{L}_2$ and $\mathcal{L}_3$ are the same as in the previous example.



$\mathcal{G}[2,3,4]$

$$\begin{aligned}
F(\mathcal{G}[2,3,4]) = \quad &= N[\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3] \\
&= \mathcal{L}_1\mathcal{L}_2\mathcal{L}_3+\mathcal{L}_1+\mathcal{L}_3 \\
&= (1+y_1)(1+y_4+y_3y_4)(1+y_6+y_6y_7+y_6y_7y_8)y_5+(1+y_1) \\
&\quad + (1+y_6+y_6y_7+y_6y_7y_8)y_1y_2y_3y_4y_5
\end{aligned}$$

Chapter 3

Lattice Paths

3.1 Connections between lattice paths, snake graphs and continued fractions

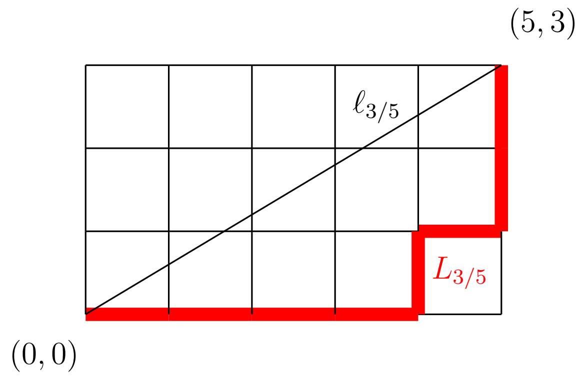
In this section, we discuss a method of associating a continued fraction to a specific lattice path, then discuss what characteristics these continued fractions share.

Let $p < q$ be integers and consider the line segment $\ell_{p/q}$ with slope p/q from the origin to the point (q, p). This line segment is the diagonal of the rectangle in $\mathbb{R}^2$ with vertices at $(0, 0)$ and (q, p) which we call the (q, p)-rectangle. In general, a lattice path is a north-east path in the $\mathbb{Z} \times \mathbb{Z}$ lattice. For our purposes, we only consider the lattice paths defined in Definition 3.1.1.

Definition 3.1.1. $L_{p/q}$ is a lattice path not exceeding the diagonal from $(0, 0)$ to (q, p). We refer to $L_{p/q}$ as a lattice path in the (q, p)-rectangle.

Each (q, p)-rectangle has finitely many lattice paths. In the case when p and q are relatively prime, every $L_{p/q}$ lies strictly below $\ell_{p/q}$.

Example 3.1.2. Depicted below is one of the lattice paths in the $(5,3)$-rectangle. The word describing the lattice path shown is $xxxxyxyy$.

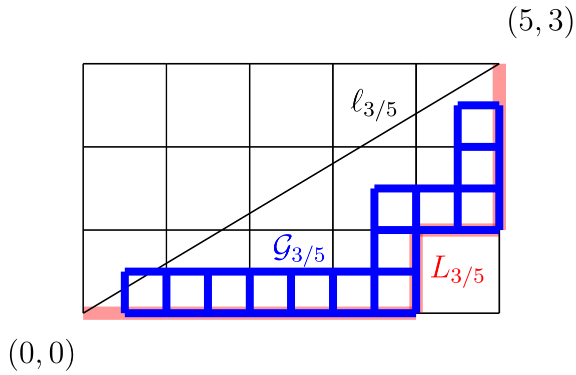


A lattice path can be described by a sequence of x's and y's called a word. An x represents a horizontal line segment of unit length and a y represents a vertical line segment of unit length. There are five x's and three y's in the word because the $(5,3)$-rectangle has length 5 and height 3. In general, a lattice path in a (q,p)-rectangle will have q many x's and p many y's. From a lattice path in a (q,p)-rectangle, we can construct what we call a lattice path snake graph.

Definition 3.1.3. A *lattice path snake graph*, $\mathcal{G}_{p/q}$, is the unique snake graph with half unit length tiles, lying on a lattice path, $L_{p/q}$, such that the south west vertex of the first tile is $(0.5,0)$ and the north east vertex of the last tile is $(q,p-0.5)$.

By this construction, any lattice path snake graph in the (q,p)-rectangle will have $2q+2p-3$ tiles.

Example 3.1.4. Starting at the point $(0.5,0)$, we construct a snake graph with square tiles of half unit length such that each tile shares at least one edge with the lattice path.



Next we associate a continued fraction to the lattice path snake graph in order to easily compute the number of perfect matchings of the snake graph. The continued fraction of a lattice path snake graph is determined by the sign function as in Section 2.2. Because the lattice path snake graphs lie on a lattice path, their shape and continued fraction is somewhat predetermined. In Lemma 3.1.5 we observe some characteristics of these continued fractions.

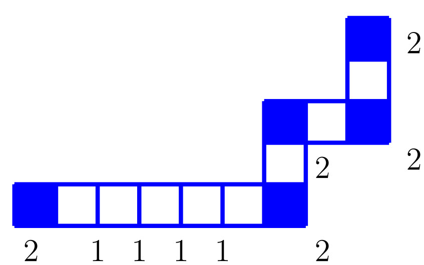
Lemma 3.1.5. *Let $[a_1, \ldots, a_n]$ be the continued fraction associated to a lattice path snake graph in the (q, p)-rectangle, then*

1. *by convention $a_1 = a_n = 2$;*
2. *$a_i \in \{2, 1\}$ for $1 \leq i \leq n$;*
3. *any maximal subsequence of all 1's has even length;*
4. *any maximal subsequence in $a_2, \ldots, a_{n-1}$ of all 2's has even length;*
5. $\displaystyle\sum_{i=1}^{n} a_i = 2q + 2p - 2.$

Remark 3.1.6. The convention which we use to define the sign function of a snake graph in Section 2.2 forces the first and last entry of the continued fraction to always be greater than one. Thus $a_1 = a_n = 2$.

In addition to using the sign function of the lattice path snake graph to determine the continued fraction, we can use the following shading process. Shade the first and last tiles in the snake graph, then shade any corner tiles. The entries in the continued fraction can then be read off the snake graph. Any shaded tile represents an entry 2 and each interior edge strictly between shaded tiles represents a single entry 1. This method ensures that the numerator of the continued fraction we obtain is equal to the number of perfect matchings of the lattice path snake graph and follows our convention of the continued fraction beginning and ending in 2.

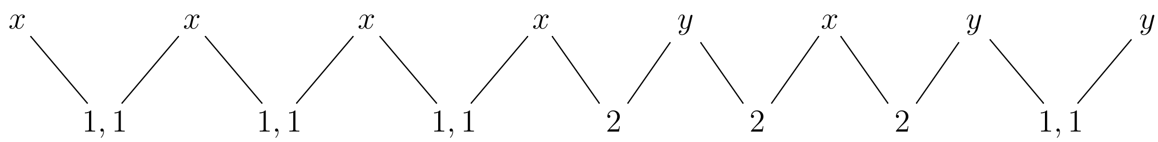
Example 3.1.7. Consider the lattice path snake graph $\mathcal{G}_{3/5}$ that we constructed in Example 3.1.4. Using the shading technique, we find a continued fraction that yields the appropriate numerator. $N[2, 1, 1, 1, 1, 2, 2, 2, 2] = 473$.



The continued fraction associated to a lattice path snake graph can also be constructed from the word associated to the lattice path. Between consecutive letters of the word we insert either a 2 or a pair of ones by the following rule. If the consecutive letters are the same (xx or yy) then we insert 1,1 between them and if the consecutive letters are different (xy or yx) then we insert a 2 between them.

This process provides a continued fraction with the appropriate numerator, however it does not necessarily satisfy our convention in Lemma 3.1.5. To fit our convention, we make a small adjustment. If $a_1 = a_2 = 1$ then we replace a_1, a_2 with 2. Similarly, if $a_{n-1} = a_n = 1$, then we replace a_{n-1}, a_n with 2. Therefore the continued fraction associated to the lattice path snake graph on the lattice path $xxxxyxyy$ is $[2, 1, 1, 1, 1, 2, 2, 2, 2]$ as seen in Example 3.1.8.

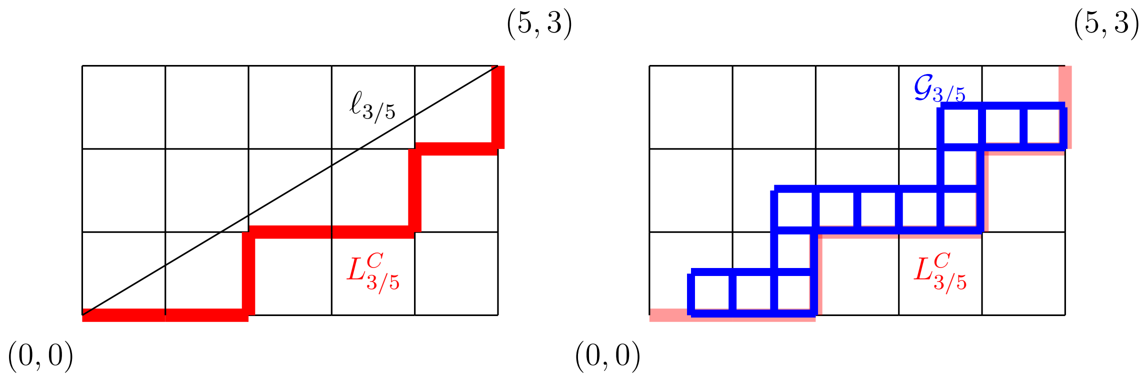
Example 3.1.8. The word for the lattice path in Example 3.1.2 is $xxxxyxyy$. This word initially yields the continued fraction $[1, 1, 1, 1, 1, 1, 2, 2, 2, 1, 1]$ which we change to $[2, 1, 1, 1, 1, 2, 2, 2, 2]$ in order to fit our convention.



3.2 Christoffel lattice paths and their analogs

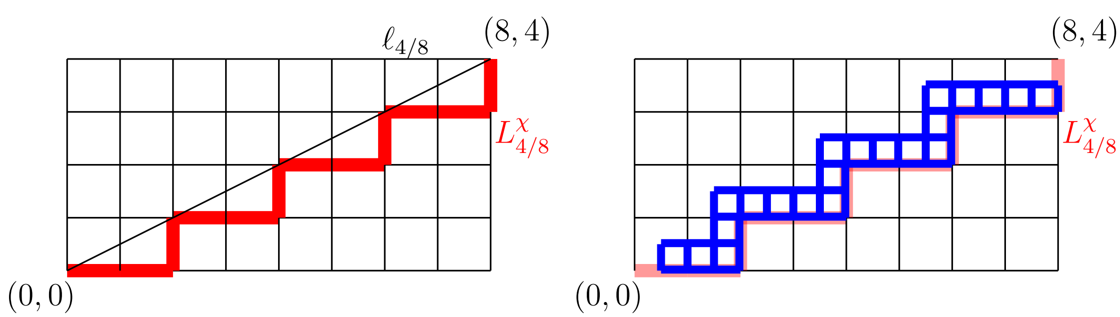
Let p and q be relatively prime. Then there is exactly one lattice path, called the *Christoffel lattice path* $L^C_{p/q}$, in the (q, p)-rectangle such that no lattice points lie strictly between the lattice path and the line segment $\ell_{p/q}$. The lattice path in Example 3.2.1 is a Christoffel lattice path and the lattice path snake graph for this Christoffel lattice path is constructed no differently than for any general lattice path.

Example 3.2.1. The figure on the left shows the Christoffel lattice path $L^C_{3/5}$ in the $(5, 3)$-rectangle. Notice $L^C_{3/5}$ lies strictly below $\ell_{3/5}$ as all lattice paths $L_{p/q}$ do when p and q are relatively prime. On the right, we have constructed the lattice path snake graph $\mathcal{G}_{3/5}$ on the lattice path $L^C_{3/5}$.



In the case when p and q are not relatively prime, there is an analogous path, $L^\chi_{p/q}$. This unique lattice path in the (q,p)-rectangle lies on or below $\ell_{p/q}$ and no lattice points lie strictly between the lattice path and the line segment $\ell_{p/q}$. The lattice path $L^\chi_{p/q}$ will intersect $\ell_{p/q}$ at $\gcd(q,p)+1$ many points, including the origin and (q,p). This is because $L^\chi_{p/q}$ is actually made up of $\gcd(q,p)$ many copies of $L^C_{a/b}$ for relatively prime a and b such that $a/b = p/q$.

Example 3.2.2. Consider the $(8,4)$-rectangle. Here the $\gcd(8,4) = 4$. Notice that the lattice path $L^\chi_{4/8}$ actually touches the line segment $\ell_{4/8}$ in five places including the origin and the endpoint $(8,4)$. Also, $L^\chi_{4/8}$ is actually four concatenated copies of $L^C_{1/2}$. We can see this in the words as well. $L^C_{1/2} = xxy$ and $L^\chi_{4/8} = xxyxxyxxyxxy$.



While the lattice path snake graph $\mathcal{G}^{\chi}_{p/q}$ is constructed on $L^{\chi}_{p/q}$ as any other lattice path snake graph, note that $\mathcal{G}^{\chi}_{p/q}$ is not the concatenation of $\gcd(q,p)$ many copies of $\mathcal{G}^{C}_{a/b}$.

3.3 Ordering lattice paths

Each (q,p)-rectangle has finitely many lattice paths, each with a corresponding lattice path snake graph and hence continued fraction. In this section we discuss the ordering on the number of perfect matchings of these lattice path snake graphs for a fixed (q,p)-rectangle by analyzing continued fractions.

Notice in Example 3.3.2 that if we consider each pair of 1's in the continued fraction as a single entry, then the entries in each continued fraction line up. In other words, if we define this as a new type of entry, the continued fractions all have the same number of these new entries.

Definition 3.3.1. Let $[a_1, a_2, \ldots, a_n]$ be the continued fraction associated to a lattice path snake graph. The sequence $a_1, a_2, ..., a_n$ can be decomposed into the subsequence $\nu_1, ..., \nu_m$ where each $\nu_i = 1, 1$ or $\nu_i = 2$ such that we have an identity of sequences $a_1, a_2, ..., a_n = \nu_1, \nu_2, ..., \nu_m$. Then each ν_i is called a *replaceable entry.*

In general the continued fraction of any lattice path snake graph in a fixed (q,p)-rectangle has $q+p-1$ replaceable entries. The sum of the entries in the continued fraction is one more than the number of tiles and there are $2p$ 2's in the continued fraction. Therefore $2q+2p-2 = 2(\text{number of pairs of 1's}) + 2(2p)$ implies there are $q+p-1$ replaceable entries. From this perspective, the continued fractions only differ by replacing pairs of 1's with 2's and vice versa.

Example 3.3.2. In this example, we order all of the lattice paths in the $(5, 4)$-rectangle by the number of perfect matchings of the associated snake graph, $\mathcal{G}$. In this section, we discuss some ways to determine orderings of the number of perfect matchings based on the continued fraction and the shape of the lattice path.

$\#Match(\mathcal{G})$	$Word$	Continued fraction associated to $\mathcal{G}$							
985	$xxyxyxyxy$	[2	2,	2,	2,	2,	2,	2,	2]
1043	$xxyxyxxyy$	[2	2,	2,	2,	2,	1, 1,	2,	2]
1045	$xxyxxyxyy$	[2	2,	2,	1, 1,	2,	2,	2,	2]
1055	$xxxyxyxyy$	[2	1, 1,	2,	2,	2,	2,	2,	2]
1103	$xxyxxyyxy$	[2	2,	2,	1, 1,	2,	1, 1,	2,	2]
1115	$xxxyyxyxy$	[2	1, 1,	2,	1, 1,	2,	2,	2,	2]
1117	$xxxyxyyxy$	[2	1, 1,	2,	2,	2,	1, 1,	2,	2]
1177	$xxxyyxxyy$	[2	1, 1,	2,	1, 1,	2,	1, 1,	2,	2]
1195	$xxxyxxyyy$	[2	1, 1,	2,	2,	1, 1,	2,	1, 1,	2]
1205	$xxyxxxyyy$	[2	2,	2,	1, 1,	1, 1,	2,	1, 1,	2]
1207	$xxxxyyxyy$	[2	1, 1,	1, 1,	2,	1, 1,	2,	2,	2]
1223	$xxxxyxyyy$	[2	1, 1,	1, 1,	2,	2,	2,	1, 1,	2]
1301	$xxxxyyyxy$	[2	1, 1,	1, 1,	2,	1, 1,	1, 1,	2,	2]
1429	$xxxxxyyyy$	[2	1, 1,	1, 1,	1, 1,	2,	1, 1,	1, 1,	2]

Definition 3.3.3. A *replacement* is an operation on the entries of a continued fraction such that a either $1, 1$ is replaced with 2 or 2 is replaced with 1,1.

$$N[\mu_1, 1, 1, \mu_2] \leftrightarrow N[\mu_1, 2, \mu_2]$$

where μ_1 and μ_2 are a sequence of entries in the continued fraction. We refer the reader to Definition A.0.2.

If either μ_1 or μ_2 are empty, rather than writing $N[1, 1, \mu_2]$ or $N[\mu_1, 1, 1]$, we keep the notation $N[\mu_1, 1, 1, \mu_2]$ and let $\mu_1 = 0, 0$ or respectively $\mu_2 = 0, 0$.

Remark 3.3.4. Here we introduce the idea of letting $\mu_i = 0, 0$ whenever μ_i is an empty sequence. In order to do this, we must introduce Definition A.0.1 and Equations (A.0.8), (A.0.9) and (A.0.10) from the Appendix.

In Lemma 3.3.5 we show that if two continued fractions differ by a single replacement, then the numerator of the continued fraction with 1,1 is greater than or equal to the numerator of the same continued fraction with 1,1 replaced with 2. Moreover we give an explicit equation for the difference in the numerators.

Lemma 3.3.5. *Let each μ_i either be a sequence of entries in $\mathbb{Z}^+$ or equal to $0, 0$. Then*

$$N[\mu_1, 1, 1, \mu_2] - N[\mu_1, 2, \mu_2] = N[\mu_1^-]N[{}^-\mu_2] \tag{3.3.1}$$

For the definition of $N[\mu^-]$, we refer the reader to Definition A.0.2 in Appendix A.

Remark 3.3.6. In the case that the replacement occurs in the first or last entry, we use $\mu_1 = 0, 0$ and respectively $\mu_2 = 0, 0$ as a place holder rather than letting μ_i

be empty. It should be clear in this case that $N[\mu_1, 1, 1, \mu_2] - N[\mu_1, 2, \mu_2] = 0$ from Equations (A.0.5) and (A.0.6).

Remark 3.3.7. It is important to note that Lemma 3.3.5 holds for positive continued fractions in general. In this section and in Chapter 4 we apply this result to analze lattice path snake graphs.

Proof. Consider the case $\mu_1 = 0, 0$. By Equation (A.0.9) and Equation (A.0.5), we know that the left side of Equation (3.3.1), $N[0, 0, 1, 1, \mu_2] - N[0, 0, 2, \mu_2]$, is equal to zero. Since the right side of Equation (3.3.1) is $N[\mu_1^-]N[^-\mu_2] = N[0]N[^-\mu_2] = 0$ in this case, the statement holds.

Next we consider the case $\mu_2 = 0, 0$. We use Equation (A.0.10) to write the left side of Equation (3.3.1) as $N[\mu_1, 1, 1, 0, 0] - N[\mu_1, 2, 0, 0] = N[\mu_1, 1, 1] - N[\mu_1, 2]$. Then by Equation (A.0.4), this is equal to zero. Since the right side of Equation (3.3.1) is $N[\mu_1^-]N[^-\mu_2] = N[\mu_1^-]N[0] = 0$, the statement holds when $\mu_2 = 0, 0$.

Next, suppose μ_1 and μ_2 are sequences in $\mathbb{Z}^+$. We will prove the statement by induction on the number of entries before the $1, 1$ in the first continued fraction. For our base case we let $\mu_1 = a_1$. We apply Equation (A.0.1) to both numerators in the expression on the left hand side of Equation (3.3.1).

$$\begin{aligned} N[a_1, 1, 1, \mu_2] - N[a_1, 2, \mu_2] &= a_1 N[1, 1, \mu_2] + N[1, \mu_2] - (a_1 N[2, \mu_2] + N[\mu_2]) \\ &= a_1 (N[1, 1, \mu_2] - N[2, \mu_2]) + N[1, \mu_2] - N[\mu_2] \end{aligned}$$

Since $N[1, 1, \mu_2] = N[2, \mu_2]$, the first term is zero. We can decompose the second term using Equation (A.0.1), then combine like terms.

$$\begin{aligned} N[a_1,1,1,\mu_2] - N[a_1,2,\mu_2] &= N[1,\mu_2] - N[\mu_2] \\ &= 1N[\mu_2] + N[^-\mu_2] - N[\mu_2] \\ &= N[^-\mu_2] \end{aligned}$$

Since $\mu_1 = a_1$, we have that $N[\mu_1^-] = N[\ ] = 1$ and therefore the right side of Equation (3.3.1) is $N[\mu_1^-]N[^-\mu_2] = N[\ ]N[^-\mu_2] = N[^-\mu_2]$. Therefore the statement holds in the base case.

Next, let μ_1 have $n > 1$ entries and assume Equation (3.3.1) holds for any μ_1 with n or less entries. We would like to prove that $N[a_0,\mu_1,1,1,\mu_2] - N[a_0,\mu_1,2,\mu_2] = N[a_0,\mu_1^-]N[^-\mu_2]$. We apply Equation (A.0.1) and then regroup the expression.

$N[a_0,\mu_1,1,1,\mu_2] - N[a_0,\mu_1,2,\mu_2]$

$$\begin{aligned} &= a_0N[\mu_1,1,1,\mu_2] + N[^-\mu_1,1,1,\mu_2] - (a_0N[\mu_1,2,\mu_2] + N[^-\mu_1,2,\mu_2]) \\ &= a_0\left(N[\mu_1,1,1,\mu_2] - N[\mu_1,2,\mu_2]\right) + N[^-\mu_1,1,1,\mu_2] - N[^-\mu_1,2,\mu_2] \end{aligned}$$

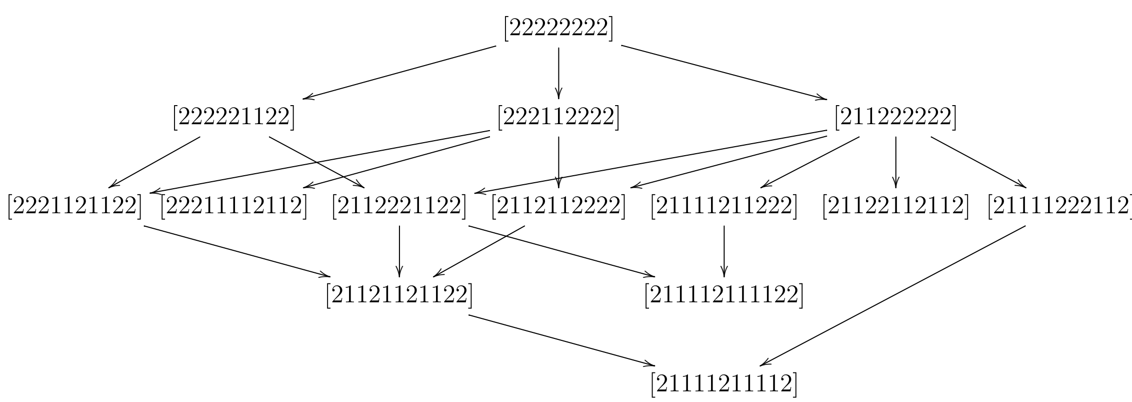
Applying our induction hypothesis to each difference, we see that this expression is equal to

$$\begin{aligned} & a_0N[\mu_1^-]N[^-\mu_2] + N[^-\mu_1^-]N[^-\mu_2] \\ =\ & (a_0N[\mu_1^-] + N[^-\mu_1^-])N[^-\mu_2] \\ =\ & N[a_0,\mu_1^-]N[^-\mu_2], \end{aligned}$$

where the last identity holds by Equation (A.0.1). Therefore Lemma 3.3.5 is proved by induction. $\square$

Clearly if $\mu_1 \neq 0, 0$ and $\mu_2 \neq 0, 0$ then $N[\mu_1, 1, 1, \mu_2] - N[\mu_1, 2, \mu_2] = N[\mu_1^-]N[^-\mu_2]$ is greater than zero. Therefore we can order the continued fractions in Example 3.3.2 by replacements.

Example 3.3.8. This poset shows the ordering stipulated by replacement. Continued fractions appearing on the same level cannot be compared by this operation. A continued fraction connected to another below it by an edge has a larger numerator that the continued fraction it is connected to. For space we have left out the commas, but every entry in the continued fractions is either 1 or 2.



In Example 3.3.8 replacement gives a partial order on all of the lattice paths. In this example, the partial order yields one connected component. However, this is not always the case.

Example 3.3.9. When ordering the set of lattice paths in the $(5, 3)$-rectangle by replacement, we obtain a partial order with two connected components. Again we have left out the commas in the poset below for space. Every entry in the continued fractions is 1 or 2.

$$\begin{tikzcd}[column sep=small] & {[22211222]} \arrow[dl] \arrow[dr] & \\ {[222111122]} & & {[21111112112]} \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & {[21122222]} \arrow[dl] \arrow[dr] & \\ {[211221122]} \arrow[dr] & & {[211112222]} \arrow[dl] \\ & {[2111121122]} & \end{tikzcd}$$

$\#Match(\mathcal{G})$	Continued fraction associated to $\mathcal{G}$
433	$[2, \quad 2, \quad 2, \quad 1,1, \quad 2, \quad 2, \quad 2]$
437	$[2, \quad 1,1, \quad 2, \quad 2, \quad 2, \quad 2, \quad 2]$
463	$[2, \quad 1,1, \quad 2, \quad 2, \quad 1,1, \quad 2, \quad 2]$
467	$[2, \quad 2, \quad 2, \quad 1,1, \quad 1,1, \quad 2, \quad 2]$
473	$[2, \quad 1,1, \quad 1,1, \quad 2, \quad 2, \quad 2, \quad 2]$
499	$[2, \quad 1,1, \quad 1,1, \quad 2, \quad 1,1, \quad 2, \quad 2]$
547	$[2, \quad 1,1, \quad 1,1, \quad 1,1, \quad 2, \quad 1,1, \quad 2]$

Example 3.3.9 highlights that ordering by replacement is not always a dependable method. Thus we consider another strategy. The following method uses a more visual perspective, looking at the shape of the lattice paths rather than analyzing a change in the continued fraction. Our goal is to provide a linear ordering of lattice paths in the (q, p)-rectangle from the *L-shaped* lattice path $L^L_{p/q}$ (with word consisting of q many $x's$ followed by p many y's, i.e. $x^q y^p$) to the Christoffel lattice path $L^C_{p/q}$ or analogously the lattice path $L^\chi_{p/q}$ when p and q are not relatively prime. First we prove that the L-shaped lattice path's snake graph has more perfect matchings than

any arbitrary lattice path snake graph with lattice of the form $L^W_{p/q} = x^{m_1}y^{n_1}x^{m_2}y^{n_2}$ such that $m_1 + m_2 = q$, $n_1 + n_2 = p$ and $m_1, m_2, n_1, n_2 \neq 0$. We call this a *W-shaped* lattice path.

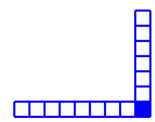
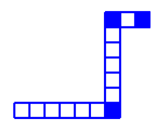
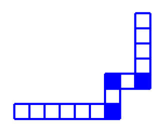
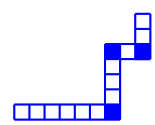
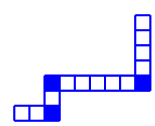
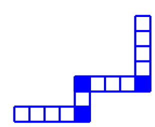
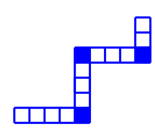
Example 3.3.10. For example in the $(5,4)$-rectangle, the L-shaped lattice path is given by the word $xxxxxyyyy = x^5y^4$. There are six W-shaped lattice paths given by the words

$$\begin{aligned} &xxxxyyyxy = x^4y^3xy, && xxxxyxyyy = x^4yxy^3, && xxxxyyxyy = x^4y^2xy^2,\\ &xxyxxxyyy = x^2yx^3y^3, && xxxyxxyyy = x^3yx^2y^3, && xxxyyxxyy = x^3y^2x^2y^2. \end{aligned}$$

Another way to describe the lattice path is by its lattice path snake graph. An L-shaped lattice path has an L-shaped lattice path snake graph with h_1 horizontal tiles, one corner tile and v_1 vertical tiles. Thus we describe $\mathcal{G}_L$ by writing its tile description, $\mathcal{G}_L = (h_1, v_1)$. Similarly, a W-shaped lattice path has a W-shaped lattice path snake graph denoted by its tile description $\mathcal{G}_W = (h_1, v_1, h_2, v_2)$. In the case of an L-shaped snake graph in a (q,p)-rectangle, we know that $h_1 = 2q-2$ and $v_1 = 2p-2$.

The tile description of a snake graph will be useful to us in the following proofs because it allows us to break up our snake graphs into straight subgraphs. If $\mathcal{G}$ is a straight snake graph with n tiles, then the number of perfect matchings of the snake graph is the numerator of the continued fraction with $n+1$ entries all of which are ones. The numerator of a continued fraction consisting only of $n+1$ many ones is the Fibonacci number F_{n+2} where $F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 2, \ldots$. Therefore for a straight snake graph $\mathcal{G}$, $\#\mathrm{Match}(\mathcal{G}) = F_{n+2}$. Thus when we consider L or W shaped snake graphs, we can graft together straight subgraphs to obtain sums of products of Fibonacci numbers rather than numerators of continued fractions.

Example 3.3.11. In the $(5,4)$-rectangle, the L-shaped lattice path yields a snake graph with more perfect matchings than any W-shaped lattice path snake graph as stated in Theorem 3.3.12.

Lattice path shape	#Match($\mathcal{G}$)	Word	Tile description	Snake graph
L	1429	x^5y^4	$(8,6)$	
W	1301	x^4y^3xy	$(6,5,1,0)$	
W	1223	x^4yxy^3	$(6,1,1,4)$	
W	1207	$x^4y^2xy^2$	$(6,3,1,2)$	
W	1205	$x^2yx^3y^3$	$(2,1,5,4)$	
W	1195	$x^3yx^2y^3$	$(4,1,3,4)$	
W	1177	$x^3y^2x^2y^2$	$(4,3,3,2)$	

Theorem 3.3.12. *Fix the (q,p)-rectangle. Let $\mathcal{G}_L$ be the lattice path snake graph on the L-shaped lattice path $L^L_{p/q} = x^qy^p$. Let $\mathcal{G}_W$ be the lattice path snake graph on a*

W-shaped lattice path $L^W_{p/q} = x^{m_1}y^{n_1}x^{m_2}y^{n_2}$ *where* $m_1 + m_2 = q$, $n_1 + n_2 = p$ *and* $m_1, m_2, n_1, n_2 \neq 0$. *Then the number of perfect matchings of* $\mathcal{G}_L$ *is greater than the number of perfect matchings of* $\mathcal{G}_W$. *i.e.*

$$\#Match(\mathcal{G}_W) < \#Match(\mathcal{G}_L) \quad \textit{for any W-shaped lattice path snake graph.}$$

Proof. In this proof, we will use the following properties of the Fibonacci numbers:

$$F_{n+1} = F_n + F_{n-1} \tag{3.3.2}$$

$$F_{m+n} = F_m F_{n+1} + F_{m-1} F_n \tag{3.3.3}$$

This proof consists of six cases. In all of these cases, $\mathcal{G}_W$ is given by (h_1, v_1, h_2, v_2) and $\mathcal{G}_L$ is given by $(h_1 + h_2, v_1 + v_2)$ as in Figure 3.3.1 where $\mathcal{G}_W$ and $\mathcal{G}_L$ are snake graphs in the same fixed p/q rectangle. Note that in order to be a snake graph in a p/q rectangle, $h_1 > 0$, $v_2 \geq 0$ must be even and $h_2, v_1 \geq 1$ must be odd, however the proof holds for integer values within the constraints regardless of parity.

Case 1: $h_1, v_2 \geq 1$ and $h_2, v_1 \geq 2$.

By grafting at the corner of $\mathcal{G}_L$, we obtain

$$\#Match(\mathcal{G}_L) = F_{h_1+h_2+2}F_{v_1+v_2+3} + F_{h_1+h_2+3}F_{v_1+v_2+2}.$$

Then by grafting in each of the three corners of $\mathcal{G}_W$ consecutively, we obtain the

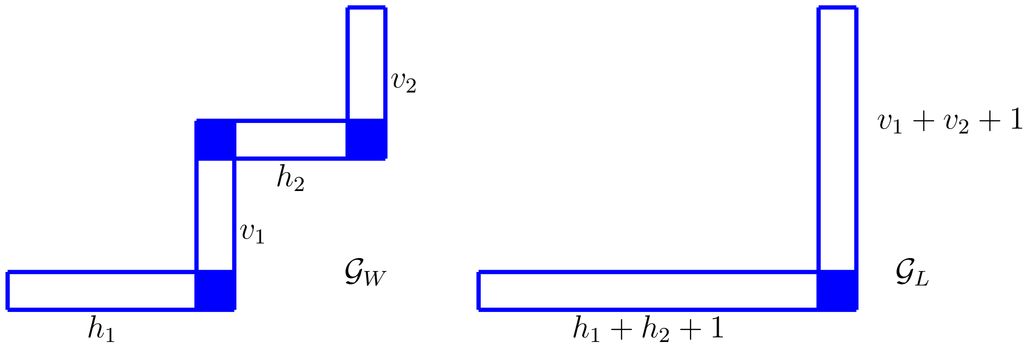


FIGURE 3.3.1: Give the tile description of the W-shaped lattice path snake graph, (h_1, v_1, h_2, v_2) first. Since both snakes lie in the (q, p)-rectangle, the tile description of the L-shaped lattice path snake graph, $(h_1 + h_2 + 1, v_1 + v_2 + 1)$, can be written in terms of the tile description of the W-shaped lattice path snake graph.

following expression for $\#Match(\mathcal{G}_W)$.

$$F_{h_1+1}(F_{v_1+1}(F_{h_2+1}F_{v_2+2} + F_{h_2+2}F_{v_2+1}) + F_{v_1+2}(F_{h_2}F_{v_2+2} + F_{h_2+1}F_{v_2+1}))$$
$$+F_{h_1+2}(F_{v_1}(F_{h_2+1}F_{v_2+2} + F_{h_2+2}F_{v_2+1}) + F_{v_1+1}(F_{h_2}F_{v_2+2} + F_{h_2+1}F_{v_2+1})$$

Then we distribute to obtain an equivalent expression.

$$F_{h_1+1}F_{v_1+1}F_{h_2+1}F_{v_2+2} + F_{h_1+1}F_{v_1+1}F_{h_2+2}F_{v_2+1} + F_{h_1+1}F_{v_1+2}F_{h_2}F_{v_2+2}$$
$$+F_{h_1+1}F_{v_1+2}F_{h_2+1}F_{v_2+1} + F_{h_1+2}F_{v_1}F_{h_2+1}F_{v_2+2} + F_{h_1+2}F_{v_1}F_{h_2+2}F_{v_2+1}$$
$$+F_{h_1+2}F_{v_1+1}F_{h_2}F_{v_2+2} + F_{h_1+2}F_{v_1+1}F_{h_2+1}F_{v_2+1}$$

Next we apply Equation (3.3.2) to F_{v_1+2} and to F_{h_1+2} every place they appear.

$$F_{h_1+1}F_{v_1+1}F_{h_2+1}F_{v_2+2} + F_{h_1+1}F_{v_1+1}F_{h_2+2}F_{v_2+1} + F_{h_1+1}F_{v_1}F_{h_2}F_{v_2+2}$$
$$+F_{h_1+1}F_{v_1}F_{h_2+1}F_{v_2+1} + F_{h_1+1}F_{v_1}F_{h_2+1}F_{v_2+2} + F_{h_1}F_{v_1}F_{h_2+1}F_{v_2+2}$$
$$+F_{h_1+1}F_{v_1}F_{h_2+2}F_{v_2+1} + F_{h_1}F_{v_1}F_{h_2+2}F_{v_2+1} + 2F_{h_1+1}F_{v_1+1}F_{h_2}F_{v_2+2}$$
$$+F_{h_1}F_{v_1+1}F_{h_2}F_{v_2+2} + 2F_{h_1+1}F_{v_1+1}F_{h_2+1}F_{v_2+1} + F_{h_1}F_{v_1+1}F_{h_2+1}F_{v_2+1}$$

We would like to compute the difference $\#Match(\mathcal{G}_L) - \#Match(\mathcal{G}_W)$, so first we will rewrite $\#Match(\mathcal{G}_L)$ into a more comparable expression. We apply Equation 3.3.3 to each Fibonacci number in the equation for the number of perfect matchings of the L-shaped snake graph $\#Match(\mathcal{G}_L) = F_{h_1+h_2+2}F_{v_1+v_2+3}+F_{h_1+h_2+3}F_{v_1+v_2+2}$.

$$\begin{aligned}\#Match(\mathcal{G}_L) &= (F_{h_1+1}F_{h_2+2} + F_{h_1}F_{h_2+1})(F_{v_1+1}F_{v_2+3} + F_{v_1}F_{v_2+2}) \\ &\quad +(F_{h_1+1}F_{h_2+3} + F_{h_1}F_{h_2+2})(F_{v_1+1}F_{v_2+2} + F_{v_1}F_{v_2+1})\end{aligned}$$

Multiplication yields the equivalent expression.

$$F_{h_1+1}F_{v_1+1}F_{h_2+2}F_{v_2+3} + F_{h_1+1}F_{v_1}F_{h_2+2}F_{v_2+2} + F_{h_1}F_{v_1+1}F_{h_2+1}F_{v_2+3}$$
$$+F_{h_1}F_{v_1}F_{h_2+1}F_{v_2+2} + F_{h_1+1}F_{v_1+1}F_{h_2+3}F_{v_2+2} + F_{h_1+1}F_{v_1}F_{h_2+3}F_{v_2+1}$$
$$+F_{h_1}F_{v_1+1}F_{h_2+2}F_{v_2+2} + F_{h_1}F_{v_1}F_{h_2+2}F_{v_2+1}$$

Then we apply Equation 3.3.2 to F_{v_2+3} and F_{h_2+3} in every place that they appear.

$$\begin{aligned}
&2F_{h_1+1}F_{v_1+1}F_{h_2+2}F_{v_2+2} + F_{h_1+1}F_{v_1+1}F_{h_2+2}F_{v_2+1} + F_{h_1+1}F_{v_1}F_{h_2+2}F_{v_2+2}\\
&+F_{h_1}F_{v_1+1}F_{h_2+1}F_{v_2+2} + F_{h_1}F_{v_1+1}F_{h_2+1}F_{v_2+1} + F_{h_1}F_{v_1}F_{h_2+1}F_{v_2+2}\\
&+F_{h_1+1}F_{v_1+1}F_{h_2+1}F_{v_2+2} + F_{h_1+1}F_{v_1}F_{h_2+2}F_{v_2+1} + F_{h_1+1}F_{v_1}F_{h_2+1}F_{v_2+1}\\
&+F_{h_1}F_{v_1+1}F_{h_2+2}F_{v_2+2} + F_{h_1}F_{v_1}F_{h_2+2}F_{v_2+1}
\end{aligned}$$

Now we compute the difference, $\#Match(\mathcal{G}_L) - \#Match(\mathcal{G}_W)$

$$\begin{aligned}
= \quad &2F_{h_1+1}F_{v_1+1}F_{h_2+2}F_{v_2+2} + F_{h_1+1}F_{v_1}F_{h_2+2}F_{v_2+2} + F_{h_1}F_{v_1+1}F_{h_2+1}F_{v_2+2}\\
&+F_{h_1}F_{v_1+1}F_{h_2+2}F_{v_2+2} - F_{h_1+1}F_{v_1}F_{h_2}F_{v_2+2} - F_{h_1+1}F_{v_1}F_{h_2+1}F_{v_2+2}\\
&-2F_{h_1+1}F_{v_1+1}F_{h_2}F_{v_2+2} - F_{h_1}F_{v_1+1}F_{h_2}F_{v_2+2} - 2F_{h_1+1}F_{v_1+1}F_{h_2+1}F_{v_2+1}
\end{aligned}$$

and simplify using Equation 3.3.2 to obtain

$$2F_{h_1+1}F_{v_1+1}F_{h_2+1}F_{v_2} + 2F_{h_1}F_{v_1+1}F_{h_2+1}F_{v_2+2}.$$

Since $h_1, v_2 \geq 1$ and $h_2, v_1 \geq 2$, we have that the above sum is greater than zero. Therefore $\#Match(\mathcal{G}_W) < \#Match(\mathcal{G}_L)$.

Remaining Cases: In the remaining cases one can graft in convenient tiles to obtain a sum of products of Fibonacci numbers. Then obtain the differences in Figure 3.3.2 in an analogous way.

Although each of these cases needs to be proved individually by grafting (proof omitted), once we have the differences in each case, we can see that each is merely a restriction of Case 1 since $F_0 = 0, F_1 = 1$ and $F_2 = 1$. Furthermore, since each of

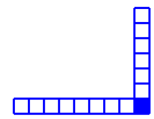
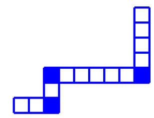
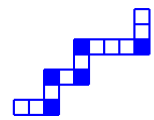
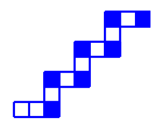
Case #	h_1	v_1	h_2	v_2	$\#Match(\mathcal{G}_L) - \#Match(\mathcal{G}_W)$
1	≥ 1	≥ 2	≥ 2	≥ 1	$2F_{h_1+1}F_{v_1+1}F_{h_2+1}F_{v_2} + 2F_{h_1}F_{v_1+1}F_{h_2+1}F_{v_2+2}$
2	≥ 1	1	≥ 2	≥ 1	$2F_{h_1+1}F_{h_2+1}F_{v_2} + 2F_{h_1}F_{h_2+1}F_{v_2+2}$
3	≥ 1	1	≥ 1	0	$2F_{h_1}F_{h_2+1}$
4	≥ 1	≥ 2	≥ 1	0	$2F_{h_1}F_{v_1+1}F_{h_2+1}$
5	≥ 1	1	1	≥ 1	$2F_{h_1+1}F_{v_2} + 2F_{h_1}F_{v_2+2}$
6	≥ 1	≥ 2	1	≥ 1	$2F_{h_1+1}F_{v_1+1}F_{v_2} + 2F_{h_1}F_{v_1+1}F_{v_2+2}$

FIGURE 3.3.2: Cases showing $\#Match(\mathcal{G}_L) - \#Match(\mathcal{G}_W) > 0$.

these differences is positive, we know that $\#Match(\mathcal{G}_W) < \#Match(\mathcal{G}_L)$. $\square$

Now we are able to prove a linear ordering from the L-shaped lattice path to the Christoffel lattice path or its analog.

Example 3.3.13. In Theorem 3.3.14 we show that gradually building the Christoffel lattice path or its analog from the L-shaped lattice path decreases the number of perfect matchings of the lattice path snake graph at each step. In this example we name and give the word for each lattice path we can compare in the $(5, 4)$-rectangle using this theorem.

Lattice path	#Match($\mathcal{G}$)	Word	Tile description	Snake graph
$L^{L}_{4/5}$	1429	x^5y^4	$(8,6)$	
$L^{W_1}_{4/5}$	1205	$x^2yx^3y^3$	$(2,1,5,4)$	
$L^{W_2}_{4/5}$	1043	$x^2yxyx^2y^2$	$(2,1,1,1,3,2)$	
$L^{W_3}_{4/5} = L^{C}_{4/5}$	985	$x^2yxyxyxy$	$(2,1,1,1,1,1,1,0)$	

Theorem 3.3.14. *Fix the* (q,p)*-rectangle. Let* $\mathcal{G}_L$ *be the lattice path snake graph on the L-shaped lattice* $L^L_{p/q} = x^q y^p$*. Then define* $\mathcal{G}_{W_i}$ *to be the lattice path snake graph on the lattice path given by* $L^{W_i}_{p/q} = x^{a_1}yx^{a_2}y\cdots x^{a_i}yx^{b_i}y^{p-i}$ *where* $a_i = \left\lceil \frac{iq}{p} \right\rceil - \sum_{j=1}^{i-1} a_j$*,* $bi = q - \sum_{j=1}^{i} a_j$ *and* $\sum_{j=1}^{0} a_j = 0$ *for* $1 \leq i \leq p-1$*. Then*

$$\#Match(\mathcal{G}_{W_{p-1}}) < \cdots < \#Match(\mathcal{G}_{W_1}) < \#Match(\mathcal{G}_L)$$

We define $L^{W_i}_{p/q}$ to be the lattice path in the (q,p)-rectangle estimating the Christoffel lattice path (or its analog) correct to i steps. More rigorously, this means that the beginning of the word for $L^{W_i}_{p/q}$ matches the beginning of the Christoffel word (or its analog). The Christoffel word (or its analog) and $L_{p/q}^{W_i}$ both begin with $x^{a_1}yx^{a_2}y\cdots x^{a_i}y$. See Example 3.3.13. In order for this to be true, we must have that the vertical distance between $\ell_{p/q}$ and the lattice point $(\sum_{j=1}^{i} a_j, i)$ is as small as possible. Thus for each $1 \leq i \leq p-1$ we need

$$0 \leq \frac{p}{q}\left(\sum_{j=1}^{i} a_j\right) - i < 1.$$

Therefore

$$\frac{iq}{p} \leq \sum_{j=1}^{i} a_j < \frac{(i+1)q}{p}.$$

Since the x-coordinate we need must be the first integer satisfying this inequality, we have that $\displaystyle\sum_{j=1}^{i} a_j = \left\lceil \frac{iq}{p} \right\rceil$ and hence $a_i = \displaystyle\left\lceil \frac{iq}{p} \right\rceil - \sum_{j=1}^{i-1} a_j$. Now we can prove Theorem 3.3.14.

Proof. We already know that $\#Match(\mathcal{G}_{W_1}) < \#Match(\mathcal{G}_L)$ by Theorem 3.3.12. Next we would like to prove that $\#Match(\mathcal{G}_{W_{i+1}}) < \#Match(\mathcal{G}_{W_i})$ for all $i = 1, \ldots, p-1$. Since $i \neq p$, we know that the lattice path snake graph $\mathcal{G}_{W_i}$ ends in a vertical segment of at least two tiles. Therefore $\mathcal{G}_{W_i}$ ends in an L-shape and we can graft in the second to last corner tile of $\mathcal{G}_{W_i}$. See Figure 3.3.3.

$$\#Match(\mathcal{G}_{W_i}) = (\#Match(\mathcal{G}_A))(\#Match(\mathcal{G}_B)) + (\#Match(\mathcal{G}_{A'}))(\#Match(\mathcal{G}_{B'})).$$

Next we replace the tiles forming $\mathcal{G}_B$ with the snake graph $\mathcal{G}_W$ and $\mathcal{G}_{B'}$ with $\mathcal{G}_{W'}$ in

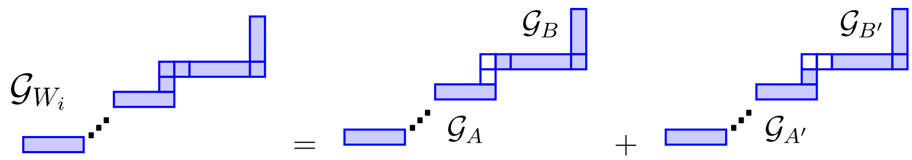


FIGURE 3.3.3: Here, we depict the grafting of $\mathcal{G}_{W_i}$ by shading the subgraphs $\mathcal{G}_A, \mathcal{G}_B$ and $\mathcal{G}_{A'}, \mathcal{G}_{B'}$. This grafting process gives us the following equation for the number of prefect matchings of $\mathcal{G}_{W_i}$.
$\#Match(\mathcal{G}_{W_i}) = (\#Match(\mathcal{G}_A))(\#Match(\mathcal{G}_B)) + (\#Match(\mathcal{G}_{A'}))(\#Match(\mathcal{G}_{B'}))$.

such a way that

$$\#Match(\mathcal{G}_{W_{i+1}}) = (\#Match(\mathcal{G}_A))(\#Match(\mathcal{G}_W)) + (\#Match(\mathcal{G}_{A'}))(\#Match(\mathcal{G}_{W'})).$$

In doing this, $\mathcal{G}_W = (h_1, 1, h_2, v_2)$ where $h_1, h_2 \geq 1$, and $v_2 \geq 0$ (even) and $\mathcal{G}_{W'} = (h_1 - 1, 1, h_2, v_2)$. Then we can compute the difference

$\#Match(\mathcal{G}_{W_i}) - \#Match(\mathcal{G}_{W_{i+1}})$

$$\#Match(\mathcal{G}_A)(\#Match(\mathcal{G}_B) - \#Match(\mathcal{G}_W))$$
$$+\#Match(\mathcal{G}_{A'})(\#Match(\mathcal{G}_{B'})) - \#Match(\mathcal{G}_{W'})).$$

It should be clear that $\#Match(\mathcal{G}_A)$ and $\#Match(\mathcal{G}_{A'})$ are positive values. We can also use the appropriate case from the table in Figure 3.3.2 to show that $\#Match(\mathcal{G}_B) - \#Match(\mathcal{G}_W)$ is positive, since $\mathcal{G}_B$ is L-shaped. Similarly, if when constructing $\mathcal{G}_{W'}$, we get that $h_1 - 1 \geq 1$, then we can also use the table in Figure 3.3.2 to show that $\#Match(\mathcal{G}_{B'})) - \#Match(\mathcal{G}_{W'})$ is positive since $\mathcal{G}_{B'}$ is L-shaped. However, if $h_1 - 1 = 0$ we need to show more work. The cases where $h_1 = 0$, $h_2 \geq 1$, and $v_2 \geq 0$ are not represented in Figure 3.3.2. However, we can create an extension using the same strategy we used to create the table. See Figure 3.3.4. According to Figure 3.3.4,

Case #	h_1	v_1	h_2	v_2	$\#Match(\mathcal{G}_{B'}) - \#Match(\mathcal{G}_{W'})$
7	0	≥ 1	≥ 3	≥ 2	$2F_{v_1+1}F_{h_2+1}F_{v_2}$
8	0	≥ 1	1	≥ 2	$2F_{v_1+1}F_{v_2}$
9	0	≥ 1	≥ 1	0	0

FIGURE 3.3.4: Additional cases showing $\#Match(\mathcal{G}_{B'}) - \#Match(\mathcal{G}_{W'}) \geq 0$.

$\#Match(\mathcal{G}_{B'})) - \#Match(\mathcal{G}_{W'}) \geq 0$ in general. Regardless, this is enough to tell us that $\#Match(\mathcal{G}_{W_i}) - \#Match(\mathcal{G}_{W_{i+1}}) > 0$. Therefore we have proved

$$\#Match(\mathcal{G}_{W_{p-1}}) < \cdots < \#Match(\mathcal{G}_{W_2}) < \#Match(\mathcal{G}_{W_1}) < \#Match(\mathcal{G}_L). \quad \square$$

Chapter 4

Markov Numbers

In this chapter we discuss a topic from number theory. Andrey Markov (alternate spelling Andrei Markoff) was a Russian mathematician born in 1856 and better known for his work in probability theory. However, his work in number theory is of particular interest to us due to its connections to cluster algebras, lattice path snake graphs and continued fractions.

4.1 Introduction to Markov numbers

Definition 4.1.1. A Markov number is any number in the triple (x, y, z) of positive integer solutions to the Diophantine equation $x^2 + y^2 + z^2 = 3xyz$, known as the Markov equation.

We consider the Markov equation rather than the more general Diophantine equation, $x^2 + y^2 + z^2 = kxyz$, because for $k \neq 1, 3$, this Diophantine equation has only the trivial solution $(0, 0, 0)$. Solutions to this Diophantine equation when $k = 1$ are multiples of 3 times solutions to the Markov equation. Hence the Markov equation is the equation of interest.

The solutions to the Markov equation are called Markov triples. There are two Markov triples with repeating entries, $(1, 1, 1)$ and $(1, 2, 1)$. The rest of the Markov triples are known to be non-singular, meaning that each entry in the triple is distinct. Figure 4.1.1 depicts some of these triples in a binary tree called the Markov tree. All of the non-singular Markov triples can be constructed from this tree. From the vertex (x, y, z) the branch leading below and to the left will be $(x, 3xy - z, y)$ and below to the right will be $(y, 3yz - x, z)$.

Every Markov number appears as the maximum of some Markov triple. Hence the first few Markov numbers are 1, 2, 5, 13, 29, 34, 89, 169, 194, 233, 433, etc. Although it is known that the underlined values in the Markov tree in Figure 4.1.1 provide a complete list of Markov numbers, it has not been proven that the underlined values are all unique. In 1913, Ferdinand Frobenius conjectured that each Markov number appears as the maximum of a *unique* Markov triple. This conjecture inspired many generations of mathematicians in several areas and is the main focus of Martin Aigner's book, [A], which we refer to for the background on Markov numbers.

Notice in Figure 4.1.2 we depict another binary tree, the Farey tree, that is combinatorially equivalent to the Markov tree in Figure 4.1.1. Each vertex in the Farey tree is a Farey triple. When starting with a triple, $\left(\frac{a}{b}, \frac{a+c}{b+d}, \frac{c}{d}\right)$, we produce the next triple to the left and right respectively by

$$\left(\frac{a}{b}, \frac{a+(a+c)}{b+(b+d)}, \frac{a+c}{b+d}\right) \text{ and } \left(\frac{a+c}{b+d}, \frac{(a+c)+c}{(b+d)+d}, \frac{c}{d}\right).$$

The operation to construct the mediant of each triple is called Farey addition, named

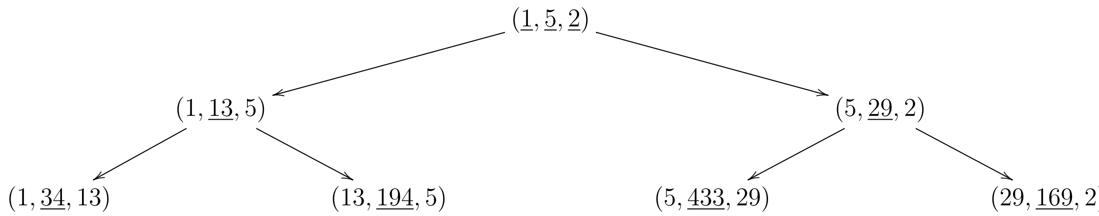


FIGURE 4.1.1: Markov Tree (non-singular triples). The underlined values are $m_{p/q}$ where p/q are values in the same position in the Farey tree.

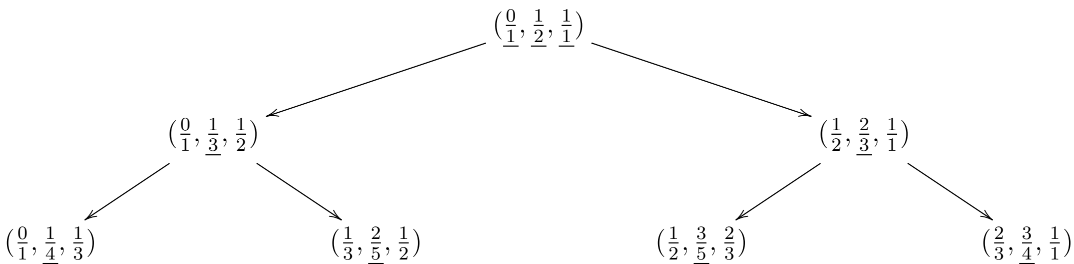


FIGURE 4.1.2: Farey Tree

after geologist John Farey[1]. It is well known that every rational number between zero and one appears as a mediant in the Farey tree exactly once. Moreover these rational numbers are all simplified.

Thus we can now denote each Markov number by its corresponding rational in $\mathbb{Q}_{[0,1]}$ by considering corresponding positions in the combinatorially equivalent trees. For example $m_{1/3} = 13$ and $m_{3/5} = 433$. It is important to note that $m_{p/q}$ denotes a Markov number only when p and q are relatively prime and $p/q \in \mathbb{Q}_{[0,1]}$.

[1] Farey numbers are an example of Stigler's law of eponomy which claims that no scientific discovery is named after its original discoverer (Appropriately, Stigler's law itself is not credited to Stigler). While John Farey published some observations on Farey numbers in 1816, they were proved by Cauchy, and in modern history are accredited to Charles Haros, who published similar results as early as 1802.

4.2 Results on Markov Numbers

In this section we prove two conjectures seen in Martin Aigner's book [A] that determine an ordering on subsets of the Markov numbers based on their corresponding rational.

Conjecture 4.2.1. [A] *(Fixed Numerator Conjecture) Let p, q and i be positive integers such that $p < q$, $\gcd(q, p) = 1$ and $\gcd(q + i, p) = 1$, then $m_{p/q} < m_{p/(q+i)}$.*

Conjecture 4.2.2. [A] *(Fixed Denominator Conjecture) Let p, q and i be positive integers such that $p + i < q$ and $\gcd(q, p) = 1$ and $\gcd(q, p + i) = 1$, then $m_{p/q} < m_{(p+i)/q}$.*

In [BBH, P] it is shown that Markov triples are related to the cluster algebra of the torus with one puncture; namely, the Markov tree is obtained from the exchange graph[2] of the cluster algebra by specializing the initial cluster variables to one. Then, via a formula from [MSW], one can express each Markov number as the number of perfect matchings of an associated graph, called a Markov snake graph. Finally, using results of [CS4, CS5], each Markov number can then be expressed as the numerator of a very particular continued fraction.

This allows us to reformulate the conjectures in terms of continued fractions. To prove the conjectures, we first show several results for continued fractions in general and then apply them to the particular case of the continued fractions of the Markov snake graphs.

Definition 4.2.3. The Markov snake graph, $\mathcal{G}_{p/q}$, is the snake graph with half unit length tiles, lying on the Christoffel lattice path $L_{p/q}$ such that the south west vertex

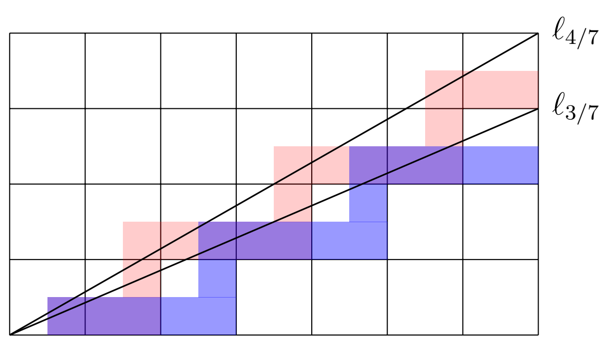
[2] See Definition 2.1.2.

of the first tile is $(0.5, 0)$ and the north east vertex of the last tile is $(q, p - 0.5)$.

Example 4.2.4. In this example, we observe that $m_{p/q} < m_{(p+1)/q}$ when $q = 7$ and $p = 3$. In the continued fraction below, we have italicized the different replacements. Notice that the $(q, p+1)$-Markov numerator has one more replaceable entry than the (q, p)-Markov numerator.

$$m_{4/7} = N[\ 2,\ \mathit{2},\ 2,\ \mathit{1},\mathit{1},\ \mathit{2},\ 2,\ \mathit{1},\mathit{1},\ \mathit{2},\ 2,\ 2\ ] = 6,466$$
$$m_{3/7} = N[\ 2,\ \mathit{1},\mathit{1},\ 2,\ \mathit{2},\ \mathit{1},\mathit{1},\ 2,\ \mathit{2},\ \mathit{1},\mathit{1},\ 2\ ] = 2,897$$

The lightly shaded snake graph is $\mathcal{G}[2, 1, 1, 2, 2, 1, 1, 2, 2, 1, 1, 2]$ and the darker snake graph is $\mathcal{G}[2, 2, 2, 1, 1, 2, 2, 1, 1, 2, 2, 2]$. Notice that the replacements occur at the beginning and end of each overlapping subgraph (except the first tile).



Notice Example 4.2.4 satisfies Aigner's Conjecture 4.2.2. In this example, we see that the continued fractions differ by some replacements and the continued fraction associated to $m_{4/7}$ has one more replaceable entry than the continued fraction associated to $m_{3/7}$.

Our goal in this section is ultimately to prove Theorem 4.2.9 which implies Aigner's conjectures. The key to proving this theorem is to analyze the difference in the numerators of the continued fractions associated to the snake graphs. The entries in the continued fractions differ by replacements and the number of replaceable entries.

We already have the exact difference in the numerators of two continued fractions where the only change is a single replacement of 1,1 with 2 from Lemma 3.3.5. In the case that the replacement isn't trivial, Lemma 3.3.5 also tells us that this difference is positive. Likewise, if we replace a 2 with 1,1, we obtain a negative difference. It follows that if the only change we made to a continued fraction was to repeatedly replace 2's with 1,1's, the numerator of the original continued fraction would be smaller than the resulting one. However, the Lemma 4.2.7 states that if the original continued fraction has one more replaceable entry, a 2 at the end, then it will remain larger even if you repeatedly replace 2's with 1,1's. First, we prove Lemma 4.2.5 which gives an equivalent expression for the difference of the two continued fractions.

Lemma 4.2.5. *[RS] Let μ_i be a sequence of entries in $\mathbb{Z}^+$ or equal to $0,0$ and let $\delta_j = \mu_1, 2, \mu_2, 2, \mu_3, \ldots, \mu_j$ and $\epsilon_j = \mu_{j+1}, 1, 1, \mu_{j+2}, 1, 1, \ldots, \mu_k$. Then*

$$N[\mu_1, 2, \mu_2, 2, \mu_3, \ldots, \mu_k, 2] - N[\mu_1, 1, 1, \mu_2, 1, 1, \mu_3, \ldots, \mu_k]$$

$$= N[\mu_1, 2, \mu_2, 2, \mu_3, \ldots, \mu_k, 1] - \sum_{j=1}^{k-1} N[\delta_j^-] N[{}^-\epsilon_j].$$

Remark 4.2.6. We cannot rule out the possibility of replacing consecutive 2's in the continued fraction. In order to have a μ_i between these two consecutive replacements, that μ_i would have to be equal to 0,0. Using $\mu_i = 0, 0$ as a placeholder does not change the value of the numerator by Equations (A.0.8), (A.0.9) and (A.0.10).

Proof. Our goal is to rewrite the left side of the equation in Lemma 4.2.5. By Equation (A.0.11), we have $N[\mu_1, 2, \mu_2, 2, \mu_3, \ldots, \mu_k, 2] = N[\mu_1, 2, \mu_2, 2, \mu_3, \ldots, \mu_k, 1] +$

$N[\mu_1, 2, \mu_2, 2, \mu_3, \ldots, \mu_k]$. Therefore the left hand side of the equation in Lemma 4.2.5 is equal to

$$N[\mu_1, 2, \mu_2, 2, \mu_3, \ldots, \mu_k, 1] + N[\mu_1, 2, \mu_2, 2, \mu_3, \ldots, \mu_k] - N[\mu_1, 1, 1, \mu_2, 1, 1, \mu_3, \ldots, \mu_k]. \tag{4.2.1}$$

Next, we focus on the last two terms of the previous expression because the continued fractions only differ by the replacements. We rewrite their difference by adding and subtracting $k-1$ placeholder terms. However, we must be careful in doing this. At each step, the two continued fractions must agree everywhere except for where the replacement is being made. We approach this by making replacements one at a time starting from the end of the continued fraction and working towards the beginning.

$$N[\mu_1, 2, \mu_2, 2, \mu_3, \ldots, \mu_k] - N[\mu_1, 1, 1, \mu_2, 1, 1, \mu_3, \ldots, \mu_k]$$

$$\begin{aligned}
= \quad & N[\mu_1, 2, \mu_2, \ldots, 2, \mu_{k-1}, 2, \mu_k] - N[\mu_1, 2, \mu_2, \ldots, 2, \mu_{k-1}, 1, 1, \mu_k] \\
& + N[\mu_1, 2, \ldots, 2, \mu_{k-2}, 2, \mu_{k-1}, 1, 1, \mu_k] - N[\mu_1, 2, \ldots, 2, \mu_{k-2}, 1, 1, \mu_{k-1}, 1, 1, \mu_k] \\
& + \ldots \\
& + N[\mu_1, 2, \mu_2, 1, 1, \ldots, 1, 1, \mu_k] - N[\mu_1, 1, 1, \mu_2, 1, 1, \ldots, 1, 1, \mu_k]
\end{aligned}$$

The expression on the right hand side of the equation above is equivalent to the left hand side because all we did was add and subtract in placeholders. It should be clear that the middle terms would all cancel leaving the expression from the left hand side behind. However it is useful to add in these placeholders, because it allows us to analyze the difference from each replacement individually. Next, since each line represents a single replacement, we can apply Lemma 3.3.5 to each line, and obtain

$$\begin{aligned}
&-N[\mu_1, 2, \mu_2, \ldots, \mu_{k-1}^-]N[^-\mu_k] \\
&-N[\mu_1, 2, \mu_2, \ldots, \mu_{k-2}^-]N[^-\mu_{k-1}, 1, 1, \mu_k] \\
&-\cdots \\
&-N[\mu_1^-]N[^-\mu_2, 1, 1, \mu_3, \ldots, \mu_k].
\end{aligned}$$

Then we introduce the notation given in the statement of the lemma. Let $\delta_j = \mu_1, 2, \mu_2, 2, \mu_3, \ldots, \mu_j$, meaning the δ's give the entries at the beginning of the continued fractions, i.e. the parts that still have 2's between the μ's. Whereas the ϵ's give the entries at the tail end of the continued fraction with 1,1's between the μ's, $\epsilon_j = \mu_{j+1}, 1, 1, \mu_{j+2}, \ldots, \mu_k$. This substitution of notation yields the following.

$$\begin{aligned}
-N[\mu_1, 2, \mu_2, \ldots, \mu_{k-1}^-]N[^-\mu_k] &= -N[\delta_{k-1}^-]N[^-\epsilon_{k-1}] \\
-N[\mu_1, 2, \mu_2, \ldots, \mu_{k-2}^-]N[^-\mu_{k-1}, 1, 1, \mu_k] &= -N[\delta_{k-2}^-]N[^-\epsilon_{k-2}] \\
&\vdots \\
-N[\mu_1^-]N[^-\mu_2, 1, 1, \mu_3, \ldots, \mu_k] &= -N[\delta_1^-]N[^-\epsilon_1]
\end{aligned}$$

Therefore

$$N[\mu_1, 2, \mu_2, \ldots, \mu_k] - N[\mu_1, 1, 1, \mu_2, \ldots, \mu_k] = -\sum_{j=1}^{k-1} N[\delta_j^-]N[^-\epsilon_j].$$

We now substitute this equality back into Equation (4.2.1) and obtain

$$N[\mu_1, 2, \mu_2, \dots, \mu_k, 2] - N[\mu_1, 1, 1, \mu_2, \dots, \mu_k] = N[\mu_1, 2, \mu_2, \dots, \mu_k, 1] - \sum_{j=1}^{k-1} N[\delta_j^-]N[^-\epsilon_j].$$

□

Our next task is to prove the difference in Lemma 4.2.5 is in fact positive, meaning that $N[\mu_1, 2, \mu_2, 2, \mu_3, \dots, \mu_k, 2] > N[\mu_1, 1, 1, \mu_2, 1, 1, \mu_3, \dots, \mu_k]$. To do this, we will use the equality we proved in Lemma 4.2.5 and induction.

Lemma 4.2.7. *[RS] Let μ_i be a sequence of entries in $\mathbb{Z}^+$ or equal to $0, 0$. Then*

$$N[\mu_1, 2, \mu_2, 2, \mu_3, \dots, \mu_k, 2] - N[\mu_1, 1, 1, \mu_2, 1, 1, \mu_3, \dots, \mu_k] > 0.$$

Proof. We prove Lemma 4.2.7 by induction. For the base case, let $k = 2$. Then we would like to prove that $N[\mu_1, 2, \mu_2, 2] - N[\mu_1, 1, 1, \mu_2] > 0$. We can rewrite the expression using the equality in Lemma 4.2.5

$$N[\mu_1, 2, \mu_2, 2] - N[\mu_1, 1, 1, \mu_2] = N[\mu_1, 2, \mu_2, 1] - N[\mu_1^-]N[^-\mu_2].$$

Then apply the grafting formula in Proposition 2.2.9 to the first term in the right side of this equation to obtain

$$= \quad N[\mu_1, 2]N[\mu_2, 1] + N[\mu_1]N[^-\mu_2, 1] - N[\mu_1^-]N[^-\mu_2].$$

The first two terms are positive, but more importantly, each is larger than the third term. Therefore the expression is positive. Note that in the case $\mu_1 = 0, 0$, the third term is zero, so the first two positive terms clearly yield a positive result.

For the induction, we assume the expression is positive for k many μ's. We would like to prove the difference is positive when there are $k+1$ many μ's (μ_0 through μ_k).

Thus we consider the expression

$$N[\mu_0, 2, \mu_1, 2, \ldots, 2, \mu_k, 2] - N[\mu_0, 1, 1, \mu_1, 1, 1, \ldots, 1, 1, \mu_k].$$

In the case where $\mu_0 = 0, 0$, by Equation (A.0.9), this expression is equal to $N[2, \mu_1, 2, \ldots, 2, \mu_k, 2] - N[1, 1, \mu_1, 1, 1, \ldots, 1, 1, \mu_k]$. Then by Equation (A.0.5), this is the same as $N[2, \mu_1, 2, \ldots, 2, \mu_k, 2] - N[2, \mu_1, 1, 1, \ldots, 1, 1, \mu_k]$. Therefore we can relabel $\mu_1' = 2, \mu_1$ and consider $N[\mu_1', 2, \ldots, 2, \mu_k, 2] - N[\mu_1', 1, 1, \ldots, 1, 1, \mu_k]$ which is positive by induction.

Next we would like to show that

$$N[\mu_0, 2, \mu_1, 2, \ldots, 2, \mu_k, 2] - N[\mu_0, 1, 1, \mu_1, 1, 1, \ldots, 1, 1, \mu_k]$$

is positive when $\mu_0 \neq 0, 0$. By Lemma 4.2.5 this positivity is equivalent to the inequality

$$N[\mu_0, 2, \mu_1, 2, \mu_2, \ldots, \mu_k, 1] > \sum_{j=0}^{k-1} N[\delta_j^-] N[{}^-\epsilon_j].$$

Our goal is to use the induction hypothesis,

$$N[\mu_1, 2, \mu_2, \ldots, 2, \mu_k, 2] \quad > \quad N[\mu_1, 1, 1, \mu_2, \ldots, 1, 1, \mu_k] \tag{4.2.2}$$

or equivalently,

$$N[\mu_1, 2, \mu_2, \ldots, \mu_k, 1] \quad > \quad \sum_{j=1}^{k-1} N[\delta_j^-] N[{}^-\epsilon_j]. \tag{4.2.3}$$

Using Proposition 2.2.9, we have

$$
\begin{aligned}
&N[\mu_0, 2, \mu_1, 2, \mu_2, \ldots, \mu_k, 1] \\
&\quad = N[\mu_0, 2]N[\mu_1, 2, \mu_2, \ldots, \mu_k, 1] + N[\mu_0]N[{}^-\mu_1, 2, \mu_2, \ldots, \mu_k, 1] \\
&\quad > N[\mu_0, 2]N[\mu_1, 2, \mu_2, \ldots, \mu_k, 1].
\end{aligned}
$$

Since $N[\mu_0, 2] = 2N[\mu_0] + N[\mu_0^-]$ by Equation (A.0.2), this is equal to

$$2N[\mu_0]N[\mu_1, 2, \mu_2, \ldots, \mu_k, 1] + N[\mu_0^-]N[\mu_1, 2, \mu_2, \ldots, \mu_k, 1]$$

and strictly greater than $N[\mu_0]N[\mu_1, 2, \mu_2, \ldots, \mu_k, 2] + N[\mu_0^-]N[\mu_1, 2, \mu_2, \ldots, \mu_k, 1]$ by Equation (A.0.12). Then we apply the induction hypothesis, Equation (4.2.2) to the first term and Equation (4.2.3) to the second term. In doing so, we obtain that $N[\mu_0, 2, \mu_1, 2, \mu_2, \ldots, \mu_k, 1]$ is greater than the following.

$$N[\mu_0]N[\mu_1, 1, 1, \mu_2, \ldots, 1, 1, \mu_k] + N[\mu_0^-]\sum_{j=1}^{k-1} N[\delta_j^-]N[{}^-\epsilon_j]$$

We are considering the case where $\mu_0 \neq 0, 0$, hence $N[\mu_0^-] \geq 1$. Also, because $N[\mu_0] > N[\mu_0^-]$ and $N[\mu_1, 1, 1, \mu_2, \ldots, 1, 1, \mu_k] > N[{}^-\mu_1, 1, 1, \mu_2, \ldots, 1, 1, \mu_k]$ we have

$$N[\mu_0, 2, \mu_1, 2, \mu_2, \ldots, \mu_k, 1] \;>\; N[\mu_0^-]N[{}^-\mu_1, 1, 1, \mu_2, \ldots, 1, 1, \mu_k] + \sum_{j=1}^{k-1} N[\delta_j^-]N[{}^-\epsilon_j].$$

Therefore

$$N[\mu_0, 2, \mu_1, 2, \mu_2, \ldots, \mu_k, 1] \;\; > \;\; \sum_{j=0}^{k-1} N[\delta_j^-]N[^-\epsilon_j]. \quad \square$$

When we eventually prove Theorem 4.2.9 and hence Conjectures 4.2.1 and 4.2.2, we will not only be replacing 2's with 1,1's but we will also be replacing 1,1's with 2's. However, replacing 1,1's with 2's yields a positive difference in the numerators of the continued fractions by Lemma 3.3.5. Therefore these kinds of replacements should only strengthen our result. To be certain, we prove Theorem 4.2.8 which states that regardless of what replacements are made ($1,1 \mapsto 2$ or $2 \mapsto 1,1$) and where in the continued fraction they occur, the numerator of the continued fraction having an extra 2 at the end will be larger than the numerator of the continued fraction with replacements without the 2 at the end.

Theorem 4.2.8. *[RS] Let μ_i be a sequence of entries in $\mathbb{Z}^+$ or equal to $0, 0$. Let $\alpha_i = 2$ or $1, 1$ for all $i = 1, \ldots, k-1$. Define α_i' by*

$$\alpha_i' = \begin{cases} 1, 1 & \textit{if } \alpha_i = 2, \\ 2 & \textit{if } \alpha_i = 1, 1. \end{cases}$$

Then $\quad N[\mu_1, \alpha_1, \mu_2, \alpha_2, \ldots, \alpha_{k-1}, \mu_k, 2] - N[\mu_1, \alpha_1', \mu_2, \alpha_2', \ldots, \alpha_{k-1}', \mu_k] > 0.$

Proof. Using Equation (A.0.11), we see that

$$N[\mu_1, \alpha_1, \mu_2, \alpha_2, \ldots, \alpha_{k-1}, \mu_k, 2] - N[\mu_1, \alpha_1', \mu_2, \alpha_2', \ldots, \alpha_{k-1}', \mu_k] \tag{4.2.4}$$

$$
\begin{aligned}
&= N[\mu_1, \alpha_1, \mu_2, \alpha_2, \ldots, \alpha_{k-1}, \mu_k, 1] \\
&\quad + N[\mu_1, \alpha_1, \mu_2, \alpha_2, \ldots, \alpha_{k-1}, \mu_k] - N[\mu_1, \alpha'_1, \mu_2, \alpha'_2, \ldots, \alpha'_{k-1}, \mu_k].
\end{aligned}
$$

Since our goal is to show that this value is positive, and the first term is clearly positive, we turn our attention to the difference given by the last two terms. Let us denote this difference by D, thus

$$D = N[\mu_1, \alpha_1, \mu_2, \alpha_2, \ldots, \alpha_{k-1}, \mu_k] - N[\mu_1, \alpha'_1, \mu_2, \alpha'_2, \ldots, \alpha'_{k-1}, \mu_k].$$

Note that the continued fractions differ only by replacements. We would like to consider these replacements one at a time. First, we start by changing all $\alpha_i = 1, 1$'s to $\alpha'_i = 2$'s and leaving the $\alpha_i = 2$'s alone. We define a map f that does just this.

$$f(\alpha_i) = \begin{cases} \alpha_i & \text{if } \alpha_i = 2, \\ \alpha'_i & \text{if } \alpha_i = 1, 1 \end{cases}$$

Then $D = D_1 + D_2$ where

$$D_1 = N[\mu_1, \alpha_1, \mu_2, \alpha_2, \ldots, \alpha_{k-1}, \mu_k] - N[\mu_1, f(\alpha_1), \mu_2, f(\alpha_2), \ldots, f(\alpha_{k-1}), \mu_k]$$

and

$$D_2 = N[\mu_1, f(\alpha_1), \mu_2, f(\alpha_2), \ldots, f(\alpha_{k-1}), \mu_k] - N[\mu_1, \alpha'_1, \mu_2, \alpha'_2, \ldots, \alpha'_{k-1}, \mu_k].$$

To compute D_1, we start with the end of the continued fraction and consecutively replace each α_i with $f(\alpha_i)$. This is the same basic process used in the proof of Lemma 4.2.5, where we add and subtract in placeholders in order to look at the difference given by each replacement individually. Notice that if $\alpha_i = 2$, then $f(\alpha_i) = 2$ and no actual change is made to the continued fraction, therefore the difference in numerators would be zero. Hence instead of summing the differences over all i, we can sum over only the i's where $\alpha_i = 1, 1$. Thus D_1 is equal to

$$\sum_{i:\alpha_i=1,1} (N[\mu_1, \ldots, \alpha_{i-1}, \mu_i, \alpha_i, \mu_{i+1}, f(\alpha_{i+1}), \ldots, \mu_k] \\ -N[\mu_1, \ldots, \alpha_{i-1}, \mu_i, f(\alpha_i), \mu_{i+1}, f(\alpha_{i+1}), \ldots, \mu_k]).$$

This expression can be simplified by applying Lemma 3.3.5 to each difference in the sum. Since we only replace $\alpha_i = 1, 1$ with $\alpha_i' = 2$, each difference is positive. Hence

$$D_1 = \sum_{i:\alpha_i=1,1} N[\mu_1, \ldots, \alpha_{i-1}, \mu_i^-]N[^-\mu_{i+1}, f(\alpha_{i+1}), \ldots, \mu_k].$$

This value is positive. However, we still need to compute D_2 by replacing each original $\alpha_i = 2$ with $\alpha_i' = 1, 1$ while leaving the rest alone. Similarly to before, we work starting from the end of the continued fraction, and separate each replacement. Thus D_2 is equal to

$$\sum_{i:\alpha_i=2} (N[\mu_1, \ldots, f(\alpha_{i-1}), \mu_i, f(\alpha_i), \mu_{i+1}, \alpha_{i+1}', \ldots, \mu_k] \\ -N[\mu_1, \ldots, f(\alpha_{i-1}), \mu_i, \alpha_i', \mu_{i+1}, \alpha_i', \ldots, \mu_k]).$$

Then Lemma 3.3.5 gives us an equivalent expression. However, this value is negative.

$$D_2 = -\sum_{i:\alpha_i=2} N[\mu_1, \dots, f(\alpha_{i-1}), \mu_i^-]N[^-\mu_{i+1}, \alpha'_{i+1}, \dots, \mu_k]$$

Therefore substituting $D = D_1 + D_2$ into Equation (4.2.4), we have shown the following equality.

$$N[\mu_1, \alpha_1, \mu_2, \alpha_2 \dots, \alpha_{k-1}, \mu_k, 2] - N[\mu_1, \alpha'_1, \mu_2, \alpha'_2 \dots, \alpha'_{k-1}, \mu_k] \tag{4.2.5}$$

$$= N[\mu_1, \alpha_1, \mu_2, \alpha_2 \dots, \alpha_{k-1}, \mu_k, 1] + \sum_{i:\alpha_i=1,1} N[\mu_1, \dots, \alpha_{i-1}, \mu_i^-]N[^-\mu_{i+1}, f(\alpha_{i+1}), \dots, \mu_k]$$

$$- \sum_{i:\alpha_i=2} N[\mu_1, \dots, f(\alpha_{i-1}), \mu_i^-]N[^-\mu_{i+1}, \alpha'_{i+1}, \dots, \mu_k].$$

In order to prove that Equation (4.2.5) is positive, we compare to the continued fraction in which every $\alpha_i = 2$. In this case $D_1 = 0$. Therefore we know that the right hand side of Equation (4.2.5) is greater than the same expression when every $\alpha_i = 2$. Hence

$$N[\mu_1, \alpha_1, \mu_2, \alpha_2 \dots, \alpha_{k-1}, \mu_k, 2] - N[\mu_1, \alpha'_1, \mu_2, \alpha'_2 \dots, \alpha'_{k-1}, \mu_k]$$

$$\geq \quad N[\mu_1, 2, \mu_2, 2 \dots, 2, \mu_k, 2] - N[\mu_1, 1, 1, \mu_2, 1, 1, \dots, 1, 1, \mu_k]$$

and this is positive by Lemma 4.2.7. □

Finally, we are ready to prove our main result. Theorem 4.2.9 is more general than Conjectures 4.2.1 and 4.2.2, but note that if p and q are relatively prime, we can apply Theorem 4.2.9 repeatedly to obtain the inequalities in Conjectures 4.2.1 and 4.2.2.

Theorem 4.2.9. *[RS] Let p and q be positive integers such that $p < q$. Then $m_{p/q} < m_{p/(q+1)}$ and $m_{p/q} < m_{(p+1)/q}$.*

Proof. We can write the continued fraction of $m_{p/q}$ as a list of $q+p-1$ replaceable entries, $a_i = 1, 1$ or 2. Whereas the continued fractions of $m_{p/(q+1)}$ and $m_{(p+1)/q}$ would have $q+p$ replaceable entries. Comparing $m_{p/q}$ with $m_{(p+1)/q}$ or $m_{p/(q+1)}$ is analogous in either case, because it depends only on the number of replaceable entries in the continued fraction. Without loss of generality, we write $m_{(p+1)/q} = N[\nu_1, \ldots \nu_{q+p-1}, 2]$ and $m_{p/q} = N[\nu'_1, \ldots, \nu'_{q+p-1}]$ where each ν_i and ν'_i represent a replaceable entry and we use the convention that each continued fraction begins and ends with 2, i.e. $\nu_1 = \nu'_1 = \nu'_{q+p-1} = 2$.

Therefore we would like to compare $N[\nu_1, \ldots \nu_{q+p-1}, 2]$ and $N[\nu'_1, \ldots, \nu'_{q+p-1}]$. Each ν_i may or may not be the same as ν'_i. We change the notation to collect subsequences of replaceable entries that agree. Each subsequence of replaceable entries that agree, becomes a μ and for each $\nu_i \neq \nu'_i$, ν_i becomes an α and ν'_i becomes an α'. In addition, if two consecutive replaceable entries do not agree, we insert a sequence $\mu = 0, 0$ between them as in Example 4.2.10. Thus we are now comparing $N[\mu_1, \alpha_1, \mu_2, \alpha_2, \ldots, \mu_k, 2]$ and $N[\mu_1, \alpha'_1, \mu_2, \alpha'_2, \ldots, \mu_k]$, where $\alpha_i = 1, 1$ implies $\alpha'_i = 2$ and vice versa. By Theorem 4.2.8, $N[\mu_1, \alpha_1, \mu_2, \alpha_2, \ldots, \mu_k, 2] > N[\mu_1, \alpha'_1, \mu_2, \alpha'_2, \ldots, \mu_k]$, hence $m_{(p+1)/q} > m_{p/q}$ and analogously $m_{p/(q+1)} > m_{p/q}$. □

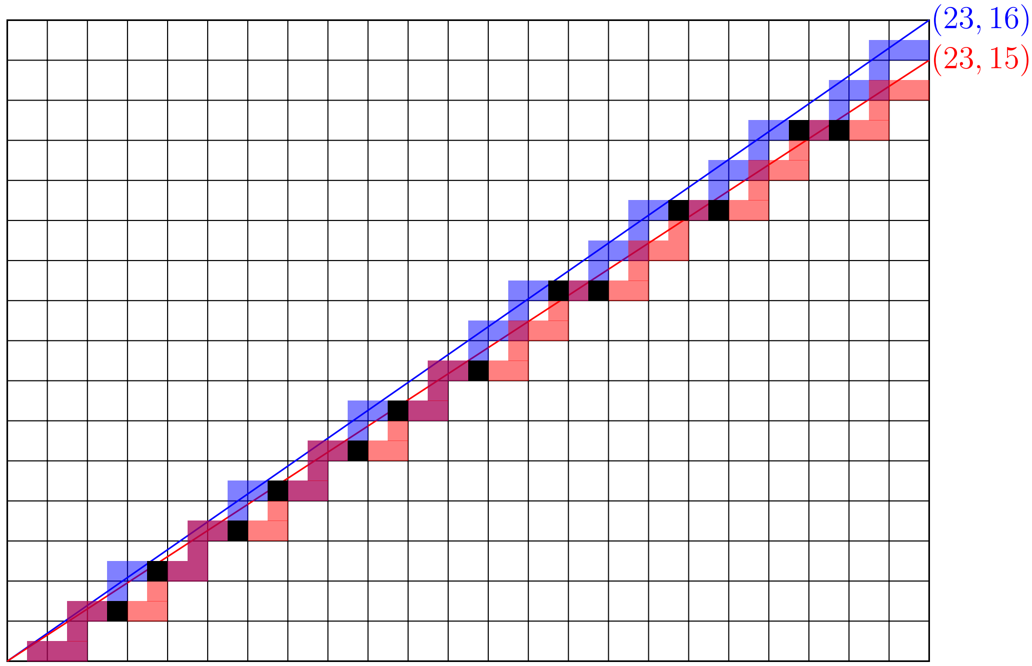
Example 4.2.10. In this example, we take the continued fractions from Example 4.2.4 and rewrite them to fit the notation in Theorem 4.2.8. Here $\mu_3 = \mu_5 = 0, 0$ because there are two consecutive replaceable entries being replaced. Once the μ's have been written, it is easy to denote the replaceable entries, α_i, in the continued fraction of $m_{4/7}$. Then the replaceable entries, α'_i in the continued fraction of $m_{3/7}$ follow.

$$m_{4/7} = N[\quad 2, \quad 2, \quad 2, \quad 1,1, \quad 0,0, \quad 2, \quad 2, \quad 1,1, \quad 0,0, \quad 2, \quad 2, \quad 2]$$
$$m_{4/7} = N[\quad \mu_1, \quad \alpha_1, \quad \mu_2, \quad \alpha_2, \quad \mu_3, \quad \alpha_3, \quad \mu_4, \quad \alpha_4, \quad \mu_5, \quad \alpha_5, \quad \mu_6, \quad 2]$$

$$m_{3/7} = N[\quad 2, \quad 1,1, \quad 2, \quad 2, \quad 0,0, \quad 1,1, \quad 2, \quad 2, \quad 0,0, \quad 1,1, \quad 2]$$
$$m_{3/7} = N[\quad \mu_1, \quad \alpha'_1 \quad \mu_2, \quad \alpha'_2, \quad \mu_3, \quad \alpha'_3, \quad \mu_4, \quad \alpha'_4, \quad \mu_5, \quad \alpha'_5, \quad \mu_6]$$

Example 4.2.11. In this example, the Markov snake graph $\mathcal{G}_{16/23}$ is shown in blue on the same graph as $\mathcal{G}_{15/23}$ in red, with their overlap in purple. The black shaded tiles represent the tiles for which a replacement in the corresponding continued fraction occurs. $m_{16/23} = 426,776,599,819,081$ and $m_{15/23} = 187,611,224,490,881$

$$m_{16/23} = N[222\ 2\ 2\,11\,22\ 2\ 2\,11\,22\ 2\ 2\,11\,22\ 2\ 222\,11\ 2\ 222\,11\ 2\ 222\,11\ 2\ 222\,2]$$
$$m_{15/23} = N[222\,11\,2\ 2\ 22\,11\,2\ 2\ 22\,11\,2\ 2\ 22\,11\,222\ 2\ 11\,222\ 2\ 11\,222\ 2\ 11\,222\ ]$$


(23, 16)
(23, 15)

Appendices

Appendix A

Continued Fraction Properties

$$N[a_1, \ldots, a_n] = a_1 N[a_2, \ldots, a_n] + N[a_3, \ldots, a_n] \tag{A.0.1}$$

$$N[a_1, \ldots, a_n] = a_n N[a_1, \ldots, a_{n-1}] + N[a_1, \ldots, a_{n-2}]. \tag{A.0.2}$$

$$N[a_1, a_2] = a_1 a_2 + 1 \tag{A.0.3}$$

$$[a_1, \ldots, a_n, 1] = [a_1, \ldots, a_n + 1] \tag{A.0.4}$$

$$N[1, 1, a_1, \ldots, a_n] = N[2, a_1, \ldots, a_n] \tag{A.0.5}$$

$$N[a_1, \ldots, a_n] = N[a_n, \ldots, a_1] \tag{A.0.6}$$

$$N[a_1, \ldots, a_{n-1}]N[a_1, \ldots, a_{n-1}] - N[a_1, \ldots, a_{n-2}]N[a_1, \ldots, a_n] = (-1)^n \tag{A.0.7}$$

Definition A.0.1. Let $a_i \in \mathbb{Z}_{\geq 0}$ then $N[a_1, \ldots, a_n, 0, 0] := N[0, 0, a_n, \ldots, a_1]$.

$$[a_1, \ldots, a_i, 0, 0, a_{i+1}, \ldots, a_n] = [a_1, \ldots, a_i, a_{i+1}, \ldots, a_n] \tag{A.0.8}$$

$$[0, 0, a_1, \ldots, a_n] = [a_1, \ldots, a_n] \tag{A.0.9}$$

$$N[a_1, \ldots, a_n, 0, 0] = N[a_1, \ldots, a_n] \tag{A.0.10}$$

$$N[a_1, \ldots, a_n, 2] = N[a_1, \ldots, a_n, 1] + N[a_1, \ldots, a_n] \tag{A.0.11}$$

$$2N[a_1, \ldots, a_n, 1] = N[a_1, \ldots, a_n, 2] + N[a_1, \ldots, a_{n-1}] \tag{A.0.12}$$

Let $a_i \in \mathbb{Z}_{\geq 0}$.

$$N[a_1, \ldots, a_i, a_{i+1}, \ldots, a_n] = N[a_1, \ldots, a_i]N[a_{i+1}, \ldots, a_n] + N[a_1, \ldots, a_{i-1}]N[a_{i+2}, \ldots, a_n] \tag{A.0.13}$$

Definition A.0.2. Define the following notation: Let $\mu = a_1, \ldots, a_n$ be a sequence of positive integers or $\mu = a_1, a_2 = 0, 0$. Then we define the following notation, $N[\mu] = N[a_1, \ldots, a_n]$.

$$\begin{aligned} N[\mu^-] &= N[a_1, \ldots, a_{n-1}] & N[{}^-\mu] &= N[a_2, \ldots, a_n] & &\text{for } n > 1 \\ N[\mu^-] &= N[\ ] = 1 & N[{}^-\mu] &= N[\ ] = 1 & &\text{for } n = 1 \end{aligned}$$

$$\begin{aligned} N[{}^-\mu^-] &= N[a_2, \ldots, a_{n-1}] & &\text{for } n > 2 \\ N[{}^-\mu^-] &= N[\ ] = 1 & &\text{for } n = 2 \end{aligned}$$

Appendix B

Dynkin Diagrams

$\mathbb{A}_n$
$$\begin{tikzcd}[column sep=small] 1 \arrow[r, no head] & 2 \arrow[r, no head] & 3 \arrow[r, no head] & \dots \arrow[r, no head] & n-1 \arrow[r, no head] & n \end{tikzcd}$$

$\mathbb{B}_n$
$$\begin{tikzcd}[column sep=small] 1 \arrow[r, equal] & 2 \arrow[r, no head] & 3 \arrow[r, no head] & \dots \arrow[r, no head] & n-1 \arrow[r, no head] & n \end{tikzcd}$$

$\mathbb{C}_n$
$$\begin{tikzcd}[column sep=small] 1 \arrow[r, no head] & 2 \arrow[r, no head] & 3 \arrow[r, no head] & \dots \arrow[r, no head] & n-1 \arrow[r, equal] & n \end{tikzcd}$$

$\mathbb{D}_n$
$$\begin{tikzcd}[column sep=small] & & & & & n-1 \\ 1 \arrow[r, no head] & 2 \arrow[r, no head] & 3 \arrow[r, no head] & \dots \arrow[r, no head] & n-2 \arrow[ur, no head] \arrow[dr, no head] & \\ & & & & & n \end{tikzcd}$$

$\mathbb{E}_6$
$$\begin{tikzcd}[column sep=small] 1 \arrow[r, no head] & 2 \arrow[r, no head] & 3 \arrow[r, no head] \arrow[d, no head] & 4 \arrow[r, no head] & 5 \\ & & 6 & & \end{tikzcd}$$

$\mathbb{E}_7$
$$\begin{tikzcd}[column sep=small] 1 \arrow[r, no head] & 2 \arrow[r, no head] & 3 \arrow[r, no head] \arrow[d, no head] & 4 \arrow[r, no head] & 5 \arrow[r, no head] & 6 \\ & & 7 & & & \end{tikzcd}$$

$\mathbb{E}_8$
$$\begin{tikzcd}[column sep=small] 1 \arrow[r, no head] & 2 \arrow[r, no head] & 3 \arrow[r, no head] \arrow[d, no head] & 4 \arrow[r, no head] & 5 \arrow[r, no head] & 6 \arrow[r, no head] & 7 \\ & & 8 & & & & \end{tikzcd}$$

$\mathbb{F}_4$
$$\begin{tikzcd}[column sep=small] 1 \arrow[r, no head] & 2 \arrow[r, equal] & 3 \arrow[r, no head] & 4 \end{tikzcd}$$

$\mathbb{G}_2$
$$\begin{tikzcd}[column sep=small] 1 \arrow[r, Rrightarrow, no head] & 2 \end{tikzcd}$$

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