5-3-2018

# Topics in Stochastic Analysis and Riemannian Foliations

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University of Connecticut, 2018


## ABSTRACT


This dissertation contains three research directions.

In the first direction, we use rough paths theory to study stochastic differential equations and stochastic partial differential equations. We first study the Taylor expansion for the solutions of differential equations driven by $p$-rough paths with $p > 2$. We provide a general convergence theorem and study the rate of convergence of the Taylor expansion. The main result is about the Castell expansion and the tail estimate with exponential decay for the remainder terms of the solutions. Our results apply to differential equations driven by continuous centered Gaussian process with finite $2D$ $\rho-$variation and fractional Brownian motion with Hurst parameter $H > 1/4$. We then give a new and simple method which allows to get a priori bounds on rough partial differential equations. The technique is based on a weak formulation of the equation and a rough version of Gronwall's lemma. The method is presented on a simple linear stochastic heat equation, but might be generalized to a wide number of situations.

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In the second direction, we study stochastic analysis on the horizontal paths space of totally geodesic Riemannian foliations. We first develop Malliavin calculus on the horizontal path space, Clark-Ocone formulas, various integration by parts formulas are obtained. Further more, we prove that the horizontal Wiener measure is quasi-invariant with respect to flows generated by suitable tangent processes. Under suitable assumptions, we further prove a log-Sobolev inequality for a natural one-parameter family of infinite-dimensional Ornstein-Uhlenbeck type operators. In particular, we also get the improved log-sobolev inequality and the equivalence of two-sided uniform Ricci curvature bounds to various functional inequalities. We also obtain concentration and tail estimates for the horizontal Brownian motion on the foliations.

In the third direction, we study Ricci flow on totally geodesic Riemannian foliations. Under the transverse Ricci flow, we prove two types of differential Harnack inequalities (Li-Yau type) for the positive solutions of the heat equation associated with the time dependent horizontal Laplacian operators. We also get a time dependent version of the generalized curvature dimension inequality. As consequences, we get parabolic Harnack inequalities and heat kernel upper bounds.

**Key words**: Rough paths theory; *A priori* estimate; Log-Sobolev inequality; Malliavin calculus; Wiener measure; Quasi-invariance; sub-Riemannian geometry; Riemannian foliation; Ricci flow; Differential Harnack inequality.

# Topics in Stochastic Analysis and Riemannian Foliations

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A Dissertation

Submitted in Partial Fulfillment of the

Requirements for the Degree of

Doctor of Philosophy

at the

University of Connecticut

2018

## APPROVAL PAGE

Doctor of Philosophy Dissertation

# Topics in Stochastic Analysis and Riemannian Foliations

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2018

Dedicated to my wife, Peifen.

# ACKNOWLEDGMENTS

First and foremost, I want to thank my advisor Professor Fabrice Baudoin. It is because of all his full-hearted support, numerous kindest encouragement, patient insight guidance during my five years working with him at Purdue and UConn, I can finish this dissertation and start my career in mathematics. It is also because of his advices and guidance that I have the courage and opportunity to explore and learn various research topics and manage to finish different research projects. I have enjoyed our discussions on both mathematics and among others. I cannot think of anything further that he could have done for me, and for all of those he has done, I am truly appreciated and thankful.

I would also like to thank my advisory committee members: Professor Maria Gordina, Professor Ambar Sengupta, and Professor Alexander Teplyaev at UConn. And also Professor Rodrigo Bañuelos and Professor Jonathon Peterson back to my four years study at Purdue where I started my journey in mathematics. I would especially like to thank Professor Samy Tindel and Professor Maria Gordina (Masha) under whose guidance and collaborative work I have learned and benefited a lot in my research, and Professor Ambar Sengupta under whose guidance I have improved a lot in teaching and for his advices and support in pursing my career. I also owe a very important debt to my recommendation letter writers, Professor Elton Hsu, Professor Ovidiu Munteanu, Professor Stephan Sturm and Professor Anton Thalmaier. I benefited a lot through our discussions and conversations regarding ongoing research project, future topics and insight of my research interest.

Also, my deep thanks to Professor Shreeram Shankar Abhyankar who passed away during my first semester at Purdue which I believe dramatically changed my career path in mathematics afterwards. In particular, without Professor S. S. Abhyankar, I may not have the chance to begin my graduate school at Purdue.

I would like to thank all my friends/colleagues who I have met and kept in touch afterwards at Purdue, UConn and in various conferences and summer schools during my graduate study. I would especially like to thank for all the precious advices, conversations and encouragements from my friends during different periods of pursuing my Ph.D and career, which have benefited my research, improved my various skills, opened my eyes, etc. I shall continue to pursue a successful career.

I would also like to take this opportunity to thank my family for their support while pursuing the Ph.D degree and for their support and love throughout my studying career. In particular, I want to thank my parents who have been trying their best effort in so many years to support me to pursue my dream. I also want to thank my parents-in-law for their support and love, so that my wife and I can pursue our career goals and keep moving forward.

Finally, I want to express my sincere thanks to my wife, Peifen. Without her unconditional love and support, I cannot accomplish what I have done during my Ph.D. She has been the first one to listen to my oral exam and various talks and provide advices, the first one to share my happiness of finishing my first research project and also the others, the first one to encourage me for whatever I have been facing with and whenever I feel frustrated, despite the fact that her major is in literature and linguistics and has no clue of my research. For sure, I cannot list everything she has done for me, but she has been with me for every single small step I have made upon completing this dissertation. I offer my sincere gratitude and thanks for all she has done for me, for all her love and support.

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# Chapter 1

# Introduction

In general, my thesis can be divided into three parts. In the first part, our goal is to study stochastic differential equations and stochastic partial differential equations (SPDEs) by using rough paths theory. In particular, we first get Taylor expansion and Castell estimates (sub-Gaussian like exponential decay for the remainder terms) for the solutions of stochastic differential equations driven by fractional Brownian motion with Hurst parameter $H > 1/4$ and general Gaussian process with i.i.d components and finite 2-$D$ $\rho$-variations. We then get *a priori* estimates for the solutions of stochastic heat equations driven by fractional Brownian motion with Hurst parameter $H > 1/3$, where our final goal is to get smoothness property of the density function of the solution. In particular, Malliavin calculus associated with fractional Brownian motion will play an important role here in order to study the smoothness of the density function. In the second part, our goal is to study stochastic analysis on the path space of sub-Riemannian manifolds, in particular, Riemannian foliations. In return, we want to study sub-Riemannian geometry by using stochastic analysis methods. We establish Malliavin calculus on the path space of Riemannian foliations and furthermore, we prove Log-Sobolev inequalities on the path space and its equivalent relations to the two sided uniform Ricci curvature bounds of Riemannian foliations. We also prove integration by parts formulas (as well as Clark-Ocone formulas) and quasi-invariance of horizontal Wiener measure on the path space of Riemannian foliations. In the third part, our goal is to study the geometric and analytic properties of Riemannian foliations by using Ricci flow, where we establish the time dependent version of Baudoin-Garofalo's [13] generalized curvature dimension inequalities under Ricci flow and we get differential Harnack inequalities for the positive solutions of heat equation associated with the time dependent horizontal Laplacian operator. In a long run, we want to connect all the three aforementioned research projects. To be precise, we want to use stochastic analysis to characterize Ricci flow and use both Ricci flow and stochastic analysis to study sub-Riemannian geometry. We also want to study rough paths and SPDEs (i.e. stochastic heat equations) on sub-Riemannian manifolds.

## 1.1 Rough paths theory

Rough paths theory is first introduced by T. Lyons [88] to study the following type differential equations driving by irregular paths,

$$y(t) = y_0 + \sum_{i=1}^{d} \int_0^t V_i\left(y(s)\right) dx^i(s). \tag{1.1.1}$$

where $V : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ and $x : [0,\infty) \to \mathbb{R}^n$ is the driving path with bounded $p$ variation, which allows us to define the solutions of (1.1.1) for any $p > 1$. To define the solution of (1.1.1) goes back to Itó's stochastic integration theory [76] where the driving process $x$ is a Brownian motion and if the driving path $x$ has bounded $p$-variation with $1 < p < 2$, Young integral is enough. However, if $p > 2$, neither Itó integral nor Young integral will work.

The main idea of rough paths theory is to consider a sequence of smooth approximation $x_n$ for driving path $x$. We hope to find a solution as the limit of the sequence $y_n$ which is independent of the choice of $x_n$, where $y_n$ is the solution corresponding to $x_n$. Usually $x$ is called a geometric rough path if its signature (2.1.1) (summation of the iterated integral against x) is the limit of the signature of $x_n$ in the $p$-variation topology (see details: [54, 53]).

This idea of geometric rough paths is generalized by M. Gubinelli first to controlled rough paths [60] and then to the non-Geometric case, namely the branched rough paths [61], which defines integrals driven by more irregular functions. The equivalence of geometric rough paths and non-geometric rough paths is proved by D. Kelly and M. Hairer [80]. One major difficulty of studying stochastic differential equations and stochastic partial differential equations is the ill posedness because of the distribution valued noise. To see this from the controlled rough paths sense, if the noise is a distribution, the solution is a functional of a distribution, namely the solution behaves like the noise, essentially we need to deal with the product of two distributions. In the extraordinary work [65, 64] of M. Hairer, the regularity structure is introduced and solved the KPZ equation in dimension one by using renormalization techniques. This regularity structure is another higher level generalization of the rough paths theory. Roughly speaking, after iterating and linearizing the equation, there will be correction terms appear, these correction terms are the more general terms compared to the correction terms between the geometric rough paths and its corresponding non-geometric rough paths, which of course can be easily seen from the correction terms between Itó integral and its corresponding Stratonovich integral. A more traditional approach, which gives a way to handle pathwise products of distributions, introduced by M. Gubinelli, P. Imkeller and N. Perkowski [62] is paracontrolled calculus originated from paraproduct and controlled rough paths theory. Recently, another new method to study rough PDEs with distribution valued drift and unbounded operators called unbounded rough drivers is introduced in [4, 33]. In the first part of my thesis, I will use rough paths and unbounded rough drivers to study stochastic differential equations of type (1.1.1) and stochastic heat equations which can be seen as a infinite dimensional version of (1.1.1).

### 1.1.1 Stochastic differential equations

The stochastic Taylor series of the solution of type (1.1.1) driving by Brownian motion was introduced by R. Azencott [2] and G. Ben Arous [21] and a Castell expansion for the solution was first proved by F. Castell [26]. A more irregular version of the above stochastic differential equation (1.1.1) is to change the driving process to be a fractional Brownian motion. A similar convergence result has been extended by F. Baudoin and X. Zhang [18] to the case where the driving process is a fractional Brownian motion with Hurst parameter $H > 1/2$, namely$1 < p < 2$. I. Bailleul proved a deterministic estimate of the remainder term of a similar Castell expansion by studying flows driven by Banach space-valued weak Geometric Hölder p-rough paths [5]. In my joint work with X. Zhang [50], under the further assumption that the vector fields $V_i$'s are analytic on the set $\{y : \|y - y_0\| \leq C\}$ for some $C > 0$, we provide a general convergence result of the Taylor expansion for the solution $y(t)$

of the differential equation (1.1.1). More precisely, we will be able to express the solution $y(t)$ of (1.1.1) as the sum of its Taylor expansion on a non-empty interval. We provide convergence criteria that enable us to express the non-empty interval in a more quantitative way and to study the rate of the convergence of Taylor series. Our result is well adapted to the stochastic differential equations driven by fractional Brownian motion with Hurst parameter $1/4 < H < 1/2$ and continuous Gaussian process with finite $2D$ $\rho-$variation with i.i.d components. After generalizing the concentration inequality in [8] to get an similar estimate for the solution of differential equations driven by continuous centered Gaussian process with finite $2D$ $\rho-$variation. We prove a Castell expansion for the solutions of (1.1.1) driven by mean zero i.i.d Gaussian process with finite $2D$ $\rho-$variation and fractional Brownian motion with Hurst parameter $H > 1/4$. Moreover, for both the Taylor expansion and the Castell expansion of the solutions in the above two cases, we prove the same type of tail estimates for the remainder terms.

### 1.1.2 Stochastic partial differential equations (SPDEs)

An infinite dimensional version of the stochastic differential equation (1.1.1) which we are going to study is of the following type:

$$\partial_t u_t(x) = \frac{\Delta}{2} u_t(x) + \sigma(u_t(x))\dot{x}_t, \tag{1.1.2}$$

which can also be viewed in the rough path sense of integral form like,

$$\delta u_{st} = \mu\left([s,t)\right) + \int_s^t \sigma(g_r) dx_r, \quad \text{where } \delta u_{st} = u_t - u_s \tag{1.1.3}$$

There have been a lot of recent results on existence and uniqueness for rough partial differential equations like (1.1.2). In particular, we recall some of the results with smooth noises in space which will be the case we are going to work on. The references [25, 90] handle stochastic PDEs by considering random flows which change the stochastic PDE into a deterministic PDE with random coefficients. In [34, 63], a variant of the rough paths theory is introduced in order to cope with PDEs of the form (1.1.2), considered in the mild sense. For linear equations like (2.4.1) (introduce in a moment), Feynman-Kac representations for the solution are available. We first review a recent method allowing us to get *a priori* estimates for rough partial differential equations, taken from [33]. Our aim here is to show how to implement the technique on a simple example. Namely, we consider the following noisy heat equation on an interval $[0,\tau] \times \mathbb{R}^d$ for $\tau > 0$ and a spatial dimension $d \geq 1$:

$$\partial_t u_t(x) = \frac{\Delta}{2} u_t(x) + \sum_{i=1}^{\infty} \beta_i u_t(x) e_i(x) dw_t^i, \tag{1.1.4}$$

where $\Delta$ stands for the Laplace operator, $\{e_i; \geq 1\}$ is an orthonormal basis of $L^2(\mathbb{R}^d)$ and $\{\beta_i; \geq 1\}$ is a family of coefficients satisfying some summability conditions. In equation (2.4.1), $\{w_i; \geq 1\}$ is also a family of noises, interpreted as $p$-variation paths with $p < 3$, which can be lifted to a rough paths $\mathbf{w}$.

We use the variational approach to rough PDEs, introduced in [4, 33], to provide a handy way to obtain $L^2(\mathbb{R}^d)$ (and more generally $L^\alpha(\mathbb{R}^d)$) estimates on the solution. Which gives a more simple and direct method compared to the variational approach in [100] where they only have the $L^2$ norm bound and our noise is the real rough case, in the sense that instead of $H > 1/2$, we consider $H > 1/3$, even though our computations are restricted to linear cases. Our final goal is to study the smoothness of the density function of the solution to the stochastic non-linear heat equation (1.1.2).

## 1.2 Stochastic analysis on Riemannian foliations

This part of my research goes into the intersections of sub-Riemanian geometry, stochastic analysis and functional inequalities. The study of stochastic analysis on manifolds dates back to the intrinsic construction of Brownian motion on a Riemannian manifold by Eells-Elworthy-Malliavin ( see for example: [41, 94]). Since then, there are numerous work done on the path space (also loop space) of Riemannian manifolds. For example, the quasi-invariance of Wiener measure on the path space of compact Riemannian manifold [36, 72], Poincaré and Log-Sobolev inequalities [45, 73], etc. As for a detailed study of stochastic analysis [37, 46, 74] and differential geometry [42, 43] on the path space of Riemannian manifolds, we refer to the aforementioned books and the references therein. Recently, the characterization of two-sided bounds of Ricci curvature by using the stochastic analysis on the path space of Riemannian manifold draws a lot of attentions, many work has been done in this direction [28, 47, 68, 97, 116].

The first successful attempt to study stochastic analysis on sub-Riemannian manifold is established by F. Baudoin [10] for sub-Riemannian manifold with transverse symmetries and satisfying the Yang-Mills condition. This is based on the Bochner-Weitzenböck formula and generalized curvature dimension inequality established in [13] on sub-Riemannian manifold with transverse symmetries and was then generalized to totally geodesic Riemannian foliations [16], which allows us to define horizontal

Laplacian and the semigroup associated with it. Furthermore, the Yang-Mills condition in [10, 16] is shown to be unnecessary by E. Grong and A. Thalmaier [57, 58].

In our first project [11], we first work on totally geodesic Riemannian foliations with bundle like metric (further details, see [9, 113]) under the framework of [16] and Yang-Mills condition. From our assumptions, the horizontal Laplacian is a hypoelliptic operator that satisfies the Hörmander's bracket generating condition. We first prove an integration by parts formula (multidimensional), for the horizontal Brownian motion, that is the diffusion generated by the horizontal Laplacian. We then introduce the relevant Malliavin derivatives on the horizontal paths space and prove a Clark-Ocone type representation for functionals of the horizontal Brownian motion. We adapt a method of E.P. Hsu [73] and another method of A. Naber [97] (see also S. Fang and B. Wu [47]), to prove in our framework a log-Sobolev inequality on the paths space of the horizontal Brownian motion. Moreover, we also provide the equivalent relations between uniform two-sided Ricci curvature bounds and the log-Sobolev inequalities which generalized the result on Riemannian manifolds [97, 116]. As corollaries, we prove concentration inequalities and large deviation principles.

In our second project [12], we work on totally geodesic Riemannian foliations with bundle like metric, but without the Yang-Mills condition. We are able to establish a more generalized version of the integration by parts formulas for both damped derivates and intrinsic derivates. We then generalized the quasi-invariance of Wiener measure on a compact Riemannian manifold, which is first proved by B. Driver [36] for the flow generated by Cameron-Martin path intersected with continuous functions and then proved for all Cameron-Martin path space by E. Hsu [72] , to our totally geodesic Riemannian foliation case. In particular, we give some examples where our results apply to. (e.g. Heisenberg group, Hopf fibration, etc. )

## 1.3 Ricci flow on Riemannian foliations

One important method to study Riemannian geometry is the Ricci flow introduced by Hamilton [66], for a detailed study of Ricci flow we refer to [29]. The study of Ricci flow on a Riemannian manifold under the curvature dimension inequality condition is recently studied by Li-Li [83]. The characterization of Ricci flow by using optimal transportation is carried out by McCann-Topping [95]. Recently, Naber-Haslhofer-Hein [67, 69] try to define the weak solution of Ricci flow through the analysis (and stochastic analysis) on the path space of a Riemannian manifold.

Lovric-Min-Oo-Ruh [85] initiated the study of Ricci flow on sub-Riemannian manifold. There are several recent study of Ricci flow on Riemannian foliations. For example: mixed curvature Ricci flow on co-dimendison one foliations [106, 105], Sasakian transverse Ricci flow [30, 108], general second order geometric flows on Riemannian folations [19]. But in all these previous work, they all have torsion free condition for the transverse Levi-Civita connection. The first non-torsion free case in the context of Ricci flow is the work by Phong-Picard-Zhang [101, 102] where they study anomaly flow with Strominger system and Fu-Yau equation on 3-dimensional complex manifolds.

In this part [49, 51], we are interested to study Ricci flow under the non-torsion free condition on complete totally geodesic Riemannian foliations with any co-dimensional leaves. In this sub-Riemannian setting, the concept of Ricci curvature lower bound is carried out in terms of the generalized curvature dimension inequality by Baudoin-Garofalo [13], which is the analogue of the curvature dimension inequality Bakry-Eméry [6] in the Riemannian setting. We prove that the sub-Riemannian structure is preserved under our flow. We then provide a time dependent version of the generalized curvature dimension inequality. Furthermore, we prove various differential Harnack inequalities for the solutions of heat equation associated with the time dependent horizontal Laplace operator. As corollaries, we get parabolic Harnack inequalities and heat kernel upper bounds. We also prove the monotonicity property for the transverse $\mathcal{F}$ functional and transverse Perelman's $\mathcal{W}$ functional for basic functions. As for general functions and long time existence of the flow, we will show later in [51].

# Chapter 2

# Rough paths theory and SPDEs

## 2.1 Rough paths theory.

Let us briefly mention the rough paths theory [88] here, a brief introduction to rough paths theory with an emphasis on the signature of the paths can be found in [7], as for a thorough and systemic study of rough paths theory and its connections to other stochastic analysis related topics, we refer to [87, 89, 54], a new look on rough paths through controlled rough paths ([60, 61]) and its connections to regularity structure ([65]) refers to [53].

Before we introduce the rough paths, let's first recall the following definition of $\alpha$-Hölder and $p$-variation paths on general metric spaces (in most cases, we deal with $\mathbb{R}^d$ equipped with the usual distance function).

**Definition 2.1.1.** [54, Definition 5.1] Let $(E, d)$ be a metric space and $x : [0, T] \to E$. A path $x : [0, T] \to E$ is said to be

(1) Hölder continuous with exponent $\alpha \geq 0$, or simply $\alpha$-Hölder, if

$$|x|_{\alpha-Hol;[0,T]} = \sup_{0\leq s<t\leq T} \frac{d(x_s, x_t)}{|x_s - x_t|^\alpha} < \infty.$$

(2) of finite $p$-variation for some $p > 0$, if

$$|x|_{p-var;[s,t]} = \left( \sup_{(t_i)\in\mathcal{D}([s,t])} \sum_i d(x_{t_i}, x_{t_{i+1}})^p \right)^{1/p} < \infty.$$

denote $\mathcal{C}^{\alpha-Hol}([0,T],E)$ the set of $\alpha$-Hölder paths $x$ and $\mathcal{C}^{p-var}([0,T],E)$ for set of continuous paths with finite $p$-variation. In particular, if $\alpha = 1$ and $p = 1$, $\mathcal{C}^{1-Hol}([0,T],E)$ denotes the set of Lipschitz or 1-Hölder paths and $\mathcal{C}^{1-var}([0,T],E)$ denotes the set of continuous paths with finite 1-variation.

**Remark 2.1.2.** As a fact [54, Theorem 5.25], for $\mathbb{R}^d$ equipped with Euclidean distance, $\mathcal{C}^{p-var}([0,T],\mathbb{R}^d)(\mathcal{C}^{1/p-Hol}([0,T],\mathbb{R}^d))$ is actually a Banach space with norm $x \mapsto |x(0)| + |x|_{p-var;[0,T]}$ $(x \mapsto |x(0)| + |x|_{1/p-Hol;[0,T]})$.

Now, we start to introduce the framework of rough paths. For each $N \in \mathbb{N}$, the truncated tensor algebra $T^N(\mathbb{R}^d) = \bigoplus_{k=0}^N (\mathbb{R}^d)^{\otimes k}$ is equipped with a straightforward vector space structure and an operation $\otimes$ defined by

$$\pi_m(g \otimes h) = \sum_{k=0}^N \pi_{m-k}(g) \otimes \pi_k(h), \quad g, h \in T^N(\mathbb{R}^d),$$

where $\pi_m$ designates the projection on the $m$-$th$ tensor level. Then $(T_N(\mathbb{R}^d), +, \otimes)$ is an associative algebra with unit element $\mathbf{1} \in (\mathbb{R}^d)^{\otimes 0}$. For a continuous path with finite variation, the truncated signature of $x$ is a $T^N(\mathbb{R}^d)$ valued two parameter path defined as

$$\mathcal{S}_N(x)_{s,t} = 1 + \sum_{k=1}^{N} \int_{\triangle^k[s,t]} dx^{\otimes k} \tag{2.1.1}$$

where

$$\begin{aligned} \int_{\triangle^k[s,t]} dx^{\otimes k} &= \sum_{I\in\{0,1,\cdots,d\}^k} \left( \int_{\triangle^k[s,t]} dx^I \right) e_{i_1} \otimes \cdots \otimes e_{i_k} \\ \int_{\triangle^k[s,t]} dx^I &= \int_{0<t_1<t_2\cdots<t_k<t} dx^{i_1}(t_1)\cdots dx^{i_k}(t_k). \end{aligned}$$

Here $(e_1.\cdots,e_n)$ is the canonical basis for $\mathbb{R}^d$ and $\triangle^k[s,t] = \{(t_1,\cdots,t_k) \in [s,t]^k, s \leq t_1 \leq t_2 \cdots \leq t_k \leq t\}$ is the subdivision of the time interval.

In particular, these elements (2.1.1) take values in the strict subset $G^N(\mathbb{R}^d) \subset T^N(\mathbb{R}^d)$ given by the *group-like* elements

$$\begin{aligned} &G^N(\mathbb{R}^d) = \exp^{\oplus}\big(L^N(\mathbb{R}^d)\big), \text{ where} \\ &L^N(\mathbb{R}^d) = \mathbb{R}^d \oplus [\mathbb{R}^d,\mathbb{R}^d] \oplus \cdots \oplus [\mathbb{R}^d,[\cdots,[\mathbb{R}^d,\mathbb{R}^d]\cdots] \end{aligned}$$

and for two elements in $T^N(\mathbb{R}^d)$, $[a,b] = a\otimes b - b\otimes a$ which is known as the Lie bracket. This set is called free nilpotent group of step $N$, and is equipped with the classical Carnot-Caratheodory norm (short for *c-c* norm) in the following sense which we simply denote by $\|\cdot\|$.

**Theorem 2.1.3.** *[54, Theorem 7.32] For every $g \in G^N(\mathbb{R}^d)$, the so-called "Carnot-Caratheodory norm" is defined as*

$$||g|| := \inf\{\int_0^1 |d\gamma| : \gamma \in \mathcal{C}^{1-var}([0,1],\mathbb{R}^d) \ \ and \ \ S_N(\gamma)_{0,1} = g\}$$

*which is finite and achieved at some minimizing path $\gamma^*$,i.e.*

$$||g|| = \int_0^1 |d\gamma^*| \quad and \quad S_N(\gamma^*)_{0,1} = g.$$

For a path $\mathbf{x} \in \mathcal{C}([0,1], G^N(\mathbb{R}^d))$, the $p$-variation norm of $\mathbf{x}$ is defined to be

$$\|\mathbf{x}\|_{p-\text{var};[0,1]} = \sup_{\Pi \subset [0,1]} \left( \sum_i \|\mathbf{x}_{t_i}^{-1} \otimes \mathbf{x}_{t_{i+1}}\|^p \right)^{1/p}$$

where the supremum is taken over all subdivisions $\Pi$ of $[0,1]$.

In particular, the space of p-rough paths is denoted by $\mathcal{C}^{p-var}([0,T], \mathbb{R}^d)$ and the signature of a p-rough paths takes values in $G^{[p]}(\mathbb{R}^d)$. To make it precise ([54, 88]),

**Definition 2.1.4.** For $p \geq 1$, $\mathbf{x} : [s,t] \to G^{[p]}(\mathbb{R}^d)$ is said to be a geometric rough path if it is the p-var limit of a sequence $S_{[p]}(x^m)$ of signatures of smooth functions $x^m$. In particular, it is an element of the space

$$\mathcal{C}_0^{p-var;[0,1]}([0,1], G^{[p]}(\mathbb{R}^d)) = \{\mathbf{x} \in \mathcal{C}([0,1], G^{[p]}(\mathbb{R}^d)); \|\mathbf{x}\|_{p-var} \leq \infty\}. \tag{2.1.2}$$

**Remark 2.1.5.** For a $\mathbb{R}^d$-valued path $x$ with finite $p$-variation, we call $S_N(x)$ the canonical lift of $x$ which is $G^N(\mathbb{R}^d)$ valued paths. The truncated signature is enough for $N = [p]$, since the higher terms can be canonically determined by the lower terms. Indeed, for any $N \geq [p]$, there exists a canonical bijection [54, Theorem 9.5]:

$$S_N : \mathcal{C}_0^{p-var}([0,T], G^{[p]}(\mathbb{R}^d)) \to \mathcal{C}_0^{p-var}([0,T], G^N(\mathbb{R}^d))$$

Moreover, for $x \in \mathcal{C}_0^{p-var}([0,T],\mathbb{R}^d)$ we have

$$\|\mathbf{x}\|_{p-var;[0,T]} \leq \|S_N(x)\|_{p-var} \leq C_N\|x\|_{p-var;[0,T]}$$

Thus, it is natural for $p$-rough paths, we just consider the truncated signature of order $[p]$, then the approximation of the path (linear approximation, geodesic approximation, etc.) is sufficient to consider the convergence of its first $[p]$-th integrals.

**Remark 2.1.6.** In most of cases, we consider $p$ rough paths with $2 \leq p < 3$ which corresponds to $\alpha$-Hölder paths ($\alpha = \frac{1}{p}$) with $\frac{1}{3} < \alpha \leq \frac{1}{2}$. In the context of $\alpha$-Hölder paths, [53, Proposition 2.4] gives a concise and more algebraic characterisation of geometric rough paths.

In the following, we will briefly mention that how to connect the rough paths theory to general Gaussian process and fractional Brownian motion, so that we can use a purely analytic method to study probability. We assume that $X_t = (X_t^1, \cdots, X_t^d)$ is a continuous, centered Gaussian process with i.i.d components on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A lot of information concerning $X$ is encoded in the rectangular increments of the covariance function $R$, which is given by

$$R_X \begin{pmatrix} s,t \\ u,v \end{pmatrix} \equiv R_{uv}^{st} \equiv \mathbb{E}[(X_t^1 - X_s^1)(X_v^1 - X_u^1)].$$

We call $2D$ $\rho$-variation of $R$ on rectangle $[s,t]^2$ the quantity

$$V_\rho(R;[s,t]^2) \equiv \sup\{(\sum_{i,j} |R_{s_j,s_{j+1}}^{t_j,t_{j+1}}|^\rho)^{1/\rho}; (s_j),(t_j) \in \mathcal{D}([s,t])\}$$

where $\mathcal{D}([s,t])$ denotes the sets of partitions of $[s,t]$. For simplicity, we denote $V_\rho(R) = V_\rho(R;[0,1]^2)$. In particular, $V_\rho(R)$ is a special case, recovered as $V_\rho = V_{\rho,\rho}$ for the mixed right $(\gamma,\rho)$-variation (see [55]) defined as below: For $\gamma, \rho \geq 1$ let

$$V_{\gamma,\rho}(R_X;[s,t][u,v]) := \sup_{\substack{(t_i)\in\mathcal{D}([s,t])\\(t'_j)\in\mathcal{D}([u,v])}} \left( \sum_{t'_j} \left( \sum_{t_i} \left| R_X \begin{pmatrix} t_i, t_{i+1} \\ t'_j, t'_{j+1} \end{pmatrix} \right|^\gamma \right)^{\frac{\rho}{\gamma}} \right)^{\frac{1}{\rho}},$$

With the concept introduced above, we now can state the following theorem [54].

**Proposition 2.1.7.** *Let $X_t = (X_t^1, ..., X_t^d)$ be a continuous, centered Gaussian process as above. If $X_t$ has finite 2-dimensional $\rho$-variation for $\rho \in [0,2)$, then $X_t$ has a lift to a geometric p-rough path provided $p > 2\rho$. Moreover, there is a unique natural lift which is the limit, in the p-var topology, of the canonical lift of piecewise linear approximations to $X$.*

Next, let's introduce the definition of fractional Brownian motion: $B_t = (B_t^1, \cdots, B_t^d)$ is a d-dimensional fractional Brownian motion indexed by $[0,1]$, with Hurst parameter $H > 1/4$, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, if the components $B^i$ are i.i.d and that each $B^i$ is centered Gaussian process satisfying

$$\mathbb{E}[(B_t^i B_s^i)] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H})$$

for $s, t \in [0,1]$. The following result is borrowed from [87].

**Proposition 2.1.8.** *Let $B_t = (B_t^1, \cdots, B_t^d)$ be a d-dimensional fractional Brownian motion with Hurst parameter $1/4 < H < 1/2$. Then $B_t$ has a lift to a geometric p-rough path provided $p > 1/H$, $p < 4$. Moreover, there is a unique natural lift in p-variation topology , of the canonical lift of piecewise linear approximations of $B_t$.*

**Remark 2.1.9.** The study of fractional Brownian motion and general Gaussian process is widely studied by using Malliavin calculus. Malliavin Calculus was first invented by P. Malliavin [92, 93] to give a probabilistic proof of Hörmander hypoellipticity theorem. For a detailed study and a thorough history of the development of Malliavin calculus, we refer to [20, 94, 99] and the references therein. To see the connection between Malliavin calculus and rough paths theory, we refer to [54, 53]

## 2.2 Unbounded rough drivers

### 2.2.1 Controlled rough paths

The purpose of the rough paths introduced in the previous section is to define integral against irregular functions, e.g. the integral form in (1.1.1). In general, we can write the differential equaiton (1.1.1) in the following form

$$dY_t = f(Y_t)dX_t \tag{2.2.1}$$

The question is how to define the integral $\int_0^t f(Y_s)dX_s$ for irregular function $X$ and function $f$ and in which sense can we extend this notion to work for partial differential equations, e.g. (1.1.3).

The controlled rough paths was introduced by M. Gubinelli [60] to define the integral against irregular functions like the form (2.2.1). Now we give a formal definition of controlled rough paths. We denote a general Banach space as $E$ and for a general $X : [0,T] \to \mathbb{R}^d$ as a $\mathbb{R}^d$ valued paths. The iterated integral of $X$, $\int_s^t X_{s,r} \otimes dX_r$ is usually denoted as $\mathbb{X}^2_{s,t}$. (e.g. in components, we have $\mathbb{X}^{2,ij}_{s,t} = \int_s^t X^i_{s,r} dX^j_r$ ). Recall the definition of the increment operator, denoted as $\delta$. Then $\delta X_{st} = X_t - X_s$. When $\mathbb{X}$ is a 2-index map defined on $[0,T]^2$, we define $\delta \mathbb{X}_{sut} = \mathbb{X}_{st} - \mathbb{X}_{su} - \mathbb{X}_{ut}$.

**Definition 2.2.1** (*Controlled* rough paths)**.** Suppose $Y$ takes values in a Banach space $V$ and $f \in \mathcal{L}(\mathbb{R}^d, E)$. Given a path $X \in \mathcal{C}^\alpha([0,T],\mathbb{R}^d)$, we say that $Y \in \mathcal{C}^\alpha([0,T],E)$ is *controlled* by $X$ if there exists $Y' \in \mathcal{C}^\alpha([0,T],\mathcal{L}(\mathbb{R}^d,E))$ so that the remainder term $Y^\sharp$ given by

$$Y^\sharp_{s,t} = \delta Y_{st} - Y'_s X_{s,t},$$

satisfies $||Y^\sharp||_{2\alpha} < \infty$. This defines the space of *controlled* rough paths,

$$(Y, Y') \in \mathcal{D}^{2\alpha}_X([0,T],E).$$

where $\mathcal{D}^{2\alpha}_X([0,T],E)$ denotes the linear space of paths equipped with the semi-norm

$$\|Y, Y'\|_{\mathcal{D}^{2\alpha}_X} =: \|Y'\|_\alpha + \|Y^\sharp\|_{2\alpha}.$$

**Remark 2.2.2.** In this frame work of controlled rough path by Gubinelli [60], the original framework of rough path theory by Lyons [88] can be recovered. Namely, the differential equation of the type (1.1.1) and (2.2.1) driving by $\gamma$-Hölder paths $(X,\mathbb{X})$ (with $1/3 < \gamma \leq 1/2$) can be well defined and the extension of multiplicative paths to any degree and the construction of a multiplicative path from an almost-multiplicative one can be proved. More details can be found in [60, Section 5, 7].

**Remark 2.2.3.** A more general definition about Banach space valued paths $(X,\mathbb{X})$, as well as more discussions can be found in [53, Chapter 4]. In general, we call any such $Y'$ the *Gubinelli* derivative of $Y$ with respect to $\mathbf{X} = (X, \mathbb{X})$.

**Remark 2.2.4.** Given a rough path $\mathbf{X} = (X, \mathbb{X})$ and path $(Y, Y') \in \mathcal{D}_X^{2\alpha}(\mathcal{L}(\mathbb{R}^d, E))$ controlled by $\mathbf{X}$, we can actually define a new controlled path $Z, Z' \in \mathcal{D}_X^{2\alpha}(\mathcal{L}(\mathbb{R}^d, E))$ by taking $Z' = Y$ and $Z$ the unique solution to

$$\delta Z_{st} = Y_s X_{s,t} + Y_s' \mathbb{X}_{s,t}^2 + Z^\sharp(s,t)$$

Moreover, $Z^\sharp$ belongs to $\mathcal{C}_2^{3\alpha}(E)$ and we call $Z$ the integral of $Y$ with respect to $\mathbf{X}$.

This more or less gives us a brief idea of how to use controlled rough paths to define solutions of the aforementioned integrals and how the idea of iteration is used in this setting. We now can introduce the newly introduced framework called *unbounded rough drivers* [4, 33].

### 2.2.2 Unbounded rough drivers

The frame work of unbounded rough drivers was first introduced in [4] by I. Bailleul and M. Gubinelli. The goal is to develop a general theory of rough PDEs aiming at extending classical PDE tools such as weak solutions, *a priori* estimates, compactness results, etc. Comparing to several recent works on the intrinsic notions of viscosity solutions [90] to rough PDEs, weak solutions [27] to linear transport equations (see also [75, 77] and the references in [4, 33]), the frame work of unbounded rough drivers has the advantage to deal with more regular PDEs in view of weak solutions and does not need explicit formulas involving the flow of rough characteristics. This frame work of unbounded rough drivers is first used to study linear symmetric hyperbolic systems of the form ([4])

$$\partial_t f + a\nabla f = 0 \tag{2.2.2}$$

where $f$ is an $\mathbb{R}^d$ valued space-time distribution on $\mathbb{R}_+ \times \mathbb{R}^d$ and $a : \mathbb{R}_+ \times \mathbb{R}^d \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^{d\times d})$ is a $d \times d$ matrix-valued family of time-dependent vector fields in $\mathbb{R}^d$. *a priori* estimates is established in [4] as well as $L^2$ theory for rough linear equations. Later on, unbounded rough drivers are used to prove a general **rough Gronwall lemma** which can be used for general rough PDEs in [33] and provide an example with application to rough conservation laws. To wit, this is a variational formulation of the PDEs that we are interested and it is algebraically convenient. In the following, we are going to give a brief introduction of unbounded rough drivers [4, 33] and rough Gronwall lemma [33].

As before, we denote $\delta$ as the increment operator. For an interval $I$ we call *control* on $I$ ( and denote it by $\omega$) any continuous superadditive map on $\triangle_I := \{s, t \in I^2 : s \leq t\}$, that is, any continuous map $\omega : \triangle_I \to [0, \infty)$ such that for all $s \leq t \leq u$ in $[0, T]$,

$$\omega(s,t) + \omega(t,u) \leq \omega(s,u) \tag{2.2.3}$$

Given a control $\omega$ on an interval $I = [a, b]$, we will use the notation $\omega(I) := \omega(a, b)$. For a fixed time interval $I$, a parameter $p > 0$, a Banach space $E$ and any continuous function $g : I \to E$ we define the norm

$$\mathcal{N}[g; V_1^p(I; E)] := \sup_{(t_i)\in\mathcal{D}(I)} \left( \sum_i |\delta g_{t_i,t_{i+1}}|^p \right)^{\frac{1}{p}} \tag{2.2.4}$$

as before $\mathcal{D}(I)$ denotes the set of all partitions of the interval $I$. In this case,

$$\omega_g(s,t) = \mathcal{N}[g; V_1^p([s,t];E)]^p$$

defines a control on $I$. We denote by $V_2^p(I;E)$ the set of continuous two-index maps $g$: $I \times I \to E$ for which there exists a control $\omega$ such that

$$|g_{st}| \leq \omega(s,t)^{\frac{1}{p}}$$

for all $s,t \in I$. We also define the space $V_{2,loc}^p(I;E)$ of maps $g : I \times I \to E$ such that there exists a countable covering $\{I_k\}_k$ of $I$ satisfying $g \in V_2^p(I_k;E)$ for any $k$.

We now introduce the definition of bounded rough driver first. Below, we denote $(\mathcal{A}, \ |\cdot|)$ as a Banach algebra with unit $\mathbf{1}_{\mathcal{A}}$.

**Definition 2.2.5.** [4, Definition 2] Let $\alpha \in [\frac{1}{3}, \frac{1}{2})$ be given. A bounded $\alpha$-rough driver in $\mathcal{A}$ is a pair $\mathbf{A} = (A^1, A^2)$ of $\mathcal{A}$-valued 2-index maps satisfying $Chen's$ relations

$$\delta A^1 = 0, \quad \text{and} \quad \delta A^2_{sut} = A^1_{su} A^1_{ut}, \tag{2.2.5}$$

and such that $A^1$ is $\alpha$-Hölder and $A^2$ is $2\alpha$-Hölder. The norm is defined by the formula

$$\|\mathbf{A}\| := \sup_{0 \leq s < t < T; t-s \leq 1} \frac{|A^1_{ts}|}{|t-s|^{\alpha}} \vee \frac{|A^2_{ts}|}{|t-s|^{2\alpha}}$$

**Example 2.2.6.** Connecting to the previous section, take $\mathcal{A}$ to be the truncated tensor algebra $\oplus_{i=0}^N (\mathbb{R}^d)^{\otimes i}$ over $\mathbb{R}^d$, for $N \geq 2$. Consider a weak geometric $\alpha$-Hölder rough paths $\mathbf{X}_{st} = 1 \oplus X_{st} \oplus \mathbb{X}_{st} \in \oplus_{i=0}^N (\mathbb{R}^d)^{\otimes i}$ with $\alpha \in [\frac{1}{3}, \frac{1}{2})$. Left multiplication by $X_{st}$ and $\mathbb{X}_{st}$ define operators $A^1$ and $A^2$ which are components of $A$, a bounded rough driver.

**Remark 2.2.7.** In particular, $\alpha$-rough driver with $\alpha \in [\frac{1}{3}, \frac{1}{2})$ is equivalent to $p$-rough driver with $p \in [2, 3)$. Then the $\alpha$-Hölder norm is replaced by the $p$-variation norm. In the following, when we introduce the unbounded rough drivers, we prefer to use the term $p$-rough driver, since $p$-variation norm is more convenient dealing with a scale sequence of Banach spaces and linear operators between those spaces.

Formally, the previous transport equation (2.2.2) can also formally fit in the framework of the rough drivers,in the sense that, if we do formal iteration, the solution can be formally represented in the following sense (For more discussions, see [4])

$$f_t = f_s + A^1_{st} f_s + A^2_{st} f_s + R_{st}$$

where $A^1_{st} = \int_s^t V_r dr$ and $A^2_{st} = \int_s^t \int_s^r V_r V_{r'} dr' dr$ with $V_r = a_r \nabla$.

However, in order to make the solution of the equation well-defined, namely in order to make sense of the nice algebraic conditions, we need to propose regularity properties in this context. Thus, we need to introduce a scale of Banach spaces $(E_n, |\cdot|)_{n\geq 0}$ with $E_{n+1}$ continuously embedded in $E_n$. For $n \geq 0$, denote $E_{-n} = E_n^*$ which is the dual space of $E_n$, equipped with its natural norm

$$|e|_{-n} := \sup_{\varphi \in E_n; |\varphi|_{E_n} \leq 1} (\varphi, e), \quad e \in E_{-n}$$

with the following continuous inclusion for all $n \geq 2$,

$$E_n \subset \cdots E_2 \subset E_1 \subset E_0.$$

Denote by $\|\cdot\|_{n,m}$ for the norm of a linear operator from $E_n$ to $E_m$. We also assume the existence of a family $(J^\eta)_{0<\eta\leq 1}$ of operators from $E_0$ to itself satisfying the following estimates

$$\|J^\eta - Id\|_{\mathcal{L}(E_{n+k},E_n)} \lesssim \eta^k, \quad \|J^\eta\|_{\mathcal{L}(E_n,E_{n+k})} \lesssim \eta^{-k}.$$

The operator $J^\eta$ plays the role of a smoothing operator here, since the element in $E_0$ may not be smooth enough. One example of the above defined Banach space is a sequence of Sobolev spaces $W^{k,n}(\mathbb{R}^d)$. With the necessary notation and background, we now in a good position to introduce unbounded rough drivers.

**Definition 2.2.8.** [33, Definition 2.2] (see also [4, Definition 5]) Let $p \in [2,3)$ be given. A continuous **unbounded $p$-rough driver** with respect to the scale $(E_n, \|\cdot\|_n)_n \in \mathbb{N}_0$, is a pair $\mathbf{A} = (A^1, A^2)$ of 2-index maps such that

$$A^1_{st} \in \mathcal{L}(E_{-n}, E_{-(n+1)}) \text{ for } n \in \{0,2\}, \quad A^2_{st} \in \mathcal{L}(E_{-n}, E_{-(n+2)}) \text{ for } n \in \{0,1\},$$

and there exists a continuous control $\omega_A$ on $[0,T]$ such that for every $s,t \in [0,T]$,

$$\begin{aligned} \|A^1_{st}\|^p_{\mathcal{L}(E_{-n},E_{-(n+1)})} &\leq \omega(s,t) \quad \text{for } n \in \{0,2\} \\ \|A^2_{st}\|^{p/2}_{\mathcal{L}(E_{-n},E_{-(n+2)})} &\leq \omega(s,t) \quad \text{for } n \in \{0,1\}, \end{aligned}$$

and,in addition, the $Chen's$ relation holds true, that is

$$\delta A^1_{sut} = 0, \quad \delta A^2_{sut} = A^1_{su}A^1_{ut}, \quad \text{for all } 0 \leq s < u < t \leq T.$$

**Example 2.2.9.** A simple example about how to implement the above definition to linear rough heat equation can be found in [33, Section 2.3].

### 2.2.3 Rough Gronwall's lemma

Below, we first recall the following result which is often referred to as *sewing lemma* in the literature, and is at the core in this setting to generalized integration. Various discussions and proofs of this *sewing lemma* in different contexts refer to [33, Lemma 2.1] and [53, Lemma 4.2]

**Lemma 2.2.10.** *Fix an interval $I$, a Banach space $E$ and a parameter $\zeta > 1$. Consider a function $h : I^3 \to E$ such that $h \in \mathrm{Im}, \delta$ and for every $s < u < t \in I$,*

$$|h_{sut}| \leq \omega(s,t)^{\zeta}, \tag{2.2.6}$$

*for some control $\omega$ on $I$. Then there exists a unique element $\Lambda h \in V_2^{\frac{1}{\zeta}}(I; E)$ such that $\delta(\Lambda h) = h$ and for every $s < t \in I$,*

$$|(\Lambda h)_{st}| \leq C_{\zeta}, \omega(s,t)^{\zeta}, \tag{2.2.7}$$

*for some universal constant $C_{\zeta}$.*

The main purpose (in [33]) is to get *apriori* estimates for a large class of rough PDEs which has the following form

$$dg_t = \mu(dt) + \mathbf{A}(dt)g_t$$

where $\mathbf{A} = (A^1, A^2)$ is an unbounded $p$-rough driver with scale $E_n$ and the drift $\mu$. The solution is understood in the sense of distributions and can be well-defined in the form of weak solutions with the chose scale $E_n$. As a consequence, we can get the following rough version of Gronwall's lemma. For a complete proof and discussion, we refer to [33, Lemma 2.7].

**Lemma 2.2.11.** *Fix a time horizon $T > 0$ and let $Q : [0,T] \to [0,\infty)$ be a path such that for some constants $C, L > 0$, $\kappa \geq 1$ and some controls $\omega_1, \omega_2$ on $[0,T]$, one has*

$$\delta Q_{st} \leq C\Big(\sup_{0\leq r\leq t} Q_r\Big), \omega_1(s,t)^{\frac{1}{\kappa}} + \omega_2(s,t), \tag{2.2.8}$$

*for every $s < t \in [0,T]$ satisfying $\omega_1(s,t) \leq L$. Then it holds*

$$\sup_{0\leq t\leq T} Q_t \leq 2\exp\left(c_{\kappa,L}, \omega_1(0,T)\right) \cdot \Big\{Q_0 + \sup_{0\leq t\leq T}\Big(\omega_2(0,t), \exp\left(-c_{\kappa,L}, \omega_1(0,t)\right)\Big)\Big\},$$

*for a strictly positive constant $c_{\kappa,L}$.*

Discussions and applications of this rough Gronwall's lemma can be found in [33, 52]

## 2.3 Stochastic differential equations driven by fractional Brownian motion with $H > 1/4$ and general Gaussian process.

In this section, we introduce the main results from [50]. In the first part, we introduce the Taylor expansion of rough differential equation (1.1.1), then we provide a general convergence result of the Taylor expansion for the solution $y(t)$ of the differential equation (1.1.1) and study the rate of the convergence of Taylor series. Our result is well adapted to the stochastic differential equations driven by fractional Brownian motion with Hurst parameter $1/4 < H < 1/2$ and continuous Gaussian process with finite $2D$ $\rho-$variation with i.i.d components. In this part, the general idea of the proof is adapting from a previous work by X. Zhang and F. Baudoin [18] where they studied stochastic differential equations of the type (1.1.1) driving by fractional Brownian motion with Hurst parameter $H > 1/2$.

In the second part, we study asymptotic properties of the solution of (1.1.1) driven by continuous centered Gaussian process with finite $2D$ $\rho-$variation and fractional Brownian motion with Hurst parameter $H > 1/4$. We first generalize the concentration inequality in [8] to get an similar estimate for the solution of differential equations driven by continuous centered Gaussian process with finite $2D$ $\rho-$variation. We prove a Castell expansion for the solutions of (1.1.1) driven by mean zero i.i.d Gaussian process with finite $2D$ $\rho-$variation and fractional Brownian motion with Hurst parameter $H > 1/4$. Moreover, for both the Taylor expansion and the Castell expansion of the solutions in the above two cases, we prove the same type of tail estimates for the remainder terms.

Before we present the main result, let us first fix some notations for this section.

1. A word $I = (i_1, \cdots, i_k) \in \{0, \cdots, d\}^k$, we denote $|I|$ the size of $I$, $k$. And $||I|| = |I|+$number of 0 in $I$, called the order of $I$.

2. For $V_0, V_1, \cdots, V_d$ vector fields on $\mathbb{R}^d$, $V_I$ is the Lie bracke of the vector fields $V_i$

$$V_I = [V_{i_1}[V_{i_2}, \cdots [V_{i_{m-1}}, V_{i_m}]] \cdots]$$

3. We denote $\Lambda_I$ as below ([26]):

$$\Lambda_I(x) = \sum_{\sigma \in \sigma_{|I|}} \frac{(-1)^{e(\sigma)}}{|I|^2 \binom{|I|-1}{e(\sigma)}} X_t^{I \circ \sigma^{-1}}$$

where $\sigma$ is a permutation of order $|I|$, $\sigma_{|I|}$ is the set of all permutations of order $|I|$. $e(\sigma)$ ([109]) is the cardinality of the set $\{i \in \{1, \cdots, k-1\}; \sigma(i) > \sigma(i+1)\}$. For a word $I$ of size $k$,

$$I \circ \sigma = (i_{\sigma(1)}, \cdots, i_{\sigma(k)}).$$

### 2.3.1 Taylor expansion for differential equations driven by p-rough paths

We recall the following differential equation (1.1.1) which we are going to study:

$$y(t) = y_0 + \sum_{i=1}^{d} \int_0^t V_i\left(y(s)\right) dx^i(s) \tag{2.3.1}$$

We assume the following Hypothesis in the whole section:

**Hypothesis 2.3.1.** (i) *The $V_i$'s are $C^\infty$ vector fields on $\mathbb{R}^n$ with bounded derivatives, and analytic on the set $\{y : \|y - y_0\| \leq C\}$ for some $C > 0$.*
(ii) *The driving path $x : [0, T] \to \mathbb{R}^d$ is p-rough path with given approximating sequence $x_n \in C^{1-var}([0, T], \mathbb{R}^d)$.*

Then the Taylor expansion of $y_t$ can be obtained from an iterative application of the change of variable formula. (Adapted from [18], more details refer to [50]). This leads to the following definition:

**Definition 2.3.2.** The Taylor expansion associated with the differential equation (1.1.1) is defined as

$$y_0 + \sum_{k=1}^{\infty} g_k(t),$$

where

$$g_k^j(t) = \sum_{|I|=k} P_I^j \int_{\triangle^k[0,t]} dx^I, \quad P_I^j = (V_{i_1} \cdots V_{i_k} \pi^j)(y_0).$$

Tt this point, the Taylor expansion associated to differential equation (1.1.1) is only a formal object. We need to address the convergence of the Taylor expansion and its relation to the solution of the rough differential equation (1.1.1) in the following.

### 2.3.2 Convergence of the Taylor expansion

In this part, we show that we can express the solution $y_t$ of the differential equation (1.1.1) as its Taylor expansion on a non-empty time interval. We prove the following result as a general convergence result of the Taylor expansion. The details about the proof refer to [50] and the strategy of the proof is similar to the proof in [18].

**Theorem 2.3.3.** *[50, Theorem 3.4] Let $y_0 + \sum_{k=1}^{\infty} g_k(t)$ be the Taylor expansion associated with the equation (1.1.1) as defined in 2.3.2. There exists $T > 0$ such that for $0 \le t < T$ the series*

$$\sum_{k=1}^{\infty} \|g_k(t)\|$$

*is convergent and*

$$y_t = y_0 + \sum_{k=1}^{\infty} g_k(t).$$

Theorem 2.3.3 is very general but gives few information concerning the radius of convergence or the rate of convergence of the Taylor expansion. In the next theorem, we study the quantitative bounds of the Taylor expansion. The crucial estimates we need to control the rate of convergence of the Taylor expansion is given by the theorem 7.16 in [7] which is proved by T. Lyons (see [86]) for the p-rough path. In the following theorem, we prove the speed of convergence of the Taylor expansion under natural assumptions on the vector fields.

**Theorem 2.3.4.** *[50, Theorem 3.6] Let* $p \geq 1$ *and we assume that there exist* $M > 0$ *and* $0 < \gamma < \frac{1}{p}$ *such that for every word* $I \in \{0, \cdots, d\}^k$

$$\|P_I\| \leq \Gamma(\gamma|I|)M^{|I|}. \tag{2.3.2}$$

*For* $r > 1$, *we define* $T_C(r) = \inf\{t, \sum_{k=1}^{\infty} r^k \|g_k(t)\| \geq C/2\}$. *Then:*

1. $T_C(r) > 0$ *and when* $t < T_C(r)$,

$$y_t = y_0 + \sum_{k=1}^{\infty} g_k(t)$$

2. *There exists a constant* $Q_{p,\gamma,M,T} > 0$ *depending on the subscript variables such that when* $t < T_C(r)$,

$$\left\| y_t - \left( y_0 + \sum_{k=1}^{N} g_k(t) \right) \right\| \leq Q_{p,\gamma,M,T} \frac{\left(MK\omega(0,T)^{1/p}\right)^N}{\Gamma((\frac{1}{p} - \gamma)N)} e^{2(MC\omega(0,T)^{\frac{1}{p}})^{\frac{p}{1-p\gamma}}}.$$

*where*

$$\omega(0,t) = \left( \sum_{j=1}^{[p]} \left\| \int dx^{\otimes j} \right\|_{\frac{p}{j}-var,[s,t]}^{1/j} \right)^p$$

*recall that* $N$ *is arbitrary here.*

*Proof.* The detailed proof can be found in [50, Theorem 3.6]. □

**Example 2.3.5.** A non-trivial Lie group example where our theorem 2.3.4 applies can be found in [18, section 2.4], where the following differential equation is considered on a connected Lie group $\mathbb{G}$ with its Lie algebra $\mathfrak{g}$

$$\begin{cases} dy_t = \sum_{i=0}^{d} V_i(y_t)dx_t^i \\ y_0 = e \end{cases}$$

where $e$ is the identity element of $\mathbb{G}$ , $V_1, V_2, \cdots, V_d \in \mathfrak{g}$ are left invariant vector field and instead of having $\beta-$Hölder path with $\beta > 1/2$ we consider $x$ as $p$-rough path with $p > 2$ satisfying Hypothesis (2.3.1) $(ii)$. The case where $\mathbb{G} = Gl(n, \mathbb{R})$, $\mathfrak{g} = M(n, \mathbb{R})$ is true as well, for further details, we refer to [18].

### 2.3.3 Castell expansion and tail estimate for differential equations driven by p-rough paths

In this part, we start to study the asymptotic expansion and tail estimates for stochastic differential equations of the type (1.1.1) driven by continuous centered Gaussian process with finite $2D$ $\rho$-variation and i.i.d components and fractional Brownian motion with Hurst parameter $H > 1/4$. The main result is the following theorem about the asymptotic Castel expansion and tail estimate for the solutions of the scaled differential equations (2.3.4) driven by continuous centered Gaussian process with finite $2D$ $\rho$-variation and i.i.d components. We can also get the same type of result for the differential equation (2.3.4) driven by fractional Brownian motion with Hurst parameter $H > 1/4$ by the same strategy. We also show the same type of remainder estimate for the Taylor expansion of the solution of (2.3.4).

**Theorem 2.3.6.** *[50, Theorem 4.1]* $\exists\ T_C(r) > 0$ *and* $\epsilon \geq 0$ *such that for* $\forall\ t < T_C(r)$,

$$y_t = \exp\left(\sum_{k=1}^{N} \epsilon^k \sum_{I\in\{0,1,\cdots,d\}^k} \Lambda_I(x)_t V_I\right)(y_0) + \epsilon^{N+1} R_{N+1}(\epsilon, t),\ \ a.s. \tag{2.3.3}$$

*is the solution of the following scaled differential equation with initial value* $y_0$*, recall that* $N$ *is arbitrary here:*

$$dy_t^\epsilon = \sum_{i=0}^{d} \epsilon V_i(y_t) dx_t^i, \tag{2.3.4}$$

*satisfying the following condition: there exists* $\alpha, c > 0,\ \forall\ 0 < \tau < T_C(r)$ *and for all* $\xi \geq 1$,

$$\mathbb{P}[\sup_{t\in[0,\tau]} ||R_{N+1}(\epsilon, t)|| \geq \xi; \tau < T_C(r)] \leq \exp\left(-c\frac{\xi^\alpha}{(C_{emb}(t))^2}\right) \tag{2.3.5}$$

*Where :*

- *The* $V_i's$ *are* $C^\infty$ *vector fields, bounded together with all their derivatives denoted as* $C_b^\infty$ *and analytic on the set* $\{y : ||y - y_0|| \leq C\}$*, for some* $C > 0$.
- *The driving path* $x = (x_t^1, \cdots, x_t^d) : [0, T] \to \mathbb{R}^d$ *is continuous, centered Gaussian process with i.i.d components and finite 2-dimensional* $\rho$*-variation for* $\rho \in [0, 2)$ *and* $x_t^0 = t$*.* $x$ *has a lift as a geometric* $p$*-rough path,* $p > 2\rho$*, with given approximating sequence* $x_n \in C^{1-var}([0, T], \mathbb{R}^d)$

**Remark 2.3.7.** The Cameron-Martin space $\bar{\mathcal{H}}$ associated with the general Gaussian process $x_t$ is defined to be the completion of the linear space of functions of the form

$$\sum_{i=1}^{n} a_i R(t_i, \cdot), \quad a_i \in \mathbb{R} \text{ and } t_i \in [0, T],$$

with respect to the inner product induced by $\langle R(t_i,\cdot), R(s_j,\cdot)\rangle_{\mathcal{H}} = R(t_i,s_j)$. The embedding coefficient from the Cameron-Martin space $\bar{\mathcal{H}}$ to the space of continuous functions with finite $q-$variation is defined as $C_{emb}(t)$ with $1/p+1/q>1$, such that $\forall\ h \in \bar{\mathcal{H}}$,

$$|h|_{q-var;[0,T]} \leq C_{emb}(t)|h|_{\bar{\mathcal{H}}}$$

In particular, $C_{emb}(t) = \sqrt{V_{1,\rho}(R_X; I\times I)}$. See details about $C_{emb}(t)$ in [55] and $I=[0,T]$.

**Note** : In the fractional Brownian motion case, $(C_{emb}(t))^2 = t^{2H}$.

**Remark 2.3.8.** When the driving path $x=(x_t^1,\cdots,x_t^d):[0,T]\to\mathbb{R}^d$ is a fractional Brownian motion with Hurst parameter $1/4<H<1/2$, and $x_t^0=t$. $x$ has a lift as a geometric p-rough path, with given approximating sequence $x_n \in C^{1-var}([0,T],\mathbb{R}^d)$. The above theorem 2.3.6 still holds true with $(C_{emb}(t))^2=t^{2H}$ and $\alpha=(2H+1)\wedge 2$. The above theorem is also true when $H>1/2$. In particular, by taking $\epsilon=1$, $x_t^\epsilon = \frac{1}{\epsilon}x_{\epsilon^{\frac{1}{H}}t}$ and follows the proof in section 5.2 in [2]. (The scaling property refers to [98]) we have:

$$y_t^{y_0} = \exp\left(\sum_{k=1}^{N}\sum_{I\in\{0,1,\cdots,d\}^k}\Lambda_I(x)V_I\right)(y_0)+t^{H(N+1)}R_{N+1}(t) \tag{2.3.6}$$

with

$$\mathbb{P}[\sup_{t\in[0,\tau]}||t^{(N+1)H}R_{N+1}(t)||\geq \xi\tau^{(N+1)H};\tau<T_C(r)]\leq \exp(-\frac{c_H\xi^{(2H+1)\wedge 2}}{t^{2H}}) \tag{2.3.7}$$

In the following, we layout the key steps of the proof and refer the details to Section 4 in [50]. Let us first recall the following definition introduced in [2].

**Definition of** $\omega(\alpha, \mathbf{c}, \zeta)$.

Let $\zeta$ be a random time, $y_t$ is said to be in $\omega(\alpha, c, \zeta)$ if and only if

$$\forall R \geq c, \mathbb{P}[\sup_{0\leq s\leq t} ||y_s|| \geq R; t < \zeta] \leq \exp(-\frac{R^\alpha}{ct}) \tag{2.3.8}$$

The following properties (see [2]) are true:

(**P1**). Let $\phi(t)$ be a continuous process on $[0, \zeta]$ with values in the space of the polynomials of degree less than q, in p Euclidean variables, with coefficients bounded by some constant A on $[0, \zeta]$. The image of $\mathcal{W}(\alpha_1, c_1, \zeta) \times \cdots \times \mathcal{W}(\alpha_p, c_p, \zeta)$ by the mapping

$$(X^1, \cdots, X^p) \to Y, \quad Y_t = \phi_t(X_t^1, \cdots, X_t^p) \tag{2.3.9}$$

is in some $\mathcal{W}(\alpha, c, \zeta)$ ,with $\alpha, c$ determined by $A, p, q, \alpha_1, c_1, \cdots, \alpha_p, c_p$.

(**P2**). if $\zeta$ is bounded by some fixed T, the image of $\mathcal{W}(\alpha, c, \zeta)$ by the mapping $X \to Y$ where $Y = \int_0^t X_u dB_u$, is in some $\mathcal{W}(\alpha, c, \zeta)$, This is also true for the mapping $X \to Y, Y_t = \int_0^t X_u du$. The driving signal $B_u$ can be Brownian motion, fractional Brownian motion with Hurst parameter $H > 1/4$ and general Gaussian process with finite $2D$ $\rho-$variation. The Brownian motion case is proved in [2] and the other two cases are proved in Proposition 4.8 and Proposition 4.10 in [50].

Now we are going to give the outline of the proof of our main theorem.

By applying Theorem 2.3.3 we know that the equation (2.3.4) admits an unique solution in the rough path sense denoted as $y_t^\epsilon$. Following the strategy in theorem

2.3.3, [18] and [26]. We know that

$$y_t^\epsilon = y_0 + \sum_{k=1}^{\infty} \epsilon^k g_k(t) = y_0 + \sum_{k=1}^{N} \epsilon^k g_k(t) + \epsilon^{N+1} M_{N+1}(\epsilon, t) \tag{2.3.10}$$

Then introduce the map $\phi$( see [26]) defined for the appropriate $d$, by

$$\phi : \mathbb{R}^d \to \mathbb{R}^n \quad by \quad (\Lambda_I)_{||I||<N+1} \to \exp(\sum_{I,||I||<N+1} \Lambda_I V_I)(y_0)$$

$\phi$ is $C^\infty$([26]) and it is clear that by Taylor expansion, we have the following:

$$\exp(\sum_{k=1}^{N} \epsilon^k \sum_{I\in\{0,\cdots,d\}^k} \Lambda_I(t) V_I)(y_0) = \phi((\epsilon^{||I||}\Lambda_I)_{||I||<N+1}) = y_0 + \sum_{k=1}^{N} \epsilon^k h_k(t) + \epsilon^{N+1} P_{N+1}(\epsilon, t) \tag{2.3.11}$$

We claim the following lemmas which are the key ingredients for our proof.

**Lemma 2.3.9.** *$g_k(t) = h_k(t)$ a.s., for $k = 1, 2, \cdots, N$.*

**Lemma 2.3.10.** *$g_j(t) \in \omega(\alpha_j, c_j, \zeta)$ for all $j = 1, \cdots, N$, and $M_{N+1}(\epsilon, t) \in \omega(\alpha_M, c_M, \zeta)$ namely*

$$\mathbb{P}(\sup_{t\in[0,\tau]} \|g_j(t)\| \geq \xi; \tau \leq \zeta) \leq \exp(-c_j \frac{\xi^{\alpha_j}}{(C_{emb}(t))^2}), \ \forall j = 1, 2, \cdots, N \tag{2.3.12}$$

$$\mathbb{P}[\sup_{t\in[0,\tau]} ||M_{N+1}(\epsilon, t)|| \geq \xi; \tau < \zeta] \leq \exp(-c_M \frac{\xi^{\alpha_M}}{(C_{emb}(t))^2}) \tag{2.3.13}$$

*with different constants $\alpha_M, \alpha_j, c_M, c_j$, with $j = 1, \cdots, N$.*

**Lemma 2.3.11.** *$P_{N+1}(\epsilon, t) \in \omega(\alpha_p, c_P, \zeta)$, for some constant $\alpha_p$ and $c_P$.*

$$\mathbb{P}[\sup_{t\in[0,\tau]} ||P_{N+1}(\epsilon, t)|| \geq \xi; \tau < \zeta] \leq \exp(-c_P \frac{\xi^{\alpha_P}}{(C_{emb}(t))^2}) \tag{2.3.14}$$

We are now ready to prove our main theorem.

*Proof.* **Proof of theorem 2.3.6**. Based on lemma 2.3.9, subtracts (2.3.10) by (2.3.11) we have the following:

$$y_t^\epsilon = \exp\left(\sum_{k=1}^{N} \epsilon^k \sum_{i_1,\cdots,\imath_k} \Lambda_I(t)V_I\right)(y_0) + \epsilon^{N+1}R_{N+1}(\epsilon,t) \tag{2.3.15}$$

then by lemma 2.3.10 and lemma 2.3.11 we deduce almost surely $R_{N+1}(\epsilon,t) = M_{N+1}(\epsilon,t) - P_{N+1}(\epsilon,t)$ which means $R_{N+1}(\epsilon,t) \in \omega(\alpha_R, c_R, \zeta)$, in particular, here we mean:

$$\mathbb{P}[\sup_{t\in[0,\tau]} ||R_{N+1}(\epsilon,t)|| \geq \xi; \tau < \zeta] \leq \exp(-c_R \frac{\xi^{\alpha_R}}{(C_{emb}(t))^2})$$

where $\alpha_R, c_R$ depends on $\alpha_M, \alpha_P, c_M, c_P$. This finish the proof. □

**Remark 2.3.12.** All the the proof above will work for the fractional Brownian motion with Hurst parameter $H > 1/4$, details see [50].

## 2.4 *A priori* estimates for rough partial differential equations.

In this section, we use a recent method developed in [33] to get a priori estimates for rough partial differential equations. In particular, we consider the following stochastic heat equation inserted with fractional Brownian motion noise on an interval $[0,\tau]\times\mathbb{R}^d$ for $\tau > 0$ and a spatial dimension $d \geq 1$:

$$\partial_t u_t(x) = \frac{\Delta}{2} u_t(x) + \sum_{i=1}^{\infty} \beta_i u_t(x) e_i(x) dw_t^i, \tag{2.4.1}$$

recall that $\Delta$ stands for the Laplace operator, $\{e_i; \geq 1\}$ is an orthonormal basis of $L^2(\mathbb{R}^d)$ and $\{\beta_i; \geq 1\}$ is a family of coefficients satisfying some summability conditions. In equation (2.4.1), $\{w_i; \geq 1\}$ is also a family of noises, interpreted as $p$-variation paths with $p < 3$, which can be lifted to a rough path $\mathbf{w}$. Below, we first layout the key assumptions about the rough path for each couple of components of the driving noise $w$ and our hypothesis on the coefficients.

**Hypothesis 2.4.1.** *Let $p \in [2,3)$ be given. We assume that the family $\{w^i;\, i \geq 1\}$ is such that there exist increments $\mathbf{w}^{\mathbf{1},i}, \mathbf{w}^{\mathbf{2},ij}$ satisfying the two following properties:*
(i) Algebraic condition: *For each $i,j \geq 1$ and $0 \leq s \leq u \leq t \leq \tau$, Chen's relation holds true:*

$$\delta \mathbf{w}^{\mathbf{1},i}_{st} = 0, \quad \textit{and} \quad \delta \mathbf{w}^{\mathbf{2},ij}_{sut} = \mathbf{w}^{\mathbf{1},i}_{su}\, \mathbf{w}^{\mathbf{1},j}_{ut}. \tag{2.4.2}$$

(ii) Analytic condition: *For all $i,j \geq 1$, we have*

$$\mathcal{N}[\mathbf{w}^{\mathbf{1},i}; V_2^p([s,t])] < \infty, \quad \textit{and} \quad \mathcal{N}[\mathbf{w}^{\mathbf{2},ij}; V_2^{p/2}([s,t])] < \infty.$$

Recall that the rough variational setting introduced in Section 2.2 ( see also[4, 33]) uses the concept of scale. A scale is defined as a sequence $(E_n, \|\cdot\|_n)_{n\in\mathbb{N}_0}$ of Banach spaces such that $E_{n+1}$ is continuously embedded into $E_n$. Besides, for $n \in \mathbb{N}_0$ we denote by $E_{-n}$ the topological dual of $E_n$. For the heat equation (2.4.1), we will consider the scale $E_n = W^{n,\infty}$. Having the concept of scale in mind, the noise $w$ should also fulfill the following hypothesis as an infinite dimensional object:

**Hypothesis 2.4.2.** *Recall that the scale $E_n$ is given by $E_n = W^{n,\infty}$. We assume that $\{\beta_i; \geq 1\}$ is a family of positive coefficients satisfying $\sum_{i\geq 1} \beta_i < \infty$. Consider an orthonormal basis $\{e_i; \geq 1\}$ of $L^2(\mathbb{R}^d)$, composed of bounded functions. The noise*

*$w$ is such that $\{w_i; \geq 1\}$ is a family of $p$-variation paths with $p < 3$, whose first and second order increments $\mathbf{w}^{\mathbf{1},i}, \mathbf{w}^{\mathbf{2},ij}$ are such that $\omega_{\mathbf{w^1}}$ and $\omega_{\mathbf{w^2}}$ below are two controls on $[0,\tau]$:*

$$\omega_{\mathbf{w^1}}(s,t) \equiv \left(\sum_{i=1}^{\infty} \beta_i \left(1 + |e_i|_{E_1}\right) \mathcal{N}[\mathbf{w}^{\mathbf{1},i}; V_2^p([s,t])]\right)^p \tag{2.4.3}$$

$$\omega_{\mathbf{w^2}}(s,t) \equiv \left(\sum_{i,j=1}^{\infty} \beta_i\beta_j |e_i|_{E_1}|e_j|_{E_1} \mathcal{N}[\mathbf{w}^{\mathbf{2},ij}; V_2^{p/2}([s,t])]\right)^{p/2}. \tag{2.4.4}$$

### 2.4.1 Linear equations with distributional drifts

In this part, we first generalize equation (2.4.1) to the following linear stochastic heat equation with $\mu$ as a distributional-valued measure:

$$dg_t = \mu(dt) + \sum_{i=1}^{\infty} \beta_i g_t e_i dw_t^i, \tag{2.4.5}$$

Given the previous **Hypothesis** (2.4.1) and **Hypothesis** (2.4.2), we then introduce a formal definition of solution to our equation (2.4.5), in terms of expansions of the increments up to a regularity order greater than 1:

**Definition 2.4.3.** Let $p \in [2,3)$ and fix an interval $I \subseteq [0,\tau]$. Let $\mu$ be a distributional-valued measure lying in $V_1^1(I; E_{-1})$. A path $g : I \to E_{-0}$ is called solution (on $I$) of equation (2.4.5) provided there exists $q < 3$ and $g^\natural \in V_{2,\mathrm{loc}}^{\frac{q}{3}}(I, E_{-1})$ such that we have:

$$\delta g_{st}(\varphi) = \sum_{i=1}^{\infty} \beta_i g_s(e_i\varphi)\mathbf{w}_{st}^{\mathbf{1},i} + \delta\mu_{st}(\varphi) + \sum_{i,j=1}^{\infty} \beta_i\beta_j g_s(e_ie_j\varphi)\mathbf{w}_{st}^{\mathbf{2},ij} + g_{st}^\natural(\varphi), \tag{2.4.6}$$

for every $s,t \in I$ satisfying $s < t$ and every $\varphi \in E_1$.

**Remark 2.4.4.** On top of (2.4.3), we will use the following expressions for $\delta g_{st}$:

$$\delta g_{st}(\varphi) = \sum_{i=1}^{\infty} \beta_i g_s(e_i\varphi)\mathbf{w}_{st}^{\mathbf{1},i} + g^\sharp(\varphi), \tag{2.4.7}$$

where $g^\sharp$ is a $V_2^{\frac{p}{2}}(E_{-1})$ increment satisfying:

$$g^\sharp_{st}(\varphi) = \delta g_{st}(\varphi) - \sum_{i=1}^{\infty} \beta_i g_s(e_i\varphi)\mathbf{w}_{st}^{\mathbf{1},i} = \delta\mu_{st}(\varphi) + \sum_{i,j=1}^{\infty} \beta_i\beta_j g_s(e_ie_j\varphi)\mathbf{w}_{st}^{\mathbf{2},ij} + g^\natural_{st}(\varphi). \tag{2.4.8}$$

**Remark 2.4.5.** Equation (2.4.6) is expressed as an expansion along the increments of $w^i$. However, according to [53, Theorem 4.10], a solution $u$ of (2.4.6) also solves the following integral equation (which has to be interpreted in the rough paths sense in time and weak sense in space):

$$\delta g_{st} = \mu\left([s,t)\right) + \sum_{i=1}^{\infty} \beta_i e_i \int_s^t g_r dw_r^i. \tag{2.4.9}$$

Furthermore, a change of variable formula (see [53, Proposition 5.6]) holds for $g$ verifying (2.4.9). Namely, for $h \in \mathcal{C}^3(\mathbb{R})$ we have (in the weak rough paths sense):

$$\delta h(g)_{st} = \int_s^t h'(g_r)\,\mu(dr) + \sum_{i=1}^{\infty} \beta_i e_i \int_s^t h'(g_r)\, g_r\, dw_r^i. \tag{2.4.10}$$

### 2.4.2 A general estimate for linear equations

In this part, we first give *a priori* estimate for the solution to equation (2.4.5). This will be a key step to give *a priori* $L^2$(or $L^\alpha$) estimates. The following proposition carries the strategy of the proof in [33, Theorem 2.5] to our setting. We give the outline proof of the following proposition, since it shows the main idea of this variational method. The details refer to [52].

**Proposition 2.4.6.** *Let* $p \in [2,3)$ *and fix an interval* $I \subseteq [0,T]$. *Let* $\mathbf{w}$ *be a rough path verifying Hypothesis 2.4.1 and 2.4.2. Consider a path* $\mu \in V_1^1(I; E_{-1})$ *such that for every* $\varphi \in E_1$, *there exists a control* $\omega_\mu$ *verifying*

$$|\delta\mu_{st}(\varphi)| \leq \omega_\mu(s,t)\, \|\varphi\|_{E_1}. \tag{2.4.11}$$

*Let* $g$ *be a solution on* $I$ *of the equation* (2.4.5), *with the following additional hypothesis:* $g$ *is controlled over the whole interval* $I$, *that is we have* $g^\natural \in V_2^{\frac{q}{3}}(I; E_{-1})$ *for* $q < 3$. *Moreover let* $S_t^g = \sup_{s \leq t} \|g_s\|_{E_{-0}}$, *and consider the following control:*

$$\omega_I(s,t) \equiv \omega_\mu(s,t)\left(\omega_{\mathbf{w}^1}^{1/p}(s,t) + \omega_{\mathbf{w}^2}^{2/p}(s,t)\right) + S_t^g\left(2\omega_{\mathbf{w}^1}^{1/p}(s,t)\omega_{\mathbf{w}^2}^{2/p}(s,t) + \omega_{\mathbf{w}^2}^{4/p}(s,t)\right). \tag{2.4.12}$$

*Then there exists a constant* $L = L_p > 0$ *(independent of* $I$*) such that if*

$$\omega_{\mathbf{w}^1}(s,t) + \omega_{\mathbf{w}^2}^2(s,t) \leq L,$$

*then for all* $s,t \in I$ *such that* $s < t$, *we have:*

$$\|g_{st}^\natural\|_{E_{-1}} \lesssim_p \omega_I(s,t). \tag{2.4.13}$$

*Proof.* Let $\omega_\natural(s,t)$ be a regular control such that $\|g_{st}^\natural\|_{E_{-1}} \leq \omega_\natural(s,t)^{\frac{3}{q}}$ for any $s,t \in I$ such that $s < t$. We divide this proof in several steps.

*Step 1: an algebraic identity.* Let $\varphi \in E_1$ be such that $\|\varphi\|_{E_3} \leq 1$. We first show that

$$\delta g^{\natural}_{sut}(\varphi) = \sum_{i=1}^{\infty} \beta_i g^{\sharp}_{su}(e_i\varphi)\mathbf{w}^{\mathbf{1},i}_{ut} + \sum_{i,j=1}^{\infty} \beta_i\beta_j \delta g_{su}(e_i e_j \varphi)\mathbf{w}^{\mathbf{2},ij}_{ut} \equiv K^1_{sut} + K^2_{sut}, \quad (2.4.14)$$

where $g^{\sharp}$ was defined in (2.4.8).

*Step 2: Bound for $K^1$.* To bound $K^1$, we need first bound $g^{\sharp}_{su}(e_i\varphi)$ term in $K^1$. Combining decomposition (2.4.8) and our assumption (2.4.4), we have:

$$|g^{\sharp}_{su}(e_i\varphi)| \leq \left[\omega_\mu(s,u) + S^g_u\, \omega^{2/p}_{\mathbf{w^2}}(s,u) + \omega^{3/p}_{\natural}(s,u)\right] |e_i|_{E_1}|\varphi|_{E_1}. \quad (2.4.15)$$

Plugging this identity into the definition of $K^1$, we have thus obtained:

$$\begin{aligned} |K^1_{sut}| &\leq |\varphi|_{E_1}\left[\omega_\mu(s,u) + S^g_u\, \omega^{2/p}_{\mathbf{w^2}}(s,u) + \omega^{3/p}_{\natural}(s,u)\right] \sum_{i=1}^{\infty} \beta_i |e_i|_{E_1} \omega^{1/p}_{\mathbf{w^1},i}(u,t) \\ &\leq |\varphi|_{E_1}\left[\omega_\mu(s,u) + S^g_u\, \omega^{2/p}_{\mathbf{w^2}}(s,u) + \omega^{3/p}_{\natural}(s,u)\right] \omega^{1/p}_{\mathbf{w^1}}(u,t). \quad (2.4.16) \end{aligned}$$

*Step 3: Bound for $K^2$ and $\delta g^{\natural}$.* The main term to treat for $K^2$ is the increment $\delta g_{su}$. To this aim, we resort to decomposition (2.4.7). This yields:

$$K^2_{sut} = \sum_{i,j,k=1}^{\infty} \beta_i\beta_j\beta_k\, g_s(e_i e_j e_k \varphi)\, \mathbf{w}^{\mathbf{1},k}_{su}\, \mathbf{w}^{\mathbf{2},ij}_{ut} + \sum_{i,j=1}^{\infty} \beta_i\beta_j\, g^{\sharp}(e_i e_j \varphi)\, \mathbf{w}^{\mathbf{2},ij}_{ut} \equiv K^{21}_{sut} + K^{22}_{sut}.$$

Hence, gathering our estimates on $K^{21}$ and $K^{22}$ we end up with:

$$|K^2_{sut}| \leq |\varphi|_{E_1}\left[S^g_t\left(\omega^{1/p}_{\mathbf{w^1}}(s,u) + \omega^{2/p}_{\mathbf{w^2}}(s,u)\right) + \omega_\mu(s,u) + \omega^{3/p}_{\natural}(s,u)\right]\omega^{2/p}_{\mathbf{w^2}}(u,t). \quad (2.4.17)$$

For $\delta g^\natural$: plugging (2.4.16) and (2.4.17) into (2.4.14), we get:

$$\left|\delta g^\natural_{sut}(\varphi)\right| \leq |\varphi|_{E_1} \left\{\omega_I(s,t) + \omega_\natural^{3/p}(s,t)\left(\omega_{\mathbf{w^1}}^{1/p}(s,t) + \omega_{\mathbf{w^2}}^{2/p}(s,t)\right)\right\}. \qquad (2.4.18)$$

*Step 4: Conclusion.* It is readily checked that $\omega_\mu$, $\omega_{\mathbf{w^1}}$, $\omega_{\mathbf{w^2}}$ and $\omega_\natural$ are controls, plus [53, Exercise 1.9], that $\omega_I$ is a control as well as $\omega_\natural^{3/p}(s,t)(\omega_{\mathbf{w^1}}^{1/p}(s,t)+\omega_{\mathbf{w^2}}^{2/p}(s,t))$. One can thus apply Lemma 2.2.10 to relation (2.4.18) and get:

$$\left|g^\natural_{st}(\varphi)\right| \leq c_p\, |\varphi|_{E_1} \left\{\omega_I(s,t) + \omega_\natural^{3/p}(s,t)\left(\omega_{\mathbf{w^1}}^{1/p}(s,t) + \omega_{\mathbf{w^2}}^{2/p}(s,t)\right)\right\}.$$

We now take $I$ such that $c_p\,(\omega_{\mathbf{w^1}}^{1/p}(s,t)+\omega_{\mathbf{w^2}}^{2/p}(s,t)) \leq \frac{1}{2}$. We obtain:

$$\|g^\natural_{st}\|_{E_{-1}} \leq 2c_p\omega_I(s,t),$$

which ends our proof. □

**Remark 2.4.7.** In order to apply Proposition 2.4.6 to the heat equation (2.4.1), we shall consider a measure $\mu$ defined by $\mu([0,t]) = \int_0^t \Delta u_s\, ds$. It is worth noting that for a noisy equation like (2.4.1), we cannot assume that $\Delta u_s$ is properly defined. This is why we consider $\mu([0,t])$ as an element of $E_{-1}$ and perform our computations with distributional increments.

### 2.4.3 $L^2$ and $L^\alpha$ type estimates

In this part, we go back to equation (2.4.1) and we will derive some a priori estimates in $L^2(\mathbb{R}^d)$ and $L^\alpha(\mathbb{R}^d)$. We start by giving some basic properties of our linear heat equation.

Let us begin by giving a precise meaning to equation (2.4.1), as a particular case of rough PDE in the weak sense.

**Definition 2.4.8.** Let $\mathbf{w}$ be a rough path satisfying Hypothesis 2.4.1 and 2.4.2. Consider the following equation:

$$du_t(x) = \frac{1}{2}\Delta u_t(x) + \sum_{i=1}^{\infty} \beta_i u_t(x) e_i dw_t^i. \tag{2.4.19}$$

We interpret this system as in Definition 2.4.3, with a measure $\mu$ given by

$$\mu([s,t)) = \int_s^t \Delta u_r \, dr.$$

As mentioned in the introduction, we are only focusing here on a priori estimates for the heat equation, which are representative of the methods at stake without being too technical. To this aim, we label the following assumption, which prevails until the end of the article:

**Hypothesis 2.4.9.** *One can construct a path $u$ on $[0,\tau]$ which solves* (2.4.19) *according to Definition 2.4.8. In addition, $u$ can be obtained as a limit of a sequence of functions $u^\varepsilon$, where $u^\varepsilon$ solves:*

$$du_t^\varepsilon(x) = \frac{1}{2}\Delta u_t^\varepsilon(x) + \sum_{i=1}^{\infty} \beta_i u_t^\varepsilon(x) e_i dw_t^{\varepsilon,i}. \tag{2.4.20}$$

*In* (2.4.20)*, the family $\{w_t^{\varepsilon,i}; \varepsilon > 0, i \geq 1\}$ is a sequence of smooth functions converging to $w$. Recalling our notations* (2.4.3) *and* (2.4.4)*, we also assume that:*

$$\lim_{\varepsilon \to 0} \omega_{\mathbf{w^1}-\mathbf{w^{1,\varepsilon}}}(0,\tau) + \omega_{\mathbf{w^2}-\mathbf{w^{2,\varepsilon}}}(0,\tau) = 0.$$

**Remark 2.4.10.** Since we assume that $u$ is obtained as a limit of smoothed paths $u^\varepsilon$ (see Hypothesis 2.4.9), all the remaining computations have to be understood as follows: we first derive our relations for $u^\varepsilon$, and we then take limits as $\varepsilon \to 0$. This step will often be implicit for sake of conciseness.

With Hypothesis 2.4.9 in hand, we now derive the equation followed by the path $u^2$ as a first step towards $L^2$ estimates.

**Proposition 2.4.11.** *Let $u$ be the solution of equation* (2.4.19) *alluded to in Hypothesis 2.4.9. We also set*

$$f_t = \|u_t\|_{L^2}^2 + \int_0^t \|\nabla u_r\|_{L^2} dr, \quad \text{and} \quad S_t^f = \sup_{s\leq t} f_s. \tag{2.4.21}$$

*Then the following holds true:*

(i) *Let $\mu^2$ be the $E_{-1}$-valued measure defined as:*

$$\delta\mu_{st}^2(\psi) = -\int_s^t |\nabla u|^2(\psi) dr - \int_s^t (u_r \nabla u_r)(\nabla\psi) dr. \tag{2.4.22}$$

*Then we have:*

$$\omega_{\mu^2}(s,t) \leq \frac{3}{2}\int_s^t \|\nabla u\|_{L^2}^2 dr + \frac{1}{2}\int_s^t \|u\|_{L^2}^2 dr \leq \frac{3}{2}\int_s^t \|\nabla u\|_{L^2}^2 dr + \frac{(t-s)S_t^f}{2}, \tag{2.4.23}$$

*provided the quantity above is finite.*

(ii) *The squared path $u^2$ admits the following representation:*

$$\delta u_{st}^2(\psi) = \delta\mu_{st}^2(\psi) + \sum_{i=1}^\infty 2\beta_i u_s^2(e_i\psi)\mathbf{w}_{st}^{\mathbf{1},i} + \sum_{j=1}^\infty\sum_{i=1}^\infty 4u_s^2(\psi e_i e_j)\beta_i\beta_j \mathbf{w}_{st}^{\mathbf{2},ij} + u_{st}^{2,\natural}(\psi), \tag{2.4.24}$$

*where $\psi$ is a generic test function, and where $u^{2,\natural}$ is an element of $V_2^{\frac{q}{3}}$ for a certain $q < 3$.*

(iii) *The increment $f$ satisfies the following relation: for $0 \le s < t \le \tau$ we have*

$$\delta f_{st} = 2\sum_{i=1}^{\infty} u_s^2(e_i)\beta_i \mathbf{w}_{st}^{\mathbf{1},i} + 4\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} u_s^2(e_i e_j)\beta_i\beta_j \mathbf{w}_{st}^{\mathbf{2},ij} + u_{st}^{2,\natural}(\mathbf{1}), \tag{2.4.25}$$

*where $\mathbf{1}$ designates the function defined on $\mathbb{R}^d$ and identically equal to 1.*

*Proof.* With Remark 2.4.10 in mind, let us divide our proof in several steps.

*Proof of (i):* Similarly to [33, Remark 2.6], and working in the scale $E_n = W^{n,\infty}(\mathbb{R}^d)$, we have

$$|(\delta\mu^2)_{st}(\psi)| \le \int_s^t \|\nabla u\|_{L^2}^2 dr \|\psi\|_{L^\infty} + \left(\int_s^t \|\nabla u\|_{L^2}^2 dr\right)^{\frac{1}{2}} \left(\int_s^t \|u\|_{L^2}^2 dr\right)^{\frac{1}{2}} \|\psi\|_{W^{1,\infty}}, \tag{2.4.26}$$

Invoking now Young's inequality (namely $AB \le \frac{A^\alpha}{\alpha} + \frac{B^\beta}{\beta}$ for two positive numbers $A, B$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$) we get our claim (2.4.23).

*Proof of (ii):* According to Definitions 2.4.3 and 2.4.8, the solution of equation (2.4.19) can be decomposed as:

$$\delta u_{st}(\psi) = \sum_{i=1}^{\infty} \beta_i u_s(e_i\psi)\mathbf{w}_{st}^{\mathbf{1},i} + \sum_{i,j=1}^{\infty} \beta_i\beta_i u_s(e_i e_j\psi)\mathbf{w}_{st}^{\mathbf{2},ij} + \delta\mu_{st}(\psi) + u_{st}^{\natural}(\psi). \tag{2.4.27}$$

As mentioned in Remark 2.4.5, $u$ can also be seen as a solution to the integral equation (2.4.9), for which the change of variable formula (2.4.10) holds true. Applying this relation (written in its weak form) to $h(z) = z^2$, we obtain:

$$\delta u^2_{st}(\psi) = 2\int_s^t \Delta u_r(u_r\psi)\,dr + 2\sum_{i=1}^{\infty}\beta_i\int_s^t u_r^2(e_i\psi)dw_r^i,$$

so that an integration by parts in the first integral above yields:

$$\delta u^2_{st}(\psi) = -2\int_s^t |\nabla u|^2(\psi)\,dr - 2\int_s^t (u_r\nabla u_r)(\nabla\psi)\,dr + 2\sum_{i=1}^{\infty}\beta_i\int_s^t u_r^2(e_i\psi)\,dw_r^i. \tag{2.4.28}$$

We now expand the rough integral in (2.4.28) along the increments of $w$. We end up with relation (2.4.24), for a certain remainder $u^{2,\natural} \in V_2^{\frac{q}{3}}(E_{-1})$.

*Proof of (ii):* Relation (2.4.25) is simply obtained from (2.4.24) by considering a sequence of test functions $\{\psi_n; n \geq 1\}$ such that $\lim_{n\to\infty}\psi_n = \mathbf{1}$ and $\lim_{n\to\infty}\nabla\psi_n = 0$. $\square$

**A priori estimate in $L^2$**

With Proposition 2.4.11 in hand, we can now derive the main estimate of this section.

**Theorem 2.4.12.** *Suppose $w$ fulfills Hypothesis 2.4.1 and 2.4.2, and let $u$ be the solution of equation* (2.4.19) *given in Hypothesis 2.4.9. For $0 \leq s < t \leq \tau$, set:*

$$\omega_1(s,t) = \omega_{\mathbf{w^1}}(s,t) + \omega^2_{\mathbf{w^2}}(s,t) + \omega_{\mathbf{w^1}}(s,t)\,\omega^2_{\mathbf{w^2}}(s,t) + \omega^4_{\mathbf{w^2}}(s,t). \tag{2.4.29}$$

*Then the following $L^2$ norm estimate for the solution $u$ holds true:*

$$S^f_\tau = \sup_{0\leq t\leq\tau}\left(\|u_r\|^2_{L^2} + \int_0^t \|\nabla u_r\|^2_{L^2}dr\right) \leq 2\exp\left(c_p\omega_1(0,\tau)\right)\|u_0\|^2_{L^2}, \tag{2.4.30}$$

*where $c_p$ is a strictly positive constant.*

**Remark 2.4.13.** Notice that $\|u_r\|_{L^2}^2$ and $\int_0^t \|\nabla u_r\|_{L^2}^2 dr$ are positive. Therefore relation (2.4.30) implies that both terms are bounded from above.

*Proof of Theorem 2.4.12.* Recall that we have obtained the following decomposition in Proposition 2.4.11:

$$\delta u_{st}^2(\psi) = \delta \mu_{st}^2(\psi) + \sum_{i=1}^{\infty} 2\beta_i u_s^2(e_i\psi)\mathbf{w}_{st}^{\mathbf{1},i} + \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} 4u_s^2(\psi e_i e_j)\beta_i\beta_j \mathbf{w}_{st}^{\mathbf{2},ij} + u_{st}^{2,\natural}(\psi), \tag{2.4.31}$$

If we now set $g = u^2$ and $\mu^g = \mu^2$, we can recast (2.4.31) as:

$$\delta g_{st}(\psi) = \delta \mu_{st}^g(\psi) + \sum_{i=1}^{\infty} 2\beta_i\, g_s(e_i\psi)\mathbf{w}_{st}^{\mathbf{1},i} + \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} 4\, g_s(\psi e_i e_j)\beta_i\beta_j \mathbf{w}_{st}^{\mathbf{2},ij} + g_{st}^{\natural}(\psi).$$

This equation is of the same form as (2.4.6), and thus we can apply Proposition 2.4.6 directly. We get the following bound for $g_{st}^\natural$, which is valid whenever $\omega_1(s,t) + \omega_2^2(s,t) \le L_p$ (recall that $p$ is the regularity index of $\mathbf{w}$):

$$\|g_{st}^\natural\|_{E_{-1}} \le c_p \omega_I(s,t), \quad \text{or equivalently} \quad \|u_{st}^{2,\natural}\|_{E_{-1}} \le c_p\, \omega_I(s,t), \tag{2.4.32}$$

where the control $\omega_I$ is defined by:

$$\omega_I(s,t) \equiv \omega_{\mu^2}(s,t)\left(\omega_{\mathbf{w^1}}^{1/p}(s,t) + \omega_{\mathbf{w^2}}^{2/p}(s,t)\right) + S_t^{u^2}\left(2\omega_{\mathbf{w^1}}^{1/p}(s,t)\omega_{\mathbf{w^2}}^{2/p}(s,t) + \omega_{\mathbf{w^2}}^{4/p}(s,t)\right), \tag{2.4.33}$$

and where we recall that we have set:

$$S_t^{u^2} = \sup_{s\le t} |u_s^2|_{E_{-0}} = \sup_{s\le t} |u_s|_{L^2}^2.$$

Let us now go back to (2.4.31), and apply this relation to $\psi = \mathbf{1}$ (notice that the function $\mathbf{1}$ obviously sits in $E_1$). It is readily checked from (2.4.22) that:

$$\delta\mu_{st}^2(\mathbf{1}) = -\int_s^t \|\nabla u\|_{L^2}^2 dr,$$

and thus, with our notation (2.4.21) in mind, relation (2.4.31) becomes:

$$\delta f_{st} = \sum_{i=1}^{\infty} 2\beta_i u_s^2(e_i)\mathbf{w}_{st}^{\mathbf{1},i} + \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} 4u_s^2(e_ie_j)\beta_i\beta_j\mathbf{w}_{st}^{\mathbf{2},ij} + u_{st}^{2,\natural}(\mathbf{1})$$

Therefore, bounding $\|u_s^2\|_{E_{-0}}$ by $S_t^f$ and invoking (2.4.32) in order to estimate $u_{st}^{2,\natural}(\mathbf{1})$, we obtain:

$$|\delta f_{st}| \le \left[2\omega_{\mathbf{w^1}}^{1/p}(s,t) + 4\omega_{\mathbf{w^2}}^{2/p}(s,t)\right] S_t^f + c_p\,\omega_I(s,t), \tag{2.4.34}$$

where $\omega_I$ is given by (2.4.33). In order to close this expression, let us further bound the term $\omega_{\mu^2}$ in the definition of $\omega_I$. Namely, according to (2.4.23), we have

$$\omega_{\mu^2}(s,t) \le \frac{3}{2}\int_s^t \|\nabla u\|_{L^2}^2 dr + \frac{(t-s)S_t^f}{2} \le c_\tau S_t^f, \tag{2.4.35}$$

where we recall that we are working on a time interval $[0,\tau]$. Plugging this inequality into the definition of $\omega_I$, we end up with:

$$\omega_I(s,t) \le c_\tau S_t^f\left(\omega_{\mathbf{w^1}}^{1/p}(s,t) + \omega_{\mathbf{w^2}}^{2/p}(s,t) + \omega_{\mathbf{w^1}}^{1/p}(s,t)\omega_{\mathbf{w^2}}^{2/p}(s,t) + \omega_{\mathbf{w^2}}^{4/p}(s,t)\right).$$

Reporting the relation above into (2.4.34), we get

$$\begin{aligned} |\delta f_{st}| &\le c_\tau S_t^f \left(\omega_{\mathbf{w^1}}^{1/p}(s,t) + \omega_{\mathbf{w^2}}^{2/p}(s,t) + \omega_{\mathbf{w^1}}^{1/p}(s,t)\,\omega_{\mathbf{w^2}}^{2/p}(s,t) + \omega_{\mathbf{w^2}}^{4/p}(s,t)\right) \\ &\le c_{\tau,p} S_t^f\, \omega_1(s,t), \end{aligned} \tag{2.4.36}$$

where $\omega_1$ is the control introduced in (2.4.29). Recall again that inequality (2.4.36) is valid when $\omega_1(s,t) + \omega_2^2(s,t) \le L_p$. It is thus also satisfied when $\omega_1(s,t) \le L_p$.

We are now in a position to directly apply our rough Grownall Lemma 2.2.11 to (2.4.36), with $Q = f$, $\kappa = 1/p$ and $\omega_2 = 0$. It is readily checked that $\omega_1$ is a control, and hence:

$$S_t^f \le 2\exp\left(c_p\, \omega_1(0,\tau)\right) f_0 = 2\exp\left(c_p\, \omega_1(0,\tau)\right) \|u_0\|_{L^2}^2, \tag{2.4.37}$$

which ends our proof. □

### $L^\alpha$ type estimates

In this part, we are going to derive some $L^\alpha$ estimates for the solution of equation (2.4.19), generalizing the case $\alpha = 2$. The method is the same as for the $L^2$ case, we will give the main results here and refer the details to [52].

**Remark 2.4.14.** We will handle the case of $L^\alpha$ estimates for an even integer $\alpha$, in order to have $u^\alpha \ge 0$ and $u^{\alpha-2} \ge 0$ in the computations below. However, notice that other values of $\alpha$ can then be reached by simple interpolation methods.

We first present an analogue of Proposition 2.4.11 which leads to the $L^\alpha$ estimates

.

**Proposition 2.4.15.** *Let $u$ be the solution of equation* (2.4.19) *alluded to in Hypothesis 2.4.9, and consider an even integer $\alpha$. We also set*

$$\ell_t = \|u_t\|_{L^\alpha}^\alpha + \int_0^t u_r^{\alpha-2} \|\nabla u_r\|^2 dr, \quad \text{and} \quad S_t^\ell = \sup_{s \le t} \ell_s.$$

*Then the following holds true:*

(i) *Let $\mu^\alpha$ be the $E_{-1}$-valued measure defined as:*

$$\delta\mu_{st}^\alpha(\psi) = -\frac{\alpha(\alpha-1)}{2} \int_s^t u_r^{\alpha-2} |\nabla u|^2(\psi) dr - \frac{\alpha}{2} \int_s^t (u_r^{\alpha-1} \nabla u_r)(\nabla\psi) dr \tag{2.4.38}$$

*Then we have:*

$$\omega_{\mu^\alpha}(s,t) \le \frac{\alpha(\alpha-1)}{4} \int_s^t u_r^{\alpha-2} |\nabla u_r|^2 dr + \frac{\alpha(t-s) S_t^\ell}{4}, \tag{2.4.39}$$

*provided the quantity above is finite.*

(ii) *The path $u^\alpha$ admits the following representation :*

$$\delta u_{st}^\alpha(\psi) = \delta\mu_{st}^\alpha(\psi) + \sum_{i=1}^\infty \alpha\beta_i u_s^\alpha(e_i\psi) \mathbf{w}_{st}^{\mathbf{1},i} + \sum_{j=1}^\infty \sum_{i=1}^\infty \alpha^2 u_s^\alpha(\psi e_i e_j) \beta_i \beta_j \mathbf{w}_{st}^{\mathbf{2},ij} + u_{st}^{\alpha,\natural}(\psi) \tag{2.4.40}$$

*where $\psi$ is a generic test function, and where $u^{\alpha,\natural}$ is an element of $V_2^{\frac{q}{3}}$ for a certain $q < 3$.*

(iii) *The increment $\ell$ satisfies the following relation: for $0 \leq s < t \leq \tau$ we have*

$$\delta\ell_{st} = \alpha \sum_{i=1}^{\infty} u_s^{\alpha}(e_i)\beta_i \mathbf{w}_{st}^{\mathbf{1},i} + \alpha^2 \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} u_s^{\alpha}(e_i e_j)\beta_i\beta_j \mathbf{w}_{st}^{\mathbf{2},ij} + u_{st}^{\alpha,\natural}(\mathbf{1}), \tag{2.4.41}$$

*where $\mathbf{1}$ designates the function defined on $\mathbb{R}^d$ and identically equal to 1.*

The proof of the above proposition is similar to the previous $L^2$ case, see details in [52]. With Proposition 2.4.15 in hand, we can derive estimate in $L^\alpha$ type spaces.

**Theorem 2.4.16.** *Suppose $w$ fulfills Hypothesis 2.4.1 and 2.4.2, and let $u$ be the solution of equation* (2.4.19) *given in Hypothesis 2.4.9. For $0 \leq s < t \leq \tau$, set:*

$$\omega_1(s,t) = \omega_{\mathbf{w^1}}(s,t) + \omega_{\mathbf{w^2}}^2(s,t) + \omega_{\mathbf{w^1}}(s,t)\,\omega_{\mathbf{w^2}}^2(s,t) + \omega_{\mathbf{w^2}}^4(s,t). \tag{2.4.42}$$

*Then for any even integer $\alpha$, the following $L^\alpha$ norm estimate for the solution $u$ holds true:*

$$\sup_{0\leq t\leq\tau}\left( \|u_r\|_{L^\alpha}^{\alpha} + \int_0^t u_r^{\alpha-2}|\nabla u_r|^2 dr \right) \leq 2\exp\left(c_p\omega_1(0,\tau)\right)\|u_0\|_{L^\alpha}^{\alpha}, \tag{2.4.43}$$

*where $c_p$ is a strictly positive constant.*

*Proof.* The proof is similar to Theorem (2.4.12), for the complete proof, see [52]. □

### 2.4.4 Examples: application to fractional Brownian motion

This section is devoted to the application of our abstract results of Section 2.4.3 to some more concrete examples of heat equations driven by an infinite dimensional fractional Brownian motion. Though our general analysis was focused on equations in $\mathbb{R}^d$, we shall treat the case of both bounded and unbounded domains.

### Equations in bounded domains

We first consider the case of an equation in a bounded domain $D$. This will enable us to compare our hypothesis with the assumptions contained in [100] for similar situations. Let us first label the conditions on our domain.

**Hypothesis 2.4.17.** *In this section, we consider an open, bounded domain $D$ with smooth boundary $\partial D$, and satisfying the cone property.*

On such a domain $D$, we wish to give conditions which are close enough to the ones produced in [100]. This is why we consider an operator $C$ given as follows:

**Hypothesis 2.4.18.** *In the remainder of the section, $C$ will stand for a linear, self-adjoint, positive trace-class operator on $L^2(D)$. This operator admits an orthonormal basis $(e_i)_{i\in\mathbb{N}_+}$ of eigenfunctions, with corresponding eigenvalues $(\lambda_i)_{i\in\mathbb{N}_+}$. It also admits an integral representation, whose generating kernel is denoted as $\kappa$. Summarizing, for all $i \geq 0$ and for almost every $x \in D$ we have:*

$$Ce_i(x) = \int_D \kappa(x,y)e_i(y)\,dy = \lambda_i e_i(x). \tag{2.4.44}$$

We can now formulate our a priori estimate in this context:

**Proposition 2.4.19.** *Let $D \subset \mathbb{R}^d$ be a domain fulfilling Hypothesis 2.4.17, together with an operator $C$ as in Hypothesis 2.4.18. On $D$, we consider the following equation:*

$$du_t(x) = \frac{1}{2}\Delta u_t(x) + \sum_{i=1}^{\infty} \lambda_i^{\nu} u_t(x) e_i(x) dB_t^i, \tag{2.4.45}$$

*where* $(B^i_t)_{t\in\mathbb{R}^+})_{i\in N^+}$ *is a sequence of one-dimensional, independent, identically distributed fractional Brownian motions with Hurst parameter* $H \in (\frac{1}{3}, 1)$*, and* $\nu \geq 0$ *is a positive parameter. For the definition of* $e_i$ *and* $\lambda_i$*, we refer to Hypothesis 2.4.18. In addition, we suppose that our operator* $C$ *and its kernel* $\kappa$ *satisfy the following conditions:*

$$A_\kappa \equiv \sup_{x\in D} \|\kappa(x,\cdot)\|_{L^2(D)} + \|\nabla\kappa(x,\cdot)\|_{L^2(D)} < \infty, \quad \textit{and} \quad \sum_{i\geq 0} \lambda_i^{\nu-1} < \infty. \tag{2.4.46}$$

*Then the results from Theorems 2.4.12 and 2.4.16 apply.*

*Proof.* By Proposition 2.1.8, we know that any finite dimensional fractional Brownian motion $(B^i)_{i\leq N}$ can be lifted as a rough path. Verifying conditions (2.4.3) and (2.4.4) will complete the proof. See details in [52]. □

**Remark 2.4.20.** With respect to [100], we have added here the assumption

$$\sup_{x\in D} \|\nabla\kappa(x,\cdot)\|_{L^2(D)} < \infty,$$

which is an artifact of our variational approach. This being said, let us recall that our method applies to rough situations (compared to the case $H > 1/2$ dealt with in [100]). We also believe that our method extends to non linear equations, with a noisy term of the form $\sum_{i=1}^{\infty} \lambda_i^\nu \sigma(u_t(x)) e_i(x) dB^i_t$ for a smooth coefficient $\sigma$.

#### Equations in $\mathbb{R}^{\mathbf{d}}$

On the whole space $\mathbb{R}^d$, choices of orthonormal basis of $L^2$ are wide. For sake of concreteness, we will stick here to a wavelet basis based on Shannon's wavelet, though a much more general setting can be found e.g in [91].

Let us start by defining the $L^2$ basis alluded to above (we refer again to [91] for proofs of general facts on wavelets).

**Lemma 2.4.21.** *Let $\psi : \mathbb{R} \to \mathbb{R}$ be defined as*

$$\psi(x) = \frac{\sin 2\pi(x-1/2)}{2\pi(x-1/2)} - \frac{\sin \pi(x-1/2)}{\pi(x-1/2)}.$$

*Then $\psi \in L^2(\mathbb{R})$, and the following holds true:*

(i) *Let us introduce a family of scaled functions $\{\psi_{j,k};\, j \geq 0,\, k \in \mathbb{Z}\}$ by:*

$$\psi_{j,k}(x) = 2^{-\frac{j}{2}} \psi\left(\frac{x - 2^j k}{2^j}\right). \tag{2.4.47}$$

*This family is an orthonormal basis of $L^2(\mathbb{R})$.*

(ii) *One can obtain an orthonormal basis of $L^2(\mathbb{R}^d)$ by tensorizing the previous basis of $L^2(\mathbb{R})$. Namely, for all $j \geq 0$ and for $n = (n_1, \cdots, n_d)$, we denote*

$$\psi_{j,n}(x) = 2^{-dj/2} \psi\left(\frac{x_1 - 2^j n_1}{2^j}, \cdots, \frac{x_d - 2^j n_d}{2^j}\right).$$

*Then $\{\psi_{j,n}(x)\}_{(j,n) \in \mathbb{Z}^{d+1}}$ is an orthonormal basis of $L^2(\mathbb{R}^d)$. In addition, it is readily checked that:*

$$|\psi_{j,k}|_{E_1} \leq 2^{\frac{jd}{2}}, \tag{2.4.48}$$

*where we recall that we work in the scale $E_n = W^{n,\infty}(\mathbb{R})$.*

**Remark 2.4.22.** A completely correct version of Lemma 2.4.21 should include a so-called father wavelet $\phi$. We omit this step for notational sake.

Under the setting of Lemma 2.4.21, here is our example of stochastic heat equation on $\mathbb{R}^d$:

**Proposition 2.4.23.** *Consider the equation*

$$du_t(x) = \frac{1}{2}\Delta u_t(x) + \sum_{j=0}^{\infty}\sum_{n\in\mathbb{Z}^d} \beta_{j,n} u_t(x)\psi_{j,n}(x) dB_t^{j,n},$$

*where $\{B^{j,n};\ j \geq 0, n \in \mathbb{Z}^d\}$ is a sequence of one-dimensional, independent, identically distributed fractional Brownian motions with Hurst parameter $H \in (\frac{1}{3}, 1)$, and $\{\beta_{j,n};\ j \geq 0, n \in \mathbb{Z}^d\}$ is a family of positive coefficients. We assume that*

$$A_\beta \equiv \sum_{j=0}^{\infty}\sum_{n\in\mathbb{Z}^d} 2^{\frac{dj}{2}}\beta_{j,n} < \infty. \tag{2.4.49}$$

*Proof.* The proof is similar to that for Proposition 2.4.19 which is to verify conditions (2.4.3) and (2.4.4). We omit the proof here and refer proof to [52]. □

# Chapter 3

# Riemannian foliations

In this section, we intend to give a brief (summarized version) introduction and recent progress of Riemannian foliations, in particular, totally geodesic Riemannian foliations, which will be the main geometry objects that we will establish stochastic analysis and geometric analysis on it in the following two sections (Section 4 and Section 5). The reason we are interested in Riemannian foliations, is that they provide a large class of spaces with sub-Riemannian structures. It is well known that, Ricci curvature, which is defined as the trace of the Riemannian curvature tensor, plays an important role in the study of Riemannian geometry. However, there is not a natural generalization of such a concept in the sub-Riemannian geometry setting. Nevertheless, the recent progress about new characterization of Ricci curvature lower bound open the eyes to the sub-Riemannian setting as well. To be precise, Bakry-Eméry [6] establish a purely analytic method which is to use curvature dimension inequality to characterize Ricci curvature lower bound. On the other hand, Sturm-Lott-Villani [84, 110] define the Ricci curvature lower bound through optimal transportation. In particular, these two notions of Ricci curvature are then extended to metric measure spaces and are shown to be equivalent in certain cases [1]. Although, the sub-Riemannian analogue of Sturm-Lott-Villani [84, 110] is not true even for the simple case like Heisenberg group [79]. The curvature dimension inequality tools turn out to be a success in the sub-Riemannian setting. The first success attempt to define Ricci curvature lower bound in a sub-Riemannian setting is carried out by Baudoin-Garofalo [13] on sub-Riemannian manifolds with transverse symmetries by using generalized curvature dimension inequalities. Under this frame work, they are able to prove Li-Yau [82] type inequality on sub-Riemannian manifolds by using the generalized curvature dimension inequality as well as other functional inequalities and heat kernel bounds. It is then pointed out by D. Elworthy [39] that sub-Riemannian manifolds with transverse symmetries are special cases of totally geodesic Riemannian foliations. Then the generalized curvature dimension inequality frame work (as well as the transverse Weitzenböck formulas) is generalized for all totally geodesic Riemannian foliations by Baudoin-Kim-Wang [16]. Thus this open a new world to study sub-Riemannian geometry from a geometric analysis point of view, in particular, armed with the recent result about sub-Laplacian comparison theorems on totally geodesic Riemannian foliations [15].

Moreover, in the Riemannian manifold setting, there is another recent striking result to study Ricci curvature by using the stochastic analysis on the path space of a Riemannian manifold by Naber-Haslhofer [97, 68], in particular, this is the first time to characterize Ricci curvature two sided bound. Later on, there are lots of work try to improve the bounds in various settings [47, 116, 28]. The characterization of two sided Ricci curvature bound through stochastic analysis on the path space of a totally geodesic Riemannian foliation is proved in [11].

The aforementioned geometric analysis and stochastic analysis aspects in the Riemannian setting are connected with Ricci flow as mentioned in the introduction part.

We will see the details about Ricci flow on totally geodesic Riemannian foliations in Section 5 which is studied in [49, 51]. The long term goal is to study sub-Riemannian geometry (structure) by using geometric analysis, stochastic analysis (path space and Wasserstein space) and Ricci flow techniques.

Now, let's start to look at Riemannian foliation itself and get the basic ideas and knowledge of it. For a detailed study and history of Riemannian foliations and related concepts introduced in the following sections, we refer to [9, 56, 96, 104, 107, 114] and the references therein.

## 3.1 Riemannian submersions

As we will see in a moment, Riemannian submersion is always closely related to Riemannian foliations, it is first a simple example of Riemannian foliation and also gives us local properties of Riemannian foliations. We thus first introduce basic properties of Riemannian submersions.

**Definition 3.1.1** (**Riemannian submersions**)**.** Let $(\mathbb{M}, g)$ and $(\mathbb{B}, j)$ be smooth and connected Riemannian manifolds. A smooth surjective map $\pi : (\mathbb{M}, g) \to (\mathbb{B}, j)$ is called a *Riemannian submersion* if its derivative maps $T_x\pi : T_x\mathbb{M} \to T_b\mathbb{B}$ are orthogonal projections with $\pi(x) = b$, i.e. for every $x \in \mathbb{M}$ with $\pi(x) = b$, the map $T_x\pi(T_x\pi)^* : T_b\mathbb{B} \to T_b\mathbb{B}$ is the identity map.

In the above setting, the kernel of the tangent map

$$T_x\pi : T_x\mathbb{M} \to T_b\mathbb{B}$$

defines the vertical space at $x$, denoted as $\mathcal{V}_x = \mathbf{Ker}(T_x\pi)$, and its complement defines a horizontal space at $x$, denoted as $\mathcal{H}_x$. Thus we have a linear isomorphism from $\mathcal{H}_x$ to $T_b\mathbb{B}$ and we also have an orthogonal decomposition of $T_x\mathbb{M}$ as

$$T_x\mathbb{M} = \mathcal{H}_x \oplus \mathcal{V}_x.$$

The vector fields belong to $\mathcal{V}$ (or $\mathcal{H}$) are called **vertical** (or **horizontal**) vector fields.

**Example 3.1.2 (Products).** Let $(\mathbb{M}_1, g_1)$ and $(\mathbb{M}_2, g_2)$ be two Riemannian manifolds and $(\mathbb{M}, g) = (\mathbb{M}_1 \times \mathbb{M}_2, g_1 + g_2)$ their Riemannian product. Then the canonical projections $p_1$ and $p_2$ on the factors $\mathbb{M}_1$ and $\mathbb{M}_2$ are Riemannian submersions. More examples of this type by changing the metrics on the product manifold $(\mathbb{M}, g)$ can be found [22, B, 9.10] (e.g. products with varying metrics on the fibres, warped products, etc.)

**Definition 3.1.3.** If $\pi : (\mathbb{M}, g) \to (\mathbb{B}, j)$ is a Riemannian submersion and $b \in \mathbb{B}$, then we call the set $\pi^{-1}(b)$ a fiber. A Riemannian submersion $\pi : (\mathbb{M}, g) \to (\mathbb{B}, j)$ is said to have totally geodesic fibers, if for every $b \in \mathbb{B}$, $\pi^{-1}(b)$ is a totally geodesic submanifold of $\mathbb{M}$.

To give a clear identification of totally geodesic property of Riemannian submersions, we first introduce the concept of basic vector fields (see [22, C, 9.23] and [9, Definition 2.7]). For a Riemannian submersion $\pi : (\mathbb{M}, g) \to (\mathbb{B}, j)$, a vector field $X$

on $\mathbb{M}$ is projectable if there exists a vector field $\bar{X}$ on $\mathbb{B}$ such that for every $x \in \mathbb{M}$ with $\pi(x) = b$, we have $T_x\pi(X(x)) = \bar{X}(b)$ and we say $X$ and $\bar{X}$ are $\pi$-related.

**Definition 3.1.4.** A vector field $X$ on $\mathbb{M}$ is called **basic** if it is projectable and horizontal.

In particular, if $\bar{X}$ is a vector field on $\mathbb{B}$, then there is a **unique** basic vector field $X$ on $\mathbb{M}$, which is $\pi$-related to $\bar{X}$. $X$ is then called the lift of $\bar{X}$. The following result is first proved Hermann [70] and one can find a concise proof in [9, Proposition 2.8].

**Proposition 3.1.5.** *The submersion $\pi$ has totally geodesic fibers if and only if the flow generated by the basic vector field induces an isometry between the fibers.*

Other characterizations of totally geodesic fibers using the horizontal gradient and horizontal laplacians can be found in [9, Theorem 2.9, 2.10].

## 3.2 Totally geodesic foliations with bundle-like metrics

With the previous Riemannian submersion example in hand, in this section we introduce what is a Riemannian foliaiton with totally geodesic leaves and bundle-like metrics. We begin with a formal definition of foliations.

**Definition 3.2.1.** Let $\mathbb{M}$ be a smooth and connected $n + m$ dimensional manifold. A $m$-dimensional foliation $\mathcal{F}$ on $\mathbb{M}$ is defined by a maximal collection of pairs $\{(U_\alpha, \pi_\alpha), \alpha \in I\}$ of open subsets $U_\alpha$ of $\mathbb{M}$ and submersions $\pi_\alpha : U_\alpha \to U_\alpha^0$ onto open subsets of $\mathbb{R}^n$ satisfying:

- $\cup_{\alpha \in I} U_\alpha = \mathbb{M}$;

- If $U_\alpha \cap U_\beta \neq \emptyset$, there exists a local diffeomorphism $\Psi_{\alpha\beta}$ of $\mathbb{R}^n$ such that $\pi_\alpha = \Psi_{\alpha\beta}\pi_\beta$ on $U_\alpha \cap U_\beta$.

The maps $\pi_\alpha$ are called disintegrating maps of $\mathcal{F}$. The connected components of the sets $\pi_\alpha^{-1}(c)$, $c \in \mathbb{R}^n$, are called the plaques of the foliation. In particular, if the leaves of the foliation are totally geodesic submanifolds of $\mathbb{M}$, we then call the foliation is equipped with totally geodesic leaves.

In our setting, we will always consider the Riemannian foliation to be equipped with totally geodesic leaves and bundle-like metrics. The concept "bundle-like metric" was first introduced by Reinhart [104] for foliated manifolds, i.e. a manifold with certain foliation structure on it. Actually, any foliation can be extended to an almost product structure, to be precise: given a Riemannian metric on $\mathbb{M}$, the set of 1 forms (i.e. vertical 1 forms, see definitions in (3.3.1)) which are zero on the orthogonal space to the tangent space of the leaf through a given point forms a $m$ dimensional linear subspace of the cotangent space at that point. Given the construction in Definition 3.2.1 and following the idea in [104], in each neighborhood $U$ of each point of $\mathbb{M}$, we can find local coordinates $\{x_1, \cdots, x_n.y_1, \cdots, y_m\}$. Thus we may choose in each flat coordinate neighborhood 1 forms $\eta_1, \cdots, \eta_m$ such that $\{dx_1, \cdots, dx_n, \eta_1, \cdots, \eta_m\}$ is a basis for the cotangent space, and vectors $X_1, \cdots, X_n$ such that $\{X_1, \cdots, X_n, \frac{\partial}{\partial y_1}, \cdots, \frac{\partial}{\partial y_m}\}$ is the dual basis for the tangent space. We then have the following definition, for details see [104, Section 1].

**Definition 3.2.2.** (**Bundle-like metric**, [104, Section 2]) A metric $g$ on a foliated manifold $(\mathbb{M}, \mathcal{F}, g)$, where $\mathcal{F}$ is the foliation equipped with $\mathbb{M}$, is called a bundle-like metric if it has the local representation,

$$g = \sum_{i,j=1}^{n} g_{ij}(x,y)dx_i dx_j + \sum_{\alpha,\beta=1}^{m} g_{\alpha,\beta}(x,y)\eta_\alpha\eta_\beta.$$

As an example, the foliation given by the fibers of a Riemannian submersion is bundle-like since the orthogonal decomposition $T_x\mathbb{M} = \mathcal{H}_x \oplus \mathcal{V}_x$ induces a splitting of the metric $g = g_{\mathcal{H}} \oplus g_{\mathcal{V}}$. In particular, $g_{\alpha,\beta}$ is the metric on the fiber.

Below we give a precise definition of totally geodesic Riemannian foliation with bundle-like metric and we fix the convention throughout the rest of the dissertation.

**Definition 3.2.3.** Let $\mathbb{M}$ be a smooth and connected $n+m$ dimensional Riemannian manifold. A $m$-dimensional foliation $\mathcal{F}$ on $\mathbb{M}$ is said to be Riemannian with a bundle-like metric if the disintegrating maps $\pi_\alpha$ are Riemannian submersions onto $U_\alpha^0$ with its given Riemannian structure. If moreover the leaves are totally geodesic sub-manifolds of $\mathbb{M}$, then we say that the Riemannian foliation is totally geodesic with a bundle-like metric.

In the following, we fix some notations associated with the Riemannian foliation. Denote subbundle $\mathcal{V}$ as the set of *vertical directions* formed by the vectors tangent to the leaves and denote the subbundle $\mathcal{H}$ as the set of *horizontal directions* which is normal to $\mathcal{V}$. We assume that the Lie algebra of vector fields generated by global $C^\infty$ sections of $\mathcal{H}$ has the full rank at each point in $\mathbb{M}$, which means that $\mathcal{H}$ satisfies the bracket generating condition, namely the (iterative) Lie bracket of the horizontal vectors will generate the whole tangent space.

We also denote $T\mathbb{M}$ as the *tangent bundle* and $T^*\mathbb{M}$ as the *cotangent bundle*. In particular, we denote $T_x\mathbb{M}$ ($T_x^*\mathbb{M}$) as the *tangent (cotangent) space* at $x \in \mathbb{M}$. Denote

$g(\cdot,\cdot)$, $g_{\mathcal{H}}(\cdot,\cdot)$, $g_{\mathcal{V}}(\cdot,\cdot)$ as the inner product on $T\mathbb{M}$ induced by the metric $g$ and its restrictions to $\mathcal{H}$ and $\mathcal{V}$ respectively. As always, for any $x \in \mathbb{M}$ denote by $g(\cdot,\cdot)_x$ (or $\langle\cdot,\cdot\rangle_x$), $g_{\mathcal{H}}(\cdot,\cdot)_x$ (or $\langle\cdot,\cdot\rangle_{\mathcal{H}_x}$), $g_{\mathcal{V}}(\cdot,\cdot)_x$ (or $\langle\cdot,\cdot\rangle_{\mathcal{V}_x}$) the inner product on the fibers $T_x\mathbb{M}$, $\mathcal{H}_x$ and $\mathcal{V}_x$ correspondingly. Let $\mathcal{C}^\infty(\mathbb{M})$ denote as the space of *smooth functions* on $\mathbb{M}$ and $\mathcal{C}_0^\infty(\mathbb{M})$ denote as the space of *smooth and compactly supported functions.* Let $\Gamma^\infty(\mathcal{E})$ denote as the space of *smooth sections* of a vector bundle $\mathcal{E}$ over $\mathbb{M}$ and $\Gamma_0^\infty(\mathcal{E})$ denote as the space of *smooth and compactly supported sections.*

## 3.3 Bott connection

In this section, we follow the geometry setting similar to [9, 10, 13, 16, 12, 49]. On the Riemannian manifold $(\mathbb{M}, g)$ there is the Levi-Civita connection that we denote by $\nabla^R$, but the suitable connection which is adapted to our study of foliations is the *Bott connection* on $\mathbb{M}$. Define the *Bott connection* [22] in the following way,

$$\nabla_X Y = \begin{cases} \pi_{\mathcal{H}}(\nabla^R_X Y), X, Y \in \Gamma^\infty(\mathcal{H}), \\ \pi_{\mathcal{H}}([X,Y]), X \in \Gamma^\infty(\mathcal{V}), Y \in \Gamma^\infty(\mathcal{H}), \\ \pi_{\mathcal{V}}([X,Y]), X \in \Gamma^\infty(\mathcal{H}), Y \in \Gamma^\infty(\mathcal{V}), \\ \pi_{\mathcal{V}}(\nabla^R_X Y), X, Y \in \Gamma^\infty(\mathcal{V}), \end{cases} \tag{3.3.1}$$

where $\pi_{\mathcal{H}}$ (resp. $\pi_{\mathcal{V}}$) is the projection on $\mathcal{H}$ (resp. $\mathcal{V}$). It is easy to check that the *Bott* connection is metric-compatible, that is, $\nabla g = 0$, and it is not torsion-free. We denote $T$ to be the torsion of the *Bott* connection $\nabla$.

For any bundle-like metric (Definition 3.2.2) $g$, we can always have a orthogonal decomposition as $g = g_{\mathcal{H}} \oplus g_{\mathcal{V}}$. In the next two sections, we will consider a family of one parameter variation of the metric $g$ as

$$g_\varepsilon = g_{\mathcal{H}} \oplus \frac{1}{\varepsilon} g_{\mathcal{V}}, \quad \text{where } \varepsilon > 0.$$

One can check that $\nabla g_\varepsilon = 0$ for every $\varepsilon > 0$. The metric $g_\varepsilon$ introduces a metric on the cotangent bundle which we still denote by $g_\varepsilon$. By using the similar notations as before we have

$$\|\eta\|_\varepsilon^2 = \|\eta\|_{\mathcal{H}}^2 + \varepsilon \|\eta\|_{\mathcal{V}}^2, \quad \forall \ \ \eta \in T^*\mathbb{M}.$$

**Definition 3.3.1.** We say that a one-form is horizontal (resp. vertical) if it vanishes on the vertical bundle $\mathcal{V}$ (resp. on horizontal bubdle $\mathcal{H}$).

The following notions of Ricci curvature and the skew-symmetric endomorphism map were first introduced in [13] within the setting for sub-Riemannian manifolds with transverse symmetry. We define our horizontal Ricci curvature $\mathfrak{Ric}_{\mathcal{H}}$ of *Bott* connection as the fiberwise symmetric linear map on one-forms such that for all smooth functions $f, g$ on $\mathbb{M}$

$$\langle \mathfrak{Ric}_{\mathcal{H}}(df), dg \rangle = \mathbf{Ric}\,(\nabla_{\mathcal{H}} f, \nabla_{\mathcal{H}} g) = \mathbf{Ric}_{\mathcal{H}}(\nabla f, \nabla g),$$

where **Ric** is the Ricci curvature of the *Bott* connection and $\mathbf{Ric}_{\mathcal{H}}$ its horizontal Ricci curvature (horizontal trace of the full curvature tensor $R$ of the *Bott* connection ). For each $Z \in \Gamma^\infty(\mathcal{V})$ there is a unique skew-symmetric endomorphism $J_Z : \mathcal{H}_x \to \mathcal{H}_x$,

$x \in \mathbb{M}$ such that for all horizontal vector fields $X, Y \in \mathcal{H}_x$

$$g_{\mathcal{H}}(J_Z(X), Y)_x = g_{\mathcal{V}}(Z, T(X, Y))_x, \tag{3.3.2}$$

where $T$ is the torsion tensor of the Bott connection $\nabla$ and we extend $J_Z$ to be 0 on $\mathcal{V}_x$. We will see that if $X, Y \in \Gamma^\infty(\mathcal{H})$, then

$$\begin{aligned} T(X,Y) &= \nabla_X Y - \nabla_Y X - [X,Y] = \pi_{\mathcal{H}}(\nabla_X Y - \nabla_Y X) - [X,Y] = \pi_{\mathcal{H}}([X,Y]) - [X,Y] \\ &= -\pi_{\mathcal{V}}([X,Y]). \end{aligned}$$

Similar computations give us the following properties for the Bott connection,

$$\begin{aligned} &\nabla_X Y \in \Gamma^\infty(\mathcal{H}) \text{ for any } X, Y \in \Gamma^\infty(\mathcal{H}), \\ &\nabla_X Y \in \Gamma^\infty(\mathcal{V}) \text{ for any } X, Y \in \Gamma^\infty(\mathcal{V}), \\ &T(X,Y) \in \Gamma^\infty(\mathcal{V}) \text{ for any } X, Y \in \Gamma^\infty(\mathcal{H}), \\ &T(U,V) \in \Gamma^\infty(\mathcal{H}) \text{ for any } U, V \in \Gamma^\infty(\mathcal{V}), \\ &T(X,U) = 0 \text{ for any } X \in \Gamma^\infty(\mathcal{H}), U \in \Gamma^\infty(\mathcal{V}), \end{aligned} \tag{3.3.3}$$

If $\{X_1, \cdots, X_n\}$ is a local orthonormal frame of horizontal vector fields, then for metric $g$ and the associated *Bott* connection $\nabla$, we have the following horizontal laplacian operator

$$L = -\nabla^*_{g_{\mathcal{H}}} \nabla_{g_{\mathcal{H}}} (\text{or} = -\nabla^*_{\mathcal{H}} \nabla_{\mathcal{H}}),$$

by using the local coordinates, we can represent $L$ by

$$L = \sum_{i=1}^{n} \nabla_{X_i}\nabla_{X_i} - \nabla_{\nabla_{X_i}X_i}. \tag{3.3.4}$$

Recall our definition of $J$. If $\{Z_1, \cdots, Z_m\}$ is a local vertical orthonormal frame, then $J$ define a $(1,1)$ tensor

$$\mathbf{J}^2 := \sum_{j=1}^{m} J_{Z_j} J_{Z_j}$$

does not depends on the choice of the frame and may be defined globally. The horizontal divergence of the torsion $T$ is also a $(1,1)$ tensor which in a local horizontal frame $\{X_1, \cdots, X_n\}$ is defined as

$$\delta_{\mathcal{H}} T(X) := -\sum_{i=1}^{n} (\nabla_{X_i} T)(X_i, X).$$

We define the horizontal gradient in a local adapted frame of a one-form $\eta$ as the $(0,2)$ tensor

$$\nabla_{\mathcal{H}} \eta = \sum_{i=1}^{n} \nabla_{X_i} \eta \otimes \theta_i,$$

where $\theta_i, i = 1, \ldots, n$ is the dual of $X_i$.

In particular, for a generic one form $\eta = \sum_{i=1}^{n} f_i \theta_i + \sum_{j=1}^{m} k_j \upsilon_j$, we have

$$\delta_{\mathcal{H}} T(\eta) = \sum_{i,j=1}^{n} \sum_{l=1}^{m} (X_i(t) \gamma_{ij}^{l}(t)) f_j \upsilon_l,$$

where $\gamma_{ij}^{l}(t)$ is from the structure equation in lemma 3.5.1. In particular, if $\delta_{\mathcal{H}} T(\cdot) = 0$, we say that it satisfies the *Yang-Mills* condition (see [22] for details).

By using the duality between the tangent and cotangent bundles with respect to the metric $g$, we can identify the $(1,1)$ tensors $\mathbf{J}^2$ and $\delta_{\mathcal{H}}T$ with linear maps on the cotangent bundle $T^*\mathbb{M}$.

Namely, let $\sharp : T^*\mathbb{M} \to T\mathbb{M}$ be the standard musical (raising an index) isomorphism which is defined as the unique vector $\omega^\sharp$ such that for any $x \in \mathbb{M}$

$$g\left(\omega^\sharp, X\right)_x = \omega(X) \text{ for all } X \in T_x\mathbb{M},$$

while in local coordinates the isomorphism $\sharp$ can be written as follows

$$\omega = \sum_{i=1}^{n+m} \omega_i dx^i \longmapsto \omega^\sharp = \sum_{j=1}^{n+m} \omega^j \partial_j = \sum_{j=1}^{n+m} \sum_{i=1}^{n+m} g^{ij} \omega_i \partial_j.$$

The inverse of this isomorphism is the (lowering an index) isomorphism $\flat : T\mathbb{M} \to T^*\mathbb{M}$ defined by

$$X^\flat = g(X, \cdot)_x, X \in T_x\mathbb{M}$$

and in local coordinates

$$X = \sum_{i=1}^{n+m} X^i \partial_i \longmapsto X^\flat = \sum_{i=1}^{n+m} X_i dx^i = \sum_{i=1}^{n+m} \sum_{j=1}^{n+m} g_{ij} X^j dx^i.$$

## 3.4 Other connections, torsions and curvatures.

In this part, we will introduce different connections related to the Bott connection, each of them play different roles in the following two sections (Section 4 and Section 5). Let us make a brief summary here. It is natural to assume our connection to be a metric connection and it is compatible with our foliation structure (see [12, Assumption 1]). The Bott connection is of course one of the connections that would belong to those connections. However, we actually need further conditions on the torsion and curvature forms associated with the connection, that's why we lay out the two connections here. One is the **Baudoin connection**, which was first introduced by Baudoin [10] to prove the Bochner-Weitzenböck formulas for sub-Laplacians on one forms and initialed stochastic analysis in this context. The term "Baudoin connection" is first used by D. Elworthy [40]. It turns out that this Baudoin connection is quite useful and works for a large class of Riemannian foliations when we want to establish geometric analysis and stochastic analysis on the foliations (see [16, 11, 57, 58, 40, 12]). For example, Bochner-Weitzenböck identities, generalized curvature dimension inequalities, gradient representation of semigroups generated by the horizontal Laplacian operator, Log-Soboloev inequalities (and other functional inequalities), etc.

Another useful and important connection is the **adjoint connection** which one can find a detailed study in [42]. In particular, we will work with the adjoint connection of the Bott connection and the adjoint connection of the Baudoin connection. The advantage of this adjoint connection is that, it is almost the same as the Bott connection (in the sense that it preserves the horizontal vector fields) and also gives us further skew-symmetry on the torsion and curvature forms.

### 3.4.1 Baudoin connection

We now introduce the following connection $\nabla^\varepsilon$ which was first discovered by F. Baudoin [9] (see also [10]). Later on, in the work by Grong-Thalmaier [57, 58], they also consider similar connections in a more general class, however, in order to prove the generalized curvature dimension inequality, they also use the (unique) same connection as in [9]. This term ”Baudoin connection” was first used in D. Elworthy [40], it seems to be a great idea to call it Baudoin connection from now on.

$$\nabla^{\varepsilon}_X Y := \nabla_X Y - \mathfrak{T}^{\varepsilon}_X Y := \nabla_X Y - T(X,Y) + \frac{1}{\varepsilon} J_Y X, \quad X, Y \in \Gamma^{\infty}(\mathbb{M}).$$

where $\mathfrak{T}^{\varepsilon}$ is the $(1,1)$ tensor defined by

$$\mathfrak{T}^{\varepsilon}_X Y = -T(X,Y) + \frac{1}{\varepsilon} J_Y X, \quad X, Y \in \Gamma^{\infty}(\mathbb{M}).$$

Then for general one-form, we denote

$$\mathfrak{T}^{\varepsilon}_{\mathcal{H}} \eta := \sum_{i=1}^{n} \mathfrak{T}^{\varepsilon}_{X_i} \eta \otimes \theta_i.$$

It is easy to check that $\nabla^{\varepsilon} g_{\varepsilon} = 0$, namely it is a metric connection. The torsion of $\nabla^{\varepsilon}$ is equal to (by definition of torsion and definition of our Baudoin connection)

$$T^{\varepsilon}(X,Y) = -T(X,Y) + \frac{1}{\varepsilon} J_Y X - \frac{1}{\varepsilon} J_X Y, \quad X, Y \in \Gamma^{\infty}(\mathbb{M}).$$

Similar to (3.3.3), we have

$$\begin{aligned} & T^{\varepsilon}(X,Y) \in \Gamma^{\infty}(\mathcal{V}) \text{ for any } X, Y \in \Gamma^{\infty}(\mathcal{H}), \\ & T^{\varepsilon}(U,V) \in \Gamma^{\infty}(\mathcal{H}) \text{ for any } U, V \in \Gamma^{\infty}(\mathcal{V}), \\ & T^{\varepsilon}(X,U) \in \Gamma^{\infty}(\mathcal{H}) \text{ for any } X \in \Gamma^{\infty}(\mathcal{H}), U \in \Gamma^{\infty}(\mathcal{V}). \end{aligned} \tag{3.4.1}$$

With the newly introduced Baudoin connection in hand, we are ready to introduce the following operator which is the key ingredient in the Bochner-Weitzenböck formula.

$$\Box_\varepsilon = -(\nabla_{g_\mathcal{H}} - \mathfrak{T}^\varepsilon_{g_\mathcal{H}})^*(\nabla_{g_\mathcal{H}} - \mathfrak{T}^\varepsilon_{g_\mathcal{H}}) - \frac{1}{\varepsilon}\mathbf{J}^2 + \frac{1}{\varepsilon}\delta_\mathcal{H} T - \mathfrak{Ric}_{g_\mathcal{H}}. \tag{3.4.2}$$

For a complete proof of how and why the operator $\Box_\varepsilon$ has the above form, one should refer to the original work [10]. Recall that we denote $L$ as the horizontal Laplacian operator, we end this subsection with the following theorem.

**Theorem 3.4.1** ([16])**.** *Let $f \in \mathcal{C}^\infty(\mathbb{M})$, we have*

$$dLf = \Box_\varepsilon df.$$

*and for $\eta = df$, we have the following Bochner's inequality*

$$\begin{aligned}
&\frac{1}{2}L\|\eta\|^2_\varepsilon - \langle \Box_\varepsilon \eta, \eta\rangle_\varepsilon \\
&= \|\nabla_{g_\mathcal{H}}\eta - \mathfrak{T}^\varepsilon_{g_\mathcal{H}}\eta\|^2_\varepsilon + \langle \mathfrak{Ric}_{g_\mathcal{H}}(\eta), \eta\rangle_\mathcal{H} - \langle \delta_\mathcal{H} T(\eta), \eta\rangle_\mathcal{V} + \frac{1}{\varepsilon}\langle \mathbf{J}^2(\eta), \eta\rangle_\mathcal{H} \\
&\geq \frac{1}{n}(\mathbf{Tr}_\mathcal{H}\nabla^{t,\sharp}_{g_\mathcal{H}}\eta)^2 - \frac{1}{4}\mathbf{Tr}_\mathcal{H}(\mathbf{J}^2_\eta) + \langle \mathfrak{Ric}_{g_\mathcal{H}}(\eta), \eta\rangle_\mathcal{H} - \langle \delta_\mathcal{H} T(\eta), \eta\rangle_\mathcal{V} + \frac{1}{\varepsilon}\langle \mathbf{J}^2(\eta), \eta\rangle_\mathcal{H}.
\end{aligned} \tag{3.4.3}$$

### 3.4.2 Adjoint connection

In general, the adjoint connection of a given connection is defined through subtracting the associated torsion term (see [42, Section 1.3] for a discussion about adjoint connections). The adjoint connection of the Baudoin connection $\nabla^\varepsilon$ (in the sense of

B. Driver [36]) is thus given by

$$\widehat{\nabla}^{\varepsilon}_X Y := \nabla^{\varepsilon}_X Y - T^{\varepsilon}(X,Y) = \nabla_X Y + \frac{1}{\varepsilon} J_X Y, \tag{3.4.4}$$

thus $\widehat{\nabla}^{\varepsilon}$ is also a metric connection. Moreover, it preserves the horizontal and vertical bundles. In particular, when $\varepsilon \to \infty$, the adjoint connection $\widehat{\nabla}^{\infty}$ is just the Bott connection $\nabla$. For later use, we record that the torsion of $\widehat{\nabla}^{\varepsilon}$ is

$$\widehat{T}^{\varepsilon}(X,Y) = -T^{\varepsilon}(X,Y) = T(X,Y) - \frac{1}{\varepsilon} J_Y X + \frac{1}{\varepsilon} J_X Y. \tag{3.4.5}$$

We can also get the same properties as in (3.4.1) for $\widehat{T}^{\varepsilon}$, since they only differ by a negative sign. The Riemannian curvature tensor of $\widehat{\nabla}^{\varepsilon}$ also can be computed easily in terms of the Riemannian curvature tensor $R$ of the Bott connection and it is given by the following lemma.

**Lemma 3.4.2.** *For $X, Y, Z \in \Gamma^{\infty}(\mathbb{M})$,*

$$\begin{aligned}\widehat{R}^{\varepsilon}(X,Y)Z =& R(X,Y)Z + \frac{1}{\varepsilon} J_{T(X,Y)} Z + \frac{1}{\varepsilon^2}(J_X J_Y - J_Y J_X)Z + \\ & \frac{1}{\varepsilon}(\nabla_X J)_Y Z - \frac{1}{\varepsilon}(\nabla_Y J)_X Z,\end{aligned}$$

*where $R$ is the curvature tensor of the Bott connection.*

*Proof.* By definition, we have $\widehat{R}^{\varepsilon}(X,Y)Z = \widehat{\nabla}^{\varepsilon}_X \widehat{\nabla}^{\varepsilon}_Y Z - \widehat{\nabla}^{\varepsilon}_Y \widehat{\nabla}^{\varepsilon}_X Z - \widehat{\nabla}^{\varepsilon}_{[X,Y]} Z$, the result follows by plugging in the definition (3.4.4). See details in [12]. □

With the above property in hand, we now can compute the Ricci curvature of the adjoint connection of the Baudoin connection $\widehat{\nabla}^{\varepsilon}$. To make it self-contained and to give the reader some idea of our setting, we also recall the proof below (the proof was given in [12]).

**Lemma 3.4.3.** *The horizontal Ricci curvature of the adjoint connection* $\widehat{\nabla}^\varepsilon$ *is given by*

$$\hat{\mathfrak{Ric}}^{\varepsilon}_{\mathcal{H}} = \mathfrak{Ric}_{\mathcal{H}} - \frac{1}{\varepsilon}\delta^*_{\mathcal{H}}T + \frac{1}{\varepsilon}\mathbf{J}^2,$$

*where* $\delta^*_{\mathcal{H}}T$ *denotes the adjoint of* $\delta_{\mathcal{H}}T$ *with respect to the metric* $g$.

*Proof.* Let $X, Y \in \Gamma^\infty(T\mathbb{M})$ and $X_1, \cdots, X_n$ be a local horizontal orthonormal frame. By the definition of the horizontal Ricci curvature and Lemma 3.4.2 we have

$$\begin{aligned}
&\hat{\mathbf{Ric}}^{\varepsilon}_{\mathcal{H}}(X,Y)\\
=&\sum_{i=1}^n g_{\mathcal{H}}(\hat{R}^\varepsilon(X_i,X)Y,X_i)\\
=&\sum_{i=1}^n g_{\mathcal{H}}(R(X_i,X)Y,X_i) + \sum_{i=1}^n g_{\mathcal{H}}\left(\frac{1}{\varepsilon}J_{T(X_i,X)}Y,X_i\right)\\
&+\sum_{i=1}^n g_{\mathcal{H}}\left(\frac{1}{\varepsilon}(\nabla_{X_i}J)_X Y - \frac{1}{\varepsilon}(\nabla_X J)_{X_i}Y, X_i\right).
\end{aligned}$$

For the first term, we have

$$\sum_{i=1}^n g_{\mathcal{H}}(R(X_i,X)Y,X_i) = \mathbf{Ric}_{\mathcal{H}}(X,Y).$$

For the second term, we easily see that

$$\begin{aligned}
\sum_{i=1}^n g_{\mathcal{H}}\left(J_{T(X_i,X)}Y,X_i\right) &= -\sum_{i=1}^n g_{\mathcal{V}}\left(T(X,X_i),T(Y,X_i)\right)\\
&= g_{\mathcal{H}}(\mathbf{J}^2X,Y).
\end{aligned}$$

For the third term, we first observe that $g_{\mathcal{H}}((\nabla_X J)_{X_i} Y, X_i) = 0$. Then, we have

$$\begin{aligned}\sum_{i=1}^n g_{\mathcal{H}}\left((\nabla_{X_i} J)_X Y, X_i\right) &= -\sum_{i=1}^n g_{\mathcal{H}}\left((\nabla_{X_i} J)_X X_i, Y\right)\\ &= -\sum_{i=1}^n g_{\mathcal{V}}\left((\nabla_{X_i} T)(X_i, Y), X\right)\\ &= g_{\mathcal{V}}\left(\delta_{\mathcal{H}} T(Y), X\right).\end{aligned}$$

□

## 3.5 Normal frames

In the previous sections, we have used the properties that we can choose $\{X_1, \cdots, X_n\}$ as local orthonormal frames and $\{Z_1, \cdots, Z_m\}$ as local orthonormal vertical frames. In this section, we first show the existence of such local orthonormal frames, then we collect their relations to Christoffel symbols associated with various connections we have considered and the structure coefficients in the Lie brackets. We first introduce the following lemma.

**Lemma 3.5.1** (Lemma 2.2 [16])**.** *Let $x_0 \in \mathbb{M}$. Around $x_0$, there exist a local orthonormal horizontal frame $\{X_1, \cdots, X_n\}$ and a local orthonormal vertical frame $\{Z_1, \cdots, Z_m\}$ such that the following structure relations hold*

$$[X_i, X_j] = \sum_{k=1}^n \omega_{ij}^k X_k + \sum_{k=1}^m \gamma_{ij}^k Z_k$$

$$[X_i, Z_k] = \sum_{j=1}^m \beta_{ik}^j Z_j,$$

*where* $\omega_{ij}^k, \gamma_{ij}^k, \beta_{ik}^j$ *are smooth functions such that:*

$$\beta_{ik}^j = -\beta_{ij}^k.$$

*Moreover, at* $x_0$ *we have*

$$\omega_{ij}^k = 0, \beta_{ij}^k = 0.$$

We record the fact (see [16]) that in this frame the Christoffel symbols of the Bott connection $\nabla$ are given by

$$\begin{cases} \nabla_{X_i} X_j = \frac{1}{2} \sum_{k=1}^n \left( \omega_{ij}^k + \omega_{ki}^j + \omega_{kj}^i \right) X_k \\ \nabla_{Z_j} X_i = 0 \\ \nabla_{X_i} Z_j = \sum_{k=1}^m \beta_{ij}^k Z_k \end{cases}$$

Thus, the Christofel symbols of the adjoint connection (3.4.4): $\widehat{\nabla}^\varepsilon = \nabla + \frac{1}{\varepsilon} J$ are given by

$$\begin{cases} \widehat{\nabla}^\varepsilon_{X_i} X_j = \frac{1}{2} \sum_{k=1}^n \left( \omega_{ij}^k + \omega_{ki}^j + \omega_{kj}^i \right) X_k \\ \widehat{\nabla}^\varepsilon_{Z_j} X_i = \frac{1}{\varepsilon} J_{Z_j} X_i = -\frac{1}{\varepsilon} \sum_{k=1}^n \gamma_{ik}^j X_k \\ \widehat{\nabla}^\varepsilon_{X_i} Z_j = \sum_{k=1}^m \beta_{ij}^k Z_k \end{cases}$$

We end this section by collecting some local coordinates representation of the curvature (torsion) terms we introduced above. In local horizontal and vertical orthonormal frames and for a generic one form $\eta = \sum_{i=1}^n f_i \theta_i + \sum_{j=1}^m k_j v_j$, we have ($\nabla_{g_\mathcal{H}} = \nabla_\mathcal{H}$)

**Lemma 3.5.2** (Theorem 3.1 and Lemma 3.2 [16])**.**

- $-(\nabla_{g_{\mathcal{H}}} - \mathfrak{T}^{\varepsilon}_{g_{\mathcal{H}}})^*(\nabla_{g_{\mathcal{H}}} - \mathfrak{T}^{\varepsilon}_{g_{\mathcal{H}}}) = \sum_{i=1}^n (\nabla_{X_i} - \mathfrak{T}^{\varepsilon}_{X_i})^2 - (\nabla_{\nabla_{X_i} X_i} - \mathfrak{T}^{\varepsilon}_{\nabla_{X_i} X_i})$

- $\delta^*_{\mathcal{H}} T(\eta) = \sum_{i,j=1}^n \sum_{k=1}^m (X_j(t)\gamma^k_{ij}(t)) g_k \theta_i,$

- $\delta_{\mathcal{H}} T(\eta) = \sum_{i,j=1}^n \sum_{l=1}^m (X_i(t)\gamma^l_{ij}(t)) f_j v_l,$

- $\mathfrak{T}^{\varepsilon}_{X_i}\eta = \sum_{j=1}^n \sum_{l=1}^m (\gamma^l_{ij} g_l \theta_j - \frac{1}{\varepsilon}\gamma^l_{ij} f_j v_l),$

- $\nabla^{\sharp}_{g_{\mathcal{H}}}(\eta) = \frac{1}{2}\sum_{i,j=1}^n ((X_i f_j)\theta_j + (X_j f_i)\theta_i) + \sum_{k=1}^m (X_i g_k) v_k,$

- $\mathbf{Ricci}(X_i, X_k) = \sum_{j=1}^n (\frac{1}{2} X_j(\omega^j_{ik} - \omega^k_{ij} - \omega^i_{kj}) - X_i \omega^j_{jk}),$

- $\mathbf{Ricci}(Z_l, X_k) = -\sum_{j=1}^n Z_l \omega^j_{jk} = 0.$

## 3.6 Examples

In this section, we present examples where we can equip totally geodesic foliation structures on the manifolds as well as put bundle-like metrics on it.

### 3.6.1 Heisenberg group

**Example 3.6.1** (**Heisenberg group**)**.** The Heisenberg group is denoted as

$$\mathbb{H}^{2n+1} = \{(x, y, z), x \in \mathbb{R}^n, y \in \mathbb{R}^n, z \in \mathbb{R}\}$$

endowed with the group law

$$(x_1, y_1, z_1) \star (x_2, y_2, z_2) := (x_1 + x_2, y_1 + y_2, z_1 + z_2 + \langle x_1, y_2\rangle_{\mathbb{R}^n} - \langle x_2, y_1\rangle_{\mathbb{R}^n}).$$

The vector fields

$$\begin{aligned} X_i &= \frac{\partial}{\partial x_i} - y_i \frac{\partial}{\partial z}, 1 \leqslant i \leqslant n, \\ Y_i &= \frac{\partial}{\partial y_i} + x_i \frac{\partial}{\partial z}, 1 \leqslant i \leqslant n, \\ Z &= \frac{\partial}{\partial z} \end{aligned}$$

form a basis for the space of left-invariant vector fields on $\mathbb{H}^{2n+1}$. We choose a left-invariant Riemannian metric on $\mathbb{H}^{2n+1}$ in such a way that $\{X_1, ..., X_n, Y_1, ..., Y_n, Z\}$ are orthonormal with respect to this metric. Note that these vector fields satisfy the following commutation relations

$$[X_i, Y_j] = 2\delta_{ij} Z, \quad [X_i, Z] = [Y_i, Z] = 0, \quad i = 1, ..., n.$$

Then, the projection map

$$\begin{aligned} \pi : \quad \mathbb{H}^{2n+1} &\longrightarrow \mathbb{R}^{2n} \\ (x, y, z) &\longmapsto (x, y) \end{aligned}$$

is a Riemannian submersion with totally geodesic fibers.

### 3.6.2 Generalized Hopf fibrations

The next class of examples encompasses all the generalized Hopf fibrations (see Chapter 9, Section H in [22]).

**Example 3.6.2 (Generalized Hopf fibrations).** Let $\mathbf{G}$ be a Lie group, and $\mathbf{H}, \mathbf{K}$

be two compact subgroups of $\mathbf{G}$ with $\mathbf{K} \subset \mathbf{H}$. Then, we have a natural fibration given by the coset map

$$\begin{aligned} \pi: \quad \mathbf{G}/\mathbf{K} &\to \mathbf{G}/\mathbf{H} \\ \alpha\mathbf{K} &\to \alpha\mathbf{H}, \end{aligned}$$

where the fiber is $\mathbf{H}/\mathbf{K}$. From [22], there exist $\mathbf{G}$-invariant metrics on respectively $\mathbf{G}/\mathbf{K}$ and $\mathbf{G}/\mathbf{H}$ that make $\pi$ a Riemannian submersion with totally geodesic fibers isometric to $\mathbf{H}/\mathbf{K}$.

**Example 3.6.3** (**Hopf fibrations**)**.** We consider the Hopf fibration as

$$\mathbb{S}^1 \to \mathbb{S}^3 \to \mathbb{S}^2.$$

with the projection map $\pi : \mathbb{S}^3 \to \mathbb{S}^2$. Then for any point $x \in \mathbb{S}^2$, $\pi^{-1}(x)$ is a $\mathbb{S}^1$ circle which gives us the totally geodesic leaves. In general, the Hopf fibrations have the following form

$$U(1) \to \mathbb{S}^{2n+1} \to \mathbb{C}P^n.$$

with the projection map $\pi : \mathbb{S}^{2n+1} \to \mathbb{C}P^n$.

### 3.6.3 The frame bundle over $\mathbb{S}^n$

**Example 3.6.4.** The Lie group $\mathbf{SO}(n+1)$ can be described as the set of matrices

$$\mathbf{SO}(n+1) = \left\{ M \in \mathbb{R}^{(n+1)\times(n+1)}, M^*M = I, \det M = 1 \right\}.$$

Its Lie algebra is

$$\mathfrak{so}(n+1) = \left\{ M \in \mathbb{R}^{(n+1)\times(n+1)}, M^* + M = 0 \right\}.$$

Let us denote by $E_{ij}$ the $(n+1)\times(n+1)$ matrix with all zero entries but the component $(i,j)$ which is 1. A basis of the Lie algebra $\mathfrak{so}(n+1)$ is given by

$$X_j = E_{j+1,1} - E_{1,j+1}, \quad 1 \le j \le n$$

$$V_{ij} = E_{ji} - E_{ij}, \quad 2 \le i < j \le n+1.$$

As usual, we identify elements of $\mathfrak{so}(n+1)$ with left-invariant vector fields on $\mathbf{SO}(n+1)$. We can identify $\mathbf{SO}(n)$ as an isotropy subgroup of $\mathbf{SO}(n+1)$:

$$\mathbf{SO}(n) = \{ M \in \mathbf{SO}(n+1), Me_1 = e_1 \},$$

where $e_1 = (1, 0, \cdots, 0)$. The quotient space $\mathbf{SO}(n+1)/\mathbf{SO}(n)$ is isomorphic to the $n$-dimensional sphere $\mathbb{S}^n$ and we have a submersion with totally geodesic fibers

$$\pi : \mathbf{SO}(n+1) \to \mathbb{S}^n,$$

given by $\pi(M) = Me_1$. This gives the frame bundle fibration

$$\mathbf{SO}(n) \to \mathbf{SO}(n+1) \to \mathbb{S}^n.$$

The horizontal space of this fibration is generated by $X_1, \cdots, X_n$.

### 3.6.4 K-contact manifold

**Example 3.6.5 (K-contact manifolds).** Another important example of Riemannian foliation is obtained in the context of contact manifolds. Let $(\mathbb{M}, \theta)$ be a $2n+1$-dimensional smooth contact manifold, where $\theta$ is a contact form, namely $\theta \wedge (d\theta)^n$ is a volume form (nonvanishing). On $\mathbb{M}$ there is a unique smooth vector field $Z$, called the Reeb vector field, satisfying

$$\theta(Z) = 1, \quad \mathcal{L}_Z(\theta) = 0,$$

where $\mathcal{L}_Z$ denotes the Lie derivative with respect to $Z$. On $\mathbb{M}$ there is a foliation, the Reeb foliation, whose leaves are the orbits of the vector field $Z$. As it is well-known (see [111]), that it is always possible to find a Riemannian metric $g$ and a $(1,1)$-tensor field $J$ on $\mathbb{M}$ so that for every vector fields $X, Y$

$$g(X, Z) = \theta(X), \quad J^2(X) = -X + \theta(X)Z, \quad g(X, JY) = (d\theta)(X, Y).$$

The triple $(\mathbb{M}, \theta, g)$ is called a *contact Riemannian manifold*. We see then that the Reeb foliation is totally geodesic with bundle-like metric if and only if the Reeb vector field $Z$ is a Killing field, that is,

$$\mathcal{L}_Z g = 0,$$

as is stated in [23, Proposition 6.4.8]. In this case, $(\mathbb{M}, \theta, g)$ is called a *K-contact Riemannian manifold*. Observe that the horizontal distribution $\mathcal{H}$ is then the kernel of $\theta$ and that $\mathcal{H}$ is bracket generating because $\theta$ is a contact form. We refer to [17, 111] for further details on this class of examples.

**Remark 3.6.6.** A triple $(\theta, J, g)$ is referred as an almost contact structure on the contact manifold $(\mathbb{M}, \theta, g)$, if

$$J(Z) = 0, \quad \text{and} \quad J^2(X) = -X + \theta(X)Z$$

the notation is the same as in the above example.

### 3.6.5 Sasakian manifold

**Example 3.6.7 (Sasakian manifolds).** A Sasakian manifold of dimension $2n+1$ is a Riemannian manifold $(\mathbb{M}^{2n+1}, g)$ satisfying that its metric cone $(C(\mathbb{M}) = \mathbb{R}_+ \times \mathbb{M}, \bar{g} = dr^2 + r^2 g)$ is Kähler, which is equivalent to say that the almost contact structure $(\theta, J, g)$ on $\mathbb{M}^{2n+1}$ is normal, which means that the almost complex structure $J$ on $C(\mathbb{M})$ is integrable. It is clear to see that, the Sasakian manifold is a special case of the contact manifold. The restriction of $\theta$ to the slice $\{r = 1\}$ is a unit length killing vector field, and its orbits define a one-dimensional foliation of $\mathbb{M}^{2n+1}$ by geodesics called the Reeb foliation. We thus can have bundle-like metric on Sasakian manifold. For details, we refer to [30, 23]

# Chapter 4

# Stochastic analysis on Riemannian foliations

Recall from the previous section, we make the convention throughout this section that $(\mathbb{M}^{n+m}, \mathcal{F}, g)$ is denoted as Riemannian foliation with totally geodesic leaves and bundle-like metrics. We will establish stochastic analysis (calculus) on the paths space of Riemannain foliation $(\mathbb{M}^{n+m}, \mathcal{F}, g)$, we present the main results in [11, 12]. In particular, we prove Clark-Ocone formulas, integration by parts formulas, Log-Sobolev inequalites, equivalent conditions to two sided uniform Ricci curvature bounds, concentration inequalities, tail estimates and quasi-invariance of horizontal Wiener measure (see (4.6.1) ) on the paths space of $\mathbb{M}^{n+m}$. We unify the assumptions from [11] and [12] in this section, to be precise, we remove the Yang-Mills condition ( i.e. $\delta_{\mathcal{H}} T = 0$) in [11]. Thus the results in [11] will be presented without assuming the Yang-Mills condition, which is the same as [12], thus the results will be more general comparing to those in [11].

## 4.1 Horizontal Brownian motion

We begin with the constructions of horizontal Brownian motion.

### 4.1.1 Construction from horizontal Dirichlet form.

Recall from the previous section, we have defined the horizontal gradient $\nabla_{\mathcal{H}} f$ of a function $f$ as the projection of the Riemannian gradient of $f$ on the horizontal bundle $\mathcal{H}$ and the vertical gradient $\nabla_{\mathcal{V}} f$ of a function $f$ as the projection of the Riemannian gradient of $f$ on the vertical bundle $\mathcal{V}$.

Consider the pre-Dirichlet form

$$\mathcal{E}_{\mathcal{H}}(f,h) = \int_{\mathbb{M}} g_{\mathcal{H}}\left(\nabla_{\mathcal{H}} f, \nabla_{\mathcal{H}} h\right)\, d\,\mathrm{Vol},\ \ f,h \in C^{\infty}(\mathbb{M}),$$

where $d\,\mathrm{Vol}$ is the Riemannian volume measure. Then there exists a unique diffusion operator $L$ on $\mathbb{M}$ such that for all $f,h \in C^{\infty}(\mathbb{M})$

$$\mathcal{E}_{\mathcal{H}}(f,h) = -\int_{\mathbb{M}} fLh\, d\,\mathrm{Vol} = -\int_{\mathbb{M}} hLf\, d\,\mathrm{Vol}\,.$$

The operator $L$ is called the *horizontal Laplacian* of the foliation. If $X_1, \cdots, X_n$ is a local orthonormal frame of horizontal vector fields, then we can write $L$ in this frame

$$L = \sum_{i=1}^{n} X_i^2 + X_0, \tag{4.1.1}$$

where $X_0$ is a smooth vector field. Observe that the subbundle $\mathcal{H}$ satisfies Hörmander's (bracket generating) condition, therefore by Hörmander's theorem the operator $L$ is locally subelliptic (for comments on this terminology introduced by Fefferman-Phong we refer to [48], see also the survey paper [78] or [38, p. 944]).

By [9, Proposition 5.1] the completeness of the Riemannian metric $g$ implies that $L$ is essentially self-adjoint on $C^\infty(\mathbb{M})$ and thus that $\mathcal{E}_\mathcal{H}$ is uniquely closable. Then we can define the semigroup $P_t = e^{\frac{t}{2}L}$ by using the spectral theorem. The diffusion process $\{W_t\}_{t\geqslant 0}$ corresponding to the semigroup $\{P_t\}_{t\geqslant 0}$ will be called the *horizontal Brownian motion* on the Riemannian foliation $(\mathbb{M}^{n+m}, \mathcal{F}, g)$. Since $\mathbb{M}$ is assumed to be compact, $1 \in \mathbf{dom}(\mathcal{E}_\mathcal{H})$ and thus $P_t 1 = 1$. This implies that $\{W_t\}_{t\geqslant 0}$ is a non-explosive diffusion.

If the horizontal Laplacian can globally be written in the form 4.1.1, for smooth horizontal vector fields $X_0, X_1, \cdots, X_n$, then $\{W_t\}_{t\geqslant 0}$ can be constructed from a stochastic differential equation on $\mathbb{M}$.

Even if the horizontal Laplacian can not globally be written in the form 4.1.1 , the horizontal Brownian motion $\{W_t\}_{t\geqslant 0}$ can still be constructed from a globally defined stochastic differential equation on a bundle over $\mathbb{M}$ (see [39, Theorem 3.8] or Corollary 4.1.3). The following section provides an explicit description of such construction that shall be used in the sequel.

### 4.1.2 Orthonormal frame bundles

In order to introduce the intrinsic construction of Brownian motion on Riemannian foliation $(\mathbb{M}^{n+m}, \mathcal{F}, g)$. We need to adapt the construction by Eells-Elworthy-Malliavin ([41, 94]) on a Riemannian manifold, where they use the orthonormal frame bundle of the Riemannian manifold. For a concise introduction of frame bundles and construction of Brownian motion on a Riemannian manifold, we refer to Chapter 2 and Chapter 3 in [74].

To make it concise, we will directly introduce the orthonormal frame bunldes. (For general frame bundles, refer to Chapter 2 in [74].) We begin with the notions of orthonormal frames. An orthonormal map at $x \in \mathbb{M}$ is an isometry $u : (\mathbb{R}^{n+m}, \langle \cdot, \cdot \rangle) \to (T_x\mathbb{M}, g)$. The orthonormal map bundle will be denoted by $\mathcal{O}(\mathbb{M})$. Since we equipt $\mathbb{M}$ with a foliation structure, we are interested in a special sub-bundle of $\mathcal{O}(\mathbb{M})$, the *horizontal frame bundle*. An isometry $u : (\mathbb{R}^{n+m}, \langle \cdot, \cdot \rangle) \longrightarrow (T_x\mathbb{M}, g)$ will be called *horizontal* if

- $u(\mathbb{R}^n \times \{0\}) \subset \mathcal{H}_x$;
- $u(\{0\} \times \mathbb{R}^m) \subset \mathcal{V}_x$.

The *horizontal frame bundle* is then defined as the set of $(x, u) \in \mathcal{O}(\mathbb{M})$ such that $u$ is horizontal. It will be denoted as $\mathcal{O}_{\mathcal{H}}(\mathbb{M})$. For notational convenience, when needed, we identify $\mathbb{R}^n$ with $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+m}$ and $\mathbb{R}^m$ with $\{0\} \times \mathbb{R}^m \subset \mathbb{R}^{n+m}$ .

In general, the connections we introduced in Chapter 3 allow to lift vector fields on $\mathbb{M}$ to vector fields on $\mathcal{O}(\mathbb{M})$. If we denote $e_1, \cdots, e_n, f_1, \cdots, f_m$ as the canonical basis of $\mathbb{R}^{n+m}$, then $u(e_1), \cdots, u(e_n), u(f_1), \cdots, u(f_m)$ form a basis for $T\mathbb{M}$. We denote by $A_i$ the vector field on $\mathcal{O}(\mathbb{M})$ such that $A_i(x, u)$ is the lift of $u(e_i)$, and we denote by $V_i$ the vector field on $\mathcal{O}(\mathbb{M})$ such that $V_i(x, u)$ is the lift of $u(f_i)$.

Fix $x_0 \in \mathbb{M}^{n+m}$, recall Lemma 3.5.1 we can pick $\{X_1 \cdots, X_n, Z_1, \cdots, Z_m\}$ to be orthonormal frames around $x_0$. If $u : \mathbb{R}^{n+m} \to T_x\mathbb{M}$ is a horizontal isometry, we can find an orthogonal matrix $e_i^j$ such that $u(e_i) = \sum_{j=1}^n e_i^j X_j$ and $u(f_i) = \sum_{j=1}^m f_i^j Z_j$. Let $\bar{X}_j$ be the vector field on $\mathcal{O}_{\mathcal{H}}(\mathbb{M})$ defined by

$$\bar{X}_j f(x, u) = \lim_{t \to 0} \frac{f(e^{tX_j}(x), u) - f(x, u)}{t}.$$

We thus can represent the horizontal lift $A_1(x,u), \cdots, A_n(x,u)$ in terms of $\bar{X}_1, \cdots, \bar{X}_n$.

**Lemma 4.1.1** (Lemma 3.4 [12]). *Let $x_0 \in \mathbb{M}$ and $(x,u) \in \mathcal{O}_{\mathcal{H}}(\mathbb{M})$, we pick up Bott connection $\nabla$ in this lemma*

$$A_i(x,u) = \sum_{j=1}^{n} e_i^j \bar{X}_j$$
$$- \sum_{j,k,l,r=1}^{n} e_i^j e_r^l \langle \nabla_{X_j} X_l, X_k \rangle \frac{\partial}{\partial e_r^k} - \sum_{j=1}^{n} \sum_{k,l,r=1}^{m} e_i^j f_r^l \langle \nabla_{X_j} Z_l, Z_k \rangle \frac{\partial}{\partial f_r^k}.$$

*In particular, at $x_0$ we have*

$$A_i(x_0,u) = \sum_{j=1}^{n} e_i^j \bar{X}_j.$$

*Proof.* Detailed proof refers to [12, Lemma 3.4]. This construction works for all the metric connection adapted to the foliation structure (see [12, Assumption 1]). This is a special case of the proof in Riemannian manifold setting, which we have a splitting of the tangent bundle $T\mathbb{M} = \mathcal{H} \oplus \mathcal{V}$. The proof in the Riemannian setting refers to [74, Proposition 2.1.3]. □

### 4.1.3 Construction from horizontal orthonormal frame bundle

With the previous Lemma 4.1.1 in hand, we are now ready to construct horizontal Brownian motion on Riemannian foliaiton $(\mathbb{M}^{n+m}, \mathcal{F}, g)$. Recall (3.3.4), we denote $L$ as our horizontal laplacian operator. Then we have

**Proposition 4.1.2.** *[12, Proposition 3.5] Let $\pi : \mathcal{O}(\mathbb{M}) \to \mathbb{M}$ be the bundle projection map. For a smooth $f : \mathbb{M} \to \mathbb{R}$, and $(x,u) \in \mathcal{O}_{\mathcal{H}}(\mathbb{M})$,*

$$\left(\sum_{i=1}^{n} A_i^2\right)(f \circ \pi)(x,u) = Lf \circ \pi(x,u).$$

As a corollary, we thus have

**Corollary 4.1.3.** *Let $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space that satisfies the usual conditions and let $(B_t)_{t\geqslant 0}$ be an adapted $n$-dimensional Brownian motion on that space. Let $(U_t)_{t\geqslant 0}$ be a solution to the Stratonovitch stochastic differential equation*

$$dU_t = \sum_{i=1}^{n} A_i(U_t) \circ dB_t^i, \quad U_0 \in \mathcal{O}_{\mathcal{H}}(\mathbb{M}),$$

*then $W_t = \pi(U_t)$ is a horizontal Brownian motion on $\mathbb{M}$. $W_t$ is a Markov process with generator $\frac{1}{2}L$. We denote $P_t = e^{\frac{1}{2}L}$ as the semigroup generated by $\frac{1}{2}L$.*

The above corollary gives us the construction of horizontal Brownian motion on the Riemannian foliation $\mathbb{M}$. It is clear that the Brownian motion is called horizontal because we have only used the lift of the horizontal vector fields: $A_1, \cdots, A_n$. What's more, the horizontal Brownian motion through the construction $B \to U \to W$ is actually on the whole manifold $\mathbb{M}$. Similar to the Riemannian case , given a horizontal Brownian motion $W$ on $\mathbb{M}$ with starting point $x$, we can also construct the anti-development of $W$, which will be a Brownian motion on the horizontal space $\mathcal{H}_x$. These aforementioned constructions actually not only work for (horizontal) Brownian motion, but also for general semimartingales, which we will make it precise later.

### 4.1.4 Stochastic parallel transport

Below, we will focus on the *stochastic parallel transport* along the (horizontal) Brownian motion on $\mathbb{M}$ and the anti-development of horizontal Brownian motion. However, the construction given below will also work for general (horizontal) semimartingales once we introduce those later.

Recall the *stochastic parallel transport* ( see for example [74]) for a general given connection $\widetilde{\nabla}$ and general function $f \in \mathcal{C}^\infty(\mathbb{M})$, for any (horizontal) Brownian motion $W_t$ on $\mathbb{M}$ with staring point $x$, the process $\widetilde{/\!/}_{0,t}^{-1} f(W_t)$ satisfies the stochastic differential equation

$$d[\widetilde{/\!/}_{0,t}^{-1} f(W_t)] = /\!/_{0,t}^{-1} \nabla_i f(W_t) \circ dW_t^i, \quad /\!/_0 = Id$$

if $\{e_i, \ldots, e_n, f_1, \cdots, f_m\}$ is a local frame for the vector bundle of $\mathbb{M}$, then $\nabla_i e_j = \widetilde{\Gamma}_{ij}^k e_k$ (same for $f_j$, $j = 1, \cdots, m$) where $\widetilde{\Gamma}_{ij}^k$ is the Christoffel symbol for the given connection $\widetilde{\nabla}$. Then $\widetilde{/\!/}_{0,t} e_i(W_0) = a_i^j(t) e_j(W_t)$, with $a_i^j(t)$ determined by the equation

$$da_i^j(t) = -\Gamma_{kl}^j(X_t) a_i^l \circ dW_t^k,$$

the above notation $\circ dW_t^k$ is in the Stratonovitch sense. This is the local formula for the *stochastic parallel transport* with $\widetilde{/\!/}_{0,t} : T_x\mathbb{M} \to T_{W_t}\mathbb{M}$.

Thus we denote by $\widetilde{/\!/}_{0,t} : T_x\mathbb{M} \to T_{W_t}\mathbb{M}$ the *stochastic parallel transport* of vector fields along the paths of $\{W_t\}_{0 \leqslant t \leqslant 1}$. However, if we need to use a stochastic parallel transport on forms. Then by duality we can define the stochastic parallel transport on one-forms as follows. We have $\widetilde{/\!/}_{0,t} : T^*_{W_t}\mathbb{M} \to T^*_x\mathbb{M}$ such that for $\alpha \in T^*_{W_t}\mathbb{M}$

$$\langle \widetilde{/\!/}_{0,t} \alpha, v \rangle = \langle \alpha, \widetilde{/\!/}_{0,t} v \rangle, \quad v \in T_x\mathbb{M}. \tag{4.1.2}$$

### 4.1.5 Anti-development of horizontal Brownian motion

With the previous construction of stochastic parallel transport. Now we focus on the stochastic parallel transport for the adjoint Baudoin connection $\widehat{\nabla}^\varepsilon$ in (3.4.4). We denote $\widehat{\Theta}_t^\varepsilon$ as the stochastic parallel transport for the adjoint Baudoin connection $\widehat{\nabla}^\varepsilon = \nabla + \frac{1}{\varepsilon} J$ along the paths of the horizontal Brownian motion $\{W_t\}_{0\leqslant t\leqslant 1}$. Since the adjoint Baudoin connection $\widehat{\nabla}^\varepsilon$ is horizontal, the map $\widehat{\Theta}_t^\varepsilon : T_x\mathbb{M} \to T_{W_t}\mathbb{M}$ is an isometry that preserves the horizontal bundle, that is, if $u \in \mathcal{H}_x$, then $\widehat{\Theta}_t^\varepsilon u \in \mathcal{H}_{W_t}$. We see then that the anti-development of $\{W_t\}_{0\leqslant t\leqslant 1}$ defined as

$$B_t := \int_0^t (\widehat{\Theta}_s^\varepsilon)^{-1} \circ dW_s,$$

is a Brownian motion in the horizontal space $\mathcal{H}_x$.

**Remark 4.1.4.** For one-forms, the process $\widehat{\Theta}_t^\varepsilon : T^*_{W_t}\mathbb{M} \to T^*_x\mathbb{M}$ is a solution to the following covariant Stratonovich stochastic differential equation

$$d[\widehat{\Theta}_t^\varepsilon \alpha(W_t)] = \widehat{\Theta}_t^\varepsilon \widehat{\nabla}^\varepsilon_{\circ dW_t} \alpha(W_t),$$

where $\alpha$ is any smooth one-form. Since $\widehat{\nabla}^\varepsilon_{\circ dW_t} = \nabla_{\circ dW_t} + \frac{1}{\varepsilon} J_{\circ dW_t} = \nabla_{\circ dW_t}$, we deduce that $\widehat{\Theta}^\varepsilon$ is actually independent of $\varepsilon$ and is therefore also the stochastic parallel transport for the Bott connection. As a consequence, the Brownian motion $(B_t)_{0\leqslant t\leqslant 1}$ and its filtration are also independent of the particular choice of $\varepsilon$. This is the reason why we can change the stochastic parallel transport associated with the Bott connection in [11] to be the stochastic parallel transport associated with the adjoint Baudoin connection $\widehat{\Theta}^\varepsilon$ in [12], which makes it consistent in the current setting.

## 4.2 Horizontal calculus of variations

In this section, we describe the space of variations of horizontal paths and introduces the notion of horizontal semimartingales. We also introduce the tangent process tangent to horizontal semimartingales and its motivations. In particular, we will see that the horizontal Brownian motion in the previous section is a special case of our horizontal semimartingales. Since the foliation structure separates us from the standard notions of variations of semimartingales on Riemannian manifolds, we first present some results about horizontal calculus of variations of deterministic paths in the current setting.

Although we are now starting to work in a more general setting for horizontal semimartingales, we keep the same notation as we did for the horizontal Brownian motion in the previous section. In particular, we consider on $\mathbb{M}$ the adjoint Baudoin connection (see (3.4.4)) $\widehat{\nabla}^\varepsilon = \nabla + \frac{1}{\varepsilon}J$. However, as we pointed it out before, the connection can be an arbitrary connection $D$ that satisfies the properties in [12, Assumption 1]. We decide to use the connection $\widehat{\nabla}^\varepsilon$ to simplify this part and also give the original motivation.

Throughout the section, we fix a point $x_0 \in \mathbb{M}$ and a point $u_0 \in \mathcal{O}_{\mathcal{H}}(\mathbb{M})$ such that $\pi(u_0) = x_0$. We denote fundamental vector fields $A_i$'s and $V_i$'s on $\mathcal{O}(\mathbb{M})$ where $A_i$ is the vector field on $\mathcal{O}(\mathbb{M})$ such that $A_i(x,u)$ is the lift of $u(e_i)$, and $V_i$ is the vector field on $\mathcal{O}(\mathbb{M})$ such that $V_i(x,u)$ is the lift of $u(f_i)$.

If $v \in \mathbb{R}^{n+m}$ is a vector, we denote by

$$Av = \sum_{i=1}^{n} v_i A_i \quad \text{and} \quad Vv = \sum_{i=1}^{m} v_{i+n} V_i, \tag{4.2.1}$$

where $A_i$ and $V_i$ are defined as above. $Av$ and $Vv$ are therefore vector fields on $\mathcal{O}(\mathbb{M})$ whose values at some $u \in \mathcal{O}(\mathbb{M})$ will be denoted respectively by $A_u v$ and $V_u v$.

We denote $W^\infty(\mathbb{R}^{n+m})$ as the space of smooth paths $v : [0, +\infty) \to \mathbb{R}^{n+m}$ such that $v(0) = 0$ and denote $W^\infty(\mathbb{M})$ as the space of smooth paths $\gamma : [0, +\infty) \to \mathbb{M}$ such that $\gamma(0) = x_0$.

The following result is for Riemannian setting (see for instance [72, Section 2]) and we remind the result to fix notations and to compare to our foliation setting.

**Lemma 4.2.1.** 1. *Let $\omega \in W^\infty(\mathbb{R}^{n+m})$ and let $(u_t)_{t\geqslant 0}$ be a solution to the differential equation*

$$du_t = \sum_{i=1}^{n} A_i(u_t) d\omega_t^i + \sum_{i=1}^{m} V_i(u_t) d\omega_t^{n+i},$$

*then $\gamma_t = \pi(u_t)$ is called the development of $\omega$ and we will concisely write $\gamma = \phi(\omega)$.*

2. *Let $\gamma \in W^\infty(\mathbb{M})$. Then there exists a unique $\omega \in W^\infty(\mathbb{R}^{n+m})$, such that if $(u_t)_{t\geqslant 0}$ is the solution to the differential equation*

$$du_t = \sum_{i=1}^{n} A_i(u_t) d\omega_t^i + \sum_{i=1}^{m} V_i(u_t) d\omega_t^{n+i},$$

*then $\gamma_t = \pi(u_t)$. The path $\omega$ is called the anti-development of $\gamma$ and we will concisely write $\omega = \phi^{-1}(\gamma)$.*

This lemma 4.2.1 extends to continuous semimartingales, in which case we speak of stochastic development and stochastic anti-development (see [74, Section 2.3]).

### 4.2.1 Horizontal paths

**Definition 4.2.2.** A smooth path $\omega : [0, +\infty) \to \mathbb{R}^{n+m}$ is called horizontal if it takes values in $\mathbb{R}^n$. The space of smooth horizontal paths such that $\omega(0) = 0$ shall be denoted as $W^\infty_{\mathcal{H}}(\mathbb{R}^{n+m})$.

**Definition 4.2.3.** A smooth path $\gamma : [0, +\infty) \to \mathbb{M}$ is called horizontal, if for every vertical smooth one-form $\theta$, $\int_\gamma \theta = 0$. The space of smooth horizontal paths such that $\gamma(0) = x_0$ shall be denoted as $W^\infty_{\mathcal{H}}(\mathbb{M})$.

**Remark 4.2.4.** Since $W^\infty_{\mathcal{H}}(\mathbb{M})$ is only made of smooth paths, this definition is readily seen to be equivalent to the more usual one: A path $\gamma$ is in $W^\infty_{\mathcal{H}}(\mathbb{M})$ if and only if $\gamma'(s) \in \mathcal{H}_{\gamma(s)}$ for every $s \geq 0$. The advantage of the above definition is that it extends to non-smooth paths like semimartingale paths.

Comparing to our definition to horizontal Brownian motion in Corollary 4.1.3, we have the following result for horizontal smooth paths.

**Proposition 4.2.5.** *1. Let $\omega^{\mathcal{H}} \in W^\infty_{\mathcal{H}}(\mathbb{R}^{n+m})$ and let $(u_t)_{t\geqslant 0}$ be a solution to the differential equation*

$$du_t = \sum_{i=1}^n A_i(u_t) d\omega_t^{\mathcal{H},i},$$

*then $\gamma_t = \pi(u_t)$ is a horizontal path on $\mathbb{M}$. The path $\gamma$ will be called the horizontal development of $\omega$ and we will concisely write $\gamma = \phi_{\mathcal{H}}(\omega)$.*

*2. Let $\gamma \in W^\infty_{\mathcal{H}}(\mathbb{M})$. Then, there exists a unique $\omega^{\mathcal{H}} \in W^\infty_{\mathcal{H}}(\mathbb{R}^{n+m})$, such that if $(u_t)_{t\geqslant 0}$ is the solution to the differential equation*

$$du_t = \sum_{i=1}^n A_i(u_t) d\omega_t^{\mathcal{H},i},$$

*then* $\gamma_t = \pi(u_t)$. *The path* $\omega^{\mathcal{H}}$ *will be called the horizontal anti-development of* $\gamma$ *and we will concisely write* $\omega^{\mathcal{H}} = \phi_{\mathcal{H}}^{-1}(\gamma)$.

*Proof.* The proof refers to [12, Proposition 4.7]. □

### 4.2.2 Paths tangent to horizontal paths

Let now $v \in W^\infty(\mathbb{R}^{n+m})$, where $W^\infty(\mathbb{R}^{n+m})$ denotes the space of smooth paths $v : [0, +\infty) \to \mathbb{R}^{n+m}$ such that $v(0) = 0$. We consider the vector field $\mathbf{D}_v$ on $W^\infty(\mathbb{M})$ defined for $\gamma \in W^\infty(\mathbb{M})$ by

$$D_v(\gamma)_s = u_s v_s,$$

where $u$ is the horizontal lift of $\gamma$ to $\mathcal{O}(\mathbb{M})$. Let $\{\zeta_t^v, t \in \mathbb{R}\}$ be the flow generated by $\mathbf{D}_v$ i.e.

$$\frac{d}{dt}(\zeta_t^v\gamma)_s = \mathbf{D}_v(\zeta_t^v\gamma)_s, \quad \zeta_0^v\gamma = \gamma.$$

One defines then

$$\xi_t^v = \phi^{-1} \circ \zeta_t^v \circ \phi, \quad t \in \mathbb{R},$$

which is a flow on $W^\infty(\mathbb{R}^{n+m})$. Theorem 2.1 in [72] describes the generator of this flow.

**Theorem 4.2.6** (Theorem 2.1, [72])**.** *Let* $v \in W^\infty(\mathbb{R}^{n+m})$ *and* $\omega \in W^\infty(\mathbb{R}^{n+m})$. *Then,*

$$\frac{d}{dt}\Big|_{t=0} \xi_t^v(\omega)_s = p_v(\omega)_s,$$

*with*

$$p_v(\omega)_t = v(t) - \int_0^t T_{u_s}^{\widehat{\nabla}^\varepsilon}(Ad\omega_s + Vd\omega_s, Av(s) + Vv(s)) - \int_0^t \left( \int_0^s \Omega_{u_\tau}^{\widehat{\nabla}^\varepsilon}(Ad\omega_\tau + Vd\omega_\tau, Av(\tau) + Vv(\tau)) \right) d\omega_s,$$

*where $u$ is the lift to $\mathcal{O}(\mathbb{M})$ of the development of $\omega$. We denoted by $T^{\widehat{\nabla}^\varepsilon}$ the torsion form of the connection $\widehat{\nabla}^\varepsilon$ and by $\Omega^{\widehat{\nabla}^\varepsilon}$ its curvature form. Of course, the statement is adapted to our current setting with connection $\widehat{\nabla}^\varepsilon$. In the original work [72], this statement is for the general connection on the Riemannina manifold.*

With the above theorem in hand, for $\omega^{\mathcal{H}} \in W_{\mathcal{H}}^\infty(\mathbb{R}^{n+m})$, we want to study the variation of horizontal paths,

$$p_v(\omega^{\mathcal{H}})_t = v(t) - \int_0^t T_{u_s}^{\widehat{\nabla}^\varepsilon}(Ad\omega_s^{\mathcal{H}}, Av(s) + Vv(s)) - \int_0^t \left( \int_0^s \Omega_{u_\tau}^{\widehat{\nabla}^\varepsilon}(Ad\omega_\tau^{\mathcal{H}}, Av(\tau) + Vv(\tau)) \right) d\omega_s^{\mathcal{H}}. \tag{4.2.2}$$

**Definition 4.2.7.** We will say that $v \in W^\infty(\mathbb{R}^{n+m})$ is tangent to the horizontal path $\gamma \in W_{\mathcal{H}}^\infty(\mathbb{M})$ if for every $s \geq 0$, $\frac{d}{dt}\mid_{t=0} \phi^{-1}(\zeta_t^v\gamma)_s \in \mathbb{R}^n$.

**Remark 4.2.8.** According to the definition, $v \in W^\infty(\mathbb{R}^{n+m})$ is tangent to the horizontal path $\gamma$ , if and only if $p_v(\omega^{\mathcal{H}})$ is horizontal, where $\omega$ is the horizontal anti-development of $\gamma$. However, even if $v \in W^\infty(\mathbb{R}^{n+m})$ is tangent to the horizontal path $\gamma$, it may not be true that for every $t \in \mathbb{R}$, $\zeta_t^v\gamma \in W_{\mathcal{H}}^\infty(\mathbb{M})$.

We then introduce the following characterization of tangent processes. We stick in the current setting with our adjoint damped connection $\widehat{\nabla}^\varepsilon$. However, the theorem we proved below has only the torsion of the Bott connection involved. In particular, the following theorem is proved in our original paper [12, Theorem 4.12] for general metric connection adapted to foliation structure and skey-symmetric (See [12, Assumption 1]). We actually found out this type of tangent process for the connection $\widehat{\nabla}^\varepsilon$ first, then realize later it is actually true for more general connections. Thus it does not depend on a particular connection we picked.

**Theorem 4.2.9.** *Let $\gamma \in W^\infty_{\mathcal{H}}(\mathbb{M})$. A path $v \in W^\infty(\mathbb{R}^{n+m})$ is tangent to the horizontal path $\gamma$ if and only if the path*

$$v(t) - \int_0^t T_{u_s}(Ad\omega^{\mathcal{H}}_s, Av(s))$$

*is horizontal, i.e. takes values in $\mathbb{R}^n$, where $\omega^{\mathcal{H}}$ is the horizontal anti-development of $\gamma$, $u$ its horizontal lift, and $T$ the torsion of the Bott connection.*

*Proof.* The path $v \in W^\infty(\mathbb{R}^{n+m})$ is tangent to the horizontal path $\gamma$ if and only if the path

$$p_v(\omega^{\mathcal{H}})_t = v(t) - \int_0^t T^{\widehat{\nabla}^\varepsilon}_{u_s}(Ad\omega^{\mathcal{H}}_s, Av(s) + Vv(s)) - \\ \int_0^t \left( \int_0^s \Omega^{\widehat{\nabla}^\varepsilon}_{u_\tau}(Ad\omega^{\mathcal{H}}_\tau, Av(\tau) + Vv(\tau)) \right) d\omega^{\mathcal{H}}_s.$$

is horizontal. Since $\widehat{\nabla}^\varepsilon$ is a horizontal metric connection, the integral

$$\int_0^t \left( \int_0^s \Omega^{\widehat{\nabla}^\varepsilon}_{u_\tau}(Ad\omega^{\mathcal{H}}_\tau, Av(\tau) + Vv(\tau)) \right) d\omega^{\mathcal{H}}_s$$

is always horizontal. Recall the torsion of the adjoint Baudoin connection $\widehat{\nabla}^\varepsilon$ in (3.4.5)

$$\widehat{T}^\varepsilon(X, Y) = T(X, Y) - \frac{1}{\varepsilon} J_Y X + \frac{1}{\varepsilon} J_X Y$$

Let us assume that $X$ is horizontal. We have then $J_X = 0$, because $\widehat{\nabla}^{\varepsilon}_{\mathcal{H}} = \nabla_{\mathcal{H}}$. Also $J_Y X$ is horizontal, because $\widehat{\nabla}^{\varepsilon}$ is adapted to the foliation $\mathcal{F}$. We deduce that the vertical part of

$$v(t) - \int_0^t T^{\widehat{\nabla}^{\varepsilon}}_{u_s}(Ad\omega^{\mathcal{H}}_s, Av(s) + Vv(s))$$

is the same as the vertical part of

$$v(t) - \int_0^t T_{u_s}(Ad\omega^{\mathcal{H}}_s, Av(s) + Vv(s)).$$

We conclude that the vertical part of $p_v(\omega^{\mathcal{H}})$ is zero if and only if the vertical part of

$$v(t) - \int_0^t T_{u_s}(Ad\omega^{\mathcal{H}}_s, Av(s) + Vv(s)).$$

is zero. According to the properties in (3.3.3) for torsion $T$, we have

$$\int_0^t T_{u_s}(Ad\omega^{\mathcal{H}}_s, Vv(s)) = 0.$$

This concludes the proof. □

### 4.2.3 Variations on the horizontal path space

In the following, we list two types of variations on the horizontal path space that are induced by tangent processes. The first one is simple, explicit and inspired by Driver [37]. The second one is more classically based on flow constructions. For details we refer to [12, Section 4.3].

**Lemma 4.2.10.** *Let $h \in W^\infty_{\mathcal{H}}(\mathbb{R}^{n+m})$. If $\omega^{\mathcal{H}} \in W^\infty_{\mathcal{H}}(\mathbb{R}^{n+m})$, then*

$$\tau_h(\omega^{\mathcal{H}})_t = h(t) + \int_0^t T_{u_s}(Ad\omega^{\mathcal{H}}_s, Ah(s)) \tag{4.2.3}$$

*is a tangent process to $\phi(\omega^{\mathcal{H}})$ where $u$, as before, denotes the lift of the horizontal development of $\omega^{\mathcal{H}}$.*

Let $v \in W^\infty(\mathbb{R}^{n+m})$, $\omega^{\mathcal{H}} \in W^\infty_{\mathcal{H}}(\mathbb{R}^{n+m})$ and assume that $v$ is tangent to the horizontal development of $\omega^{\mathcal{H}}$. Recall that

$$p_v(\omega^{\mathcal{H}})_t = v(t) - \int_0^t T^{\widehat{\nabla}^\varepsilon}_{u_s}(Ad\omega^{\mathcal{H}}_s, Av(s) + Vv(s)) - \\ \int_0^t \left( \int_0^s \Omega^{\widehat{\nabla}^\varepsilon}_{u_\tau}(Ad\omega^{\mathcal{H}}_\tau, Av(\tau) + Vv(\tau)) \right) d\omega^{\mathcal{H}}_s.$$

According to our definition (3.4.4): $\widehat{\nabla}^\varepsilon = \nabla + \frac{1}{\varepsilon}J$. We then have

$$p_v(\omega^{\mathcal{H}})_t = v_{\mathcal{H}}(t) + \int_0^t (\frac{1}{\varepsilon}J_{Vv(s)})_{u_s}(Ad\omega^{\mathcal{H}}_s) - \int_0^t \left( \int_0^s \Omega^{\widehat{\nabla}^\varepsilon}_{u_\tau}(Ad\omega^{\mathcal{H}}_\tau, Av(\tau) + Vv(\tau)) \right) d\omega^{\mathcal{H}}_s.$$

More concisely, we have therefore

$$p_v(\omega^{\mathcal{H}})_s = v_{\mathcal{H}}(s) + \int_0^s q_v(\omega^{\mathcal{H}})_u d\omega^{\mathcal{H}}_u,$$

where $q_v(\omega^{\mathcal{H}})_u \in \mathfrak{so}(n)$ is defined in such a way that

$$\int_0^s q_v(\omega^{\mathcal{H}})_u d\omega^{\mathcal{H}}_u \\ = \int_0^t (\frac{1}{\varepsilon}J_{Vv(s)})_{u_s}(Ad\omega^{\mathcal{H}}_s) - \int_0^t \left( \int_0^s \Omega^{\widehat{\nabla}^\varepsilon}_{u_\tau}(Ad\omega^{\mathcal{H}}_\tau, Av(\tau) + Vv(\tau)) \right) d\omega^{\mathcal{H}}_s.$$

As a consequence, with the above notations, one has therefore for every $h \in W^{\infty}_{\mathcal{H}}(\mathbb{R}^{n+m})$,

$$p_{\tau_h(\omega^{\mathcal{H}})}(\omega^{\mathcal{H}})_s = h(s) + \int_0^s q_{\tau_h(\omega^{\mathcal{H}})}(\omega^{\mathcal{H}})_u d\omega^{\mathcal{H}}_u,$$

We are now ready to introduce two relevant variations of horizontal paths.

**Definition 4.2.11.** Let $h \in W^{\infty}_{\mathcal{H}}(\mathbb{R}^{n+m})$.

1. For $t \in \mathbb{R}$, we define a map $\rho^h_t : W^{\infty}_{\mathcal{H}}(\mathbb{R}^{n+m}) \to W^{\infty}_{\mathcal{H}}(\mathbb{R}^{n+m})$ as follows

$$(\rho^h_t \omega^{\mathcal{H}})_s = \int_0^s e^{t q_{\tau_h(\omega^{\mathcal{H}})}(\omega^{\mathcal{H}})_u} d\omega^{\mathcal{H}}_u + th(s). \tag{4.2.4}$$

2. For $t \in \mathbb{R}$, we define a map $\nu^h_t : W^{\infty}_{\mathcal{H}}(\mathbb{R}^{n+m}) \to W^{\infty}_{\mathcal{H}}(\mathbb{R}^{n+m})$ as the flow generated by $p_{\tau_h}$:

$$\frac{d}{dt}(\nu^h_t \omega^{\mathcal{H}})_s = p_{\tau^h(\nu^h_t \omega^{\mathcal{H}})}(\nu^h_t \omega^{\mathcal{H}})_s, \quad \nu^h_0 \omega^{\mathcal{H}} = \omega^{\mathcal{H}}.$$

**Remark 4.2.12.** $\rho^h_t$ is **not** a flow on $W^{\infty}_{\mathcal{H}}(\mathbb{R}^{n+m})$, but is a convenient explicit one-parameter variation, since we observe that $\rho^h_0 \omega^{\mathcal{H}} = \omega^{\mathcal{H}}$ and

$$\frac{d}{dt} |_{t=0} (\rho^h_t \omega^{\mathcal{H}})_s = p_{\tau_h(\omega^{\mathcal{H}})}(\omega^{\mathcal{H}})_s.$$

We have then the following result, which is immediate in view of Theorem 4.2.6, since

$$\frac{d}{dt} |_{t=0} (\rho^h_t \omega^{\mathcal{H}})_s = \frac{d}{dt} |_{t=0} (\nu^h_t \omega^{\mathcal{H}})_s = p_{\tau_h(\omega^{\mathcal{H}})}(\omega^{\mathcal{H}})_s.$$

**Proposition 4.2.13** (Variation of horizontal paths along tangent processes)**.** *Let*

*$h \in W^\infty_{\mathcal{H}}(\mathbb{R}^{n+m})$. For every $\gamma \in W^\infty_{\mathcal{H}}(\mathbb{M})$,*

$$\frac{d}{dt}\,|_{t=0}\, \phi_{\mathcal{H}} \circ \rho^h_t \circ \phi^{-1}_{\mathcal{H}}(\gamma)_s = \frac{d}{dt}\,|_{t=0}\, \phi_{\mathcal{H}} \circ \nu^h_t \circ \phi^{-1}_{\mathcal{H}}(\gamma)_s = u_s \tau_h(\omega^{\mathcal{H}})_s,$$

*where $u$ is the lift of $\gamma$ and $\omega^{\mathcal{H}}$ its horizontal development.*

### 4.2.4 Horizontal semimartingales

We now extend our horizontal paths framework and horizontal Brownian motion to general horizontal semimartingales. Let $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space that satisfies the usual conditions.

**Definition 4.2.14.** A $\mathbb{R}^{n+m}$-valued $\mathcal{F}$ adapted continuous seminartingale $(W_t)_{t\geq 0}$ is called horizontal if $\forall t \geq 0$,

$$\mathbb{P}\left(W_t \in \mathbb{R}^n\right) = 1.$$

The space of horizontal semimartingales such that $W_0 = 0$ shall be denoted $SW_{\mathcal{H}}(\mathbb{R}^{n+m})$.

**Definition 4.2.15.** A $\mathbb{M}$-valued $\mathcal{F}$ adapted continuous seminartingale $(M_t)_{t\geq 0}$ is called horizontal if for every vertical smooth one-form $\theta$, and every $t \geq 0$ the Stratonovitch stochastic line integral $\int_{M[0,t]} \theta = 0$ almost surely. The space of of horizontal semimartingales such that $M_0 = x_0$ shall be denoted $SW_{\mathcal{H}}(\mathbb{M})$.

**Remark 4.2.16.** For the definition of Stratonovitch stochastic line integrals, see [74, Definition 2.4.1]

Now according to Lemma 4.2.5, we then have the following result for general horizontal semimartingales. In particular, we can see that the horizontal Brownian motion constructed in Corollary 4.1.3 is a special case of the following result. In addition, the stochastic parallel transport along the path of the horizontal Brownian motion and anti-development of the horizontal Brownian motion is also adapted to general horizontal semimartingales.

**Proposition 4.2.17.**

1. *Let* $(W_t)_{t\geq 0} \in SW_{\mathcal{H}}(\mathbb{R}^{n+m})$ *and let* $(U_t)_{t\geqslant 0}$ *be the solution to the Stratonovitch stochastic differential equation*

$$dU_t = \sum_{i=1}^{n} A_i(U_t) \circ dW_t^i, \quad U_0 = u_0,$$

*then* $M_t = \pi(U_t)$ *is a horizontal semimartingale on* $\mathbb{M}$. $M$ *will be called the stochastic horizontal development of* $W$ *and we will concisely write* $M = \phi_{\mathcal{H}}(W)$.

2. *Let* $(M_t)_{t\geq 0} \in SW_{\mathcal{H}}(\mathbb{M})$. *Then there exists a unique* $(W_t)_{t\geq 0} \in SW_{\mathcal{H}}(\mathbb{R}^{n+m})$ *, such that if* $(U_t)_{t\geqslant 0}$ *is the solution to the Stratonovitch stochastic differential equation*

$$dU_t = \sum_{i=1}^{n} A_i(U_t) \circ dW_t^i, \quad U_0 = u_0,$$

*then* $M_t = \pi(U_t)$ *. The path* $W$ *will be called the stochastic horizontal anti-development of* $M$ *and we will concisely write* $W = \phi_{\mathcal{H}}^{-1}(M)$.

As mentioned in our paper [12], we define the tangent processes to horizontal semimartingales using Theorem 4.2.9, since we stick in the horizontal setting.

**Definition 4.2.18.** ([12, Definition 4.22]) Let $(M_t)_{t\geq 0} \in SW_{\mathcal{H}}(\mathbb{M})$. A semimartingale

$S \in SW(\mathbb{R}^{n+m})$ is said to be tangent to $(M_t)_{t\geq 0}$ if the semimartingale

$$S_t - \int_0^t T_{U_s}(A \circ dW_s, AS(s))$$

is well defined (for the Stratonovitch integral part) and horizontal, i.e. takes values in $\mathbb{R}^n$. $W$ denotes here the stochastic horizontal anti-development of $M$ and $U$ its stochastic lift.

## 4.3 Malliavin calculus on the horizontal paths space

### 4.3.1 Damped parallel transport

The following damped parallel transport was first introduced in [10] in the sub-Riemannian setting, which was motivated from the work by Fang-Malliavin [46] to define gradients on the path space of Riemannian manifold. We define a *damped parallel transport* $\tau_t^\varepsilon : T^*_{W_t}\mathbb{M} \to T^*_x\mathbb{M}$ by the formula

$$\tau_t^\varepsilon = \mathcal{M}_t^\varepsilon \Theta_t^\varepsilon, \tag{4.3.1}$$

where the process $\Theta_t^\varepsilon : T^*_{W_t}\mathbb{M} \to T^*_x\mathbb{M}$ is the stochastic parallel transport of one-forms with respect to the Baudoin connection $\nabla^\varepsilon = \nabla - \mathfrak{T}^\varepsilon$ along the paths of $\{W_t\}_{0\leqslant t\leqslant 1}$. Since $\mathfrak{T}^\varepsilon$ is skew-symmetric, $\Theta_t^\varepsilon$ is an isometry for the Riemannian metric $g_\varepsilon$. The multiplicative functional $\mathcal{M}_t^\varepsilon : T^*_x\mathbb{M} \to T^*_x\mathbb{M}$, $t \geqslant 0$, is defined as the solution to the following ordinary differential equation

$$\frac{d\mathcal{M}_t^\varepsilon}{dt} = -\frac{1}{2}\mathcal{M}_t^\varepsilon \Theta_t^\varepsilon \left( \frac{1}{\varepsilon}\mathbf{J}^2 - \frac{1}{\varepsilon}\delta_{\mathcal{H}} T + \mathfrak{Ric}_{\mathcal{H}} \right) (\Theta_t^\varepsilon)^{-1}, \tag{4.3.2}$$
$$\mathcal{M}_0^\varepsilon = \mathbf{Id}.$$

It is easy to check by using the chain rules that $\tau_t^\varepsilon : T^*_{W_t}\mathbb{M} \to T^*_x\mathbb{M}$ is a solution of the following covariant Stratonovich stochastic differential equation

$$d[\tau_t^\varepsilon \alpha(W_t)] \tag{4.3.3}$$
$$= \tau_t^\varepsilon \left( \nabla_{\circ dW_t} - \mathfrak{T}^\varepsilon_{\circ dW_t} - \frac{1}{2}\left( \frac{1}{\varepsilon}\mathbf{J}^2 - \frac{1}{\varepsilon}\delta_{\mathcal{H}} T + \mathfrak{Ric}_{\mathcal{H}} \right) dt \right) \alpha(W_t),$$
$$\tau_0 = \mathbf{Id},$$

where $\alpha$ is any smooth one-form.

Also observe that $\mathcal{M}_t^\varepsilon$ is invertible and that its inverse is the solution of the following ordinary differential equation

$$\frac{d(\mathcal{M}_t^\varepsilon)^{-1}}{dt} = \frac{1}{2}\Theta_t^\varepsilon \left( \frac{1}{\varepsilon}\mathbf{J}^2 - \frac{1}{\varepsilon}\delta_{\mathcal{H}} T + \mathfrak{Ric}_{\mathcal{H}} \right) (\Theta_t^\varepsilon)^{-1}(\mathcal{M}_t^\varepsilon)^{-1}. \tag{4.3.4}$$

In particular, it implies that $\tau_t^\varepsilon$ is invertible.

### 4.3.2 Gradient representation of semigroup $P_t$

From now on and throughout this chapter, we will assume that for every horizontal one-form $\eta_1 \in \Gamma^\infty(\mathcal{H}^*)$ and every vertical one-from $\eta_2 \in \Gamma^\infty(\mathcal{V}^*)$,

$$\langle \mathfrak{Ric}_{\mathcal{H}} \eta_1, \eta_1 \rangle_{\mathcal{H}} \geq -K \|\eta_1\|_{\mathcal{H}}^2, \quad -\langle \mathbf{J}^2(\eta_1), \eta_1 \rangle_{\mathcal{H}} \leq \kappa \|\eta_1\|_{\mathcal{H}}^2, \quad -\langle \delta_{\mathcal{H}} T(\eta_2), \eta_2 \rangle_{\mathcal{V}} \geq -\beta \|\eta_2\|_{\mathcal{V}}^2, \tag{4.3.5}$$

with $K \geq 0, \kappa > 0$, $\beta > 0$. Recall that the semigroup $P_t$ is defined as the semigroup generated by the operator $\frac{1}{2}L$ where $L$ is the horizontal Laplacian. This is because that the completeness of the metric implies the operator $L$ is essentially self-adjoint on the space of smooth and compactly supported functions. Similarly, we can also define semigroup for the operator (3.4.2) $\Box_\varepsilon$

$$\Box_\varepsilon = -(\nabla_{g_{\mathcal{H}}} - \mathfrak{T}^\varepsilon_{g_{\mathcal{H}}})^*(\nabla_{g_{\mathcal{H}}} - \mathfrak{T}^\varepsilon_{g_{\mathcal{H}}}) - \frac{1}{\varepsilon}\mathbf{J}^2 + \frac{1}{\varepsilon}\delta_{\mathcal{H}} T - \mathfrak{Ric}_{g_{\mathcal{H}}}.$$

The property of essentially self-adjoint for $\Box_\varepsilon$ on smooth one forms was first proved under Yang-Mills condition ($\delta_{\mathcal{H}} T = 0$) for sub-Riemannian manifold with transverse symmetries in [10] and then for totally geodesic Riemannian foliations in [16]. Then the Yang-Mills condition is removed in [59] by using spectral theory. Thus we can define $Q_t^\varepsilon$ as the semigroup generated by $\frac{1}{2}\Box_\varepsilon$. The following result was first proved in [16, Lemma 4.1] under Yang-Mills condition. But, we will not need to assume the Yang-Mills condition anymore.

**Proposition 4.3.1.** *Let $\varepsilon > 0$. If $f \in C_0^\infty(\mathbb{M})$, then for every $t \geq 0$,*

$$dP_t f = Q_t^\varepsilon df.$$

**Lemma 4.3.2** (Theorem 4.6 and Corollary 4.7 in [10], Theorem 2.7 in [59]). *For $f \in C_0^\infty(\mathbb{M})$, the process*

$$N_s = \tau_s^\varepsilon (dP_{1-s} f)(W_s), \quad 0 \leqslant s \leqslant 1, \tag{4.3.6}$$

*is a martingale, where $dP_{1-s}f$ denotes here the exterior derivative of the function $P_{1-s}f$. As a consequence, for every $0 \leqslant t \leqslant 1$,*

$$dP_t f(x) = \mathbb{E}_x(\tau_t^\varepsilon df(W_t)). \tag{4.3.7}$$

*Proof.* See proof in [12, Lemma 7.13]. □

### 4.3.3 Damped, intrinsic and directional derivatives

We make the following convention: fix the time interval $I$ to be $[0,1]$ and denote $W = C_x([0,1], \mathbb{M})$ as the space of continuous functions on $\mathbb{M}$, or simply denoted as $C_x(\mathbb{M})$. For $\omega \in C_x(\mathbb{M})$, the coordinate functional is defined as $W_t(\omega) =: \omega(t),\ 0 \le t \le 1$. Thus we call $(w_t)_{0 \leqslant t \leqslant 1}$ the coordinate process on $C_x(\mathbb{M})$. Now, let us recall what is a cylinder function.

**Definition 4.3.3.** A function $F : C_x(\mathbb{M}) \to \mathbb{R}$ is called a $C^k$*-cylinder function* if there exists a partition $\pi := \{0 = t_0 < t_1 < t_2 < \cdots < t_n \leqslant 1\}$ of the interval $[0,1]$ and $f \in C^k(\mathbb{M})$ such that

$$F(w) = f(w_{t_1}, ..., w_{t_n}) \text{ for all } w \in C_x(\mathbb{M}). \tag{4.3.8}$$

The function $F$ is called a *smooth cylinder function* on $C_x(\mathbb{M})$ of $\mathbb{M}$, if there exists a partition $\pi$ and $f \in C^{\infty}(\mathbb{M})$ such that (4.3.8) holds.

We denote by $\mathcal{F}C^{k}\left(C_x(\mathbb{M})\right)$ the space of $C^k$-cylinder functions, and by $\mathcal{F}C^{\infty}\left(\mathbb{M})\right)$ the space of $C^{\infty}$-cylinder functions. For more discussions of such cylinder functions, see [12, Remark 7.3].

**Definition 4.3.4.** For $F = f\left(w_{t_1}, ..., w_{t_n}\right) \in \mathcal{F}C^{\infty}\left(C_x(\mathbb{M})\right)$ we define

- **Intrinsic gradient (Intrinsic derivative):**

$$D_t^{\varepsilon} F = \sum_{i=1}^{n} \mathbf{1}_{[0,t_i]}(t) \Theta_{t_i}^{\varepsilon} d_i f(W_{t_1}, \cdots, W_{t_n}), \quad 0 \le t \le 1$$

- **Damped gradient (Damped Malliavin derivative):**

$$\tilde{D}_t^{\varepsilon} F = \sum_{i=1}^{n} \mathbf{1}_{[0,t_i]}(t) (\tau_t^{\varepsilon})^{-1} \tau_{t_i}^{\varepsilon} d_i f(X_{t_1}, \cdots, X_{t_n}), \quad 0 \le t \le 1.$$

- **Directional Derivative** For an $\mathcal{F}$-adapted and $T_x\mathbb{M}$-valued semimartingale $(v(t))_{0 \leqslant t \leqslant 1}$ such that $v(0) = 0$, we define the directional derivative

$$\mathbf{D}_v F = \sum_{i=1}^{n} \left\langle d_i f(W_{t_1}, \cdots, W_{t_n}), \widehat{\Theta}_{t_i}^{\varepsilon} v(t_i) \right\rangle$$

Observe that from this definition $\widetilde{D}_t^{\varepsilon} F \in T_{W_t}^{*}\mathbb{M}$.

### 4.3.4 Clark-Ocone formulas

We begin with the definition of horizontal Cameron Martin space.

**Definition 4.3.5.** An $\mathcal{F}_t$-adapted absolutely continuous $\mathcal{H}_x$-valued process $(\gamma(t))_{0\leqslant t\leqslant 1}$ such that $\gamma(0)=0$ and $\mathbb{E}_x\left(\int_0^1 \|\gamma'(t)\|_{\mathcal{H}}^2 dt\right) < \infty$ will be called a *horizontal Cameron-Martin process.* The space of horizontal Cameron-Martin processes will be denoted by $\mathcal{CM}_{\mathcal{H}}(\mathbb{M},\Omega)$.

Then we layout the following propositions as direct consequences of the gradient representation (4.3.7) for one-dimensional cylinder function. These two propositions are first proved for sub-Riemannian manifolds with transverse symmetries and $\delta_{\mathcal{H}}T = 0$ in [10]. We reproduce the arguments in [12, Lemma 7.14] for totally geodesic Riemannian foliations without $\delta_{\mathcal{H}}T = 0$.

**Proposition 4.3.6.** *Let $\varepsilon > 0$ and $0 \le t \le 1$. For $f \in C^\infty(\mathbb{M})$, and $\gamma \in \mathcal{CM}_{\mathcal{H}}(\mathbb{M},\Omega)$*

$$\mathbb{E}_x\left(f(X_t)\int_0^t \langle \gamma'(s), dB_s\rangle_{\mathcal{H}}\right) = \mathbb{E}_x\left(\left\langle \tau_t^\varepsilon df(X_t), \int_0^t (\tau_s^{\varepsilon,*})^{-1}\widehat{\Theta}_s^\varepsilon \gamma'(s)ds\right\rangle\right).$$

**Proposition 4.3.7.** *Let $\varepsilon > 0$. For every $f \in C^\infty(\mathbb{M})$, and every $t > 0$,*

$$f(X_t) = P_t f(x) + \int_0^t \left\langle \mathbb{E}_x((\tau_s^\varepsilon)^{-1}\tau_t^\varepsilon df(X_t)|\mathcal{F}_s), \widehat{\Theta}_s^\varepsilon dB_s\right\rangle$$

Out goal is to prove a Clark-Ocone formula for multidimensional cylinder function similar to Proposition 4.3.7. We first generalize the gradient representation (4.3.7) to multi-dimensional case. Namely, we have

**Proposition 4.3.8.** *Let $F = f(W_{t_1},\cdots,W_{t_n}) \in \mathcal{F}C^\infty(C_x(\mathbb{M}))$. We have*

$$d\mathbb{E}_x(F) = \mathbb{E}_x\left(\sum_{i=1}^n \tau_{t_i}^\varepsilon d_i f(W_{t_1},\cdots,W_{t_n})\right).$$

*Proof.* The proof in our current setting can be found in [12, Proposition 7.16]. This gradient representation was first proved in [11, Proposition 3.3] with $\delta_{\mathcal{H}} T = 0$ in the sub-Riemannian setting and was first proved in [73] on Riemannian path space. $\square$

**Proposition 4.3.9 (Clark-Ocone formula).** *Let $\varepsilon > 0$. Let $F = f(W_{t_1}, \cdots, W_{t_n})$, $f \in C^\infty(\mathbb{M})$. Then*

$$F = \mathbb{E}_x(F) + \int_0^1 \langle \mathbb{E}_x(\tilde{D}_s^\varepsilon F | \mathcal{F}_s), \widehat{\Theta}_s^\varepsilon dB_s \rangle.$$

*Proof.* We will still proceed with our proof by induction. When $n = 1$, it is true from Proposition 4.3.7. Now let us proceed to the case $n$ by assuming that the formula is true for the case $n - 1$. We first represent the function $F = f(W_{t_1}, \cdots, W_{t_n})$ conditioned on starting from $W_{t_1}$ and use the case $n = 1$. We have

$$\begin{aligned} F &= \mathbb{E}_{W_{t_1}}(f(W_{t_1}, \cdots, W_{t_n})) \\ &\quad + \int_{t_1}^{t_n} \langle \mathbb{E}_x(\sum_{i=2}^n \mathbf{1}_{[t_1, t_i]}(t)(\tau_s^\varepsilon)^{-1} \tau_{t_i - t_1}^\varepsilon d_i f(W_{t_1}, \cdots, W_{t_n})) | \mathcal{F}_s), \widehat{\Theta}_s^\varepsilon dB_s \rangle. \end{aligned} \tag{4.3.9}$$

By using the Markov property, we can then write

$$\mathbb{E}_{W_{t_1}}(f(W_{t_1}, \cdots, W_{t_n})) = g(W_{t_1}), \text{ where } g(y) = \mathbb{E}_y(f(y, \cdots, W_{t_n - t_1}))$$

By using Proposition 4.3.8, we have

$$\begin{aligned} g(W_{t_1}) &= \mathbb{E}_x(f(W_{t_1}, \cdots, W_{t_{n-1}}, W_{t_n})) + \int_0^{t_1} \langle \mathbb{E}_x((\tau_s^\varepsilon)^{-1} \tau_{t_1}^\varepsilon dg(W_{t_1}) | \mathcal{F}_s), \widehat{\Theta}_s^\varepsilon dB_s \rangle. \\ &= \mathbb{E}_x(f(W_{t_1} \cdots, W_{t_{n-1}}, W_{t_n})) + \int_0^{t_1} \langle \mathbb{E}_x((\tau_s^\varepsilon)^{-1} \tau_{t_1}^\varepsilon d_1 f(W_{t_1} \cdots, W_{t_n}) | \mathcal{F}_s), \widehat{\Theta}_s^\varepsilon dB_s \rangle. \\ &\quad + \int_0^{t_1} \langle \mathbb{E}_x((\tau_s^\varepsilon)^{-1} \tau_{t_1}^\varepsilon \sum_{i=2}^n \mathbb{E}_{X_{t_1}}(\tau_{t_i - t_1}^\varepsilon d_i f(W_{t_1} \cdots, W_{t_n})) | \mathcal{F}_s), \widehat{\Theta}_s^\varepsilon dB_s \rangle. \end{aligned}$$

Thus by applying the multiplicative property of $\tau_t^\varepsilon$ and the Markov property, we obtain

$$\begin{aligned}\mathbb{E}_{W_{t_1}}(f(W_{t_1},\cdots,W_{t_n})) = \mathbb{E}_x(f(W_{t_1}\cdots,W_{t_{n-1}},W_{t_n}))\\ +\int_0^{t_1}\langle\mathbb{E}_x((\tau_s^\varepsilon)^{-1}\tau_{t_1}^\varepsilon d_1 f(W_{t_1}\cdots,W_{t_n})|\mathcal{F}_s),\widehat{\Theta}_s^\varepsilon dB_s\rangle.\\ +\int_0^{t_1}\langle\mathbb{E}_x(\sum_{i=2}^n(\tau_s^\varepsilon)^{-1}\tau_{t_i}^\varepsilon d_i f(W_{t_1}\cdots,W_{t_n})|\mathcal{F}_s),\widehat{\Theta}_s^\varepsilon dB_s\rangle.\end{aligned} \tag{4.3.10}$$

The proof is then completed by combining (4.3.9) and (4.3.10). □

### 4.3.5 Integration by parts formulas for damped derivatives

As an application of the previous Clark-Ocone formula, we get the the following integration by parts formula for the damped (Malliavin) gradient. The proof follows by induction on $n$.

**Theorem 4.3.10 (Integration by parts for the damped gradient).** *Suppose $F\in\mathcal{F}C^\infty(C_x(\mathbb{M}))$ and $\gamma\in\mathcal{CM}_\mathcal{H}(\mathbb{M},\Omega)$, then*

$$\mathbb{E}_x\left(\int_0^1\langle\widetilde{D}_s^\varepsilon F,\widehat{\Theta}_s^\varepsilon\gamma'(s)\rangle ds\right)=\mathbb{E}_x\left(F\int_0^1\langle\gamma'(s),dB_s\rangle_\mathcal{H}\right). \tag{4.3.11}$$

*Proof.* This result was first proved in [11] for stochastic parallel transport associated with Bott connection and $\delta_\mathcal{H}T=0$. The complete proof of this theorem is stated in [12, Theorem 7.11] for general adjoint connection compatible with foliation structure without assuming $\delta_\mathcal{H}T=0$. □

**Remark 4.3.11.** The above integration by parts (IBP) formula for the damped

Malliavin derivative is closely related to the integration by parts formula for the directional derivative (see Driver's IBP formula [36] in the Riemannian case).

### 4.3.6 Integration by parts formulas for directional derivatives

Comparing to the previous integration by parts formula for the damped gradient, we wonder if there is such an integration by parts formula for the directional derivative as well. Since this one is more directly related to the quasi-invariance of the horizontal Wiener measure. The integration by parts formula for the directional derivative is first proved in [36] and later in [72] in the direction of Cameron-Martin paths. However, in our current foliation setting, the directional derivative is in the direction of certain tangent process not a deterministic path. We first introduce this type of tangent process which we have defined before.

**Definition 4.3.12.** An $\mathcal{F}_t$-adapted $T_x\mathbb{M}$-valued continuous semimartingale $(v(t))_{0\leqslant t\leqslant 1}$ such that $v(0)=0$ and $\mathbb{E}_x\left(\int_0^1 \|v(t)\|^2 dt\right) < \infty$ will be called a *tangent process* if the process

$$v(t) - \int_0^t (\widehat{\Theta}_s^\varepsilon)^{-1} T(\widehat{\Theta}_s^\varepsilon \circ dB_s, \widehat{\Theta}_s^\varepsilon v(s))$$

is a horizontal Cameron-Martin process. The space of tangent processes will be denoted by $TW_{\mathcal{H}}(\mathbb{M},\Omega)$.

**Remark 4.3.13.** By Remark 4.1.4 the stochastic parallel transport $\widehat{\Theta}_s^\varepsilon$ is independent of $\varepsilon$, therefore the notion of a tangent process is itself independent of $\varepsilon$ as well.

**Remark 4.3.14.** As the torsion $T$ is a vertical tensor, then an $\mathcal{F}_t$-adapted and $T_x\mathbb{M}$-valued continuous semimartingale $\{v(t)\}_{0\leqslant t\leqslant 1}$ such that

$$\mathbb{E}_x\left(\int_0^1 \|v(t)\|^2 dt\right) < \infty, \quad v(0) = 0$$

is in $TW_{\mathcal{H}}(\mathbb{M})$ if and only if

1. The horizontal part $v_{\mathcal{H}} \in \mathcal{CM}_{\mathcal{H}}(\mathbb{M}, \Omega)$;

2. The vertical part $v_{\mathcal{V}}$ is given by

$$v_{\mathcal{V}}(t) = \int_0^t (\widehat{\Theta}_s^\varepsilon)^{-1} T(\widehat{\Theta}_s^\varepsilon \circ dB_s, \widehat{\Theta}_s^\varepsilon v_{\mathcal{H}}(s)).$$

We then have the following main result.

**Theorem 4.3.15 (Integration by parts for the directional derivatives).** *Suppose $F \in \mathcal{F}C^\infty\left(C_x(\mathbb{M})\right)$ and $v \in TW_{\mathcal{H}}(\mathbb{M}, \Omega)$, then*

$$\mathbb{E}_x\left(\mathbf{D}_v F\right) = \mathbb{E}_x\left(F \int_0^1 \left\langle v'_{\mathcal{H}}(t) + \frac{1}{2}(\widehat{\Theta}_t^\varepsilon)^{-1}\mathfrak{Ric}_{\mathcal{H}}\widehat{\Theta}_t^\varepsilon v_{\mathcal{H}}(t), dB_t\right\rangle_{\mathcal{H}}\right).$$

*Proof.* The complete proof we refer to [12, Theorem 7.12]. However, we list the key steps and the motivations here to give another presentation of the proof. The original motivation is to prove the integration by parts formula for the directional derivative using the integration by parts formula for the damped Malliavin derivative. The main idea is still to use induction on $n$, in particular we give the complete proof for the induction part which is not included in [12, Theorem 7.12]. The key steps are:

**Step 1**.

We prove the integration by parts formula for directional derivative in dimension

$n = 1$. Namely, For $v \in TW_{\mathcal{H}}(\mathbb{M}, \Omega)$ and $f \in C^{\infty}(\mathbb{M})$, we prove [12, Lemma 7.19]

$$\mathbb{E}_x \left( \left\langle df(W_1), \widehat{\Theta}_1^{\varepsilon} v(1) \right\rangle \right) = \mathbb{E}_x \left( f(W_1) \int_0^1 \left\langle v'_{\mathcal{H}}(t) + \frac{1}{2} (\widehat{\Theta}_t^{\varepsilon})^{-1} \mathfrak{Ric}_{\mathcal{H}} \widehat{\Theta}_t^{\varepsilon} v_{\mathcal{H}}(t), dB_t \right\rangle_{\mathcal{H}} \right).$$

In order to prove this Lemma, we use our integration by parts formula Proposition 4.3.6, where for $f \in C^{\infty}(\mathbb{M})$, and $h \in \mathcal{CM}_{\mathcal{H}}(\mathbb{M}, \Omega)$ we have

$$\mathbb{E}_x \left( \left\langle \tau_t^{\varepsilon} df(X_t), \int_0^t (\tau_s^{\varepsilon,*})^{-1} \widehat{\Theta}_s^{\varepsilon} h'(s) ds \right\rangle \right) = \mathbb{E}_x \left( f(X_t) \int_0^t \langle h'(s), dB_s \rangle_{\mathcal{H}} \right).$$

Then for the L.H.S, we set

$$v(t) = (\widehat{\Theta}_t^{\varepsilon})^{-1} \tau_t^{\varepsilon,*} \int_0^t (\tau_s^{\varepsilon,*})^{-1} \widehat{\Theta}_s^{\varepsilon} h'(s) ds,$$

then by Itô formula, we then can represent $h(t)$ in terms of $v(t)$, where we have

$$v(t) = (\widehat{\Theta}_t^{\varepsilon})^{-1} \tau_t^{\varepsilon,*} \int_0^t (\tau_s^{\varepsilon,*})^{-1} \widehat{\Theta}_s^{\varepsilon} \circ dM_t,$$

$$dM_t = dh(t) + \frac{1}{\varepsilon} (\widehat{\Theta}_t^{\varepsilon})^{-1} J_{\widehat{\Theta}_t^{\varepsilon} v(t)} \widehat{\Theta}_t^{\varepsilon} dB_t + \frac{1}{2} (\widehat{\Theta}_t^{\varepsilon})^{-1} \left( \mathfrak{Ric}_{\mathcal{H}} \right) \widehat{\Theta}_t^{\varepsilon} h(t) dt.$$

Then converting Stratonovich integral to Itô form. We will prove **Step 1** with the identity in **Step 2**.

**Step 2**.

In order to prove the above result for dimension $n = 1$, we need the following identity, [12, Lemma 7.18],

$$\mathbb{E}_x \left( \left\langle \tau_1^{\varepsilon} df(W_1), \int_0^1 (\tau_s^{\varepsilon,*})^{-1} \widehat{\Theta}_s^{\varepsilon} \left( \mathcal{O}_s dB_s - \frac{1}{2} T_{\mathcal{O}_s}^{\varepsilon} ds \right) \right\rangle \right) = 0,$$

where $(\mathcal{O}_s)_{0\leqslant s\leqslant 1}$ is a continuous and $\mathcal{F}$-adapted process taking values in the space of skew-symmetric endomorphisms of $\mathcal{H}_x$ such that $\mathbb{E}\left(\int_0^1 \|\mathcal{O}_s\|^2 ds\right) < +\infty$ with $\|\mathcal{O}_s\|^2 = \mathbf{Tr}(\mathcal{O}_s^*\mathcal{O}_s)$ . For $f \in C^\infty(\mathbb{M})$, we have where $T^\varepsilon_{\mathcal{O}_s}$ is the tensor given in a horizontal frame $e_1, \cdots, e_n$ by

$$T^\varepsilon_{\mathcal{O}_s} = \sum_{i=1}^n (\widehat{\Theta}^\varepsilon_s)^{-1} T^\varepsilon(e_i, \widehat{\Theta}^\varepsilon_s \mathcal{O}_s (\widehat{\Theta}^\varepsilon_s)^{-1} e_i).$$

**Step 3**. Induction on $n$.

The case $n = 1$ is in **Step 1**. Let $F = f(W_{t_1}, \cdots, W_{t_n})$, with $n \geqslant 2$ and assume that Theorem 4.3.15 holds for $n-1$. We have

$$\begin{aligned}
&\mathbb{E}_x\left(F\int_0^T \left\langle v'_{\mathcal{H}}(t) + \frac{1}{2}(\hat{\Theta}^\varepsilon_t)^{-1}\mathfrak{Ric}_{\mathcal{H}}\hat{\Theta}^\varepsilon_t v_{\mathcal{H}}(t), dB_t\right\rangle_{\mathcal{H}}\right)\\
=&\mathbb{E}_x\left(F\int_0^{t_n} \left\langle v'_{\mathcal{H}}(t) + \frac{1}{2}(\hat{\Theta}^\varepsilon_t)^{-1}\mathfrak{Ric}_{\mathcal{H}}\hat{\Theta}^\varepsilon_t v_{\mathcal{H}}(t), dB_t\right\rangle_{\mathcal{H}}\right)\\
=&\mathbb{E}_x\left(F\int_0^{t_1} \left\langle v'_{\mathcal{H}}(t) + \frac{1}{2}(\hat{\Theta}^\varepsilon_t)^{-1}\mathfrak{Ric}_{\mathcal{H}}\hat{\Theta}^\varepsilon_t v_{\mathcal{H}}(t), dB_t\right\rangle_{\mathcal{H}}\right)\\
&+\mathbb{E}_x\left(F\int_{t_1}^{t_n} \left\langle v'_{\mathcal{H}}(t) + \frac{1}{2}(\hat{\Theta}^\varepsilon_t)^{-1}\mathfrak{Ric}_{\mathcal{H}}\hat{\Theta}^\varepsilon_t v_{\mathcal{H}}(t), dB_t\right\rangle_{\mathcal{H}}\right)\\
=&\mathbb{E}_x\left(\mathbb{E}_{X_{t_1}}(F)\int_0^{t_1} \left\langle v'_{\mathcal{H}}(t) + \frac{1}{2}(\hat{\Theta}^\varepsilon_t)^{-1}\mathfrak{Ric}_{\mathcal{H}}\hat{\Theta}^\varepsilon_t v_{\mathcal{H}}(t), dB_t\right\rangle_{\mathcal{H}}\right)\\
&+\mathbb{E}_x\left(\mathbb{E}\left(F\int_{t_1}^{t_n} \left\langle v'_{\mathcal{H}}(t) + \frac{1}{2}(\hat{\Theta}^\varepsilon_t)^{-1}\mathfrak{Ric}_{\mathcal{H}}\hat{\Theta}^\varepsilon_t v_{\mathcal{H}}(t), dB_t\right\rangle_{\mathcal{H}} |\mathcal{F}_{t_1}\right)\right).
\end{aligned}$$

By the Markov property we have

$$\mathbb{E}_{W_{t_1}}(F) = g(W_{t_1}), \text{ where } g(y) = \mathbb{E}_y(f(y, W_{t_2-t_1}, \cdots, W_{t_n-t_1})).$$

Thus by **Step 1**

$$\mathbb{E}_x\left(\mathbb{E}_{X_{t_1}}(F)\int_0^{t_1}\left\langle v'_{\mathcal{H}}(t)+\frac{1}{2}(\hat{\Theta}_t^\varepsilon)^{-1}\mathfrak{Ric}_{\mathcal{H}}\hat{\Theta}_t^\varepsilon v_{\mathcal{H}}(t),dB_t\right\rangle_{\mathcal{H}}\right)=\mathbb{E}_x\left(\left\langle dg(X_{t_1}),\hat{\Theta}_{t_1}^\varepsilon v(t_1)\right\rangle\right).$$

From **Step 1** (see details [12, Lemma 7.19]), we also have

$$v(t)=(\hat{\Theta}_t^\varepsilon)^{-1}\tau_t^{\varepsilon,*}\int_0^t(\tau_s^{\varepsilon,*})^{-1}\hat{\Theta}_s^\varepsilon\circ dM_t,$$

namely

$$\hat{\Theta}_{t_1}^\varepsilon v(t_1)=\tau_{t_1}^{\varepsilon,*}\int_0^{t_1}(\tau_s^{\varepsilon,*})^{-1}\hat{\Theta}_s^\varepsilon\circ dM_t$$

Now according to Lemma 4.3.8

$$dg(y)=\mathbb{E}_y\left(\sum_{i=1}^n\tau_{t_i-t_1}^\varepsilon d_if(y,W_{t_2-t_1},\cdots,W_{t_n-t_1})\right).$$

Using now the fact that

$$\begin{aligned}&\mathbb{E}_{W_{t_1}}\left(\tau_{t_i-t_1}^\varepsilon d_if(y,W_{t_2-t_1},\cdots,W_{t_n-t_1})\right)\\&=(\tau_{t_1}^\varepsilon)^{-1}\mathbb{E}\left(\tau_{t_i}^\varepsilon d_if(W_{t_1},\cdots,W_{t_n})\mid\mathcal{F}_{t_1}\right),\end{aligned}$$

we conclude

$$\begin{aligned}&\mathbb{E}_x\left(\mathbb{E}_{X_{t_1}}(F)\int_0^{t_1}\left\langle v'_{\mathcal{H}}(t)+\frac{1}{2}(\hat{\Theta}_t^\varepsilon)^{-1}\mathfrak{Ric}_{\mathcal{H}}\hat{\Theta}_t^\varepsilon v_{\mathcal{H}}(t),dB_t\right\rangle_{\mathcal{H}}\right)\\&=\mathbb{E}_x\left(\sum_{i=1}^n\langle d_if(W_{t_1},\cdots,W_{t_n}),\tau_{t_i}^{\varepsilon,*}\int_0^{t_1}(\tau_s^{\varepsilon,*})^{-1}\hat{\Theta}_s^\varepsilon\circ dM_t\rangle\right).\end{aligned}$$

Using now the induction hypothesis for $n-1$, we have

$$\begin{aligned}
&\mathbb{E}\left(F\int_{t_1}^{t_n}\left\langle v'_{\mathcal{H}}(t)+\frac{1}{2}(\hat{\Theta}_t^{\varepsilon})^{-1}\mathfrak{Ric}_{\mathcal{H}}\hat{\Theta}_t^{\varepsilon}v_{\mathcal{H}}(t),dB_t\right\rangle_{\mathcal{H}}|\mathcal{F}_{t_1}\right)\\
&=\mathbb{E}_{W_{t_1}}\left(\sum_{i=1}^{n}\langle d_if(W_{t_1},\cdots,W_{t_n}),\hat{\Theta}_{t_i}^{\varepsilon}v(t_i)\rangle\right)\\
&=\mathbb{E}_{W_{t_1}}\left(\sum_{i=1}^{n}\langle d_if(W_{t_1},\cdots,W_{t_n}),\tau_{t_i}^{\varepsilon,*}\int_{t_1}^{t_i}(\tau_s^{\varepsilon,*})^{-1}\hat{\Theta}_s^{\varepsilon}\circ dM_t\rangle\right).
\end{aligned}$$

combining the above two cases we thus have

$$\begin{aligned}
&\mathbb{E}_x\left(F\int_0^T\left\langle v'_{\mathcal{H}}(t)+\frac{1}{2}(\hat{\Theta}_t^{\varepsilon})^{-1}\mathfrak{Ric}_{\mathcal{H}}\hat{\Theta}_t^{\varepsilon}v_{\mathcal{H}}(t),dB_t\right\rangle_{\mathcal{H}}\right)\\
&=\mathbb{E}_x\left(\sum_{i=1}^{n}\langle d_if(W_{t_1},\cdots,W_{t_n}),\tau_{t_i}^{\varepsilon,*}\int_0^{t_i}(\tau_s^{\varepsilon,*})^{-1}\hat{\Theta}_s^{\varepsilon}\circ dM_t\rangle\right).\\
&=\mathbb{E}_x\left(\sum_{i=1}^{n}\langle d_if(W_{t_1},\cdots,W_{t_n}),\hat{\Theta}_{t_i}^{\varepsilon}v(t_i)\rangle\right)=\mathbf{D}_vF
\end{aligned}$$

This completes the proof of Theorem 4.3.15. □

We then have the following Corollary.

**Corollary 4.3.16.** *([12, Corollary 7.20]) Let $F,G\in\mathcal{F}C^{\infty}\left(C_x(\mathbb{M})\right)$ and $v\in TW_{\mathcal{H}}(\mathbb{M})$. We have*

$$\mathbb{E}_x(F\mathbf{D}_vG)=\mathbb{E}_x(G\mathbf{D}_v^*F),$$

*where*

$$\mathbf{D}_v^*=-\mathbf{D}_v+\int_0^1\left\langle v'_{\mathcal{H}}(t)+\frac{1}{2}(\widehat{\Theta}_t^{\varepsilon})^{-1}\mathfrak{Ric}_{\mathcal{H}}\widehat{\Theta}_t^{\varepsilon}v_{\mathcal{H}}(t),dB_t\right\rangle_{\mathcal{H}}.$$

## 4.4 Functional inequalities

The following functional inequalities are proved in [11] under the assumption $\delta_{\mathcal{H}} T = 0$. We will present the results here in a more general setting without assuming $\delta_{\mathcal{H}} T = 0$ but instead assuming that it is bounded. We will reproduce some arguments under our new assumptions if necessary, otherwise we refer to [11].

### 4.4.1 Log-Sobolev inequalities

In this part, we present a family of log-Sobolev inequalities on the horizontal path space of $\mathbb{M}$ using the method introduced by Hsu [73] (see also [24]). We make the following assumption that for every horizontal one-form $\eta_1$ and every vertical one-form $\eta_2$,

$$|\langle \mathfrak{Ric}_{\mathcal{H}} \eta_1, \eta_1 \rangle_{\mathcal{H}}| \le K \|\eta_1\|_{\mathcal{H}}^2, \quad |\langle \mathbf{J}^2 \eta_1, \eta_1 \rangle_{\mathcal{H}}| \le \kappa \|\eta_1\|_{\mathcal{H}}^2, \quad |\langle \delta_{\mathcal{H}} T(\eta_2), \eta_2 \rangle_{\mathcal{V}}| \le \beta \|\eta_2\|_{\mathcal{V}}^2 \tag{4.4.1}$$

**Theorem 4.4.1.** *For every cylindric function $G \in \mathcal{F}C^\infty(C_x(\mathbb{M}))$, we have the following log-Sobolev inequality.*

$$\mathbb{E}_x(G^2 \ln G^2) - \mathbb{E}_x(G^2) \ln \mathbb{E}_x(G^2) \le 2 e^{3T(K + \frac{\beta}{\varepsilon} + \frac{\kappa}{\varepsilon})} \mathbb{E}_x \left( \int_0^T \|D_s^\varepsilon G\|_\varepsilon^2 ds \right).$$

*Proof.* The proof follows by proving the following lemmas. □

**Lemma 4.4.2.** *We have the following inequality*

$$\mathbb{E}_x(G^2 \ln G^2) - \mathbb{E}_x(G^2) \ln \mathbb{E}_x(G^2) \le 2 \mathbb{E}_x \left( \int_0^T \|\tilde{D}_s^\varepsilon G\|_\varepsilon^2 ds \right)$$

*Proof.* Let us consider the martingale $N_s = \mathbb{E}(G^2|\mathcal{F}_s)$. Applying Itô's formula to $N_s \ln N_s$ and taking expectation yields

$$\mathbb{E}_x(N_t \ln N_t) - \mathbb{E}_x(N_0 \ln N_0) = \frac{1}{2}\mathbb{E}_x\left(\int_0^t \frac{d[N]_s}{N_s}\right),$$

where $[N]$ is the quadratic variation of $N$. From Proposition 4.3.9, we have

$$dN_s = 2\left\langle \mathbb{E}\left(G\tilde{D}_s^\varepsilon G \mid \mathcal{F}_s\right), \widehat{\Theta}_s^\varepsilon dB_s\right\rangle.$$

Thus we have from Cauchy-Schwarz inequality

$$\begin{aligned}\mathbb{E}_x(N_t \ln N_t) - \mathbb{E}_x(N_0 \ln N_0) &\leq 2\mathbb{E}_x\left(\int_0^t \frac{\|\mathbb{E}\left(G\tilde{D}_s^\varepsilon G \mid \mathcal{F}_s\right)\|_\varepsilon^2}{N_s} ds\right) \\ &\leq 2\mathbb{E}_x\left(\int_0^T \|\tilde{D}_s^\varepsilon G\|_\varepsilon^2 ds\right).\end{aligned}$$

□

**Lemma 4.4.3.** *With $G = f(W_{t_1}, \cdots, W_{t_n})$ we have the following inequality*

$$\mathbb{E}_x\left(\int_0^T \|\tilde{D}_s^\varepsilon G\|_\varepsilon^2 ds\right) \leq e^{T(K+\frac{\beta}{\varepsilon}+\frac{\kappa}{\varepsilon})} \sum_{l=1}^n \frac{t_l - t_{l-1}}{T} \| \sum_{i=l}^n (\tau_{t_l}^\varepsilon)^{-1} \tau_{t_i}^\varepsilon d_i f(W_{t_1}, \cdots, W_{t_n})\|_\varepsilon^2$$

*Proof.* From definition 4.3.4, we have

$$\begin{aligned}\tilde{D}_s^\varepsilon G &= \sum_{i=1}^n \sum_{l=1}^i \mathbf{1}_{[t_{l-1}, t_l]}(s)(\tau_s^\varepsilon)^{-1}\tau_{t_i}^\varepsilon d_i f(W_{t_1}, \cdots, W_{t_n}) \\ &= \sum_{l=1}^n \sum_{i=l}^n \mathbf{1}_{[t_{l-1}, t_l]}(s)(\tau_s^\varepsilon)^{-1}\tau_{t_i}^\varepsilon d_i f(W_{t_1}, \cdots, W_{t_n}).\end{aligned} \tag{4.4.2}$$

Thus we have

$$\mathbb{E}_x\left(\int_0^T \|\tilde{D}_s^\varepsilon G\|_\varepsilon^2 ds\right) = \sum_{l=1}^n \int_{t_{l-1}}^{t_l} \|\sum_{i=l}^n (\tau_s^\varepsilon)^{-1}\tau_{t_l}^\varepsilon(\tau_{t_l}^\varepsilon)^{-1}\tau_{t_i}^\varepsilon d_i f(W_{t_1},\cdots,W_{t_n})\|_\varepsilon^2 ds$$

By our definition of $\tau_t^\varepsilon$ and the assumption 4.4.1, we have

$$\begin{aligned}&\|(\tau_s^\varepsilon)^{-1}\tau_{t_l}^\varepsilon\sum_{i=l}^n(\tau_{t_l}^\varepsilon)^{-1}\tau_{t_i}^\varepsilon d_i f(W_{t_1},\cdots,W_{t_n})\|_\varepsilon^2\\ &\leq e^{(K+\frac{\beta}{\varepsilon}+\frac{\kappa}{\varepsilon})(t_l-s)}\|\sum_{i=l}^n(\tau_{t_l}^\varepsilon)^{-1}\tau_{t_i}^\varepsilon d_i f(W_{t_1},\cdots,W_{t_n})\|_\varepsilon^2.\end{aligned} \tag{4.4.3}$$

Hence

$$\mathbb{E}_x\left(\int_0^T \|\tilde{D}_s^\varepsilon G\|_\varepsilon^2 ds\right) \leq \sum_{l=1}^n \int_{t_{l-1}}^{t_l} e^{(K+\frac{\beta}{\varepsilon}+\frac{\kappa}{\varepsilon})(t_l-s)}ds\|\sum_{i=l}^n(\tau_{t_l}^\varepsilon)^{-1}\tau_{t_i}^\varepsilon d_i f(W_{t_1},\cdots,W_{t_n})\|_\varepsilon^2.$$

We now use the elementary inequality $\frac{e^{sc}-1}{c} \leq se^c$ to get :

$$\mathbb{E}_x\left(\int_0^T \|\tilde{D}_s^\varepsilon G\|_\varepsilon^2 ds\right) \leq e^{T(K+\frac{\beta}{\varepsilon}+\frac{\kappa}{\varepsilon})}\sum_{l=1}^n \frac{t_l-t_{l-1}}{T}\|\sum_{i=l}^n(\tau_{t_l}^\varepsilon)^{-1}\tau_{t_i}^\varepsilon d_i f(W_{t_1},\cdots,W_{t_n})\|_\varepsilon^2.$$

□

We conclude the proof by the following bound.

**Lemma 4.4.4.** *With $G = f(W_{t_1},\cdots,W_{t_n})$, we have*

$$\sum_{l=1}^n \frac{t_l-t_{l-1}}{T}\|\sum_{i=l}^n(\tau_{t_l}^\varepsilon)^{-1}\tau_{t_i}^\varepsilon d_i f(W_{t_1},\cdots,W_{t_n})\|_\varepsilon^2 \leq e^{2T(K+\frac{\beta}{\varepsilon}+\frac{\kappa}{\varepsilon})}\mathbb{E}_x\left(\int_0^T \|D_s^\varepsilon G\|_\varepsilon^2 ds\right)$$

*Proof.* From Definition 4.3.4, we have

$$D_t^\varepsilon F = \sum_{i=1}^n \mathbf{1}_{0,t_i}(t)\Theta_{t_i}^\varepsilon d_i f(W_{t_1},\cdots,W_{t_n})$$

Also recall from equation (4.3.1) that $\tau_t^\varepsilon = \mathcal{M}_t^\varepsilon \Theta_t^\varepsilon$. Let us denote $z_l = \sum_{i=l}^n \Theta_{t_i}^\varepsilon d_i f(W_{t_1},\cdots,W_{t_n})$. we then have

$$\begin{aligned}
&\|\sum_{i=l}^n (\tau_{t_l}^\varepsilon)^{-1}\tau_{t_i}^\varepsilon d_i f(W_{t_1},\cdots,W_{t_n})\|_\varepsilon^2 \\
=&\|(\Theta_{t_l}^\varepsilon)^{-1}\sum_{i=l}^n (\mathcal{M}_{t_l}^\varepsilon)^{-1}\mathcal{M}_{t_i}^\varepsilon \Theta_{t_i}^\varepsilon f(W_{t_1},\cdots,W_{t_n})\|_\varepsilon^2 \\
=&\left\|(\Theta_{t_l}^\varepsilon)^{-1}\left(z_l + \sum_{i=l+1}^n [(\mathcal{M}_{t_l}^\varepsilon)^{-1}\mathcal{M}_{t_i}^\varepsilon - (\mathcal{M}_{t_l}^\varepsilon)^{-1}\mathcal{M}_{t_{i-1}}^\varepsilon] z_i\right)\right\|_\varepsilon^2 \\
=&\left\| z_l + \sum_{i=l+1}^n [(\mathcal{M}_{t_l}^\varepsilon)^{-1}\mathcal{M}_{t_i}^\varepsilon - (\mathcal{M}_{t_l}^\varepsilon)^{-1}\mathcal{M}_{t_{i-1}}^\varepsilon] z_i\right\|_\varepsilon^2
\end{aligned}$$

From equation (4.3.2) and (4.4.1) we have

$$\|[(\mathcal{M}_{t_l}^\varepsilon)^{-1}\mathcal{M}_{t_i}^\varepsilon - (\mathcal{M}_{t_l}^\varepsilon)^{-1}\mathcal{M}_{t_{i-1}}^\varepsilon] z_i\|_\varepsilon^2 \le \left(\frac{K+\frac{\beta}{\varepsilon}+\frac{\kappa}{\varepsilon}}{2}\int_{t_{i-1}}^{t_i} e^{\frac{1}{2}(K+\beta+\frac{\kappa}{\varepsilon})(s-t_l)}ds\right)^2 \|z_i\|_\varepsilon^2$$

Thus by Cauchy-Schwarz inequality, with $c = (K+\frac{\beta}{\varepsilon}+\frac{\kappa}{\varepsilon})$ and $\lambda = \frac{c}{2}e^{\frac{c}{2}}$

$$\|\sum_{i=l}^n (\tau_{t_l}^\varepsilon)^{-1}\tau_{t_i}^\varepsilon d_i f(W_{t_1},\cdots,W_{t_n})\|_\varepsilon^2 \le (1+\lambda)\|z_l\|_\varepsilon^2 + (1+\frac{1}{\lambda})\frac{c^2}{4}\|\int_{t_l}^T e^{\frac{1}{2}c(s-t_l)}g_s ds\|_\varepsilon^2$$

where $g_s = \|z_l\|_\varepsilon$ for $s \in [t_{l-1}, t_l)$. We easily deduce from that (see the argument in

Lemma 4.3 in [73] for more details),

$$\sum_{l=1}^{n}\frac{t_l-t_{l-1}}{T}\|\sum_{i=l}^{n}(\tau_{t_l}^{\varepsilon})^{-1}\tau_{t_i}^{\varepsilon}d_i f(W_{t_1},\cdots,W_{t_n})\|_{\varepsilon}^2 \le e^{2T(K+\frac{\beta}{\varepsilon}+\frac{\kappa}{\varepsilon})}\int_0^T g_s^2 ds.$$

We then complete the proof by observing that

$$\int_0^T g_s^2 ds = \mathbb{E}_x\left(\int_0^T \|D_s^{\varepsilon}G\|_{\varepsilon}^2 ds\right).$$

□

### 4.4.2 Improved Log-Sobolev inequalities

In this part, we give the previous Log-Sobolev inequality an improved upper bound. This improved upper bound was first proved in [97] and then generalized to have the following form in [47]. We begin with the following lemma.

**Lemma 4.4.5.** *According to our definition* (4.3.4)*, we have the following relation*

$$\tilde{D}_t^{\varepsilon}F = \Theta_t^{\varepsilon,-1}D_tF + \Theta_t^{\varepsilon,-1}\int_t^T -\frac{1}{2}\mathcal{M}_{s,t}^{\varepsilon}\Theta_{s,t}^{\varepsilon}\left(\frac{1}{\varepsilon}\mathbf{J}^2 - \frac{1}{\varepsilon}\delta_{\mathcal{H}}T + \mathfrak{Ric}_{\mathcal{H}}\right)(\Theta_{s,t}^{\varepsilon})^{-1}D_sF ds \tag{4.4.4}$$

*Proof.*

$$
\begin{aligned}
&\mathbb{E}_x(\int_0^T < \tilde{D}_t^\varepsilon F, \widehat{\Theta}_t^\varepsilon \gamma'(t) > dt) \\
&= \mathbb{E}_x(\int_0^T < \sum_{i=1}^n \mathbf{1}_{[0,t_i]}(t)(\tau_t^\varepsilon)^{-1}\tau_{t_i}^\varepsilon d_i F, \widehat{\Theta}_t^\varepsilon \gamma'(t) > dt) \\
&= \mathbb{E}_x(\int_0^T < \sum_{i=1}^n \mathbf{1}_{[0,t_i]}(t)\mathcal{M}_{t,t_j}^\varepsilon \Theta_{t_i}^\varepsilon d_i F, (\Theta_t^{\varepsilon,-1})^* \widehat{\Theta}_t^\varepsilon \gamma'(t) > dt) \\
&= \mathbb{E}_x(\int_0^T < \sum_{i=1}^n \mathbf{1}_{[0,t_i]}(t)(I + \int_0^{t_i - t} d\mathcal{M}_{t,s}^\varepsilon / ds)\Theta_{t_i}^\varepsilon d_i F, (\Theta_t^{\varepsilon,-1})^* \widehat{\Theta}_t^\varepsilon \gamma'(t) > dt) \\
&= \mathbb{E}_x(\int_0^T < \Theta_t^{\varepsilon,-1} D_t F, \widehat{\Theta}_t^\varepsilon \gamma'(t) > dt) \\
&+ \mathbb{E}_x(\int_0^T < \int_t^T \frac{d\mathcal{M}_{s,t}^\varepsilon}{ds} D_s F ds, (\Theta_t^{\varepsilon,-1})^* \widehat{\Theta}_t^\varepsilon \gamma'(t) > dt)
\end{aligned}
\tag{4.4.5}
$$

which finishes the proof. □

We are in a good position to prove the following result.

**Theorem 4.4.6.** *Let $0 < t < T$, Set*

$$\Lambda(t,T) \quad = 1 + \frac{K_1}{K_2}(1 - e^{-\frac{K_2(T-t)}{2}}) + \frac{K_1}{K_2}(1 - e^{-\frac{K_2 t}{2}}) \tag{4.4.6}$$

$$+(\frac{K_1}{K_2})^2[(1 - e^{-\frac{K_2 t}{2}}) + \frac{1}{2}(e^{-\frac{K_2(T+t)}{2}} - e^{-\frac{K_2(T-t)}{2}})] \tag{4.4.7}$$

*Then we have*

$$\int_0^T |\tilde{D}_t^\varepsilon F|_\varepsilon^2 dt \le \int_0^T \Lambda(t,T)|D_t F|_\varepsilon^2 dt$$

*At the same time, all the estimates and analysis for function $\Lambda(t,T)$ works for our case which of course obtain the better estimate. Where $K_2 = -K - \frac{\beta}{\varepsilon} - \frac{\kappa}{\varepsilon}$ and $|\langle \mathfrak{Ric}_\mathcal{H} \eta_1, \eta_1 \rangle_\mathcal{H} - \frac{1}{\varepsilon}\langle \delta_\mathcal{H} T(\eta_2), \eta_2 \rangle + \frac{1}{\varepsilon}\langle \mathbf{J}^2(\eta_1), \eta_1 \rangle_\mathcal{H}| \le K_1$.*

*Proof.* According to the relation in Lemma 4.4.5, we have

$$\begin{aligned}
|\tilde{D}_t^\varepsilon F|_\varepsilon^2 =& \langle \Theta_t^{\varepsilon,-1} D_t F, \Theta_t^{\varepsilon,-1} D_t F\rangle_\varepsilon \\
& - \langle \Theta_t^{\varepsilon,-1} D_t F, \Theta_t^{\varepsilon,-1} \int_t^T \mathcal{M}_{s,t}^\varepsilon \Theta_{s,t}^\varepsilon \left(\frac{1}{\varepsilon}\mathbf{J}^2 - \frac{1}{\varepsilon}\delta_{\mathcal{H}} T + \mathfrak{Ric}_{\mathcal{H}}\right) (\Theta_{s,t}^\varepsilon)^{-1} D_s F ds\rangle_\varepsilon \\
& + \frac{1}{4} \langle \Theta_t^{\varepsilon,-1} \int_t^T \mathcal{M}_{s,t}^\varepsilon \Theta_{s,t}^\varepsilon \left(\frac{1}{\varepsilon}\mathbf{J}^2 - \frac{1}{\varepsilon}\delta_{\mathcal{H}} T + \mathfrak{Ric}_{\mathcal{H}}\right) (\Theta_{s,t}^\varepsilon)^{-1} D_s F ds, \\
& \Theta_t^{\varepsilon,-1} \int_t^T \mathcal{M}_{s,t}^\varepsilon \Theta_{s,t}^\varepsilon \left(\frac{1}{\varepsilon}\mathbf{J}^2 - \frac{1}{\varepsilon}\delta_{\mathcal{H}} T + \mathfrak{Ric}_{\mathcal{H}}\right) (\Theta_{s,t}^\varepsilon)^{-1} D_s F ds\rangle_\varepsilon
\end{aligned} \tag{4.4.8}$$

Since $\Theta_t$ is $\langle,\rangle_\varepsilon$ isometry, so the above equality gives us

$$\begin{aligned}
|\tilde{D}_t^\varepsilon F|_\varepsilon^2 \quad & = |D_t F|_\varepsilon^2 - < D_t F, \int_t^T \mathcal{M}_{s,t}^\varepsilon \Theta_{s,t}^\varepsilon \left(\frac{1}{\varepsilon}\mathbf{J}^2 - \frac{1}{\varepsilon}\delta_{\mathcal{H}} T + \mathfrak{Ric}_{\mathcal{H}}\right) (\Theta_{s,t}^\varepsilon)^{-1} D_s F ds >_\varepsilon \\
& + \frac{1}{4} | \int_t^T \mathcal{M}_{s,t}^\varepsilon \Theta_{s,t}^\varepsilon \left(\frac{1}{\varepsilon}\mathbf{J}^2 - \frac{1}{\varepsilon}\delta_{\mathcal{H}} T + \mathfrak{Ric}_{\mathcal{H}}\right) (\Theta_{s,t}^\varepsilon)^{-1} D_s F ds|_\varepsilon^2 \\
& := Q_1 + Q_2 + Q_3
\end{aligned}$$

Recall that $||\tau_t^\varepsilon||_\varepsilon \le e^{\frac{1}{2}(K + \frac{\beta}{\varepsilon} + \frac{\kappa}{\varepsilon})t}$, namely

$$||\mathcal{M}_{s,t}^\varepsilon||_\varepsilon \le e^{\frac{1}{2}(K + \frac{\beta}{\varepsilon} + \frac{\kappa}{\varepsilon})(s-t)}$$

$$\langle \mathfrak{Ric}_{\mathcal{H}} \eta_1, \eta_1 \rangle_{\mathcal{H}} \ge -K \|\eta_1\|_{\mathcal{H}}^2, \quad -\langle \mathbf{J}^2(\eta_1), \eta_1\rangle_{\mathcal{H}} \le \kappa \|\eta_1\|_{\mathcal{H}}^2, \quad -\langle \delta_{\mathcal{H}} T(\eta_2), \eta_2\rangle_{\mathcal{V}} \ge -\beta \|\eta_2\|_{\mathcal{V}}^2 \tag{4.4.9}$$

Now take $K_2 = -K - \frac{\beta}{\varepsilon} - \frac{\kappa}{\varepsilon}$ and $|\langle \mathfrak{Ric}_{\mathcal{H}} \eta_1, \eta_1\rangle_{\mathcal{H}} - \frac{1}{\varepsilon}\langle \delta_{\mathcal{H}} T(\eta_2), \eta_2\rangle_{\mathcal{V}} + \frac{1}{\varepsilon}\langle \mathbf{J}^2(\eta), \eta\rangle_{\mathcal{H}}| \le K_1$, we get the same estimate like Theorem 3.1 in [47] following the estimates for $Q_1, Q_2, Q_3$. □

### 4.4.3 Concentration inequalities

In the following, we study concentration inequalities for the horizontal Brownian motion $(W_t)_{0\le t\le 1}$ with the fixed staring point $x \in \mathbb{M}$. For $\varepsilon > 0$, we denote by $d_\varepsilon$ the distance associated with the Riemannian metric $g_\varepsilon$. To get a concentration bound for the distance $d_\varepsilon$, we adapt an argument from Ledoux [81] (see also [71]). The following lemma is a consequence of Herbst argument (see [81] page 148) applied to the log-Sobolev inequality in Lemma 4.4.2. Recall we still assume the following bound

**Lemma 4.4.7.** *Let $\varepsilon > 0$. Let $F \in \mathbf{Dom}(\tilde{D}^\varepsilon)$. If there is a constant $C > 0$ such that*

$$\int_0^T \|\tilde{D}_s^\varepsilon F\|^2 ds < C,$$

*almost surely, then for every $r \ge 0$,*

$$\mathbb{P}_x\left(F - \mathbb{E}_x(F) \ge r\right) \le \exp\left(-\frac{r^2}{2\sigma^2}\right),$$

*where $\sigma^2 = \mathbb{E}_x\left(\int_0^T \|\tilde{D}_s^\varepsilon F\|^2 ds\right)$.*

With this lemma in hand, we then can prove the following concentration inequality.

**Proposition 4.4.8.** *Let $\varepsilon > 0$. We have for every $T > 0$ and $r \ge 0$*

$$\mathbb{P}_x\left(\sup_{0\le t\le T} d_\varepsilon(X_t, x) \ge \mathbb{E}_x\left[\sup_{0\le t\le T} d_\varepsilon(X_t, x)\right] + r\right) \le \exp\left(-\frac{r^2}{2Te^{\left(K+\frac{\beta}{\varepsilon}+\frac{\kappa}{\varepsilon}\right)T}}\right). \tag{4.4.10}$$

*In particular, if we keep consistent with our previous convention, we will take $T = 1$.*

*Proof.* Let

$$F = f(W_{t_1}, \cdots, W_{t_n}) = \max_{1\le i\le n} d_\varepsilon(W_{t_i}, x)$$

where $0 \le t_1 \le \cdots \le t_n$ is a partition of $[0,T]$.

By using the arguments of [81, p.196], we obtain for a certain partition $(A_j)_{1\le j\le n}$ of the path space,

$$\begin{aligned}\|\tilde{D}^\varepsilon_s F\|_\varepsilon &\le \sum_{l=1}^n \mathbf{1}_{(t_{l-1},t_l]}(s) \sum_{i=l}^n e^{\frac{1}{2}(K+\frac{\beta}{\varepsilon}+\frac{\kappa}{\varepsilon})(t_i-s)} \|d_i f(W_{t_1}, \cdots, W_{t_n})\|_\varepsilon \\ &\le e^{\frac{1}{2}(K+\frac{\beta}{\varepsilon}+\frac{\kappa}{\varepsilon})T} \sum_{l=1}^n \mathbf{1}_{(t_{i-1},t_i]}(s) \sum_{i=l}^n \mathbf{1}_{A_i} \le e^{\frac{1}{2}(K+\frac{\beta}{\varepsilon}+\frac{\kappa}{\varepsilon})T}\end{aligned}$$

We then use the previous lemma and finish the proof by monotone convergence when the mesh of the partition goes to zero. □

The previous proposition easily implies that

$$\limsup_{r\to\infty} \frac{1}{r^2} \ln \mathbb{P}_x \left( \sup_{0\le t\le T} d_\varepsilon(W_t, x) \ge r \right) \le -\frac{1}{2Te^{\left(K+\frac{\beta}{\varepsilon}+\frac{\kappa}{\varepsilon}\right)T}}$$

Under further assumptions we can also provide a lower bound.

**Proposition 4.4.9.** *Assume that* (4.4.9) *is satisfied with* $K = 0$ *and moreover that for any vector field* $Z$*,*

$$-\frac{1}{4}\mathbf{Tr}_{\mathcal{H}}(J_Z^2) \ge \rho_2 \|Z\|^2_{\mathcal{V}},$$

*where* $\rho_2 > 0$*. Then for every* $\varepsilon, T > 0$*,*

$$\liminf_{r\to\infty} \frac{1}{r^2} \ln \mathbb{P}_x \left( \sup_{0\le t\le T} d_\varepsilon(X_t, x) \ge r \right) \ge -\frac{1}{T}\left( \frac{D}{n} + \frac{4\varepsilon^2}{T}\frac{3D}{2\rho_2 n} \ln(2) \right)$$

*where* $D = \left(1 + \frac{3\kappa}{2(\rho_2 - \beta)}\right) n.$

*Proof.* See proof in [11, Proposition 6.3] □

**Proposition 4.4.10.** *Assume that* (4.4.9) *is satisfied with* $K = 0$ *and moreover that for any vector field* $Z$*,*

$$-\frac{1}{4}\mathbf{Tr}_{\mathcal{H}}(J_Z^2) \geq \rho_2 \|Z\|_{\mathcal{V}}^2,$$

*where* $\rho_2 > 0$*. Then for every* $T > 0$*,*

$$\lim\sup_{r \to +\infty} \frac{1}{r^2} \ln \mathbb{P}_x \left( \sup_{0 \leq t \leq T} d(X_t, x) \geq r \right) \leq -\frac{1}{2T}$$

*and*

$$\lim\inf_{r \to +\infty} \frac{1}{r^2} \ln \mathbb{P}_x \left( \sup_{0 \leq t \leq T} d(X_t, x) \geq r \right) \geq -\frac{D}{2nT},$$

*where* $D = \left(1 + \frac{3\kappa}{2(\rho_2 - \beta)}\right) n.$

*Proof.* See proof in [11, Proposition 6.4]. □

## 4.5 Equivalent conditions to two-sided uniform Ricci curvature bounds

In this part, we start to generalize the similar result Theorem 1.1 in [116], to prove the uniform Ricci curvature bound by using the Log-Sobolev inequality was first proved by A. Naber [97]. In this part, we use the method in [116] to get similar results in our Riemannian foliation setting. Before we state and prove the main results, let us first prove the following useful lemma.

**Lemma 4.5.1.** *For* $\forall x \in \mathbb{M}$*, there exists* $f \in \mathcal{C}_0^\infty(\mathbb{M})$*, such that* $df_x = (df)_{\mathcal{H}_x}$*,* $\|\nabla_{\mathcal{H}} df - \mathfrak{T}^\varepsilon_{\mathcal{H}} df\|^2_\varepsilon = 0$.

*Proof.* Let us fix point $x_0 \in \mathbb{M}$, we claim that (similar to Theorem 2.20 [14]), we can find a function $f \in \mathcal{C}_0^\infty(\mathbb{M})$ such that

- $df_{x_0} = (df)_{\mathcal{H}_{x_0}}$;
- $(df)_{\mathcal{V}_{x_0}} = 0$;
- $\nabla^2_{\mathcal{H}} f(x_0) = 0$;
- $Z_i X_m f = -\frac{2}{\varepsilon}\sum_{i=1}^n \gamma_{ij}^m (df)_i$

Given the above conditions, we can directly get

$$\begin{aligned}\|\nabla_{\mathcal{H}} df - \mathfrak{T}^\varepsilon_{\mathcal{H}} df\|^2_\varepsilon = \|\nabla_{\mathcal{H}} df\|^2_{\mathcal{H}} + \varepsilon\|\nabla_{\mathcal{H}} df\|^2_{\mathcal{V}} + \varepsilon \sum_{i,j=1}^n (\frac{1}{\varepsilon}\sum_{l=1}^m \gamma_{ij}^l f_j)^2 + 2\varepsilon \sum_{i=1}^n\sum_{l=1}^m (X_i g_l)\left(\frac{1}{\varepsilon}\sum_{j=1}^n \gamma_{ij}^l f_j\right) \\ - 2\sum_{i,j=1}^n\sum_{l=1}^m (X_i f_j)\gamma_{ij}^l g_l + \sum_{i,j=1}^n\sum_{k,l=1}^m \gamma_{ij}^k\gamma_{ij}^l g_l g_k\end{aligned}$$

where in general we $df = \sum_{i=1}^n f_i\theta_i + \sum_{l=1}^m g_l\nu_l$, $\{\theta_1, \cdots, \theta_n, \nu_1, \cdots, \nu_m\}$ is the dual frame of $\{X_1, \cdots, X_n, Z_1, \cdots, Z_m\}$. At the point $x_0$, since $df_{x_0} = (df)_{\mathcal{H}_{x_0}}$, so $g_l = 0$ for $l = 1, \cdots, m$. So the function we choose directly gives us the condition that

$$\|\nabla_{\mathcal{H}} df - \mathfrak{T}^\varepsilon_{\mathcal{H}} df\|^2_\varepsilon = 0.$$

□

Now we introduce the following results under the Yang-Mills condition, namely

$$\delta_{\mathcal{H}} T(\cdot) = 0.$$

**Theorem 4.5.2.** *Given $K > 0$ and $\kappa > 0$ as two constants, for any $p, q \in [1,2]$, the following statements are equivalent to each other:*

*1 For any horizontal 1-form $\eta$,*

$$|\langle \mathfrak{Ric}_{\mathcal{H}}\eta, \eta\rangle_{\mathcal{H}}| \le K\|\eta\|_{\mathcal{H}}^2, \quad |\langle \mathbf{J}^2\eta, \eta\rangle_{\mathcal{H}}| \le \kappa\|\eta\|_{\mathcal{H}}^2 \tag{4.5.1}$$

*and we further denote $\tilde{K} = K + \frac{\kappa}{\varepsilon}$ where $\varepsilon$ corresponds to the parameter in the metric $g_\varepsilon = g_{\mathcal{H}} \oplus \frac{1}{\varepsilon} g_{\mathcal{V}}$.*

*2 For any $f \in C_0^\infty(\mathbb{M})$, $T > 0$, and $x \in \mathbb{M}$, with $A = \int_0^T \frac{\tilde{K}}{2} e^{1/2\tilde{K}s} ds = e^{1/2\tilde{K}T} - 1$*

$$|dP_T f|_\varepsilon^p(x) \le \mathbb{E}\left[(1+A)^p |df|_\varepsilon^p(W_T^x)\right]$$
$$|df(x) - \frac{1}{2}dP_T f(x)|_\varepsilon^q \le \mathbb{E}\left[(1+A)^{q-1} \times \left(|df(x) - \frac{1}{2}\Theta_T^\varepsilon df(W_T^x)|_\varepsilon + \frac{A}{2^q}|df(W_T^x|_\varepsilon^q\right)\right]$$

*3 For any cylindrical function $F = f(W_{t_1}, \cdots, W_{t_n})$, $x \in \mathbb{M}$ and $T > 0$,*

$$|d\mathbb{E}[F]|_\varepsilon^q \le \mathbb{E}\left[(1+A)^{q-1}\left(|D_0^\varepsilon F|_\varepsilon^q + \int_0^T |D_s^\varepsilon F|_\varepsilon^q \frac{\tilde{K}}{2} e^{1/2\tilde{K}s} ds\right)\right]$$

*4 Log-Sobolev inequality, for $A(t,T) = \int_t^T \frac{\tilde{K}}{2} e^{1/2\tilde{K}s} ds$*

$$\mathbb{E}_x(F^2 \ln F^2) - \mathbb{E}_x(F^2)\ln \mathbb{E}_x(F^2) \le 2\int_0^T (1+A(t,T))\mathbb{E}_x\left(|D_t F|_\varepsilon^2 + \int_t^T |D_s^\varepsilon F|_\varepsilon^2 \frac{\tilde{K}}{2} e^{1/2\tilde{K}s} ds\right) dt$$

*5 Poincaré inequality*

$$\mathbb{E}\left[\{\mathbb{E}(F|\mathcal{F}_t)\}^2\right] - (\mathbb{E}(F))^2 \le \int_0^T (1+A(t,T))\mathbb{E}_x\left(|D_t F|_\varepsilon^2 + \int_t^T |D_s^\varepsilon F|_\varepsilon^2 \frac{\tilde{K}}{2} e^{1/2\tilde{K}s} ds\right) dt$$

*Proof.* Throughout the proof, for any given $\varepsilon$, we denote $|\cdot|_\varepsilon$ as the norm for $g_\varepsilon$.

$(\mathbf{1}) \Rightarrow (\mathbf{3})$

Using the relation in lemma 4.4.5, we have

$$d\mathbb{E}[f(X_{t_1},\cdots,X_{t_n})] = \mathbb{E}_x[\sum_{i=1}^n \tau^\varepsilon_{t_i} d_i f(X_{t_1},\cdots,X_{t_n})] = \mathbb{E}_x[\tilde{D}^\varepsilon_0 F]$$
$$= \mathbb{E}_x\left[\Theta_0^{\varepsilon,-1} D^\varepsilon_0 F + \Theta_0^{\varepsilon,-1}\int_0^T -\frac{1}{2}\mathcal{M}^\varepsilon_{s,0}\Theta^\varepsilon_{s,t}\left(\frac{1}{\varepsilon}\mathbf{J}^2 + \mathfrak{Ric}_{\mathcal{H}}\right)(\Theta^\varepsilon_{s,0})^{-1} D^\varepsilon_s F ds\right]$$

Given the two sided bound in (4.5.1), we have

$$|\langle \mathfrak{Ric}_{\mathcal{H}}\eta,\eta\rangle_{\mathcal{H}} + \frac{1}{\varepsilon}\langle \mathbf{J}^2(\eta),\eta\rangle_{\mathcal{H}}| \le \tilde{K}\|\eta\|^2_{\mathcal{H}} = (K+\frac{\kappa}{\varepsilon})\|\eta\|^2_{\mathcal{H}}, \quad |\mathcal{M}^\varepsilon_t| \le e^{1/2\tilde{K}t}$$

we have

$$|d\mathbb{E}_x[F]|^q_\varepsilon \le \left[\mathbb{E}|D^\varepsilon_0 F|_\varepsilon + \mathbb{E}\int_0^T |D^\varepsilon_s F|_\varepsilon \frac{\tilde{K}}{2} e^{1/2\tilde{K}s} ds\right]^q$$
$$\le \mathbb{E}\left[\left(|D^\varepsilon_0 F|^q_\varepsilon + \frac{(\int_0^T |D^\varepsilon_s F|_\varepsilon \frac{\tilde{K}}{2} e^{1/2\tilde{K}s} ds)^q}{A^{q-1}}\right)(1+A)^{q-1}\right], \quad A = \int_0^T \frac{\tilde{K}}{2} e^{1/2\tilde{K}s} ds$$
$$\le \mathbb{E}\left[\left(|D^\varepsilon_0 F|^q_\varepsilon + \int_0^T |D^\varepsilon_s F|^q_\varepsilon \frac{\tilde{K}}{2} e^{1/2\tilde{K}s} ds\right)(1+A)^{q-1}\right]$$

$(\mathbf{3}) \Rightarrow (\mathbf{2})$.

For the first inequality, take $p = q$, and $F = f(X_T)$, then we have $|D^\varepsilon_s F|_\varepsilon \le |df(X_T)|_\varepsilon$ for all $s \in [0,T]$, then the first inequality follows directly from $(\mathbf{3})$. Then we take

$F = f(x) - \frac{1}{2}f(X_T)$, $\mathbb{E}F = f(x) - \frac{1}{2}P_T f(x)$ and

$$|D_0^\varepsilon F|_\varepsilon = |df(x) - \frac{1}{2}\Theta_T^\varepsilon df(X_T^x)|_\varepsilon$$
$$|D_s^\varepsilon F|_\varepsilon \leq \frac{1}{2}|df(X_T^x)|_\varepsilon, \quad s \in [0,T]$$

then the second inequality follows immediately from (**3**).

(**2**) $\Rightarrow$ (**1**)

We first get a generalized version of Theorem 2.2.4 in [117], since

$$P_t|df|_\varepsilon^p = |df|_\varepsilon^p + \frac{pt}{2}|df|_\varepsilon^{p-2}\frac{L}{2}|df|_\varepsilon^2 + o(t)$$
$$|dP_t f|_\varepsilon^p = |df|_\varepsilon^p + pt|df|_\varepsilon^{p-2}\langle d\frac{L}{2}f, df\rangle_\varepsilon + o(t)$$

and according to Theorem 5.2.1, we know that $dLf = \square^\varepsilon df$, so we have

$$\langle \mathfrak{Ric}_\mathcal{H} df, df\rangle_\mathcal{H} + \frac{1}{\varepsilon}\langle \mathbf{J}^2(df), df\rangle_\mathcal{H} + \|\nabla_\mathcal{H} df - \mathfrak{T}_\mathcal{H}^\varepsilon df\|_\varepsilon^2 - \langle \delta_\mathcal{H} T(df), df\rangle_\mathcal{V} = \lim_{t\to 0}(\frac{P_t|df|_\varepsilon^p - |dP_t f|_\varepsilon^p}{pt/2})$$

since we assume the Yang-Mills condition $\delta_\mathcal{H} T(\cdot) = 0$ here, thus we have

$$\langle \mathfrak{Ric}_\mathcal{H} df, df\rangle_\mathcal{H} + \frac{1}{\varepsilon}\langle \mathbf{J}^2(df), df\rangle_\mathcal{H} + \|\nabla_\mathcal{H} df - \mathfrak{T}_\mathcal{H}^\varepsilon df\|_\varepsilon^2 = \lim_{t\to 0}(\frac{P_t|df|_\varepsilon^p - |dP_t f|_\varepsilon^p}{pt/2})$$

combining with the first inequality in (**2**), we have

$$\begin{aligned}
&-\langle \mathfrak{Ric}_\mathcal{H} df, df\rangle_\mathcal{H} - \frac{1}{\varepsilon}\langle \mathbf{J}^2(df), df\rangle_\mathcal{H} - \|\nabla_\mathcal{H} df - \mathfrak{T}_\mathcal{H}^\varepsilon df\|_\varepsilon^2 \\
&= \lim_{T\to 0}(\frac{|dP_T f|_\varepsilon^p - P_T|df|_\varepsilon^p}{pT/2}) \\
&\leq \lim_{T\to 0}\frac{[(1+A)^p - 1]|df|_\varepsilon^P(X_T^x)}{pT/2} = \tilde{K} = K + \frac{\kappa}{\varepsilon}.
\end{aligned}$$

where according to Lemma 4.5.1, we can choose $f \in C_0^\infty(\mathbb{M})$ with $|df(x)|_{\mathcal{H}} = 1$, and $\|\nabla_{\mathcal{H}} df - \mathfrak{T}_{\mathcal{H}}^{\varepsilon} df\|_{\varepsilon}^2 = 0$ thus we have

$$\langle \mathfrak{Ric}_{\mathcal{H}} df, df \rangle_{\mathcal{H}} + \frac{1}{\varepsilon} \langle \mathbf{J}^2(df), df \rangle_{\mathcal{H}} \geq -K - \frac{\kappa}{\varepsilon},$$

this implies for any horizontal 1-form $\eta \in \Gamma_{\mathcal{H}}^\infty(T_x^*\mathbb{M})$ with $|\eta| = 1$ and $\|\nabla_{\mathcal{H}}\eta - \mathfrak{T}_{\mathcal{H}}^{\varepsilon}\eta\|_{\varepsilon}^2 = 0$, we have

$$\langle \mathfrak{Ric}_{\mathcal{H}} \eta, \eta \rangle_{\mathcal{H}} + \frac{1}{\varepsilon} \langle \mathbf{J}^2(\eta), \eta \rangle_{\mathcal{H}} \geq -K - \frac{\kappa}{\varepsilon}.$$

since the above bound works for any given $\varepsilon$, thus taking $\varepsilon \to 0$ and $\varepsilon \to \infty$ gives us the desired lower bound

$$\langle \mathfrak{Ric}_{\mathcal{H}} \eta, \eta \rangle_{\mathcal{H}} \geq -K, \quad \langle \mathbf{J}^2(\eta), \eta \rangle_{\mathcal{H}} \geq -\kappa.$$

Now we prove that the second inequality in (**2**) implies the upper bound. Taking $q = 2$, we have

$$|df(x) - \frac{1}{2} dP_T f(x)|_{\varepsilon}^2 \leq \mathbb{E}\left[(1+A)\left(|df(x) - \frac{1}{2}\Theta_T^{\varepsilon} df(X_T^x)|_{\varepsilon}^2 + \frac{A}{4}|df(X_T^x)|_{\varepsilon}^2\right)\right]$$

$$\Longrightarrow \frac{|dP_T f(x)|_{\varepsilon}^2 - P_T|df(x)|_{\varepsilon}^2}{4T/2} \leq \frac{2}{T}\mathbb{E}\left[\langle df(x), dP_T f(x) - \mathbb{E}[\Theta_T^{\varepsilon} df(X_T^x)]\rangle_{\varepsilon}\right]$$
$$+ \frac{2}{T}\mathbb{E}\left[A|df(x) - \frac{1}{2}\Theta_T^{\varepsilon} df(X_T^x)|_{\varepsilon}^2 + \frac{A(1+A)}{4}|df(X_T^x)|_{\varepsilon}^2\right]$$

since we have

$$\begin{aligned}
&\mathbb{E}\left[\langle df(x), dP_T f(x) - \mathbb{E}[\Theta_T^\varepsilon df(X_T^x)]\rangle_\varepsilon\right] \\
&= \mathbb{E}\left[\langle df(x), \mathbb{E}[\tau_T^\varepsilon df(x) - \Theta_T^\varepsilon df(X_T^x)]\rangle_\varepsilon\right] \\
&= \mathbb{E}\left[\langle df(x), \mathbb{E}[(\mathcal{M}_T^\varepsilon - I)\Theta_T^\varepsilon df(X_T^x)]\rangle_\varepsilon\right] \\
&= \mathbb{E}\left[\langle df(x), \mathbb{E}[(\int_0^T d\mathcal{M}_t^\varepsilon)\Theta_T^\varepsilon df(X_T^x)]\rangle_\varepsilon\right] \\
&= -\frac{1}{2}\mathbb{E}\left[\langle df(x), [\int_0^T \mathcal{M}_t^\varepsilon \Theta_t^\varepsilon (\frac{1}{\varepsilon}\mathbf{J}^2 + \mathfrak{Ric}_{\mathcal{H}})(\Theta_t^\varepsilon)^{-1} dt \Theta_T^\varepsilon df(X_T^x)]\rangle_\varepsilon\right] \\
&= -\frac{1}{2}\int_0^T \langle df(x), \mathcal{M}_t^\varepsilon \Theta_t^\varepsilon (\frac{1}{\varepsilon}\mathbf{J}^2 + \mathfrak{Ric}_{\mathcal{H}})(\Theta_t^\varepsilon)^{-1} \Theta_T^\varepsilon df(X_T^x)\rangle_\varepsilon dt \\
&= -\frac{1}{2}T\left[\langle \mathfrak{Ric}_{\mathcal{H}} df, df\rangle_{\mathcal{H}} + \frac{1}{\varepsilon}\langle \mathbf{J}^2(df), df\rangle_{\mathcal{H}}\right] + o(T)
\end{aligned}$$

taking limit $t \to 0$, so we have

$$\begin{aligned}
&-\langle \mathfrak{Ric}_{\mathcal{H}} df, df\rangle_{\mathcal{H}} - \frac{1}{\varepsilon}\langle \mathbf{J}^2(df), df\rangle_{\mathcal{H}} - \|\nabla_{\mathcal{H}} df - \mathfrak{T}_{\mathcal{H}}^\varepsilon df\|_\varepsilon^2 \\
&\leq \tilde{K}|df|_\varepsilon^2 + \limsup_{T\to 0} \frac{2}{T} E\left[\langle df(x), dP_T f(x) - \mathbb{E}[\Theta_T^\varepsilon df(X_T^x)]\rangle_\varepsilon\right] \\
&\leq \tilde{K}|df|_\varepsilon^2 - 2\langle \mathfrak{Ric}_{\mathcal{H}} df, df\rangle_{\mathcal{H}} - 2\frac{1}{\varepsilon}\langle \mathbf{J}^2(df), df\rangle_{\mathcal{H}}
\end{aligned}$$

So we have proved that

$$\langle \mathfrak{Ric}_{\mathcal{H}} df, df\rangle_{\mathcal{H}} + \frac{1}{\varepsilon}\langle \mathbf{J}^2(df), df\rangle_{\mathcal{H}} \leq \tilde{K}|df|_\varepsilon^2 + \|\nabla_{\mathcal{H}} df - \mathfrak{T}_{\mathcal{H}}^\varepsilon df\|_\varepsilon^2 \leq K + \frac{\kappa}{\varepsilon}$$

where the second step in the above inequality follows from Lemma 4.5.1: $f \in C_0^\infty(\mathbb{M})$ with $|df(x)|_{\mathcal{H}} = 1$ and $\|\nabla_{\mathcal{H}} df - \mathfrak{T}_{\mathcal{H}}^\varepsilon df\|_\varepsilon^2 = 0$. Similarly as before, we take $\varepsilon \to \infty$ and $\varepsilon \to 0$ to get the upper bound in (4.5.1).

$(\mathbf{1}) \Rightarrow (\mathbf{4})$

Recall the proof from $((\mathbf{1}) \Rightarrow (\mathbf{3}))$, we conclude that

$$\begin{aligned}\mathbb{E}_x(F^2 \ln F^2) - \mathbb{E}_x(F^2) \ln \mathbb{E}_x(F^2) &\leq 2\mathbb{E}_x \left( \int_0^T \|\tilde{D}_s^\varepsilon F\|_\varepsilon^2 ds \right) \\ &\leq 2 \int_0^T (1 + A(t,T)) \mathbb{E}_x \left( |D_t F|_\varepsilon^2 + \int_t^T |D_s^\varepsilon F|_\varepsilon^2 \frac{\tilde{K}}{2} e^{1/2 \tilde{K} s} ds \right) dt\end{aligned}$$

$(\mathbf{5}) \Rightarrow (\mathbf{1})$

Take $F(\gamma) = f(\gamma_T)$, (5) implies that

$$P_T f^2(x) - (P_T f(x))^2 \leq \int_0^T (1 + A(t,T))^2 \mathbb{E}_x[|df(X_T)|_\varepsilon^2] dt$$

Similar to [117] Theorem 2.2.4, we can show that:

$$\langle \mathfrak{Ric}_\mathcal{H} df, df \rangle_\mathcal{H} + \frac{1}{\varepsilon} \langle \mathbf{J}^2(df), df \rangle_\mathcal{H} + \|\nabla_\mathcal{H} df - \mathfrak{T}_\mathcal{H}^\varepsilon df\|_\varepsilon^2 = \lim_{t \to 0} \frac{1}{t/2} \left( \frac{P_t f^2(x) - (P_t f)^2(x)}{2t/2} - |dP_t f(x)|_\varepsilon^2 \right)$$

thus we have

$$\begin{aligned}&\langle \mathfrak{Ric}_\mathcal{H} df, df \rangle_\mathcal{H} + \frac{1}{\varepsilon} \langle \mathbf{J}^2(df), df \rangle_\mathcal{H} + \|\nabla_\mathcal{H} df - \mathfrak{T}_\mathcal{H}^\varepsilon df\|_\varepsilon^2 \\ &= \lim_{t \to 0} \frac{1}{T/2} \left( \frac{P_T f^2(x) - (P_T f)^2(x)}{2T/2} - |dP_T f(x)|_\varepsilon^2 \right) \\ &\leq \lim_{t \to 0} \frac{1}{T/2} \left( \frac{1}{T} \int_0^T (1 + A(t,T))^2 \mathbb{E}_x[|df(X_T)|_\varepsilon^2] dt - |dP_T f(x)|_\varepsilon^2 \right) \\ &\leq \lim_{t \to 0} \frac{1}{T/2} \left( P_T |df|_\varepsilon^2 - |dP_T f|_\varepsilon^2(x) + \frac{2|df|_\varepsilon^2}{T} \int_0^T A(t,T) dt \right) + 0(t) \\ &= 2 \left( \langle \mathfrak{Ric}_\mathcal{H} df, df \rangle_\mathcal{H} + \frac{1}{\varepsilon} \langle \mathbf{J}^2(df), df \rangle_\mathcal{H} + \|\nabla_\mathcal{H} df - \mathfrak{T}_\mathcal{H}^\varepsilon df\|_\varepsilon^2 \right) + \tilde{K} |df|_\varepsilon^2(x)\end{aligned}$$

this implies

$$\langle \mathfrak{Ric}_{\mathcal{H}} df, df\rangle_{\mathcal{H}} + \frac{1}{\varepsilon}\langle \mathbf{J}^2(df), df\rangle_{\mathcal{H}} \geq -\tilde{K} - \|\nabla_{\mathcal{H}} df - \mathfrak{T}^{\varepsilon}_{\mathcal{H}} df\|^2_{\varepsilon} \geq -\tilde{K}$$

Similar as before, we can prove the lower bound in (4.5.1).
To prove the upper bound, we take $F(\gamma) = f(\gamma_\delta) - \frac{1}{2} f(\gamma_T)$ for $\delta \in (0, T)$, the proof follows directly similar to [116].

$$|D^{\varepsilon}_t F|_{\varepsilon} = |df(X_\delta) - \frac{1}{2}(\Theta^{\varepsilon}_{\delta})^{-1}\Theta^{\varepsilon}_T df(X_T)|_{\varepsilon}\mathbf{1}_{[0,\delta)}(t) + \frac{1}{2}|df(X_T)|_{\varepsilon}\mathbf{1}_{[\delta,T)}(t)$$

Then (**5**) implies

$$\begin{aligned} I_\delta &:= \mathbb{E}\left[f(X_\delta) - \frac{1}{2}\mathbb{E}(f(X_T)|\mathcal{F}_\delta)\right]^2 - \left(P_\delta f(x) - \frac{1}{2}P_T f(x)\right)^2 \\ &\leq \delta\mathbb{E}\left[(1 + A(0,T))\left(|df(X_\delta) - \frac{1}{2}(\Theta^{\varepsilon}_{\delta})^{-1}\Theta^{\varepsilon}_T df(X_T)|^2_{\varepsilon} + \frac{A(0,T)}{4}|df(X_T)|^2_{\varepsilon}\right)\right] + c\delta^2 \\ &:= J_\delta, \quad \delta \in (0,T). \end{aligned}$$

The R.H.S gives us

$$\lim_{\delta\to 0}\frac{J_\delta}{\delta} = \mathbb{E}\left[(1 + A(0,T))\left(|df(X_\delta) - \frac{1}{2}(\Theta^{\varepsilon}_{\delta})^{-1}\Theta^{\varepsilon}_T df(X_T)|^2_{\varepsilon} + \frac{A(0,T)}{4}|df(X_T)|^2_{\varepsilon}\right)\right] \tag{4.5.2}$$

Now the L.H.S gives

$$\frac{I_\delta}{\delta} = \frac{P_\delta f^2 - (P_\delta f)^2}{\delta} + \frac{1}{4\delta}[\mathbb{E}\left[\mathbb{E}(f(X_T)|\mathcal{F}_\delta)]^2 - (P_T f)^2(x)\right] + \frac{\mathbb{E}[f(X_T)[P_\delta f(x) - f(X_\delta)]]}{\delta}$$

Let $f \in \mathcal{C}_0^\infty(\mathbb{M})$ satisfy the Neumann boundary condition and Lemma 4.5.1, we have

$$\lim_{\delta\to 0}\frac{P_\delta f^2-(P_\delta f)^2}{\delta}=|\nabla_{\mathcal{H}}f|^2(x)=\langle df,df\rangle_{\mathcal{H}}=|df_{|\mathcal{H}}|^2.$$

recall the Clark-Ocone formula

$$F=\mathbb{E}_x(F)+\int_0^T\langle\mathbb{E}_x(\tilde{D}_s^\varepsilon F|\mathcal{F}_s),\widehat{\Theta}_{0,s}^\varepsilon dB_s\rangle.$$

appling to funtion $\mathbb{E}(f(X_T)|\mathcal{F}_\delta)]$ gives us

$$\mathbb{E}(f(X_T)|\mathcal{F}_\delta)]=P_Tf(x)+\int_0^\delta\langle\mathbb{E}(\tau_{s,T}^\varepsilon df(X_T)|\mathcal{F}_s),\widehat{\Theta}_{0,s}^\varepsilon dB_s\rangle$$

Then we have

$$\mathbb{E}[\mathbb{E}(f(X_T)|\mathcal{F}_\delta)]^2=(P_Tf(x))^2+\int_0^\delta\mathbb{E}|\tau_{s,T}^\varepsilon df(X_T)|_\varepsilon^2ds$$

so we have

$$\lim_{\delta\to 0}\frac{1}{4\delta}[\mathbb{E}\left[\mathbb{E}(f(X_T)|\mathcal{F}_\delta)]^2-(P_Tf)^2(x)\right]=\frac{1}{4}|\mathbb{E}[\tau_{0,T}^\varepsilon df(X_T)]|_\varepsilon^2=\frac{1}{4}|dP_Tf(x)|_\varepsilon^2$$

By Itó formula, we have

$$\begin{aligned}P_\delta f(x)-f(X_\delta)&=P_\delta f(x)-f(x)-\int_0^\delta\frac{1}{2}Lf(X_s)ds-\int_0^\delta\langle\nabla_{\mathcal{H}}f(X_s),\widehat{\Theta}_{0,s}^\varepsilon dB_s\rangle\\&=o(\delta)-\int_0^\delta\langle\nabla_{\mathcal{H}}f(X_s),\widehat{\Theta}_{0,s}^\varepsilon dB_s\rangle\end{aligned}$$

thus we have

$$\lim_{\delta\to 0}\frac{\mathbb{E}[f(X_T)[P_\delta f(x)-f(X_\delta)]]}{\delta}=-\langle df_{|\mathcal{H}},dP_tf(x)\rangle_\varepsilon$$

so we have

$$\lim_{\delta\to 0}\frac{I_\delta}{\delta}=|df_{|\mathcal{H}}-\frac{1}{2}dP_Tf(x)|_\varepsilon^2.$$

This combines with (4.5), we proved the second inequality in (**2**) for $q=2$ and $df=df_{|\mathcal{H}}$, similar to the proof for (**2**) $\Rightarrow$ (**1**), we can get the upper bound as well. □

## 4.6 Quasi-invariance of horizontal Wiener measure

In the end of this Chapter, we present the quasi-invariance of the horizontal Wiener measure, i.e. the law of the horizontal Brownian motion path, along the flow generated by suitable *tangent processes*. We follows relatively close to the framework by B. Driver [36] and E. Hsu [72] (see also [31, 32, 44]). The complete presentation of this part is in[12, Section 5], however, we add complementary parts to [12, Section 5] which are the sub-Riemannian analogue of the Riemannian case in [36, 72] and are omitted in [12, Section 5].

In this section, we will see another reason why we need to use the adjoint damped connection $\widehat{\nabla}^\varepsilon$. It is because that it satisfies the Driver anti-symmetry condition,

- For any $X,Y,Z\in\Gamma^\infty(\mathbb{M})$, we have $\langle T^{\widehat{\nabla}^\varepsilon}(X,Y),Z\rangle=-\langle T^{\widehat{\nabla}^\varepsilon}(Z,Y),X\rangle$.

### 4.6.1 Horizontal Wiener measure

We first introduce some notation and introduce the concept of horizontal Wiener measure for which we establish quasi-invariance. We will mainly follow the framework in [35, 37].

**Notation 4.6.1.** *We work in the probability space* $(C_0(\mathbb{R}^n), \mathcal{B}, \mu_{\mathcal{H}})$, *where* $C_0(\mathbb{R}^n)$ *is the space of continuous functions* $\omega^{\mathcal{H}} : [0,1] \to \mathbb{R}^n$ *such that* $\omega^{\mathcal{H}}(0) = 0$, $\mathcal{B}$ *is the Borel* $\sigma$*-field on* $C_0(\mathbb{R}^n)$, *and* $\mu_{\mathcal{H}}$ *is the Wiener measure. The coordinate process* $(\omega_t^{\mathcal{H}})_{0\leqslant t\leqslant 1}$ *is therefore a Brownian motion in* $\mathbb{R}^n$. *The usual completion of the natural filtration generated by* $(\omega_t^{\mathcal{H}})_{0\leqslant t\leqslant 1}$ *will be denoted by* $\mathcal{B}_t$. *We use the subscripts or superscripts* $\mathcal{H}$, *because, as in the previous sections,* $\mathbb{R}^n$ *is identified with the subspace* $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+m}$. *The* $\mathbb{R}^{n+m}$ *valued process* $(\omega_t^{\mathcal{H}}, 0)$ *will then be referred to as a horizontal Brownian motion. The process* $(W_t)_{0\leq t\leq 1}$ *constructed through the corollary 4.1.3 is the horizontal Brownian motion and the law* $\mu_W$ *of the horizontal Brownian motion on* $\mathbb{M}$ *will be referred to as the horizontal Wiener measure on* $\mathbb{M}$. *Therefore,* $\mu_W$ *is a probability measure on the space* $C_{x_0}(\mathbb{M})$ *of continuous paths* $w : [0,1] \to \mathbb{M}$, $w(0) = x_0$.

**Remark 4.6.2.** If the horizontal Laplacian can globally be written in the Hörmander's forms 4.1.1, then from Corollary 5.4 in [112], the support of the horizontal Wiener measure $\mu_W$ in the supremum topology is $C_{x_0}(\mathbb{M})$ itself.

As before, if $h \in \mathbb{R}^{n+m}$ is a vector, we denote by $Ah = \sum_{i=1}^n h_i A_i$ and $Vh = \sum_{i=1}^m h_{i+n} V_i$, where $A_i$ and $V_i$ are the fundamental vector fields on $\mathcal{O}(\mathbb{M})$. We consider the solution $(U_t)_{0\leqslant t\leqslant 1}$ to the Stratonovitch stochastic differential equation

$$dU_t = A_{U_t} \circ d\omega_t^{\mathcal{H}}, \qquad U_0 \in \mathcal{O}_{\mathcal{H}}(\mathbb{M}),$$

and the horizontal Brownian motion on $\mathbb{M}$ given by $W_t = \pi(U_t)$. The horizontal Itô map $\mathcal{I}_{\mathcal{H}}$ is $\mu_{\mathcal{H}}$ defined as the map $\mathcal{I}_{\mathcal{H}} : \omega^{\mathcal{H}} \to U$ and the horizontal stochastic development map $\phi_{\mathcal{H}}$ is $\mu_{\mathcal{H}}$ defined as the map $\phi_{\mathcal{H}} : \omega^{\mathcal{H}} \to W$. We refer to Definition 2.5 in [35] and the associated comments for a discussion of the Itô and stochastic development map in the classical Riemannian setting and to the previous section for explicit constructions in our setting.

### 4.6.2 Tangent processes to the horizontal Brownian motion

We now introduce the relevant class of tangent processes to the horizontal Brownian motion, which is a special case of the tangent process to the horizontal semimartingales we introduced in the previous section. To prove quasi-invariance, we restrict the class of tangent processes (in the sense of Definition 4.2.18) to the following class.

**Definition 4.6.3.** We define the *horizontal Cameron-Martin space* denoted by $\mathcal{CM}_{\mathcal{H}}(\mathbb{R}^{n+m})$ as the space of absolutely continuous $\mathbb{R}^n$-valued (deterministic) functions $(h(t))_{0\leqslant t\leqslant 1}$ such that $h(0) = 0$ and

$$\int_0^1 |h'(t)|^2_{\mathbb{R}^n} dt < \infty.$$

**Definition 4.6.4.** A $\mathcal{B}$-adapted $\mathbb{R}^{n+m}$-valued continuous semimartingale $(v(t))_{0\leqslant t\leqslant 1}$ such that $v(0) = 0$ and $\mathbb{E}\left(\int_0^1 |v(t)|^2_{\mathbb{R}^{n+m}} dt\right) < \infty$ will be called a *tangent process* to the horizontal Brownian motion if the process

$$v(t) - \int_0^t T_{U_s}(A \circ d\omega_s^{\mathcal{H}}, Av(s))$$

is a horizontal Cameron-Martin path, where $T$ denotes the torsion form of the Bott connection. The space of tangent processes to the horizontal Brownian motion will be denoted by $TW_{\mathcal{H}}(\mathbb{M})$.

**Remark 4.6.5.** In Definition 4.6.4 we denote by $T$ the torsion of the Bott connection. Observe that since $T$ is a vertical tensor, a $\mathcal{B}$-adapted and $\mathbb{R}^{n+m}$-valued continuous semimartingale $(v(t))_{0\leqslant t\leqslant 1}$ such that $v(0) = 0$ and $\mathbb{E}\left(\int_0^1 |v(t)|^2_{\mathbb{R}^{n+m}} dt\right) < \infty$ is in $TW_{\mathcal{H}}(\mathbb{M})$ if and only if

1. The horizontal part $v_{\mathcal{H}}$ is in $\mathcal{CM}_{\mathcal{H}}(\mathbb{R}^{n+m})$;

2. The vertical part $v_{\mathcal{V}}$ is given by

$$v_{\mathcal{V}}(t) = \int_0^t T_{U_s}(A \circ d\omega^{\mathcal{H}}_s, Av_{\mathcal{H}}(s)).$$

As a consequence, for any $h \in \mathcal{CM}_{\mathcal{H}}(\mathbb{R}^{n+m})$,

$$\tau^h(\omega^{\mathcal{H}})_t = h(t) + \int_0^t T_{U_s}(A \circ d\omega^{\mathcal{H}}_s, Ah(s)) \tag{4.6.1}$$

is a tangent process to the horizontal Brownian motion.

**Notation 4.6.6.** *If $v \in TW_{\mathcal{H}}(\mathbb{M})$ is a tangent process, we denote*

$$p_v(\omega^{\mathcal{H}})_t = v(t) - \int_0^t T^{\widehat{\nabla}^\varepsilon}_{U_s}(A \circ d\omega^{\mathcal{H}}_s, Av(s) + Vv(s)) - \int_0^t \left(\int_0^s \Omega^{\widehat{\nabla}^\varepsilon}_{U_\tau}(A \circ d\omega^{\mathcal{H}}_\tau, Av(\tau) + Vv(\tau))\right) \circ d\omega^{\mathcal{H}}_s,$$

*where $\Omega^{\widehat{\nabla}^\varepsilon}$ is the curvature form of $\widehat{\nabla}^\varepsilon$.*

This definition comes from the formula (4.2.2) where $d\omega^{\mathcal{H}}$ is simply formally replaced by the Stratonovitch differential $\circ d\omega^{\mathcal{H}}$.

Since $\widehat{\nabla}^\varepsilon$ is a horizontal metric connection, the stochastic integral $\int_0^s \Omega_{U_\tau}^{\widehat{\nabla}^\varepsilon}(A \circ d\omega_\tau^{\mathcal{H}}, Av(\tau) + Vv(\tau))$ restricts to $\mathbb{R}^n$ as a skew-symmetric endomorphism of $\mathbb{R}^n$. Also, from the proof of theorem 4.2.9, we have

$$\int_0^t T_{U_s}^{\widehat{\nabla}^\varepsilon}(A \circ d\omega_s^{\mathcal{H}}, Av(s) + Vv(s)) =$$
$$\int_0^t T_{U_s}(A \circ d\omega_s^{\mathcal{H}}, Av(s)) - \int_0^t \frac{1}{\varepsilon} J_{Vv(s)}(A \circ d\omega_s^{\mathcal{H}})_{U_s},$$

As a consequence, $p_v(\omega^{\mathcal{H}})_t$ is actually a horizontal process, that is, it is $\mathbb{R}^n$-valued.

We can rewrite $p_v(\omega^{\mathcal{H}})_t$ by using Itô's integral, and we obtain

$$p_v(\omega^{\mathcal{H}})_t = v_{\mathcal{H}}(t) + \frac{1}{2}\int_0^t \left(\mathfrak{Ric}_{\mathcal{H}}^{\widehat{\nabla}^\varepsilon}\right)_{U_s}(Av(s) + Vv(s))ds$$
$$+ \int_0^t J_{Vv(s)}(A \circ d\omega_s^{\mathcal{H}})_{U_s} - \int_0^t \left(\int_0^s \Omega_{U_\tau}^{\widehat{\nabla}^\varepsilon}(A \circ d\omega_\tau^{\mathcal{H}}, Av(\tau) + Vv(\tau))\right) d\omega_s^{\mathcal{H}},$$

where $\mathfrak{Ric}_{\mathcal{H}}^{\widehat{\nabla}^\varepsilon}$ is the horizontal Ricci curvature of the connection $\widehat{\nabla}^\varepsilon$. We can further simplify this expression as follows.

$$J_{Vv(s)}(A \circ d\omega_s^{\mathcal{H}})_{U_s} =$$
$$\sum_{i=1}^n J_{Vv(s)}(A_i)_{U_s} \circ d\omega_s^i =$$
$$J_{Vv(s)}(Ad\omega_s^{\mathcal{H}})_{U_s} + \frac{1}{2}\sum_{i=1}^n A_i J_{Vv(s)}(A_i)_{U_s} ds$$
$$- \frac{1}{2}\sum_{i=1}^n J_{T(A_i, Av_{\mathcal{H}}(s))}(A_i)_{U_s} ds$$

As a result, we see that

$$p_v(\omega^{\mathcal{H}})_t = v_{\mathcal{H}}(t) + \frac{1}{2}\sum_{i=1}^{n}\int_0^t A_i J_{Vv(s)}(A_i)_{U_s} ds - \frac{1}{2}\sum_{i=1}^{n}\int_0^t J_{T(A_i, Av_{\mathcal{H}}(s))}(A_i)_{U_s} ds$$
$$+ \frac{1}{2}\int_0^t \left(\mathfrak{Ric}_{\mathcal{H}}^{\widehat{\nabla}^{\varepsilon}}\right)_{U_s}(Av(s)+Vv(s))ds + \int_0^t J_{Vv(s)}(Ad\omega_s^{\mathcal{H}})_{U_s}$$
$$- \int_0^t \left(\int_0^s \Omega_{U_\tau}^{\widehat{\nabla}^{\varepsilon}}(A\circ d\omega_\tau^{\mathcal{H}}, Av(\tau)+Vv(\tau))\right) d\omega_s^{\mathcal{H}}.$$

More concisely, one can thus write

$$p_v(\omega^{\mathcal{H}})_t = \int_0^t q_v(\omega^{\mathcal{H}})_s d\omega_s^{\mathcal{H}} + \int_0^t r_v(\omega^{\mathcal{H}})_s ds, \tag{4.6.2}$$

where $q_v$ is a $\mathfrak{so}(n)$-valued adapted process and $r_v$ is an $\mathbb{R}^n$-valued adapted process such that $\int_0^t |r_v(s)|^2_{\mathbb{R}^n} ds < +\infty$ a.e. The process $p_v$ is therefore an adapted vector field on $C_0(\mathbb{R}^n)$ in the sense of [35, Definition 3.2].

### 4.6.3 Quasi-invariance of the horizontal Wiener measure

We are now ready to construct the first relevant variation of the horizontal Brownian motion paths.

**Notation 4.6.7.** *Let $h \in \mathcal{CM}_{\mathcal{H}}(\mathbb{R}^{n+m})$. For $t \in \mathbb{R}$, we define then a map $\rho_t^h : C_0(\mathbb{R}^n) \to C_0(\mathbb{R}^n)$, which is $\mu_{\mathcal{H}}$ a.s. well defined, as follows*

$$(\rho_t^h \omega_{\mathcal{H}})_s = \int_0^s e^{tq_{\tau_h(\omega^{\mathcal{H}})}(\omega^{\mathcal{H}})_u} d\omega_u^{\mathcal{H}} + t\int_0^s r_{\tau_h(\omega^{\mathcal{H}})}(\omega^{\mathcal{H}})_u du. \tag{4.6.3}$$

**Remark 4.6.8.** As in the deterministic case, observe that $\rho^h$ is **not** the flow generated by $p_{\tau_h}$ on $C_0(\mathbb{R}^n)$. This variation is similar to [37, Theorem 7.28]. Let us however observe that $\mu_{\mathcal{H}}$ a.s., $\rho_0^h \omega^{\mathcal{H}} = \omega^{\mathcal{H}}$ and

$$\frac{d}{dt} |_{t=0} \, (\rho_t^h \omega^{\mathcal{H}})_s = p_{\tau_h(\omega^{\mathcal{H}})}(\omega^{\mathcal{H}})_s.$$

One has then the following analogue of [37, Theorem 7.28] (see [35] for the details).

**Theorem 4.6.9** (Differential of the horizontal stochastic development map and quasi-invariance of the horizontal Wiener measure)**.** *Let $h \in \mathcal{CM}_{\mathcal{H}}(\mathbb{R}^{n+m})$.*

- *For every $t \in \mathbb{R}$ the law under $\mu_{\mathcal{H}}$ of the semimartingale $((\rho_t^h \omega_{\mathcal{H}})_s)_{s\in[0,1]}$ is equivalent to $\mu_{\mathcal{H}}$ and the following Radon-Nikodym density is explicitly given by*

$$\begin{aligned}&\frac{d(\rho_t^h)_* \mu_{\mathcal{H}}}{d\mu_{\mathcal{H}}}(\omega^{\mathcal{H}})\\ &= \exp\left( t \int_0^1 \left\langle r_{\tau_h(\omega^{\mathcal{H}})}(\omega^{\mathcal{H}})_s, e^{tq_{\tau_h(\omega^{\mathcal{H}})}(\omega^{\mathcal{H}})_s} d\omega_s^{\mathcal{H}} \right\rangle - \frac{t^2}{2} \int_0^1 |r_{\tau_h(\omega^{\mathcal{H}})}(\omega^{\mathcal{H}})_s|^2_{\mathbb{R}^n} ds \right).\end{aligned}$$

- *For every $t \in \mathbb{R}$ the law under $\mu_{\mathcal{H}}$ of the semimartingale $(\phi_{\mathcal{H}}(\rho_t^h \omega_{\mathcal{H}})_s)_{0\le s\le 1}$ is equivalent to $\mu_W$ and the following Radon-Nikodym density is explicitly given by*

$$\begin{aligned}&\frac{d(\phi_{\mathcal{H}} \rho_t^h \phi_{\mathcal{H}}^{-1})_* \mu_W}{d\mu_W}(w)\\ &= \exp\left( t \int_0^1 \left\langle r_{\tau_h(\omega^{\mathcal{H}})}(\omega^{\mathcal{H}})_s, e^{tq_{\tau_h(\omega^{\mathcal{H}})}(\omega^{\mathcal{H}})_s} d\omega_s^{\mathcal{H}} \right\rangle - \frac{t^2}{2} \int_0^1 |r_{\tau_h(\omega^{\mathcal{H}})}(\omega^{\mathcal{H}})_s|^2_{\mathbb{R}^n} ds \right),\end{aligned}$$

  *where $\omega^{\mathcal{H}} = \phi_{\mathcal{H}}^{-1}(w)$.*

- *There exists a version of $\phi_{\mathcal{H}}((\rho_t^h \omega_{\mathcal{H}}))_s$ which is continuous in $(s, t)$ differentiable in $t$ and such that $\mu_{\mathcal{H}}$ a.s.*

$$\frac{d}{dt} |_{t=0} \phi_{\mathcal{H}}((\rho_t^h \omega_{\mathcal{H}}))_s = U_s \tau_h(\omega^{\mathcal{H}})$$

*Proof.* The first part is simply Girsanov theorem in the form of [36, Lemma 8.2]. The second part comes from 4.2.13 and is similar to [37, Theorem 7.28]. □

We now turn to the discussion of the stochastic flow generated by $p_{\tau^h}$.

**Notation 4.6.10.** *For a fixed $h \in \mathcal{CM}_{\mathcal{H}}(\mathbb{R}^{n+m})$ we denote by $S\mathbb{M}_{\mathcal{H}}(h)$ the space of continuous and $\mathcal{B}$-adapted $\mathbb{R}^n$-valued semimartingales $z$ that can be written as*

$$z_t = \int_0^t a_s ds + \int_0^t \sigma_s d\omega_s^{\mathcal{H}}, \quad 0 \leqslant t \leqslant 1,$$

*where $a$ is a $\mathbb{R}^n$-valued progressively $\mathcal{B}$-measurable process such that there exists a deterministic constant $C$ such that:*

$$|a_t|_{\mathbb{R}^n} \leqslant C(1 + |h'(t)|_{\mathbb{R}^n}),$$

*and where $\sigma$ is a progressively $\mathcal{B}$-measurable process taking values in the space of isometries of $\mathbb{R}^n$.*

Observe that by the Girsanov theorem in the form of [36, Lemma 8.2], the law of $z \in S\mathbb{M}_{\mathcal{H}}(h)$ is equivalent to the law $\mu_{\mathcal{H}}$ of the horizontal Brownian motion. We are now in position to prove that $p_{\tau^h}$ generates a flow in the horizontal path space for which the horizontal Wiener measure on $\mathbb{R}^{n+m}$ is quasi-invariant. The following statement is similar to in [72, Theorem 3.1] which we present in [12].

**Theorem 4.6.11.** *Let $h \in \mathcal{CM}_{\mathcal{H}}(\mathbb{R}^{n+m})$. There exists a unique family of semimartingales $\{\nu_t^h, t \in \mathbb{R}\}$ such that*

- *$\nu_t^h \in S\mathbb{M}_{\mathcal{H}}(h)$ for all $t \in \mathbb{R}$ and $\nu_0^h \omega^{\mathcal{H}} = \omega^{\mathcal{H}}$, $\mu_{\mathcal{H}}$ a.e.; hence the law of $\nu_t^h$ is equivalent to $\mu_{\mathcal{H}}$;*

- *For $\mu_{\mathcal{H}}$-almost every $\omega^{\mathcal{H}}$, the function $t \to \nu_t^h \omega^{\mathcal{H}}$ is a $C_0(\mathbb{R}^n)$-valued continuous function;*
- *$\mu_{\mathcal{H}}$-almost surely, $\nu_{t_1}^h \circ \nu_{t_2}^h(\omega^{\mathcal{H}}) = \nu_{t_1+t_2}^h(\omega^{\mathcal{H}})$, for every $(t_1, t_2) \in \mathbb{R} \times \mathbb{R}$;*
- *There exists a continuous version of $\{p_{\tau^h \nu_t^h}(\nu_t^h), t \in \mathbb{R}\}$ such that $\mu_{\mathcal{H}}$-almost surely, $\{\nu_t^h, t \in \mathbb{R}\}$ satisfies the equation*

$$\nu_t^h(\omega^{\mathcal{H}}) = \omega^{\mathcal{H}} + \int_0^t p_{\tau^h(\nu_s^h(\omega^{\mathcal{H}}))}(\nu_s^h(\omega^{\mathcal{H}}))ds. \tag{4.6.4}$$

**Remark 4.6.12.** In the previous theorem, the word unique is understood in the sense of Proposition 3.3 in [72].

We are now finally in position to prove quasi-invariance properties for the horizontal Wiener measure with respect to a suitable flow. The following statement is similar to [72, Theorem 4.1].

**Theorem 4.6.13** (Quasi-invariance of the horizontal Wiener measure)**.** *Let $h \in \mathcal{CM}_{\mathcal{H}}(\mathbb{R}^{n+m})$. The $\mu_W$ a.s. well-defined flow $\zeta_t^h = \phi_{\mathcal{H}} \circ \nu_t^h \circ \phi_{\mathcal{H}}^{-1} : C_{x_0}(\mathbb{M}) \to C_{x_0}(\mathbb{M})$, $t \in \mathbb{R}$, is generated by $U\tau^h \phi_{\mathcal{H}}^{-1}$ and for every $t \in \mathbb{R}$ the distribution of $\zeta_t^h$ under $\mu_W$ is equivalent to $\mu_W$. More precisely, there exists a family of measurable maps*

$$\zeta_t^h : C_{x_0}(\mathbb{M}) \to C_{x_0}(\mathbb{M}), \quad t \in \mathbb{R},$$

*with the following properties.*

- *For every fixed $t \in \mathbb{R}$, the law $\mu_{\zeta_t^h}$ of $\zeta_t^h$ is equivalent to the horizontal Wiener* *measure $\mu_W$ and the Radon-Nikodym derivative is given by*

$$\frac{d\mu_{\zeta_t^h}}{d\mu_X}(w) = \frac{d\mu_{\nu_t^h}}{d\mu_{\mathcal{H}}}(\phi_{\mathcal{H}}^{-1} w), \quad w \in C_{x_0}(\mathbb{M}).$$

- *For $\mu_W$-almost every $w \in C_{x_0}(\mathbb{M})$, the function $t \mapsto \zeta_t^h w$ is a $C_{x_0}(\mathbb{M})$-valued continuous differentiable function;*

- *For $\mu_W$-almost every $w \in C_{x_0}(\mathbb{M})$, there is a continuous version of $t \mapsto U_t \tau^h(\phi_{\mathcal{H}}^{-1} \zeta_t^h w)$ such that $\zeta_v^t w$ satisfies the differential equation*

$$\frac{d\zeta_t^h w}{dt} = U_t \tau^h(\phi_{\mathcal{H}}^{-1} \zeta_t^h w);$$

- *$\mu_W$-almost surely,*

$$\zeta_{t_1}^h \circ \zeta_{t_2}^h = \zeta_{t_1+t_2}^h, \quad \textit{for all} \ \ (t_1, t_2) \in \mathbb{R} \times \mathbb{R}.$$

*Proof.* The result follows from Theorem 4.6.11. For details, we refer to the proof of Theorem 4.1 in [72]. □

**Remark 4.6.14.** It is well known that a quasi-invariance result yields an integration by parts formula (as we proved in the previous section) on the path space of the underlying diffusion, see B. Driver [36] and then E. Hsu [72] (see also [31, 32, 44]).

As an direct corollary of our Theorem 4.6.9, we can prove the following integration by parts formula following the proof of Theorem 7.32 in [37]. Then similar to what we have done before, we use induction on $n$ for the cylinder function.

**Lemma 4.6.15.** *Let $h \in \mathcal{CM}_{\mathcal{H}}(\mathbb{R}^{n+m})$. For $f \in C^\infty(\mathbb{M})$,*

$$\mathbb{E}\left(\langle df(W_1), U_1 \tau_h(\omega^{\mathcal{H}})\rangle\right) = \mathbb{E}\left(f(W_1)\int_0^1 \left\langle h'(t) + \frac{1}{2}(\mathfrak{Ric}_{\mathcal{H}})_{U_t} h(t), d\omega_t^{\mathcal{H}}\right\rangle_{\mathbb{R}^n}\right),$$

*where $\mathbb{E}$ is computed under $\mu_{\mathcal{H}}$ and.$\mathfrak{Ric}_{\mathcal{H}}$ is the horizontal Ricci curvature of the Bott connection (seen as an operator on $\mathbb{R}^n$) .*

## 4.7 Examples

### 4.7.1 Riemannian submersions.

To compare our result in the foliation setting to the standard Riemannian manifold setting. We look at our result in the case where the foliation on $\mathbb{M}$ comes from a totally geodesic submersion $\pi : (\mathbb{M}, g) \to (\mathbb{B}, j)$ (see Example 3.1.1). This will recover the classical Riemannian result by B. Driver [36] from our quasi-invariance result.

In the submersion setting, the horizontal lift of curves on the base manifold is crucial.

**Definition 4.7.1.** Let $\bar{\gamma} : [0, +\infty) \to \mathbb{B}$ be a $C^1$ curve. Let $x \in \mathbb{M}$, such that $\pi(x) = \gamma(0)$. Then, there exists a unique $C^1$ horizontal curve $\gamma : [0, +\infty) \to \mathbb{M}$ such that $\gamma(0) = x$ and $\pi(\gamma(t)) = \bar{\gamma}(t)$. The curve $\gamma$ is called the horizontal lift of $\bar{\gamma}$ at $x$.

The above definition of horizontal lift may classically be extended to Brownian motion paths on $\mathbb{B}$ by using stochastic calculus. (We can follow the similar framework from [36, Theorem 3.2] where the horizontal stochastic lift is the lift of the Brownian motion of a Riemannian manifold to the orthonormal frame bundle.

Since the submersion has totally geodesic fibers, $\pi$ is harmonic and the projected process:

$$W_t^{\mathbb{B}} = \pi(W_t)$$

is, under $\mu_{\mathcal{H}}$, a Riemannian Brownian motion on $\mathbb{B}$ started at $\pi(x_0)$. The submersion $\pi$ induces a map $C_{x_0}(\mathbb{M}) \to C_{\pi(x_0)}(\mathbb{B})$ that we still denote by $\pi$. Let now $h$ be a Cameron-Martin path in $\mathbb{R}^n$ and consider the $\mu_{\mathcal{H}}$ a.s. well-defined flow $\zeta_t^h : C_{x_0}(\mathbb{M}) \to C_{x_0}(\mathbb{M})$, $t \in \mathbb{R}$, defined in Theorem 4.6.13. By using the ($\mu_{\mathcal{H}}$ a.s. well defined) horizontal stochastic lift $H : C_{\pi(x_0)}(\mathbb{B}) \to C_x(\mathbb{M})$, one can then construct a unique $\mu_{\mathcal{H}}$ a.s. well-defined flow $\tilde{\zeta}_t^h : C_{\pi(x_0)}(\mathbb{B}) \to C_{\pi(x_0)}(\mathbb{B})$, $t \in \mathbb{R}$, so that we have

$$H \circ \tilde{\zeta}_t^h \circ \pi = \zeta_t^h,$$

for a diagram presentation, see [12, Section 5.5]. (Recall, in the classical Riemannian setting with stochastic horizontal lift to the orthonormal frame bundle, we actually have the same relation but the flow $\zeta_t^h$ will be the flow on the orthonormal frame bundle.)

From Theorem 4.6.13, the flow $\tilde{\zeta}_t^h$ lets the law of $W^{\mathbb{B}}$ quasi-invariant. Since the connection $\widehat{\nabla}^\varepsilon$ projects down to the Levi-Civita connection on $\mathbb{B}$, the flow $\tilde{\zeta}_t^h$ provides a version of the flow considered by E. Hsu [72, Theorem 4.1]. We recover therefore the Driver's quasi-invariance result [36] on the manifold $\mathbb{B}$.

### 4.7.2 Heisenberg group

We first further explain our above Riemannian submersion example to a computable example, which is for the previous example in Chapter 3 Heisenberg group 3.6.1. In that example, the Bott connection is trivial: $\nabla X_i = \nabla Y_j = \nabla Z = 0$ and its torsion is given by

$$T(X_i, Y_j) = -2\delta_{ij} Z, \quad T(X_i, Z) = T(Y_i, Z) = 0.$$

Let now $C_0(\mathbb{R}^{2n})$ be the Wiener space of continuous functions $[0,1] \to \mathbb{R}^{2n}$ vanishing at 0. We denote by $(B_t, \beta_t)_{0 \leqslant t \leqslant 1}$ the coordinate maps on $C_0(\mathbb{R}^{2n})$ and by $\mu_{\mathcal{H}}$ the Wiener measure on $C_0(\mathbb{R}^{2n})$, so that $(B_t, \beta_t)_{0 \leqslant t \leqslant 1}$ is a $2n$-dimensional Brownian motion under $\mu_{\mathcal{H}}$. By using the submersion $\pi$, the Brownian motion $(B_t, \beta_t)_{0 \leqslant t \leqslant 1}$ can be horizontally lifted to the horizontal Brownian motion on $\mathbb{H}^{2n+1}$ which is given explicitly by

$$W_t = \left( B_t, \beta_t, \sum_{i=1}^n \int_0^t B_t^i d\beta_t^i - \beta_t^i dB_t^i \right).$$

Let $h = (h_1, h_2)$ be a Cameron-Martin path in $\mathbb{R}^{2n}$ and consider the Cameron-Martin flow $\tilde{\zeta}_t^h : C_0(\mathbb{R}^{2n}) \to C_0(\mathbb{R}^{2n})$, $t \in \mathbb{R}$, explicitly given by

$$\tilde{\zeta}_t^h(B, \beta) = (B, \beta) + th, namely$$

$$\zeta_t^h(W) = \left( B + th_1, \beta + th_2, \sum_{i=1}^n \int_0^{\cdot} (B_u^i + th_1^i(u)) d(\beta_u^i + th_2^i(u)) - (\beta_u^i + th_2^i(u)) d(B_u^i + th_1^i(u)) \right).$$

One can compute the generator of this flow: (see details in [12, Section 5])

$$\begin{aligned} &\frac{d}{dt}|_{t=0} \zeta_t^h(W) \\ =& \sum_{i=1} h_1^i X_i(W) + \sum_{i=1}^n h_2^i Y_i(W) \\ &+ \int_0^{\cdot} T\left( \sum_{i=1}^n X_i \circ dB_u^i + \sum_{i=1}^n Y_i \circ d\beta_u^i, \sum_{i=1}^n h_1^i(u) X_i + \sum_{i=1}^n h_2^i(u) Y_i \right) \end{aligned}$$

This is nothing else but the formula 4.6.1 written in the parallel frame $\{X_i, Y_j, Z\}$.

### 4.7.3 K-contact manifold

Now we compute the tangent process for our K-contact manifold, Example 3.6.5. We assume that the Riemannian foliation on $\mathbb{M}$ is the Reeb foliation of a K-contact structure. The Reeb vector field on $\mathbb{M}$ will be denoted by $R$ and the almost complex structure by $\mathbf{J}$. The torsion of the Bott connection is then

$$T(X,Y) = \langle \mathbf{J}X, Y\rangle_{\mathcal{H}} R.$$

Therefore with the previous notation, one has

$$J_Z X = \langle Z, R\rangle \mathbf{J} X.$$

and the vertical part of a tangent process is given by

$$\begin{aligned} v_{\mathcal{V}}(t) &= -\int_0^t (\widehat{\Theta}^{\varepsilon}_s)^{-1} T(\widehat{\Theta}^{\varepsilon}_s \circ dB_s, \widehat{\Theta}^{\varepsilon}_s v_{\mathcal{H}}(s)) \\ &= \int_0^t ((\widehat{\Theta}^{\varepsilon}_s)^{-1} R) \langle \mathbf{J} \widehat{\Theta}^{\varepsilon}_s v_{\mathcal{H}}(s), \widehat{\Theta}^{\varepsilon}_s \circ dB_s\rangle_{\mathcal{H}}. \end{aligned}$$

### 4.7.4 Lie group $G(\rho)$ case

In this part, we look at 3-dimensional Lie group model spaces. Compute the horizontal Brownian motion on these spaces and the tangent process $v \in TW_{\mathcal{H}}(\mathbb{M})$ which generate flow $\widehat{\Theta}^{\varepsilon} v$ under which we have quasi-invariance of the horizontal Wiener measure. In another word, we have $v(s) = \frac{d}{dt}|_{t=0} \zeta^h_t(W)(s)$. In particular, we use another method to do the computations. We will compute the stochastic parallel transport and have the form of horizontal Brownian motion.

**Example 4.7.2.** Given a number $\rho \in \mathbb{R}$, suppose that $G(\rho)$ is simply a connected three-dimensional Lie group whose Lie algebra $\mathfrak{g}$ admits a basis $\{X, Y, Z\}$ satisfying

(i) $[X, Y] = Z$,

(ii) $[X, Z] = -\rho Y$,

(iii) $[Y, Z] = \rho X$.

Then for $\rho = 0$, $G(\rho)$ is the Heisenberg group with $n = 1$ (see Example 3.6.1); For $\rho = 1$, $G(\rho)$ is $\mathbf{SU}(2)$, and for $\rho = -1$, $G(\rho)$ is $\mathbf{SL}(2)$. Following our work in the previous section, the horizontal distribution $\mathcal{H}$ is generated by $\{X, Y\}$ and the vertical distribution is generated by $\{Z\}$. Consider $\{\theta_1, \theta_2, \nu\}$ as the dual basis of $\{X, Y, Z\}$, then for every one form $\eta$, we can write $\eta = f_1\theta_1 + f_2\theta_2 + g\nu$. We then can identify $\eta$ with a column vector

$$\begin{pmatrix} f_1 \\ f_2 \\ g \end{pmatrix}. \tag{4.7.1}$$

As for the one-parameter family of the Riemannian metric $g_\varepsilon$ on $G(\rho)$, the adjoint damped connection is given by

$$\hat{\nabla}^\varepsilon_X Y := \nabla^\varepsilon_X Y - T^\varepsilon(X, Y) = \nabla_X Y + \frac{1}{\varepsilon} J_X Y,$$

together with the structure coefficients for the Lie bracket, we then have

$$\mathbf{J}_Z(X) = -Y, \quad \mathbf{J}_Z(Y) = X, \quad \mathbf{J}_X = \mathbf{J}_Y = 0$$

so we get

$$\frac{1}{\varepsilon}\mathbf{J} = \begin{pmatrix} 0 & \frac{1}{\varepsilon} & 0 \\ -\frac{1}{\varepsilon} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In the following, we then consider $\rho = 0$ (or $\rho \neq 0$) with $\varepsilon \to \infty$, namely two different cases. However, we will not deal with the case with $\varepsilon \not\to \infty$, namely $\varepsilon$ as a fixed constant, since the computation is more complicated.

**$\mathfrak{T}^{\infty}$ and $\rho = 0$ case**

This is the Heisenberg group case for $\rho = 0$. And the adjoint damped connection is the same as the Bott connection after taking $\varepsilon \to \infty$. Follow the similar computations as in [10], by taking $\varepsilon \to \infty$, we then have the following:

$$\mathfrak{Ric}_{\mathcal{H}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathfrak{T}^{\infty}_X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathfrak{T}^{\infty}_Y = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.7.2}$$

where the above $\mathfrak{T}^{\infty}_X$ and $\mathfrak{T}^{\infty}_Y$ are actions on one-forms. By using the Lie bracket condition in 4.7.2 for $\rho = 0$, we then have

$$\nabla_X Y = \nabla_X Z = \nabla_Y X = \nabla_Y Z = \nabla_Z X = \nabla_Z Y = 0$$

Recall our definition of stochastic parallel transport, since the initial condition for $\widehat{\Theta}^{\infty}$ is the identity matrix so we know that

$$\widehat{\Theta}_t^{\infty} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.7.3}$$

Recall that the anti-development of our horizontal Brownian motion $(W_t)_{0\leq t\leq 1}$,

$$B_t = \int_0^t \widehat{\Theta}_t^{\infty} \circ dW_s,$$

is a Brownian motion in the horizontal space $\mathcal{H}_x$. If we denot $B_t = (B_t^1, B_t^2, 0)$ as the Brownian motion on the horizontal space $\mathcal{H}_x$. Then we know $W_t^1 = B_t^1,\ W_t^2 = B_t^2$ and input 4.7.2 in our definition for $\tau_t^{\infty}$, for any one-form $\alpha = \alpha_1\theta_1 + \alpha_2\theta_2 + \alpha_3\nu$ we have the following

$$d[\tau_t^{\infty}\alpha(W_t)] = \tau_t^{\infty}\left(\nabla_{\circ dX_t} - \mathfrak{T}_{\circ dW_t}^{\infty}\right)\alpha(W_t), \quad \tau_0^{\infty} = \mathbf{Id}, \tag{4.7.4}$$

Recall that $\{\theta_1, \theta_2, \nu\}$ is the dual basis of $\{X, Y, Z\}$, then for any one form $\alpha(W_t) = \alpha_1\theta_1 + \alpha_2\theta_2 + \alpha_3\nu$, we have

$$(\nabla_X - \mathfrak{T}_X^{\infty})\alpha(X_t) = -\alpha_1\mathfrak{T}_X^{\infty}\theta_1 - \alpha_2\mathfrak{T}_X^{\infty}\theta_2 - \alpha_3\mathfrak{T}_X^{\infty}\nu = -\alpha_3\theta_2$$

$$(\nabla_Y - \mathfrak{T}_Y^{\infty})\alpha(X_t) = -\alpha_1\mathfrak{T}_Y^{\infty}\theta_1 - \alpha_2\mathfrak{T}_Y^{\infty}\theta_2 - \alpha_3\mathfrak{T}_Y^{\infty}\nu = \alpha_3\theta_1$$

so we then get

$$d\tau_t^{\infty}\alpha(X_t) = \tau_t^{\infty}(-\alpha_3\theta_2 \circ dB_t^1 + \alpha_3\theta_1 \circ dB_t^2) \tag{4.7.5}$$

namely we have

$$d\tau_t^{\infty}\theta_1(X_t) = 0, \quad d\tau_t^{\infty}\theta_2(X_t) = 0 \tag{4.7.6}$$

$$d\tau_t^{\infty}\nu(X_t) = \tau_t^{\infty}(-\theta_2 \circ dB_t^1 + \theta_1 \circ dB_t^2) \tag{4.7.7}$$

thus we know that

$$\tau_t^{\infty}\theta_1(X_t) = \theta_1(X_0), \quad \tau_t^{\infty}\theta_2(X_t) = \theta_2(X_0) \tag{4.7.8}$$

$$\tau_t^{\infty}\nu(X_t) = -\theta_2 B_t^1 + \theta_1 B_t^2 + \nu \tag{4.7.9}$$

which means that

$$\tau_t^{\infty} = \begin{pmatrix} 1 & 0 & B_t^2 \\ 0 & 1 & -B_t^1 \\ 0 & 0 & 1 \end{pmatrix} \quad \tau_t^{\infty,*} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ B_t^2 & -B_t^1 & 1 \end{pmatrix}, \quad (\tau_t^{\infty,*})^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -B_t^2 & B_t^1 & 1 \end{pmatrix} \tag{4.7.10}$$

the tangent process tangent to the horizontal Brownian motion can be computed from the integration by parts formula in the following form, where $\gamma \in \mathcal{CM}_{\mathcal{H}}(\mathbb{M})$.

$$v(t) = \tau_t^{\infty,*} \int_0^t (\tau_s^{\infty,*})^{-1} \widehat{\Theta}^{\infty} \gamma'(s) ds$$

thus according to the above computations, we have

$$v(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ B_t^2 & -B_t^1 & 1 \end{pmatrix} \times \begin{pmatrix} \gamma(t)_1 \\ \gamma(t)_2 \\ B_t^1\gamma(t)_2 - B_t^2\gamma(t)_1 + \int_0^t \gamma(s)_1 dB_s^2 - \int_0^t \gamma(s)_2 dB_s^1 \end{pmatrix} \tag{4.7.11}$$

if we assume $\gamma(0)_i = 0$ for $i = 1, 2, 3$, we then have a nice representation formula for $v(t)$ in the following form.

$$v(t) = \begin{pmatrix} \gamma(t)_1 \\ \gamma(t)_2 \\ \int_0^t \gamma(s)_1 dB_s^2 - \int_0^t \gamma(s)_2 dB_s^1 \end{pmatrix} \tag{4.7.12}$$

This family of tangent processes is the same as we we did horizontal variations for the Heisenberg group in the early example by taking $n = 1$.

### $\mathfrak{T}^\infty$ and $\rho \neq 0$ case

For $\rho \neq 0$, we still follow the same procedure in the $\rho = 0$ case, since everything is the same except that condition 4.7.2 will give us non-zero $\mathfrak{Ric}_{\mathcal{H}}$, namely

$$\mathfrak{Ric}_{\mathcal{H}} = \begin{pmatrix} \rho & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.7.13}$$

and the stochastic differential equation for $\tau_t^\infty$ is,

$$d[\tau_t^\infty \alpha(X_t)] = \tau_t^\infty \left( \nabla_{\circ dX_t} - \mathfrak{T}^\infty_{\circ dX_t} - \frac{1}{2}\mathfrak{Ric}_{\mathcal{H}} dt \right) \alpha(X_t), \quad \tau_0^\infty = \mathbf{Id}, \tag{4.7.14}$$

we thus get the following,

$$d\tau_t^{\infty}\theta_1(X_t) = -\frac{\rho}{2}\tau_t^{\infty}\theta_1(X_t)dt, \quad d\tau_t^{\infty}\theta_2(X_t) = -\frac{\rho}{2}\tau_t^{\infty}\theta_2(X_t)dt \tag{4.7.15}$$

$$d\tau_t^{\infty}\nu(X_t) = \tau_t^{\infty}(-\theta_2 \circ dB_t^1 + \theta_1 \circ dB_t^2) \tag{4.7.16}$$

by solving the above equations, we have

$$\tau_t^{\infty}\theta_1(X_t) = e^{-\frac{\rho}{2}t}\theta_1, \quad \tau_t^{\infty}\theta_2(X_t) = e^{-\frac{\rho}{2}t}\theta_2 \tag{4.7.17}$$

$$\tau_t^{\infty}\nu(X_t) = \int_0^t e^{-\frac{\rho}{2}s}dB_s^2\theta_1 - \int_0^t e^{-\frac{\rho}{2}s}dB_s^1\theta_2 + \nu \tag{4.7.18}$$

according to the above three solutions, we then have

$$\tau_t^{\infty} = \begin{pmatrix} e^{-\frac{\rho}{2}t} & 0 & \int_0^t e^{-\frac{\rho}{2}s}dB_s^2 \\ 0 & e^{-\frac{\rho}{2}t} & -\int_0^t e^{-\frac{\rho}{2}s}dB_s^1 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.7.19}$$

thus we have

$$\tau_t^{\infty,*} = \begin{pmatrix} e^{-\frac{\rho}{2}t} & 0 & 0 \\ 0 & e^{-\frac{\rho}{2}t} & 0 \\ \int_0^t e^{-\frac{\rho}{2}s}dB_s^2 & -\int_0^t e^{-\frac{\rho}{2}s}dB_s^1 & 1 \end{pmatrix}, \quad (\tau_t^{\infty,*})^{-1} = e^{\rho t}\begin{pmatrix} e^{-\frac{\rho}{2}t} & 0 & 0 \\ 0 & e^{-\frac{\rho}{2}t} & 0 \\ -\int_0^t e^{-\frac{\rho}{2}s}dB_s^2 & \int_0^t e^{-\frac{\rho}{2}s}dB_s^1 & e^{-\rho t} \end{pmatrix} \tag{4.7.20}$$

Plug in the above solution into the formula for $v(t) \in TW_{\mathcal{H}}(\mathbb{M})$, we have

$$v(t) = \begin{pmatrix} e^{-\frac{\rho}{2}t} & 0 & 0 \\ 0 & e^{-\frac{\rho}{2}t} & 0 \\ \int_0^t e^{-\frac{\rho}{2}s} dB_s^2 & -\int_0^t e^{-\frac{\rho}{2}s} dB_s^1 & 1 \end{pmatrix} \times \underbrace{\int_0^t \begin{pmatrix} e^{\frac{\rho}{2}s}\gamma'(s)_1 \\ e^{\frac{\rho}{2}s}\gamma'(s)_2 \\ -e^{\rho s}\gamma'(s)_1 \int_0^s e^{-\frac{\rho}{2}\tau} dB_\tau^2 + e^{\rho s}\gamma'(s)_2 \int_0^s e^{-\frac{\rho}{2}\tau} dB_\tau^1 \end{pmatrix} ds}_{Q_t}$$

by using the integration by parts formula for the above integral part, we have

$$Q_t = \begin{pmatrix} e^{\frac{\rho}{2}t}\gamma(t)_1 - \frac{\rho}{2}\int_0^t \gamma(s)_1 e^{\frac{\rho}{2}s} ds \\ e^{\frac{\rho}{2}t}\gamma(t)_2 - \frac{\rho}{2}\int_0^t \gamma(s)_2 e^{\frac{\rho}{2}s} ds \\ A_t + B_t \end{pmatrix} \tag{4.7.21}$$

where

$$A_t = -e^{\frac{\rho}{2}t}\gamma(t)_1 \int_0^t e^{-\frac{\rho}{2}s} dB_s^2 + \int_0^t \gamma(s)_1 dB_s^2 + \frac{\rho}{2}\int_0^t \gamma(s)_1 e^{\frac{\rho}{2}s} \int_0^s e^{-\frac{\rho}{2}\tau} dB_\tau^2 ds \tag{4.7.22}$$

$$B_t = e^{\frac{\rho}{2}t}\gamma(t)_2 \int_0^t e^{-\frac{\rho}{2}s} dB_s^1 - \int_0^t \gamma(s)_2 dB_s^1 - \frac{\rho}{2}\int_0^t \gamma(s)_2 e^{\frac{\rho}{2}s} \int_0^s e^{-\frac{\rho}{2}\tau} dB_\tau^1 ds \tag{4.7.23}$$

thus further computation gives us the following expresion for $v(t)$

$$v(t) = \begin{pmatrix} \gamma(t)_1 - \frac{\rho}{2} e^{-\frac{\rho}{2}t} \int_0^t \gamma(s)_1 e^{\frac{\rho}{2}} ds \\ \gamma(t)_2 - \frac{\rho}{2} e^{-\frac{\rho}{2}t} \int_0^t \gamma(s)_2 e^{\frac{\rho}{2}} ds \\ \int_0^t \gamma(s)_1 dB_s^2 - \int_0^t \gamma(s)_2 dB_s^1 + C_t \end{pmatrix} \tag{4.7.24}$$

where

$$C_t = \frac{\rho}{2} \int_0^t \gamma(s)_1 e^{\frac{\rho}{2}s} \int_0^s e^{-\frac{\rho}{2}\tau} dB_\tau^2 ds - \frac{\rho}{2} \int_0^t \gamma(s)_2 e^{\frac{\rho}{2}s} \int_0^s e^{-\frac{\rho}{2}\tau} dB_\tau^1 ds \tag{4.7.25}$$

$$-\frac{\rho}{2} \int_0^t e^{-\frac{\rho}{2}s} dB_s^2 \int_0^t \gamma(s)_1 e^{\frac{\rho}{2}s} ds + \frac{\rho}{2} \int_0^t e^{-\frac{\rho}{2}s} dB_s^1 \int_0^t \gamma(s)_2 e^{\frac{\rho}{2}s} ds$$

**Remark 4.7.3.** In particular, if we take $\rho = 1$, this gives us the tangent process on $\mathbf{SU}(2)$, namely, this is the tangent process on the Hopf fibration $S^1 \to S^3 \to S^2$ in Example 3.6.3, which can generate a flow such that, under which the Wiener measure of the horizontal Brownian motion on the Hopf fibration is quasi-invariant.

# Chapter 5

# Ricci flow on Riemannian foliations

In this chapter, we consider $\mathbb{M}$ as complete general Riemannian foliations with totally geodesic leaves and bundle-like metrics. The main results are presented in [49, 51]. We will study a pair of equations which are the transverse Ricci flow for the *Bott* connection

$$\begin{cases} \frac{\partial g_{\mathcal{H}}}{\partial t} = -2\mathbf{Ric}_{\mathcal{H}}, \\ \frac{\partial g_{\mathcal{V}}}{\partial t} = 0, \end{cases} \tag{5.0.1}$$

and the heat equation associated with the time dependent horizontal Laplacian operator $L^t$ $(= \Delta_{g_{\mathcal{H}(t)}})$ for metric $g(t) = g_{\mathcal{H}}(t) \oplus g_{\mathcal{V}}$.

$$(L^t - \frac{\partial}{\partial t})u(x,t) = 0, \quad x \in \mathbb{M}, t \in [0,T]. \tag{5.0.2}$$

Since our metric $g(t)$ evolves in time, similar to our previous Lemma 3.5.1, we can actually prove that there exists orthonormal frames depending on time $t$, namely a time dependent version of Lemma 3.5.1( see Lemma 5.1.3 below). We denote

$\{X_1(t), \cdots, X_n(t)\}$ as the horizontal time dependent orthonormal frames and $\{Z_1, \cdots, Z_m\}$ as the vertical time independent orthonormal frames. In particular, we can represent a generic one-form as $\eta = \sum_{i=1}^n f_i\theta_i(t) + \sum_{j=1}^m k_j v_j$, where $\{\theta_1(t), \cdots, \theta_n(t), v_1, \cdots, v_m\}$ is the dual coframe of $\{X_1(t), \cdots, X_n(t), Z_1, \cdots, Z_m\}$, which is similar to the static case as we did in the previous chapters.

Then for metric $g(t)$ and the associated *Bott* connection $\nabla^t$, we have the following time dependent horizontal laplacian operator

$$L^t = -\nabla^*_{g_{\mathcal{H}}(t)}\nabla_{g_{\mathcal{H}}(t)}(\text{or} = -\nabla^*_{\mathcal{H}}\nabla_{\mathcal{H}}),$$

by using the local coordinates from Lemma 5.1.3, we can represent $L^t$ by

$$L^t = \sum_{i=1}^n \nabla^t_{X_i(t)}\nabla^t_{X_i(t)} - \nabla^t_{\nabla^t_{X_i(t)}X_i(t)}.$$

Recall our definition of $J$ in chapter 3. If $\{Z_1, \cdots, Z_m\}$ is a local vertical frame, then $J$ define a $(1,1)$ tensor which is also the same as in the static case,

$$\mathbf{J}^2 := \sum_{j=1}^m J_{Z_j}J_{Z_j}.$$

This does not depend on the choice of the frame and may be defined globally. The horizontal divergence of the torsion $T$ is also a $(1,1)$ tensor which in a local horizontal frame $\{X_1(t), \cdots, X_n(t)\}$ is defined as

$$\delta_{\mathcal{H}}T(X) := -\sum_{i=1}^n (\nabla^t_{X_i(t)}T)(X_i(t), X).$$

In particular, for a generic one form $\eta = \sum_{i=1}^n f_i\theta_i(t) + \sum_{j=1}^m k_j v_j$, we have

$$\delta_{\mathcal{H}} T(\eta) = \sum_{i,j=1}^n \sum_{l=1}^m (X_i(t)\gamma_{ij}^l(t)) f_j v_l,$$

where $\gamma_{ij}^l(t)$ is from the structure equation in Lemma 5.1.3. Recall that, if $\delta_{\mathcal{H}} T(\cdot) = 0$, we say that it satisfies the *Yang-Mills* condition. With all the above definition in hand, we are ready to introduce the following time dependent $\square_\varepsilon$ operator

$$\square_\varepsilon^t = -(\nabla_{g_{\mathcal{H}}(t)} - \mathfrak{T}^\varepsilon_{g_{\mathcal{H}}(t)})^*(\nabla_{g_{\mathcal{H}}(t)} - \mathfrak{T}^\varepsilon_{g_{\mathcal{H}}(t)}) - \frac{1}{\varepsilon}\mathbf{J}^2(t) + \frac{1}{\varepsilon}\delta_{\mathcal{H}} T(t) - \mathfrak{Ric}_{g_{\mathcal{H}}(t)}.$$

Here we denote $\mathbf{J}^2(t)$ and $\delta_{\mathcal{H}} T(t)$ to emphasize their dependence on time in the current setting, although the local representation of $\mathbf{J}^2$ is time independent.

Now we introduce the *carré du champ* operator [6] associated with the time dependent horizontal Laplacian operator $L^t$. For $f, g \in \mathcal{C}^\infty(\mathbb{M})$,

$$\begin{aligned} \Gamma^t(f,g) &= \frac{1}{2}(L^t(fg) - fL^t g - gL^t f) = g_{\mathcal{H}}(\nabla_{g_{\mathcal{H}}(t)} f, \nabla_{g_{\mathcal{H}}(t)} g) = \langle df, dg\rangle_{g_{\mathcal{H}}(t)}, \\ \Gamma^{\mathcal{V}}(f,g) &= g_{\mathcal{V}}(\nabla_{\mathcal{V}} f, \nabla_{\mathcal{V}} g) = \langle df, dg\rangle_{g_{\mathcal{V}}}. \end{aligned} \tag{5.0.3}$$

Their iteration are defined as

$$\begin{aligned} \Gamma_2^t(f,g) &= \frac{1}{2}(L^t\Gamma^t(f,g) - \Gamma^t(L^t f, g) - \Gamma^t(f, L^t g)), \\ \Gamma_2^{\mathcal{V}}(f,g) &= \frac{1}{2}(L^t\Gamma_{\mathcal{V}}(f,g) - \Gamma_{\mathcal{V}}(L^t f, g) - \Gamma_{\mathcal{V}}(f, L^t g)). \end{aligned} \tag{5.0.4}$$

Both $\square_\varepsilon^t$ and the *carré du champ* operators will play important role in the Bochner's identity and the generalized curvature dimension inequality in the following sections.

**Remark 5.0.1.** Note that, $\Gamma^t(f,g)$ depends on time $t$ only for the horizontal metric, thus when we take time derivative of $\Gamma^t(f,g)$, we will have $\frac{\partial}{\partial t}(\Gamma^t(f,g)) = \frac{\partial g_{\mathcal{H}(t)}}{\partial t}(df, dg) = \frac{\partial g_{\mathcal{H}(t)}}{\partial t}(\nabla_{g_{\mathcal{H}(t)}} f, \nabla_{g_{\mathcal{H}(t)}} g)$, which is the same as in the Riemannian setting for the gradient of a function.

## 5.1 Structure under transverse Ricci flow

### 5.1.1 General setting for the evolution equaiton

In general, we consider the following transverse geometric flow equation on $(\mathbb{M}^{n+m}, \mathcal{F}, g_0)$,

$$\begin{cases} \frac{\partial}{\partial t} g_{\mathcal{H}}(t) = h(t), \quad g(0) = g_0. \\ \frac{\partial}{\partial t} g_{\mathcal{V}} = 0, \end{cases} \tag{5.1.1}$$

where $h(t)$ satisfies the following properties:

1 . For any $X \in \Gamma^\infty(\mathcal{V})$ or $Z \in \Gamma^\infty(\mathcal{V})$, we have $h(X, Z) = 0$,

2 . $h(t)$ is a symmetric $(0, 2)$ tensor.

Similar consideration can also be found in [105], where the authors studied partial Ricci flow for co-dimension one foliation. In particular, if we consider the specific transverse Ricci flow for $h(t) = -2\mathbf{Ric}_{\mathcal{H}}(t)$ where $\mathbf{Ric}_{\mathcal{H}}(t)$ is the horizontal Ricci curvature for metric $g(t)$. Locally we have the projection $\Pi : U \in \mathbb{M} \to \tilde{U}$, where $\tilde{U}$ is the local Riemannian quotient. If we denote the Ricci flow on the local Riemannian quotient as

$$\frac{\partial g_{\tilde{U}}}{\partial t} = -2\mathbf{Ric}_{\tilde{U}}. \tag{5.1.2}$$

We denote the transverse Ricci flow on $\mathbb{M}$ as

$$\begin{cases} \frac{\partial g_{\mathcal{H}}}{\partial t} = -2\mathbf{Ric}_{\mathcal{H}}, \\ \\ \frac{\partial g_{\mathcal{V}}}{\partial t} = 0. \end{cases} \tag{5.1.3}$$

Then the pull-back $\Pi^*$ of (5.1.2) will give us (5.1.3) directly following from Lemma 5.1.6, since the bundle-like metric will allow us to patch the metric on $\mathbb{M}$ so that the pull-back metric can be globally defined. In particular, if the Riemannian foliation comes from a Riemannian submersion, then the Ricci flow (5.1.2) is just the Ricci flow on the base manifold $\mathbb{B}$. In particular, we will always consider Riemannian submersion as a specific example for our result of Riemannian foliations.

**Remark 5.1.1.** Recall that from Chapter 3, we define

$$\langle \mathfrak{Ric}_{\mathcal{H}}(df), dg\rangle = \mathbf{Ric}\,(\nabla_{\mathcal{H}} f, \nabla_{\mathcal{H}} g) = \mathbf{Ric}_{\mathcal{H}}(\nabla f, \nabla g),$$

where **Ric** is the Ricci curvature of the *Bott* connection and $\mathbf{Ric}_{\mathcal{H}}$ its horizontal Ricci curvature. In the later computations, when we deal with one forms, even though we still consider the evolution equation is $\frac{\partial}{\partial t} g(t) = h(t)$ where $h$ is a $(0,2)$ tensor, but we will also use $\mathfrak{Ric}_{\mathcal{H}}$ on one forms since we we have the above identity, although in a time dependent version. This is not stated clearly in the first version of the paper [49].

We end this section with the following theorem which is the foundation of our later results.

**Theorem 5.1.2.** *The transverse flow* (5.1.1) *and transverse Ricci flow* (5.1.3) *have unique short-time solution.*

*Proof.* See proof in [49, Theorem 4.2] □

### 5.1.2 Time dependent normal frames

Similar to the previous Lemma 3.5.1, we now have a time dependent version.

**Lemma 5.1.3.** *Let $x \in \mathbb{M}$. Around $x$, at any time $t$, there exists a local orthonormal horizontal frame $\{X_1(t), \cdots, X_n(t)\}$ and a local orthonormal vertical frame $\{Z_1, \cdots, Z_m\}$ such that the following structure relations hold*

$$[X_i(t), X_j(t)] = \sum_{k=1}^{n} \omega_{ij}^k(t) X_k(t) + \sum_{k=1}^{m} \gamma_{ij}^k(t) Z_k$$

$$[X_i(t), Z_k] = \sum_{j=1}^{m} \beta_{ik}^j(t) Z_j,$$

*where $\omega_{ij}^k(t), \gamma_{ij}^k(t), \beta_{ik}^j(t)$ are smooth functions in space and time such that:*

$$\beta_{ik}^j(t) = -\beta_{ij}^k(t).$$

*Moreover, at $x$, we have*

$$\omega_{ij}^k(t) = 0, \beta_{ij}^k(t) = 0.$$

*Proof.* See proof in [49, Lemma 4.11]. To point it out, the proof need Lemma 5.1.8 which will be proved in the next section. Namely, transverse flow (5.1.1) preserves the Riemannian submersion structure so that we can have local orthonormal frames. □

We record the fact that in this frame the Christoffel symbols of the Bott connection

$\nabla^t$ are given by

$$\begin{cases} \nabla^t_{X_i(t)} X_j(t) = \frac{1}{2}\sum_{k=1}^n \left(\omega_{ij}^k(t) + \omega_{ki}^j(t) + \omega_{kj}^i(t)\right) X_k(t) \\ \nabla^t_{Z_j} X_i(t) = 0 \\ \nabla^t_{X_i(t)} Z_j = \sum_{k=1}^m \beta_{ij}^k(t) Z_k \end{cases}$$

**Remark 5.1.4.** Similar to the static case, we can also have the above Christoffel symbols for the time dependent adjoint connection of the damped connection as we did in Chapter 3, which will appear in [51], since we do not bother with the long time existence of the transverse flow and the results herein do not need those relations.

### 5.1.3 Bundle-like metrics preserved under the transverse flow

We first show that the time dependent Bott connection is actually a metric connection.

**Lemma 5.1.5.** *The time-dependent Bott connection is a metric connection,which means for any vector fields $X, Y, Z \in T\mathbb{M}$*

$$\nabla^t_X g(Y,Z) = g(\nabla^t_X Y, Z) + g(\nabla^t_X Z, Y) \tag{5.1.4}$$

*Proof.* Since this is true at $t = 0$, it is enough to check

$$\frac{\partial}{\partial t}(\nabla^t_X g(Y,Z)) = \frac{\partial}{\partial t}(g(\nabla^t_X Y, Z) + g(\nabla^t_X Z, Y))$$

at each fixed time $t$, we can find only space dependent vector fields

$$\text{L.H.S} = \frac{\partial}{\partial t}(\nabla^t_X g(Y,Z)) = \nabla^t_X h(Y,Z)$$

$$\begin{aligned}
\text{R.H.S} &= \frac{\partial}{\partial t}(g(\nabla^t_X Y, Z) + g(\nabla^t_X Z, Y)) \\
&= h(\nabla^t_X Y, Z) + h(\nabla^t_X Z, Y) + g(\frac{\partial}{\partial t}\nabla^t_X Y, Z) + g(\frac{\partial}{\partial t}\nabla^t_X Z, Y)
\end{aligned}$$

Since the *Bott* connection is not torsion free, the connection evolving in time has the following format, (the torsion free version can be found in [115], the proof for our situation is similar but more computations).

$$\begin{aligned}
g(\frac{\partial}{\partial t}\nabla^t_X Y, Z) = &\frac{1}{2}\{(\nabla^t_X h)(Y,Z) - (\nabla^t_Z h)(X,Y) + (\nabla^t_Y h)(Z,X)\} \\
&+ \frac{1}{2}\{g(X, \frac{\partial}{\partial t}([Z,Y] + T(Z,Y))) - g(Y, \frac{\partial}{\partial t}([X,Z] + T(X,Z))) \\
&- g(Z, \frac{\partial}{\partial t}([Y,X] + T(Y,X)))\}
\end{aligned}$$

so we simplify the R.H.S by using the above formula

$$\text{R.H.S} = h(\nabla^t_X Y, Z) + h(\nabla^t_X Z, Y) + (\nabla^t_X h)(Y,Z) = \nabla^t_X h(Y,Z)$$

This finishes the proof. □

**Lemma 5.1.6.** *Under the transverse Ricci flow* (5.1.3) (*transverse flow* (5.1.1)), *Riemannian foliation with bundle-like metric is preserved.*

*Proof.* See proof in [49, Lemma 4.6]. □

**Remark 5.1.7.** The proof is actually for the general case for the transverse flow (5.1.1). Then for the transverse Ricci flow (5.1.3), the properties follow directly, since $h(t) = -2\mathbf{Ric}_{\mathcal{H}}$ satisfies our assumptions on $h(t)$.

It is important that in order to prove the bundle-like metrics preserveed for Riemannian foliations under the transver flow, we, of course, need to prove this property for Riemannian submersion first. Namely, we first prove the following local version of our result.

**Lemma 5.1.8.** *If the Riemannian foliation comes from a Riemannian submersion,* $\Pi : \mathbb{M} \to \mathbb{B}$ *with* $\mathcal{F}$ *as the totally geodesic foliation, then the transverse flow* (5.1.1) *preserves the Riemannian submersion structure.*

*Proof.* See proof in [49, Lemma 4.9] □

### 5.1.4 Totally geodesic leaves preserved under the transverse flow

Now we turn to the study of totally geodesic leaves structure of the Riemannian foliations. Recall from the static case, Proposition 3.1.5, in order to prove this proposition, we just need to show that for any horizontal vector field $X$ and for any vertical vector field $Z_1, Z_2$, (where $\mathcal{L}$ is the Lie derivative)

$$(\mathcal{L}_X g)(Z_1, Z_2) = -2g(X, \nabla^R_{Z_1} Z_2) = 0 \quad \text{iff} \quad \nabla^R_{Z_1} Z_2 \quad \text{is always vertical}$$

So in the current setting, we need to prove this argument, however, for time dependent Bott connection $\nabla^t$ and vector field $X(t)$ which is orthogonal to $Z_1, Z_2$. We thus have the following property.

**Lemma 5.1.9.** *Given transverse (Ricci* (5.1.3)*) flow* (5.1.1)*, it will preserve the totally geodesic foliation condition, which means for any time-dependent vector fields* $X_i(t) \in \mathcal{H}$ *and any* $Z_j \in \mathcal{V}$*, any basic time-dependent vector fields induces an isometry between fibers.*

*Proof.* See proof in [49, Lemma 4.13] □

We end this section with the following property.

**Lemma 5.1.10.** *If* $(\mathbb{M}, \mathcal{F}, g_0)$ *satisfies the Yang-Mills condition* $\delta_{\mathcal{H}} T = 0$ *at time* $t = 0$*, then the transverse* (*Ricci* (5.1.3)) *flow* (5.1.1) *preserves the Yang-Mills condition.*

*Proof.* By the definition of $\delta_{\mathcal{H}} T(\cdot)$ and $\mathbf{Ric}(\cdot,\cdot)$, it is easy to check that $\delta_{\mathcal{H}} T(\cdot) = 0$ is equivalent to $\mathbf{Ric}(X, Z) = 0$ if $X \in \mathcal{H}, Z \in \mathcal{V}$. Since $\mathbf{Ric}(X, Z) = 0$ for $X \in \mathcal{H}, Z \in \mathcal{V}$ for any time $t$, so we are done. □

# 5.2 Time dependent generalized curvature dimension inequality.

## 5.2.1 Bochner-Weizenböck identity

Recall that ,we are always dealing with $(\mathbb{M}^{n+m}, \mathcal{F}, g_0)$ as a $n+m$-dimensional complete manifold, equipped with bundle-like metric $g_0$ and totally geodesic foliation structure $\mathcal{F}$. Similar to the static case, we denote $g_{\varepsilon,0} = g_{\mathcal{H},0} \oplus \frac{1}{\varepsilon} g_{\mathcal{V},0}$ as the one parameter family variation of the initial metric $g_0 = g_{\mathcal{H},0} \oplus g_{\mathcal{V},0}$. In this part, we assume that the metric evolves in time under the general transverse flow, we denote the evolution equation as

$$\begin{cases} \frac{\partial}{\partial t} g_{\mathcal{H}}(t) = h(t), \quad g_\varepsilon(0) = g_{\varepsilon,0} = g_{\mathcal{H},0} \oplus \frac{1}{\varepsilon} g_{\mathcal{V},0}, \\ \frac{\partial}{\partial t} g_{\mathcal{V}} = 0. \end{cases} \tag{5.2.1}$$

Assuming that $h(t)$ is a symmetric $(0, 2)$ tensor and $h(X, Z) = 0$ if $X \in \Gamma^\infty(\mathcal{V})$ or $Z \in \Gamma^\infty(\mathcal{V})$.

We then get a time dependent Bochner-Weitzenböck formula for operator

$$\Box_\varepsilon^t = -(\nabla_{g_\mathcal{H}(t)} - \mathfrak{T}^\varepsilon_{g_\mathcal{H}(t)})^*(\nabla_{g_\mathcal{H}(t)} - \mathfrak{T}^\varepsilon_{g_\mathcal{H}(t)}) - \frac{1}{\varepsilon}\mathbf{J}^2(t) + \frac{1}{\varepsilon}\delta_\mathcal{H} T(t) - \mathfrak{Ric}_{g_\mathcal{H}(t)},$$

which is introduced at the beginning of this chapter. Before we give the statement of our results, let us recall the *caré du champ* operator as we defined in the previous section. If we take $f = g$ in our definition (5.0.3), (5.0.4), then for $f \in \mathcal{C}^\infty(\mathbb{M})$, we use $\Gamma^t(f) = \Gamma^t(f, f)$ for simplicity, then we have

$$\Gamma^t(f) = \|\nabla_{g_\mathcal{H}(t)} f\|_\mathcal{H} = \|df\|_\mathcal{H}, \quad \Gamma^\mathcal{V}(f) = \|\nabla_\mathcal{V} f\|_\mathcal{V} = \|df\|_\mathcal{V}$$

and their iteration are given by

$$\Gamma_2^t(f) = \frac{1}{2}(L^t\Gamma^t(f) - 2\Gamma^t(L^t f, f)), \quad \Gamma_2^\mathcal{V}(f) = \frac{1}{2}(L^t\Gamma_\mathcal{V}(f) - 2\Gamma_\mathcal{V}(L^t f, f))$$

Then we have the following results.

**Theorem 5.2.1.** *Let $f \in \mathcal{C}^\infty(\mathbb{M})$, under the general transverse flow* (5.2.1)

$$dL^t f = \Box_\varepsilon^t df,$$

*and for $\eta = df$, we have the following time dependent Bochner's inequality*

$$\begin{aligned}&\frac{1}{2}L^t\|\eta\|_\varepsilon^2 - < \Box_\varepsilon^t \eta, \eta >_\varepsilon \\ &\geq \frac{1}{n}(\mathbf{Tr}_\mathcal{H}\nabla^{t;\sharp}_{g_\mathcal{H}(t)}\eta)^2 - \frac{1}{4}\mathbf{Tr}_\mathcal{H}(\mathbf{J}_\eta^2) + < \mathfrak{Ric}_{g_\mathcal{H}(t)}(\eta), \eta >_\mathcal{H} - < \delta_\mathcal{H} T(\eta), \eta >_\mathcal{V} + \frac{1}{\varepsilon} < \mathbf{J^2}(\eta), \eta >_\mathcal{H} .\end{aligned} \tag{5.2.2}$$

*Proof.* The proof of this theorem follows the same like [16, Theorem 3.1] by using our time dependent orthonormal frames in Lemma 5.1.3. In particular, we prove Theorem 5.2.1 under general transverse flow (5.1.1). □

### 5.2.2 Time dependent generalized curvature dimension inequality

Under further assumption that for any horizontal one form $\eta_1$, vertical one form $\eta_2$ and constants $\rho_1, \kappa, \rho_2 > 0$

$$\begin{aligned} < \mathfrak{Ric}_{\mathcal{H}}(\eta_1), \eta_1 >_{\mathcal{H}} &\geq \rho_1 \|\eta_1\|^2_{\mathcal{H}}, \\ - < \mathbf{J^2}\eta_{\mathbf{1}}, \eta_{\mathbf{1}} > &\leq \kappa \|\eta_1\|^2_{\mathcal{H}}, \\ -\frac{1}{4}\mathbf{Tr}_{\mathcal{H}}(\mathbf{J}^{\mathbf{2}}_{\eta_{\mathbf{2}}}) &\geq \rho_2 \|\eta_2\|^2_{\mathcal{V}}, \end{aligned} \tag{5.2.3}$$

and Yang-Mills condition $\delta_{\mathcal{H}} T(\cdot) = 0$. We can prove a time dependent version of the generalized curvature dimension inequality [16, Theorem 5.1] in the following sense.

**Theorem 5.2.2.** *For every* $f, g \in \mathcal{C}^\infty(\mathbb{M})$ *and* $\varepsilon > 0$,

$$\Gamma^t_2(f,f) + \varepsilon \Gamma^{\mathcal{V}}_2(f,f) \geq \frac{1}{n}(L^t f)^2 + (\rho_1 - \frac{\kappa}{\varepsilon})\Gamma^t(f,f) + \rho_2 \Gamma^{\mathcal{V}}(f,f),$$

*and*

$$\Gamma^t(f, \Gamma^{\mathcal{V}}(f)) = \Gamma^{\mathcal{V}}(f, \Gamma^t(f)).$$

*Proof.* The proof follows the same to the static case of [16, Theorem 5.1] by using our time dependent orthonormal frames in Lemma 5.1.3. However, we only prove the case under the transverse Ricci flow (5.1.3) for simplicity. Although, one can also get a version of Theorem 5.2.2 under general transverse flow (5.1.1) which need some bounded condition for $h(t)$. □

### 5.2.3 Entropy inequality and gradient estimates

In this section, we focus to prove entropy inequality and gradient estimates after the previous theorems we proved, namely, Theorem 5.2.1 and Theorem 5.2.2.

We will use the semigroup representation for the solution of heat equation (5.0.2).

$$(L^t - \frac{\partial}{\partial t})u(x,t) = 0, \quad x \in \mathbb{M}, t \in [0,T].$$

We denote $P_{s,T}f$ for the solution at time $T$ with initial condition $f$ at time $s$. We refer similar representations in [67, 83] and the discussions therein. In fact, we have

$$\partial_t P_{s,t} f = L^t P_{s,t} f, \quad \partial_s P_{s,t} f = -P_{s,t} L^s f$$

we can get the following $L^\infty$ global parabolic comparison theorem which is a time dependent version of [13, Proposition 4.5].

**Proposition 5.2.3.** *Suppose* $\mathbb{M}$ *is stochastic complete with* $P_t 1 = 1$*. Let* $u, v : \mathbb{M} \times [0,T] \to \mathbb{R}$ *be smooth functions such that for every* $T > 0$*,* $\sup_{t\in\{0,T\}} \|u(\cdot,t)\|_\infty < \infty$*,* $\sup_{t\in\{0,T\}} \|v(\cdot,t)\|_\infty < \infty$*; If the inequality*

$$L^t u + \frac{\partial}{\partial t} u \geq v,$$

*holds on* $\mathbb{M}\times[0,T]$*, then we have*

$$P_{s,T}(u(\cdot,T))(x) \geq u(x,s) + \int_s^T P_{\tau,s}(v(\cdot,\tau))(x)d\tau$$

*Proof.* See proof [49, Proposition 5.1]. □

Before we introduce our time dependent entropy inequality, we first introduce the following two functions. For $f \in \mathcal{C}_b^\infty(\mathbb{M})$, we denote

$$\Phi_1(x,t) = (P_{s,T-t}f)(x)\Gamma^t(\ln P_{s,T-t}f)(x)$$

$$\Phi_2(x,t) = (P_{s,T-t}f)(x)\Gamma^{\mathcal{V}}(\ln P_{s,T-t}f)(x)$$

By direct computations similar to [13, Lemma 5.1], we have

**Lemma 5.2.4.**

$$L^t\Phi_1 + \frac{\partial}{\partial t}\Phi_1 = 2P_{s,T-t}\Gamma_2(\ln P_{s,T-t}f) + P_{s,T-t}(\frac{\partial}{\partial t}g_{\mathcal{H}})(\nabla_{\mathcal{H}}\ln P_{s,T-t}f, \nabla_{\mathcal{H}}\ln P_{s,T-t}f)$$
$$L^t\Phi_2 + \frac{\partial}{\partial t}\Phi_2 = 2(P_{s,T-t}f)\Gamma_2^{\mathcal{V}}(\ln P_{s,T-t}f)$$

Then we have

**Theorem 5.2.5.** *Given a function* $f \in \mathcal{C}_b^\infty(\mathbb{M})$ *and* $\varepsilon > 0$*, we let* $f_\varepsilon = f + \varepsilon$*.*

$$a(T)P_{s,T}(f_\varepsilon\Gamma^t(\ln f_\varepsilon)) + b(T)P_{s,T}(f_\varepsilon\Gamma^{\mathcal{V}}(\ln f_\varepsilon)) - (P_{s,T}f_\varepsilon)(a(s)\Gamma^t(\ln P_{s,T}f_\varepsilon) + b(s)\Gamma^{\mathcal{V}}(\ln P_{s,T}f_\varepsilon))$$
$$\geq \int_s^T (a' - 2\kappa\frac{a^2}{b} - \frac{4a\gamma}{d})\Phi_1 d\tau + \int_s^T (b' + 2\rho_2 a)\Phi_2 d\tau + (\frac{4}{d}\int_s^T a\gamma d\tau)L^t P_{s,T}f_\varepsilon$$
$$- (\frac{2}{d}\int_s^T a\gamma^2 d\tau)P_{s,T}f_\varepsilon.$$

*Proof.* We give the outline of the proof, details refer to [49, Theorem 3.3]. Denote $\Phi(t)$ as

$$\Phi(t) = a(t)\Phi_1(t) + b(t)\Phi_2(t)$$

then compute

$$L^t\Phi + \frac{\partial}{\partial t}\Phi$$

plugging in Bochner's inequality (5.2.2) and transverse Ricci flow $\frac{\partial}{\partial t}g_{\mathcal{H}} = -2\mathbf{Ric}_{\mathcal{H}}$ to further simply the computations, then the proof follows by applying Proposition 5.2.3. □

With this entropy inequality in hand, we are ready to prove the following type Li-Yau gradient estimates.

**Theorem 5.2.6.** *Let* $b : [0, T] \to [0, \infty)$ *be non-increasing* $\mathcal{C}^2$ *function such that, with*

$$\gamma =^{def} \frac{d}{4}(\frac{b''}{b'} + \frac{\kappa}{\rho_2}\frac{b'}{b}), \quad b(t) = (T-t)^3,$$

*we have*

$$\Gamma^t(\ln P_{s,t}f) + \frac{2\rho_2}{3}t\Gamma^{\mathcal{V}}(\ln P_{s,t}f) \le (1 + \frac{3\kappa}{2\rho_2})\frac{L^tP_{s,t}f}{P_{s,t}f} + \frac{d(1+\frac{3\kappa}{2\rho_2})^2}{2(t-s)}.$$

*Proof.* First choose $\mathcal{C}^1$ function $a$, such that $b' + 2\rho_2 a = 0$, and with our choice of function of $b$ and $\gamma$, plugging into theorem (5.2.5), we get the desired result. □

**Remark 5.2.7.** Note that, we still assume the Yang-Mills condition $\delta_{\mathcal{H}}T(\cdot) = 0$ in the above theorem. This condition is not necessary, if we further assume (5.3.1) as we did in the next section, the only difference for the above theorem is just $b' + 2\rho_2 a + 2\beta a = 0$ in the proof and the computations follow directly. As a consequence, we can get parabolic Harnack inequality and heat kernel upper bound which is similar to the static case as in [13].

## 5.3 Differential Harnack inequality under Ricci flow.

In the following, we will forget the semigroup representation of the solution $u(x,t)$ to the heat equation (5.0.2), we directly work with $u(x,t)$, which is the solution to the heat equation at time $t$ associated with the time dependent horizontal Laplacian, namely we consider

$$(\frac{\partial}{\partial t} - L^t)u(x,t) = 0, \quad x \in \mathbb{M}, t \in [0,T].$$

Under the assumption,

$$\begin{aligned} -k_1 g_{\mathcal{H}}(x,t) \leq \mathfrak{Ric}_{\mathcal{H}}(x,t) &\leq k_2 g_{\mathcal{H}}(x,t), \\ - < \mathbf{J^2}\eta_{\mathbf{1}}, \eta_{\mathbf{1}} > &\leq \kappa \|\eta_1\|^2_{\mathcal{H}}, \\ -2 < \delta_{\mathcal{H}} T(\eta_2), \eta_2 >_{\mathcal{V}} &\geq -2\beta \|\eta_2\|^2_{\mathcal{V}}, \\ -\frac{1}{4}\mathbf{Tr}_{\mathcal{H}}(\mathbf{J}^{\mathbf{2}}_{\eta_{\mathbf{2}}}) &\geq \rho_2 \|\eta_2\|^2_{\mathcal{V}}, \end{aligned} \tag{5.3.1}$$

**Remark 5.3.1.** Comparing with assumption (5.2.3), we do not assume the Yang-Mills condition $\delta_{\mathcal{H}} T(\cdot) = 0$, instead we have a bound on it and we assume two sided bounds on the horizontal Ricci curvature.

In this setting, it is more clear and convenient to think that the metric evolves in time and the heat spreads over the manifold at the same time. For more details, we refer to [3]. We are going to get a time dependent version of gradient estimates for

$u(x,t)$. We will use similar methods as in [118, 3]. So we recall the following lemma first.

**Lemma 5.3.2** (lemma 2.1 [3].)**.** *Given* $\tau \in (0, T]$*, there exists a smooth function* $\bar{\Psi} : [0, \infty) \times 0, T \to \mathbb{R}$ *satisfying the following requirements:*

- *1. The support of* $\bar{\psi}(r,t)$ *is a subset of* $[0,\rho] \times [0,T]$*, and* $0 \le \bar{\Psi}(r,t) \le 1$ *in* $[0,\rho] \times [0,T]$*.*
- *2. The equalities* $\bar{\Psi}(r,t) = 1$ *and* $\frac{\partial \bar{\Psi}}{\partial r} = 0$ *hold in* $[0, \frac{\rho}{2}] \times [\tau, T]$ *and* $[0, \frac{\rho}{2}] \times [0, T]$ *respectively.*
- *3. The estimate* $|\frac{\partial \bar{\Psi}}{\partial t}| \le \frac{\bar{C}\bar{\Psi}^{\frac{1}{2}}}{\tau}$ *is satisfied on* $[0,\infty) \times [0,T]$ *for some constants* $\bar{C} > 0$*, and* $\bar{\Psi}(r,0) = 0$ *for all* $r \in [0, \infty)$*.*
- *4. The inequalities* $-\frac{C_l \bar{\Psi}^l}{\rho} \le \frac{\partial \bar{\Psi}}{\partial r} \le 0$ *and* $|\frac{\partial^2 \bar{\Psi}}{\partial r^2}| \le \frac{C_l \bar{\Psi}^l}{\rho^2}$ *hold on* $[0,\infty) \times [0,T]$ *for every* $l \in (0,1)$ *with some constant* $C_l$ *dependent on* $l$*.*

### 5.3.1 Key lemma for differential Harnack inequality

Before we prove the main theorem, we prove the following lemma which will play an important role later, this can be viewed as an analogue of [3, Lemma 2.6].

**Lemma 5.3.3.** *Suppose* $(\mathbb{M}, g(x,t))_{t \in [0,T]}$ *is a complete solution to the transverse Ricci flow* (5.0.1)*. Given assumption* (5.3.1)*, for any horizontal one form* $\eta_1$ *and vertical one form* $\eta_2$*. Recall our assumption* (5.3.1)

$$
\begin{aligned}
&-k_1 g_{\mathcal{H}}(x,t) \le \mathbf{Ric}_{\mathcal{H}}(x,t) \le k_2 g_{\mathcal{H}}(x,t), \quad - < \mathbf{J^2}\eta_\mathbf{1}, \eta_\mathbf{1} > \le \kappa \|\eta_\mathbf{1}\|^\mathbf{2}_{\mathcal{H}} \\
&-2 < \delta_{\mathcal{H}} T(\eta_2), \eta_2 >_{\mathcal{V}} \ge -2\beta \|\eta_2\|^2_{\mathcal{V}}, \quad -\frac{1}{4}\mathbf{Tr}_{\mathcal{H}}(\mathbf{J}^\mathbf{2}_{\eta_\mathbf{2}}) \ge \rho_\mathbf{2} \|\eta_\mathbf{2}\|^\mathbf{2}_{\mathcal{V}}
\end{aligned}
$$

*for some* $k_1, k_2, \rho_2, \beta, \kappa > 0$. *Suppose that* $u : \mathbb{M} \times [0,T] \to \mathbb{R}$ *is a smooth positive function satisfying the heat equation* (5.0.2)

$$(L^t - \frac{\partial}{\partial t})u(x,t) = 0, \quad x \in \mathbb{M}, \quad t \in [0,T].$$

*For some positive function* $b(t)$ *and some function* $\alpha(t), \phi(t)$, *define* $f = \log u$ *and*

$$F = \Gamma^t(f) + b(t)\Gamma^{\mathcal{V}}(f) - \alpha(t)f_t - \phi(t). \tag{5.3.2}$$

*the estimate*

$$\begin{aligned}(L^t - \frac{\partial}{\partial t})F + 2\Gamma^t(f,F) \geq & (-2\alpha(t)k_1 - \frac{\kappa}{\varepsilon} - \frac{4a\alpha(t)\eta(t)}{n})\Gamma^t(f) \\ & + (2\rho_2 - 2\beta - b'(t))\Gamma^{\mathcal{V}}(f) + (\alpha'(t) + \frac{4a\alpha(t)\eta(t)}{n})f_t \\ & + \phi'(t) - \frac{\alpha(t)n}{2c}\max\{k_1^2, k_2^2\}\end{aligned}$$

*holds for any* $a, c > 0$ *such that* $a + c = \frac{1}{\alpha(t)}$.

*Proof.* The complete proof can be found in [49, Lemma 6.2]. The result is also true for super transverse Ricci flow

$$\frac{\partial}{\partial t}g_{\mathcal{H}} \geq -2\mathbf{Ric}_{\mathcal{H}} \tag{5.3.3}$$

The proof follows the same as we did for the above lemma, necessary steps are given in [49, Lemma 6.2] □

**Remark 5.3.4.** In the above lemma, if we choose

$$
\begin{aligned}
&-2\alpha(t)k_1-\frac{\kappa}{\varepsilon}-\frac{4a\alpha(t)\eta(t)}{n}=\frac{d'}{d}\\
&2\rho_2-2\beta-b'(t)=\frac{d'}{d}b(t)\\
&\alpha'(t)+\frac{4a\alpha(t)\eta(t)}{n}=-\frac{d'}{d}\alpha(t)\\
&\phi'(t)-\frac{\alpha(t)n}{2c}\max\{k_1^2,k_2^2\}=-\frac{d'}{d}\phi(t)
\end{aligned}
$$

we can further get

$$
(L^t-\frac{\partial}{\partial t})F+2\Gamma^t(f,F)\geq\frac{d'}{d}F \tag{5.3.4}
$$

this ensures that we can assume $\Psi F>0$ in the proof of the our main theorem (Theorem 5.3.5), since otherwise $F<0$ will give us the result directly, see [103] for details. As the the assumptions of function $d$, see in the next section.

### 5.3.2 Differential Harnack inequality for $u(x,t)$.

We are now ready to introduce our differential Harnack inequality for solution $u(x,t)$.

For a given $\mathcal{C}^1$ function $d(t):[0,\infty)\to[0,\infty)$ such that $d(0)=0$, $\lim_{t\to 0}\frac{d(t)}{d'(t)}=0$, $\frac{d'}{d}>0$, $\frac{\int_0^t d(s)ds}{d(t)}>0$ and some integrable conditions [103], we can prove the following Li-Yau type inequality Theorem 5.3.5. This theorem is a generalization of [103, Theorem 2.1] and [3, Theorem 2.7].

**Theorem 5.3.5.** *Suppose* $(\mathbb{M},g(x,t))_{t\in[0,T]}$ *is a complete solution to the transverse Ricci flow* (5.0.1). *Given assumption* (5.3.1). *Let $u$ be the positive solution to heat equation* (5.0.2) *associated with the time dependent horizontal Laplacian operator* $L^t$.

*We have the following differential Harnack inequality:*

$$\Gamma^t(\log u) + b(t)\Gamma^{\mathcal{V}}(\log u) - \alpha(t)(\log u)_t \leq \phi(t) + B, \tag{5.3.5}$$

*where we have*

$$b(t) = \frac{2(\rho_2 - \beta)\int_0^t d(s)ds}{d(t)}, \quad \alpha(t) = \frac{e^{2k_1 t}\int_0^t e^{-2k_1 s} d(s)(\frac{d'(s)}{d(s)} + \frac{\kappa}{\varepsilon} + \frac{t_1 d_2}{\rho})ds}{d(t)},$$

$$\phi(t) = \frac{\alpha(t) n \max\{k_1^2, k_2^2\}\int_0^t d(s)ds}{2cd(t)}, \quad \eta(t) = -\frac{nk_1}{2a} - \frac{n}{4a\alpha(t)}(\frac{d'(t)}{d(t)} + \frac{\kappa}{\varepsilon} + \frac{t_1 d_2}{\rho})),$$

$$B = \frac{1}{4}(\frac{1}{\rho} + F(r_{\varepsilon,\gamma}) + \frac{\rho}{t} + \rho\bar{k} - \frac{\rho d'(t_1)\Psi(t_1)}{d_2 t_1 d(t_1)}), \quad d_2 = \max\{3d_1, C_{1/2}, 3C_{1/2}^2, \bar{C}\}$$

*where $F(r_{\varepsilon,\gamma})$ comes from Corollary 2.9 [15], certain constant comes from the sub-Laplacian comparison theorem. $d_1$ and $d_2$ are some constants, $t_1$ is the time when the maximum of $\Psi F$ (5.3.6) is attained.*

*Proof.* The complete proof refers to [49, Theorem 3.7]. We give the set up and the main idea here, the strategy is a mix of the idea from Q. Zhang [118] and X. Cao [3]. We still keep our convention as in the previous lemma for $f$ and $F$. Let us pick $\tau \in (0, T]$ and fix $\bar{\Psi}(x,t)$ satisfyting the conditions of Lemma 5.3.2. Define $\psi : \mathbb{M} \times 0, T \to \mathbf{R}$ by setting

$$\Psi(x,t) = \bar{\psi}(dist(x, x_0, t), t) = \bar{\psi}(r_\varepsilon(x), t) \tag{5.3.6}$$

where we denote $r_\varepsilon(x) = dist(x, x_0, t)$ as the Riemannian distance from $x$ to a fixed point $x_0$ at time $t$ associate with the metric $g_\varepsilon(t)$. We will show our estimate at $(x, \tau)$ for $x \in \mathbb{M}$ such that $r_\varepsilon(x) < \frac{\rho}{2}$. The main step is to estimate $(\frac{\partial}{\partial t} - L^t)(\Psi F)$ and analyze the result at a point where the function $\psi F$ attains its maximum. □

**Remark 5.3.6.** For the above theorem, if we further investigate it on a Sasakian manifold. Then by taking the limit as $\varepsilon \to 0$ for the metric $g_\varepsilon$, we will actually approximate the sub-Riemannian distance $r_0$ by the Riemannian distance $r_\varepsilon$. We then get a more specific upper bound where we can replace $F(r_{\varepsilon,\gamma})$ by the constant proved in [15, Theorem 3.2].

### 5.3.3 Comparison to generalized curvature dimension inequality

As a direct consequence of Theorem 5.3.5, we have the following

**Corollary 5.3.7.** *Under the assumption of Theorem 5.3.5, taking* $d(t) = t^2$, $c = \frac{\alpha(t)}{2}$ *, we have for all* $t$*,*

$$\begin{aligned}\Gamma^t(\log u) + \frac{2(\rho_2-\beta)}{3}\Gamma^{\mathcal{V}}(\log u) \le \alpha(t)(\log u)_t + \frac{n\max\{k_1,k_2\}}{2} \\ + \frac{1}{4}(\frac{1}{\rho} + F(r_\varepsilon,\gamma) + \rho\bar{k}) + \frac{\rho}{4t},\end{aligned} \tag{5.3.7}$$

*where*

$$\alpha(t) = (2k_1 + \frac{\kappa}{\varepsilon} + \frac{t_1 d_2}{\rho})\frac{e^{2k_1 t}}{4k_1^3 t^2} + 1 - (2k_1 + \frac{\kappa}{\varepsilon} + \frac{t_1 d_2}{\rho})(\frac{1}{2k_1} + \frac{1}{2k_1^2 t} + \frac{1}{4k_1^3 t^2}).$$

**Remark 5.3.8.** Of course, different function $d(t)$ will give us different estimates. But $d(t) = t^2$, is a suitable function such that we have the coefficient $\frac{2(\rho_2-\beta)}{3}$, which is the same coefficient in our Theorem 5.2.6 and the static setting result in [13] without assuming the Yang-Mills condition. Thus we can directly compare the right hand side to see how the estimates is affected under the Ricci flow, the same for the parabolic Harnack inequality in the following.

### 5.3.4 Parabolic Harnack inequality

We use the above corollary 5.3.7 to get the parabolic Harnack inequality.

**Theorem 5.3.9.** *Under the assumptions of Theorem 5.3.5 and conditions in Corollary 5.3.7, suppose $u$ is the solution to the time dependent heat equation, we have the estimates*

$$u(x,s) \le u(y,t)(\frac{t}{s})^{\frac{\varrho}{4}} \exp\left(\frac{T\hat{\alpha}(k_1,T)}{4(t-s)} r^2(x,y) + A(t-s)\right), \tag{5.3.8}$$

*where*

$$A = \frac{n\bar{k}}{2} + \frac{1}{4}(\frac{1}{\rho} + F(r_\varepsilon,\gamma) + \rho\bar{k}), \quad \bar{k} = \max\{k_1,k_2\},$$

$$\hat{\alpha}(k_1,T) =: \begin{cases} C\frac{e}{2k_1}, & \text{if } k_1 \le \frac{1}{T^2}, \\ C\max\{\frac{e}{2k_1}, \frac{e^{2k_1T}}{4k_1^3T^2}\}, & \text{if } k_1 \ge \frac{1}{T^2}. \end{cases}, \quad C = (2k_1 + \frac{\kappa}{\varepsilon} + \frac{t_1d_2}{\rho}),$$

*holds for all $(x,s) \in \mathbb{M} \times (0,T)$ and $(y,t) \in \mathbb{M} \times (0,T)$ such that $s < t$ and $r(x,y) \le \rho$ where $r$ is the sub-Riemannian (cc) distance.*

*Proof.* See the complete proof in [49, Theorem 3.11]. □

**Remark 5.3.10.** A detailed analysis of function $\alpha(t)$ may give us more concise estimates, where a constant function $\alpha$ version of this parabolic Harnack inequality can be found in [3, Theorem 2.11]. If we further expand $\alpha(t)$ in a Taylor expansion in higher order terms of $t$, we can get a similar version like [103, Theorem 4.1]. In the current setting, we give one explicit analysis of $\alpha(t)$ which we use in the statements of our theorem, other cases can also be done similarly. Recall that, we have $C = (2k_1 + \frac{\kappa}{\varepsilon} + \frac{t_1 d_2}{\rho})$,

$$\alpha(t) = C[\frac{e^{2k_1 t}}{4k_1^3 t^2} - (\frac{1}{2k_1} + \frac{1}{2k_1^2 t} + \frac{1}{4k_1^3 t^2})] + 1 \leq C\frac{e^{2k_1 t}}{4k_1^3 t^2}$$

since $1-(2k_1+\frac{\kappa}{\varepsilon}+\frac{t_1 d_2}{\rho})(\frac{1}{k_1}+\frac{1}{2k_1^2 t}+\frac{1}{64k_1^3 t^2})$ is monotone increasing on $(0,T]$, but its value is negative. Note that $\frac{e^{2k_1 t}}{4k_1^3 t^2}$ is decreasing on $[0, \sqrt{\frac{1}{k_1}})$ and increasing on $[\sqrt{\frac{1}{k_1}}, \infty)$. so we have

$$\alpha(t) \leq \hat{\alpha}(k_1, T) =: \begin{cases} C\frac{e}{2k_1}, & \text{if } k_1 \leq \frac{1}{T^2}, \\ C\max\{\frac{e}{2k_1}, \frac{e^{2k_1 T}}{4k_1^3 T^2}\}, & \text{if } k_1 \geq \frac{1}{T^2}. \end{cases}$$

As for the lower bound for $\alpha(t)$, we make Taylor expansion for $e^{2k_1 t}$ up to $4^{th}$ order, we then get

$$\alpha(t) = 1 + \frac{C}{3}t + o(t) \geq 1, \quad \text{for } t \in (0,T], \text{and } C = (2k_1 + \frac{\kappa}{\varepsilon} + \frac{t_1 d_2}{\rho})$$

thus we have $\frac{1}{\alpha(t)} \leq 1$ for $\forall t \in (0,T]$.

### 5.3.5 Heat kernel upper bounds.

We end this section with the following heat kernel upper bounds.

**Corollary 5.3.11.** *Given the assumption in Theorem 5.3.5. Let $p(x,y,t)$ be the heat kernel for $u(t)$ on $\mathbb{M}$. For every $x, y, z \in \mathbb{M}$ and $0 < s < t < T$, we have*

$$p(x,y,s) \leq u(x,z,t)(\frac{t}{s})^{\frac{\varrho}{4}} \exp\left(\frac{T\hat{\alpha}(k_1,T)}{4(t-s)} r^2(x,y) + A(t-s)\right).$$

*Proof.* The proof is similar to Corollary 7.2 in [13]. □

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