5-5-2017

Existence of Diffusions on 4N Carpets

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University of Connecticut, 2017

ABSTRACT

Following the methods used by Barlow and Bass to prove the existence of a diffusion on the Sierpinski Carpet, we establish the existence of a diffusion for a class of planar fractals which are not post critically finite. We conjecture that specific resistance estimates hold on our class of fractals. We further conjecture that these resistance estimates imply the existence of the spectral dimension for our class of fractals. Under these assumptions we use the methods of Barlow and Bass to establish heat kernel asymptotics. From there, we can use the techniques of Barlow, Bass, Kumagai, and Teplyaev to show the uniqueness of the diffusion up to scalar multiples. As a corollary, we conjecture the existence and uniqueness of a diffusion on the Octagasket, thus partially answering a question posed by Strichartz.

Existence of Diffusions on $4N$ Carpets

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M.A. University of Wisconsin at Madison

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A Dissertation

Submitted in Partial Fulfillment of the

Requirements for the Degree of

Doctor of Philosophy

at the

University of Connecticut

2017

APPROVAL PAGE

Doctor of Philosophy Dissertation

Existence of Diffusions on $4N$ Carpets

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2017

ACKNOWLEDGMENTS

I would like to thank my wife Zeynep, and my son Owen. You now have my undivided attention. I would also like to thank my parents Ulysses and Bernice, and my brothers, Samuel and Nathaniel. Little pieces of our experiences are in these pages, though it may be difficult to see where, or how that could be.

I would also like to thank my commitee members, Professor Rogers and Professor Gordina. Your insight and suggestions will stay with me as I go forward. Finally, I would like to thank Professor Teplyaev. You picked the perfect project, and it has taken two years for me to appreciate that.

Contents

Chapter 1

Introduction

A significant part of the motivation for asking whether there exists a diffusion on a given fractal space comes from mathematical physics. In principle, the existence of a diffusion on a space implies the existence of a Laplacian on the space. With a Laplacian, one can investigate solutions of partial differential equations and eventually, physical phenomena. Here, we take a step in this direction. Proceeding from an empirical point of view, one can build finite physical versions of the Sierpinski gasket, or other fractal, and directly measure some of its properties. Rammal and Toulouse describe and motivate this point of view in [89]; the reader should also consider the work by Bellissard, Ghez, Wang, Rammal, and Pannetier in [14]. Their analysis leads one to consider, not just the Hausdorff dimension of the fractal d_f, but also two quantities d_s, and d_w. The quantity d_s is called the spectral dimension and in some sense describes the density of states of the fractal, but is not, in a purely mathematical sense, a dimension in the way that the Hausdorff dimension is. The quantity d_w is referred to as the walk dimension and is given by the equation

$$d_w = 2\frac{d_f}{d_s}. \tag{1.0.1}$$

In [18] Barlow and Perkins constructed a Brownian motion on the Sierpinski gasket in $\mathbb{R}^2$ and in addition obtained estimates on the symmetric transition density $p_t(x, y)$. A fundamental feature of the results and techniques used is the geometry of the gasket. In particular, it is a post-critically-finite set. In this setting one has access to a technique called spectral decimation. The process constructed has the property of decimation invariance. Furthermore, the process constructed hits points, so the analysis, even in case of higher dimensional analogues, gives more detailed results. In particular, the process is locally invariant with respect to local isometries and is unique up to a linear change of time.

In [5, 6, 7, 8, 9, 11] Barlow and Bass construct a Brownian motion on a class of fractals called Generalized Sierpinski Carpets. In the case of the standard Sierpinski carpet one sees an immediate challenge in dealing with the fact that the set is infinitely ramified and although the process does hit points, this is difficult to prove. As a result, the techniques of [18] do not appear to work. For the Sierpinski carpet in $\mathbb{R}^2$ the authors use an encircling technique to derive an elliptic Harnack inequality that forms the basis of subsequent results, and eventually heat kernel estimates. For generalized Sierpinski carpets in $\mathbb{R}^d$ the encircling technique does not work, and is replaced by a coupling estimate, which is used to derive the elliptic Harnack inequality. The higher dimensional analogues of the results in the planar case can then be derived.

Heat Kernel estimates have been an area of great interest on manifolds, fractals, and graphs. Consider, for instance [39, 37, 38, 11, 18, 27, 65], and the references therein. Once diffusions have been constructed on a given fractal, several of the above authors have considered the corresponding transition density estimates of the following form:

$$p(x,y,t) \le \frac{c_1}{t^{\alpha/\beta}} \exp\left(-c_2\left(\frac{\mathrm{dist}(x,y)^{\beta}}{t}\right)^{1/(\beta-1)}\right) \tag{1.0.2}$$

It should be noted that in [39, 37, 38] an axiomatic approach is taken. In particular the authors give results on general metric-measure spaces equipped with a Dirichlet form which allow one to reduce the process of deriving heat kernel estimates to one of checking if specific conditions are held in the space. We do not take that approach here, but build the results concretely following [5]. Still, much of the same themes exist between these results and those found in the axiomatic approach. What is significant is that the generality of these results allow one to approach the establishment of heat kernel estimates more efficiently and abstractly.

This abstraction plays a key role in the techniques used to establish uniqueness in [12]. Therein Barlow, Bass, Kumagai, and Teplyaev proved that the diffusions constructed by Barlow and Bass in [11] and Kusuoka and Zhou, in [73], on the Generalized Sierpinski carpets are unique up to scalar time change. We apply the same techniques to establish uniqueness of our diffusion on $4N$ carpets.

1.1 Summary of Results

We begin by following the proofs and ideas in [5]. First, an Elliptic Harnack inequality, uniform in n, is proved using the symmetry of the $4N$ carpet in Theorem 2.3.2. This is then used to analyze our processes, which is a Brownian motion where time is renormalized on each level. In section 2.5 we then establish the tightness of the processes so that we may take a weak limit as $n \to \infty$ in Theorem 2.6.6.

Barlow and Bass proved the existence of the spectral dimension d_s in [13], we did not establish the corresponding result. In section 2.7 we conjecture the necessary resistance estimates hold. As our process is non-degenerate, we may then derive the transition density estimates via the methods of [8] in section 2.8. These estimates will be a crucial step in establishing the uniqueness of the diffusion later

To show that the process is unique one follows the ideas and proofs of Barlow, Bass, Kumagai, and Teplyaev in [12]. There the authors developed an abstract theory of Dirichlet forms which are invariant with respect to the local symmetries of the space. The first key ingredient is a folding map φ_S which is defined in Definition 3.1.8. This folding map is a key component in Definition 3.2.1 which gives the criteria for a diffusion which is invariant with respect to the local symmetries of the space. In Theorem 3.3.1 we show that the diffusion constructed in the previous chapter satisfies this definition. One can then relate the properties of the diffusion X on the space to the diffusion $Z = \varphi(X)$, as is done in section 3.2. The key parts of the theory reduce the abstract question of uniqueness of a diffusion on a space to showing that the space has a volume doubling measure, appropriately scaling first exit times, satisfies the elliptic Harnack inequality, and satisfies resistance estimates. One can then use a result of Grigor'yan and Telcs [37] to show that the process satisfies sub-Gaussian heat kernel estimates. These estimates can then be used to show that the process is unique up to a rescaling of time. As these results all apply to the $4N$-carpet under consideration we recover the uniqueness of the diffusions on these spaces. The case $N = 2$ is an immediate corrollary, and settles the question on the Octacarpet.

The expert will find that beyond the extension of the elliptic Harnack inequality this work does not show essentially new techniques in the study of diffusions or of fractals. The primary purpose was to give a self contained treatment of the subject matter.

Chapter 2

The Existence of a Non-Degenerate Diffusion on $4N$-carpets

2.1 Definitions and Notation

Fix $N \in \mathbb{N}$. Consider a regular closed polygon with $4N$ sides and its interior inscribed in the unit square $[0,1]^2$. Call this polygon V_0. Write q_1 through q_{4N} for the vertices and number them, in order, proceeding counterclockwise. We denote the length of a side of the polygon l_V. Label the side joining q_i and q_{i+1} as L_i and let L_{4N} be the side joining q_{4N} to q_1. Let $F_i(x) = l_V^{-1}(x - q_i) + q_i$. That is, each F_i is a contraction of the $4N$-gon to a $4N$-gon which is l_V times smaller with q_i as a fixed point. Here l_V is the length factor of the fractal, namely the factor by which each copy of the $4N$-gon is scaled when applying one of the contraction mappings. In the case of the Octacarpet we have $N = 2$ and $l_V^{-1} = 1 - \frac{\sqrt{2}}{2}$. Write

$$F_w = F_{w_1} \circ F_{w_2} \circ \cdots \circ F_{w_m} \tag{2.1.1}$$

for $w = (w_1, \ldots, w_m)$ where w is called a word of length m. Where applicable we sometimes write $|w| = m$ to indicate an unspecified word of length m. Let $V_1 = \bigcup_{i=1}^{4N} F_i(V_0)$ and in general let

$$V_{n+1} = \bigcup_i F_i(V_n).$$

The $4N$ carpet is the set

$$V = \bigcap_{n=0}^{\infty} V_n \tag{2.1.2}$$

and we write $\mathcal{S}_n$ for the set of $4N$-gons of with side length l_V^{-n}.

For some later results it will be useful to consider the infinite blow up of the $4N$ carpet which we define as follows:

$$\widetilde{V} = \bigcup_{n=0}^{\infty} l_V^n V. \quad \text{Here } rA = \{rx : x \in A\} \tag{2.1.3}$$

Let μ_n be Lebesgue measure on V_n normalized to have total mass 1 and let μ be the weak limit of μ_n. Denote the Lebesgue measure of a set A as $|A|$. With this definition for a set A, $\mu(A) = |A \cap V_n|/|V_n|$. See [5] for convergence and it's relationship to the fractal dimension d_f. We write m_V for the ratio of areas of a $4N$-gon of level n and $n+1$.

Some of the results will require the use of a Brownian motion W^n on V_n with normal reflection on some parts of the boundary, and absorption on other parts of the boundary. Denote the reflecting portion of the boundary of V_n as $\partial_r V_n$ and the absorbing portion of the boundary $\partial_a V_n$. It will always be the case that $\partial V_n =$

$\partial_r V_n \bigcup \partial_a V_n$. Let A be a Borel set. For any process X we use the notation

$$\tau = \tau(X) = \inf\{t : X_t \in \partial_a V_0\} \tag{2.1.4}$$

$$T_A = T_A(X) = \inf\{t : X_t \in A\} \tag{2.1.5}$$

A key element of the proofs that follow is an estimation of the probability of particular paths taken by the reflecting Brownian motion. The "move" that will be useful is the move from L_i to L_{i+1}. Following the exposition in [11] call the move in question a "Knight's move" (note that this is analogous to the "slide move" used by Barlow and Bass in [11]). The key differences between what is done here and the work done by Barlow and Bass in [5] are in section 2.2. The main ideas of the proofs are the same but the exposition is slightly different because of the differences in the geometries of the spaces considered. With these changes the rest of the proofs in [5] apply without much modification starting in section 2.3.

2.2 Some results Leading to an Elliptic Harnack Inequality

Our strategy will involve two processes, one denoted X, and the other denoted W^n. The process X is a Brownian motion in a single $4N$-gon with normal reflection in $L_1 \bigcup L_{4N}$ and absorption on the rest of the boundary. We will call this $4N$-gon V_0 as it is consistent with our notation. However the reader should see that the conclusions concerning probabilities of hitting portions of the boundary apply to any $4N$-gon in V_n as well. The process W^n is a reflecting Brownian motion with absorption on ∂V_n. The probabilities that X is absorbed in various L_i will be used to show that the process W^n takes certain paths with positive probability. The proofs in this section are thematically similar to those found in [11] but some clarifications have been made given the slight differences in the geometry of the spaces considered.

Let $S_i = \inf\{t \geq 0 : X_t \in m_i\}$ where m_i is the line joining vertices q_i and q_{2N+i}. Write $p_i(x_0) = \mathbb{P}(X_\tau \in L_i | X_0 = x_0)$.

Lemma 2.2.1. *If $x_0 \in L_1$, we have $\partial_r V_0 = L_1 \bigcup L_{4N}$, and $\partial_a V_0 = \partial V_0 - \partial_r V_0$, then $\mathbb{P}^{x_0}(X_\tau) \in L_2 \bigcup \cdots \bigcup L_{2N} \geq \frac{1}{2}$.*

<u>Proof</u>

We follow the proof ideas in [11] Theorem 2.1. Then by symmetry about the line m_1 we have for $2 \leq i \leq 2N$.

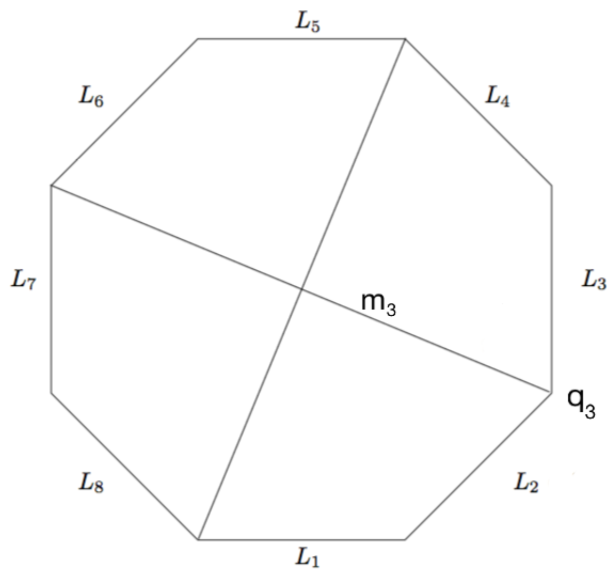
$$\begin{aligned}
p_{4N-i+1}(x_0) &= \mathbb{P}^{x_0}(X_t \in L_{4N-i+1}, S_1 < \tau) && (2.2.1)\\
&= \mathbb{E}^{x_0} 1_{S_1 < \tau} \mathbb{P}^{X_{S_1}}(X_\tau \in L_{4N-i+1}) && (2.2.2)\\
&= \mathbb{E}^{x_0} 1_{S_1 < \tau} \mathbb{P}^{X_{S_1}}(X_\tau \in L_i) && (2.2.3)\\
&= \mathbb{P}^{x_0}(X_\tau \in L_i, S_1 < \tau) && (2.2.4)\\
&\leq p_i(x_0) && (2.2.5)
\end{aligned}$$

where 2.2.3 uses symmetry. Thus $\sum_{2N+1}^{4N-1} p_i(x_0) = \sum_2^{2N} p_{4N-i+1}(x_0) \leq \sum_2^{2N} p_i(x_0)$. This shows that the process is more likely to be absorbed on the right half of the polygon rather than the left half. ∎

The ideas in the following proof are due to Barlow and Bass in [11] Theorem 2.1 but the details differ.

Lemma 2.2.2. *Let X be a Brownian motion in V_0 with normal reflection in $L_r = L_1 \cup L_{4N}$ and absorption on $\partial V_0 - \partial_r V_0$. As usual, $X_0 = x \in L_1$. Fix $2 < i < 2N$,*

Figure 2.2.1: The definitions of m_3 and q_3 within the Octacarpet



and write $j = i + 1$*.*

$$\mathbb{P}^x(X_\tau \in L_i) \geq \mathbb{P}^x(X_\tau \in L_j) \tag{2.2.6}$$

<u>Proof</u>

Recall that we denote the plane that separates L_i and L_j by m_j. Let $T_0 = 0$ and define a sequence of stopping times $T_k, k \in \mathbb{N}$ as follows:

$$T_{k+1} = \inf\{t > T_k : X_t \in m_j, X_s \in L_r \text{ for some } s, T_k < s < t\}. \tag{2.2.7}$$

Write $p_i(x) = \mathbb{P}^x(X_\tau \in L_i)$, $p_j(x) = \mathbb{P}^x(X_\tau \in L_j)$. Definite k_0 by $T_{k_0} < \infty$ and $T_{k+1} = \infty$. Since the mean squared displacement of Brownian motion is proportional to $t^{1/2}$ we know that k_0 is almost surely finite. That is, the process is absorbed before hitting plane m_j a total of $k_0 - 1$ times. We may thus write

$$p_j(x) = \sum_{k=0}^{\infty} \mathbb{P}^x(X_\tau \in L_j, T_k < \tau < T_{k+1}) \tag{2.2.8}$$
$$= \sum_{k=0}^{\infty} \mathbb{E}^x 1_{(\tau > T_k)} \mathbb{P}^{X_{T_k}}(X_\tau \in L_j, \tau < T_1) \tag{2.2.9}$$

knowing that a.s. the sum is finite. but may have arbitrarily many terms in it. For $y \in m_j$, and $T_k < \infty$, and we write $\xi_k = \inf\{t > T_k, X_t \in L_r\}$ then

$$\mathbb{P}^y(X_\tau \in L_j, T_k < \tau < T_{k+1}) = \mathbb{P}^y(X_\tau \in L_j, T_k < \tau < \xi_k) + \mathbb{P}^y(X_\tau \in L_j, \xi_k < \tau < T_{k+1}) \tag{2.2.10}$$

$$\leq \mathbb{P}^y(X_\tau \in L_i, T_k < \tau < \xi_k) + \mathbb{P}^y(X_\tau \in L_i, \xi_k < \tau < T_{k+1}) \tag{2.2.11}$$

$$= \mathbb{P}^y(X_\tau \in L_i, T_k < \tau < T_{k+1}) \tag{2.2.12}$$

The inequalities above establish the case where $k > 1$. If $k = 0$ then it is impossible for absorbtion on L_j to occur. Repeated application of 2.2.12 gives

$$p_j(x) = \sum_{k=0}^{\infty} \mathbb{P}^x(X_\tau \in L_j, T_k < \tau < T_{k+1}) \tag{2.2.13}$$
$$= \sum_{k=0}^{\infty} \mathbb{E}^x 1_{(\tau > T_k)} \mathbb{P}^{X_{T_k}}(X_\tau \in L_j, \tau < T_1) \tag{2.2.14}$$
$$\leq \sum_{k=0}^{\infty} \mathbb{E}^x 1_{(\tau > T_k)} \mathbb{P}^{X_{T_k}}(X_\tau \in L_i, \tau < T_1) \tag{2.2.15}$$
$$= \sum_{k=0}^{\infty} \mathbb{P}^x(X_\tau \in L_i, T_k < \tau < T_{k+1}) \tag{2.2.16}$$
$$\leq p_i(x). \tag{2.2.17}$$

■

Lemma 2.2.3. *If* $x \in L_1$, $\mathbb{P}^x(X_\tau \in L_2) > \frac{1}{2(2N-1)} > 0$.

Proof

Following the notation used before we write $p_i(x) = \mathbb{P}^x(X_\tau \in L_i)$. Note that by construction of the process

$$\sum_{i=2}^{4N-1} p_i(x) = 1. \tag{2.2.18}$$

Lemma 2.2.1 gives

$$\sum_{i=2}^{2N} p_i(x) \geq \frac{1}{2}. \tag{2.2.19}$$

Lemma 2.2.2 gives the following inequalities,

$$p_2(x) \geq p_3(x) \geq \cdots \geq p_{2N-1}(x) \geq p_{2N}(x). \tag{2.2.20}$$

This implies that $p_2(x) \geq \frac{1}{2(2N-1)}$ ■

2.3 An Elliptic Harnack Inequality

The exposition and proofs are very similar to those found in [5] with a few small simplifications in the proof of Lemma 2.3.1. We denote by W^n the reflecting Brownian motion on V_n. For a given ϵ and $n \geq 2$. Write $A(\epsilon) = \{x \,|\, x \in V_0 \text{ and } |x-y| > \epsilon \text{ for } y \in \partial V_0\}$. Let $G_n(\epsilon) = V_n \cap A(\epsilon)$.

Lemma 2.3.1. *Given $\epsilon > 0$, for any n so that $l_V^{n-2} < \epsilon$ there exists $\delta_\epsilon > 0$ such that if $x, y \in G_n(\epsilon)$ and $\gamma(t), 0 \leq t \leq 1$, is a continuous curve from y to $\partial_a V_n$ contained in V_n then*

$$\mathbb{P}^x(W^n \text{ hits } \gamma \text{ before time } \tau) > \delta_\epsilon \tag{2.3.1}$$

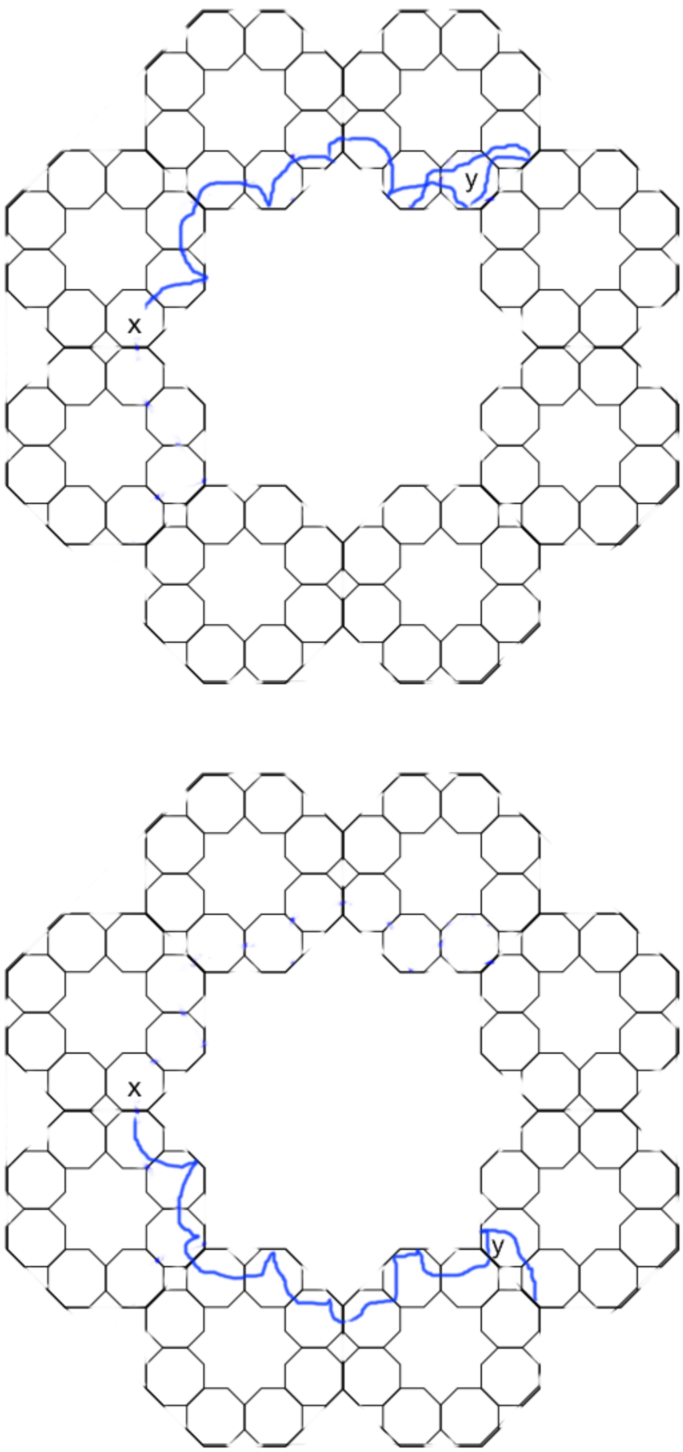
Proof

Given ϵ, choose n so that $l_V^{n-2} < \epsilon$. Take x and y to be on the boundaries of scale n $4N$-gons $\mathcal{O}_x$ and $\mathcal{O}_y$, where $\mathcal{O}_x, \mathcal{O}_y \subset G_n(\epsilon)$. We require no more than $(2N)^n$ moves for the process to go from $\mathcal{O}_x$ to $\mathcal{O}_y$. Once $\mathcal{O}_y$ has been reached we need at most $4N$ more moves to "cut off" y. To see that no generality was lost, consider the case where $\mathcal{O}_x$ is a $4N$-gon of scale n and $\mathcal{O}_y$ is a $4N$-gon of scale m, where $n < m$. We still work with moves at scale n and we simply encircle y in a $4N$-gon at scale n. To handle the situation where one or more of x and y are not boundary points we need only note that cutting off the appropriate boundary achieves the desired effect. In the case where $\mathcal{O}_y$ is not completely contained in $G_n(\epsilon)$ then at most $4N$ sides of $\mathcal{O}_y$ are in $G_n(\epsilon)$. In this case, the path from y to $\partial_a V_n$ can be cut off with at most $4N$ moves, once one has reached $\mathcal{O}_y$. Thus there are a total of at most $(2N)^n + 4N$ moves required and each has a probability of occurring that is greater than $(2(2N-1))^{-1}$. Denote by υ the smallest integer greater than $\frac{\log \epsilon}{\log l_V} + 2$. We set

$$\delta_\epsilon = (2(2N-1))^{-(2N)^{\upsilon} - 4N} \tag{2.3.2}$$

■

FIGURE 2.3.1: Two examples of ways in which a path can begin at x and "cutoff" y



Let $A \subseteq \partial_a V$. We denote our harmonic functions on V_n by

$$h_n(x, A) = \mathbb{P}^x(W^n_\tau \in A). \tag{2.3.3}$$

Theorem 2.3.2. *For each $0 < \epsilon \leq l_V^{-1}$ and any n such that $l_V^{-n+2} < \epsilon$ there exists a constant θ_ϵ independent of n such that*

$$\theta_\epsilon^{-1} \leq \frac{h_n(x, A)}{h_n(y, A)} \leq \theta_\epsilon \text{ for all } x, y \in G_n(\epsilon). \tag{2.3.4}$$

Proof

This argument is due to Barlow and Bass in [5] Theorem 3.1. Let $M_t = h_n(W^n_{t\wedge\tau}, A)$. Then M is a P^y martingale, bounded by 0 and 1. It is continuous and since W^n is a reflecting Brownian motion, M is adapted to the filtration of W^n . Let $\eta < 1$ and write $\zeta = \inf\{t \geq 0 : M_t < \eta h_n(y, A)\} \wedge \tau$. We now have

$$\begin{aligned} h_n(y, A) &= \mathbb{E}^y h_n(W^n_\zeta, A) && (2.3.5)\\ &= \mathbb{E}^y(h_n(W^n_\tau, A); \zeta = \tau) + \mathbb{E}^y(h_n(W^n_\zeta, A); \zeta < \tau) && (2.3.6)\\ &\leq \mathbb{P}^y(\zeta = \tau) + \eta h_n(y, A)\mathbb{P}^y(\zeta < \tau). && (2.3.7) \end{aligned}$$

If we rearrange this we have

$$\mathbb{P}^y(\zeta < \tau) \leq \frac{1 - h_n(y, A)}{1 - \eta h_n(y, A)} < 1. \tag{2.3.8}$$

Thus $\mathbb{P}^y(\zeta = \tau) > 0$. So there is at least one curve γ such that $\gamma(0) = y$, $\gamma(1) \in \partial_a V_n$ and $h_n(\gamma(t), A) \geq \eta h_n(y, A)$ for $0 \leq t \leq 1$. We now consider the process W^n starting at x. By Lemma 2.3.1 the path taken by W^n intersects Γ with positive probability. Write $S = \inf\{t \geq 0 : W_t^n \in \{\gamma(s), 0 \leq s \leq 1\}\}$.

$$h_n(x, A) = \mathbb{E}^x h_n(W_{S\wedge\tau}^n, A) \tag{2.3.9}$$

$$\geq \mathbb{E}^x(h_n(W_S^n, A); S < \tau) \tag{2.3.10}$$

$$\geq \delta_\epsilon \eta h_n(y, A) \tag{2.3.11}$$

Since η is arbitrary we can take the limit as $\eta \to 1$ to establish the inequality in that case. Using the same argument with the roles of x and y reversed gives the result with $\theta_\epsilon = \delta_\epsilon^{-1}$. ■

We now introduce some additional notation: for x contained in a $4N$-gon $\mathcal{O}_{n_1}$, let $D_n(x)$ be the union of $4N$-gons O_{n_i} and $\mathcal{O}_{n_1} \in V_n$ such that $x \in O_{n_1}$ and O_{n_i} share a side. We also set some notation for exit times of the process from $D_n(x)$.

$$\sigma_r(x) = \sigma_r^W(x) = \inf\{t : W_t \notin D_r(x)\} \tag{2.3.12}$$

$$\sigma_{r+1}^n(W) = \inf\{t \geq \sigma_r^n(W) : W_t \in \partial D_n(W_{\sigma_r^n(W)})\}. \tag{2.3.13}$$

Furthermore for $f : V_n \to \mathbb{R}, x \in V_n$ write

$$\mathrm{Osc}_m(x, f) = \sup_{y\in D_m(x)\cap V_n} f(y) - \inf_{y\in D_m(x)\cap V_n} f(y) \tag{2.3.14}$$

for the oscillation of f in $D_m(x)$. Using Theorem 2.3.2 we can deduce that bounded Harmonic functions in V_n are uniformly Hölder continuous away from the boundary. The ideas are due to Moser [83], Krylov and Safonov [63], and Barlow and Bass [11]. In order to do this we need a few preliminary results which follow directly from Theorem 2.3.2.

Corollary 2.3.3. *Let* $x \in V_n, r \geq 1, y \in D_r(x)$ *with* $d(y, \partial D_r(x)) > \epsilon l_V^{-r}$. *Then*

$$\theta_\epsilon'^{-1} \leq \mathbb{P}^x(W_n(\sigma_r') \in A)/\mathbb{P}^y(W_n(\sigma_r) \in A) \leq \theta_\epsilon' \text{ for all } A \subseteq \partial D_r(x) \tag{2.3.15}$$

Proof

Here we simply use the argument from Theorem 2.3.2 for each $4N$-gon of scale r and rescale by l_V^r. Write θ_r' and δ_r for the constants that result from applying the argument from Theorem 2.3.2 and Lemma 2.3.1 to a $4N$-gon of scale r. For any fixed value of N the number of $4N$-gons in $D_r(x)$ is finite. From this we can deduce the value of θ_ϵ'. ■

Since θ_ϵ and θ_ϵ', from Theorem 2.3.2 and Corollary 2.3.3 respectively, may be different, write θ_ϵ for the larger of the two. Also, let $\epsilon_0 = \frac{1}{2} - l_V^{-1}$, and $p = 1 - (4\theta_{\epsilon_0}^{-1})$. The following argument is from Barlow and Bass [5].

Lemma 2.3.4. *Let* f *be bounded and harmonic on* V_n. *Let* $x \in V_n$ *where* $D_m(x) \subseteq [-1, 1]^2$. *Then*

$$Osc_{m+1}(x, f) \leq p\,Osc_m(x, f). \tag{2.3.16}$$

Proof

Note that $d(y, \partial D_m(x)) \geq l_V^{-m}\left(\frac{1}{2} - l_V^{-1}\right)$ for all $y \in D_{m+1}(x)$. By considering the function $g = af - b$ for constants $a, b \in \mathbb{R}$ it is enough to consider $-1 \leq f \leq 1$, and $\mathrm{Osc}_m(x, f) = 2$. Write $T = \inf\{t \geq 0 : W_t^n \in \partial D_m(x)\}$ and $A = \{y \in \partial D_m(x) : f(y) \leq 0\}$. Since either $\mathbb{P}^x(W_T^n \in A^c) \geq 1/2$ or $\mathbb{P}^x(W_T^n \in A) \geq 1/2$ multiplying by

-1 allows us to assume the latter. For $y \in D_{m+1}(x)$ we have

$$
\begin{aligned}
f(y) &= \mathbb{E}^y f(W_T^n) && (2.3.17)\\
&= \mathbb{E}^y(f(W_T^n); W_T^n \in A^c) + \mathbb{E}^y(f(W_T^n); W_T^n \in A) && (2.3.18)\\
&\leq \mathbb{P}^y(W_T^n \in A^c) && (2.3.19)\\
&= 1 - \mathbb{P}^y(W_T^n \in A). && (2.3.20)
\end{aligned}
$$

By Corollary 2.3.3 $\mathbb{P}^y(W_T^n \in A) \geq \theta_{\epsilon_0}^{-1}\mathbb{P}^x(W_T^n \in A)$, this gives, for $y \in D_{m+1}(x)$,

$$
\begin{aligned}
f(y) &\leq 1 - \theta_{\epsilon_0}^{-1}\mathbb{P}^x(W_T^n \in A) && (2.3.21)\\
&\leq 1 - \frac{1}{2}\theta_{\epsilon_0}^{-1}. && (2.3.22)
\end{aligned}
$$

This implies that $\mathrm{Osc}_{m+1}(x, f) \leq 2 - \frac{1}{2}\theta_{\epsilon_0}^{-1} = p\mathrm{Osc}_m(x, f)$. ■

Recall that we defined $A(\epsilon) = \{x \,|\, x \in V_0 \text{ and } |x - y| > \epsilon \text{ for } y \in \partial V_0\}$. In addition, recall that $G_n(\epsilon) = V_n \cap A(\epsilon)$ and we write $G_(\epsilon) = V \cap A(\epsilon)$.

Theorem 2.3.5. *Fix $\epsilon > 0$. There exist constants β_1,and c_ϵ such that if f is bounded and harmonic on V_n, then*

$$|f(x) - f(y)| \leq c_\epsilon |x - y|^{\beta_1} \|f\|_\infty \text{ for all } x, y \in G_n(\epsilon) \tag{2.3.23}$$

Proof

This result is due to Barlow and Bass [5]. Let $x, y \in G_n(\epsilon)$. Choose m such that

$$\frac{1}{2} l_V^{-(m+1)} < |x - y| \leq \frac{1}{2} l_V^{-m} \tag{2.3.24}$$

and r such that $2l_V^{-r} \leq \epsilon \leq 2l_V^{-(r-1)}$. This implies that $y \in D_m(x)$, and $D_r(x) \cap \partial_a V_n =$

$\emptyset$. If $m \geq r$ then

$$|f(x) - f(y)| \leq \mathrm{Osc}_m(x, f) \tag{2.3.25}$$

$$\leq p^{m-r}\mathrm{Osc}_r(x, f)(\text{using Lemma 2.3.4}) \tag{2.3.26}$$

$$\leq 2p^{-r}p^m \|f\|_\infty \tag{2.3.27}$$

$$\leq c_\epsilon |x - y|^{\beta_1} \|f\|_\infty \tag{2.3.28}$$

where c_ϵ only depends on r, and $\beta_1 = \log p^{-1}/\log l_V$. If $m \leq r$ then $|x-y| \geq \epsilon/4l_V^2$ and $|f(x) - f(y)| \leq 2\|f\|_\infty$. If we adjust c_ϵ then the two cases together give the result. ■

2.4 Technical Inequalities Related to the Process W^n

With the exception of a few definitions the ideas, proofs, and conclusions in this section are the same as that in [5]. The reader, familiar with those arguments, will see that the same conclusions follow through the same proofs. Let $n \geq 1, \tau = \tau(W^n)$ be the first time that W^n exits V_n or is absorbed. We set the following notation:

$$g_n(x) = \mathbb{E}^x \tau \text{ for } x \in V_n \tag{2.4.1}$$

$$\alpha_n = \sup_{x \in V_n} g_n(x) \tag{2.4.2}$$

$$\beta_n = \inf_{x \in G_n(1/2)} g_n(x) \tag{2.4.3}$$

$$\gamma_n = \sup_{x \in G_n(1/2)} g_n(x) \tag{2.4.4}$$

Trivially one has the inequalities

$$\beta_n \leq \gamma_n \leq \alpha_n. \tag{2.4.5}$$

For $x \in V_n$ let $y = (y_1, y_2)$ be the "center" of $D_n(x)$ defined in the following way: as D_n consists of a $4N$-gon $\mathcal{O}$ at scale n, and any $4N$-gons at the same scale which touch it. Then we take the "center" of $D_n(x)$ to be the center of $\mathcal{O}$.

Let $x \in V_n$ and let $y = (y^{(1)}, y^{(2)})$ be the center of $D_m(x)$. Define the function $\psi : \mathbb{R}^2 \to \mathbb{R}_+$ by

$$\psi(x^{(1)}, x^{(2)}) = (l_V^m |x^{(1)} - y^{(1)}|, l_V^m |x^{(2)} - y^{(2)}|). \tag{2.4.6}$$

Let $Y_t = \psi(W^n(t l_V^{-2m}))$. Thus Y a reflecting Brownian motion on V_{n-m} and equal in law to W^{n-m} under $\mathbb{P}^{\psi(x)}$. It is then immediate that

$$l_V^{-2m} \beta_{n-m} \leq \mathbb{E}^x \sigma^m(W^n) \leq l_V^{-2m} \gamma_{n-m}. \tag{2.4.7}$$

By scaling we see that from $G_n(1/2)$ the process must cross at least $(2N - 1)$ $4N$-gons in S_1 to leave V_n. This implies $\beta_n \geq \frac{l_V}{2} l_V^{-2} \beta_{n-1}$. If one uses the fact that

$$\frac{1}{2} l_V^r - 1 \geq \frac{1}{2} l_V^{r-1} \tag{2.4.8}$$

we have

$$\beta_n \geq (2 l_V)^{-1} l_V^{-r} \beta_{n-r} \tag{2.4.9}$$

Using the knight's move and scaling we can find m and $\eta > 0$, where η does not depend on n, so that for all $x \in V_n, \mathbb{P}^x(\sigma_m > \tau) > \eta$. This gives the following bound for $g_n(x)$,

$$g_n(x) = \mathbb{E}^x(\tau, \tau > \sigma_m) + \mathbb{E}^x(\tau, \tau \le \sigma_m) \tag{2.4.10}$$

$$\le \mathbb{E}^x(1_{\tau > \sigma} \mathbb{E}^{W^n_\sigma} \tau) + \mathbb{E}^x \sigma \tag{2.4.11}$$

$$\le (1 - \eta)\alpha_n + m l_V^{-2} \gamma_{n-1}. \tag{2.4.12}$$

If we then take the supremum of the left hand over x we have

$$\alpha_n \le \frac{m}{l_V^2 \eta}. \tag{2.4.13}$$

Set $r_n = \gamma_n / \beta_n$ and let $x_1, x_2 \in G_n(1/2)$ be the points for which $\beta_n = g_n(x_1), \gamma_n = g_n(x_2)$. Since $\sigma_1 < \tau$ P^{x_i}-a.s. we have

$$g_n(x_i) = \mathbb{E}^{x_i} \sigma_1 + \mathbb{E}^{x_i} g_n(W^n_{\sigma_1}). \tag{2.4.14}$$

Lemma 2.4.1. *There exists a constant $c_2 > 0$ such that*

$$c_2^{-1} \mathbb{E}^{x_1} g_n(W^n_{\sigma_1}) \le \mathbb{E}^{x_2} g_n(W^n_{\sigma_1}) \le c_2 \mathbb{E}^{x_1} g_n(W^n_{\sigma_1}). \tag{2.4.15}$$

This result is due to Barlow and Bass, see [5] Lemma 4.1. Let $A = \{y \in \bigcup_{Q \in S_1} \partial Q : D_1(y) = D_1(x)\}$. Using the knight move estimate there exists $m \ge 1, \eta > 0$ such that $\mathbb{P}^{x_2}(\sigma_m \in A, \sigma_m < \tau) > \eta$. The map $y \mapsto \mathbb{E}^y g_n(W^n_{\sigma_1})$ is harmonic in $D_1(x_1)$ by Theorem 2.3.2. Here write $\theta = \theta_{1/2}$,

$$\theta^{-1}\mathbb{E}^{x_1} g_n(W^n_{\sigma_1} \leq \mathbb{E}^y g_n(W^n_{\sigma_1}) \leq \theta \mathbb{E}^{x_1} g_n(W^n_{\sigma_1}) \text{ for all } y \in A. \tag{2.4.16}$$

Therefore,

$$\mathbb{E}^{x_2} g_n(X_{\sigma_1}) = \mathbb{E}^{x_w}(\tau - \sigma_1) \tag{2.4.17}$$

$$\geq \mathbb{E}^{x_2}(\tau - \sigma_m; W^n(\sigma_m) \in A, \sigma_m < \tau) \tag{2.4.18}$$

$$= \mathbb{E}^{x_2}(1_{(W^n(\sigma_m)\in A, \sigma_m<\tau)} \mathbb{E}^{W^n(\sigma_m)} \tau) \tag{2.4.19}$$

$$\geq \eta\theta^{-1}\mathbb{E}^{x_1} g_n(W^n_{\sigma_1}). \tag{2.4.20}$$

If we reverse the roles of x_1 and x_2 we get the other inequality of 2.4.15. ■

Using 2.4.7, 2.4.14, and 2.4.15 we can bound γ_n

$$\gamma_n \leq l_V^{-2}\gamma_{n-1} + \mathbb{E}^{x_2} g_n(W^n_{\sigma_1}) \tag{2.4.21}$$

$$\leq l_V^{-2} r_{n-1}\beta_{n-1} + c_2 \mathbb{E}^{x_1} g_n(W^n_{\sigma_1}) \tag{2.4.22}$$

$$\leq (r_{n-1} \vee c_2)(l_V^{-2}\beta_{n-1} + \mathbb{E}^{x_1} g_n(W^n_{\sigma_1})). \tag{2.4.23}$$

In addition $\beta_n \geq l_V^{-2}\beta_{n-1} + \mathbb{E}^{x_1} g_n(W^n_{\sigma_1})$. So $\gamma_n \leq (r_{n-1} \vee c_2)\beta_n$, and thus $r_n \geq r_{n-1} \vee c_2$. If we then write $c_2 = r_0 \vee c_2$, we have $\gamma_n \leq c_3\beta_n$, for $n \geq 1$.

Theorem 2.4.2. *There exist constants $c_1, c_4, c_5 > 0$ independent of n such that*

$$\beta_n \leq \gamma_n \leq \alpha_n \leq c_4\beta_n \quad \textit{for } n \geq 0 \tag{2.4.24}$$

$$c_5 l_V^{-r}\alpha_{n-r} \leq \alpha_n \leq c_1^r \alpha_{n-r} \quad \textit{for } n \geq 0, 0 \leq r \leq n. \tag{2.4.25}$$

Proof

TThis proof is due to Barlow and Bass [5], see Proposition 4.2. he first inequality, 2.4.24 follows from 2.4.5, 2.4.9, and 2.4.13. The other set of inequalities follow from 2.4.5, 2.4.13. Using 2.4.24 and 2.4.9 on has

$$\alpha_n \geq \beta_n \geq (2l_V)^{-1} l_V^{-r} \beta_{n-r} \geq (2l_V c_4)^{-1} l_V^{-r} \alpha_{n-r}. \tag{2.4.26}$$

$\blacksquare$.

The following Lemma is due to Barlow and Bass [5] Lemma 4.3.

Lemma 2.4.3. *There exists a constant* $c_6 \in (0,1)$ *with*

$$\mathbb{P}^x(\tau \leq s) \leq c_6 + s\alpha_n^{-1} \quad \text{for } s \geq 0, x \in G_n(1/2). \tag{2.4.27}$$

Proof

Let $t > 0$. Then for $\tau \leq t + (\tau - t)1_{(\tau > t)}$, we have

$$\mathbb{E}^x \tau \leq t + \mathbb{E}^x 1_{(\tau > t)} \mathbb{E}^{W_\tau^n} \tau \tag{2.4.28}$$

$$\leq t + \alpha_n \mathbb{P}^x(\tau > t). \tag{2.4.29}$$

This shows that $\alpha_n \mathbb{P}^x(\tau \leq t) \leq \alpha_n + t - \mathbb{E}^x \tau$. This combined with the inequalities $\mathbb{E}^x \tau \geq \beta_n \geq c_4^{-1} \alpha_n$, gives $\mathbb{P}^x(\tau \leq t) \leq \alpha_n^{-1}(\alpha_n + t - c_4^{-1}\alpha_n)$. $\blacksquare$

Lemma 2.4.4. *This proof is due to Barlow and Bass [5] Lemma 1.1. Let* $X, Y_1, \ldots, Y_n$ *be non-negative random variables satisfying*

1. $X \geq \sum_{i=1}^n Y_i$
2. $\mathbb{P}(Y_i \leq x | \sigma(Y_j, j \leq i-1)) \leq p + bx, \quad i = 1, \ldots, n, \quad x \geq 0.$

Proof

Let Z be a random variable with distribution function $G(x)$ defined by

$$G(x) = p + bx, \quad 0 \leq x \leq (1-p)/b, \quad G(0-) = 0. \tag{2.4.30}$$

Then $\mathbb{E}(e^{uY_i}|\sigma(Y_j, j \leq i-1)) \leq \mathbb{E}(e^{-uZ})$. Write $q = 1-p$, and we have

$$\mathbb{E}e^{-uZ} = p + \int_0^{q/b} e^{-ux} b dx \tag{2.4.31}$$

$$= p + bu^{-1}(1 - e^{-uq/b}). \tag{2.4.32}$$

So,

$$P(X \leq x) = \mathbb{P}(e^{-uX} \geq e^{-ux}) \tag{2.4.33}$$

$$\leq e^{ux}\mathbb{E}e^{-uX} \tag{2.4.34}$$

$$\leq e^{ux}(p + bu^{-1}(1 - e^{-uq/b}))^n \tag{2.4.35}$$

$$\leq p^n \exp\left(ux + \frac{bn}{pu}\right). \tag{2.4.36}$$

Setting $u = (bn/px)^{1/2}$. ■

This result is due to Barlow and Bass, see proposition 4.4 [5].

Theorem 2.4.5. *There exist constants $\gamma, c_1 1, c_1 2 > 0$ such that*

$$\mathbb{P}^x(\tau \leq \alpha_n s) \leq c_1 2 e^{-c_1 1 s^{-\gamma}} \quad \textit{for } s \geq 0, n \geq 3, x \in G_n(1/2) \tag{2.4.37}$$

Proof Let $x \in G_n(1/2)$, let $3 \leq r \leq n$, and let $N = \min\{i : \sigma_i^r \geq \tau\}$. Assume

$N \geq \frac{1}{2} l_V^r$. Let $m_r = \frac{1}{2} l_V^r - 2 \geq \frac{1}{3} l_V^r$, and let $Y_i = \sigma_{i+1}^r - \sigma_i^r$ for $i = 1, \ldots, m_r$. Then

$$\tau \geq \sum_{i=1}^{m_r} Y_i. \tag{2.4.38}$$

By scaling, the $\mathbb{P}^x$ law of σ_1^r is the $\mathbb{P}^y$ law of $l_V^{2r} \tau(W^{n-r})$ for some $y \in G_{n-r}(1/2)$. By Lemma 2.4.3

$$\mathbb{P}^x(Y_i \leq t | \sigma(Y_j, j \leq i-1)) \tag{2.4.39}$$

$$\leq \sup_{y \in G_n(1/2)} \mathbb{P}^y(\tau(W^{n-r}) < l_V^{2r} t) \tag{2.4.40}$$

$$\leq c_6 + t l_V^{2r} \alpha_{n-r}^{-1} \tag{2.4.41}$$

By Lemma 2.4.4,

$$\mathbb{P}^x(\tau \leq \alpha_n s) \leq \exp\left(2 \left(\frac{l_V^{2r} m_r \alpha_n s}{\alpha_{n-r} c_6} \right)^{1/2} - m_r \log c_6^{-1} \right) \tag{2.4.42}$$

$$\leq \exp(c_7 (c_1 l_V^3)^{r/2} s^{1/2} - c_8 l_V^r). \tag{2.4.43}$$

Choose $c_9 \geq (c_1 l_V)^{1/2}$. Then $c_9 > 1$. Let $f_1(r) = c_7 s^{1/2} (c_9 l_V)^r$, and $f_2(r) = c_8 l_V^r$. Now if $r_0 = \frac{\log(c_8 c_7^{-1} s^{-1/2})}{\log c_9}$, $f_1(r_0) = f_2(r_0) = c_1 0 s^{-\gamma}$, where $\gamma = \frac{\log l_V}{2 \log c_9}$. There exists a constant $s_0 > 0$ such that, if $0 < s \leq s_0$, then $r_0 \geq 5$. If $s < s_0$, then let $r = [r_0 - 1]$, which makes $r_0 - 2 \leq r \leq r_0 - 1$. Then $f_1(r) - f_2(r) = f_2(r_0)((c_9 l_V)^{r_0 - r} - l_V^{r_0 - r}) \leq -c_1 1 s^{-\gamma}$, where $c_1 1 > 0$. By choosing $c_{12} = \exp(c_1 1 s_0^\gamma)$ we have the desired result. ■

Corollary 2.4.6. *For all* $x \in V_n, 3 \leq r \leq n, s \geq -, k \geq 1$,

$$\mathbb{P}^x(\sigma_{k+1}^r(W^n) - \sigma_k^r(W^n) \leq \alpha_{n-r} s) \leq c_{12} \exp(-c_1 1 (9^r s)^{-\gamma}). \tag{2.4.44}$$

Proof

This follows from 2.4.5 by scaling. ■

2.5 Tightness of the Process

Here we will directly use the ideas of [5] as the proofs require no changes. The main point is that estimates of W^n which are uniform in n, can be used to construct a process on $V = \cap_n V_n$.

Definition 2.5.1. Let $X_t^n = W_{\alpha_n t}^n, t \geq 0$. Let $\mathbb{P}_n^x$ be the probability distribution on $\mathcal{C}(\mathbb{R}_+, V)$ corresponding to X^n with $X_0^n = 0$. Write X for the coordinate process on $\mathcal{C}(\mathbb{R}_+, V)$.

Theorem 2.5.2. *Let x_n be a sequence with $x_n \in V_n$. Then $\{\mathbb{P}_n^{x_n}, n \geq 1\}$ is tight in the space of cadlag functions from $\mathbb{R}_+$ to V_0, which we denote by $\mathcal{D}(\mathbb{R}_+, V_0)\}$.*

Proof

This proof is due to Barlow and Bass [5] Theorem 5.1. Using 2.4.6 one has

$$\sup_{i \geq 1} \mathbb{P}_n^{x_n}(\sigma_{i+1}^r(X) - \sigma_i^r(X) \leq s) c_{12} \exp(-c_{11}(9^r s)^\gamma), \tag{2.5.1}$$

for $3 \leq r \leq n$. By Ethier and Kurtz [24], Lemma 3.8.1, Proposition, 3.8.3, and Theorem 3.7.2 we have $\mathbb{P}_n^{x_n}$ is tight in $\mathcal{D}(\mathbb{R}_+, V_0)$. Since X is $\mathbb{P}_n^{x_0}$ a.s. continuous, by [24] Theorem 10.2, that if Q is any limit point of $\{\mathbb{P}_n^{x_n}, n \geq 1\}$ then X is Q-a.s. continuous. This implies that $\{\mathbb{P}_n^{x_n}, n \geq 1\}$ is tight in $\mathcal{C}(\mathbb{R}_+, V_0)$. ■

Definition 2.5.3. For $x \in V_n, n \geq 0$, f bounded, let

$$U_n f(x) = \mathbb{E}_n^x \int_0^\tau f(X_s) ds. \tag{2.5.2}$$

We have the immediate bound

$$U_n f(x) \leq \|f\|_\infty \mathbb{E}_n^x \tau \leq \|f\|_\infty . \tag{2.5.3}$$

By scaling and 2.4.25 we have

$$\mathbb{E}_n^x(\sigma_1^r(X)) \leq l_V^{2r} \frac{\alpha_{n-r}}{\alpha_n} \leq c_5^{-1} l_V^{-r}. \tag{2.5.4}$$

We first examine $\mathbb{E}_n^x \tau = U_n 1(x)$ as $x \to \partial_a V_n$.

Theorem 2.5.4. *There exist constants β_2 and $c_{13} > 0$ such that*

$$\mathbb{E}_n^x \tau \leq c_{13}(d(x, \partial_a V))^{\beta_2} \quad n \geq 0, x \in V_n. \tag{2.5.5}$$

Proof

This proof is due to Barlow and Bass in [11] Theorem 5.2. Fix $n \geq 0.$, let $H_r' = V_n - G_n(l_V^{-r})$. Because some of the $4N$-gons will be overlap H_r' but not be contained within it, let $H_r = \{\bigcup \mathcal{O}_i | \mathcal{O}_i \cap H' \neq \varnothing\}$. Let

$$h_r = \sup_{x \in H_r} \mathbb{E}_n^x \tau. \tag{2.5.6}$$

Let $S = \sigma_1^r(X) \wedge \tau$, and note $X_S \in H_{r-1} \mathbb{P}^x$-a.s. if $x \in H_r$. Using the Knight's move estimates there exists a constant $\delta > 0$, which independent of n and r such that

$$\mathbb{P}_n^x(S = \tau) > \delta \text{ for all } x \in H_r. \tag{2.5.7}$$

By definition $S \leq \tau$

$$\mathbb{E}_n^x \tau = \mathbb{E}_n^x S + \mathbb{E}_n^x 1_{(S<\tau)} \mathbb{E}^{X_s} \tau \tag{2.5.8}$$

$$\leq \mathbb{E}_n^x S + (1-\delta) h_{r-1}. \tag{2.5.9}$$

By taking the supremum over x and using 2.5.4 we have

$$h_r \leq c_5^{-1} l_V^{-r} + (1-\delta) h_{r-1}. \tag{2.5.10}$$

Recall that $h_0 \leq 1$. Letting $c_{13} = 1 + c_5^{-1}(l_V(1-\delta) - 1)^{-1}$ we have

$$h_r \leq c_{13}(1-\delta)^r - (c_{13}-1) l_V^{-r} \leq c_{13}(1-\delta)^r. \tag{2.5.11}$$

The desired result follows. ■

Theorem 2.5.5. *There exist constants β_2, c_{14} such that*

$$|U_n f(x) - U_n f(y)| \leq c_{14} \|f\|_\infty |x-y|^{\beta_2} \tag{2.5.12}$$

for all boudned f on $V_n, x, y \in V_n$.

Proof

This result is Theorem 5.3 from [5]. Let f be bounded on $V_n, x, y \in V_n$ with $|x-y| = \delta$, and chose r so that

$$\frac{1}{4} l_v^{-(r+1)} \leq |x-y| < \frac{1}{4} l_V^{-r}. \tag{2.5.13}$$

If $d(x, \partial_a V) \leq 2l_V\delta$, then by 2.5.3, and Theorem 2.5.4,

$$|U_n f(x) - U_n f(y)| \leq c_{13} \|f\|_\infty ((2l_V\delta)^{\beta_2} + ((2l^V + 1)\delta)^{\beta_2}) \tag{2.5.14}$$

$$\leq c \|f\|_\infty \delta^{\beta_2}. \tag{2.5.15}$$

If, on the other hand, $d(x, \partial_a V) \leq 2l_V\delta$ then $D_r(x) \subseteq V_0$. Set $S = \inf\{t \geq 0 : X_t \in D_r(x)^c\}$. Since $S \leq \tau$ for $z \in D_r(x)$

$$U_n f(z) = \mathbb{E}_n^z \int_0^S f(X_t)dt + \mathbb{E}_n^z \int_S^\tau f(X_t)dt \tag{2.5.16}$$

$$= \mathbb{E}_n^z + \mathbb{E}_n^z U_n f(X_S). \tag{2.5.17}$$

As $\mathbb{E}_n U_n f(X_s)$ is harmonic in $D_r(x), d(y, \partial D_r(x)) \geq \frac{1}{4} l_V^{-r}$, using Theorem 2.3.5

$$|\mathbb{E}_n^x U_n f(X_S) - \mathbb{E}_n^y U_n f(X_S)| \leq c_{13}|x-y|^{\beta_1} \|f\|_\infty. \tag{2.5.18}$$

Using 2.5.3 and 2.5.4

$$\mathbb{E}_n^z \int_0^S f(X_t)dt \leq \|f\|_\infty \mathbb{E}_n^z S \leq c_5^{-1} l_V^{-r} \|f\|_\infty. \tag{2.5.19}$$

This gives

$$|U_n f(x) - U_n f(y)| \leq (2c_5^{-1} l_V^{-r} + c_{13}|x-y|^{\beta_1}) \|f\|_\infty \tag{2.5.20}$$

$$\leq c|x-y|^{\beta'} \|f\|_\infty \tag{2.5.21}$$

for suitable β'. Since we've considered all values of $d(y, \partial D_r(x))$ the result follows. ■

For each n, X under $\mathbb{P}_n^x$ is a time change of a reflecting Brownian motion on a

Lipschitz domain. As a result, U_n has a symmetric potential kernel density $u_n(x,y)$ with respect to μ_n,and u_n is jointly continuous in x and y away from the diagonal.

Theorem 2.5.6. *For each $\epsilon > 0$ there exists $M_\epsilon < \infty$ such that $u_n(x,y) \leq M_\epsilon$ for all $n \geq 0, x, y \in G_n(\epsilon)$ with $|x-y| > \epsilon$.*

Proof

This result is due to Barlow and Bass in [5], see Theorem 5.4. Fix $\epsilon > 0$, $n \geq 1$, and let $x \in V_n, y \in G_n(\epsilon)$ where $|x-y| > \epsilon$. Choose r so that $6l_V^{-r} < \epsilon \leq 6l_V^{-r+1}$. Then $D_r(x) \subseteq B(x, 3l_V^{-r} \subseteq B(x,\epsilon)$, and $D_r(y) \subseteq V_n, D_r(x) \cap D_r(y) = \varnothing$. First,

$$\int_{V_n} u_n(x,y)\mu_n(dy) \leq \mathbb{E}_n^x \tau, \tag{2.5.22}$$

second, $\mu(D_{r+1}(y)) \geq c_{4N} l_V^{-2(r+1)}$. So there exists $y_0 \in D_{r+1}(y)$ with

$$u_n(x, y_0) \leq c_{4N} l_V^{2(r+1)} \mathbb{E}_n^x \tau. \tag{2.5.23}$$

Recall that $u_n(x,\cdot)$ is harmonic in $D_r(y)$, and so by 2.3.3

$$u_n(x,z) \leq \theta u_n(x,y_0) \text{ for all } z \in D_{r+1}(y). \tag{2.5.24}$$

If we take $z = y$ and note that $\mathbb{E}_n^x \tau \leq 1$, and that r depends only on ϵ then combining 2.5.23, and 2.5.24 gives the result. ∎

2.6 The Construction of the Limiting Process

Thus far we have a sequence of processes $\{X^n, \mathbb{P}_n^x\}$ with state space V_n and we intend to take a weak limit as $n \to \infty$ to get a process on V. We use the following conventions, identify points on $\partial_a V_0$ as Δ and $f(\Delta) = 0$ for all functions f. For $x \in V_0 - V_n$ let $\mathbb{P}_n^x$ be the probability measure for the process that is standard Brownian motion until the time t where it hits V_n and then behave like X_t^n after that time. The proofs in this section are from [5], and require no modification.

If $\tau_n = \inf\{t > 0 : X_t \in V_n\}$, Q^x is standard Weiner measure on paths in $\mathbb{R}^2$, $A \in \mathcal{F}_{\tau_n}$, then define

$$\mathbb{P}_n^x(A \cap (B \circ \theta_{\tau_n})) = \mathbb{E}_Q^x(\mathbb{P}^{X_{\tau_n}}(B); A). \tag{2.6.1}$$

In this way we have defined $\mathbb{P}_n^x$ on $\mathcal{F}_\infty$. Now extend U_n to all of V_0.

Theorem 2.6.1. *Suppose $f : V_0 \to \mathbb{R}$ is bounded. Then there exists $\omega(\delta)$ that tends to 0 as $\delta \to 0$ such that*

$$\sup_n \sup_{\substack{x,y \in V_0 \\ |x-y|<\delta}} |U_n f(x) - U_n f(y)| \leq \omega(\delta) \|f\|_\infty \tag{2.6.2}$$

Proof

This is Theorem 6.1 from [5]. If $\epsilon, \eta > 0$ and B is a $4N$-gon of side r and $\tau_B = \inf\{t : X_t \notin B\}$, then there exists $\delta > 0$(independent of r) such that

$$\mathbb{E}_Q^x \tau_B < \epsilon \text{ and } Q^x(|X_{\tau_B} - x| > \eta) < \epsilon \tag{2.6.3}$$

whenever $x \in B$ and $d(x, B^c) < \delta$ by the arguments of [87] in chapter 2 section 3.

Let $\epsilon > 0$. We wish to show that for each x_0 there exists a $\delta(x_0)$ such that

$$|U_n f(x) - U_n f(x_0)| \leq 4\epsilon \|f\|_\infty \tag{2.6.4}$$

whenever $|x - x_0| < \delta$. Then the result follows by the compactness of V_0.

Suppose $x_0 = \Delta$. Choose η small enough that $\mathbb{E}^x_\tau < \epsilon$ if $x \in V_n$ and $d(x, \Delta) < 2\eta$. This choice can be made independently of n by Theorem 2.5.4. Now choose $\delta \in (0, \eta)$ small enough that 2.6.3 is satisfied. If $d(x, \Delta) < \delta$ and $x \in V_n$, then $\mathbb{E}^x_n \tau < \epsilon$. On the other hand if $x \notin V_n$ then

$$\mathbb{E}^x_n \tau \leq \mathbb{E}^x_Q \tau_n + \mathbb{E}^x_Q \mathbb{E}^{X_{t_n}}_n \tau \tag{2.6.5}$$

$$\epsilon + \epsilon \sup_{x \in V_n} \mathbb{E}^x_n \tau + \sup_{\substack{x \in V_n \\ d(x,\Delta) < \eta + \delta}} \mathbb{E}^x_n \tau \leq 3\epsilon. \tag{2.6.6}$$

Then using 2.5.3 and the definition of Δ, we have

$$|U_n f(x) - U_n f(x_0)| = |U_n f(x)| \leq \|f\|_\infty \mathbb{E}^x_n \tau \leq 3\epsilon \|f\|_\infty . \tag{2.6.7}$$

Suppose $x_0 \notin V$, then there exists m and η such that $B_\eta(x_0) \cap V_n = \varnothing$ when $n \geq m$. Write $\rho = \inf\{t : X \not\in B_\eta(x_0)\}$. Choose η to be small enough that $\sup_{x \in B_\eta(x_0)} \mathbb{E}^x_Q \rho < \epsilon$. Let g be a bounded function, then $\mathbb{E}^x_Q g(X_\rho)$ is harmonic in x in the interior of $B_\eta(x_0)$. Furthermore there exists $\delta < \eta/2$ such that

$$|\mathbb{E}^x_Q g(X_\rho) - \mathbb{E}^{x_0}_Q g(X_\rho) \leq \epsilon \|g\|_\infty \tag{2.6.8}$$

if $|x - x_0| < \delta$. For $x \in B_\eta(x_0)$ we have

$$U_n f(x) = \mathbb{E}_Q^x \int_0^\rho f(X_s) ds + \mathbb{E}_Q^x U n f(X_\rho) \tag{2.6.9}$$

From this we see that

$$|U_n f(x) - U_n f(x_0)| \leq 2 \|f\|_\infty \sup_{x \in B_\eta(x_0)} \mathbb{E}_Q^x \rho + |\mathbb{E}_Q^x U_n f(X_\rho) - \mathbb{E}_Q^{x_0} U_n f(X_p)| \tag{2.6.10}$$

$$\leq 2\epsilon \|f\|_\infty + \epsilon \|U_n f\|_\infty \tag{2.6.11}$$

$$\leq 3\epsilon \|f\|_\infty . \tag{2.6.12}$$

Now, suppose $x_0 \in V$. If we pick η so that if $|x - x_0| < 2\eta, x \in V_n$, then

$$|U_n f(x) - U_n f(x_0)| < \epsilon \|f\|_\infty . \tag{2.6.13}$$

By theorem 2.5.5 this choice is independent of n. Now pick $\delta < \eta$ to satisfy 2.6.3. If $x \in V_n$ then 2.6.4 is true, and if $x \notin V_n$ and $|x - x_0| < \delta$, then

$$|U_n f(x) - U_n f(x_0)| \leq |\mathbb{E}_Q^x \int_0^{\tau_n} f(X_s) ds| + \mathbb{E}_Q^x U_n f(X_{\tau_n}) - U_n f(x_0)| \tag{2.6.14}$$

$$\leq \|f\|_\infty \mathbb{E}_Q^x \tau_n + 2 \|U_n f\|_\infty \mathbb{P}_Q^x(|X_{\tau_n} - x| \geq \eta) + \sup_{\substack{y \in V_n \\ |y - x_0| < 2\eta}} |U_n f(y) - U_n f(x)| \tag{2.6.15}$$

$$\leq 4\epsilon \|f\|_\infty . \tag{2.6.16}$$

∎

The tightness estimate for $\mathbb{P}_n^x$ when $x \notin V_n$ is routine and thus omitted.

Definition 2.6.2. We define the λ resolvent of U for bounded f and by

$$U_n^\lambda f(x) = \mathbb{E}_n^x \int_0^\tau e^{-\lambda s} f(X_s) ds \tag{2.6.17}$$

We then have $U_n^0 = U_n$ and for any λ and analogously to the case for U_n we have

$$|U_n^\lambda f(x)| \le \|f\|_\infty \mathbb{E}_n^x \tau \le \|f\|_\infty . \tag{2.6.18}$$

Using [21] V. 5.10, we have the identity

$$U_n^\lambda f = \sum_{i=0}^\infty (\beta - \lambda)^i (U_n^\beta)^{i+1} f, \quad \beta \ge 0, |\beta - \lambda| < 1. \tag{2.6.19}$$

Theorem 2.6.3. *Suppose* $f : V_0 \to \mathbb{R}, \|f\|_\infty$ *, and* $\lambda > 0$*. Then*

$$\lim_{\delta \downarrow 0} \sup_n \sup_{\substack{x,y \in V_n \\ |x-y|<\delta}} |U_n^\lambda f(x) - U_n^\lambda f(y)| = 0. \tag{2.6.20}$$

<u>Proof</u>

This argument is due to Barlow and Bass in [5] Theorem 6.2. Assume that $\lambda \le 1/2$. By theorem 2.5.5,

$$\sup_{\substack{x,y \in V_n \\ |x-y|<\delta}} |U_n g(x) - U_n g(y)| \le \|g\|_\infty \omega(\delta), \tag{2.6.21}$$

where $\omega(\delta) \to 0$ as $\delta \to 0$ independently of n. By applying this to $g = (U_n)^i f$ and using 2.6.18 we have

$$\sup_{\substack{x,y \in V_n \\ |x-y|<\delta}} |(U_n)^{i+1} f(x) - (U_n)^{i+1} f(y)| \le \|f\|_\infty \omega(\delta). \tag{2.6.22}$$

Let $\epsilon > 0$. Choose i_0 so that $\sum_{i=i_0}^{\infty} \lambda^i < \epsilon/4$, and choose δ so that $\omega(\delta) < \epsilon/2i_0$. We then have (via 2.6.18)

$$\sum_{i=i_0}^{\infty} \lambda^i |(U_n)^{i+1} f(x) - (U_n)^{i+1} f(y)| \leq \epsilon \left\| f \right\|_\infty /2. \tag{2.6.23}$$

Combining 2.6.19 and 2.6.22 with $\beta = 0$ gives

$$\sup_{\substack{x,y \in V_n \\ |x-y|<\delta}} |U_n^\lambda f(x) - U_n^\lambda f(y)| \leq (\epsilon/2 + i_0 \omega(\delta)) \left\| f \right\|_\infty < \epsilon \left\| f \right\|_\infty . \tag{2.6.24}$$

The above proof gives the result for $\lambda \leq 1/2$. For $\lambda \in [m, m + 1/2]$ repeat the proof using U_n^β with $\beta = m$. ■

Theorem 2.6.4. *There is a subsequence n_j for which $U_{n_j}^\lambda f$ converges uniformly for each $\lambda \in [0, \infty)$ if f is continuous on V_0. Furthermore, the limit U^λ satisfies the resolvent identity and $\left\| U^\lambda \right\|_\infty \leq \lambda^{-1}$.*

Proof

This is Proposition 6.3 from [5]. Let $\{f_m\}_m$ be a sequence of continuous function on V_0 such that $\|f_m\| \leq 1$ and the closure of the linear span of $\{f_m\}$ consists of all continuous functions on V_0. Let λ_i be a countable dense subset of $[0, \infty)$. Fix i and m. By the Arzela-Ascoli theorem there exists a subsequence of $U_n^{\lambda_i} f_m$ converges uniformly. By using a diagonalization argument we can choose a subsequence n_j so that $U_{n_j}^{\lambda_i} f_m$ converges uniformly for each i and m, and we call this limit $U^{\lambda_i} f$. Note that each $U_n^{\lambda_i}$ satisfies the inequality $\left\| U_n^{\lambda_i} \right\|_\infty \leq \lambda^{-1}$ and thus $U^{\lambda_i} \leq \lambda^{-1}$. We may then conclude that $U_n^{\lambda_i} f$ converges uniformly and we call the limit $U^{\lambda_i} f$ for f

continuous on V_0. Each U_n^λ satisfies the resolvent equation

$$U_n^\lambda - U_n^\beta = (\beta - \lambda) U_n^\lambda U_n^\beta. \tag{2.6.25}$$

A limiting argument shows that U^λ also does if $\lambda = \lambda_i$ for some i. Further, assuming that $\beta, \lambda \in \{\lambda_i\}$ then by 2.6.25

$$\left\| U_n^\lambda - U_n^\beta \right\|_\infty \leq \frac{\beta - \lambda}{\lambda \beta} \tag{2.6.26}$$

and similarly for $U^\lambda - U^\beta$. It then follows that for all $\lambda \in [0, \infty)$, and all f continuous on V_0 that $U_{n_j}^\lambda f$ converges uniformly. We'll call this limit $U^\lambda f$, and U^λ satisfies the resolvent equation and $\left\| U^\lambda \right\|_\infty \leq \lambda^{-1}$. ■

The following lemma is elementary.

Lemma 2.6.5. *Suppose g and g_m, for $m = 1, 2, \ldots$ are functions on V_0 for which $g_m(y_m) \to g(y)$ whenever $y_m \to y$. Then g is continuous and $g_m \to g$ uniformly.*

This next result comes from Barlow and Bass in [5], see Lemma 6.4.

Theorem 2.6.6. *Suppose $x_j \to x$ so that $\{x_j\}, \{x\} \in V_0$. Then $\{\mathbb{P}_{n_j}^{x_j}\}$ converges weakly and we call the limit $\mathbb{P}^x$.*

Proof

By Theorem 2.5.2 and the remark following 2.6.1 it suffices to show that any two limit points agree. Let f be a continuous function on V_0. If a subsequence of $\mathbb{P}_{n_j}^{x_j}$ converges weakly to a limit which we call $\mathbb{P}'$ then

$$U_{n_j}^\lambda f(x_j) = \mathbb{E}_{n_j}^{x_j} \int_0^\infty e^{-\lambda s} f(X_s^{n_j}) ds \to \mathbb{E}' \int_0^\infty e^{\lambda s} f(X_s) ds. \tag{2.6.27}$$

However, by the equicontinuity of $U^{\lambda}_{n_j} f$, we have $U^{\lambda}_{n_j} f(x_j) \to U^{\lambda} f(x)$. If $\mathbb{P}''$ is limit point of a subsequence of $\mathbb{P}^{x_j}_{n_j}$ we have

$$\mathbb{E}' \int_0^{\infty} e^{\lambda s} f(X_s) ds = U^{\lambda} f(X) = \mathbb{E}'' \int_0^{\infty} e^{-\lambda s} f(X_s) ds. \tag{2.6.28}$$

By the uniqueness of the Laplace transform we have

$$\mathbb{E}' f(X_s) = \mathbb{E}'' f(X_s) \tag{2.6.29}$$

for almost every s. Since f is continuous and X_s is continuous a.s. under both $\mathbb{P}'$ and $\mathbb{P}''$, we have equality for all s. A standard limiting argument gives 2.6.21. for bounded f, and it follows that the one-dimensional distributions of X_t are equal under both $\mathbb{P}'$ and $\mathbb{P}''$.

Write $\mathbb{E}' f(X_s)$ by $\mathbb{P}_s f(x)$. We have $\mathbb{P}^{n_j}_s f(x_j) \to \mathbb{P}_s f(x)$ for $x_j \to x$. By 2.6.5 $\mathbb{P}_s f$ is continuous on V_0 thus the convergence is uniform. Let $s < t$ be times, and f and g be continuous functions. Using the Markov property of $\mathbb{P}^{x}_{n_j}$

$$\mathbb{E}^{x_j}_{n_j} g(X_s) f(X_t) = \mathbb{E}^{x_j}_{n_j}((\mathbb{P}^{n_j}_{t-s} f) g)(X_s) \tag{2.6.30}$$

$$= \mathbb{E}^{x_j}_{n_j}((\mathbb{P}_{t-s} f) g)(X_s) + \mathbb{E}^{x_j}_{n_j}((\mathbb{P}_{t-s} f - \mathbb{P}^{n_j}_{t-s} f) g)(X_s). \tag{2.6.31}$$

The first term converges to $\mathbb{E}'((\mathbb{P}_{t-s} f) g)(X_s)$ by argument above. The second term tends to 0 by by the uniform convergence of $\mathbb{P}^{n_j}_{t-s} f$. Repeating the argument shows that the finite dimensional distributions under $\mathbb{P}^{x_j}_{n_j}$ converge. Tightness gives the result. ∎

Corollary 2.6.7. *If f is continuous, $\mathbb{P}_t f$ is continuous.*

Theorem 2.6.8. $\{\mathbb{P}^x\}$ *is a strong Markov family of probability measures.*

Proof

This proof is due to Barlow and Bass in [5] Proposition 6.7. For each n, we have $\mathbb{P}_t^n \mathbb{P}_s^n = \mathbb{P}_{t+s}^n$ since $\{\mathbb{P}_n^x\}$ is a Markov process. If f is continuous then by 2.6.7 $\mathbb{P}_s^{n_j} f \to \mathbb{P}_s f$ uniformly and $\mathbb{P}_0 f$ is continuous, $\mathbb{P}_{t+s}^{n_j} f \to \mathbb{P}_{t+s} f$, so we have

$$\left\|\mathbb{P}_t^{n_j}\mathbb{P}_s^{n_j} f - \mathbb{P}_t\mathbb{P}_s f\right\|_\infty \leq \|\mathbb{P}_s^{n_j} f - \mathbb{P}_s f\|_\infty + \left\|\mathbb{P}_t^{n_j}\mathbb{P}_s f - \mathbb{P}_t\mathbb{P}_s f\right\|_\infty \to 0 \tag{2.6.32}$$

as $j \to \infty$. This gives the result for functions, and a limiting argument gives the result for all bounded and measurable functions f.

Because $\mathbb{P}_{n_j}^x$ are tight and $\mathbb{P}^x$ is the weak limit, then $\mathbb{P}^x$(The Paths of X_t are continuous) = 1 and and $\mathbb{P}^x(X_0 = x) = 1$. Thus $\mathbb{P}_t f(x) = \mathbb{E}^x f(X_t) \to f(x)$ as $t \to 0$ if f is continuous. By if f is continuous then by 2.6.7 $\mathbb{P}_t f$ is continuous. By the proof of [21] Th. I. 9.4 $\{X_t, \mathbb{P}^x\}$ is a strong Markov Process. ■

Theorem 2.6.9. *The process X_t is nowhere degenerate or more precisely*

$$\mathbb{P}^x(X_t = x \textit{ for all } t) \neq 1. \tag{2.6.33}$$

Proof

The ideas for this proof come from [5]. Let $\tau_\epsilon = \inf\{t : X_t \notin [0, 1-\epsilon]^2\}$. Write

$X_s^{(i)}, i = 1, 2$ for the $x^{(i)}$ coordinates of X_s. Then for any x and all t and ϵ we have

$$\mathbb{P}^x(\tau_\epsilon > t) = \mathbb{P}^x(\sup_{s \le t}(X_s^{(1)} \vee X_s^{(2)} \le 1 - \epsilon) \tag{2.6.34}$$

$$\le \lim_{j\to\infty} \sup \mathbb{P}^x_{n_j}(\sup_{s \le t}(X_s^{(1)} \vee X_s^{(2)} \le 1 - \epsilon/2) \tag{2.6.35}$$

$$= \lim_{j\to\infty} \sup \mathbb{P}^x_{n_j}(\tau_{\epsilon/2} > t) \tag{2.6.36}$$

$$= \lim_{j\to\infty} \sup \mathbb{P}^x_{n_j}(\tau_{\epsilon/2} > t). \tag{2.6.37}$$

We thus have

$$\mathbb{E}^x\tau_\epsilon = \int_0^\infty \mathbb{P}^x(\tau_\epsilon > s)ds \tag{2.6.38}$$

$$\le \limsup \int_0^\infty \mathbb{P}^x_{n_j}(\tau_{\epsilon/2} > s)ds \tag{2.6.39}$$

$$= \limsup \mathbb{E}^x_{n_j}\tau_{\epsilon/2} \tag{2.6.40}$$

$$\le \limsup \mathbb{E}^x_{n_j}\tau \tag{2.6.41}$$

$$\le 1. \tag{2.6.42}$$

By the monotone convergence theorem $\mathbb{E}^x\tau \le 1$, and $\tau < \infty$ a.s. ∎

Theorem 2.6.10. *If $x \in V, \mathbb{P}^x(X_t \notin V$ for some $t < \tau) = 0$.*

Proof

This result is due to Barlow and Bass, [5] Proposition 6.8. If $x \in V \subset V_n$ and $n > m, \mathbb{P}^x_n(X_t$ ever hits $V_0 - V_m) = 0$. It is routine to show that using the regularity of the sets $V_0 - V_m$ and taking limits shows that $\mathbb{P}^x(X_t$ ever hits $V_0 - V_m) = 0$ for all m. This gives the result. ∎

This next result is Proposition 6.9 in [5] and we include it as it is an important step toward the study of the Heat Kernel associated to the process that we have created.

Theorem 2.6.11. *If f is bounded, then $U^\lambda f$ is continuous on V.*

Proof

This result, also due to Barlow and Bass is Proposition 6.9 of [5]. Suppose f is continuous. Then

$$\sup_{\substack{x,y\in V_n \\ |x-y|<\delta}} |U_n^\lambda f(x) - U_n^\lambda f(y)| \leq \omega(\delta)\, \|f\|_\infty \tag{2.6.43}$$

where as before $\omega(\delta) \to 0$ as $\delta \to 0$, independent of n. If we take the limit along the special subsequence n_j gives

$$\sup_{\substack{x,y\in V_n \\ |x-y|<\delta}} |U^\lambda f(x) - U^\lambda f(y)| \leq \omega(\delta)\, \|f\|_\infty\,. \tag{2.6.44}$$

Since the bound $\omega(\delta)\, \|f\|_\infty$ does not depend on the modulus of continuity of f a limiting argument shows that we have 2.6.44. ■

2.7 Two Conjectures about Resistance Estimates and the existence of the Spectral Dimension d_s

For a given $4N$-gon define the resistance constant R_n by

$$R_n^{-1} = \inf\left\{ \int_{l_V^n V_n} |\nabla f|^2 dx : f = 0 \text{ on } x_1 = 0, f = 1 \text{ on } x_1 = l_V^n \right\}. \tag{2.7.1}$$

With this definition R_n is the resistance between two opposite faces of $l_V^n V_n$. In [7] Barlow and Bass show that for Sierpinski Carpets with $d = 2$ there exists a constant

ρ_V and constants c_1, and c_2 for which

$$c_1\rho_V^n \leq R_n \leq c_2\rho_V^n \tag{2.7.2}$$

Here the constants don't depend on n. They then use this result to show that the spectral dimension d_s exists in [13]. We will refer to ρ_V the resistance scale factor of the fractal V. The proof takes advantage of network modifications and a subadditivity argument. Since we have not established the analogous result we state it here as a conjecture that will be assumed for future results.

Conjecture 2.7.1. Let V be a $4N$-gon.

1. There exist constants ρ_V, c_1, c_2 such that $c_1\rho_V^n \leq R_n \leq c_2\rho_V^n$
2. There exists a constant c_3 such that $c_3^{-1}R_nR_m \leq R_{n+m} \leq c_3R_nR_m$ for all $n, m \geq 0$

Conjecture 2.7.2. If conjecture 2.7.1 holds on a given $4N$-gon V then the spectral dimension d_s exists for V.

Assuming both of these conjectures, we may now define the time scale factor $t_V = m_V\rho_V$ and with these we may define the fractal dimension, the walk dimension, and the spectral dimension by

$$d_f = \log m_V / \log l_V, \tag{2.7.3}$$

$$d_w = \log t_F / \log l_V, \tag{2.7.4}$$

$$d_s = 2d_f/d_w = 2\log m_V / \log t_V. \tag{2.7.5}$$

2.8 Transition Density Estimates

Through this section we follow the exposition of Barlow and Bass in [8]. The following is Lemma 3.1. In addition, special emphasis should be given to results that involve the spectral dimension d_s as these are proven under the assumption that one can establish 2.7.2 and the existence of the spectral dimension.

Lemma 2.8.1. *There is a constant c_α, independent of r and constants $\beta_r \in [c_\alpha^{-2} t_V^r, c_\alpha^2 t_V^r]$, where $r \in \mathbb{Z}$ such that if Q^x is equal to the $P^{l_V^r x}$ law of $l_V^{-r} X(t\beta_r)$, then (Q^x, X_t) is also a Brownian motion on $\widetilde{V}$.*

Proof

There exists a sequence $n_j \to \infty$ such that for each $x \in \widetilde{V}$, P^x is the weak limit of $P^x_{n_j}$. Since $c_\alpha^{-2} t_V^r \leq \beta_n^r \leq c_\alpha^2 t_V^r$ there is a subsequence of the subsequence, also called n_j, such that $\beta^r_{n_j+r}$ converges; Write $\lim_{j\to\infty} \beta^r_{n_j+r} = \beta_r$. Assume $n_1 + r \geq 0$. Using the continuity of the paths of X and the properties of the subsequence n_j, the law of $l_V^{-r} W^{n_j}(t\alpha_{n_j}\beta^r_{n_j+r})$ starting at $l_V^r x$ converges to the $P^{l^r x}$ law of $l_V^{-r} W^{n_j}(t l_V^{2r} \alpha_{n_j+r}) = Q^x$. The law starting at $l_V^r x$ of $l_V^{-r} W^{n_j+r}(\alpha_{n_j+r} t) = l_V^{-r} W^{n_j}(t l_V^{2r} \alpha_{n_j+r})$ is equal to the law of $W^{n_j+r}(\alpha_{n_j+r} t)$ starting at x, by Brownian scaling. This implies that Q^x is the weak limit of the law of $W^{m_j}(\alpha_{m_j} t)$ starting at x where $m_j = n_j + r$ is independent of x. ■

What follows is Proposition 3.3 in [8].

Proposition 2.8.2. *There exist c_3 and $c_4 > 0$ such that*

$$\mathbb{P}^x(\sigma_0(x) \leq s) \leq \exp(-c_4 s^{-1/(d_w - 1)}), \quad x \in V \tag{2.8.1}$$

Proof

We begin with the proof of Proposition 2.4.5. Note that since $\widetilde{V}$ is homogenous it applies to $\sigma_0(x)$ as well as τ. This gives

$$\mathbb{P}_n^x(\sigma_0(x) \leq s) \leq \exp\left(2\left(\frac{k^{2r} m_r \alpha_n s}{\alpha_{n-r} c_6}\right)^{1/2} - m_r \log c_6^{-1}\right). \tag{2.8.2}$$

Here $k = l_V$ and $m_r = \frac{1}{2} l^r - 2$. Recall that

$$c_1^{-1}(t_V/l_V^2)^n \leq \alpha_n \leq c_1(t_V/l_V^2)^n. \tag{2.8.3}$$

This gives

$$\mathbb{P}_n^x(\sigma_0(x) \leq s) \leq \exp(c_7((l_V t_V)^r s)^{1/2} - c_8 l_F^r), 3 \leq r \leq n. \tag{2.8.4}$$

Let $r = [\log(c_8^2/4c_7^2 s)/\log(t_V/l_V)]$. Because $t_V > 1$, there exists a c_9 sufficiently small so that if $s \leq c_9$ then, $r = r(s) \geq 3$. Using this r we have

$$\mathbb{P}_n^x(\sigma_0(x) \leq s) \leq \exp(-c_8 l_V^r/2), \tag{2.8.5}$$

$$\leq \exp(-c^4 s^{-1/(d_w-1)}), \quad n \geq r, s \leq c_9. \tag{2.8.6}$$

If we let $n \geq \infty$ using the special subsequence n_j and using 2.6.6 we have 2.8.5 but for $\mathbb{P}$. By definition we have $\mathbb{P}^x(\sigma_0(x) \leq s) \leq 1$ so we can find c_3 such that

$$\mathbb{P}^x(\sigma_0(x) \leq s) \leq c_3 \exp(-c_4 s^{-1/(d_w-1)}), s \geq 0. \tag{2.8.7}$$

■

Theorem 2.8.3. *1. There exist c_3 and $c_4 > 0$ such that for any $r \in \mathbb{Z}$*

$$\mathbb{P}^x(\sigma_r(x) \leq t) \leq c_3 \exp(-c_4(t_V^r l_V)^{-1/(d_w-1)}). \tag{2.8.8}$$

2. *For any* $\lambda > 0$

$$\mathbb{P}^x\left(\sup_{s\le t}|X_s - X_0| > \lambda\right) \le c_3 \exp(-c_1 1(\lambda^{d_w}/t)^{1/(d_w-1)}). \tag{2.8.9}$$

<u>Proof</u>

This argument is due to Barlow and Bass in [8], see Theorem 3.4.

1. Write $y = l_V x$. Let Q be another Brownian motion on $\widetilde{V}$. By 2.8.1

$$\mathbb{P}^x(\sigma_r(x) \le t) = Q^y(\sigma_0(y) \le t/\beta_{-r}). \tag{2.8.10}$$

   By 2.8.2 and the fact that $c_1^{-2} \le \beta_n^r \le c_1^2 t_V^r$ we have our result. ■

2. Let $r = [-\log\lambda/\log l_V]$. Since $B(x, \frac{1}{2}l_V^{-r}) \subset D_r(x) \subset B(y, 3l_V^{-r})$ for any $y \in D_r(x)$, there is a constant c_{10} such that $D_r(x) \subset B(x, c_{10}\lambda)$. Thus

$$\mathbb{P}^x\left(\sup_{s\le t}|X_s - X_0| > c_{10}\lambda\right) \le \mathbb{P}^x(\sigma_r(x) \le t). \tag{2.8.11}$$

   We then use the bound from the previous part of the proof and replace λ with $c_{10}^{-1}\lambda$. ■

Recall that $\mathbb{P}_n^x$ is the law on Ω induced by $X_t^n = W_{\alpha_n t}^n$. Thus X_t^n has a symmetric Green's function $u_n(x, y)$ with respect to μ_n for $f \ge 0$ and $x \in V_n$.

$$\mathbb{E}_n^x \int_0^\tau f(X_s)ds = \int f(y)u_n(x, y)\mu_n(dy). \tag{2.8.12}$$

For all $\epsilon > 0$, $u_n(x, y)$ is Hölder continuous and bounded on the set $\{(x, y) : |x - y| > \epsilon\}$.

Theorem 2.8.4. *Suppose $\epsilon > 0$. Then there exists M such that*

$$|u_n(x,y)| \leq M \text{ for } n > 0, |x-y| > \epsilon, x, y \in V_n. \tag{2.8.13}$$

Proof

The following argument is due to Barlow and Bass in [5], see Theorem7.1. Let $\epsilon \in (0, \frac{1}{8}l_V^2)$ and write $G_n = G_n(\epsilon/2)$. Note that the case where $x, y \in G_n$ is Theorem 2.5.6. So, suppose $x \in G_n, y \in V_n - G_n$, and $|x-y| > \epsilon$. If $z \in G_n \cap \overline{(V_n - G_n)}$, then $|z-x| > \epsilon > 2$. Then $u_n(y,x)$ is harmonic in y and 0 for $y \in \Delta$. So using the maximum principle and Theorem 2.5.6 we have

$$u_n(y,x) \leq \sup_{z \in G_n \cap \overline{(V_n - G_n)}} u_n(z,x) \leq M. \tag{2.8.14}$$

Finally, consider the case $x, y \in V_n - G_n, |x-y| \geq \epsilon$. Let $A \subseteq B_{\epsilon/2}(x)$ be a neighborhood of x such that $A \cap G_n \varnothing$. Since $U_n(y,A)$ is harmonic the maximum principle gives

$$U_n(y,A) \leq \sup_{|y-x|=\epsilon} U_n(y,A), \tag{2.8.15}$$

For this reason we restrict to the case $|x-y| = \epsilon$. We may also assume that $x^{(2)}, y^{(2)} \geq 1/2$. If $x^{(1)}, y^{(1)} \leq 1/2$, let x', y', A' be the reflection of x, y and A across the line $[(0,1/2),(1,1/2)]$. Otherwise reflect across the line $[(0,1),(1,0)]$. This puts $x', y' \in G_n$.

$$\mathbb{E}_n^y \int_0^{\sigma_1} 1_A(X_s)ds = \mathbb{E}_n^{y'} \int_0^{\sigma_1} 1_{A'}(X_s)ds \tag{2.8.16}$$

$$\leq \mathbb{E}_n^{y'} \int_0^{\tau} 1_{A'}(X_s)ds \tag{2.8.17}$$

$$= U_n(y', A') \tag{2.8.18}$$

$$\leq M\mu_n(A') \tag{2.8.19}$$

$$= M\mu_n(A). \tag{2.8.20}$$

From our previous results there exists δ, independent of n, such that $\mathbb{P}_n^y(\sigma_1 = \tau) > \delta$. Write $N = \sup_{\substack{y \in V_n - G_n \\ |x-y|=\epsilon}} U_n(y, A)$. In the case where $|y - x| = \epsilon, y \in V_n - G_n$ we have

$$U_n(y, A) = \mathbb{E}_n^y \int_0^{\tau} 1_A(X_s)ds \tag{2.8.21}$$

$$= \mathbb{E}_n^y \int_0^{\sigma_1} 1_A(X_s)ds + \mathbb{E}_n^y U_n(X_{\sigma_1}, A) \tag{2.8.22}$$

$$\leq M\mu_n(A) + N\mathbb{P}^y(\sigma_1 < \tau) \tag{2.8.23}$$

$$\leq M\mu_n(A) + (1 - \delta)N, \tag{2.8.24}$$

where we have used the Markov Property and the maximum principal. Taking the supremum over $y \in V_n - G_n$ gives $N \leq M\mu_n(A) + (1 - \delta)N$. We then have

$$U_n(y, A) \leq N \leq (M/\delta)\mu_n(A). \tag{2.8.25}$$

Letting A shrink to x we have $u_n(y, x) \leq M/\delta$ because $u_n(y, x)$ is continuous.

Theorem 2.8.5. *Fix $\epsilon > 0$. There exists $K, \alpha, \beta > 0$ independent of ϵ such that*

$$|u_n(x, y_1) - u_n(x, y_2)| \leq K\epsilon^{-\beta}|y_1 - y_2|^{\alpha}, \tag{2.8.26}$$

whenever $n > 0$, $|x - y_1|, |x - y_2| > \epsilon$, and $x, y_1, y_2 \in V_n$.

Proof

This is Theorem 7.2 of [5]. Fix $\epsilon \in (0, 1/8l_V^2)$, and write $G_n = G_n(l_V^{-2})$. The function $u_n(x, y)$ is harmonic in y away from the diagonal, that is, $|x - y| > \epsilon$, and so by 2.3.5 we have the result for $y_1, y_2 \in G_n$. Suppose at least one of $y_1, y_2 \in V_n - G_n$. Further, suppose $|y_1 - y_2| < \epsilon/4$.

The function $u_n(x, y_1) - u_n(x, y_2)$ is harmonic in x so by the maximum and minimum principals, we can restrict to the case where $\epsilon/2 \leq |x - y_1|, |x - y_2| \leq \epsilon$.

For $i = 1, 2$ let A_i be neighborhoods of y_i contained in $B_{\epsilon/2}(y_i)$. Assume $x^{(2)} \geq 1/2$, and let $x', y_1', y_2', A_1', A_2', D_1'(x)$ be the reflections of x, y_1, y_2, A_1, A_2 and $D_1(x)$ across either the line $[(0, 1/2), (1, 1/2)]$ or $[(0, 1), (1, 0)]$ as in the proof of 2.8.4. Let $S = \inf\{t : X_t \notin D_1'(x) \cap (0, 1]^2\}$. Provided the A_i are sufficiently small, we have

$$|\mu_n(A_1')^{-1}U_N(z, A_1') - \mu_n(A_2')| \leq 2|u_n(z, y_1') - u_n(z, y_2')| \tag{2.8.27}$$

$$\leq 2K\epsilon^{-\beta}|y_1 - y_2|^{\alpha} \tag{2.8.28}$$

uniformly for $z \in G_n$. Because

$$\mathbb{E}_n^{x'} \int_0^S 1_{A_i'}(X_s)ds = U_n(x', A_i') - \mathbb{E}^{x'} U_n(X_S, A_i'), \tag{2.8.29}$$

we have the bound

$$|\mu_n(A_1')^{-1}\mathbb{E}_n^{x'}\int_0^S 1_{A_i'}(X_s)ds - \mu_n(A_2')^{-1}\mathbb{E}_n^{x'}\int_0^S 1_{A_2'}(X_s)ds| \leq 4K\epsilon^{-\beta}|y_1 - y_2|^\alpha. \tag{2.8.30}$$

Let $f(z) = \mu_n(A_1)^{-1}U_n(z, A_1) - \mu_n(A_2)^{-1}U_n(z, A_2)$. Since f is harmonic in z for $|z - y_1|, |z - y_2| \geq \epsilon/2$, we may apply the maximum and minimum principals to get

$$\sup_{|z-y_1|,|z-y_2|\geq\epsilon/2} |f(z)| \leq \sup_{\epsilon\geq|z-y_1|,|z-y_2|\geq\epsilon/2} |f(z)| \tag{2.8.31}$$

Write $\theta = \sup_{\epsilon\geq|z-y_1|,|z-y_2|\geq\epsilon/2} |f(z)|$. As in 2.8.4 there exists $\delta > 0$ independent of n such that $\mathbb{P}_n^x(\sigma_1 = \tau) > \delta$ for $x \in V_n - G_n$. By the strong Markov Property we have

$$\mathbb{E}_n^x \int_0^\tau 1_{A_i}(X_s)ds = \mathbb{E}_n^x \int_0^{\sigma_1} 1_{A_i}(X_s)ds + \mathbb{E}_n^x(X_{\sigma_1}, A_i) \tag{2.8.32}$$

$$= \mathbb{E}_n^{x'} \int_0^{\sigma_1} 1_{A_i'}(X_s)ds + \mathbb{E}_n^x[U_n(X_{\sigma_1}, A_i); \sigma_1 < \tau]. \tag{2.8.33}$$

Let z be chosen so that $\epsilon \geq |z - y_1, |z - y_2| \geq \epsilon/2$. Then by 2.8.30 and 2.8.32 we have

$$|f(z)| \leq 4K\epsilon^{-\beta}|y_1 - y_2|^\alpha + \mathbb{E}_n^x[f(X_{\sigma_1}); \sigma_1 < \tau] \tag{2.8.34}$$

$$\leq 4K\epsilon^{-\beta}|y_1 - y_2|^\alpha + \theta(1 - \delta). \tag{2.8.35}$$

If we take the supremum over z and use the definition of θ we have

$$\theta \leq 4K\delta^{-1}\epsilon^{-\beta}|y_1 - y_2|\alpha. \tag{2.8.36}$$

Since $f(x) \leq \theta$, we can let A_1, A_2 shrink to y_1 and y_2 respectively and use continuity of u_n to get

$$||u_n(x,y_1) - u_n(x,y_2)| \leq 4K\delta^{-1}\epsilon^{-\beta}|y_1 - y_2|\alpha. \qquad (2.8.37)$$

■

Theorem 2.8.6. *There exists a symmetric function $u(x,y)$ which is bounded and Hölder continuous on $\{(x,y)|x-y| > \epsilon\}$ for each ϵ and which is the Green's Function for $\{X_t, \mathbb{P}^x\}$: if $f \geq 0, x \in V$*

$$\mathbb{E}^x \int_0^\tau f(X_s)ds = \int u(x,y)f(y)\mu(dy). \qquad (2.8.38)$$

Proof

This is Theorem 7.3 of [5] and the proof is due to Barlow and Bass. Let f be a continuous function on V. It can be extended to a continuous function on V_0. Then as $n_j \to \infty$

$$\int u_{n_j}(x,y)f(y)\mu_{n_j}(dy) = \mathbb{E}^x_{n_j} \int_0^\tau f(X_s)ds \to \mathbb{E}^x \int_0^\tau f(X_s)ds. \qquad (2.8.39)$$

If g is continuous

$$\int g(y)\mu(dy) \to \int g(y)\mu(dy) \qquad (2.8.40)$$

as $n \to \infty$. If $n \geq r$,

$$\mathbb{E}_n^x \int_0^\tau f(X_s)ds \leq \|f\|_\infty \sup_n \mathbb{E}_n^x \sigma_r + \mathbb{E}_n^x \mathbb{E}_n^{X_{\sigma_r}} \int_0^\tau f(X_s)ds \tag{2.8.41}$$

$$\leq \|f\|_\infty \sup_n \mathbb{E}_n^x \sigma_r + \sup_{|z-x|>l_v^{-r}/2} \mathbb{E}_n^z \int_0^\tau f(X_s)ds. \tag{2.8.42}$$

The proof is now a routine application of limits using the above equalities, and inequalities, Theorem 2.8.4, and 2.8.32. ■

What follows is Theorem 7.4 of [5], the proof, due to Barlow and Bass relies on a result by Fukushima.

Theorem 2.8.7. *There exists a symmetric function $p_t(x,y)$ which is the transition density of X (killed at time τ) with respect to μ, where*

$$p_t(x,y) = p_t(y,x) \quad \text{for all } x,y \in V \tag{2.8.43}$$

$$\mathbb{P}^x(X_t \in A, t < \tau) = \int_A p_t(x,y)\mu(dy) \quad \text{for all } A \subseteq V. \tag{2.8.44}$$

Proof

By Fukushima [28] Theorem 4.3.4 the transition semigroup P_t of X is absolutely continuous with respect to μ. Since P_t is self-adjoint with respect to μ, the symmetry of its density p_t follows by [101] Corollary 1.2. ■

Remark 2.8.8. If $x \neq y$, $\mathbb{E}^y \int_0^\tau 1_{\{x\}}(X_s)ds = \int_{\{x\}} u(y,z)\mu(dz) = 0$. We can see that if we begin X_t at x then the process leaves $\{x\}$ immediatly. By the strong Markov property we have

$$\mathbb{E}^x \int_0^\tau 1_{\{x\}}(X_s)ds = 0. \tag{2.8.45}$$

Another consequence of 2.8.45 is that we can define $u(x,x)$ without violating 2.8.4

and 2.8.5

$$u(x,x) = \limsup_{y \to x} u(x,y). \tag{2.8.46}$$

Theorem 2.8.9. *There exists c such that $u(x,y) \leq c$ whenver $x, y \in V$.*

Proof

The ideas in this proof are those of Barlow and Bass, See Theorem 3.1 [6]. Fix $x, y \in V$ where $x \neq y$. Let $\epsilon > 0$, and $A = A(\epsilon) = V_0 \cap B_\epsilon(x)$. Observe $\mu_n(A) \leq 4Nl_V^{-1}\mu(A)$. The goal is to estimate

$$q_{nr}(y, A) = E_n^y \int_{\sigma_{r+1}(x)}^{\sigma_r(x)} 1_A(X_s) ds$$

for $n > r \geq 0$. By the strong Markov property

$$q_{nr}(y, A) \leq \sup_{z \in \partial_{D_{r+1}(x)}} E_n^z \int_0^{\sigma_r(x)} 1_A(X_s) ds. \tag{2.8.47}$$

By the definition of $\sigma_r(x)$, $q_{nr}(y, A) = q_{nr}(y, A \cap D_r(x))$ and $q_{nr}(y, A) = 0$ if $y \notin D_r(x)$. Consider two cases $r \leq 4$ and $r > 4$. If $r \leq 4$. then by 2.8.47,

$$q_{nr}(y, A) \leq \sup_{z \in \partial D_{r+1}(X)} E_n^z \int_0^\tau 1_A(X_s) ds.$$

Notice that if $\partial D_r(x) \cap \partial D_{r-1}(x) \neq \varnothing$ then

$$\inf_{r,x} l_V^r \operatorname{dist}(\partial D_r(x), \partial D_{r-1}(x)) > 0. \tag{2.8.48}$$

If $A \subseteq D_{r+1}(x)$ then by 2.8.48 there exists $\delta_1 > 0$ (independent of r and x) such that $\mathrm{dist}(\partial D_{r+1}(x), D_{r+2}(x)) > \delta_1$.

Then by 3.2 from [1, Sect. 7]

$$\sup_{z\in\partial D_{r+1}(x)} E_n^z \int_0^\tau 1_A(X_s)ds \le \mu_n(A) \sup_{\substack{z\in\partial D_{r+1}(x)\\ w\in A}} u_n(z,w) \le 4Nl_V^{-1}\mu(A)c(\delta_1). \tag{2.8.49}$$

Conversely, if $A \not\subseteq D_{r+1}(x)$ then $\mu(A) \ge \delta_2 > 0$ for some constant δ_2 independent of x and r. Thus

$$E_n^z \int_0^\tau 1_A(X_s)ds \le E_n^z \tau \tag{2.8.50}$$

$$\le 1 \tag{2.8.51}$$

$$\le \delta_2^{-1}\mu(A). \tag{2.8.52}$$

This gives the desired bound

$$q_{nr}(y,A) \le c\mu(A) \tag{2.8.53}$$

in the case that $0 \le r \le 4$ and $r \le n$. The next step is to use scaling to generalize 2.8.53. Let $r > 4$ and $p = r-3$. Suppose that $D_{r+1}(x) \subseteq [0, l_V^{-r+1})^2$. The law of $W^n(t)$ started at x is the same as the law of $l_V^{-p}W^{n-p}(m_V^p t)$ starting at $l_V^{-p}x$ by Lemma 2.8.1. Thus X_t under P_n^x has the same law as $l_V^{-p}X\left(tm_V^p\frac{\alpha_n}{\alpha_{n-p}}\right)$ under $P_{n-p}^{l_v^p x}$.

Write $\theta_{np} = m_V^p \alpha_n / \alpha_{n-p}$, then

$$q_{nr}(y, A) = E_{n-p}^{l_V^p y} \int_{\theta_{np}\sigma_4(l_V^p x)}^{\theta_{np}\sigma_3(l_V^p x)} 1_A(l_V^{-p} X(t\theta_{np}))dt \tag{2.8.54}$$

$$= \theta^{-1} E_{n-p}^{l_V^p y} \int_{\sigma_4(l_V^p x)}^{\sigma_3(l_V^p x)} 1_A(l_V^{-p} X_s)ds \tag{2.8.55}$$

$$= \theta_{np}^{-1} q_{n-p, l_V}(l_V^p y, l_V^p A). \tag{2.8.56}$$

Then 2.8.53 gives

$$q_{nr}(y, A) \leq \theta_{np}^{-1} c\mu(l_V^p A).$$

Since $\mu(l_V^p A) \leq l_V^{pd_f} \mu(A)$. Rewriting θ_{np} one has

$$q_{nr}(y, A) \leq c m_V^{-2p} \alpha_{n-p} \alpha_n^{-1} l_V^{pd_f} \mu(A) \tag{2.8.57}$$

$$\leq c'(\beta_n^{-r}) l^{rd_f} \mu(A) \tag{2.8.58}$$

$$\leq c''(m_V/t_V)^r \mu(A) \tag{2.8.59}$$

Let $n \to \infty$ along the sequence n_j. Now since $U1_{\partial A} \cong 0$ by the remark following 2.8.45

$$E^y \int_{\sigma_{r+1}(x)}^{\sigma_r(x)} 1_A(X_s)ds \leq c''(m_V/t_V)^r \mu(A). \tag{2.8.60}$$

We must remove the restriction $D_r(x) \subseteq [0, l_V^{-r+1})^2$ and consider $D_r(x) \nsubseteq [0, l_V^{-r+1})^2$. Fortunately, one can perform an appropriate set of rotations, translations, and reflections to find $\widehat{x}$, $\widehat{y}$, and $\widehat{A}$ so that

$$q_{nr}(y, A) \leq q_{nr}(\widehat{y}, \widehat{A}) = E_n^{\widehat{y}} \int_{\sigma_{r+1}(\widehat{x})}^{\sigma(\widehat{x})} 1_{\widehat{A}}(X_s)ds$$

where $D_r(\widehat{x}) \subset [0, l_V^{-r+1})^2$ and $\mu_n(A) \geq \mu_n(\widehat{A})$. One may then apply the argument 2.8.57 and 2.8.60 with $q_{nr}(\widehat{y}, \widehat{A})$. Summing over r gives

$$\sum_{r=0}^{\infty} E^y \int_{\sigma_{r+1}(x)}^{\sigma_r(x)} 1_A(X_s)ds = E^y \int_0^{\tau} 1_A(X_s)ds \tag{2.8.61}$$

$$\leq c'' \sum_{r=0}^{\infty} (m_V/t_V)^r \mu(A) \tag{2.8.62}$$

$$\leq c''' \mu(A) \tag{2.8.63}$$

for all ϵ. Since $u(x,y)$ is continuous if $x \neq y$ (see 3.3), one has $u(x,y) \leq c$. Using 2.8.46 concludes the proof. ■

To move from considering functions defined on the fractal V to considering $\widetilde{V}$ one must proceed with λ resolvents rather than Green's functions. Let L_t^y be the local time of X_t at y. One then has

$$\int_0^t f(X_s)ds = \int_{\widetilde{V}} f(y) L_t^y \mu(dy).$$

Let $A \subseteq \mathbb{R}^2$, and write

$$R_A = \inf\{t \geq 0 : X_t \in A^c\}.$$

For $\lambda \geq 0$ we have the following definitions

$$u_A^\lambda(x,y) = E^x \int_0^{R_A} e^{\lambda s} dL_s^y = E^x L_{R_A \wedge R_\lambda}^y,$$

$$U_A^\lambda f(x) = E^x \int_0^{R_A} e^{\lambda s} f(X_s)ds, \text{ for } f \geq 0.$$

also, let R_λ be an independent negative exponential random variable with mean λ^{-1}. One additional notational clarification is the following:

$$u_A(x,y) = u_A^0(x,y), u^\lambda(x,y) = u_{\widetilde{V}}^\lambda(x,y),$$

and define U_A and U^λ similarly.

Lemma 2.8.10. *Let $x \in \widetilde{V}$. Then*

$$E^z L^y_{\sigma(x)} \leq c(l_V^{-m})^{(d_w-d_f)}, \quad z, y \in D_m(x)$$

Proof

This is Proposition 3.6 of [8]. Suppose that $m \geq 5$, and $r \geq m$. Then by 2.8.9 if $n \geq r$ and $p = r-3$, then

$$E_n^y \int_{\sigma_{r+1}(x)}^{\sigma_r(x)} 1_A(X_s)ds \leq c l_V^{-2p} \alpha_{n-p} \alpha_n^{-1} l^{pd_f} \mu(A).$$

Since

$$\beta_n^r = \frac{\alpha_n l^{2r}}{\alpha_{n-r}} \tag{2.8.64}$$

one has

$$E_n^y \int_{\sigma_{r+1}(x)}^{\sigma_r(x)} 1_A(X_s)ds \leq c(\beta_n^r)^{-1} l_V^{rd_f} \mu(A) \leq c(m_V/t_V)^r \mu(A).$$

Letting $n \to \infty$ along the subsequence n_j gives

$$E^y \int_A (L^z_{\sigma_r(x)} - L^z_{\sigma_{r+1}(x)})\mu(dz) = E^y \int_{\sigma_{r+1}(x)}^{\sigma_r(x)} 1_A(X_s)ds \tag{2.8.65}$$

$$\leq c(m_V/t_V)^r \mu(A). \tag{2.8.66}$$

Since L_t^z is continuous in t and z, we may set $A = B_\epsilon(x) \cap \widetilde{V}$. Dividing by $\mu(A)$

and letting $\epsilon \to 0$ gives

$$E^y L^x_{\sigma_r(x)} - E^y L^x_{\sigma_{r+1}(x)} \leq c(m_V/t_V)^r.$$

As above, we sum over r noting that $m_V < l_V^2 < t_V$ to get

$$E^y L^x_{\sigma_m(x)} \leq \sum_{r=m}^{\infty} c(m_V/t_V)^r = c(l_V^{-m})^{(d_w - d_f)}.$$

This gives the case where $m \geq 5$. A similar scaling argument as 2.8.3 addresses the case of $m < 5$. ■

The following result is due to Barlow and Bass, see [8] Lemma 4.1 and 4.3 and Proposition 4.4. The proofs are included for completeness.

Lemma 2.8.11. *For $x \in \widetilde{V}, r \in \mathbb{Z}$,*

$$c_{14}^{-1}(m_V/t_V)^r \leq u_{D_r(x)}(x,x) \leq c_{14}(m_V/t_V)^r.$$

<u>Proof</u>

Write $A = D_r(x)$ then lemma 3.2 gives

$$c_{14}^{-1} t_V^{-r} \leq \mathbb{E}^x \sigma_r(x) = \mathbb{E}^x \int_A L^y_{\sigma_V(x)} \mu(dy) = \int_A u_A(x,y)\mu(dy) \tag{2.8.67}$$

$$\leq \int_A u_A(x,x)\mu(dy) \leq c_{14} m_V^{-r} u_A(x,x). \tag{2.8.68}$$

■

Lemma 2.8.12. *Suppose $A \subseteq B \subseteq \widetilde{V}$, and A is bounded, and $\sup_x u_B^\lambda(x,y) < \infty$. For $x, y \in \widetilde{V}$ one has t*

$$u_A(x,y) = u_B^\lambda(x,y) + \mathbb{E}^x(1_{(R_\lambda \le R_A)} u_A(X_{R_\lambda}, y)) - \mathbb{E}^x(1_(R_\lambda > R_A) u_B^\lambda(X_{R_A}, y)).$$

<u>Proof</u>

Note $R_A \le R_B$. This gives

$$u_A(x,y) = \mathbb{E}^x(L_{R_A}^y; R_\lambda \le R_A) + \mathbb{E}^x(L_{R_A}^y; R_\lambda > R_A) \tag{2.8.69}$$

$$= \mathbb{E}^x(L_{R_\lambda \wedge R_B}^y; R_\lambda \le R_A) + \mathbb{E}^x(1_{(R_\lambda \le R_A)} \mathbb{E}^{X_{R_\lambda}} L_{R_A}^y) \tag{2.8.70}$$

$$+ \mathbb{E}^x(L_{R_\lambda \wedge R_B}^y; R_\lambda > R_A) - \mathbb{E}^x(L_{R_\lambda \wedge R_B}^y - L_{R_A}^y; R_\lambda > R_A) \tag{2.8.71}$$

$$= u_B^\lambda(x,y) + \mathbb{E}^x(1_{(R_\lambda \le R_A)} u_A(X_{R_\lambda}, y)) - \mathbb{E}^x(1_{(R_\lambda > R_A)} u_B^\lambda(X_{R_A}, y)). \tag{2.8.72}$$

Lemma 2.8.13. *There exists $c_{15} > 0$ such that for all $\lambda > 0, x, y, \in \widetilde{V}$,*

$$c_{15}^{-1} \lambda^{d_s/2-1} \le \sup_y u^\lambda(x,x) \le c_{15} \lambda^{d_s/2-1}$$

<u>Proof</u>

Let $x \in \widetilde{V}$ and fix $\lambda > 0$. Choose r so that $c_{16} t_V^{r+1} > \lambda > c_{16} t_V^r$. Set $A = D_r(x)$, and $B = D_m(x)$ where $m < r$. Now, 2.8.11 gives

$$u_B^\lambda(x,x) \le u_B(x,x) \le c_{16}(m_V/t_V)^m < \infty.$$

2.8.12 gives

$$u_B^\lambda(x,x) \le u_A(x,x) + \mathbb{E}^x(1_{(R_\lambda > R_A)} u_B^\lambda(X_{R_A}, x)) \tag{2.8.73}$$

$$\le u_A(x,x) + \mathbb{P}^x(R_\lambda > R_A) u_B^\lambda(x,x). \tag{2.8.74}$$

Note $R_A = \sigma_r(x)$ so by Corollary 3.5 one has

$$u_B^\lambda(x,x) \leq 2u_A(x,x) \leq 2c_{16}(m_V/t_V)^r \leq c_{15}\lambda^{d_s/2-1}.$$

Letting $m \downarrow \infty$ give the upper bound. For the lower bound choose r so that $c_{17}t_V^{r-1} < \lambda \leq c_{17}t_V^r$. As before let $A = D_r(x)$. Let $B = \widetilde{V}$, now by lemma 4.3 one has

$$u_A(x,x) \leq u^\lambda(x,x) + \mathbb{P}^x(R_\lambda \leq R_A)u_A(x,x) \tag{2.8.75}$$

$$\leq u^\lambda(x,x) + \frac{1}{2}u_A(x,x), \tag{2.8.76}$$

2.8.11 gives the lower bound. In addition the middle inequality is immediate from 2.8.11. ■

Theorem 2.8.14. *Mercer's Theorem*

Let

$$T_K f(x) = \int_a^b K(x,s)f(s)ds.$$

Suppose K is a continuous symmetric non-negative definite kernel. Then there exists an orthonormal basis $\{\varphi_i\}_i$ of $L^2[a,b]$ consisting of eigenfunctions of T_K such that the corresponding sequence of eigenvalues $\{\lambda_i\}_i$ is non-negative. The eigenfunctions corresponding to non-zero eigenvalues are continuous on $[a,b]$ and K has the representation

$$K(s,t) = \sum_{j=1}^{\infty} \gamma_j \varphi_j(x)\varphi_j(y).$$

By Mercer's expansion theorem there is a non-increasing sequence of real numbers

$\gamma_j > 0$ and an orthonormal sequence of functions φ_j in $L^2(A, \mu)$ so that

$$\overline{u}^{-\lambda}(x, y) = \sum_{j=1}^{\infty} \gamma_j \varphi_j(x) \varphi_j(y), \tag{2.8.77}$$

$$\overline{U}^{\lambda} f(x) = \sum_{j=1}^{\infty} \gamma_j (f, \varphi_j) \varphi_j(x), \quad \in L^2(A, \mu). \tag{2.8.78}$$

These sums, 2.8.77, and 2.8.78, converge uniformly and in L^2. Write $\lambda_j = \gamma_j^{-1} - \lambda$. Define

$$\overline{p}(t, x, y) = \sum_{j=1}^{\infty} e^{-\lambda_j t} \varphi_j(x) \varphi_j(y), \quad x, y \in D_r(x_0) \cap \widetilde{V} \tag{2.8.79}$$

Lemma 2.8.15. *1. $\overline{p}(t, x, y)$ is non-increasing in t.*

2. If $t > 0$ and $x, y \in A$, then $\overline{p}(t, x, y) \leq \overline{p}(t, x, x)^{1/2} \overline{p}(t, y, y)^{1/2}$.

3. There exists $c_a > 0$ which is independent of r such that

$$\sup_{x,y} \overline{p}(t, x, y) \leq c_a t^{-d_s/2}$$

Proof

This proof is due to Barlow and Bass, see [8] Lemma 5.2. 1 is immediate from the definition of p. 2 follows from Cauchy-Schwarz. Using 2 one can reduce 3 to the case $x = y$. One then has

$$u^{\alpha}(x, x) \geq u_A^{\alpha}(x, x) = \int_0^{\infty} e^{\alpha s} \overline{p}_A(s, x, x) ds \geq \overline{p}_A(t, x, x) \alpha^{-1} (1 - e^{-\alpha t}).$$

If we set $\alpha = t^{-1}$ then by Lemma 2.8.13 we have

$$\overline{p}_A(t,x,x) \le c_b \alpha u^{\alpha}(x,x) \le c_b c_c t^{-d_s/2}$$

. ■

Theorem 2.8.16. *The transition density $\overline{p}(t,x,y)$ is Hölder continuous:*

$$|\overline{p}(t,x,y) - \overline{p}(t,x',y)| \le c_d t^{-1} |x-x'|^{d_w-d_f}$$

Proof

This result is Theorem 5.3 of [8] and the proof is due to Barlow and Bass. Fix t and y. Write $R(x) = \sum_{j=1}^{\infty} (\lambda+\lambda_j) e^{-\lambda_j t} \varphi_j(x)\varphi_j(y)$. Notice that

$$\sup_{a\ge 0} (\lambda + a) e^{-at/2} \le \lambda \vee 2t^{-1}.$$

Using Cauchy-Schwarz and 2.8.15

$$\begin{aligned}
|R(x)| &\le \left(\sum (\lambda+\lambda_j) e^{-\lambda_j t} \varphi_j^2(x)\right)^{1/2} \left(\sum (\lambda+\lambda_j) e^{-\lambda_j t} \varphi_j^2(y)\right)^{1/2} && (2.8.80)\\
&\le \left((\lambda \vee 2t^{-1}) \sum e^{-\lambda_j t/2} \varphi_j^2(x)\right)^{1/2} \left((\lambda \vee 2t^{-1}) \sum e^{-\lambda_j t/2} \varphi_j^2(y)\right)^{1/2} && (2.8.81)\\
&= (\lambda \vee 2t^{-1}) \overline{p}(t/2,x,x)^{1/2} \overline{p}(t/2,y,y)^{1/2} && (2.8.82)\\
&\le c_d \lambda (1 \vee 2(\lambda t)^{-1}) t^{d_s/2}. && (2.8.83)
\end{aligned}$$

In addition

$$\overline{U}^{\lambda} R(x) = \sum (\lambda+\lambda_j) e^{-\lambda_j t} (\overline{U}^{\lambda} \varphi_j(x)) \varphi(y)_j = \overline{p}(t,x,y)$$

.

The Hölder continuity of U^λ gives

$$|\overline{p}(t,x,y) - \overline{p}(t,x',y)| \le c_e|x-x'|^{d_w-d_f}(\lambda t)^{-d_s/2}\lambda(1 \vee 2(\lambda t)^{-1}.$$

Setting $\lambda = t^{-1}$ gives the result as the Hölder continuity in y is immediate from the symmetry of $\overline{p}(t,x,y)$. ■

The theorem below summarizes the properties of the transition function and is essentially due to Barlow and Bass in [8].

Theorem 2.8.17. *1. For each $x, p(t,x,x)$ is decreasing in t.*

2. $p(t,x,y) \le (p(t,x,x))^{1/2}(p(t,y,y))^{1/2}$.

3. $p(t,x,y)$ is symmetric in x and y.

4. Assuming Conjectures 2.7.1, and 2.7.2 we have the following:
$p(t,x,y) \le ct^{-d_s/2}$ and
$p(t,x,y)$ is jointly continuous in x, y, and t with the following bounds

$$|p(t,x,y) - p(t,x',y)| \le c_f t^{-1}|x-x'|^{d_w-d_f},$$

$$|p(t,x,y) - p(s,x,y)| \le c_g(s \wedge t)^{-1-d_s/2}.$$

Proof

These all follow from the corresponding results for $p_{D_r(x_0)}(t,x,y)$ by letting $r \to -\infty$.

Theorem 2.8.18. *Assuming Conjectures 2.7.1, and 2.7.2 there exist constants c_4 and c_5 such that*

$$p(t,x,y) \le c_4 t^{-d_s/2} \exp\left(-c_5\left(\frac{(|x-y|)_w^d}{t}\right)^{\frac{1}{d_w-1}}\right), \quad x,y \in \widetilde{V}. \tag{2.8.84}$$

Proof

This result is due to Barlow and Bass, see Theorem 6.2 of [11]. Fix $x \neq y$ and t. Let $\epsilon < \frac{1}{6}|x-y|, C_x = B(x,\epsilon) \cap \widetilde{V}, C_y = B(y,\epsilon) \cap \widetilde{V}, v_x = \mu|_{C_x}, v_y = \mu|_{C_y}, A_1 = \{z : |z-x| \leq \frac{1}{2}|z-y|\} \cap \widetilde{V}, A_2 = A_1^c \cap \widetilde{V}$. Further let

$$S = \inf\{t : |X_t - X_0| > \frac{1}{3}|x-y|\}. \tag{2.8.85}$$

Then

$$P^{v_x}(X_t \in C_y) = P^{v_x}(X_t \in C_y, X_{t/2} \in A_1) + P^{v_x}(X_t \in C_y, X_{t/2} \in A_2) \qquad I_1 + I_2. \tag{2.8.86}$$

For $z \in C_x$, by Theorem 2.8.3

$$\mathbb{P}^z(X_{t/2} \in A_2) \leq \mathbb{P}^z(S < t/2) \leq c_6 \exp(-c_7(|z-x|^{d_w}/t)^{1/(d_w-1)}). \tag{2.8.87}$$

Write $q(z) = \mathbb{P}(X_t \in C_y | X_{t/2} = z)$ then by Theorem 2.8.17

$$q(z) = \int_{C_y} p(t/2, z, w)\mu(dw) \leq c_1 t^{-d_s/2}\mu(C_y). \tag{2.8.88}$$

We may then write

$$\begin{aligned} I_2 &= \mathbb{E}^{v_x}(q(X_{t/2}); X_{t/2} \in A_2) & (2.8.89)\\ &\leq c_8 \mu(C_x)\mu(C_y) t^{-d_s/2} \exp(-c_5(|z-x|^{d_w}/t)^{1/(d_w-1)}). & (2.8.90) \end{aligned}$$

By the symmetry of $p(t,x,y)$,

$$\mathbb{P}^{v_x}(X_t \in C_y, X_{t/2} \in A_1) = \mathbb{P}^{v_y}(X_t \in C_x, X_{t/2} \in A_1), \tag{2.8.91}$$

thus we can bound I_1 as we did I_2.

$$I_1 + I_2 \leq \mathbb{P}^{v_x}(X_t \in C_y) \tag{2.8.92}$$

$$\leq c_4 \mu(C_x)\mu(C_y) t^{-d_s/2} \exp(-c_5(|x-y|^{d_w}/t)^{1/(d_w-1)}). \tag{2.8.93}$$

If we divide both sides by $\mu(C_x)\mu(C_y)$, and let $\epsilon \to \infty$, by the continuity of $p(t,x,y)$ in each variable. ■

Proposition 2.8.19. *Assuming conjectures 2.7.1 and 2.7.2, there exists* $c_1 > 0$ *such that*

$$p(t,x,x) \geq c_1 t^{-d_s/2} \tag{2.8.94}$$

Proof

This result is due to Barlow, and Bass and we inlcude the proof for completeness. By Theorem 2.8.3

$$\mathbb{P}^x(\sigma_r(x) \leq t) \leq c_2 \exp(-c_3(t_F^r t)^{-1/(d_w-1)}). \tag{2.8.95}$$

Fix s and choose a so that $c_2 \exp(-c_3 a^{-1/(d_w-1)}) \leq 1/2$. Let $r = [\log(a/s)/\log t_F]$. We have

$$\mathbb{P}^x(X_s \in D_r(x)) \geq \mathbb{P}^x(\sigma_r(x) > s) \geq \frac{1}{2}. \tag{2.8.96}$$

Furthermore,

$$\mu(D_r(x)) \le c_{4N} m_F^{-r} \le c_5 s^{d_s/2} \tag{2.8.97}$$

where c_{4N} is the number of $4N$-gons in of scale r which make up $D_r(x)$. By Cauchy-Schwarz we have,

$$\frac{1}{4} \le [\mathbb{P}^x(X_s \in D_r(x))]^2 \tag{2.8.98}$$

$$= \left(\int_{D_r(x)} p(s,x,y)\mu(dy)\right)^2 \tag{2.8.99}$$

$$\le \mu(D_r(x)) \int_{D_r(x)} p(s,x,y)^2 \mu(dy) \tag{2.8.100}$$

$$\le \mu(D_r(x)) p(2s,x,x). \tag{2.8.101}$$

This implies that $p(2s,x,x) \ge (4\mu(D_r(x)))^{-1}$. Combining this with 2.8.97 gives the result.

Proposition 2.8.20. *Assuming conjectures 2.7.1 and 2.7.2 there exist constants c_{10} and c_{11} such that*

$$p(t,x,y) \ge c_{11} t^{-d_s/2} \textit{ for } |x-y| \le c_{10} t^{1/d_w}. \tag{2.8.102}$$

Proof

This argument is due to Barlow and Bass. See Proposition 7.2 in [6].Write $c_{10} = (\frac{1}{2}c_1 c_{6.2}^{-1})^{1/(d_w-d_f)}$. If $a \le c_{10} t^{1/d_w}$ then $c_{6.2} t^{-1} a^{d_w-d_f} \le \frac{1}{2} c_1 t^{-d_s/2}$. By Theorem 2.8.17, if $|x-y| \le c_{10} t^{1/d_w}$ then

$$p(t,x,y) \ge p(t,x,x) - |p(t,x,y - p(t,x,x)| \tag{2.8.103}$$

$$\le c_1 t^{-d_s/2} - c_{6.2} t^{-1} |x-y|^{d_w - d_f} \tag{2.8.104}$$

$$\le \frac{1}{2} c_1 t^{-d_s/2}. \tag{2.8.105}$$

Write $d(x,y)$ for the length of the shortest path in $\widetilde{V}$ connecting points x and y in $\widetilde{V}$.

The following result is due to Barlow and Bass. Like the previous, it comes from [6].

Lemma 2.8.21. *There exists a constant c_{17} depending on V_1, such that for $x, y \in \widetilde{V}$*

$$|x-y| \le d(x,y) \le c_{17}|x-y|. \tag{2.8.106}$$

<u>Proof</u>

For $x \in \widetilde{V}$ let $\varphi_n(x)$ denote the q_{4N} vertex of the $4N$-gons $\mathcal{O}$ in $\mathcal{S}_n$ containing x. Write

$$H_n = \bigcup \{\partial S : S \in \mathcal{S}_n, S \in \widetilde{V}\} \tag{2.8.107}$$

and note that $H_n \subset \widetilde{V}$.

For $x, y \in H_n$ write $d_n(x,y)$ for the length of the shortest path in H_n connecting x and y. Write

$$c_{16} = \sup\{d_1(0,y) : y \in V \cap H_1\} \tag{2.8.108}$$

Let $x \in V$. By scaling we have

$$d_n(\psi_{n+1}(x), \psi_n(x)) \leq l_V^{-n} c_{16}. \tag{2.8.109}$$

This gives

$$d(x, 0) \leq \sum_{n=0}^{\infty} d_n(\psi n + 1(x), \psi_n(x)) \leq 2c_{16}. \tag{2.8.110}$$

We also have

$$d(\psi_n(x), x) \leq 2c_{16} l_V^{-n}. \tag{2.8.111}$$

Now we consider $x, y \in \widetilde{V}$ and choose m so that $y \in D_m(x) - D_{m+1}(x)$. Then $|x - y| \geq c l_V^{-m}$. Let z be the center of $D_m(x)$. By center, what is meant is that z is the center of the $4N$-gon $\mathcal{O} \subset D_m(x)$ which contains x. We have

$$d(x, y) \leq d(x, z) + d(z, y) \leq 4c_{16} l_V^{-m}. \tag{2.8.112}$$

Rearranging gives the result. ■

The forthcoming argument for the lower bound comes from [26]. Also see Theorem 7.4 in [6].

Theorem 2.8.22. *Assuming conjectures 2.7.1 and 2.7.2, there exist* c_{18}, c_{19} *such that*

$$p(t, x, y) \geq c_{18} t^{-d_s/2} \exp(-c_{19} |x - y|^{d_w}/t)^{1/(d_w - 1)}, \quad x, y \in \widetilde{V}. \tag{2.8.113}$$

Proof

Write $D = c_{17}|x - y|$. Using Proposition 2.8.20 the result is immediate if $D \leq$

$c_{17}c_{10}t^{1/d_w}$, where we write $c_{10} = c_{10}c_{17}$. There exists c_{21} depending on c_{20} and d_w such that if n is the largest integer less than or equal to $c_{21}t^{-1/(d_w-1)}D^{d_w/(d_w-1)}$ then $n \geq 4$ and $3D/n \leq c_{10}(t/n)^{t/d_w}$. Let $x_0 = x, x_n = y$, and choose $x_1, x_2, \ldots, x_{n-1} \in \widetilde{V}$ such that $d(x_{i+1}, x_i) \leq 2D/n$. Let $\epsilon = D/n$ and $B_i = B(x_i, \epsilon) \cap \widetilde{V}$. If $z \in B_i$,

$$|x_{i-1} - z| \leq 2D/n + \epsilon \leq 3D/n \leq c_{10}(t/n)^{1/d_w}, \tag{2.8.114}$$

so that $p(t/n, x_{i-1}, z) \geq c_{11}(t/n)^{-d_s/2}$. We now have the bound

$$p(t,x,y) \geq \int_{B_1} \cdots \int_{B_{n-1}} p(t/n, x, y_1) \cdots p(t,n,y_{n-2},y_{n_1}) p(t,n,y_{n-1},y) \mu(d_1) \cdots \mu(d_{y_{n-1}}) \tag{2.8.115}$$

$$\geq (\prod_{i=1}^{n-1} \mu(B_i)) c_{11}^n (t/n)^{-d_s n/2} \tag{2.8.116}$$

$$\geq c_{19}^n (D/n)^{d_f(n-1)} (t/n)^{-d_s n/2}. \tag{2.8.117}$$

Using the fact that $d_s/2 = d_f/d_w$ and the above choice of n, $(D/n)/(t/n)^{1/d_w}$ is bounded above and below by positive constants that don't depend on D and t, we have

$$p(t,x,y) \geq c_{21}^n c_{22} (t/n)^{-d_f/d_w} \tag{2.8.118}$$

$$\geq c_{21}^n c_{23} t^{-d_s/2} \tag{2.8.119}$$

$$= c_{23} t^{-d_s/2} \exp(-n \log c_{21}^{-1}), \text{ where } c_{21} < 1. \tag{2.8.120}$$

Substituting in the choice of n in 2.8.120

We now may collect the previous results into the following theorem.

Theorem 2.8.23. *Assuming conjectures 2.7.1 and 2.7.2, there is a function* $p(t,x,y), 0 <$

$t < \infty, x, y \in \widetilde{V}$, such that

1. *$p(t, x, y)$ is the transition density of X with respect to μ.*

2. *$p(t, x, y) = p(t, y, x)$ for all x, y, t.*

3. *$(t, x, y) \to p(t, x, y)$ is jointly continuous on $(0, \infty) \times \widetilde{V} \times \widetilde{V}$.*

4. *There exist constants $c_1, c_2, c_3, c_4 > 0$ and d_w such that writing $d_s = 2d_f/d_w$ we have*

$$c_1 t^{-d_s/2} \exp(-c_2(|x-y|^{d_w}/t)^{1/(d_w-1)}) \le p(t, x, y) \le c_3 t^{-d_s/2} \exp(-c_4(|x-y|^{d_w}/t)^{1/(d_w-1)}) \tag{2.8.121}$$

5. *$p(t, x, y)$ is Hölder continuous of order $d_w - d_f$ in x and y and C^∞ in t on $(0, \infty) \times \widetilde{V} \times \widetilde{V}$. Moreover, there exists a constant c_5 such that*

$$|p(t, x, y) - p(t, x', y)| \le c_5 t^{-1} |x - x'|^{d_w - d_f}, \text{ for } t > 0, x, x', y \in \widetilde{V}, \tag{2.8.122}$$

   and for each $k \ge 1, \partial^k p(t, x, y)/\partial t^k$ is Hölder continuous of order $d_w - d_f$ in each space variable.

Chapter 3

The Uniqueness of the process

The uniqueness of the diffusion on generalized Siérpinski carpets is proved in [12]. In this work we present a partial proof for other fractals with the assumption of resistance estimates (See section 2.7 and in particular 2.7.1, 2.7.2, 3.4.1). Our proof is not complete but we will explain the steps which are missing. The presentation follows [12]. Therein, the authors prove results about abstract Dirichlet forms which are invariant under the group of symmetries of the space. They also demonstrate that these results can be applied to the specific constructions of Barlow-Bass construction, see [5, 6, 7, 8, 9], and Kusuoka-Zhou [73]. It turns out that with minimal alteration the same techniques apply for our class of fractals. For completeness, we include their presentation as it demonstrates techniques which apply to a broader class of fractals than those in our planar setting.

3.1 General Properties of Dirichlet Forms

For the sake of brevity we summarize some general properties of Dirichlet Forms. For proofs of the following propositions see [12]. For definitions relating to Dirichlet forms the reader should see [22] or [30]. Let V be a compact metric space and m a Radon measure on V. For any Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(V, m)$ write $\mathcal{E}_1(u, u) = \mathcal{E}(u, u) + \|u\|_2^2$. The functions in $\mathcal{F}$ are defined up to quasi-everywhere equivalence. Furthermore quasi-continuous modifications are used where applicable. Write $\langle \cdot, \cdot \rangle$ for the inner product in $L^2(V, m)$ and $\langle \cdot, \cdot \rangle_S$ for the inner product on a subset $S \subset V$.

Theorem 3.1.1. *Suppose that $(\mathcal{A}, \mathcal{F}), (\mathcal{B}, \mathcal{F})$ are local regular conservative irreducible Dirichlet forms on $L^2(V, m)$ and that*

$$\mathcal{A}(u, u) \leq \mathcal{B}(u, u) \quad \text{for all } u \in \mathcal{F}. \tag{3.1.1}$$

Let $\delta > 0$, and $\mathcal{E} = (1+\delta)\mathcal{B} - \mathcal{A}$. Then $(\mathcal{E}, \mathcal{F})$ is a regular local conservative irreducible Dirichlet form on $L^2(V, m)$.

For the rest of the section assume that $(\mathcal{E}, \mathcal{F})$ is a local regular Dirichlet form on $L^2(V, m)$ for which $1 \in \mathcal{F}$, and $\mathcal{E}(1, 1) = 0$. Write T_t for the semigroup associated with $\mathcal{E}$ and X for the diffusion.

Lemma 3.1.2. *The semigroup T_t is recurrent and conservative.*

The next set of results require a bit of additional notion. Let D be a Borel subset of V. Write T_D for the hitting time of D, and τ_D for the first exit time of D. Formally we write:

$$T_D = T_D^X = \inf\{t \geq 0 : X_t \in D\}, \quad \tau_D = \tau_D^X = \inf\{t \geq 0 : X_t \notin D\}. \tag{3.1.2}$$

Let $\overline{T}$ be the semigroup of the process X which is killed when it exits D, and write $\overline{X}$ for the killed process. We use the function $q(x)$ for the probability that starting at x the process doesn't leave D in finite time, or

$$q(x) = \mathbb{P}^x(\tau_D = \infty) \tag{3.1.3}$$

$$E_D = \{x : q(x) = 0\} \tag{3.1.4}$$

$$Z_D = \{x : q(x) = 1\} \tag{3.1.5}$$

Lemma 3.1.3. *If D be a Borel subset of V, then $m(D-(E_D \bigcup Z_D)) = 0$. In addition, E_D and Z_D are invariant sets for the killed process $\overline{X}$, and Z_D is an invariant set for the process X.*

In what follows there will be instances in which it is useful to consider two definitions of harmonic functions, one based on Dirichlet forms, and one defined probabilistically. Let D be a Borel subset of V and let $h : V \to \mathbb{R}$. We say that a function is harmonic in D in the probabilistic sense if $h(X_{t\wedge\tau_{D'}})$ is a uniformly integrable martingale under $\mathbb{P}^x$ for q.e. x whenever $D' \subset D$ is relatively open. We say that a function is harmonic in the Dirichlet form sense if $h \in \mathcal{F}$ and $\mathcal{E}(h, u) = 0$ whenever $u \in \mathcal{F}$ is continuous and supported in D.

Proposition 3.1.4. *1. Let $(\mathcal{E}, \mathcal{F})$ and D satisfy the above conditions, and let $h \in \mathcal{F}$ be bounded. Then h is harmonic in a domain D in the probabilistic sense if and only if it is harmonic in the Dirichlet form sense.*

2. If h is a bounded Borel measurable function in D, and D' is a relatively open subset of D, then $h(X_{t\wedge\tau_{D'}})$ is a martingale under $\mathbb{P}^x$ for q.e. $x \in E_D$ if and only if $h(x) = \mathbb{E}^x(h(X_{\tau_{D'}})$ for q.e. $x \in E_D$.

A function is said to be caloric in the probabilistic sense if $u(t, x) = \mathbb{E}^x[f(X_{t\wedge\tau_D})]$

for some bounded Borel function $f : V \to \mathbb{R}$. We view $u(t, x)$ as the solution to the heat equation with boundary data f outside of D, and initial data $f(x)$ inside of D. A function $u : \mathbb{R}_+ \times V \to \mathbb{R}$ is caloric in the Dirichlet Form sense if there exists a function h which is harmonic in D and a bounded Borel function $f_D : V \to \mathbb{R}$ which vanishes outside of D for which $u(t, x) = h(x) + \overline{T}_t f_D$.

Proposition 3.1.5. *Let $(\mathcal{E}, \mathcal{F})$ and D satisfy the above conditions, and let $f \in \mathcal{F}$ be bounded and $t \geq 0$. Then*

$$\mathbb{E}^x[f(X_{t\wedge\tau_D})] = h(x) + \overline{T}_t f_D \quad q.e \tag{3.1.6}$$

where $h(x) = \mathbb{E}^x[f(X_{\tau_D})]$ is the harmonic function that coincides with f on D^c and $f_D(x) = f(x) - h(x)$.

Since $\mathcal{E}$ is regular, $\mathcal{E}(f, f)$ can be written in terms of an energy measure $\Gamma(f, f)$. If $f \in cal f_b$ then $\Gamma(f, f)$ is the unique smooth Borel measure on V for which we have.

$$\int_V g d\Gamma(f, f) = 2\mathcal{E}(f, fg) - \mathcal{E}(f^2, g), \quad g \in \mathcal{F}_f \tag{3.1.7}$$

Lemma 3.1.6. *If $\mathcal{E}$ is a local regular Dirichlet form with domain $\mathcal{F}$, then for any $f \in \mathcal{F} \cap L^\infty(V)$ we have $\Gamma(f, f)(A) = 0$, where $A = \{x \in V : f(x) = 0\}$.*

Lemma 3.1.7. *Given an m-symmetric Feller process on V, the corresponding Dirichlet form $(\mathcal{E}, \mathcal{F})$ is regular.*

For brevity we give notation for different types of interior and boundary subsets of V. For $A \subset V$ write $\text{int}_V(A)$ for the interior of A with respect to the metric space (V, d), $\partial_V(A) = \overline{A} - \text{int}_V(A)$. In contrast, for $U \subset \mathbb{R}^d$ write U^0 for the interior of U

with respect to the usual topology in $\mathbb{R}^d$; $\partial U = \overline{U} - U^0$ for the usual boundary of U. Denote the set of $4N$-gons of scale n by $\mathcal{S}_n$, so that

$$\mathcal{S}_n = \mathcal{S}_n(V) = \{\mathcal{O} \cap V : \mathcal{O} \in \mathcal{O}_n(V)\}. \tag{3.1.8}$$

Let A be a finite union of elements of $\mathcal{S}_n$, that is, $A = \bigcup_{i=1}^k S_i = V \cap V \cap \mathcal{O}_i$ for $\mathcal{O} \in \mathcal{O}_n(V)$. Then define $\operatorname{int}_r(A) = V \cap (\bigcup_{i=1}^k \mathcal{O}_i)^0$, and $\partial_r(A) = A - \operatorname{int}_r(A)$. Then $\operatorname{int}_r(A) = A - \partial(\bigcup_{i=1}^k \mathcal{O}_i)$.

Definition 3.1.8. We begin by defining three maps which, together, will be composed to form our folding map. Let $\varphi_F : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as

$$\varphi_F((x_1, x_2)) = (|x_1|, x_2). \tag{3.1.9}$$

Let $\varphi_R : \mathbb{R}^2 \to \mathbb{R}^2$ be rotation by $\frac{\pi}{2N}$ radians about the origin. Let S' be the $4N$-gon in V_{n-1} which contains S, and write $z = (z_1, z_2)$ for the center of S'. For $x \in S'$ We may define $\varphi'_{(S',S)} : S' \to S$ by

$$\varphi'_{(S',S)}(x) = z + \varphi_R^{-k_1} \circ (\varphi_F \circ \varphi_R)^{2N} \circ \varphi_F \circ \varphi_R^{k_1}(x - z). \tag{3.1.10}$$

Here k_1 is chosen so that applying φ_R k_1 times places S so that it's left edge touches the y-axis in the first quadrant. If $x \in S_n \subset \cdots \subset S_0$, where $S_0, \ldots, S_n$ be $4N$-gons in $V_0, \ldots, V_n$ respectively. Then we define $\varphi_S : V \to S$ by

$$\varphi_S(x) = \varphi'_{(S_{n-1},S_n)} \circ \cdots \circ \varphi'_{(S_1,S_2)} \circ \varphi'_{(S_0,S_1)}(x). \tag{3.1.11}$$

Lemma 3.1.9. *1. The function φ_S is the identity on S and for each $S' \in \mathcal{S}_n$, $\varphi_S : S' \to S$ is an isometry.*

2. If $S_1, S_2 \in \mathcal{S}_n$ then

$$\varphi_{S_1} \circ \varphi_{S_2} = \varphi_{S_1} \tag{3.1.12}$$

3. Let $x, y \in V$. If there exists $S_1 \in \mathcal{S}_n$ such that $\varphi_{S_1}(x) = \varphi_{S_1}(y)$, then $\varphi_S(x) = \varphi_S(y)$ for every $S \in \mathcal{S}_n$.

4. Let $S \in \mathcal{S}_n$ and $S' \in S_{n+1}$. If $x, y \in V$ and $\varphi_S(x) = \varphi_S(y)$ then $\varphi_{S'}(x) = \varphi_{S'}(y)$.

Proof

1. If we write $\varphi_S(B)$ for the union of the set $\varphi_S(x)$ for $x \in B$, then we have

$$\begin{aligned}
\varphi_S(S) &= \bigcup_{x\in S} \varphi'_{(S',S)}(x) && (3.1.13)\\
&= \bigcup_{x\in S} \left[z + \varphi_R^{-k_1} \circ (\varphi_F \circ \varphi_R)^{2N} \circ \varphi_F \circ \varphi_R^{k_1}(x - z)\right] && (3.1.14)\\
&= \bigcup_{x\in S} [z + x - z] && (3.1.15)\\
&= S. && (3.1.16)
\end{aligned}$$

   To see that if $S' \in \mathcal{S}_n$ then $\varphi_S : S' \to S$ is an isometry notice that $\varphi'_{(S',S)}$ is built up of functions that are isometries.

2. This follows from the previous part.

3. By construction each $S \in \mathcal{S}_n$ is mapped onto one $4N$-gon of scale n. The conclusion follows.

4. Assume $S' \subset S$. Write $\widetilde{S}$ for the $4N$-gon of scale $n - 1$ which contains S. We

have

$$\varphi_{S'}(x) = \varphi'_{(S,S')} \circ \varphi'_{(S'',S')} \circ \cdots \circ \varphi'_{(S_0,S_1)}(x) \tag{3.1.17}$$

$$= \varphi'_{(S,S')} \circ \varphi_S(x) \tag{3.1.18}$$

$$= \varphi'_{(S,S')} \circ \varphi_S(y) \tag{3.1.19}$$

$$= \varphi_{S'}(y). \tag{3.1.20}$$

For the case where S' is not contained in S we use part 3 and a similar argument.

■

For $S \in \mathcal{S}_n, f : S \to \mathbb{R}$ and $g : V \to \mathbb{R}$ we define the unfolding and restriction operators by the following,

$$U_S f = f \circ \varphi_S, \quad R_S g = g|_S \tag{3.1.21}$$

Using the above lemma we can see that for $S_1, S_2 \in \mathcal{S}_n$

$$U_{S_2} R_{S_2} U_{S_1} R_{S_1} = U_{S_1} R_{S_1} \tag{3.1.22}$$

We now give some results concerning Dirichlet forms which are invariant with respect to the local symmetries of V.

3.2 The Theory of V-invariant Dirichlet Forms

For this section let $(\mathcal{E}, \mathcal{F})$ be a local regular Dirichlet form on $L^2(V, \mu)$. Let $S \in \mathcal{S}_n$. Write

$$\mathcal{E}^S(g, g) = \frac{1}{m_V^n}\mathcal{E}(U_S g.U_S g). \tag{3.2.1}$$

Define the domain of $\mathcal{E}^S$ to be $\mathcal{E}^S = \{g \mid g : S \to \mathbb{R}, U_S g \in \mathcal{F}\}$. Also we will write $\mu_S, \mu|_S$.

Definition 3.2.1. Let $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form on $L^2(V, \mu)$. We say that $\mathcal{E}$ is a V-invariant Dirichlet form, i.e. that $\mathcal{E}$ is invariant with respect to all of the local symmetries of V if the following hold:

1. If $S \in \mathcal{S}_n(V)$, then $U_S R_S f \in \mathcal{F}$ for any $f \in \mathcal{F}$.

2. Let $n \geq 0$ and S_1, S_2 be any two elements of $\mathcal{S}_n$, and let Φ be any isometry of $\mathbb{R}^d$ which maps S_1 to S_2. If $f \in \mathcal{F}^{S_2}$, then $f \circ \Phi \in \mathcal{F}^{S_1}$ and

$$\mathcal{E}^{S_3}(f \circ \Phi, f \circ \Phi) = \mathcal{E}^{S_2}(f, f). \tag{3.2.2}$$

3. For all $f \in \mathcal{F}$,

$$\mathcal{E}(f, f) = \sum_{S \in \mathcal{S}_n(V)} \mathcal{E}^S(R_s f, R_s f). \tag{3.2.3}$$

We write $\mathfrak{E}$ for the set of V-invariant, non-zero, local, regular, conservative Dirichlet forms. We also give a definition of a scale invariant Dirichlet form.

Definition 3.2.2. If $\Psi_{\mathcal{O}}, \mathcal{O} \in \mathcal{O}_1(V_1)$ are the affine maps which define V_1, and if $(\mathcal{E}, calf)$ is a Dirichlet form on $L^2(V, \mu)$ and that

$$f \circ \Psi_{\mathcal{O}} \in \mathcal{F} \text{ for all } \mathcal{O} \in \mathcal{O}_1(V_1), f \in \mathcal{F}, \tag{3.2.4}$$

the we define replication of $\mathcal{E}$ by

$$\mathcal{R}\mathcal{E}(f,f) = \sum_{\mathcal{O} \in \mathcal{O}_1(V_1)} \mathcal{E}(f \circ \Psi_{\mathcal{O}}, f \circ \Psi_{\mathcal{O}}). \tag{3.2.5}$$

If 3.2.4 holds then $(\mathcal{E}, \mathcal{F})$ is scale invariant, and there exists $\lambda > 0$ such that

$$\mathcal{R}\mathcal{E} = \lambda \mathcal{E} \tag{3.2.6}$$

Lemma 3.2.3. *Let $(\mathcal{A}, \mathcal{F}), (\mathcal{B}, \mathcal{F}) \in \mathfrak{E}$ and $\mathcal{A} \geq \mathcal{B}$. Then $\mathcal{C} = (1+\delta)\mathcal{A} - \mathcal{B} \in \mathfrak{E}$ for any $\delta > 0$.*

Proof

The conditions of Definition 3.2.1 hold, so the this lemma follows from Theorem 3.1.1. ■

Proposition 3.2.4. *If $\mathcal{E} \in \mathfrak{E}$ and $S \in \mathcal{S}_n(V)$, then $(\mathcal{E}^S, \mathcal{F}^S)$ is a local regular Dirichlet form on $L^2(S, \mu_S)$.*

Proof

This argument is due to Barlow, Bass, Kumagai, and Teplyaev see [12] Proposition 2.20. If $u, v \in \mathcal{F}^S$ with compact support and v is constant in a neighborhood of the support of u, then $U_s u, U_s v \in \mathcal{F}$. Since $\mathcal{E}$ is local we conclude $\mathcal{E}(U_s u, U_s v) = 0$. Using 3.2.1 we conclude that $\mathcal{F}^S(u, v) = 0$.

Since $\mathcal{S}$ is local by Theorem 3.1.1 it is Markovian. As $1 \in \mathcal{F}, \mathcal{S}(1,1) = 0$ by 3.2.1

$\mathcal{S}$ is conservative.

If $h \in \mathcal{F}$ then by 3.2.1 $\mathcal{E}^S(R_S h, R_S h) \leq \mathcal{E}(h,h)$. Let $f \in \mathcal{F}^S$, then $U_s f \in \mathcal{F}$. Since $\mathcal{E}$ is regular, given $\epsilon > 0$ there exists $g \in \mathcal{F}$ which is continuous such that $\mathcal{E}_1(U_S f - g, U_S f - g) < \epsilon$. Then $R_S U_S f - R_S g = f - R_S g$ on S. This gives

$$\mathcal{E}_1^S(f - R_S g, f - R_S g) = \mathcal{E}_1^S(R_S U_S f - R_S g, R_S U_S f - R_S g) \tag{3.2.7}$$

$$\leq \mathcal{E}_1(U_S f - g, U_S f - g) < \epsilon. \tag{3.2.8}$$

From the continuity of $R_S g$, we see that $\mathcal{F}^S \cap C(S)$ is dense in $\mathcal{F}^S$ in the $\mathcal{E}_1^S$ norm. Similarly one can show that $\mathcal{F}^S \cap C(S)$ is dense in $C(S)$ in the supremum norm. This shows that E^S is regular.

Assume that f_m is Cauchy with respect to $\mathcal{E}_1^S$, then $U_S f_m$ is Cauchy with respect to $\mathcal{E}_1$. Thus $U_S f_m$ converges with respect to $\mathcal{E}_1$, and it follows $R_S(U_S f_m) = f_m$ converges with respect to $\mathcal{E}_1^S$. This shows that $\mathcal{E}^S$ is closed. Taken together we have shown that $(\mathcal{E}^S, \mathcal{F}^S)$ is a local regular Dirichlet form on $L^2(S, \mu_S)$. ■

Fix n and for functions f on V define

$$\Theta f = \frac{1}{m_V^n} \sum_{S \in \mathcal{S}_n(V)} U_S R_S f. \tag{3.2.9}$$

If $S_1, S_2 \in \mathcal{S}_n$ then we have $U_{S_2} R_{S_2} U_{S_1} R_{S_1} = U_{S_1} R_{S_1}$. This implies that $Theta^2 = Theta$ making $Theta$ a projection operator. Further it is bounded on $C(V)$ and $L^2(V, \mu)$ and $Theta : \mathcal{F} \to \mathcal{F}$.

The following Proposition is 2.21 from [12].

Proposition 3.2.5. *Let $\mathcal{E}$ be a local regular Dirichlet form on V, denote its semigroup by T_t, and assume that $U_S R_S f \in \mathcal{F}$ when $S \in \mathcal{S}_n(V)$ and $f \in \mathcal{F}$. Then the* *following are equivalent:*

1. *For all* $f \in \mathcal{F}$, $\mathcal{E}(f,f) = \sum_{S \in \mathcal{S}_n(V)} \mathcal{E}^S(R_S f, R_S f)$.

2. *For all* $f, g \in \mathcal{F}$,

$$\mathcal{E}(\Theta f, g) = \mathcal{E}(f, \Theta g). \tag{3.2.10}$$

3. *For any* $f \in L^2(V, \mu)$ *and* $t \geq 0$ $T_t \Theta f = \Theta T_t f$ *a.e.*

Proof

To prove that $1 \Rightarrow 2$ note that 1 implies

$$\mathcal{E}(f,g) = \sum_{T \in \mathcal{S}_n(V)} \mathcal{E}^T(R_T f, R_T g) = \frac{1}{m_V^n} \sum_{T \in \mathcal{S}_n(V)} \mathcal{E}(U_T R_T f, U_T R_T g). \tag{3.2.11}$$

Using the definition of Θ, 3.1.22, and 3.2.11 we have

$$\begin{aligned}
\mathcal{E}(\Theta f, g) &= \frac{1}{m_V^n} \sum_{S \in \mathcal{S}_n(V)} \mathcal{E}(U_S R_S f, g) && (3.2.12)\\
&= \frac{1}{m_V^{2n}} \sum_{S \in \mathcal{S}_n(V)} \sum_{T \in \mathcal{S}_V} \mathcal{E}(U_T R_T f, U_T R_T g) && (3.2.13)\\
&= \frac{1}{m_V^{2n}} \sum_{S \in \mathcal{S}_n(V)} \sum_{T \in \mathcal{S}_n(V)} \mathcal{E}(U_S R_S f, U_T R_T g). && (3.2.14)
\end{aligned}$$

A similar calculation shows that $\mathcal{E}(f, \Theta g)$ is equal to the above line with the summations reversed. To prove that 2$\Rightarrow$3 we let $\mathcal{L}$ be the generator corresponding to $\mathcal{E}$. Let

$f \in \mathcal{D}(\mathcal{L})$ and $g \in \mathcal{F}$ and write $\langle f, g \rangle$ for the inner product $\int_V f g d\mu$. We have

$$\langle \Theta \mathcal{L} f, g \rangle = \langle \mathcal{L} f, \Theta g \rangle \tag{3.2.15}$$

$$= -\mathcal{E}(f, \Theta g) \tag{3.2.16}$$

$$= -\mathcal{E}(\Theta f, g) \tag{3.2.17}$$

using the fact that Θ is self-adjoint in the L^2 sense and the assumption from 2. This is equivalent to

$$\Theta f \in \mathcal{D}(\mathcal{L}) \text{ and } \Theta \mathcal{L} f = \mathcal{L} \Theta f. \tag{3.2.18}$$

This implies that any bounded Borel function of $\mathcal{L}$ commutes with Θ by [90] Theorem 13.33. Specifically, the L^2 semi-group T_t of $\mathcal{L}$ commutes with Θ in the L^2 sense which implies 3. To prove 3⇒2 let $f, g \in \mathcal{F}$. We have

$$\mathcal{E}(\Theta f, g) = \lim_{t \to 0} t^{-1} \langle (I - T_t) \Theta f, g \rangle \tag{3.2.19}$$

$$= \lim_{t \to 0} t^{-1} \langle \Theta (I - T_t) f, g \rangle \tag{3.2.20}$$

$$= \lim_{t \to 0} t^{-1} \langle (I - T - t) f, \Theta g \rangle \tag{3.2.21}$$

$$= \lim_{t \to 0} \langle f, (I - T_t) \Theta g \rangle \tag{3.2.22}$$

$$= \mathcal{E}(f, \Theta g) \tag{3.2.23}$$

We now prove that 2⇒1. Assume f, g bounded. Define

$$N_n(X) = \sum_{S \in \mathcal{S}_n(V)} 1_S(x) \tag{3.2.24}$$

is the number of $4N$-gons whose interiors intersect V and contain the point x. Observe,

$$\sum_{S\in\mathcal{S}_n(V)} \frac{1_S(x)}{N_n(x)} = 1 \quad x \in \mathcal{F}. \tag{3.2.25}$$

The proof is longer so we present it as a series of steps.

Step 1. If $\Theta f = f$ then $\Theta(hf) = f(\Theta h)$. We begin with 3.1.22, and sum over $S \in \mathcal{S}(V)$ and divide by m_V^n to get

$$U_T R_T f = U_T R_T \Theta(f) = \Theta f = f. \tag{3.2.26}$$

We have $R_S(f_1, f_2) = R_S(f_1)R_S(f_2)$ and $U_S(g_1 g_2) = U_S(g_1)U_S(g_2)$. Thus

$$\Theta(hf) = \frac{1}{m_V^n}\sum_{S\in\mathcal{S}}(U_S R_S f)(U_S R_S h) = \frac{1}{m_V^n}\sum_{S\in\mathcal{S}_n} f(U_S R_S h) = f(\Theta h). \tag{3.2.27}$$

As a special case one has

$$\Theta(f^2) = f\Theta f = f^2. \tag{3.2.28}$$

Step 2. Compute the adjoints of R_S and U_S. Since R_S maps $C(V)$ to $C(S)$, R_S^* maps finite measures on S to finite measures on V.

$$\int f d(R_S^*\nu) = \int R_S f d\nu \tag{3.2.29}$$

$$= \int 1_S(X) f(x)\nu(dx) \tag{3.2.30}$$

from which it follows that

$$R_S^*\nu(dx_= 1_S(x)\nu(dx). \tag{3.2.31}$$

Similarly, U_S maps $C(S)$ to $C(V)$ so U_S^* maps finite measures on V to finite measures on S. Assume ν is a finite measure on V. Using 3.2.25 we have

$$\int_S fd(U_S^*\nu) = \int_V U_S f d\nu \tag{3.2.32}$$

$$= \int_V f \circ \varphi_S(x)\nu(dx) \tag{3.2.33}$$

$$= \int_V \left(\sum_{T\in\mathcal{S}_n} \frac{1_T(x)}{N_n(X)} \right) f \circ \varphi_(x)\nu(dx) \tag{3.2.34}$$

$$= \sum_T \int_T \frac{f \circ \varphi_S(x)}{N_n(X)} \nu(dx). \tag{3.2.35}$$

Let $\varphi_{T,S} : T \to S$ be the restriction of φ_S to T. Note this is one to one and onto. If κ is a measure on T, define its pull-back $\varphi_{T,S}^*\kappa$ to be the measure on S defined by

$$\int_S fd(\varphi_{T,S}^*\kappa) = \int_T (f \circ \varphi_{T,S})d\kappa. \tag{3.2.36}$$

Write

$$\nu_T(dx) = \frac{1_T(x)}{N_n(x)}\nu(dx). \tag{3.2.37}$$

This allows us to rewrite 3.2.35 as

$$\int_S fd(U_S^*\nu) = \sum_T \int_T f\varphi_{T,S}^*(\nu_T)(dx), \tag{3.2.38}$$

and we can conclude

$$U_S^*\nu = \sum_{T\in\mathcal{S}_n} \varphi_{T,S}^*(\nu_T). \tag{3.2.39}$$

Step 3. We show that if ν is a finite measure on V such that $\Theta^*\nu = \nu$ and $S \in \mathcal{S}_n$, then

$$\nu(V) = m_V^n \int_S \frac{1}{N_n(x)} \nu(dx). \tag{3.2.40}$$

First note that $\varphi^*_{T,R}(\nu_T)$ is a measure on R, and by 3.2.31, and 3.2.39

$$\begin{aligned}
\Theta^*\nu &= \frac{1}{m_V^n} \sum_{R\in\mathcal{S}_n} R_R^* U_R^* \nu && (3.2.41)\\
&= \frac{1}{m_V^n} \sum_{R\in\mathcal{S}_n} \sum_{T\in\mathcal{S}_n} \int 1_R(x) \varphi^*_{T,R}(\nu_T)(dx) && (3.2.42)\\
&= \frac{1}{m_V^n} \sum_{R} \sum_{T} \int \varphi^*_{T,R}(\nu_T)(dx) && (3.2.43)
\end{aligned}$$

However, using 3.2.25 we have

$$\nu(dx) = \sum_R \frac{1_R(x)}{N_n(x)} \nu(dx) = \sum_R \nu_R(dx) \tag{3.2.44}$$

Both ν_R and $m_V^{-n} \sum_T \varphi^*_{T,R}(\nu_T)$ are supported on R and if $\Theta^*\nu = \nu$ then

$$\nu_R = m_V^{-n} \sum_{T\mathcal{S}_n} \varphi^*_{T,R}(\nu_T) \tag{3.2.45}$$

for each R. Thus

$$\int_S \frac{1}{N_n(x)}\nu(dx) = \nu_S(V) \tag{3.2.46}$$

$$= m_V^{-n}\sum_T \int 1_V(x)\varphi^*_{T,S}(\nu_T)(dx) \tag{3.2.47}$$

$$= m_V^{-n}\sum_T \int 1_V \circ \varphi_{T_S}(x)\nu_T(dx) \tag{3.2.48}$$

$$= m_V^{-n}\sum_T \int \nu_T(dx) \tag{3.2.49}$$

$$= m_V^{-n}\sum_T \int \frac{1_T(x)}{N_n(x)}\nu(dx) = m_V^{-n}\int \nu(dx) = m_V^{-n}\nu(V). \tag{3.2.50}$$

$$\tag{3.2.51}$$

If we multiply both sides by m_V^n then this is 3.2.40.

Step 4. Here we prove that if $\Theta f = f$, then

$$\Theta^*(\Gamma(f,f)) = \Gamma(f,f). \tag{3.2.52}$$

Let $h \in C(F)\cap\mathcal{F}$, and using Step 1 we have

$$\int_V h\Theta^*(\Gamma(f,f))(dx) = \int_V \Theta h(x)\Gamma(f,f)(dx) \tag{3.2.53}$$

$$= 2\mathcal{E}(f, f\Theta h) - \mathcal{E}(f^2, \Theta h) \tag{3.2.54}$$

$$= 2\mathcal{E}(f, \Theta(fh)) - \mathcal{E}(\Theta f^2, h) \tag{3.2.55}$$

$$= 2\mathcal{E}(\Theta f, fh) - \mathcal{E}(f^2, h) \text{ using 2} \tag{3.2.56}$$

$$= 2\mathcal{E}(f, fh) - \mathcal{E}(f^2, h) \tag{3.2.57}$$

$$= \int_V h\Gamma(f,f)(dx). \tag{3.2.58}$$

Step 5. All that remains is to prove 1. If $g \in \mathcal{F} \cap L^\infty(V)$ and $A = \{x \in V : g(x) = 0\}$, then $\Gamma(g, g)(A) = 0$ by Lemma 3.1.6. We may then apply this to $g = f - U_S R_S f$ and using the inequality

$$|\Gamma(f, f)(B)^{1/2} - \Gamma(U_S R_S f, U_S R_S f)(B)^{1/2} \leq \Gamma(g, g)(B)^{1/2} \tag{3.2.59}$$

$$\leq \Gamma(g, g)(S)^{1/2} = 0 \quad \text{for all } B \subset S. \tag{3.2.60}$$

Using the argument on page 111 in [30] we have

$$1_s(x)\Gamma(f, f)(dx) = 1_S(x)\Gamma(U_S R_S f, U_S R_S f)(dx) \tag{3.2.61}$$

for any $f \in \mathcal{F}$ and $S \in \mathcal{S}_n(V)$. If we begin from 3.1.22

$$U_T R_T U_S R_S f = U_S R_S f, \tag{3.2.62}$$

sum over $T \in \mathcal{S}_n$, and divide by m_V^n we have

$$\Theta^*(\Gamma(U_S R_S f, U_S R_S f))(dx) = \Gamma(U_S R_S f, U_S R_S f)(dx). \tag{3.2.63}$$

We now use Step 3 with $\nu = \Gamma(U_S R_S f, U_S R_S f)$ and get

$$\mathcal{E}(U_S R_S f, U_S R_S f) = \Gamma(U_S R_S f, U_S R_S f)(V) \tag{3.2.64}$$

$$= m_V^n \int_S \frac{1}{N_n(x)} \Gamma(U_S R_S f, U_S R_S f)(dx). \quad \text{here we use Step 3} \tag{3.2.65}$$

We now divide both sides by m_V^n, use the definition of $\mathcal{E}^S$,and 3.2.61 and obtain

$$\mathcal{E}^S(R_S f, R_S f) = \int_S \frac{1}{N_n(x)} \Gamma(f,f)(dx). \tag{3.2.66}$$

If we now sum over $S \in \mathcal{S}_n$ and use 3.2.25 we find

$$\sum_S \mathcal{E}^S(R_S f, R_S f) = \int \Gamma(f,f)(dx) \tag{3.2.67}$$

$$= \mathcal{E}(f,f) \tag{3.2.68}$$

This proves 1. ■

Corollary 3.2.6. *If $\mathcal{E} \in \mathfrak{E}, f \in \mathcal{F}, S \in \mathcal{S}_n(V)$, and $\Gamma_S(R_S f, R_S f)$ is the energy measure of $\mathcal{E}^S$ then*

$$\Gamma_S(R_S f, R_S f)(dx) = \frac{1}{N_n(x)} \Gamma(f,f)(dx), \quad x \in S \tag{3.2.69}$$

For clarity we include some results about sets of capacity zero for V-invariant Dirichlet forms. Let $A \subset V$ and $S \in cals_n$. Define

$$\Theta(A) = \varphi_S^{-1}(\varphi_S(A)). \tag{3.2.70}$$

This makes $\Theta(A)$ the union of all sets that can be obtained by A by local reflections. One can check that $\Theta(A)$ does not depend on S, and

$$\Theta(A) = \{x : \Theta(1_A)(x) > 0\}. \tag{3.2.71}$$

(

Lemma 3.2.7. *If $\mathcal{E} \in \mathfrak{E}$ then*

$$Cap(A) \leq Cap(\Theta(A)) \leq m_V^{2n} Cap(A) \quad \textit{for all Borel sets } A \subset V. \tag{3.2.72}$$

Proof

This result and the Corollary that follows is due to Barlow, Bass, Kumagai, and Teplyaev in [12]. The first inequality follows from the fact that $A \subset \Theta(A)$. We now prove the second. Assume A is open. If $u \in \mathcal{F}$ and $u \geq 1$ on A, then $m_v^n \Theta u \geq 1$ on $\Theta(A)$. This implies the second inequality as $\mathcal{E}(\Theta u, \Theta u) \leq \mathcal{E}(u, u)$, since Θ is an orthogonal projection with respect to $\mathcal{E}$. Explicitly we have $\mathcal{E}(\Theta f, h) = \mathcal{E}(f, \Theta g)$. ■

Corollary 3.2.8. *If $\mathcal{E} \in \mathfrak{E}$, then $Cap(A) = 0$ if and only if $Cap(\Theta(A)) = 0$. In addition, if f is quasi-continuous, then Θf is quasi-continuous.*

Proof

This follows from Lemma 3.2.7 and the fact that Θ preserves continuity of functions on Θ invariant sets. ■

3.3 The Process Constructed in the Plane is in $\mathfrak{E}$

The work in the first part of this paper has been to show the existence of a process on V but the associated Dirichlet form was not the focus. It should also be noted that the diffusion constructed was on the unbounded fractal $\widetilde{V}$. The first step to showing that the Dirichlet forms are V-invariant is discussing them on V. As before W_t^n is the normally reflecting Brownian motion on V_n. Write $X_t^n = W_{a_n t}^n$ for a suitable sequence $\{a_n\}$. Again, we assume conjectures 2.7.1 and 2.7.2, that there are resistance bounds for the sequence,

$$C(m_V\rho_V/l_V^2)^n \leq a_n \leq C'(m_V\rho_V/l_V^2)^n. \tag{3.3.1}$$

We showed that the laws of X^n were tight. We denoted the λ-resolvent for X^n by U_n^λ on V_n and showed that we had resolvent tightness. This allowed us to conclude that there existed a special subsequence n_j, for which $U_n^\lambda f$ converged uniformly on V, if f was continuous on V_0, and the $\mathbb{P}^x$ law of X^{n_j} converged weakly for each x. The explicit Dirichlet form for X^n was never given so we express it here.

$$\mathcal{E}_n(f,f) = a_n \int_{V_n} |\nabla f(x)|^2 \mu_n(dx) \tag{3.3.2}$$

on $L^2(V,\mu_n)$. If X is the limiting process and T_t is the limiting semigroup of X, then define

$$\mathcal{E}_U(f,f) = \sup_{t>0} \frac{1}{t}\langle f - T_t f, f\rangle. \tag{3.3.3}$$

The domain $\mathcal{F}$ is the set of $f \in L^2(V,\mu)$ for which the supremum is finite. The main fact that we need is that if U_n^λ is the λ-resolvent operator for X^n and f is bounded on F_0 then $U_n^\lambda f$ is equicontinuous on V. We've shown this for $\widetilde{V}$ and the proof is similar so we provide a sketch of the ideas.

Fix x_0 and suppose x, y in $B(x_0,r)\cap V_n$. Write S_r^n for the first exist from $B(x_0,r)\cap V_n$, then

$$\begin{aligned} U_n^\lambda f(x) &= \mathbb{E}^x \int_0^\infty e^{-\lambda t} f(X_t^n)dt \qquad (3.3.4)\\ &= \mathbb{E}^x \int_0^{S_r^n} e^{-\lambda t} f(X_t^n)dt + \mathbb{E}^x(e^{-\lambda S_r^n}-1)U_n^\lambda f(X_{S_r^n}^n) + \mathbb{E}^x U_n^\lambda f(X_{S_r^n}^n) \qquad (3.3.5) \end{aligned}$$

The first term in 3.3.5 is bounded by $\|f\|_\infty \mathbb{E}S_r^n$, and the second term is bounded by $\lambda \left\|U_n^\lambda f\right\|_\infty \mathbb{E}S_r^n \leq \|f\|_\infty \mathbb{E}^x S_r^n$. The same estimates hold when x is replaced by y. This gives the estimate

$$|U_n^\lambda f(x) - U_n^\lambda f(y)| \leq |\mathbb{E}^x U_n^\lambda f(X^n_{S^n)r}) - \mathbb{E}^y U_n^\lambda f(X^n_{S_r^n})| + \delta_n(r) \tag{3.3.6}$$

where $\delta_n(r) \to 0$ as $r \to 0$ uniformly in n by Proposition 2.4.5. The function $z \to \mathbb{E}^z U_n^\lambda f(X^n_{S_r^n})$ is harmonic in the ball of radius $r/2$ about x_0. Using the uniform elliptic Harnack inequality for X_t^n and the uniform modulus of continuity for harmonic functions, one may take $r = |x-y|^{1/2}$ and use the estimate for $\delta_n(r)$ to show the equicontinuity. One can then show that the limiting resolvent $U^\lambda f$ is continuous when f is bounded.

Theorem 3.3.1. *The Dirichlet form $\mathcal{E}_U$ in $\mathfrak{E}$.*

This is Theorem 3.1 of [12]. Fix our special subsequence n_j. Each X^n is conservative, so we have $T_t^n 1 = 1$. By the choice of our sequence $\{n_j\}$, $T_t^{n_j} f \to T_n f$ uniformly for each continuous f. Thus $T_t 1 = 1$. Thus X is conservative, and $\mathcal{E}_U(1,1) = \sup_t \langle 1 - T_t 1, 1\rangle = 0$. The regularity of $\mathcal{E}_U$ follows from Lemma 3.1.7 and the fact that the process constructed is μ-symmetric Feller. The process is local because of [30] Theorem 4.5.1.

The process is non-degenerate so $\mathcal{E}_U$ is non-zero. Fix l and let $S \in S_l(V)$. Clearly $U_S R_S f \in \mathcal{F}$ if $f \in \mathcal{F}$. We now show that Θ_l and T_t commute. Here when we say, Θ_l, we mean as defined by 3.2.9 with $\mathcal{S}_n(V)$ replaces by $\mathcal{S}_l(V)$. Write $\langle f, g\rangle_n = \int_{V_n} f(x)g(x)\mu_n(dx)$. The infinitesimal generator is a constant times the Laplacian, and this commutes with Θ_l. This gives

$$\langle \Theta^\lambda f, \Theta_l g \rangle = \langle \Theta_l U^\lambda f, g \rangle. \tag{3.3.7}$$

Suppose that f, g are continuous, and f is non-negative. We have

$$\langle \Theta_l U_n^\lambda f, g \rangle = \langle U_n^\lambda f, \Theta_l g \rangle_n \tag{3.3.8}$$

Letting $n \to \infty$ along the special subsequence $\{n_j\}$ gives

$$\langle U^\lambda f, \Theta g \rangle = \langle \Theta_l U^\lambda f, g \rangle. \tag{3.3.9}$$

The function f is assumed to be continuous so $\Theta_l f$ is continuous so the right hand side of 3.3.7 converges to $\langle U^\lambda \Theta_l f, g \rangle$. The process X_t has continuous paths so the function $t \mapsto T_t f$ is continuous. By the uniqueness of the Laplace transform

$$\langle \Theta_l T_t f, g \rangle = \langle T_t \Theta_l f, g \rangle. \tag{3.3.10}$$

A limit argument and linearity allow one to extend this equality to all of $f \in L^2(V)$. Proposition $\mathcal{E}_U \in \mathfrak{E}$. ∎

3.4 Diffusions of V-invariant Dirichlet Forms

In this section we fix a Dirichlet form $\mathcal{E} \in \mathfrak{E}$, let X be the associated diffusion, T_t be the associated semigroup, and $\mathbb{P}^x = \mathbb{P}^{x,\mathcal{E}}, x \in V - \mathcal{N}_0$ be the associated probability laws. Here $\mathcal{N}_0$ is the properly exceptional set for X.

Theorem 3.4.1. *Let $S \in \mathcal{S}_n(V)$ and $Z = \varphi_S(X)$. Then Z is a μ_S-symmetric Markov* *process with Dirichlet form* $(\mathcal{E}^S, \mathcal{F}^S)$*, and semigroup* $T_t^Z f = R_S T_t U_S f$*. Write* $\widetilde{P}^y$ *for the laws of* Z*, which are defined up to a properly exception set* $\mathcal{N}_n^Z$ *for* Z*. There exists a properly exceptional set* $\mathcal{N}_2$ *for* X *such that for any Borel set* $A \subset V$

$$\widetilde{\mathbb{P}}^{\varphi_S(x)}(Z_t \in A) = \mathbb{P}^x(X_t \in \varphi_S^{-1}(A)), x \in V - \mathcal{N}_n \tag{3.4.1}$$

Proof

This result is due to Barlow, Bass, Kumagai, and Teplyaev. See Theorem 4.1 of [12]. For simplicity write $\varphi = \varphi_S$. Step 1. Show that there exists a properly exceptional set $\mathcal{N}_n$ for X such that

$$\mathbb{P}^x(X_t \in \varphi^{-1}(A)) = T_t 1_{\varphi^{-1}(A)}(x) \tag{3.4.2}$$

$$= T_t 1_{\varphi_{-1}(A)}(y) \tag{3.4.3}$$

$$= \mathbb{P}^y(X_t \in \varphi^{-1}(A)) \tag{3.4.4}$$

whenever $A \subset S$ is Borel, $\varphi_X = \varphi(y)$, and $x, y \in V - \mathcal{N}_2$. It is sufficient to prove 3.4.4 for a countable base (A_m) of the Borel σ-field on V. Let $f_m = 1_{A_m}$. As $T_t 1_{\varphi^{-1}(A_m)} = T_t U_S f_m$, it is sufficient to prove that there exists a properly exceptional set $\mathcal{N}_n$ such that for $m \in \mathbb{N}$

$$T_t U_S f_m(x) = T_t U_S f_m(y) \quad \text{if } x, y \in V - \mathcal{N}_2 \text{ and } \varphi(x) = \varphi(y) \tag{3.4.5}$$

By 3.1.22 we have $\Theta(U_S f) = U_S f$. Proposition 3.2.5 we have

$$\Theta T_t U_S f = T_t \Theta U_S f_m \tag{3.4.6}$$

$$= T_t U_S f \tag{3.4.7}$$

for $f \in L^2$ where equality holds in the L^2 sense. By Corollary 3.2.8 $\Theta_t U_S f_m$ is quasi-continuous. Note that as $\Theta_t U_S f_m = T_t U_S f_m, \mu$-a.e, then using [30] Lemma 2.1.4 we can conlude that they are q.e. By the definition of Θ we have $\Theta(T_t U_S f_m)(x) = \Theta(T_t U_S f_m)(y)$ whenever $\varphi(x) = \varphi(y)$, thus there is a properly exceptional set $\mathcal{N}_{2,n}$ such that 3.4.5 holds. We then take $\mathcal{N}_2 = \bigcup_m \mathcal{N}_{2,m}$ to get 3.4.4. By Theorem 10.13 of [25], Z is a Markov process with semigroup $T_t^Z f = R_S T_t(U_S f)$. We may then take $\mathcal{N}_2^Z = \varphi(\mathcal{N}_n)$. By 3.4.5 we have $U_S R_S T_t U_S f = T_t U_S f$. We then obtain

$$\langle T_t^Z f, g\rangle_S = \langle R_S T_t U_S f, g\rangle_S \tag{3.4.8}$$

$$= m_V^{-n}\langle U_S R_S T_t U_S f, U_S g\rangle \tag{3.4.9}$$

$$= m_V^{-n}\langle T_t U_S f, U_S g\rangle. \tag{3.4.10}$$

This equals $m^{-n}\langle U_S f, T_t U_S g\rangle$. If we reverse the calculation above, we conclude

$$\langle f, T_t^Z g\rangle = m_V^{-n}\langle U_S f, T_t U_S g\rangle \tag{3.4.11}$$

which shows that Z is μ_S-symmetric. We now calculate the Dirichlet form of Z.

$$t^{-1}\langle T_t^Z f - f, f\rangle_S = m_V^{-n} t^{-1}\langle T_t U_S f - U_S f, U_S f\rangle. \tag{3.4.12}$$

If we take the limit at $t \to 0$ and use [30] Lemma 1.3.4, then Z has Dirichlet form

$$\mathcal{E}_Z(f,f) = m_V^{-n}\mathcal{E}(U_S f U_S f) = \mathcal{E}^S(f,f). \quad \blacksquare \tag{3.4.13}$$

Because it will be used multiple times we state but do not prove [30] Theorem 4.2.7

Theorem 3.4.2. *Let M_1 and M_2 be two m-symmetric Hunt processes on X possessing* *a common regular Dirichlet space on $L^2(X,m)$. Then M_1 and M_2 are equivalent.*

The next set of Lemmas are due to Barlow, Bass, Kuamgai, and Teplyaev in [12]. For reference see Lemma 4.2, 4.3 and Proposition 4.5.

Lemma 3.4.3. *Let $S, S' \in \mathcal{S}_n$, $Z = \varphi_S(X)$, and Φ be an isometry of S onto S'. Then if $x \in S - \mathcal{N}$,*

$$\mathbb{P}^x(\Phi(Z) \in \cdot) = \mathbb{P}^{\Phi(x)}(Z \in \cdot) \tag{3.4.14}$$

Proof

Using Theorem 3.4.1, and Definition 3.2.9, Z and $\Phi(Z)$ have the same Dirichlet form. The result then follows by [30] Theorem 4.2.7. ■

If $S, S' \in \mathcal{S}_n(V)$ and there exist $\mathcal{O}, \mathcal{O}' \in \mathcal{O}_n(V)$ such that $\mathcal{O} \cap \mathcal{O}'$ is a $(d-1)$-dimensional set and $S = \mathcal{O} \cap V, S' = \mathcal{O}' \cap V$ then we say that S and S' are adjacent. In this setting let H be the plane separating S, S'. For any plane $H \subset \mathbb{R}^2$, let $g_H : \mathbb{R}^d \to \mathbb{R}^d$ be reflection in H.

Lemma 3.4.4. *Let $S_1, S_2 \in \mathcal{S}_n(V)$ be adjacent, let $D = S_1 \bigcup S_2$,let $B = \partial_r(S_1 \bigcup S_2)$, and let H be the hyperplane separating S_1 and S_2. Then there exists a properly exceptional set $\mathcal{N}$ such that if $x \in H \cap D - \mathcal{N}$, the processes $(X_t, 0 \le t \le T_B)$ and $(g_H(X_t), 0 \le t \le T_B)$ have the same law under $\mathbb{P}^x$.*

Proof

Let $f \in \mathcal{F}$ have support in the interior of D. Then by Definition 3.2.1 and Proposition 3.2.4 we have

$$\mathcal{E}(f,f) = \mathcal{E}^{S_1}(R_{S_1}f, R_{S_1}f) + \mathcal{E}^{S_2}(R_{S_2}f, R_{S_2}f). \tag{3.4.15}$$

and

$$\mathcal{E}(f,f) = \mathcal{E}(f \circ g_H, f \circ g_H) \tag{3.4.16}$$

Thus $(g_H(X_t), 0 \le t \le T_B)$ has the same Dirichlet form as $(X_t, 0 \le t \le T_B)$, and thus have the same law if we exclude a V-invariant set of capacity zero by [30] Theorem 4.2.7.

Beyond this point we consider the process $Z = \varphi_S(X)$ for some $S \in \mathcal{S}_n$, where $n \ge 0$. For notational simplicity we take $n = 0$ and $S = V$. This presents no problems as the previous arguments were scale invariant.

Definition 3.4.5. Let $1 \le i, j \le d$, and $i \ne j$ and let

$$H_i(t) = \{x = (x_1, \ldots, x_d) : x_i = t\}, \quad t \in \mathbb{R} \tag{3.4.17}$$

$$L_i = H_i(0) \cup [0, 1/2]^d \tag{3.4.18}$$

$$M_{ij} = \{x \in [0,1]^d : x_i 0, 1/2 \le x_j \le 1, \text{ and } 0 \le x_k \le 1/2 \text{ for } k \ne j\}. \tag{3.4.19}$$

Also let

$$\partial_e S = S \cap \bigcup_{i=1}^{d} H_i(1), \quad D = S - \partial_e S. \tag{3.4.20}$$

We wish to use the sets E_D and Z_D as in 3.1.3.

Proposition 3.4.6. *There exists a constant q_0, depending only on the dimension d such that*

$$\widetilde{\mathbb{P}}^x(T^Z_{L_j} < \tau^Z_D) \ge q_0, \quad x \in L_i \cap E_D, \tag{3.4.21}$$

$$\widetilde{\mathbb{P}}^x(T^Z_{M_i j} < \tau^Z_D) \ge q_0, \quad x \in L_i \cap E_D. \tag{3.4.22}$$

These inequalities hold for any $n \geq 0$ assuming that we adjust Definition 3.4.5 to account for the n.

Proof

If we use Lemma 3.4.3 then this result follows the reflection arguments of Proposition 3.5 through Lemma 3.10 of [11].

It should be noted that for polygons in the plane one doesn't need both "corners", and "slides". In fact only slides are needed for our process in $\mathbb{R}^2$. We include the above result and the following ones for completeness to show that the techniques allow one to draw conclusions about processes on fractals in arbitrary dimensions.

Definition 3.4.7. Fix $n \geq 0$. We call $A \subset \mathbb{R}^d$ a level n half-face if there exists $i \in \{1, \ldots, d\}$ and $a = (a_1, \cdots, a_d) \in \frac{1}{2}\mathbb{Z}^d$ with $a_i \in \mathbb{Z}$ such that

$$A = \{x : x_i = a_i l_V^{-n}, a_j l_V^{-n} \leq x_j \leq (a_j + 1/2) l_V^{-n} \text{ for } j \neq i\}. \tag{3.4.23}$$

For A write $\iota(A) = i$, and let $\mathcal{A}^{(n)}$ be the collection of level n half-faces, and

$$\mathcal{A}_V^{(n)} = \{A \in \mathcal{A}^{(n)} : A \subset V_n\}. \tag{3.4.24}$$

Define a graph struchture on $\mathcal{A}_V^{(n)}$ be setting $\{A, B\}$ to be an edge if

$$\dim(A \cap B) = d - 2 \text{ and } A \cup B \subset \mathcal{O} \text{ for some } \mathcal{O} \in \mathcal{O}_n. \tag{3.4.25}$$

Let $E(\mathcal{A}_V^{(n)}$ be the set of edges in $\mathcal{A}_V^{(n)}$. Note that the graph $\mathcal{A}_V^{(n)}$ is connected. Call an edge $\{A, B\}$ an $i - j$ corner if $\iota(A) = i, \iota(B) = j$, and $i \neq j$. Call $\{A, B\}$ an $i - j$ slide if $\iota(A) = \iota(B) = i$ and the line joining the centers of A and B is parallel to the x_j axis. The move L_i, L_j is an $i - j$ corner; the move (L_i, M_{ij}) is an $i - j$ slide.

Definition 3.4.8. Let (A_0, A_1) be an edge in $E(\mathcal{A}_V^{(n)}$, and $\mathcal{O}_0$ be a $4N$-gon in $\mathcal{O}_n(V)$ such that $A_0 \cup A_1 \subset \mathcal{O}_*$. Let v_* the unique vertex of $\mathcal{O}_*$ such that $v_* \in A_0$, and let R be the union of all $4N$-gons in $\mathcal{O}_n$ which contain v_*. Then there exist distinct $S_i \in \mathcal{N}, 1 \leq i \leq m$, such that $V \cap R = \bigcup_{i=1}^m S_i$. Let $D = V \cap R^0$. Thus

$$\overline{D} = V \cap R = \bigcup_{i=1}^{m} S_i. \tag{3.4.26}$$

Let S_* be a single S_i and set $Z = \varphi_{S_*}(X)$.

Write

$$\tau = \tau_D^X = \inf\{t \geq 0 : X_t \notin D\} = \inf\{t : Z_t \in \partial_r R\}. \tag{3.4.27}$$

Let

$$E_D = \{x \in D : \mathbb{P}^x(\tau < \infty) = 1\} \tag{3.4.28}$$

Our next goal is to find a lower bound for

$$\inf_{x \in A_0 \cap E_D} \mathbb{P}^x(T_{A_1}^X \leq \tau). \tag{3.4.29}$$

Propsition 4.5 gives

$$\inf_{y \in A_0 \cap E_D} \widetilde{\mathbb{P}}^x(T_{A_1}^X \leq \tau) \geq q_0. \tag{3.4.30}$$

By definition Z hits A_1 if and only if X hits $\Theta(A_1)$. Ideally one could use symmetry to prove that if $x \in A_0 \cap E_D$ then for some $q_1 > 0$

$$\mathbb{P}^x(T^X_{A_1} \leq \tau) \geq q_1 \widetilde{\mathbb{P}}^x(T^Z_{A_1} \leq \tau) \geq q_1 q_0. \tag{3.4.31}$$

This is proven in section 3 of [11], and treated in more generality in section 4 of [12].

Proposition 3.4.9. *Let (A_0, A_1) and Z as in Definition 3.4.8. Ther exists a constant $q_1 > 0$, depnding only on d,such that if $x \in A_0 \cap E_D$ and $T_0 \leq \tau$ is a finite $(\mathcal{F}^Z_t)$ stopping time, then*

$$\mathbb{P}^x(X_{T_0} \in S | \mathcal{F}^Z_{T_0}) \geq q_1. \tag{3.4.32}$$

From this it follows that

$$\mathbb{P}^x(T^X_{A_1} \leq \tau) \geq q_1 \widetilde{\mathbb{P}}^x(T^Z_{A_1} \leq \tau) \geq q_1 q_0. \tag{3.4.33}$$

The next Proposition corresponds to Proposition 4.14 [12] Let I be a face of V_0 and let $V' = V - I$.

Proposition 3.4.10. *There exists a set $\mathcal{N}$ of capacity 0 such that if $x \notin \mathcal{N}$, then $\mathbb{P}^x(\tau_{V'}) = 1$.*

<u>Proof</u>

Let A be the set of x such that when the process starts at x, it never leaves x. We proceed via proof by contradiction. Assume that $V - A$ does not have positive measure. Then $T_t f(x) = f(x)$ for almost every x, thus

$$\frac{1}{t}\langle f - T_t f, f \rangle = 0 \tag{3.4.34}$$

If we take the supremum over $t > 0$, we obtain $\mathcal{E}(f, f) = 0$. This holds for every $f \in L^2$. This contradicts the fact that $\mathcal{E}$ is non-zero. If $\mu(E_S \cap S) = 0$ for every $S \in \mathcal{S}_n(V)$ and $n \geq 1$ then $\mu(V - A) = 0$. So there must be an n and $S \in \mathcal{S}_n(V)$ such that $\mu(E_S \cap S) > 0$. Fix $\epsilon > 0$. Since μ is a double measure for every Borel subset H of V almost every point of H is a point of density for H, by [99] Corollary IX.1.3. Therefor we can find $k \geq 1$ so that there exists $S' \in S_{n+k}(V)$ such that

$$\frac{\mu(E_S \mathcal{S}')}{\mu(S')} > 1 - \epsilon \tag{3.4.35}$$

Now let $S'' \in S_{n+k}$ be adjacent to S' and containined in S. Let g be the map that reflects $S' \cap S''$ accross $S' \cap S''$. Define

$$J_i(S') = \bigcup \{T : T \in S_{n+k+i}, T \subset \int_r (S')\}, \tag{3.4.36}$$

and also define $J_i(S'')$ accordingly. We may now choose i large enough so that

$$\mu(E_S \cap J_i(S')) > (1 - 2\epsilon)\mu(S'). \tag{3.4.37}$$

Let $x \in E_S \cap J_i(S')$. As $x \in E_S$, the process started at x will leave S' with probability 1. We can this find a finite sequence of moves at level $n+k+i$ so that X started at x will exit S' by hitting $S' \cap S''$. By 3.4.9 the probability of X following this sequence of moves is positive, thus

$$\mathbb{P}^x(X(\tau_{S'} \in S' \cap S'') > 0. \tag{3.4.38}$$

If the process starts from $x \in E_S$, then the process can never leave E_S by definition. This means that X will leave S' through $B = E_S \cap S' \cap S''$ with positive probability.

By symmetry, X_t started from $g(x)$ will leave S' in B with positive probability. Using the strong Markov property, starting from $g(x)$, the process will leave S with positive probability. This implies that $g(x) \in E_S$. But this means that $g(E_S \cap J_i(S')) \subset E_S \cap J_i(S'')$ so by 3.4.37 we have

$$\mu(E_S \cap J_i(S'')) > (1 - 2\epsilon)\mu(S''). \tag{3.4.39}$$

If we iterate this argument we conclude that $S_j \in S_{n+k}(V)$ with $S_j \subset S$,

$$\mu(E_S \cap S_j) \geq \mu(E_S \cap J_i(S_j)) \geq (1 - 2\epsilon)\mu(S_j). \tag{3.4.40}$$

Summing over the S_i gives

$$\mu(E_S \cap S) \geq (1 - 2\epsilon)\mu(S). \tag{3.4.41}$$

Letting ϵ go to 0 gives $\mu(E_S \cap S(= \mu(S)$. This means that starting from almost every point of S the process will leave S. By symmetry this is true for every element of $\mathcal{S}_n(V)$ isomorphic to S. It is then the case that starting at almost any $x \in V$ there is a positive probability of exiting V' by 3.4.9. This implies that $E_{V'}$ has full measure. The function $1_{E_{V'}}$ is invariant so $T_t 1_{E_{V'}} = 1$ a.e. Then [30] Lemma 2.1.4 implies that $T_t(1 - 1_{E_{V'}}) = 0$ q.e. Write $\mathcal{N}$ for the set of x where $T_t 1_{E_{V'}}(X) \neq 1$ for some rational t. If $x \notin \mathcal{N}$, then $\mathbb{P}^x(X_t \in E_{V'} = 1$ if t is rational. The Markov Property implies $x \in E_{V'}$. ■

Lemma 3.4.11. *Let $U \subset V$ be open and non-empty. Then, $\mathbb{P}^x(T_U < \infty) = 1$q.e.*

Proof

This result is due to Barlow, Bass Kumagai, and Teplyaev; for reference see Lemma

4.15 of [12]. This is immediate by 3.4.9 and Proposition 3.4.10. ■

3.4.1 Coupling and the Elliptic Harnack Inequality

Lemma 3.4.12. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let X and Z be random variables taking values in separable metric spaces E_1 and E_2 respectively. Assume each of E_1 and E_2 has the Borel σ-field. Then there exists $F : E_x \times [0,1] \to E_1$ that is jointly measurable such that if U is a random variable whose distribution is uniform on $[0,1]$ which is independent of Z and $\widetilde{X} = F(Z,U)$, then (X,Z) and $(\widetilde{X}, Z)$ have the same law.*

Proof

This result is due to Barlow, Bass, Kumagai, and Teplyaev in [12]. First suppose $E_1 = E_2 = [0,1]$. Let $\mathbb{Q}$ denote the rational numbers. For each $r \in [0,1] \cap \mathbb{Q}, \mathbb{P}(X \leq r|Z)$ is a $\sigma(Z)$-measurable random variable. So there exists a Borel measurable function h_r such that $\mathbb{P}(X \leq r|Z) = h_r(Z)$ a.s. For $r < s$ let $A_{rs} = \{z : h_r(z) > h_s(z)\}$. If $C = \bigcup_{r<s; r,s\in\mathbb{Q}} A_{rs}$, then $\mathbb{P}(Z \in C) = 0$. For $z \notin C, h_r(z)$ is non-decreasing in r for $r \in \mathbb{Q}$. For $x \in [0,1]$ define $g_x(z)$ to be equal to x if $z \in C$ and equal to $\inf_{s>x, s\to x; s\in\mathbb{Q}} h_s(z)$ otherwise. For each z, let $f_x(z)$ be the right continuous inverse to $g_x(z)$. Last, let $F(z,x) = f_x(z)$.

We now check the distributions of (X,Z) and $(\widetilde{X}, Z)$ have the same law.

$$\mathbb{P}(X \leq x, Z \leq z) = \mathbb{E}[\mathbb{P}(X \leq x|Z); Z \leq Z] \tag{3.4.42}$$

$$= \lim_{s>x, s\in\mathbb{Q}, s\to x} \mathbb{E}[\mathbb{P}(X \leq x|Z); Z \leq z] \tag{3.4.43}$$

$$= \lim \mathbb{E}[h_s(Z); Z \leq z] \tag{3.4.44}$$

$$= \mathbb{E}[g_x(Z); Z \leq z] \tag{3.4.45}$$

while

$$\mathbb{P}(\widetilde{X} \le x, Z \le r) = \mathbb{E}[\mathbb{P}(F(Z,U) \le x|Z); Z \le z) \tag{3.4.46}$$
$$= \mathbb{E}[\mathbb{P}(f_U(Z) \le x|Z); Z \le z] \tag{3.4.47}$$
$$= \mathbb{E}[\mathbb{P}(U \le g_x(Z)|Z); Z \le z] \tag{3.4.48}$$
$$= \mathbb{E}[g_x(Z); Z \le z]. \tag{3.4.49}$$

For the general case of E_1 and E_2, let ψ_i be bimeasurable one to one maps from E_i to $[0,1], i = 1, 2$. If we apply the above calculation to $\overline{X} = \psi_1(X)$ and $\overline{Z} = \psi_2(Z)$ to obtain a function $\overline{F}$. Then we have

$$F(z,u) = \psi_1^{-1} \circ \overline{F}(\psi_2(z), u) \tag{3.4.50}$$

as the required function.

We call $x, y \in V$, m-associated and write $x \sim_m y$, if $\phi_S(x) = \phi_S(y)$ for some $S \in \mathcal{S}_m$. Incidentally, by Lemma 3.1.9 if $x \sim_m y$ then $x \sim_{m+1} y$.

The following result is proposition 4.17 of [12].

Proposition 3.4.13. *Let $x_1, x_2 \in V$ with $x_1 \sim_n x_2$, where $x_1 \in S_1 \in \mathcal{S}_n(V)$ and $x_2 \in S_2 \in \mathcal{S}_n(V)$. Also, let $\Phi = \phi_{S_1}|_{S_2}$. Then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ carrying processes $X_k, k = 1, 2$ and Z with the following properties:*

1. *Each X_k is an $\mathcal{E}$-diffusion started at x_k.*

2. *We have $Z = \phi_{S_2}(X_2) = \Phi \circ \phi_{S_1}(X_1)$.*

3. *Given Z X_1 and X_2 are conditionally independent.*

Proof

Let Y be the diffusion corresponding to the Dirichlet form $\mathcal{E}$ and let Y_1, Y_2 be processes such that Y_i is equal in law to Y started at x_i. Let $Z_1 = \Phi \circ \phi_{S_1}(Y_1)$ and $Z_2 = \phi_{S_2}(Y_2)$. Since the Dirichlet form for $\phi_{S_i}(Y)$ is $\mathcal{E}^{S_i}$ and Z_1, Z_2 have the same starting point, then Z_1 and Z_2 are equal in law. Employing Lemma 3.4.12 to find functions F_1 and F_2 such that $F_i(Z_i, U), Z_i)$ is equal in law to $(Y_i, Z_i), i = 1, 2$ if U is a uniform random variable on $[0, 1]$.

Now consider a probability space supporting a process Z with the same law as Z_i and two independent random variables U_1, U_2 independent of Z which are uniform in $[0, 1]$. Let $X_i = F_i(Z, U_i), i = 1, 2$. We will show that the three above properties are satisfied.

First X_i is equal in law to $F_i(Z_i, U_i)$ which is equal in law to $Y_i, i = 1, 2$. This proves . Similarly (X_i, Z) is equal in law to $(F(Z_i, U_i), Z_i)$ which is equal in law to (Y_i, Z_i). As $Z_1 = \Phi \circ \phi_{S_1}(Y_1)$ and $Z_2 = \phi_{S_2}(Y_2)$, by equality in law we have $Z = \Phi \circ \phi_{S_1}(Y_1)$ and $Z = \phi_{S_2}(Y_2)$. This proves . The last part follows from the fact that $X_i = F_i(Z, U_i), i = 1, 2$ and Z, U_1 and U_2 are independent. ∎

Given a pair of $\mathcal{E}$-diffusions $X_1(t)$ and $X_2(t)$ we define their couple time as

$$T_C(X_1, X_2) = \inf\{t \geq 0 : X_1(t) = X_2(t)\} \tag{3.4.51}$$

Theorem 3.4.14. *Let $r > 0, \epsilon > 0$ and $r' = r/l_V^2$. There exist constants q_3 and δ depending only on V such that we have the following:*

1. *Suppose $x_1, x_2 \in V$ with $\|x_1 - x_2\|_\infty < r'$ and $x_1 \sim_m x_2$ for some $m \geq 1$. There exist $\mathcal{E}$-diffusions $X_i(t), i = 1, 2$ with $X_i(0) = x_i$ such that writing*

$$\tau_i = \inf\{t \geq 0 : X_i(t) \notin B(x_1, r)\}, \tag{3.4.52}$$

one has

$$\mathbb{P}(T_C(X_1, X_2) < \tau_1 \wedge \tau_2) > 1 - \epsilon \tag{3.4.53}$$

Using Propositions 3.4.9 and 3.4.13 our result follows form the arguments given in [11] pp694-701. ■

We now outline the steps taken in [12] to prove an Elliptic Harnack inequality in our context of abstract Dirichlet Forms. These results correspond to Lemma 4.19, Proposition 4.20, Lemma 4.21, and Proposition 4.22 of [12].

Lemma 3.4.15. *Let $\mathcal{E}$ be in $\mathfrak{E}, r \in (0,1)$, if h be bounded and harmonic in $B = B(x_0, r)$, then there exists $\theta > 0$ such that*

$$|h(x) - h(y)| \leq C \left(\frac{|x-y|}{r} \right)^{\theta} (\sup_B |h|), \quad x, y \in B(x_0, r/2), x \sim_m y. \tag{3.4.54}$$

Proof This follows from the coupling Theorem 3.4.14 and by standard arguments. See [11]. ■

Proposition 3.4.16. *Let $\mathcal{E} \in \mathfrak{E}$ and h be bounded and harmonic in $B(x_0, r)$. Then there exists a set $\mathcal{N}$ of $\mathcal{E}$-capacity 0 such that*

$$|h(x) - h(y)| \leq C \left(\frac{|x-y|}{r} \right)^{\theta} (\sup_B |h|), \quad x, y \in B(x_0, r/2), x, y \in B(x_0, r/2) - \mathcal{N}. \tag{3.4.55}$$

Proof

Write $B = B(x_0, r)$ and $B' = B(x_0, r/2)$. By Lusin's theorem there exist open sets $G_n \downarrow$ such that $\mu(G_n) \downarrow 0$, and h restricted to $G_n^c \cap B$ is continuous. First we show that h restricted to G_n^c satisfies 3.4.54 except when one or both of x or y are in $\mathcal{N}_n$.

Which is a set of measure 0. If $G = \bigcap_n G_n$, then h is Hölder continuous in $G^c - \bigcup \mathcal{N}$. Thus h is Hölder continuous on all of B' outside of a set E of measure 0.

Fix n and let $H = G_n^c$. Let x, y be points of density for H. Let S_x and S_y be isometries of an element S_k such that $x \in S_x, y \in S_y$ and $\mu(S_x \cap H)/\mu(S_x) \geq 2/3$ and similarly for S_y. Let Φ be the isometry taking S_x to S_y. Then the measure of $\Phi(S_x \cap H)$ must be at least two thirds the measure of S_y. So $\mu(S_y \cap H) \cap (\Phi(S_x \cap H)) \geq \frac{1}{2} S^y$. Thus there must exist points $x_k \in S_x \cap H$ and $y_k = \Phi(x_k) \in S_y \cap H$ that are m-associated for some m. Inequality 3.4.54 holds for each pair x_k, y_k. If we take k sufficiently large then we get sequences of $x_k \in H$ which tend to x and $y_k \in H$ which tend to y. Since h restricted to H is continuous we have 3.4.54 for our given x and y.

We now know that h is continuous a.e. on B'. We must now show that h is continuous q.e without modification. Let x, y be two points in B' for which $h(X_{t \wedge \tau_B})$ is a martingale under $\mathbb{P}^x$ and $\mathbb{P}^y$. The set of points $\mathcal{N}$ where this fails has $\mathcal{E}$-capacity zero. Let $R = |x - y| < r$ and let $\epsilon > 0$. Since $\mu(E) = 0$, by [30] Lemma 4.1.1 for each $t, T_t 1_E(x) = T_t(x, E) = 0$ for m-a.e. x. By [30] Lemma 2.1.4 since $T_t 1_E$ is in the domain of $\mathcal{E}$ we have $T_t 1_E = 0$ q.e. Now, enlarge $\mathcal{N}$ to include the nulls sets where $T_t 1_E \neq 0$ for some rational t. It thus follows that if $x, y \notin \mathcal{N}$, then with probability one with respect to both $\mathbb{P}^x$ and $\mathbb{P}^y$ we have $X_t \notin E$ for rational t. If we now choose balls B_x, B_y with radii in $[R/4, R/3]$ and centered at x and y respectively such that $\mathbb{P}^x(X_{\tau_{B_x}} \in \mathcal{N}) = \mathbb{P}^y(X_{\tau_{B_y}} \in \mathcal{N}) = 0$, then by the continuity of paths we can choose t rational and small enough that $\mathbb{P}^x(\sup_{s \leq t} |X_s - X_0| > R/4) < \epsilon$ and the same with x

replaced by y. We then have

$$|h(x)-h(y)| = |\mathbb{E}^x h(X_{t\wedge\tau_{B_x}}) - \mathbb{E}^y h(X_{t\wedge\tau_{B_y}})| \tag{3.4.56}$$

$$\leq |\mathbb{E}^x[h(X_{t\wedge\tau_{B_x}}); t<\tau_{B_x}] - \mathbb{E}^y[h(X_{t\wedge\tau_{B_y}}); t<\tau_{B_y}]| + 2\epsilon\,\|h\|_\infty \tag{3.4.57}$$

$$\leq C(R/r)^\infty \|h\|_\infty + 4\epsilon\,\|h\|_\infty \tag{3.4.58}$$

since $\mathbb{P}^x(X_t \in \mathcal{N}) = 0$ and similarly for $\mathbb{P}^y$, further points in B_x are at most $2R$ from points in B_y, and finally $X_{t\wedge\tau_{B_{x}i}}$ and $X_{t\wedge\tau_{B_y}}$ are not in E a.s. Since ϵ is arbitrary we have that except for x, y in a set of capacity 0 we have 3.4.54. ■

We say that X satisfies the elliptic Harnack inequality (EHI) if there exists a constant c such that the following holds: for any ball $B(x,R)$ whenever u is a non-negative harmonic function on $B(x,R)$ then there is a quasi-continuous modification $\widetilde{u}$ of u which satisfies

$$\sup_{B(x,R/2)} \widetilde{u} \leq c \inf_{B(x,R/2)} \widetilde{u} \tag{3.4.59}$$

Proposition 3.4.17. *EHI holds for $\mathcal{E}$, with constants depending only on V.*

Proof

Given the previous results of this section, this follows by the same arguments used in sections 2.2 and 2.3, and in particular Theorem 2.3.2. ■

Corollary 3.4.18. *If $\mathcal{E} \in \mathfrak{E}$ then the following hold:*

1. *The Dirchlet form $\mathcal{E}$ is irreducible*

2. *If $\mathcal{E}(f,f) = 0$ then f is a.e. constant.*

Proof

This result is due to Barlow, Bass, Kumagai, and Teplyaev in [12]. If A is an invariant set, then $T_t 1_A = 1_A$, or 1_A is harmonic on V. By EHI, either 1_A is never 0 except for a set of capacity 0, or else it is 0 q.e. From this it follows that $\mu(A)$ is either 0 or 1. So $\mathcal{E}$ is irreducible. This proves the first part.

Suppose f is a function such that $\mathcal{E}(f,f) = 0$, and that f is not a.e. constant. Then using the contraction property and scaling we can assume without loss of generality that $0 \leq f \leq 1$. We can further assume that there exist $0 < a < b < 1$ such that the sets $A = \{x : f(x) < a\}$ and $B = \{x : f(x) > b\}$ both have positive measure. Let $g = b \wedge (a \vee f)$; then $\mathcal{E}(g,g) = 0$. By lemma 1.3.4 in [30] for any $t > 0$

$$\mathcal{E}^{(t)}(g,g) = t^{-1}\langle g - T_t g, g\rangle = 0. \tag{3.4.60}$$

This implies that $\langle g, T_t g\rangle = \langle g, g\rangle$. The semigroup property implies that $T_t^2 = T_{2t}$, and thus we have $\langle T_t g, T_t g\rangle = \langle g, T_{2t} g\rangle = \langle g, g\rangle$. This implies that $\langle g - T_t g, g - T_t g\rangle = 0$. From this we conclude that $g(x) = \mathbb{E}^x g(X_t)$ a.e. This implies that the sets A and B are invariant for (T_t) which contradicts the irriducibility of $\mathcal{E}$. ∎

Definition 3.4.19. Given a Dirichlet form $(\mathcal{E}, \mathcal{F})$ on V define the effective resistance between subsets A_1 and A_2 of V by

$$R_{\text{eff}}(A_1, A_2)^{-1} = \inf\{\mathcal{E}(f,f) : f \in \mathcal{F}, f|_A = 0, f|_{A_2} = 1\}. \tag{3.4.61}$$

Let

$$A(t) = \{x \in V : x_1 = t\}, t \in [0,1]. \tag{3.4.62}$$

For $\mathcal{E} \in \mathfrak{E}$ set

$$\|\mathcal{E}\| = R_{\text{eff}}(A(0), A(1))^{-1}. \tag{3.4.63}$$

Let $\mathfrak{E}_1 = \{\mathcal{E} \in \mathfrak{E} : \|\mathcal{E}\| = 1\}$.

Lemma 3.4.20. *If $\mathcal{E} \in \mathfrak{E}$ then $\|\mathcal{E}\| > 0$.*

Proof

Write $\mathcal{H}$ for the set of functions u on V such that $u = i$ on $A(i), i = 0, 1$. First, note that $\mathcal{F} \cap \mathcal{H}$ is not empty. The regularity of $\mathcal{E}$ guarantees that there is a continuous function $u \in \mathcal{F}$ such that $u \leq 0$ on the face $A(0)$ and $u \geq 1$ one the opposite face $A(1)$. Then the Markov property for Dirichlet forms says $0 \wedge (u \vee 1) \in \mathcal{F} \cap \mathcal{H}$.

Second, observe that by Proposition 3.4.10 and the symmetry $T_{A(0)} < \infty$ a.s. This implies that $(\mathcal{E}, \mathcal{F}_{A(0)})$ is a transient Dirichlet form, see [30] Lemma 1.6.5 and Theorem 1.6.2. Denote $\mathcal{F}_{A(0)} = \{f \in \mathcal{F} : f|_{A(0)} = 0\}$. Thus $\mathcal{F}_{A(0)}$ is a Hilbert space with norm $\mathcal{E}$. Let $u \in \mathcal{F} \cap \mathcal{H}$ and h be its orthogonal projection onto the orthogonal complement of $\mathcal{F}_{A(0)\cup A(1)}$ in this Hilbert space. One can then see that $\mathcal{E}(h, h) = \|\mathcal{E}\|$.

Suppose that $\|\mathcal{E}\| = 0$, then $h = 0$ by Corollary 3.4.18. By definition h is harmonic in the complement of $A(0) \cup A(1)$ in the Dirichlet sense and thus also in the probabilistic sense by Proposition 3.1.4. We may then write $h(x) = \mathbb{P}^x(X_{T_{A(0)\cup A(1)}} \in A(1))$. By the symmetries of V, the fact that $h = 0$ contradicts the fact that $T_{A(1)} < \infty$ by Proposition 3.4.10.

3.4.2 Resistance Estimates

Let $\mathcal{E} \in \mathfrak{E}_1$, $S \in \mathcal{S}_n$, and $\gamma = \gamma_n(\mathcal{E})$ be the conductance across S. Where conductance has the following definition, if $S = \mathcal{O} \cap V$ for $\mathcal{O} \in \mathcal{O}_n(V)$. Then

$$\gamma_n = \inf\{\mathcal{E}^S(u,u) : u\mathcal{F}^S, u|_a = 0, u|_{a'} = 1\} \tag{3.4.64}$$

Where a and a' are opposite faces in $\mathcal{O}$. Note that γ_n does not depend on S and that $\gamma_0 = 1$. We write $v_n = v_n^{\mathcal{E}}$ for the minimizing function.

Conjecture 3.4.1. We assume, as in section 2.7, the existence of ρ_V and that it satisfies the following estimate.

$$\mathcal{C}_1 \rho_V^n \leq \gamma_n(\mathcal{E}_U) \leq \mathcal{C}_2 \rho_V^n \tag{3.4.65}$$

The following is Proposition 4.25 of [12].

Proposition 3.4.21. *Assume conjecture 3.4.1. Let $\mathcal{E} \in \mathfrak{E}_1$. Then for $n, m \geq 0$,*

$$\gamma_{n+m}(\mathcal{E}) \geq \mathcal{C}_1 \gamma_m(\mathcal{E}) \rho_V^n. \tag{3.4.66}$$

Proof

Consider the case $m = 0$. We compare the energry of v_0 with that of a function constructed from v_n and the minimizing function on a network where each $4N$-gon side l_V^{-n} is replaced by a diagonal crosswire. Write D_n for the network of diagonal crosswires obtained by joining each vertex of a $4N$-gon $\mathcal{O} \in \mathcal{O}_n$ to a vertex at the center of the $4N$-gon by a wire of unit resistance. Let R_n^D be the resistance across two opposite facts of V in this network and let f_n be the minimizing potential function.

Fix a $4N$-on $\mathcal{O} \in \mathcal{O}_n$ and let $S = \mathcal{O} \cap V$. Let $\{x_i\}_{i=1}^N$ be the set of vertices of

$\mathcal{O}$ and for each i let $A_{ij}, j = 1, \ldots, d$ be the faces containing x_i. Let A'_{ij} be the face opposite to A_{ij}. Let w_{ij} be the function, congruent to v_n, which is 1 on A_{ij} and zero on A'_{ij}. Set

$$u_i = \min\{w_{i1}, \ldots, w_{id}\}. \tag{3.4.67}$$

We have $u_i(x_i) = 1$ and $u_i = 0$ on $\bigcup_j A'_{ij}$. This lets us obtain

$$\mathcal{E}(u_i, u_i) \leq \sum_j \mathcal{E}(w_{ij}, w_{ij}) = d\gamma_n. \tag{3.4.68}$$

Write $a_i = f(x_i)$, and $\overline{a} = N^{-1} \sum_i a_i$. The energy of f_n in S is

$$\mathcal{E}_D^S(f_n, f_n) = \sum_i (a_i - \overline{a})^2 \tag{3.4.69}$$

Define a function $g_S : S \to \mathbb{R}$ by

$$g_S(y) = \overline{a} + \sum_i (a_i - \overline{a}) u_i(y). \tag{3.4.70}$$

Then

$$\mathcal{E}^S(g_S, g_S) \leq C\mathcal{E}(u_1, u_1) \sum_i (a_i - \overline{a})^2 \leq C\gamma_n \mathcal{E}_D^S(f_n, f_n). \tag{3.4.71}$$

By the definition of g_S the two $4N$-gons have a common face A and that $S_i = \mathcal{O}_i \cap V$, so $g_{S_1} = g_{S_2}$ on A. Define $g : V \to \mathbb{R}$ by taking $g(x) = g_S(x)$ for $x \in S$. If we sum over $\mathcal{O} \in \mathcal{O}_n(V)$ we see that $\mathcal{E}(g, g) \leq C\gamma_n (R_n^D)^{-1}$. But, since the function g is zero on one face of V and 1 on the opposite face, we have

$$1 = \gamma_0 = \mathcal{E}(v_0, v_) \leq \mathcal{E}(g, g) \leq \gamma_n (R_n^D)^{-1} \leq C\gamma_n \rho_V^{-n} \tag{3.4.72}$$

This completes the proof for $m = 0$ the proof when $m \geq 1$ is identical except that we use $4N$-gons $S \in \mathcal{S}_m$ and use sub-$4N$-gons with sides of length l_V^{-n-m}. ■

The next sequence of Lemmas lead to Proposition 4.28 of [12], starting from Lemma 4.26 of [12].

Lemma 3.4.22. *Assuming conjecture 3.4.1 we have*

$$\mathcal{C}_1 \gamma_n \leq \gamma_{n+1} \leq \mathcal{C}_2 \gamma_n \tag{3.4.73}$$

<u>Proof</u>

The left-hand inequality follows from 3.4.80. To prove the right hand one let $n = 0$. By Propositions 3.4.9 and 3.4.10 we see that $v_0 \geq C_3 > 0$ on $A(l_V^{-1})$. Let $w = (v_0 \wedge C_3)/C_3$. Choose a $4N$-gon $\mathcal{O} \in \mathcal{O}_1(V_1)$ between the hyperplanes $A_1(0)$ and $A_1(l_V^{-1}$. See definition 3.4.62. Then we have

$$\gamma_1 = \mathcal{E}^{V_1}(v_1, v_1) \leq \mathcal{E}^{V_1} \leq \mathcal{E}^{V_1}(w, w) \leq \mathcal{E}(w, w) \tag{3.4.74}$$

$$= C_3^{-2} \mathcal{E}(v_0 \wedge_0 C_3), v_0 \wedge C_3) \tag{3.4.75}$$

$$\leq C_3^{-2} \mathcal{E}(v_0, v_0) \tag{3.4.76}$$

$$= c_4 \gamma_0. \tag{3.4.77}$$

The case $n \geq 0$ is similar but we work with $4N$-gons $S \in \mathcal{S}_n$. ■

Set

$$\alpha = \log m_V / log l_V, \quad \beta_0 = \log(m_V \rho_V)/\log l_V. \tag{3.4.78}$$

Note that $\beta_0 \geq 2$, which makes $\rho_V m_V \geq l_V^2$. Write $H_0(r) = r^{\beta_0}$. By 3.4.80 for $n, k \geq 0$ we have

$$\frac{\gamma_n m_V^n}{\gamma_{n+k} m_V^{n+k}} \leq C \rho_V^{-k} m_V^{-k}. \tag{3.4.79}$$

We have $\rho_V m_V \geq l_V^2 > 1$ so there exists $k \geq 1$ such that

$$\gamma_n m_V^n < \gamma_{n+k} m_V^{n+k}, n \geq 0. \tag{3.4.80}$$

Fix this value of k and define the time scale function H for $\mathcal{E}$.

$$H(l_V^{-nk}) = \gamma_{nk}^{-1} m_V^{-nk}, n \geq 0, \tag{3.4.81}$$

and define H by linear interpolation on each interval $(l_V^{-(n+1)k}, l_V^{-nk})$. Also, set $H(0) = 0$.

The properties of H are summarized in the following lemma.

Lemma 3.4.23. *Assume conjecture 3.4.1. There exist constants C_i, and β', depending only on V, such that the following hold.*

1. *The function H is strictly increasing and continuous on* $[0, 1]$.
2. *For any $n, m \geq 0$ one has*

$$H(l_V^{-nk-mk}) \leq C_1 H(l_V^{-nk} H_0(l_V^{-mk}). \tag{3.4.82}$$

3. *For* $n \geq 0$,

$$H(l^{-(n+1)k} \leq H(l_V^{-nk}) \leq C_2 H(l_V^{-(n+1)k}). \tag{3.4.83}$$

4. *One has,*

$$C_3(t/s)^{\beta_0} \leq \frac{H(t)}{H(s)} \leq C_4(t/s)^{\beta'} \textit{ for } 0 < s \leq t < 1. \tag{3.4.84}$$

5. H *satisfies the 'fast time growth' condition of [37]. Explicitly* H *satisfies 'time doubling':*

$$H(2r) \leq C_5 H(r) \textit{ for } 0 \leq r \leq 1/2. \tag{3.4.85}$$

6. *For* $r \in [0, 1]$,

$$H(r) \leq C_6 H_0(r). \tag{3.4.86}$$

Proof

The first three items are follow from the definitions of H and H_0. To show 3.4.84 we use 3.4.82 and have

$$\frac{H(l_V^{-kn}}{H(l_V^{-kn-km}} \geq C_7 \frac{H(l^{kn})}{H(l_V^{-kn})(H_0(l_V^{-km})} = C_7 l_V^{km\beta_0} = C_7 \left(\frac{l_V^{-kn}}{l_v^{-kn-km}} \right)^{\beta_0}, \tag{3.4.87}$$

and interpolating using 3.4.83 gives the lower bound. To show the upper bound we use 3.4.73 to obtain

$$\frac{H(l_V^{-kn})}{H(l_V^{-kn-km})} \leq C_8^{km} = l_V^{km\beta'} = \left(\frac{l_V^{-kn}}{l_V^{-kn-km}}\right)^{\beta'} \tag{3.4.88}$$

where $\beta' = \log C_8 / \log l_V$. Now using 3.4.83 gives 3.4.84. Note 3.4.85 follows from 3.4.84. If we set $n = 0$ in 3.4.28 then this gives 3.4.86. ■

We say that $\mathcal{E}$ satisfies the condition RES(H, c_1, c_1) if for all $x_0 \in V, r \in (0, l_V^{-1})$, we have

$$c_1 \frac{H(r)}{r^\alpha} \leq R_{\text{eff}}(B(x_0, r), B(x_0, 2r)^c) \leq c_2 \frac{H(r)}{r^\alpha}. \tag{3.4.89}$$

The next Proposition is a generalization of our earlier assumption about resistance scaling. For reference, this is Proposition 4.28 of [12].

Proposition 3.4.24. *Assuming conjecture 3.4.1 there exist constants C_1, C_2 depending only on V, such that $\mathcal{E}$ satisfies RES(H, C_1, C_2).*

Proof

Let k be the smallest integer so that $l_V^{-k} \leq \frac{1}{2} d^{-1/2}$. Note that if $\mathcal{O} \in \mathcal{O}_k$ then $d(x, y) \leq d^{1/2} l_V^{-k} \leq \frac{1}{2} R$. Write $B_0 = B(x_0, R)$ and $B_1 = B(x_0, 2R)^c$.

First we prove the uppoer bound. Let $S_0, S_1, \ldots, S_n$ be such that $S_n \subset B_1$ and S_i is adjacent to S_{i+1} for $i = 0, \ldots, n-1$. Let f be the harmonic function in $V - (S_0 \cup B_1)$ which is 1 on S_0 and 0 on B_1. Let $A_0 = S_0 \cap S_1$, and A_1 be the face of S_1 opposite to A_0. Then using the lower bounds for the slides and corner moves, one can show that there exists $C_1 \in (0, 1)$ such that $f \geq C_1$ on A_1. Thus $g = \frac{(f - C_1)_+}{1 - C1}$ satisfies $\mathcal{E}^{S_1}(g, g) \geq \gamma_k$. This implies that

$$R_{\text{eff}}(S_0, B_1)^{-1} = \mathcal{E}(f, f) \geq \mathcal{E}^{S_1}(f, f) \geq (1 - C_1)^{-2} \gamma_k. \tag{3.4.90}$$

By the monotonicity of the resistance we have

$$R_{\text{eff}}(B_0, B_1) \le R_{\text{eff}}(S_0, B_1) \le C_2\gamma_k^{-1}, \tag{3.4.91}$$

which gives the desired upper bound. Let $n = k+1$ and let $S \in \mathcal{O}_n$. We use the definitions of v_n, w_{ij}, and u_i from Proposition 3.4.21. The symmetry of v_n shows that we have $w_{ij} \ge \frac{1}{2}$ on the half of S which is closer to A_{ij}. This implies that $u_i(x) \ge \frac{1}{2}$ if $\|x - x_i\|_\infty \le \frac{1}{2}l^{-n})V$.

Let $y \in l_V^{-n}\mathbb{Z} \cap V$, and let $V(y)$ be the union of the $4N$-gons in $\mathcal{O}_n$ containing y. We consider functions congruent to $2u_o \wedge 1$ in each of the $4N$-gons in $V(y)$, and construct a function g_i such that $g_i = 0$ on $V - V(y), g_i(z) = 1$ for $z \in V$ with $\|z - y\|_\infty \le \frac{1}{2}l_V^{-n}$, and $\mathcal{E}(g_i, g_i) \le C\gamma_n$. Choose $y_1, \dots, y_m$ so that $B_0 \subset \bigcup_i V(y_i)$. We may take $m \le C_5$. Then if $h = 1 \wedge (\sum_i g_i)$ we have $h = 1$ on B_0 and $h = 0$ on B_1. This gives

$$R_{\text{eff}}(B_0, B_1)^{-1} \le \mathcal{E}(h,h) \le \mathcal{E}\left(\sum g_i, \sum g_i\right) \le C_6\gamma_n. \tag{3.4.92}$$

This gives the lower bound. ■

3.4.3 Heat Kernel Estimates

We remind the reader that the contents of this section depend on the assumptions contained in section 2.7.

For this subsection we write h for the inverse of H, and $V(x,r) = \mu(B(x,r))$. We say that V satisfies volume doubling (VD) if there exists a constant C_1 such that

$$V(x, 2R) \le c_1 V(x,R) \quad \text{for all } x \in V, 0 \le R \le 1. \tag{3.4.93}$$

We say that $p_t(x, y)$ satisfies $HK(H; \eta_1, \eta_2), c_0)$ if for $x, y \in V, 0 < t \leq 1$,

$$p_t(x, y) \geq c_0^{-1} V(X, h(t))^{-1} \exp(-c_0 (H(d(x, y))/t)^{\eta_1}), \tag{3.4.94}$$

$$p_t(x, y) \geq c_0 V(X, h(t))^{-1} \exp(-c_0^{-1} (H(d(x, y))/t)^{\eta_2}). \tag{3.4.95}$$

The following equivalence is proved in [37].

Theorem 3.4.25. *Let $H : [0, 1] \to [0, \infty)$ be a strictly increasing function with $H(1) \in (0, \infty)$ that satisfies 3.4.85 and 3.4.84. Then the following are equivalent:*

1. *The Dirichlet form $(\mathcal{E}, \mathcal{F})$ satisfies VD, EHI, and REH(H, c_1, c_2) for some $c_1, c_2 > 0$.*

2. *The Dirichlet form $(\mathcal{E}, \mathcal{F})$ satisfies HK$(H; \eta_1, \eta_2, c_0)$ for some $\alpha, \eta_1, \eta_2, c_0 > 0$.*

Theorem 3.4.26. *Assuming conjecture 3.4.1 the process X has transition density $p_t(x, y)$ which satisfies HK$(H; \eta_1, \eta_2, C)$, where $\eta_1 = 1/(\beta_0 - 1)$, and $\eta_2 = 1/(\beta' - 1)$, and the constant C depends only on V.*

Proof

This proof is due to Barlow, Bass, Kumagai, and Teplyaev. See Theorem 4.30 [12]. This follows from Theorem 3.4.25 and Propositions 3.4.17 and 3.4.24. ∎

Let

$$J_r(f) = r^{\alpha} \int_V \int_{B(x,r)} |f(x) - f(y)|^2 d\mu(x) d\mu(y), \tag{3.4.96}$$

$$N_H^r(ff) = H(r)^{-1} J_r(f), \quad N_H(f) = \sup_{0<r\leq 1} N_H^r(f), \tag{3.4.97}$$

$$W_H = \{f \in L^2(V, \mu) : N_H(f) < \infty\}. \tag{3.4.98}$$

Let $r_j = l^{-kj}$, where k is from the definition of H. The following is Theorem 4.1 of [69]

Theorem 3.4.27. *Assume conjecture 3.4.1. Suppose p_t satisfies $HK(H, \eta_1, \eta_2, C_0)$, and H satisfies 3.4.85 and 3.4.84, then*

$$C_1\mathcal{E}(f,f) \le \limsup_{j\to\infty} N_H^{r_j}(f) \le N_H(f) \le C_2\mathcal{E}(f,f) \quad \textit{for all } f \in W_H, \tag{3.4.99}$$

where the constants C_i depend only on the constants in 3.4.84,3.4.85, and in HK$(H;\eta_1,\eta_2,C_0)$. Further

$$\mathcal{F} = W_H \tag{3.4.100}$$

Theorem 3.4.28. *Let $(\mathcal{E},\mathcal{F}) \in \mathfrak{E}_1$ and assume conjecture 3.4.1.*

1. *There exists constants $C_1, C_2 > 0$ such that for all $r \in [0,1]$,*

$$C_1H_0(r) \le H(r) \le C_2H_0(r). \tag{3.4.101}$$

2. *$W_H = W_{H_0}$, and there exists constants C_3, C_4 such that*

$$C_3N_{H_0}(F) \le \mathcal{E}(f,f) \le C_4N_{H_0}(f) \quad \textit{for all } f \in W_H. \tag{3.4.102}$$

3. *$\mathcal{F} = W_{H_0}$.*

<u>Proof</u>

This result is due to Barlow, Bass, Kumagai, and Teplyaev. See Theorem 4.32 [12].

We have $H(r) \leq C_2 H_0(r)$ by Lemma 3.4.23 so

$$N_H(f) \geq C_2^{-1} N_{H_0}(f). \tag{3.4.103}$$

Recall that the diffusions and their domains constructed earlier were denoted $(\mathcal{E}_U, \mathcal{F}_U)$. By 3.4.103 and 3.4.100 we have $\mathcal{F} \subset \mathcal{F}_U$. In particulate $v_0^{\mathcal{E}} \in \mathcal{F}_U$. Let,

$$A = \limsup_{k \to \ infty} \frac{H(r_k)}{H_0(r_k)} \tag{3.4.104}$$

Note that $A \leq C_2$. Let $f \in \mathcal{F}$, then we have by Theorem 3.4.27,

$$\mathcal{E}_U(f, f) \leq C_3 \limsup_{j \to \infty} H_0(r_j)^{-1} J_{r_j}(f) = C_3 \limsup_{j \to \infty} \frac{H(r_j)}{H_0(r_j)} H(r_j)^{-1} J_{r_j}(f) \tag{3.4.105}$$

$$\leq C_3 \limsup_{j \to \infty} A N_H^{r_j}(f) \leq C_4 A \mathcal{E}(f, f). \tag{3.4.106}$$

If we take $f = v_0^{\mathcal{E}}$ gives

$$1 \leq \mathcal{E}_U(v_0^{\mathcal{E}}, v_0^{\mathcal{E}}) \leq C_4 A \mathcal{E}(v_0^{\mathcal{E}}, v_0^{\mathcal{E}}) = C_4 A. \tag{3.4.107}$$

This implies that $A \geq C_5 = C_4^{-1}$. By Lemma 3.4.23 we have for $n, m \geq 0$,

$$\frac{H(r_{n+m})}{H_0(r_n + m)} \geq C_6^{-1} \geq C_5 / C_6, \tag{3.4.108}$$

which implies 3.4.101. The remainder of the theorem follows from Theorem 3.4.27. ■

Remark 3.4.29. Theorem 3.4.28 implies that $p_t(x, y)$ satisfies HK(H_0, η_1, η_1, C) with $\eta_1 = 1/(\beta_0 - 1)$.

3.4.4 Uniqueness of the Process

Definition 3.4.30. Let $W = W_{H_0}$ be as defined in 3.4.96. Let $\mathcal{A}, \mathcal{B} \in \mathfrak{E}$. We say that $\mathcal{A} \leq \mathcal{B}$ if

$$\mathcal{B}(u,u) - \mathcal{A}(u,u) \geq 0 \text{ for all } u \in W. \tag{3.4.109}$$

For $\mathcal{A}, \mathcal{B} \in \mathfrak{E}$ define

$$\sup(\mathcal{B}|\mathcal{A}) = \sup\left\{\frac{\mathcal{B}(f,f)}{\mathcal{A}(f,f)} : f \in W\right\}, \tag{3.4.110}$$

$$\inf(\mathcal{B}|\mathcal{A}) = \inf\left\{\frac{\mathcal{B}(f,f)}{\mathcal{A}(f,f)} : f \in W\right\} \tag{3.4.111}$$

$$h(\mathcal{A},\mathcal{B}) = \log\left(\frac{\sup(\mathcal{B}|\mathcal{A})}{\inf(\mathcal{B}|\mathcal{A})}\right); \tag{3.4.112}$$

Note that h is Hilbert's projective metric. Further we have $h(\theta\mathcal{A},\mathcal{B}) = h(\mathcal{A},\mathcal{B})$ for any $\theta \in (0,\infty)$. Also, $h(\mathcal{A},\mathcal{B}) = 0$ if and only if $\mathcal{A}$ is a non-zero constant multiple of $\mathcal{B}$.

The final two Theorems are due to Barlow, Bass, Kumagai, and Teplyaev in [12]. Together these establish the uniqueness – up to scalar multiples of the time parameter – of the diffusion we've constructed.

Theorem 3.4.31. *Assumning conjecture 3.4.1 there exists a constant C_V, which depends only on V, such that if $\mathcal{A}, \mathcal{B} \in \mathfrak{E}$ then*

$$h(\mathcal{A},\mathcal{B}) \leq C_V \tag{3.4.113}$$

Proof Let $A' = \mathcal{A}/\|\mathcal{A}\|, B' = \mathcal{B}/\|\mathcal{B}\|$. Then $h(\mathcal{A},\mathcal{B}) = h(\mathcal{A}',\mathcal{B}')$. By Theorem 3.4.28 there exist C_i depending only on V such that 3.4.102 holds for both $\mathcal{A}'$ and $\mathcal{B}'$.

This implies

$$\frac{\mathcal{B}'(f,f)}{\mathcal{A}(f,f)} \leq \frac{C_2}{C_2} \text{ for } f \in W. \tag{3.4.114}$$

This means that $\sup(\mathcal{B}'|\mathcal{A}) \leq C_2/C_1$. Similarly one has the bound $\inf(\mathcal{B}'|\mathcal{A}') \geq C_1/C_2$. Thus we obtain $h(\mathcal{A}', \mathcal{B}') \leq 2\log(C_2/C_1)$. ■

Theorem 3.4.32. *Let $V \subset \mathbb{R}$ be a $4N$ carpet. Assuming conjecture 3.4.1 then, up to scalar multiples $\mathfrak{E}$ consists of at most one element. Furthermore, this element satisfies scale invariance.*

Proof

We've shown earlier that $\mathfrak{E}$ is non-empty. Let $\mathcal{A}, \mathcal{B} \in \mathfrak{E}$, and $\lambda = \inf(\mathcal{B}|\mathcal{A})$. Let $\delta > 0$ and $\mathcal{C} = (1+\delta)\mathcal{B} - \lambda\mathcal{A}$. By Theorem 3.1.1 $\mathcal{C}$ is a local regular Dirichlet form on $L^2(v, \mu)$ and $\mathcal{C} \in \mathfrak{E}$. Since

$$\frac{\mathcal{C}(f,f)}{\mathcal{A}(f,f)} = (1+\delta)\frac{\mathcal{B}(f,f)}{\mathcal{A}(f,f)} - \lambda, \quad f \in W, \tag{3.4.115}$$

we have

$$\sup(\mathcal{C}|\mathcal{A}) = (1+\delta)\sup(\mathcal{B}|\mathcal{A}) - \lambda \tag{3.4.116}$$

$$\inf(\mathcal{C}|\mathcal{A}) = (1+\delta)\inf(\mathcal{B}|\mathcal{A}) - \lambda = \delta\lambda \tag{3.4.117}$$

From which it follows that for any $\delta > 0$, we have

$$e^{h(\mathcal{A},\mathcal{C})} = \frac{(1+\delta)\sup(\mathcal{B}|\mathcal{A}) - \lambda}{\delta\lambda} \geq \frac{1}{\delta}(e^{h(\mathcal{A},\mathcal{B})} - 1) \tag{3.4.118}$$

If $h(\mathcal{A}, \mathcal{B}) > 0$ the quantity is not bounded as $\delta \to 0$. This contradicts Theorem 3.4.31. It must therefore be the case that $h(\mathcal{A}, \mathcal{B}) = 0$. ■

Corollary 3.4.33. *Let V be a $4N$-gon for which conjectures 2.7.1, 2.7.2 (3.4.1) hold. If X is a continuous non-degenerate symmetric strong Markov process which is a Feller process, whose state space is V, and whose Dirichlet form is invariant with respect to the local symmetries of V, then the law of X under $\mathbb{P}^x$ is uniquely defined up to scalar multiples of the time parameter, for each $x \in V$.*

Bibliography

[1] S. Alexander and R. Orbach, *Density of states on fractals: "fractons".* J.Physique (Paris) Lett. *43* (1982) L625L631.

[2] M.T. Barlow, *Random walks, electrical resistance and nested fractals.* Asymptotic Problems in Probability Theory, 131157. Longman Scientific, Harlow UK, 1990

[3] M.T. Barlow, *Random walks and diffusions on fractals.* Proc. Int. Congress Math. Kyoto 1990, 10251035. Springer, Tokyo, 1991.

[4] M.T. Barlow, *Harmonic analysis on fractal spaces.* Astèrisque *206* (1992) 345-368.

[5] M. T. Barlow and R. F. Bass, *The construction of Brownian motion on the Sierpinski carpet.* Ann. Inst. H. Poincaré *25* (1989) 225257.

[6] M. T. Barlow and R. F. Bass, *Local times for Brownian motion on the Sierpinski carpet.* Probab. Th. Rel. Fields *85* (1990) 91104.

[7] M. T. Barlow and R. F. Bass, *On the resistance of the Sierpinski carpet.* Proc. R. Soc. London A. *431* (1990) 345360.

[8] M.T. Barlow and R.F. Bass, *Transition densities for Brownian motion on the Sierpinski carpet.* Probab. Th. Rel. Fields *91* (1992) 307330.

[9] M.T. Barlow and R.F. Bass, *Coupling and Harnack inequalities for Sierpinski carpets.* Bull. A.M.S. *29* (1993) 208212.

[10] R.F. Bass, *Diffusions on the Sierpinski Carpet.* Preprint.

[11] M.T. Barlow and R.F. Bass, *Brownian motion and harmonic analysis on Sierpinski carpets.* Canad. J. Math. *51* (4) (1999) 673-744.

[12] M.T. Barlow, R.F. Bass, T. Kumagai, A. Teplyaev, *Uniqueness of Brownian motion on Sierpiǹski carpets* J. Eur. Math. Soc *12* (2010) 655-701.

[13] M. T. Barlow, R. F. Bass, and J. D. Sherwood, *Resistance and spectral dimension of Sierpinski carpets.* J. Phys. *A 23* (1990) L253L258.

[14] J. Bellissard, J.M. Ghez, W. Wang, R. Rammal, and B. Pannetier *Band Speectrum for an Electron on a Sierpinski Gasket in a Magnetic Field* Solid State Comm., 64 (1987), 1291-1294.

[15] M.T. Barlow and B.M. Hambly, *Transition density estimates for Brownian motion on scale irregular Sierpinski gaskets.* Ann. I.H. Poincaré, to appear.

[16] M.T. Barlow, K. Hattori, T. Hattori, and H. Watanabe, *Weak homogenization of anisotropic diffusion on pre-Sierpinski carpets.* Preprint.

[17] M.T. Barlow and J. Kigami, *Localized eigenfunctions of the Laplacian on p.c.f. self-similar sets.* To appear J. Lond. Math. Soc.

[18] M. T. Barlow and E. A. Perkins, *Brownian motion on the Sierpinski gasket.* Probab. Th. Rel. Fields *79* (1988) 543623.

[19] R.F. Bass, *Probabilistic Techniques in Analysis.* Springer, New York, 1995.

[20] R.F. Bass and P. Hsu, *Some potential theory for reflecting Brownian motion in Hölder and Lipschitz domains.* Ann. Prob. *19* (1991) 486508.

[21] R.M. Blumenthal and R.K. Getoor, *Markov Processes and Potential Theory.* Academic Press, New York, 1968.

[22] Z.Q. Chen and M. Fukushima *Symmetric Markov Processes, Time Change, and Boundary Theory* Princeton University Press, Princeton, 2012.

[23] E.A. Carlen, S. Kusuoka, and D.W. Stroock, *Upper bounds for symmetric Markov transition functions.* Ann. Inst. H. Poincaré *23* (1987) 245287.

[24] S. N. Ethier and T.G. Kurtz *Markov Processes: Characterization and Convergence* Wiley, New York, 1986.

[25] E.B. Dynkin *Markov Processes. I.* Springer, Berlin(1965)

[26] E.B. Fabes and D.W. Stroock, *A new proof of Mosers parabolic Harnack inequality via the old ideas of Nash.* Arch. Mech. Rat. Anal. *96* (1986) 327338.

[27] P.J. Fitzsimmons, B.M. Hambly, and T. Kumagai, *Transition density estimates for Brownian motion on affine nested fractals* . Comm. Math. Phys. *165* (1995) 595620.

[28] M. Fukushima, *Dirichlet Forms and Markov Processes.* North-Holland/Kodansha Toyko, 1980.

[29] M. Fukushima, *Dirichlet forms, diffusion processes, and spectral dimensions for nested fractals.* Ideas and methods in stochastic analysis, stochastics and applications, 151161. Cambridge Univ. Press, Cambridge, 1992.

[30] M. Fukushima, Y. Oshima, and M. Takeda *Dirichlet Forms and Symmetric Markov Processes* DeGruyter, Berlin, 1992.

[31] M. Fukushima and T. Shima, *On a spectral analysis for the Sierpinski gasket.* Potential Analysis *1* (1992) 135.

[32] M. Fukushima and T. Shima, *On discontinuity and tail behaviours of the integrated density of states for nested pre-fractals.* Comm. Math. Phys. *163* (1994) 461471.

[33] A. Aharony, Y. Gefen, and B. Mandelbrot, *Phase transitions on fractals: III.* Infinitely ramified lattices. J. Phys. *A 17* (1984) 12771289.

[34] S. Goldstein, *Random walks and diffusion on fractals.* Percolation theory and ergodic theory of infinite particle systems, 121129. Springer, New York,1987.

[35] R.F. Green, *A simple model for the construction of Brownian motion on Sierpinskis hexagonal gasket.* Technical report, Univ. of Minnesota, Duluth, 1989.

[36] A. Grigor'yan *The heat equation on noncompact Riemannian manifolds* Mat. Sb. *182* (1991), no.1, 55-87

[37] A. Grigor'yan, and A. Telcs, *Two-sided estimates of heat kernels on metric measure spaces* Annals of Probability *40* (2012) 1212-1284.

[38] A. Grigor'yan and A. Telcs *Sub-Gaussian estimates of heat kernels on infinite graphs* Duke Math. J. *109* (2001), no. 3, 451510.

[39] A. Grigor'yan, and J. Hu *Heat kernels and Green functions on metric measure spaces* Canad. J. Math. *60* (2014) no.3 641-699.

[40] B.M. Hambly, *Brownian motion on a homogeneous random fractal.* Probab.Th. Rel. Fields *94* (1992) 138.

[41] B.M. Hambly, *Brownian motion on a homogeneous random recursive Sierpinski gasket.* Ann. Probab., to appear.

[42] B.M. Hambly, T. Kumagai, *Heat kernel estimates and homogenization for asymptotically lower dimensional processes on some nested fractals*Ṗotential Analysis, to appear.

[43] K. Hattori, T. Hattori, and H. Watanabe, *Gaussian field theories and the spectral dimensions.* Prog. Th. Phys. Supp. *92* (1987) 108143

[44] K. Hattori, T. Hattori, and H. Watanabe, *New approximate renormalisation method on fractals.* Phys Rev. *A 32* (1985) 37303733.

[45] K. Hattori, T. Hattori, and S. Kusuoka, *Self-avoiding paths on the preSierpinski gasket.* Probab. Th. Rel. Fields *84* (1990) 126.

[46] K. Hattori and T. Hattori, *Self-avoiding process on the Sierpinski gasket.* Probab. Th. Rel. Fields *88* (1991) 405428.

[47] K. Hattori, T. Hattori, and S. Kusuoka, *Self-avoiding paths on the threedimensional Sierpinski gasket.* Publ. Res. Inst. Math. Sci. *29* (1993) 455509.

[48] K. Hattori, T. Hattori, and H. Watanabe, *Asymptotically one-dimensional diffusions on the Sierpinski gasket and the abc-gaskets.* Probab. Th. Rel. Fields *100* (1994) 85116.

[49] T. Hattori, *Asympotically one-dimensional diffusions on scale-irregular gaskets.* Preprint.

[50] T. Hattori and S. Kusuoka, *The exponent for mean square displacement of self-avoiding random walk on Sierpinski gasket.* Probab. Th. Rel. Fields *93* (1992) 273284.

[51] S. Havlin and D. Ben-Avraham, *Diffusion in disordered media.* Adv. Phys. *36* (1987) 695798.

[52] J. Kigami, *A harmonic calculus on the Sierpinski space.* Japan J. Appl. Math. *6* (1989) 259290.

[53] J. Kigami, *A harmonic calculus for p.c.f. self-similar sets.* Trans. A.M.S. *335* (1993) 721755.

[54] J. Kigami, *Harmonic metric and Dirichlet form on the Sierpinski gasket.* Asymptotic Problems in Probability Theory, 201218. Longman Scientific, Harlow UK, 1990.

[55] J. Kigami, *Effective resistance for harmonic structures on P.C.F. self-similar sets.* Math. Proc. Camb. Phil. Soc. *115* (1994) 291303.

[56] J. Kigami, *Hausdorff dimension of self-similar sets and shortest path metric.* J. Math. Soc. Japan *47* (1995) 381404.

[57] J. Kigami, *Harmonic calculus on limits of networks and its application to dendrites.* J. Functional Anal. *128* (1995) 4886.

[58] J. Kigami, *Analysis on Fractals.* Cambridge University of Press, Cambridge, UK (2001)

[59] J. Kigami and M. Lapidus, *Weyls spectral problem for the spectral distribution of Laplacians on P.C.F. self-similar fractals.* Comm. Math. Phys. *158* (1993) 93125.

[60] S.M. Kozlov, *Harmonization and homogenization on fractals.* Comm. Math. Phys. *153* (1993) 339357.

[61] W.B. Krebs, *A diffusion defined on a fractal state space.* Stoch. Proc. Appl. *37* (1991) 199212.

[62] W.B. Krebs, *Hitting time bounds for Brownian motion on a fractal.* Proc. A.M.S. *118* (1993) 223232.

[63] N.V Krylov and M.V. Safonov, *An Estimate of the Probability that a diffusion Process hits a Set of Positive Measure*. Soviet Math. Doklady, *20* (1979) 253-255.

[64] T. Kumagai, *Construction and some properties of a class of non- symmetric diffusion processes on the Sierpinski gasket.* Asymptotic Problems in Probability Theory, 219247. Longman Scientific, Harlow UK, 1990.

[65] T. Kumagai, *Estimates of the transition densities for Brownian motion on nested fractals.* Probab. Th. Rel. Fields *96* (1993) 205224.

[66] T. Kumagai, *Regularity, closedness, and spectral dimension of the Dirichlet forms on p.c.f. self-similar sets.* J. Math. Kyoto Univ. *33* (1993) 765786.

[67] T. Kumagai, *Rotation invariance and characterization of a class of self-similar diffusion processes on the Sierpinski gasket.* Algorithms, fractals and dynamics, 131142. Plenum, New York, 1995.

[68] T. Kumagai, *Short time asymptotic behaviour and large deviation for Brownian motion on some affine nested fractals.* Publ. RIMS, to appear.

[69] T. Kumagai and K.-T. Sturm *Construction of diffusion processes on fractals, d-sets, and general metric measure spaces.* J. Math. Kyoto Univ *45*, 307-327 (2005)

[70] T. Kumagai and S. Kusuoka, *Homogenization on nested fractals.* Probab. Th. Rel. Fields *104* (1996) 375398.

[71] S. Kusuoka, *A diffusion process on a fractal.* Symposium on Probabilistic Methods in Mathematical Physics, Taniguchi, Katata, 251274. Academic Press, Amsterdam, 1987.

[72] S. Kusuoka, *Dirichlet forms on fractals and products of random matrices.* Publ. RIMS Kyoto Univ., *25* (1989) 659680.

[73] S. Kusuoka and X.Y. Zhou, *Dirichlet form on fractals: Poincaré constant and resistance.* Probab. Th. Rel. Fields *93* (1992) 169196.

[74] S. Kusuoka and X.Y. Zhou, *Waves on fractal-like manifolds and effective energy propagation.* Preprint.

[75] T. Lindstrm, *Brownian motion on nested fractals.* Mem. A.M.S. *420* (1990).

[76] T. Lindstrm, *Brownian motion penetrating the Sierpinski gasket.* Asymptotic Problems in Probability Theory, 248-278. Longman Scientific, Harlow UK, 1990.

[77] T. Lindvall and L.C.G. Rogers, Coupling of multi-dimensional diffusions by reflection. Ann. Prob. *14* (1986) 860872.

[78] V. Metz, *Potential theorie auf dem Sierpinski gasket.* Math. Ann. *289* (1991) 207237.

M2 V. Metz, *Renormalization of finitely ramified fractals.* Proc. Roy. Soc. Edinburgh Ser *A 125* (1995) 10851104.

[79] V. Metz, *How many diffusions exist on the Vicsek snowflake?* Acta Appl. Math. *32* (1993) 224241.

[80] V. Metz, *Renormalization on fractals.* Proc. Inter. Conf. Potential Th. *94*, 413422. de Gruyter, Berlin, 1996.

[81] V. Metz, *Renormalization contracts on nested fractals.* C.R. Acad. Sci. Paris *332* (1996) 10371042.

[82] V. Metz and K.-T. Sturm, *Gaussian and non-Gaussian estimates for heat kernels on the Sierpinski gasket.* Dirichlet forms and stochastic processes, 283289. de Gruyter, Berlin, 1995.

[83] J. Moser, *On Harnacks inequality for elliptic differential equations.* Comm. Pure Appl. Math. *14* (1961) 577591.

[84] M. Okada, T. Sekiguchi, and Y. Shiota, *Heat kernels on infinite graph networks and deformed Sierpinski gaskets.* Japan J. App. Math. *7* (1990) 527554.

[85] H. Osada, *Isoperimetric dimension and estimates of heat kernels of pre Sierpinski carpets.* Probab. Th. Rel. Fields *86* (1990) 469490.

[86] K. Pietruska-Paluba, *The Lifchitz singularity for the density of states on the Sierpinski gasket.* Probab. Th. Rel. Fields *89* (1991) 134.

[87] S.C. Port and C.J. Stone *Brownian Motion and Classical Potential Theory* Academic Press, New York, 1978.

[88] R. Rammal, *Spectrum of harmonic excitations on fractals.* J. de Physique *45* (1984) 191-206.

[89] R. Rammal and G. Toulouse, *Random walks on fractal structures and percolation clusters.* J. Physique Lettres *44* (1983) L13L22.

[90] W. Rudin, *Functional Analysis* McGraw-Hill (1991).

[91] C. Sabot, *Existence et unicite de la diffusion sur un ensemble fractal*. C.R. Acad. Sci. Paris (1995) 10531059.

[92] T. Shima, *On eigenvalue problems for the random walk on the Sierpinski pre-gaskets.* Japan J. Appl. Ind. Math. *8* (1991) 127141.

[93] T. Shima, *The eigenvalue problem for the Laplacian on the Sierpinski gasket.* Asymptotic Problems in Probability Theory , 279-288. Longman Scientific, Harlow UK, 1990.

[94] T. Shima, *On Lifschitz tails for the density of states on nested fractals.* Osaka J. Math. *29* (1992) 749-770.

[95] T. Shima, *On eigenvalue problems for Laplacians on P.C.F. self-similar sets.* Japan J. Indust. Appl. Math. *13* (1996) 123.

[96] W. Sierpinski, *Sur une courbe dont tout point est un point de ramification.* C.R. Acad. Sci. Paris *160* (1915) 302305.

[97] W. Sierpinski, *Sur une courbe cantorienne qui contient une image biunivoque et continue de toute courbe donnée.* C.R. Acad. Sci. Paris *162* (1916) 629632.

[98] B. Steinhurst, *Diffusions and Laplacians on Laakso, Barlow-Evans, and other Fractals* Ph.D. Dissertation (2010)

[99] A. Torchinsky, *Real-Variable Methods in Harmonic Analysis* Academic Press, Orlando FL (1986)

[100] H. Watanabe, *Spectral dimension of a wire network.* J. Phys. *A. 18* (1985) 28072823.

[101] R. Wittmann *Natural Densities for Markov Transition Probabilities* Prob. Th. and related Fields, Vol. *73*, (1986), 1-10.

[102] L.-Y. Yuan and R. Tao, *Experimental study of the conductivity exponent for some Sierpinski carpets.* Phys Lett. *A 116* (1986) 284286.

[103] L.-Y. Yuan and R. Tao, *Studies on the scaling exponents of conductivity for Sierpinski carpets.* J. Phys. *C 21* (1986) 401409.