5-17-2016

Modules from Tilted to Cluster-Tilted Algebras

Stephen M. Zito
University of Connecticut - Storrs, stephen.zito@uconn.edu

Modules from Tilted to Cluster-Tilted Algebras

Stephen M. Zito, Ph.D.

University of Connecticut, 2016

ABSTRACT

We study the module categories of a tilted algebra C and the corresponding cluster-tilted algebra $B = C \ltimes E$ where E is the C-C-bimodule $\mathrm{Ext}^2_C(DC, C)$. We investigate how various properties of a C-module are affected when considered in the module category of B. We give a complete classification of the projective dimension of a C-module inside $\mathrm{mod}\, B$. If a C-module M satisfies $\mathrm{Ext}^1_C(M, M) = 0$, we show two sufficient conditions for M to satisfy $\mathrm{Ext}^1_B(M, M) = 0$. In particular, if M is indecomposable and $\mathrm{Ext}^1_C(M, M) = 0$, we prove M always satisfies $\mathrm{Ext}^1_B(M, M) = 0$. Furthermore, we study which τ_C-rigid C-modules are also τ_B-rigid B-modules. In the special case M is an indecomposable τ_C-rigid C-module, we prove a necessary and sufficient condition for M to be a τ_B-rigid B-module.

Modules from Tilted To Cluster-Tilted Algebras

Stephen M. Zito

M.S. University of Connecticut, 2012

M.S. Fairfield University, 2010

B.S. Sacred Heart University, 2008

A Dissertation

Submitted in Partial Fulfillment of the

Requirements for the Degree of

Doctor of Philosophy

at the

University of Connecticut

2016

APPROVAL PAGE

Doctor of Philosophy Dissertation

Modules from Tilted to Cluster-Tilted Algebras

Presented by

Stephen M. Zito, B.S. Math., M.S. Math.

Major Advisor ______________________________
Ralf Schiffler

Associate Advisor ______________________________
Jerzy Weyman

Associate Advisor ______________________________
Thomas Roby

University of Connecticut

2016

ACKNOWLEDGMENTS

It has been six years since I entered the graduate math program at Uconn. It has been a long and difficult journey, but I am happy to graduate with a degree from such a prestigious university. I would like to acknowledge all the people that have guided and supported me throughout my tenure here.

I would like to thank my advisor Ralf Schiffler. His wealth of knowledge and infinite patience has been invaluable in completing this work. I can recall numerous visits to his office to show off a new proof, and he would find errors or gaps in the argument. However, he would always guide me in the right direction for fixing the mistakes. I can not express enough gratitude for him.

I would also like to give thanks to the rest of my advising committee Jerzy Weyman and Thomas Roby. Through the courses they have taught, talks and presentations they have given, and the always helpful comment or suggestion, I've been able to improve my paper significantly. It was an absolute pleasure knowing I could always stop by Monique Roy's office for a friendly chat.

Words can not adequately describe the level of support I have received from my parents, Jane and Steve Zito. Completing this degree would not have been possible without them. My sister Stephanie and her husband Jeremy are my closest friends and were always by my side every step of the way.

Finally, I must acknowledge three very special individuals whose encouragement and support was limitless. Gary Valentine has been my coach and mentor for seven years in the sport of weightlifting. It has been an honor to train with him. I have known Richard Baines for twelve years and our relationship goes beyond friendship. I consider him family. I can not count all the wonderful memories I have shared with Katherine Goodrich. Our late night talks always provided relief and comfort from a stressful day of math. Every day she makes me smile.

Contents

Ch. 1. Introduction and Preliminaries 1

1.1 Tilted Algebras . . . 5
1.2 Cluster categories and cluster-tilted algebras . . . 8
1.3 Relation Extensions . . . 9
1.4 Induction and coinduction functors . . . 9
1.5 Standard results . . . 11
1.6 Auslander-Reiten Theory . . . 13
1.7 Induced and coinduced modules in cluster-tilted algebras . . . 15
1.8 τ-rigid modules . . . 16

Ch. 2. Homological Dimensions 20

2.1 Projective dimension 0 and 2 . . . 20
2.2 Projective dimension 1 . . . 22

Ch. 3. Extensions 27

3.1 Main result . . . 27
3.2 Corollaries . . . 30

Ch. 4. τ-rigid modules 33

4.1 Main Results . . . 33
4.2 Partial Tilting Modules and τ_B-rigidity . . . 37
4.3 Projective Covers and τ_B-rigidity . . . 44

Ch. 5. Examples 47

5.1 Homological dimensions . . . 50
5.2 Extensions . . . 51
5.3 τ-rigid modules . . . 53

Bibliography 55

Chapter 1

Introduction and Preliminaries

Representation theory is a branch of mathematics that studies abstract algebraic objects by representing their elements as matrices and the operations between these elements as multiplication of matrices. This enables us to translate questions from an abstract algebraic setting to a linear algebra setting, a subject which is well understood. One can then use well-developed techniques of linear algebra such as Gaussian elimination, eigenvalue theory, and vector space bases to solve these questions. Representation theory was introduced more than 100 years ago by Ferdinand Georg Frobenius as a tool for studying the algebraic structure of finite groups. Since then, representation theory has seen connections to numerous branches of mathematics including algebraic geometry, module theory, analytic number theory, differential geometry, operator theory, algebraic combinatorics and topology. In particular, we are interested in studying the representation theory of cluster-tilted algebras which are finite dimensional associative algebras that were introduced by Buan, Marsh, and Reiten in [15] and, independently, by Caldero, Chapoton, and Schiffler in [18] for type $\mathbb{A}$.

One motivation for introducing these algebras came from Fomin and Zelevinsky's cluster algebras [20]. Cluster algebras were developed as a tool to study dual canonical bases and total positivity in semisimple Lie groups, and cluster-tilted algebras were constructed as a categorification of these algebras. To every cluster in an acyclic cluster algebra one can associate a cluster-tilted algebra, such that the indecomposable rigid modules over the cluster-tilted algebra correspond bijectively to the cluster variables outside the chosen cluster. Many people have studied cluster-tilted algebras in this context, see for example [12, 15, 16, 17, 19, 22].

The second motivation came from classical tilting theory. Tilted algebras are the endomorphism algebras of tilting modules over hereditary algebras, whereas cluster-tilted algebras are the endomorphism algebras of cluster-tilting objects over cluster categories of hereditary algebras. This similarity in the two definitions lead to the following precise relation between tilted and cluster-tilted algebras, which was established in [3].

There is a surjective map

$$\{\text{tilted algebras}\} \longmapsto \{\text{cluster-tilted algebras}\}$$

$$C \longmapsto B = C \ltimes E$$

where E denotes the C-C-bimodule $E = \operatorname{Ext}^2_C(DC, C)$ and $C \ltimes E$ is the trivial extension.

This result allows one to define cluster-tilted algebras without using the cluster category. It is natural to ask how the module categories of C and B are related and several results in this direction have been obtained, see for example [4, 5, 6, 11, 13]. In this work, we investigate how various properties of a C-module are affected when the same module is viewed as a B-module via the standard embedding. We let M be a right C-module and define a right $B = C \ltimes E$ action on M by

$$M \times B \to M \;\;,\;\; (m, (c, e)) \mapsto mc.$$

Our first main result is on the projective dimension of a C-module when viewed as a B-module.

Here, τ_C^{-1} and Ω_C^{-1} denote respectively the inverse Auslander-Reiten translation and first cosyzygy of a C-module. In Chapter 2 we show the following.

Theorem 1.0.1. (Theorem 2.2.5). *Let C be a tilted algebra, $E = \operatorname{Ext}_C^2(DC, C)$, and $B = C \ltimes E$ the corresponding cluster-tilted algebra.*

(a) *If* $\operatorname{pd}_C M = 0$, *then* $\operatorname{pd}_B M = 0$ *if and only if* $\operatorname{id}_C M \leq 1$. *Otherwise,* $\operatorname{pd}_B M = \infty$.

(b) *If* $\operatorname{pd}_C M = 2$, *then* $\operatorname{pd}_B M = \infty$.

(c) *Let* $\operatorname{pd}_C M = 1$ *with minimal projective resolution* $0 \to P_1 \to P_0 \to M \to 0$. *Then* $\operatorname{pd}_B M = 1$ *if and only if* $\operatorname{id}_C M \leq 1$ *and* $\tau_C^{-1}\Omega_C^{-1}P_0 \cong \tau_C^{-1}\Omega_C^{-1}P_1$. *Otherwise,* $\operatorname{pd}_B M = \infty$.

Our second main result is on C-modules that satisfy $\operatorname{Ext}_C^1(M, M) = 0$. These are known as *rigid* modules. Here, our result holds in a more general setting with C an algebra of global dimension equal to 2. We determine two sufficient conditions to guarantee when a rigid C-module remains rigid when viewed as a B-module, i.e., $\operatorname{Ext}_B^1(M, M) = 0$. Here, τ_C and Ω_C denote respectively the Auslander-Reiten translation and first syzygy of a C-module. The following result is shown in Chapter 3.

Theorem 1.0.2. (Theorem 3.1.2). *Let M be a rigid C-module with a projective cover $P_0 \to M$ and an injective envelope $M \to I_0$ in* $\operatorname{mod} C$.

(a) *If* $\operatorname{Hom}_C(\tau_C^{-1}\Omega_C^{-1}P_0, M) = 0$, *then M is a rigid B-module.*

(b) *If* $\operatorname{Hom}_C(M, \tau_C\Omega_C I_0) = 0$, *then M is a rigid B-module.*

As an immediate consequence, in the case C is tilted, we obtain an affirmative answer to whether an indecomposable rigid C-module remains rigid as a B-module.

Corollary 1.0.3. (Corollary 3.2.2). *Let C be a tilted algebra with B the corresponding cluster-titled algebra. Suppose M is an indecomposable, rigid C-module. Then M is a rigid B-module.*

Our third main result deals with C-modules that satisfy $\operatorname{Hom}_C(M, \tau_C M) = 0$ otherwise know as *τ_C-rigid modules*. We investigate under what conditions a τ_C-rigid module M is also a τ_B-rigid module. The following result is shown in Chapter 4.

Theorem 1.0.4. (Theorem 4.2.1). *Let M be a partial tilting C-module such that $\operatorname{pd}_C \tau_C M \leq 1$. Then M is τ_B-rigid if and only if* $\operatorname{Hom}_C(\tau_C^{-1}\Omega_C^{-1}M, \operatorname{Gen} M) = 0$.

In the special case where C is a tilted algebra and B is the corresponding cluster-tilted algebra, we have a complete classification determining when an indecomposable τ_C-rigid module M is also τ_B-rigid.

Corollary 1.0.5. (Corollary 4.2.2). *Let C be a tilted algebra such that $C = \operatorname{End}_A T$ where A is hereditary and T is a tilting A-module. Let $B = C \ltimes E$ be the corresponding cluster-tilted algebra and let M be an indecomposable τ_C-rigid module. Then M is τ_B-rigid if and only if* $\operatorname{Hom}_C(\tau_C^{-1}\Omega_C^{-1}M, \operatorname{Gen} M) = 0$.

We now set the notation for the remainder of this paper. All algebras are assumed to be finite dimensional over an algebraically closed field k. Suppose $Q = (Q_0, Q_1)$ is a connected quiver without oriented cycles where Q_0 denotes a finite set of vertices and Q_1 denotes a finite set of oriented arrows. By kQ we denote the path algebra of Q. If Λ is a k-algebra then denote by $\operatorname{mod}\Lambda$ the category of finitely generated right Λ-modules and by $\operatorname{ind}\Lambda$ a set of representatives of each isomorphism class of indecomposable right Λ-modules. Given $M \in \operatorname{mod}\Lambda$, the projective dimension of M in $\operatorname{mod}\Lambda$ is denoted $\operatorname{pd}_\Lambda M$ and its injective dimension by $\operatorname{id}_\Lambda M$. We denote by $\operatorname{add} M$ the smallest additive full subcategory of $\operatorname{mod}\Lambda$ containing M, that is, the full subcategory of $\operatorname{mod}\Lambda$ whose objects are the direct sums of direct summands of the module M. As mentioned before, we let τ_Λ and τ_Λ^{-1} be the Auslander-Reiten translations in $\operatorname{mod}\Lambda$. We let D be the standard duality functor $\operatorname{Hom}_k(-, k)$. Also mentioned before, ΩM and $\Omega^{-1}M$ will denote the first syzygy and first cosyzygy of M. Finally, let gl.dim stand for the global dimension of an algebra. For more details and exact definitions, we direct the reader to [8] and [24].

1.1 Tilted Algebras

Tilting theory is one of the main themes in the study of the representation theory of algebras. Given a k-algebra A, one can construct a new algebra B in such a way that the corresponding module categories are closely related. The main idea is that of a tilting module.

Definition 1.1.1. Let A be an algebra. An A-module T is a *partial tilting module* if the following two conditions are satisfied:

(1) $\mathrm{pd}_A\, T \leq 1$.

(2) $\mathrm{Ext}^1_A(T, T) = 0$.

A partial tilting module T is called a *tilting module* if it also satisfies the following additional condition:

(3) There exists a short exact sequence $0 \to A \to T' \to T'' \to 0$ with T' and $T'' \in \mathrm{add}\, T$.

We recall that an A-module M is *faithful* if its right annihilator

$$\mathrm{Ann}\, M = \{a \in A \mid Ma = 0\}.$$

vanishes. It follows easily from (3) that any tilting module is faithful. We will need the following characterization of faithful modules. Define *Gen* M to be the class of all modules X in $\mathrm{mod}\, A$ generated by M, that is, the modules X such that there exists an integer $d \geq 0$ and an epimorphism $M^d \to X$ of A-modules. Here, M^d is the direct sum of d copies of M. Dually, we define *Cogen* M

to be the class of all modules Y in $\operatorname{mod} A$ cogenerated by M, that is, the modules Y such that there exist an integer $d \geq 0$ and a monomorphism $Y \to M^d$ of A-modules.

Lemma 1.1.2. [8, VI, Lemma 2.2.]. *Let A be an algebra and M an A-module. The following are equivalent:*

(a) *M is faithful.*

(c) *A is cogenerated by M.*

(d) *DA is generated by M.*

Partial tilting modules induce torsion pairs in a natural way. We consider the restriction to a subcategory C of a functor F defined originally on a module category, and we denote it by $F|_C$. Also, let S be a subcategory of a category C. We say S is a *full subcategory* of C if, for each pair of objects X and Y of S, $\operatorname{Hom}_S(X, Y) = \operatorname{Hom}_C(X, Y)$.

Definition 1.1.3. A pair of full subcategories $(\mathcal{T}, \mathcal{F})$ of $\operatorname{mod} A$ is called a *torsion pair* if the following conditions are satisfied:

(a) $\operatorname{Hom}_A(M, N) = 0$ for all $M \in \mathcal{T}$, $N \in \mathcal{F}$.

(b) $\operatorname{Hom}_A(M, -)|_{\mathcal{F}} = 0$ implies $M \in \mathcal{T}$.

(c) $\operatorname{Hom}_A(-, N)|_{\mathcal{T}} = 0$ implies $N \in \mathcal{F}$.

Consider the following full subcategories of $\operatorname{mod} A$ where T is a partial tilting module.

$$\mathcal{T}(T) = \{M \in \operatorname{mod} A \mid \operatorname{Ext}^1_A(T, M) = 0\}$$

$$\mathcal{F}(T) = \{M \in \operatorname{mod} A \mid \operatorname{Hom}_A(T, M) = 0\}$$

Then $(\mathcal{T}(T), \mathcal{F}(T))$ is a torsion pair in $\operatorname{mod} A$ called the *induced torsion pair* of T. Considering the endomorphism algebra $C = \operatorname{End}_A T$, there is an induced torsion pair, $(\mathcal{X}(T), \mathcal{Y}(T))$, in $\operatorname{mod} C$.

$$\mathcal{X}(T) = \{M \in \operatorname{mod} B \mid M \otimes_C T = 0\}$$

$$\mathcal{Y}(T) = \{M \in \operatorname{mod} B \mid \operatorname{Tor}_1^C(M, T) = 0\}$$

We now state the definition of a tilted algebra.

Definition 1.1.4. Let A be a hereditary algebra with T a tilting A-module. Then $C = \operatorname{End}_A T$ is called a *tilted algebra*.

The following proposition describes several facts about tilted algebras which we will use throughout the paper. Let A be an algebra and M, N be two indecomposable A-modules. A *path* in $\operatorname{mod} A$ from M to N is a sequence

$$M = M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \to \ldots \xrightarrow{f_s} M_s = N$$

where $s \geq 0$, all the M_i are indecomposable, and all the f_i are nonzero nonisomorphisms. In this case, M is called a *predecessor* of N in $\operatorname{mod} A$ and N is called a *successor* of M in $\operatorname{mod} A$.

Proposition 1.1.5. [8, VIII, Lemma 3.2.]. *Let A be a hereditary algebra, T a tilting A-module, and $C = \operatorname{End}_A T$ the corresponding tilted algebra. Then*

(a) $\operatorname{gl.dim} C \leq 2$.

(b) *For all $M \in \operatorname{ind} C$,* $\operatorname{id}_C M \leq 1$ *or* $\operatorname{pd}_C M \leq 1$.

(c) *For all $M \in \mathcal{X}(T)$,* $\operatorname{id}_C M \leq 1$.

(d) *For all $M \in \mathcal{Y}(T)$,* $\operatorname{pd}_C M \leq 1$.

(e) $(\mathcal{X}(T), \mathcal{Y}(T))$ *is splitting, which means that every indecomposable C-module belongs to either* $\mathcal{X}(T)$ *or* $\mathcal{Y}(T)$*.*

(f) $\mathcal{Y}(T)$ *is closed under predecessors and* $\mathcal{X}(T)$ *is closed under successors.*

We also need the following characterization of split torsion pairs.

Proposition 1.1.6. [8, VI, Proposition 1.7] *Let* $(\mathcal{T}, \mathcal{F})$ *be a torsion pair in* $\operatorname{mod} A$*. The following are equivalent:*

(a) $(\mathcal{T}, \mathcal{F})$ *is split.*

(b) *If* $M \in \mathcal{T}$*, then* $\tau_A^{-1} M \in \mathcal{T}$*.*

(c) *If* $N \in \mathcal{F}$*, then* $\tau_A N \in \mathcal{F}$*.*

1.2 Cluster categories and cluster-tilted algebras

Let $A = kQ$ and let $\mathcal{D}^b(\operatorname{mod} A)$ denote the derived category of bounded complexes of A-modules as summarized in [14]. The *cluster category* C_A is defined as the orbit category of the derived category with respect to the functor $\tau_{\mathcal{D}}^{-1}[1]$, where $\tau_{\mathcal{D}}$ is the Auslander-Reiten translation in the derived category and [1] is the shift. Cluster categories were introduced in [14], and in [18] for type $\mathbb{A}$, and were further studied in [2, 21, 22, 23]. They are triangulated categories [21], that are 2-Calabi Yau and have Serre duality [14].

An object T in C_A is called *cluster-tilting* if $\operatorname{Ext}^1_{C_A}(T, T) = 0$ and T has $|Q_0|$ non-isomorphic indecomposable direct summands. The endomorphism algebra $\operatorname{End}_{C_A} T$ of a cluster-tilting object is called a *cluster-tilted algebra* [15].

The following theorem was shown in [22]. It characterizes the homological dimensions of a cluster-tilted algebra.

Theorem 1.2.1. [22]. *Cluster-tilted algebras are 1-Gorenstein, that is, every projective module has injective dimension at most 1 and every injective module has projective dimension at most 1.*

As an important consequence, the projective dimension and the injective dimension of any module in a cluster-tilted algebra are simultaneously either infinite, or less than or equal to 1 (see [22, Section 2.1]).

1.3 Relation Extensions

Let C be an algebra of global dimension at most 2 and let E be the C-C-bimodule $E = \mathrm{Ext}^2_C(DC, C)$.

Definition 1.3.1. The *relation extension* of C is the trivial extension algebra $B = C \ltimes E$, whose underlying C-module structure is $C \oplus E$, and multiplication is given by $(c, e)(c', e') = (cc', ce' + ec')$.

Relation extensions were introduced in [3]. In the special case where C is a tilted algebra, we have the following result.

Theorem 1.3.2. [3]. *Let C be a tilted algebra. Then $B = C \ltimes \mathrm{Ext}^2_C(DC, C)$ is a cluster-titled algebra. Moreover all cluster-tilted algebras are of this form.*

1.4 Induction and coinduction functors

A fruitful way to study cluster-tilted algebras is via induction and coinduction functors. Recall, D denotes the standard duality functor.

Definition 1.4.1. Let C be a subalgebra of B such that $1_C = 1_B$, then

$$- \otimes_C B : \mathrm{mod}\, C \to \mathrm{mod}\, B$$

is called the *induction functor*, and dually

$$D(B \otimes_C D-) : \operatorname{mod} C \to \operatorname{mod} B$$

is called the *coinduction functor*. Moreover, given $M \in \operatorname{mod} C$, the corresponding induced module is defined to be $M \otimes_C B$, and the coinduced module is defined to be $D(B \otimes_C DM)$.

We can say more in the situation when B is a split extension of C. Call a C-C-bimodule E *nilpotent* if, for $n \geq 0$, $E \otimes_C E \otimes_C \cdots \otimes_C E = 0$, where the tensor product is performed n times.

Definition 1.4.2. Let B and C be two algebras. We say B is a *split extension* of C by a nilpotent bimodule E if there exists a short exact sequence of B-modules

$$0 \to E \to B \underset{\sigma}{\overset{\pi}{\rightleftarrows}} C \to 0$$

where π and σ are algebra morphisms, such that $\pi \circ \sigma = 1_C$, and $E = \ker \pi$ is nilpotent.

In particular, relation extensions are split extensions. The next proposition shows a precise relationship between a given C-module and its image under the induction and coinduction functors.

Proposition 1.4.3. [25, Proposition 3.6]. *Suppose B is a split extension of C by a nilpotent bimodule E. Then, for every $M \in \operatorname{mod} C$, there exists two short exact sequences of B-modules:*

(a) $0 \to M \otimes_C E \to M \otimes_C B \to M \to 0$

(b) $0 \to M \to D(B \otimes_C DM) \to D(E \otimes_C DM) \to 0$

The next two results give information on the projective cover and the minimal projective presentation of an induced module.

Lemma 1.4.4. [7, Lemma 1.3]. *Suppose B is a split extension of C by a nilpotent bimodule E. Let M be a C-module. If $f : P \to M$ is a projective cover in* $\operatorname{mod} C$*, then $f \otimes_C 1_B : P \otimes_C B \to M \otimes_C B$ is a projective cover in* $\operatorname{mod} B$.

Lemma 1.4.5. [7]. *Suppose B is a split extension of C by a nilpotent bimodule E. Let M be a C-module. If $P_1 \to P_0 \to M \to 0$ is a projective presentation, then $P_1 \otimes_C B \to P_0 \otimes_C B \to M \otimes_C B \to 0$ is a projective presentation. Furthermore, if the first is minimal, then so is the second.*

The following is a crucial result needed in chapter 2.

Lemma 1.4.6. [7, Lemma 2.2]. *For a C-module M, we have* $\operatorname{pd}_B(M \otimes_C B) \leq 1$ *if and only if* $\operatorname{pd}_C M \leq 1$ *and* $\operatorname{Hom}_C(DE, \tau_C M) = 0$.

1.5 Standard results

In this section we list several standard results which hold over arbitrary k-algebras of finite dimension. We begin with a result on the projective dimension of arbitrary modules related by a short exact sequence.

Lemma 1.5.1. [8, Appendix, Proposition 4.7]. *Let A be a finite dimensional k-algebra and suppose* $0 \to L \to M \to N \to 0$ *is a short exact sequence in* $\operatorname{mod} A$.

(a) $\operatorname{pd}_A N \leq \max(\operatorname{pd}_A M, 1 + \operatorname{pd}_A L)$*, and equality holds if* $\operatorname{pd}_A M \neq \operatorname{pd}_A L$.

(b) $\operatorname{pd}_A L \leq \max(\operatorname{pd}_A M, -1 + \operatorname{pd}_A N)$*, and equality holds if* $\operatorname{pd}_A M \neq \operatorname{pd}_A N$.

(c) $\operatorname{pd}_A M \leq \max(\operatorname{pd}_A L, \operatorname{pd}_A N)$*, and equality holds if* $\operatorname{pd}_A N \neq 1 + \operatorname{pd}_A L$.

The next result, which relates the Ext and Tor functors, will be needed in Chapter 2.

Proposition 1.5.2. [8, Appendix, Proposition 4.11] *Let A be a finite dimensional k-algebra. For all modules Y and Z in* $\operatorname{mod} A$*, we have*

$$D\operatorname{Ext}^1_A(Y, DZ) \cong \operatorname{Tor}^A_1(Y, Z).$$

The following proposition gives information about certain compositions of morphisms.

Proposition 1.5.3. [8, IV, Exercise 7.1] *Let A be a finite dimensional k-algebra. Suppose that $f\colon M \to N$ is a morphism in* $\operatorname{mod} A$*. The following are equivalent:*

(a) *For every surjective morphism $h\colon L \to N$, there exists $g\colon M \to L$ such that $f = h \circ g$.*

(b) *For every surjective morphism $h\colon L \to N$ with L projective there exists $g\colon M \to L$ such that $f = h \circ g$.*

(c) *f factors through a projective A-module.*

For our next two statements we need two definitions. We say a a submodule S of a module M is *superfluous* if, whenever $L \subseteq M$ is a submodule with $L + S = M$, then $L = M$. An epimorphism $f\colon M \to N$ is *minimal* if $\ker f$ is superfluous in M. In particular, any projective cover is minimal.

Lemma 1.5.4. [8, I, Lemma 5.6] *Let A be a finite dimensional k-algebra and M an A-module. Then an epimorphism $f : P \to M$ is minimal if and only if for any morphism $g : N \to P$, the surjectivity of $f \circ g$ implies the surjectivity of g.*

Corollary 1.5.5. *If $g\colon M \to N$ and $f\colon N \to L$ are epimorphisms and f and g are minimal, then $f \circ g$ is minimal.*

Proof. Cleary, $f \circ g$ is surjective. Thus, we must show that $\ker f \circ g$ is superfluous. Let $h\colon X \to M$ be a morphism such that $f \circ g \circ h$ is surjective. Since $f \circ g \circ h = f \circ (g \circ h)$ and f is minimal, we know by Lemma 1.5.4 that $g \circ h$ is surjective. Since g is minimal, we may use Lemma 1.5.4 again to say h is surjective. Thus, $f \circ g \circ h$ is surjective and a final application of Lemma 1.5.4 says that $f \circ g \circ h$ is minimal. □

1.6 Auslander-Reiten Theory

The Auslander-Reiten formulas are among the most powerful tools in the representation theory of finite-dimensional algebras. Since we will be using these formulas throughout this paper, we present them here for convenient reference. We will also state a corollary and a useful lemma involving the Auslander-Reiten translations.

Assume A is a finite-dimensional algebra. Let $\nu = D\mathrm{Hom}_A(-, A)$ be the *Nakayama functor* and $\nu^{-1} = \mathrm{Hom}_A(DA, -)$ be the *inverse Nakayama functor*. Given an A-module M, we define its *Auslander−Reiten translate* $\tau_A M$ as follows. Let

$$P_1 \xrightarrow{g_1} P_0 \xrightarrow{g_0} M \to 0$$

be a minimal projective presentation of M. Applying ν to this sequence one obtains $\tau_A M$ as the kernel of νg_1. In particular, there exists an exact sequence

$$0 \to \tau_A M \to \nu P_1 \xrightarrow{\nu g_1} \nu P_0$$

which is an injective presentation of $\tau_A M$. Dually, let

$$0 \to M \xrightarrow{f_0} I_0 \xrightarrow{f_1} I_1$$

be a minimal injective presentation of M. Applying ν^{-1} to this sequence one obtains the *inverse Auslander−Reiten translate* $\tau_A^{-1}M$ as the cokernel of $\nu^{-1}f_1$. In particular, there exists an exact sequence

$$\nu^{-1}I_0 \xrightarrow{\nu^{-1}f_1} \nu^{-1}I_1 \longrightarrow \tau_A^{-1}M \longrightarrow 0$$

which is a projective presentation of $\tau_A^{-1}M$. For two A-modules M and N, let $\mathcal{P}(M,N)$ (respectively $\mathcal{I}(M,N)$) denote the subset of $\mathrm{Hom}_A(M,N)$ consisting of all morphisms that factor through a projective (respectively injective) A-module. We define $\underline{\mathrm{Hom}}_A(M,N)$ to be the quotient space

$$\underline{\mathrm{Hom}}_A(M,N) = \mathrm{Hom}_A(M,N)/\mathcal{P}(M,N)$$

of $\mathrm{Hom}_A(M,N)$. Dually, we define $\overline{\mathrm{Hom}}_A(M,N)$ to be the quotient space

$$\overline{\mathrm{Hom}}_A(M,N) = \mathrm{Hom}_A(M,N)/\mathcal{I}(M,N)$$

of $\mathrm{Hom}_A(M,N)$.

With the preceding notation, we may now state the Auslander-Reiten Formulas.

Theorem 1.6.1 (The Auslander-Reiten Formulas)**.** *Let A be a finite-dimensional algebra and M, N be two A-modules. Then there exist isomorphisms*

$$\mathrm{Ext}_A^1(M,N) \cong D\underline{\mathrm{Hom}}_A(\tau^{-1}N, M) \cong D\overline{\mathrm{Hom}}_A(N, \tau M).$$

Corollary 1.6.2. *Let A be a finite-dimensional algebra and M, N be two A-modules.*

(a) *if* $\mathrm{pd}_A\, M \leq 1$ *and N is arbitrary, then there exists an isomorphism*

$$\mathrm{Ext}_A^1(M,N) \cong D\mathrm{Hom}_A(N, \tau M).$$

(b) *if* $\mathrm{id}_A N \leq 1$ *and* M *is arbitrary, then there exists an isomorphism*

$$\mathrm{Ext}^1_A(M, N) \cong D\mathrm{Hom}_A(\tau^{-1}N, M).$$

The next lemma provides a useful criterion for a module to have projective, or injective, dimension at most 1.

Lemma 1.6.3. [8, IV, Lemma 2.7]. *Let* M *be a module in* $\mathrm{mod}\, A$.

(a) $\mathrm{pd}_A M \leq 1$ *if and only if* $\mathrm{Hom}_A(DA, \tau_A M) = 0$.

(b) $\mathrm{id}_A M \leq 1$ *if and only if* $\mathrm{Hom}_A(\tau_A^{-1} M, A) = 0$.

1.7 Induced and coinduced modules in cluster-tilted algebras

In this section we cite several properties of the induction and coinduction functors particularly when C is an algebra of global dimension at most 2 and $B = C \ltimes E$ is the trivial extension of C by the C-C-bimodule $E = \mathrm{Ext}^2_C(DC, C)$. In the specific case when C is also a tilted algebra, B is the corresponding cluster-tilted algebra.

Proposition 1.7.1. [25, Proposition 4.1]. *Let* C *be an algebra of global dimension at most 2. Then*

(a) $E \cong \tau_C^{-1} \Omega_C^{-1} C$.

(b) $DE \cong \tau_C \Omega_C DC$.

(c) $M \otimes_C E \cong \tau_C^{-1} \Omega_C^{-1} M$.

(d) $D(E \otimes_C DM) \cong \tau_C \Omega_C M$.

The next two results use homological dimensions to extract information about induced and coinduced modules.

Proposition 1.7.2. [25, Proposition 4.2]. *Let C be an algebra of global dimension at most 2, and let $B = C \ltimes E$. Suppose $M \in \operatorname{mod} C$, then*

(a) $\operatorname{id}_C M \leq 1$ *if and only if* $M \otimes_C B \cong M$.

(b) $\operatorname{pd}_C M \leq 1$ *if and only if* $D(B \otimes_C DM) \cong M$.

Lemma 1.7.3. [25, Lemma 4.4]. *Let C be an algebra of global dimension 2 and M a C-module.*

(a) $\operatorname{pd}_C N = 2$ *for all nonzero* $N \in \operatorname{add}(M \otimes_C E)$.

(b) $\operatorname{id}_C N = 2$ *for all nonzero* $N \in \operatorname{add}(D(E \otimes_C DM))$.

We end this section with a lemma which tells us what the projective cover of a projective C-module is in $\operatorname{mod} B$.

Lemma 1.7.4. [3, Lemma 2.7] *Let C be an algebra of global dimension at most 2 and $B = C \ltimes E$. Suppose P is a projective C-module. Then the induced module, $P \otimes_C B$, is a projective cover of P in* $\operatorname{mod} B$.

1.8 τ-rigid modules

Following [1] we state the following definition.

Definition 1.8.1. A Λ-module M is τ_Λ-*rigid* if $\operatorname{Hom}_\Lambda(M, \tau_\Lambda M) = 0$. A τ_Λ-rigid module M is τ_Λ-*tilting* if the number of pairwise, non-isomorphic, indecomposable summands of M equals the number of isomorphism classes of simple Λ-modules.

It follows from the Auslander-Reiten formulas that any τ_Λ-rigid module is rigid and the converse holds if the projective dimension is at most 1. In particular, any partial tilting module is a τ_Λ-rigid module, and any tilting module is a τ_Λ-tilting module. Thus, we can regard τ_Λ-tilting theory as a generalization of classic tilting theory.

The following theorem provides a characterization of τ_Λ-rigid modules.

Proposition 1.8.2. [10, Proposition 5.8]. *For X and Y in $\operatorname{mod}\Lambda$, we have $\operatorname{Hom}_\Lambda(X, \tau_\Lambda Y) = 0$ if and only if $\operatorname{Ext}^1_\Lambda(Y, \operatorname{Gen} X) = 0$.*

The following observations are useful.

Proposition 1.8.3. [1, Proposition 2.4]. *Let X be in $\operatorname{mod}\Lambda$ with a minimal projective presentation $P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} X \to 0$.*

(a) *For Y in $\operatorname{mod}\Lambda$, we have an exact sequence*

$$0 \to \operatorname{Hom}_\Lambda(Y, \tau_\Lambda X) \to D\operatorname{Hom}_\Lambda(P_1, Y) \xrightarrow{D(d_1,Y)} D\operatorname{Hom}_\Lambda(P_0, Y) \xrightarrow{D(d_0,Y)} D\operatorname{Hom}_\Lambda(X, Y) \to 0.$$

(b) *$\operatorname{Hom}_\Lambda(Y, \tau_\Lambda X) = 0$ if and only if the morphism $\operatorname{Hom}_\Lambda(P_0, Y) \xrightarrow{(d_1,Y)} \operatorname{Hom}_\Lambda(P_1, Y)$ is surjective.*

(c) *X is τ_Λ-rigid if and only if the morphism $\operatorname{Hom}_\Lambda(P_0, X) \xrightarrow{(d_1,X)} \operatorname{Hom}_\Lambda(P_1, X)$ is surjective.*

The following lemma is very useful in applications. We need several preliminary definitions. Let U be a Λ-module. We define

$${}^\perp(\tau_\Lambda U) = \{X \in \operatorname{mod}\Lambda \mid \operatorname{Hom}_\Lambda(X, \tau_\Lambda U) = 0\}.$$

Also, we say a module $X \in \operatorname{Gen} U$ is *Ext-projective* if $\operatorname{Ext}^1_\Lambda(X, \operatorname{Gen} U) = 0$. We denote by $P(\operatorname{Gen} U)$ the direct sum of one copy of each indecomposable Ext-projective module in $\operatorname{Gen} U$ up to isomorphism. Finally, we say a morphism $f\colon A \to B$ is a *left* Gen M*-approximation* if $B \in \operatorname{Gen} M$ and, whenever $g\colon A \to C$ is a morphism with $X \in \operatorname{Gen} M$, there is some $h\colon B \to X$ such that $h \circ f = g$.

Lemma 1.8.4. [1, Lemma 2.20]. *Let T be a τ_Λ-rigid module. If U is a τ_Λ-rigid module satisfying ${}^\perp(\tau_\Lambda T) \subseteq {}^\perp(\tau_\Lambda U)$, then there is an exact sequence*

$$U \xrightarrow{f} T' \to C \to 0$$

satisfying the following conditions.

- *f is a minimal left* Gen T*-approximation of U.*
- *T' is in* add T*, C is in* add $P(\operatorname{Gen} T)$*, and* $\operatorname{add} T' \cap \operatorname{add} C = 0$.

We will also need the following special case of Lemma 1.8.4.

Lemma 1.8.5. [1, Proposition 2.23]. *Let T be a τ_Λ-tilting module. Assume that U is a τ_Λ-rigid module such that* $\operatorname{Gen} T \subseteq {}^\perp(\tau_\Lambda U)$*. Then there exists an exact sequence*

$$U \xrightarrow{f} T^0 \to T^1 \to 0$$

such that

- *f is a minimal left* Gen T*-approximation of U.*
- *T^0 and T^1 are in* add T *and satisfy* $\operatorname{add} T^0 \cap \operatorname{add} T^1 = 0$.

We now return to the situation where the algebra B is a split extension of the algebra C by a nilpotent bimodule E. The induction functor can be used to derive information about the Auslander-Reiten translation of a C-module M inside the module category of B. The next theorem tells us exactly when the Auslander-Reiten translation remains the same, i.e., $\tau_C M \cong \tau_B M$ as B-modules.

Theorem 1.8.6. [9, Theorem 2.1]. *Let M be an indecomposable non-projective C-module. The following are equivalent:*

(a) *The almost split sequences ending with M in* $\operatorname{mod} C$ *and* $\operatorname{mod} B$ *coincide.*

(b) $\tau_C M \cong \tau_B M$.

(c) $\operatorname{Hom}_C(E, \tau_C M) = 0$ *and* $M \otimes_C E = 0$.

Having information about the Auslander-Reiten translation of an induced module is very useful.

Lemma 1.8.7. [7, Lemma 2.1]. *Let M be a C-module. Then*

$$\tau_B(M \otimes_C B) \cong \operatorname{Hom}_C({}_B B_C, \tau_C M) \cong \tau_C M \oplus \operatorname{Hom}_C(E, \tau_C M)$$

where the isomorphisms are isomorphisms of C-modules.

Deducing information about $\tau_B M$ is generally more difficult but we have an answer in the following special case.

Lemma 1.8.8. [9, Corollary 1.3]. *Assume $M \otimes_C E = 0$, then we have*

$$\tau_B M \cong \tau_C M \oplus \operatorname{Hom}_C(E, \tau_C M)$$

where the isomorphism is an isomorphism of C-modules.

We also have the following important fact which will be used extensively in subsequent chapters.

Lemma 1.8.9. [9, Corollary 1.2]. *$\tau_C M$ and $\tau_B(M \otimes_C B)$ are submodules of $\tau_B M$.*

Chapter 2

Homological Dimensions

In this section let C be an algebra of global dimension 2, $E = \mathrm{Ext}^2_C(DC, C)$, and $B = C \ltimes E$ be the relation extension. We investigate what happens to the projective dimension of a C-module M when viewed as a B-module. In the special case when C is a tilted algebra and B is the corresponding cluster-tilted algebra, we provide a complete classification. First, we prove a lemma which provides a useful criteria for a C-module to have projective or injective dimension at most 1 in an algebra of global dimension 2.

2.1 Projective dimension 0 and 2

Lemma 2.1.1. *Let M be a C-module. Then,*

(a) $\mathrm{pd}_C\, M \leq 1$ *if and only if* $\mathrm{Hom}_C(\tau_C^{-1}\Omega_C^{-1}C, M) = 0$.

(b) $\mathrm{id}_C\, M \leq 1$ *if and only if* $\mathrm{Hom}_C(M, \tau_C\Omega_C DC) = 0$.

Proof. We prove (a) with the proof of (b) being similar. Assume $\mathrm{pd}_C\, M \leq 1$. Consider the short exact sequence

$$0 \to C \to I_0 \to \Omega_C^{-1} C \to 0$$

where I_0 is an injective envelope of C. Apply $\mathrm{Hom}_C(M, -)$ to obtain an exact sequence

$$\mathrm{Ext}_C^1(M, I_0) \to \mathrm{Ext}_C^1(M, \Omega_C^{-1} C) \to \mathrm{Ext}_C^2(M, C).$$

Now, $\mathrm{Ext}_C^1(M, I_0) = 0$ because I_0 is injective and $\mathrm{Ext}_C^2(M, C) = 0$ because $\mathrm{pd}_C\, M \leq 1$. Since the sequence is exact, $\mathrm{Ext}_C^1(M, \Omega_C^{-1} C) = 0$. By Theorem 1.6.1, $D\mathrm{Hom}_C(\tau_C^{-1} \Omega_C^{-1} C, M) \cong \mathrm{Ext}_C^1(M, \Omega_C^{-1} C)$. Thus, $0 = \mathrm{Ext}_C^1(M, \Omega_C^{-1} C) \cong D\mathrm{Hom}_C(\tau_C^{-1} \Omega_C^{-1} C, M)$.

Conversely, assume $\mathrm{Hom}_C(\tau_C^{-1} \Omega_C^{-1} C, M) = 0$. Then we have

$$D\underline{\mathrm{Hom}}_C(\tau_C^{-1} \Omega_C^{-1} C, M) \cong \mathrm{Ext}_C^1(M, \Omega_C^{-1} C) = 0$$

by Theorem 1.6.1. We then have $\mathrm{Ext}_C^2(M, C) \cong \mathrm{Ext}_C^1(M, \Omega_C^{-1} C) = 0$. Since C has global dimension equal to 2, this implies $\mathrm{pd}_C\, M \leq 1$. □

The following corollary will be used in Chapter 4.

Corollary 2.1.2. *Let M be a C-module such that* $\mathrm{pd}_C\, M \leq 1$. *Then* $\mathrm{Hom}_C(\tau_C^{-1} \Omega_C^{-1} M, M) = 0$.

Proof. Let $f \colon P \to M$ be a projective cover of M in $\mathrm{mod}\, C$. Apply the functor $- \otimes_C E$ to obtain a surjective morphism $f \otimes_C 1_E \colon P \otimes_C E \to M \otimes_C E$. Apply $\mathrm{Hom}_C(-, M)$ to obtain the exact sequence

$$0 \to \mathrm{Hom}_C(M \otimes_C E, M) \xrightarrow{\overline{f \otimes_C 1_E}} \mathrm{Hom}_C(P \otimes_C E, M).$$

Now, Proposition 1.7.1 says $M \otimes_C E \cong \tau_C^{-1} \Omega_C^{-1} M$ and $P \otimes_C E \cong \tau_C^{-1} \Omega_C^{-1} P$. Thus, we have that $\mathrm{Hom}_C(P \otimes_C E, M) = 0$ by Lemma 2.1.1 and we conclude $\mathrm{Hom}_C(\tau_C^{-1} \Omega_C^{-1} M, M) = 0$.

□

We begin with the case where M is a projective C-module.

Proposition 2.1.3. *Let M be a projective C-module. Then $\operatorname{pd}_B M = 0$ if and only if $\operatorname{id}_C M \leq 1$.*

Proof. Assume $\operatorname{pd}_B M = 0$. By Proposition 1.4.3 we have a short exact sequence

$$0 \to \tau_C^{-1}\Omega_C^{-1}M \to M \otimes_C B \to M \to 0$$

where $M \otimes_C B$ is a projective cover by Lemma 1.4.4. This implies $M \otimes_C B \cong M$ and $\tau_C^{-1}\Omega_C^{-1}M = 0$. By Proposition 1.7.2, we conclude $\operatorname{id}_C M \leq 1$.

Conversely, assume $\operatorname{id}_C M \leq 1$. Then Proposition 1.7.2 implies $M \otimes_C B \cong M$ and we conclude M is a projective B-module. □

The case where the projective dimension of M is equal to 2 holds in a more general setting which we explicitly state.

Proposition 2.1.4. *Let C be an algebra of global dimension 2 with B a split extension by a nilpotent bimodule E. If M is a C-module with $\operatorname{pd}_C M = 2$, then $\operatorname{pd}_B M \geq 2$.*

Proof. By Lemma 1.4.6, we have $\operatorname{pd}_B(M \otimes_C B) \geq 2$. This implies the existence of a non-zero morphism $f : DB \to \tau_B(M \otimes_C B)$ by Lemma 1.6.3. By Lemma 1.8.9, we have an injective morphism $i : \tau_B(M \otimes_C B) \to \tau_B M$. Thus, there is a non-zero morphism $i \circ f : DB \to \tau_B M$. By Lemma 1.6.3 again, we have $\operatorname{pd}_B M \geq 1$. □

2.2 Projective dimension 1

The case where the projective dimension of M is equal to 1 is the most restrictive.

Proposition 2.2.1. *Let M be a C-module with $\operatorname{pd}_C M = 1$ and a minimal projective resolution $0 \to P_1 \to P_0 \to M \to 0$ in $\operatorname{mod} C$. Then $\operatorname{id}_C M \leq 1$ and $\tau_C^{-1}\Omega^{-1}P_1 \cong \tau_C^{-1}\Omega^{-1}P_0$ if and only if $\operatorname{pd}_B M = 1$.*

Proof. Assume $\operatorname{id}_C M \leq 1$ and $\tau_C^{-1}\Omega_C^{-1}P_1 \cong \tau_C^{-1}\Omega_C^{-1}P_0$. Since $\operatorname{id}_C M \leq 1$, by Proposition 1.7.2, we have $M \otimes_C E \cong \tau_C^{-1}\Omega_C^{-1}M = 0$ and $M \otimes_C B \cong M$. Using Lemma 1.4.6, we need to to show $\operatorname{Hom}_C(DE, \tau_C M) = 0$. Apply $- \otimes_C E$ to the minimal projective resolution of M to obtain the exact sequence

$$\operatorname{Tor}_1^C(P_1, E) \to \operatorname{Tor}_1^C(M, E) \to P_1 \otimes_C E \to P_0 \otimes_C E \to M \otimes_C E \to 0. \tag{1}$$

Now, $\operatorname{Tor}_1^C(P_1, E) = 0$ because P_1 is projective and we showed $M \otimes_C E = 0$. Also, Proposition 1.7.2 says $P_1 \otimes_C E \cong \tau_C^{-1}\Omega_C^{-1}P_1 \cong \tau_C^{-1}\Omega_C^{-1}P_0 \cong P_0 \otimes_C E$. Since (1) is exact, we know $\operatorname{Tor}_1^C(M, E) = 0$. By Proposition 1.5.2 and Theorem 1.6.1, we have

$$0 = \operatorname{Tor}_1^C(M, E) \cong D\operatorname{Ext}_C^1(M, DE) \cong \overline{\operatorname{Hom}}_C(DE, \tau_C M).$$

Since $\operatorname{pd}_C M = 1$ by assumption, we may use Corollary 1.6.2 to say

$$0 = \overline{\operatorname{Hom}}_C(DE, \tau_C M) \cong \operatorname{Hom}_C(DE, \tau_C M).$$

Conversely, assume $\operatorname{pd}_B M = 1$. If $\operatorname{pd}_B(M \otimes_C B) > 1$ then we have a non-zero composition of morphisms, $DB \to \tau_B(M \otimes_C B) \to \tau_B M$, guaranteed by Lemma 1.6.3 and Lemma 1.8.9 . By Lemma 1.6.3, this contradicts $\operatorname{pd}_B M = 1$. Thus, $\operatorname{pd}_B(M \otimes_C B) = 1$ and Proposition 1.5.2, Lemma 1.4.6, and Corollary 1.6.2 imply

$$0 = \operatorname{Hom}_C(DE, \tau_C M) \cong D\operatorname{Ext}_C^1(M, DE) \cong \operatorname{Tor}_1^C(M, E).$$

Next, consider the short exact sequence of Propositions 1.4.3 and Proposition 1.7.1

$$0 \to \tau_C^{-1}\Omega_C^{-1}M \to M \otimes_C B \to M \to 0$$

in $\operatorname{mod} B$. Since $\operatorname{pd}_B(M \otimes_C B)$ and $\operatorname{pd}_B M$ are equal to 1, Lemma 1.5.1 implies $\operatorname{pd}_B(\tau_C^{-1}\Omega_C^{-1}M) \leq 1$. By Lemma 1.7.3, we know $\operatorname{pd}_C(\tau_C^{-1}\Omega_C^{-1}M) = 2$ or $\tau_C^{-1}\Omega_C^{-1}M = 0$. However, Proposition 2.1.4 implies $\operatorname{pd}_B(\tau_C^{-1}\Omega_C^{-1}M) \geq 2$. Thus, $\tau_C^{-1}\Omega_C^{-1}M = 0$ and $M \otimes_C B \cong M$. Returning to sequence (1), since $M \otimes_C B \cong M$ we have $M \otimes_C E = 0$. Also, we have shown that $\operatorname{Tor}_1^C(M, E) = 0$. Since the sequence is exact, we have $P_1 \otimes_C E \cong P_0 \otimes_C E$ and Proposition 1.7.1 implies $\tau_C^{-1}\Omega_C^{-1}P_1 \cong \tau_C^{-1}\Omega_C^{-1}P_0$. Finally, since $M \otimes_C E = 0$, Proposition 1.7.2 tells us that $\operatorname{id}_C M \leq 1$. □

If M is a C-module which satisfies the conditions of Proposition 2.2.1, then the following corollary tells us what a minimum projective presentation is in $\operatorname{mod} B$.

Corollary 2.2.2. *Let M be a C-module with minimal projective resolution*

$$0 \to P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0.$$

If $\operatorname{pd}_B M = 1$*, then* $0 \to P_1 \otimes_C B \xrightarrow{f_1 \otimes_C 1_B} P_0 \otimes_C B \xrightarrow{f_0 \otimes_C 1_B} M \to 0$ *is a minimal projective resolution in* $\operatorname{mod} B$*.*

Proof. By Lemma 1.4.5, we know that $P_1 \otimes_C B \xrightarrow{f_1 \otimes_C 1_B} P_0 \otimes_C B \xrightarrow{f_0 \otimes_C 1_B} M \otimes_C B \to 0$ is a minimal projective presentation of $M \otimes_C B$ in $\operatorname{mod} B$. By Proposition 2.2.1, we know $\operatorname{id}_C M \leq 1$. By Proposition 1.7.2 we have $M \otimes_C B \cong M$ and our statement follows. □

In the situation where C is an algebra of global dimension 2 and B is a split extension by a nilpotent bimodule E, we prove that the global dimension of B is strictly greater then the global dimension of C. We need a lemma.

Lemma 2.2.3. *Let M be a projective C-module such that* $\operatorname{id}_C M = 2$. *Then*

$$\operatorname{pd}_B M = \operatorname{pd}_B(\tau_C^{-1}\Omega_C^{-1} M) + 1 \geq 3.$$

Proof. Consider the short exact sequence $0 \to \tau_C^{-1}\Omega_C^{-1} M \to M \otimes_C B \to M \to 0$ guaranteed by Proposition 1.4.3 and Proposition 1.7.1. We have $\operatorname{pd}_B(M \otimes_C B) = 0$ and $\operatorname{pd}_B(\tau_C^{-1}\Omega_C^{-1} M) \geq 2$ by Proposition 2.1.4. Our statement then follows from Lemma 1.5.1. □

Corollary 2.2.4. *Let C be an algebra of global dimension 2 and B a split extension by a nilpotent bimodule E. Then* $\operatorname{gl.dim.} B > \operatorname{gl.dim.} C$.

Proof. This follows immediately from Lemma 2.2.3. □

We conclude this section with a complete classification of the projective dimension of a C-module when viewed as a B-module in the special case C is tilted and B is the corresponding cluster-titled algebra.

Theorem 2.2.5. *Let C be a tilted algebra, $E = \operatorname{Ext}_C^2(DC, C)$, and $B = C \ltimes E$ the corresponding cluster-tilted algebra.*

(a) *If* $\operatorname{pd}_C M = 0$, *then* $\operatorname{pd}_B M = 0$ *if and only if* $\operatorname{id}_C M \leq 1$. *Otherwise,* $\operatorname{pd}_B M = \infty$.

(b) *If* $\operatorname{pd}_C M = 2$, *then* $\operatorname{pd}_B M = \infty$.

(c) *Let* $\operatorname{pd}_C M = 1$ *with minimal projective resolution* $0 \to P_1 \to P_0 \to M \to 0$. *Then* $\operatorname{pd}_B M = 1$ *if and only if* $\operatorname{id}_C M \leq 1$ *and* $\tau_C^{-1}\Omega_C^{-1} P_0 \cong \tau_C^{-1}\Omega_C^{-1} P_1$. *Otherwise,* $\operatorname{pd}_B M = \infty$.

Proof. Part (a) follows from Proposition 2.1.3. If the conditions for M are not met, then Lemma 2.2.3 and the 1-Gorenstein property of a cluster-tilted algebra, (Theorem 1.2.1), shows $\operatorname{pd}_B M = \infty$. Part (b) follows from Proposition 2.1.4 and the 1-Gorenstein property. Finally, part (c) follows from Proposition 2.2.1 and the 1-Gorenstein property. □

For an illustration of this theorem, see Examples 5.1.1, 5.1.2, and 5.1.3 in chapter 5.

Chapter 3

Extensions

In this section, we study C-modules which have no self-extension, i.e., $\mathrm{Ext}^1_C(M, M) = 0$. These modules are typically referred to as rigid modules. We investigate under what conditions does a rigid C-module remain a rigid B-module. Unless otherwise stated, we assume that C is an algebra of global dimension 2 and $B = C \ltimes E$ is a split extension by a nilpotent bimodule E.

3.1 Main result

To prove our main result we first need an easy lemma. We recall from Lemma 1.7.4 that if P is a projective C-module, then $P \otimes_C B$ is a projective cover of P in $\operatorname{mod} B$.

Lemma 3.1.1. *Let M be a C-module with $f : P_0 \to M$ a projective cover in* $\operatorname{mod} C$. *Suppose $g\colon P_0 \otimes_C B \to P_0$ is a projective cover of P_0 in* $\operatorname{mod} B$. *Then $f \circ g\colon P_0 \otimes_C B \to M$ is a projective cover of M in* $\operatorname{mod} B$.

Proof. Clearly, $f \circ g$ is surjective. Thus, we need to show $\ker f \circ g$ is superfluous. This follows easily from Corollary 1.5.5 since f and g are both minimal. □

Theorem 3.1.2. *Let M be a rigid C-module with a projective cover $P_0 \to M$ and an injective envelope $M \to I_0$ in* $\operatorname{mod} C$.

(a) *If* $\operatorname{Hom}_C(\tau_C^{-1}\Omega^{-1}P_0, M) = 0$*, then M is a rigid B-module.*

(b) *If* $\operatorname{Hom}_C(M, \tau_C\Omega I_0) = 0$*, then M is a rigid B-module.*

Proof. We prove case (a) with case (b) being dual. In $\operatorname{mod} B$, consider the following short exact sequence of M

$$0 \to \Omega_B^1 M \xrightarrow{f} P_0 \otimes_C B \to M \to 0.$$

Apply $\operatorname{Hom}_B(-, M)$ to obtain

$$0 \to \operatorname{Hom}_B(M, M) \to \operatorname{Hom}_B(P_0 \otimes_C B, M) \xrightarrow{\overline{f}} \operatorname{Hom}_B(\Omega_B^1 M, M) \to \operatorname{Ext}_B^1(M, M) \to 0. \qquad (1)$$

Since (1) is exact, we need to show that $\overline{f}$ is surjective. This will imply that $\operatorname{Ext}_B^1(M, M) = 0$. In $\operatorname{mod} C$, consider the sequence

$$0 \to \Omega_C^1 M \xrightarrow{g} P_0 \xrightarrow{a} M \to 0.$$

Apply $\operatorname{Hom}_C(-, M)$ to obtain

$$0 \to \operatorname{Hom}_C(M, M) \to \operatorname{Hom}_C(P_0, M) \xrightarrow{\overline{g}} \operatorname{Hom}_C(\Omega_C^1 M, M) \to \operatorname{Ext}_C^1(M, M). \qquad (2)$$

Since M is a rigid C-module by assumption and (2) is exact, we have $\overline{g}$ is surjective. Next, in $\operatorname{mod} B$, consider the following commutative diagram guaranteed by Lemma 3.1.1 and the universal property of the kernel.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^1_B M & \xrightarrow{f} & P_0 \otimes_C B & \xrightarrow{a \circ w} & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle z} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & \Omega^1_C M & \xrightarrow{g} & P_0 & \xrightarrow{a} & M & \longrightarrow & 0. \end{array} \tag{3}$$

Here, id is the identity map, w is a projective cover of P_0, and z is induced by the universal property of the kernel. By the Snake Lemma, we know $\ker z \cong \ker w$. Thus, Proposition 1.4.3 and Proposition 1.7.1 implies that $\ker z \cong \ker w \cong \tau_C^{-1}\Omega_C^{-1}P_0$. Thus, we have an exact sequence

$$0 \to \tau_C^{-1}\Omega_C^{-1}P_0 \xrightarrow{i} \Omega^1_B M \xrightarrow{z} \Omega^1_C M \to \operatorname{coker} z \to 0.$$

Since the morphism w is surjective and id is clearly injective, we may use the Snake Lemma again to say that $\operatorname{coker} z = 0$. Apply $\operatorname{Hom}_B(-, M)$ to obtain an exact sequence

$$0 \to \operatorname{Hom}_B(\Omega^1_C M, M) \xrightarrow{\bar{z}} \operatorname{Hom}_B(\Omega^1_B M, M) \to \operatorname{Hom}_B(\tau_C^{-1}\Omega_C^{-1}P_0, M). \tag{3.1}$$

Since $\operatorname{Hom}_B(\tau_C^{-1}\Omega_C^{-1}P_0, M) = 0$ by assumption, we have that $\bar{z}$ is an isomorphism. To show $\overline{f}$ is surjective, let $h \in \operatorname{Hom}_B(\Omega^1_B M, M)$. Since $\bar{z}$ is an isomorphism, we know there exists a morphism $j \in \operatorname{Hom}_B(\Omega^1_C M, M)$ such that $h = j \circ z$.

$$\begin{array}{ccc} \Omega^1_B M & & \\ \downarrow{\scriptstyle z} & \searrow^{h = j \circ z} & \\ \Omega^1_C M & \xrightarrow{j} & M \end{array}$$

Since $\overline{g}$ is surjective, there exists a morphism $l \in \mathrm{Hom}_B(P_0, M)$ such that $j = l \circ g$.

$$\begin{array}{ccc} \Omega^1_C M & \xrightarrow{j=l\circ g} & M \\ \downarrow{\scriptstyle g} & \nearrow{\scriptstyle l} & \\ P_0 & & \end{array}$$

Thus, we have $h = l \circ g \circ z$.

$$\begin{array}{ccc} \Omega^1_B M & & \\ \downarrow{\scriptstyle z} & \searrow{\scriptstyle h=l\circ g\circ z} & \\ \Omega^1_C & & \\ \downarrow{\scriptstyle g} & & \\ P_0 & \xrightarrow{l} & M \end{array}$$

From our commutative diagram (3), we know $g \circ z = w \circ f$. Thus, we have the following commutative diagram.

$$\begin{array}{ccc} \Omega^1_B M & & \\ \downarrow{\scriptstyle f} & \searrow{\scriptstyle h=l\circ w\circ f} & \\ P_0 \otimes_C B & & \\ \downarrow{\scriptstyle w} & & \\ P_0 & \xrightarrow{l} & M \end{array}$$

This gives $h = l \circ w \circ f$ and we conclude that $\overline{f}$ is surjective. □

For an illustration of this theorem, see Examples 5.2.1 and 5.2.2 in chapter 5.

3.2 Corollaries

We now examine several corollaries of our main result. For the first corollary, we say M is a *partial cotilting module* if $\mathrm{id}_C\, M \leq 1$ and $\mathrm{Ext}^1_C(M, M) = 0$ and *cotilting* if the number of pairwise, non-isomorphic, indecomposable summands of M equals the number of isomorphism classes of simple C-modules.

Corollary 3.2.1. *If M is a partial tilting or cotilting C-module, then M is a rigid B-module.*

Proof. We assume M is a partial tilting module. The proof for the case M is a partial cotilting module is dual. Since $\operatorname{pd}_C M \leq 1$, we have that Lemma 2.1.1 implies $\operatorname{Hom}_C(\tau_C^{-1}\Omega_C^{-1}C, M) = 0$. The statement now follows from Theorem 3.1.2. □

The next result holds in the specific case where C is tilted and B is cluster-tilted.

Corollary 3.2.2. *Let C be a tilted algebra with B the corresponding cluster-titled algebra. Suppose M is an indecomposable, rigid C-module. Then M is a rigid B-module.*

Proof. Let M be an indecomposable, rigid C-module. By Proposition 1.1.5(b), we have that $\operatorname{pd}_C M \leq 1$ or $\operatorname{id}_C M \leq 1$. Since M is rigid, we have M is partial tilting or partial cotilting. By Corollary 3.2.1, our statement follows. □

We now state the converse to Theorem 3.1.2. We note that if M is a C-module which is rigid as a B-module, then M is trivially a rigid C-module.

Proposition 3.2.3. *Assume C is an algebra of global dimension 2. Let M be a C-module with a projective cover $g\colon P_0 \to M$ and an injective envelope $h\colon M \to I_0$ in $\operatorname{mod} C$. Suppose M is a rigid B-module.*

(a) *If* $\operatorname{Ext}_B^1(P_0, M) = 0$*, then* $\operatorname{Hom}_C(\tau_C^{-1}\Omega_C^{-1}P_0, M) = 0$.

(b) *If* $\operatorname{Ext}_B^1(M, I_0) = 0$*, then* $\operatorname{Hom}_C(M, \tau_C\Omega_C I_0) = 0$.

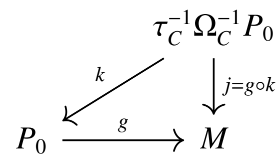
Proof. We prove case (a) with case (b) being dual. Consider the following sequence in $\operatorname{mod} B$ guaranteed by Proposition 1.4.3 and Proposition 1.7.1.

$$0 \to \tau_C^{-1}\Omega_C^{-1}P_0 \xrightarrow{f} P_0 \otimes_C B \to P_0 \to 0.$$

Apply $\operatorname{Hom}_B(-, M)$ to obtain

$$0 \to \operatorname{Hom}_B(P_0, M) \to \operatorname{Hom}_B(P_0 \otimes_C B, M) \xrightarrow{\overline{f}} \operatorname{Hom}_B(\tau_C^{-1}\Omega_C^{-1}P_0, M) \to \operatorname{Ext}_B^1(P_0, M).$$

Since the sequence is exact and $\operatorname{Ext}_B^1(P_0, M) = 0$ by assumption, we have that $\overline{f}$ is surjective. This implies that any morphism of B-modules, $j : \tau_C^{-1}\Omega_C^{-1}P_0 \to M$, factors through the projective B-module $P_0 \otimes_C B$. Thus, we may use Proposition 1.5.3. Since $g \colon P_0 \to M$ is a surjective morphism, Proposition 1.5.3 implies the existence of a morphism $k \colon \tau_C^{-1}\Omega_C^{-1}P_0 \to P_0$ such that $j = g \circ k$.



But $\operatorname{pd}_C P_0 = 0$ and Lemma 2.1.1 implies $\operatorname{Hom}_B(\tau_C^{-1}\Omega_C^{-1}P_0, P_0) = 0$. Thus k must be the 0 morphism. This forces j to also be the 0 morphism. Since j was arbitrary we conclude that $\operatorname{Hom}_B(\tau_C^{-1}\Omega_C^{-1}P_0, M) = 0$ which further implies $\operatorname{Hom}_C(\tau_C^{-1}\Omega_C^{-1}P_0, M) = 0$ by restriction of scalars.

□

Chapter 4

τ-rigid modules

In this section we study modules which are τ-rigid. We investigate under what conditions a τ-rigid C-module remains a τ-rigid B-module. We assume C is an algebra of global dimension 2 and $B = C \ltimes E$ where E=$\mathrm{Ext}^2_C(DC, C)$. Specific cases will be explicitly stated. We begin with determining when the Auslander-Reiten translation of a C-module remains unchanged in $\mathrm{mod}\, C$ and $\mathrm{mod}\, B$, i.e., $\tau_C M \cong \tau_B M$ as B-modules.

4.1 Main Results

Proposition 4.1.1. *Let M be a C-module. Then $\tau_C M \cong \tau_B M$ if and only if* $\mathrm{pd}_C\, \tau_C M \leq 1$ *and* $\mathrm{id}_C\, M \leq 1$

Proof. By Theorem 1.8.6, we know $\tau_C M \cong \tau_B M$ if and only if $\mathrm{Hom}_C(E, \tau_C M) = 0$ and $M \otimes_C E = 0$. Using Lemma 2.1.1, we know that $\mathrm{pd}_C\, \tau_C M \leq 1$ if and only if $\mathrm{Hom}_C(\tau_C^{-1}\Omega_C^{-1}C, \tau_C M) = 0$. Since Proposition 1.7.1 gives $E \cong \tau_C^{-1}\Omega_C^{-1}C$, this is equivalent to $\mathrm{Hom}_C(E, \tau_C M) = 0$. Using Proposition 1.7.2, we have $M \otimes_C E = 0$ if and only if $\mathrm{id}_C\, M \leq 1$. Our result follows. □

The next two results deal with the situations where the assumptions on the previous proposition are relaxed.

Proposition 4.1.2. *Let M be a τ_C-rigid C-module. If* $\mathrm{id}_C M \leq 1$*, then M is τ_B-rigid.*

Proof. Since $\mathrm{id}_C M \leq 1$, Proposition 1.7.2 implies $M \otimes_C E = 0$. By Lemma 1.8.8, we have $\tau_B M \cong \tau_C M \oplus \mathrm{Hom}_C(E, \tau_C M)$ as C-modules. Now, we want to show that $\mathrm{Hom}_B(M, \tau_B M) = 0$. Since any B-module homomorphism is also a C-module homomorphism, it suffices to show that $\mathrm{Hom}_C(M, \tau_C M) = 0$ and $\mathrm{Hom}_C(M, \mathrm{Hom}_C(E, \tau_C M)) = 0$. Using the adjoint isomorphism, we know that $\mathrm{Hom}_C(M, \mathrm{Hom}_C(E, \tau_C M)) \cong \mathrm{Hom}_C(M \otimes_C E, \tau_C M)$. Since $M \otimes_C E = 0$, we conclude $\mathrm{Hom}_C(M, \mathrm{Hom}_C(E, \tau_C M)) = 0$. Certainly, M being τ_C-rigid implies $\mathrm{Hom}_C(M, \tau_C M) = 0$. Thus, we conclude M is τ_B-rigid. □

Proposition 4.1.3. *Let M be a τ_C-rigid C-module. If* $\mathrm{pd}_C \tau_C M \leq 1$*, then the induced module $M \otimes_C B$ is τ_B-rigid.*

Proof. Consider the following short exact sequence guaranteed by Proposition 1.4.3 and Proposition 1.7.1.

$$0 \to \tau_C^{-1} \Omega_C^{-1} M \to M \otimes_C B \to M \to 0.$$

Apply $\mathrm{Hom}_B(-, \tau_B(M \otimes_C B))$ to obtain the exact sequence

$$\mathrm{Hom}_B(M, \tau_B(M \otimes_C B)) \to \mathrm{Hom}_B(M \otimes_C B, \tau_B(M \otimes_C B)) \to \mathrm{Hom}_B(\tau_C^{-1} \Omega_C^{-1} M, \tau_B(M \otimes_C B)). \quad (1)$$

We wish to show that $\mathrm{Hom}_B(M \otimes_C B, \tau_B(M \otimes_C B)) = 0$. Using Lemma 1.8.7, we know that $\tau_B(M \otimes_C B) \cong \tau_C M \oplus \mathrm{Hom}_C(E, \tau_C M)$ as C-modules. Since $\mathrm{pd}_C \tau_C M \leq 1$, Lemma 2.1.1 implies $\mathrm{Hom}_C(E, \tau_C M) = 0$. Thus, $\tau_B(M \otimes_C B) \cong \tau_C M$. Since M is a τ_C-rigid module, we have that $\mathrm{Hom}_B(M, \tau_B(M \otimes_C B)) = 0$.

Next, consider $f: P_0 \to M$, a projective cover of M in $\operatorname{mod} C$. Apply the functor $- \otimes_C E$ to obtain a surjective morphism $f \otimes_C 1_E \colon P_0 \otimes_C E \to M \otimes_C E$. This gives a short exact sequence

$$0 \to \ker f \otimes_C 1_E \to P_0 \otimes_C E \xrightarrow{f \otimes_C 1_E} M \otimes_C E \to 0.$$

Apply $\operatorname{Hom}_C(-, \tau_C M)$ to obtain the exact sequence

$$0 \to \operatorname{Hom}_C(M \otimes_C E, \tau_C M) \xrightarrow{\overline{f \otimes_C 1_E}} \operatorname{Hom}_C(P_0 \otimes_C E, \tau_C M).$$

We know from Proposition 1.7.1 that $P_0 \otimes_C E \cong \tau_C^{-1}\Omega_C^{-1} P_0$ and $M \otimes_C E \cong \tau_C^{-1}\Omega_C^{-1} M$. Thus, any non-zero morphism from $\tau_C^{-1}\Omega_C^{-1} M$ to $\tau_C M$ would imply a non-zero morphism from $\tau_C^{-1}\Omega_C^{-1} P_0$ to $\tau_C M$ because $\overline{f \otimes_C 1_E}$ is injective. Since $\operatorname{pd}_C \tau_C M \leq 1$, this is a contradiction by Lemma 2.1.1. Thus, $\operatorname{Hom}_B(\tau_C^{-1}\Omega_C^{-1} M, \tau_B(M \otimes_C B)) = 0$. Since we have shown that $\operatorname{Hom}_B(M, \tau_B(M \otimes_C B))$ and $\operatorname{Hom}_B(\tau_C^{-1}\Omega_C^{-1}, \tau_B(M \otimes_C B))$ are equal to 0, we conclude $\operatorname{Hom}_B(M \otimes_C B, \tau_B(M \otimes_C B)) = 0$ by sequence (1). □

We now turn our attention to C-modules which are projective. We derive a a necessary and sufficient condition for a projective C-module to be τ_B-rigid.

Proposition 4.1.4. *Let P be a projective C-module with $\overline{P}$ a projective cover of $\tau_C^{-1}\Omega_C^{-1} P$ in* $\operatorname{mod} C$. *Then P is τ_B-rigid if and only if* $\operatorname{Hom}_C(\overline{P}, P) = 0$.

Proof. In $\operatorname{mod} B$, consider the following short exact sequence guaranteed by Proposition 1.4.3 and Proposition 1.7.1

$$0 \to \tau_C^{-1}\Omega_C^{-1} P \xrightarrow{f} P \otimes_C B \xrightarrow{g} P \to 0.$$

Since $\overline{P} \otimes_C B$ is a projective cover of $\tau_C^{-1}\Omega_C^{-1} P$ in $\operatorname{mod} B$ by Lemma 3.1.1, we have a minimal projective presentation

$$\overline{P} \otimes_C B \xrightarrow{h} P \otimes_C B \xrightarrow{g} P \to 0.$$

By Proposition 1.8.3, we know that P is τ_B-rigid if and only if $\operatorname{Hom}_B(P \otimes_C B, P) \xrightarrow{\overline{h}} \operatorname{Hom}_B(\overline{P} \otimes_C B, P)$ is surjective. Assume $\operatorname{Hom}_C(\overline{P}, P) = 0$. As a C module, $\overline{P} \otimes_C B \cong (\overline{P} \otimes_C C) \oplus (\overline{P} \otimes_C E)$. Now, $\overline{P} \otimes_C C \cong \overline{P}$ and Proposition 1.7.1 implies $\overline{P} \otimes_C E \cong \tau_C^{-1}\Omega_C^{-1}\overline{P}$. We have $\operatorname{Hom}_C(\overline{P}, P) = 0$ by assumption and $\operatorname{Hom}_C(\tau_C^{-1}\Omega_C^{-1}\overline{P}, P) = 0$ by Lemma 2.1.1. Thus, $\operatorname{Hom}_B(\overline{P} \otimes_C B, P) = 0$ and clearly $\overline{h}$ will be surjective. We conclude P is τ_B-rigid.

Conversely, assume P is τ_B-rigid. Then $\overline{h}$ must be a surjective morphism, i.e., given any morphism $j \in \operatorname{Hom}_B(\overline{P} \otimes_C B, P)$, there exists a morphism $k \in \operatorname{Hom}_B(P \otimes_C B, P)$ such that $j = k \circ h$.

$$\begin{array}{ccc} & & \overline{P} \otimes_C B \\ & \swarrow^{h} & \downarrow {\scriptstyle j = k \circ h} \\ P \otimes_C B & \xrightarrow{k} & P \end{array}$$

But h must factor through $\tau_C^{-1}\Omega_C^{-1}P$, and $\operatorname{Hom}_B(\tau_C^{-1}\Omega_C^{-1}P, P) = 0$ by Lemma 2.1.1. This implies that j must be the 0 morphism, and thus $\operatorname{Hom}_B(\overline{P} \otimes_C B, P) = 0$. Since $\overline{P} \otimes_C B$ is the projective cover of $\overline{P}$, we must have $\operatorname{Hom}_B(\overline{P}, P) = 0$. By restriction of scalars, $\operatorname{Hom}_C(\overline{P}, P) = 0$. □

Next, we examine the special case where M is a semisimple C-module. We recall that a module M is *semisimple* if it is a direct sum of simple modules.

Proposition 4.1.5. *Let M be a τ_C-rigid semisimple C-module with $f\colon P_0 \to M$ a projective cover and $g\colon M \to I_0$ an injective envelope in* $\operatorname{mod} C$.

(a) *If* $\operatorname{Hom}_C(\tau_C^{-1}\Omega_C^{-1}P_0, M) = 0$*, then M is τ_B-rigid.*

(b) *If* $\operatorname{Hom}_C(M, \tau_C\Omega_C I_0) = 0$*, then M is τ_B-rigid.*

Proof. We prove (a) with the proof of (b) being dual. By assumption, we have M is τ_C-rigid and $\operatorname{Hom}_C(\tau_C^{-1}\Omega_C^{-1}P_0, M) = 0$. Thus, we know from Theorem 3.1.2 that M is a rigid B-module. Since M is semisimple, we have that $\operatorname{Gen} M = \operatorname{add} M$. Thus, we have

$$\operatorname{Ext}^1_B(M, \operatorname{Gen} M) = \operatorname{Ext}^1_B(M, \operatorname{add} M) = \operatorname{Ext}^1_B(M, M) = 0.$$

By Proposition 1.8.2, we conclude M is τ_B-rigid. □

4.2 Partial Tilting Modules and τ_B-rigidity

As in section 4.1, we assume C is an algebra of global dimension 2 and let $B = C \ltimes E$ where $E = \operatorname{Ext}^2_C(DC, C)$. The next three results deal with C-modules M which are partial tilting modules. Our main result gives a necessary and sufficient condition for a partial tilting module to remain τ_B-rigid in the special case where $\operatorname{pd}_C \tau_C M \leq 1$.

Theorem 4.2.1. *Let M be a partial tilting C-module such that* $\operatorname{pd}_C \tau_C M \leq 1$. *Then M is τ_B-rigid if and only if* $\operatorname{Hom}_C(\tau_C^{-1}\Omega_C^{-1}M, \operatorname{Gen} M) = 0$.

Proof. Assume $\operatorname{Hom}_C(\tau_C^{-1}\Omega_C^{-1}M, \operatorname{Gen} M) = 0$. Consider the following short exact sequence guaranteed by Proposition 1.4.3 and Proposition 1.7.1

$$0 \to \tau_C^{-1}\Omega_C^{-1}M \xrightarrow{h} M \otimes_C B \to M \to 0. \tag{1}$$

Apply $\operatorname{Hom}_B(-, \operatorname{Gen} M)$ to obtain the exact sequence

$$\operatorname{Hom}_B(\tau_C^{-1}\Omega_C^{-1}M, \operatorname{Gen} M) \to \operatorname{Ext}^1_B(M, \operatorname{Gen} M) \to \operatorname{Ext}^1_B(M \otimes_C B, \operatorname{Gen} M).$$

Since $\operatorname{pd}_C \tau_C M \leq 1$, we know from Proposition 4.1.3 that $M \otimes_C B$ is a τ_B-rigid module. Thus, because $\operatorname{Gen} M \subseteq \operatorname{Gen}(M \otimes_C B)$, Proposition 1.8.2 implies $\operatorname{Ext}^1_B(M \otimes_C B, \operatorname{Gen} M) = 0$. Since $\operatorname{Hom}_C(\tau_C^{-1}\Omega_C^{-1}M, \operatorname{Gen} M) = 0$ we have $\operatorname{Hom}_B(\tau_C^{-1}\Omega_C^{-1}M, \operatorname{Gen} M) = 0$. Thus, sequence (1) yields $\operatorname{Ext}^1_B(M, \operatorname{Gen} M) = 0$, and Proposition 1.8.2 implies M is τ_B-rigid.

Assume M is τ_B-rigid. Again, because $\operatorname{pd}_C \tau_C M \leq 1$, we know $M \otimes_C B$ is a τ_B-rigid module by Proposition 4.1.3. Since $M \otimes_C B$ is τ_B-rigid and $\tau_B(M \otimes_C B)$ is a submodule of $\tau_B M$ by Lemma 1.8.9, we have ${}^{\perp}(\tau_B M) \subseteq {}^{\perp}(\tau_B(M \otimes_C B))$. Thus, Lemma 1.8.4 guarantees an exact sequence

$$M \otimes_C B \xrightarrow{f} M' \xrightarrow{g} C \to 0$$

where $M' \in \operatorname{add} M$ and $C \in \operatorname{add} P(\operatorname{Gen} M)$. Next, consider the short exact sequence

$$0 \to \ker g \xrightarrow{i} M' \xrightarrow{g} C \to 0.$$

We know that $f : M \otimes_C B \to \ker g$ is a surjective morphism. Considering f as a morphism of C-modules, we have a surjective morphism $f : M \oplus \tau_C^{-1}\Omega_C^{-1}M \to \ker g$ where the following decomposition $M \otimes_C B \cong M \oplus \tau_C^{-1}\Omega_C^{-1}M$ is given by Proposition 1.7.1. Now, consider the Hom space $\operatorname{Hom}_C(\tau_C^{-1}\Omega_C^{-1}M, \ker g)$. If this Hom space were not equal to 0, then the injectivity of i would imply a non-zero morphism from $\tau_C^{-1}\Omega_C^{-1}M$ to M'. But M' is partial tilting and we would have a contradiction to Corollary 2.1.2. But we can not have a surjective morphism from M to $\ker g$ because this would imply $\ker g \in \operatorname{Gen} M$ and would contradict $C \in \operatorname{add} P(\operatorname{Gen} M)$. Thus, $C = 0$ and we have a short exact sequence

$$0 \to \ker f \to M \otimes_C B \xrightarrow{f} M' \to 0.$$

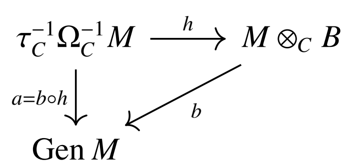
Apply $\mathrm{Hom}_B(-, \mathrm{Gen}\, M)$ to obtain an exact sequence

$$0 \to \mathrm{Hom}_B(M', \mathrm{Gen}\, M) \xrightarrow{\overline{f}} \mathrm{Hom}_B(M \otimes_C B, \mathrm{Gen}\, M) \to \mathrm{Hom}_B(\ker f, \mathrm{Gen}\, M).$$

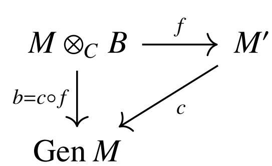
Now, Lemma 1.8.4 says that f is a left $\mathrm{Gen}\, M$-approximation of $M \otimes_C B$. This implies that $\overline{f}$ is surjective and the exactness of the sequence further implies $\overline{f}$ is an isomorphism. Returning to sequence (1), we apply $\mathrm{Hom}_B(-, \mathrm{Gen}\, M)$ to obtain an exact sequence

$$0 \to \mathrm{Hom}_B(M, \mathrm{Gen}\, M) \to \mathrm{Hom}_B(M \otimes_C B, \mathrm{Gen}\, M) \xrightarrow{\overline{h}} \mathrm{Hom}_B(\tau_C^{-1}\Omega_C^{-1} M, \mathrm{Gen}\, M) \to 0$$

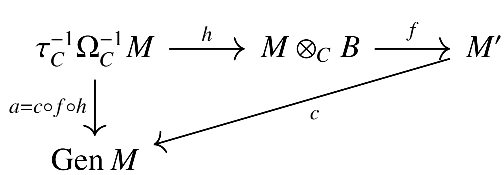
where $\mathrm{Ext}^1_B(M, \mathrm{Gen}\, M) = 0$ by Proposition 1.8.2. Since $\overline{h}$ is a surjective morphism, given any morphism $a \in \mathrm{Hom}_B(\tau_C^{-1}\Omega_C^{-1} M, \mathrm{Gen}\, M)$, there exists a morphism $b \in \mathrm{Hom}_B(M \otimes_C B, \mathrm{Gen}\, M)$ such that $a = b \circ h$.



Since we have a morphism b from $M \otimes_C B$ to a module in $\mathrm{Gen}\, M$, we may use $\overline{f}$ to say there exists a morphism $c \in \mathrm{Hom}_B(M', \mathrm{Gen}\, M)$ such that $b = c \circ f$.



So we have $a = b \circ h = c \circ f \circ h$.



But $\mathrm{Hom}_B(\tau_C^{-1}\Omega_C^{-1}M, M') = 0$ by Lemma 2.1.1 and a must be the 0 morphism. Since a was arbitrary, we conclude $\mathrm{Hom}_B(\tau_C^{-1}\Omega_C^{-1}M, \mathrm{Gen}\, M) = 0$ and our result follows. □

For an illustration of this theorem, see Examples 5.3.1 and 5.3.2 in chapter 5.

In the special case where C is a tilted algebra and B is the corresponding cluster-tilted algebra, we have a complete classification determining when an indecomposable τ_C-rigid module is also τ_B-rigid.

Corollary 4.2.2. *Let C be a tilted algebra such that $C = \mathrm{End}_A T$ where A is hereditary and T is a tilting A-module. Let B be the corresponding cluster-tilted algebra and let M be an indecomposable τ_C-rigid module. Then M is τ_B-rigid if and only if* $\mathrm{Hom}_C(\tau_C^{-1}\Omega_C^{-1}M, \mathrm{Gen}\, M) = 0$.

Proof. Since M is indecomposable and C is tilted, we know from Proposition 1.1.5 that $M \in \mathcal{X}(T)$ or $M \in \mathcal{Y}(T)$. Assume $M \in \mathcal{Y}(T)$. Since $(\mathcal{Y}(T), \mathcal{X}(T))$ is split, we know $\tau_C M \in \mathcal{Y}(T)$ by Proposition 1.1.6. Also, by Proposition 1.1.5, $\mathrm{pd}_C\, M \leq 1$ and $\mathrm{pd}_C\, \tau_C M \leq 1$. Since M is τ_C-rigid by assumption, we have M is a partial tilting module. Our result follows from Theorem 4.2.1.

Next, assume $M \in \mathcal{X}(T)$. Then Proposition 1.1.5 says $\mathrm{id}_C\, M \leq 1$. Thus, $\tau_C^{-1}\Omega_C^{-1}M = 0$ and certainly $\mathrm{Hom}_C(\tau_C^{-1}\Omega_C^{-1}M, \mathrm{Gen}\, M) = 0$. Also, Proposition 4.1.2 says M is τ_B-rigid. Our result follows. □

The case where M is a tilting C-module follows from the following proposition.

Proposition 4.2.3. *Let M be a τ_C-rigid module which is faithful. Then M is τ_B-rigid if and only if* $\mathrm{id}_C\, M \leq 1$.

Proof. If $\mathrm{id}_C\, M \leq 1$, then M is τ_B-rigid by Proposition 4.1.2. Next, assume M is τ_B-rigid and suppose $\mathrm{id}_C\, M = 2$. Then Lemma 2.1.1 implies $\mathrm{Hom}_C(M, \tau_C\Omega_C(DC)) \neq 0$. Consider the following short exact sequence in mod B guaranteed by Proposition 1.4.3 and Proposition 1.7.1

$$0 \to DC \to DB \xrightarrow{f} \tau_C \Omega_C(DC) \to 0.$$

Apply $\mathrm{Hom}_B(M, -)$ to obtain the exact sequence

$$\mathrm{Hom}_B(M, DC) \to \mathrm{Hom}_B(M, DB) \xrightarrow{\overline{f}} \mathrm{Hom}_B(M, \tau_C \Omega_C(DC)) \to \mathrm{Ext}^1_B(M, DC) \to \mathrm{Ext}^1_B(M, DB).$$

Now, $\mathrm{Ext}^1_B(M, DB) = 0$ because DB is an injective B-module. Also, because M is a faithful C-module, Lemma 1.1.2 tells us that DC is generated by M. Thus, because M is τ_B-rigid, we know $\mathrm{Ext}^1_B(M, DC) = 0$ by Proposition 1.8.2. This implies that $\overline{f}$ is a surjective morphism. Thus, given any morphism $g \in \mathrm{Hom}_B(M, \tau_C \Omega_C(DC))$, there exists a morphism $h \in \mathrm{Hom}_B(M, DB)$ such that $g = f \circ h$.

Next, consider an injective envelope $j \colon M \to I_0$ of M in mod C. Now, I_0 may or may not be an injective B-module but j is still an injective map in mod B. Since DB is an injective B-module, there exists a morphism $k \colon I_0 \to DB$ such that $h = k \circ j$.

$$\begin{array}{ccccc} 0 & \longrightarrow & M & \xrightarrow{j} & I_0 \\ & & \downarrow{\scriptstyle h} & \swarrow{\scriptstyle k} & \\ & & DB & & \end{array}$$

Thus, we have $g = f \circ h = f \circ k \circ j$.

$$\begin{array}{ccc} M & \xrightarrow{g} & \tau_C \Omega_C(DC) \\ \downarrow{\scriptstyle j} & \nearrow{\scriptstyle f \circ k} & \\ I_0 & & \end{array}$$

But I_0 is an injective C-module and Lemma 2.1.1 implies $\mathrm{Hom}_C(I_0, \tau_C \Omega_C(DC)) = 0$ and subsequently $\mathrm{Hom}_B(I_0, \tau_C\Omega_C(DC)) = 0$. This forces $g = f \circ k \circ j = 0$. Since g was arbitrary, we conclude $\mathrm{Hom}_B(M, \tau_C\Omega_C(DC)) = 0$. But we showed $\mathrm{Hom}_C(M, \tau_C\Omega_C(DC)) \neq 0$, which implies $\mathrm{Hom}_B(M, \tau_C\Omega_C(DC)) \neq 0$, and we have a contradiction. Thus, the assumption $\mathrm{id}_C M = 2$ must be false, and we conclude $\mathrm{id}_C M \leq 1$. □

Corollary 4.2.4. *Suppose M is a tilting C-module. Then M is τ_B-tilting if and only if* $\mathrm{id}_C M \leq 1$.

Proof. Since M is a tilting C-module, it is faithful by Lemma 1.1.2, and our result follows from Proposition 4.2.3. □

For an illustration of this corollary, see Examples 5.3.3 and 5.3.4 in chapter 5.

We may generalize the preceding result in the special case that the algebra C is tilted.

Proposition 4.2.5. *Let C be a tilted algebra such that $C = \mathrm{End}_A T$ where A is hereditary and T is a tilting A-module. Let B be the corresponding cluster-tilted algebra. If M is τ_C-tilting, then M is τ_B-tilting if and only if* $\mathrm{id}_C M \leq 1$.

Proof. Assume $\mathrm{id}_C M \leq 1$. Since M is τ_C-rigid, we know from Proposition 4.1.2 that M is also τ_B-rigid. Next, assume M is τ_B-tilting and suppose $\mathrm{id}_C M = 2$. Then at least one indecomposable summand of M, say M_i, has injective dimension equal to 2 in $\mathrm{mod}\, C$. By Proposition 1.1.5, we know $M_i \in \mathcal{Y}(T)$. By Proposition 1.1.6, we know $(\mathcal{Y}(T), \mathcal{X}(T))$ is split which implies $\tau_C M_i \in \mathcal{Y}(T)$ and Proposition 1.1.5 gives $\mathrm{pd}_C\, \tau_C M_i \leq 1$. Thus, by Proposition 4.1.3, we have that $M_i \otimes_C B$ is τ_B-rigid.

By Lemma 1.8.9, we know $\tau_B(M_i \otimes_C B)$ is a submodule of $\tau_B M_i$. Thus, $\mathrm{Gen}\, M \subseteq {}^{\perp}(\tau_B(M_i \otimes_C B))$. By Lemma 1.8.5, there exists an exact sequence

$$M_i \otimes_C B \xrightarrow{f} M^0 \xrightarrow{g} M^1 \to 0 \tag{1}$$

where f is a minimal left $\operatorname{Gen} M$-approximation of $M_i \otimes_C B$, M^0 and M^1 are in $\operatorname{add} M$, and we have $\operatorname{add} M^0 \cap \operatorname{add} M^1 = 0$. Next, consider the following short exact sequence

$$0 \to \ker g \to M^0 \xrightarrow{g} M^1 \to 0. \tag{2}$$

We have a surjective morphism $f\colon M_i \otimes_C B \to \ker g$. Using Proposition 1.7.1, we have a surjective morphism $f\colon M_i \oplus \tau_C^{-1}\Omega_C^{-1}M \to \ker g$ in $\operatorname{mod} C$. Now, we must have $\operatorname{Hom}_C(\tau_C^{-1}\Omega_C^{-1}M_i, \ker g) \neq 0$ for every indecomposable summand of $\ker g$. Otherwise, we would have a surjective morphism from M_i to a summand of $\ker g$ say $\ker g_i$. This would imply $\ker g_i \in \operatorname{Gen} M_i$. Thus, (2) would imply $\operatorname{Ext}_B^1(M, \operatorname{Gen} M_i) \neq 0$ unless (2) splits. But (2) can not split because Lemma 1.8.5 states $\operatorname{add} M^0 \cap \operatorname{add} M^1 = 0$. Thus, $\operatorname{Ext}_B^1(M, \operatorname{Gen} M_i) \neq 0$ would give a contradiction to Proposition 1.8.2.

Since $\operatorname{Hom}_C(\tau_C^{-1}\Omega_C^{-1}M_i, \ker g) \neq 0$ for every summand of $\ker g$, Corollary 2.1.2 says that $\operatorname{pd}_C(\ker g) = 2$ for every summand of $\ker g$. Since C is a tilted algebra, Proposition 1.1.5 says $\ker g \in \mathcal{X}(T)$. Returning to (1), we know f is a left $\operatorname{Gen} M$-approximation of $M_i \otimes_C B$. Thus, given any non-zero morphism $h\colon M_i \otimes_C B \to M_i$, there exists a morphism $j\colon M^0 \to M_i$ such that $h = j \circ f$.

$$\begin{array}{ccc} & & M_i \otimes_C B \\ & \swarrow^{f} & \downarrow{\scriptstyle h = j \circ f} \\ M^0 & \xrightarrow{\ j\ } & M_i \end{array}$$

But f must factor through $\ker g$ which implies a non-zero morphism from $\ker g$ to M_i. Since $\ker g \in \mathcal{X}(T)$ and $M_i \in \mathcal{Y}(T)$, we have a contradiction by Proposition 1.1.5. Thus, we have $\operatorname{Hom}_B(M_i \otimes_C B, M_i) = 0$ but this is a contradiction by Proposition 1.4.3. So $\operatorname{id}_C M_i \leq 1$ and, since M_i was arbitrary, we conclude $\operatorname{id}_C M \leq 1$. $\square$

4.3 Projective Covers and τ_B-rigidity

The next three results use information about a module's projective cover to determine whether or not the module is τ_B-rigid.

Proposition 4.3.1. *Let M be a τ_C-rigid module with $f: P_0 \to M$ a projective cover in* $\operatorname{mod} C$*. If* $\operatorname{Hom}_C(\tau_C^{-1}\Omega_C^{-1}P_0, \operatorname{Gen} M) = 0$*, then M is τ_B-rigid.*

Proof. We modify the proof of Theorem 3.1.2 by replacing $\operatorname{Hom}_C(\tau_C^{-1}\Omega_C^{-1}P_0, M) = 0$ with the assumption $\operatorname{Hom}_C(\tau_C^{-1}\Omega_C^{-1}P_0, \operatorname{Gen} M) = 0$. The concluding statement is now $\operatorname{Ext}^1_B(M, \operatorname{Gen} M) = 0$ and we conclude by Proposition 1.8.2 that M is τ_B-rigid. □

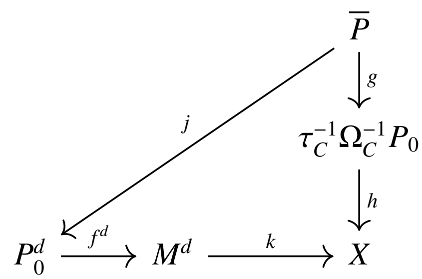
Corollary 4.3.2. *If M is τ_C-rigid, and* $\operatorname{pd}_C X \leq 1$ *for every module $X \in$* $\operatorname{Gen} M$*, then M is τ_B-rigid.*

Proof. Since $\operatorname{pd}_C X \leq 1$ for every module $X \in \operatorname{Gen} M$, we have $\operatorname{Hom}_C(\tau_C^{-1}\Omega_C^{-1}C, \operatorname{Gen} M) = 0$ by Lemma 2.1.1. Our result follows from Proposition 4.3.1. □

Corollary 4.3.3. *Let M be τ_C-rigid with $f: P_0 \to M$ a projective cover in* $\operatorname{mod} C$*. If P_0 is τ_B-rigid, then M is τ_B-rigid.*

Proof. Consider $g : \overline{P} \to \tau_C^{-1}\Omega_C^{-1}P_0$ a projective cover in $\operatorname{mod} C$. Since P_0 is τ_B-rigid by assumption, we know $\operatorname{Hom}_C(\overline{P}, P_0) = 0$ by Proposition 4.1.4. Suppose there exists a morphism $h: \tau_C^{-1}\Omega_C^{-1}P_0 \to X$ with $X \in \operatorname{Gen} M$. This also gives a morphism $h \circ g : \overline{P} \to X$ because $\overline{P}$ is a projective C-module. Since $X \in \operatorname{Gen} M$, we have a surjective morphism $k: M^d \to X$. Combining with the fact P_0 is a projective cover of M, we have a surjective morphism $k \circ f^d : P_0^d \to X$. However, since $\overline{P}$ is a projective C-module, we have an induced morphism $j: \overline{P} \to P_0^d$ such that

$h \circ g = k \circ f^d \circ j$ and the following diagram commutes.



But $\mathrm{Hom}_C(\overline{P}, P_0) = 0$ and j must be the 0 morphism. If g is non-zero then we must have that h is also the 0 morphism. Since h was arbitrary, we conclude $\mathrm{Hom}_C(\tau_C^{-1}\Omega_C^{-1}P_0, X) = 0$ and Proposition 4.3.1 implies M is τ_B-rigid. □

We have the following corollary in the special case that M is partial tilting and the projective dimension of $\tau_C M$ is not necessarily less than or equal to 1.

Corollary 4.3.4. *Let M be a partial tilting C-module with $f : P_0 \to M$ a projective cover in* $\mathrm{mod}\, C$*. If* $\mathrm{Hom}_C(\Omega_C(\tau_C^{-1}\Omega_C^{-1}P_0), M) = 0$*, then M is τ_B-rigid.*

Proof. Consider the following short exact sequence in $\mathrm{mod}\, C$

$$0 \to \Omega_C^1(\tau_C^{-1}\Omega_C^{-1}P_0) \to P_1 \to \tau_C^{-1}\Omega_C^{-1}P_0 \to 0 \qquad (1)$$

where P_1 is a projective cover of $\tau_C^{-1}\Omega_C^{-1}P_0$. Apply $\mathrm{Hom}_C(-, M)$ to obtain the exact sequence

$$\mathrm{Hom}_C(\tau_C^{-1}\Omega_C^{-1}P_0, M) \to \mathrm{Hom}_C(P_1, M) \to \mathrm{Hom}_C(\Omega_C^1(\tau_C^{-1}\Omega_C^{-1}P_0), M).$$

Since M is a partial tilting module we know $\mathrm{pd}_C\, M \leq 1$. Thus, $\mathrm{Hom}_C(\tau_C^{-1}\Omega_C^{-1}P_0, M) = 0$ by Lemma 2.1.1. Also, $\mathrm{Hom}_C(\Omega_C^1(\tau_C^{-1}\Omega_C^{-1}P_0), M) = 0$ by asumption. Since the sequence is exact, we have

$\mathrm{Hom}_C(P_1, M) = 0$. Since P_1 is a projective C-module, this further implies that $\mathrm{Hom}_C(P_1, \mathrm{Gen}\, M) = 0$. Apply $\mathrm{Hom}_C(-, \mathrm{Gen}\, M)$ to sequence (1) to obtain the exact sequence

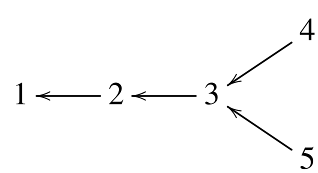
$$0 \to \mathrm{Hom}_C(\tau_C^{-1}\Omega_C^{-1} P_0, \mathrm{Gen}\, M) \to \mathrm{Hom}_C(P_1, \mathrm{Gen}\, M).$$

Since $\mathrm{Hom}_C(P_1, \mathrm{Gen}\, M) = 0$ and the sequence is exact, we have $\mathrm{Hom}_C(\tau_C^{-1}\Omega_C^{-1} P_0, \mathrm{Gen}\, M) = 0$. By Proposition 4.3.1, we have that M is τ_B-rigid. □

Chapter 5

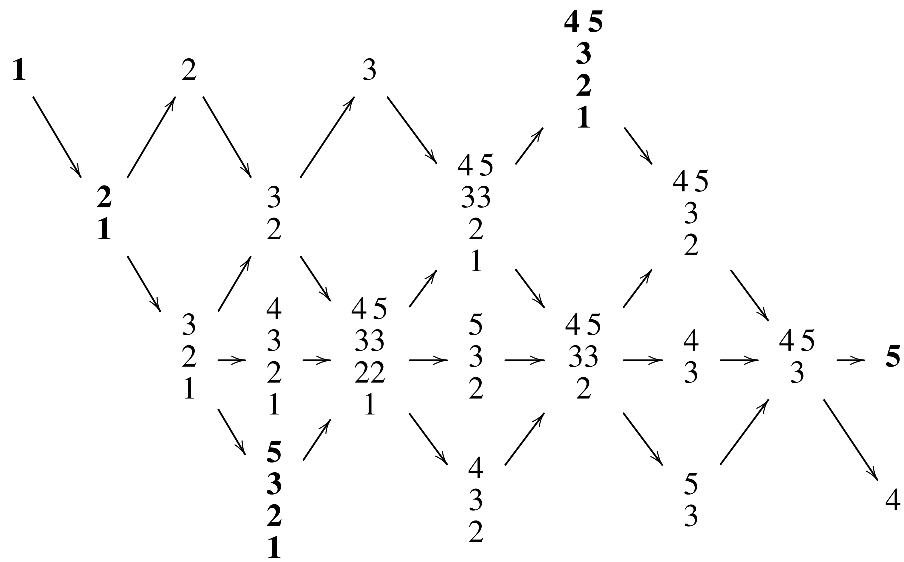
Examples

In this chapter we illustrate our main results from chapters 2, 3, and 4 with several examples. We will use the following throughout this chapter. Let A be the path algebra of the following quiver:



Since A is a hereditary algebra, we may construct a tilted algebra. To do this, we need an

A-module which is tilting. Consider the Auslander-Reiten quiver of A which is given by:



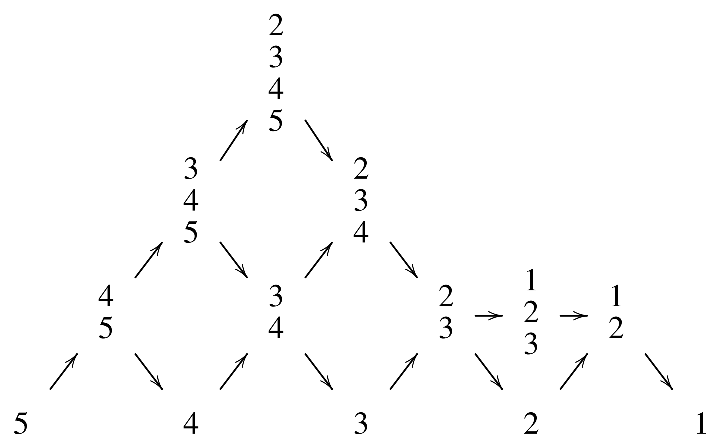
Let T be the tilting A-module

$$T = 5 \oplus \begin{array}{c} 4\,5 \\ 3 \\ 2 \\ 1 \end{array} \oplus \begin{array}{c} 5 \\ 3 \\ 2 \\ 1 \end{array} \oplus \begin{array}{c} 2 \\ 1 \end{array} \oplus 1$$

The corresponding titled algebra $C = \mathrm{End}_A T$ is given by the bound quiver

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \xrightarrow{\gamma} 4 \longrightarrow 5 \qquad \alpha\beta\gamma = 0$$

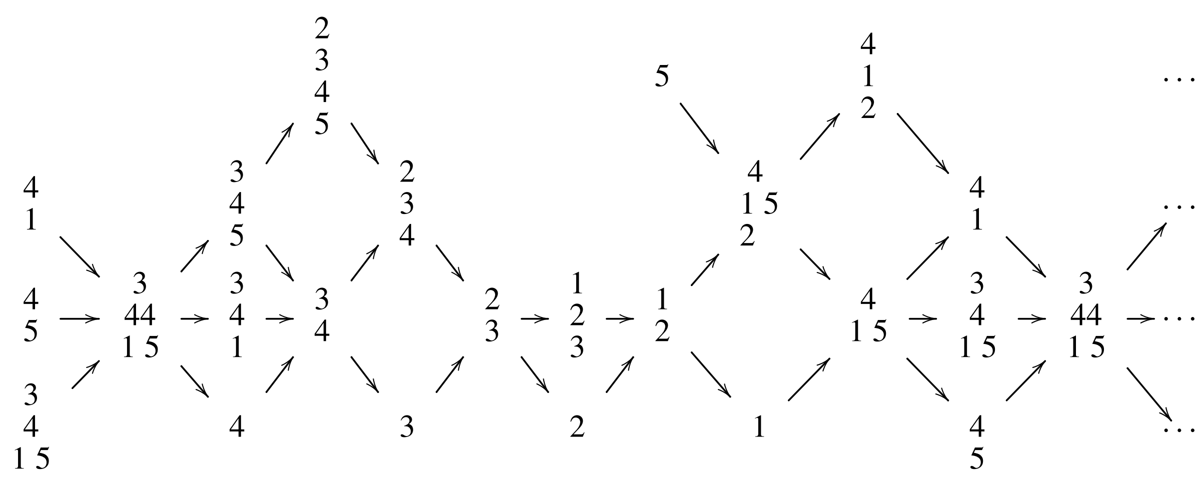
Then, the Auslander-Reiten quiver of C is given by:



The corresponding cluster-tilted algebra $B = C \ltimes \mathrm{Ext}^2_C(DC, C)$ is given by the bound quiver

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \xrightarrow{\gamma} 4 \longrightarrow 5, \quad \delta: 4 \to 1 \qquad \alpha\beta\gamma = \beta\gamma\delta = \gamma\delta\alpha = \delta\alpha\beta = 0$$

Then, the Auslander-Retien quiver of B is given by:



5.1 Homological dimensions

We wish to illustrate Theorem 2.2.5 with an example for each case. We will use Lemma 2.1.1 frequently so we note that

$$\tau_C^{-1}\Omega_C^{-1}C = \begin{smallmatrix}1\\2\end{smallmatrix} \oplus 1 \quad , \quad \tau_C\Omega_C(DC) = \begin{smallmatrix}3\\4\end{smallmatrix} \oplus 4 \ .$$

Example 5.1.1. We'll start with the projective dimension equal to 2. In $\operatorname{mod} C$, consider the module $M = \begin{smallmatrix}1\\2\end{smallmatrix}$. Since $\operatorname{Hom}_C(\tau_C^{-1}\Omega_C^{-1}C, M) \neq 0$, Lemma 2.1.1 says $\operatorname{pd}_C M = 2$. Thus, Theorem 2.2.5 says $\operatorname{pd}_B M = \infty$ and we have the following projective resolution in $\operatorname{mod} B$

$$\cdots \to \begin{smallmatrix}4\\1\,5\\2\end{smallmatrix} \to \begin{smallmatrix}3\\4\\1\,5\end{smallmatrix} \to \begin{smallmatrix}1\\2\\3\end{smallmatrix} \to \begin{smallmatrix}1\\2\end{smallmatrix} \to 0\ .$$

Example 5.1.2. Next, let's examine the projective case, i.e., projective dimension equal to 0. In $\operatorname{mod} C$, consider the module $M = 5$. Then M is the projective C-module at vertex 5. Since $\operatorname{Hom}_C(M, \tau_C\Omega_C(DC)) = 0$, Lemma 2.1.1 says $\operatorname{id}_C M \leq 1$. Thus, Theorem 2.2.5 says $\operatorname{pd}_B M = 0$. Now, consider $N = \begin{smallmatrix}4\\5\end{smallmatrix}$ in $\operatorname{mod} C$. Then N is a projective C-module. Now, we have that $\operatorname{Hom}_C(N, \tau_C\Omega_C(DC)) \neq 0$. Thus, Lemma 2.1.1 says $\operatorname{id}_C N = 2$. Finally, Theorem 2.2.5 states $N = \begin{smallmatrix}4\\5\end{smallmatrix}$ is not a projective B-module and $\operatorname{pd}_B N = \infty$ with the following projective resolution in $\operatorname{mod} B$

$$\cdots \to \begin{smallmatrix}3\\4\\1\,5\end{smallmatrix} \to \begin{smallmatrix}1\\2\\3\end{smallmatrix} \to \begin{smallmatrix}4\\1\,5\\2\end{smallmatrix} \to \begin{smallmatrix}4\\5\end{smallmatrix} \to 0\ .$$

Example 5.1.3. Finally, let's examine the case where the projective dimension is equal to 1. Consider the C-module $M = \begin{smallmatrix}3\\4\end{smallmatrix}$. Since $\operatorname{Hom}_C(\tau_C^{-1}\Omega_C^{-1}C, M) = 0$, Lemma 2.1.1 says $\operatorname{pd}_C M \leq 1$ with projective resolution

$$0 \to 5 \to \begin{smallmatrix}3\\4\\5\end{smallmatrix} \to \begin{smallmatrix}3\\4\end{smallmatrix} \to 0\ .$$

Denote $P_1 = 5$ and $P_0 = \begin{matrix} 3 \\ 4 \\ 5 \end{matrix}$. Since $\operatorname{Hom}_C(M, \tau_C \Omega_C(DC)) \neq 0$, Lemma 2.1.1 says $\operatorname{id}_C M = 2$. Also, note that $\tau_C^{-1}\Omega_C^{-1} P_1 \ncong \tau_C^{-1}\Omega_C^{-1} P_0$ because $\tau_C^{-1}\Omega_C^{-1} P_1 = 0$ while $\tau_C^{-1}\Omega_C^{-1} P_0 = \begin{matrix} 1 \\ 2 \end{matrix}$. Thus, Theorem 2.2.5 says $\operatorname{pd}_B M = \infty$ and we have the following projective resolution in $\operatorname{mod} B$

$$\cdots \to \begin{matrix} 2 \\ 3 \\ 4 \\ 5 \end{matrix} \to 5 \oplus \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \to \begin{matrix} 3 \\ 4 \\ 1\,5 \end{matrix} \to \begin{matrix} 3 \\ 4 \end{matrix} \to 0 .$$

Next, consider the C-module $N = \begin{matrix} 2 \\ 3 \\ 4 \end{matrix}$. Since $\operatorname{Hom}_C(\tau_C^{-1}\Omega_C^{-1} C, N) = 0$, Lemma 2.1.1 says that $\operatorname{pd}_C N = 1$ with minimal projective resolution

$$0 \to 5 \to \begin{matrix} 2 \\ 3 \\ 4 \\ 5 \end{matrix} \to \begin{matrix} 2 \\ 3 \\ 4 \end{matrix} \to 0 .$$

Denote $P_1' = 5$ and $P_0' = \begin{matrix} 2 \\ 3 \\ 4 \\ 5 \end{matrix}$. Since $\operatorname{Hom}_C(N, \tau_C \Omega_C(DC)) = 0$, Lemma 2.1.1 says that $\operatorname{id}_C N \leq 1$. Also, note that $\tau_C^{-1}\Omega_C^{-1} P_1' \cong \tau_C^{-1}\Omega_C^{-1} P_0' = 0$. Thus, Theorem 2.2.5 says $\operatorname{pd}_B N = 1$ and Corollary 2.2.2 implies the minimal projective resolution in $\operatorname{mod} C$ is the same as the minimal projective resolution in $\operatorname{mod} B$.

5.2 Extensions

In this section, we will illustrate Theorem 3.1.2 with two examples.

Example 5.2.1. Consider the C-module $M = \begin{matrix} 4 \\ 5 \end{matrix} \oplus \begin{matrix} 1 \\ 2 \end{matrix}$. To use Theorem 3.1.2 we need several preliminary calculations. We have a projective cover and an injective envelope in mod C

$$f\colon \begin{matrix}4\\5\end{matrix} \oplus \begin{matrix}1\\2\\3\end{matrix} \to M \ , \ g\colon M \to \begin{matrix}2\\3\\4\\5\end{matrix} \oplus \begin{matrix}1\\2\end{matrix} .$$

Let us denote $\begin{matrix}4\\5\end{matrix} \oplus \begin{matrix}1\\2\\3\end{matrix}$ by P_0 and $\begin{matrix}2\\3\\4\\5\end{matrix} \oplus \begin{matrix}1\\2\end{matrix}$ by I_0. Then we have

$$\tau_C^{-1}\Omega_C^{-1}P_0 = \begin{matrix}1\\2\end{matrix} \ , \ \tau_C\Omega_C I_0 = 4 .$$

It is easily seen that $\mathrm{Ext}_C^1(M, M) = 0$ but $\mathrm{Ext}_B^1(M, M) \neq 0$ with self-extension

$$0 \to \begin{matrix}1\\2\end{matrix} \to \begin{matrix}4\\1\,5\\2\end{matrix} \to \begin{matrix}4\\5\end{matrix} \to 0$$

Note that $\mathrm{Hom}_C(\tau_C^{-1}\Omega_C^{-1}P_0, M) \neq 0$ and $\mathrm{Hom}_C(M, \tau_C\Omega_C I_0) \neq 0$ in accordance with Theorem 3.1.2.

Example 5.2.2. Consider the C-module $N = 5 \oplus 3$. We have a projective cover and an injective envelope in mod C

$$f\colon 5 \oplus \begin{matrix}3\\4\\5\end{matrix} \to M \ , \ g\colon M \to \begin{matrix}2\\3\\4\\5\end{matrix} \oplus \begin{matrix}1\\2\\3\end{matrix} .$$

Denote $5 \oplus \begin{matrix}3\\4\\5\end{matrix}$ by P_0 and $\begin{matrix}2\\3\\4\\5\end{matrix} \oplus \begin{matrix}1\\2\\3\end{matrix}$ by I_0. Then we have

$$\tau_C^{-1}\Omega_C^{-1}P_0 = 1 \ , \ \tau_C\Omega_C I_0 = 0.$$

Now, we have $\mathrm{Ext}_C^1(M, M) = 0$ and $\mathrm{Ext}_B^1(M, M) = 0$. Note, $\mathrm{Hom}_C(\tau_C^{-1}\Omega_C^{-1}P_0, M) = 0$ and $\mathrm{Hom}_C(M, \tau_C\Omega_C I_0) = 0$ in accordance with Theorem 3.1.2.

5.3 τ-rigid modules

In this section we will illustrate Theorem 4.2.1 and Corollary 4.2.4. We will start with Theorem 4.2.1.

Example 5.3.1. Consider the C-module $M = \begin{matrix}1\\2\\3\end{matrix} \oplus \begin{matrix}3\\4\end{matrix}$. Then M is partial tilting, $\mathrm{pd}_C\, \tau_C M = 0$, and $\tau_C^{-1}\Omega_C^{-1} M = 1$. Since $1 \in \mathrm{Gen}\, M$, we have $\mathrm{Hom}_C(\tau_C^{-1}\Omega_C^{-1} M, \mathrm{Gen}\, M) \neq 0$. Note that $\tau_B M = \begin{matrix}3\\44\\1\,5\end{matrix}$ and $\mathrm{Hom}_B(M, \tau_B M) \neq 0$ in accordance with Theorem 4.2.1.

Example 5.3.2. Consider the C-module $N = \begin{matrix}3\\4\\5\end{matrix} \oplus \begin{matrix}3\\4\end{matrix} \oplus 4$. Then N is partial tilting, $\mathrm{pd}_C\, \tau_C N = 0$, and $\tau_C^{-1}\Omega_C^{-1} N = \begin{matrix}1\\2\end{matrix} \oplus 1$. It is easily seen that $\mathrm{Hom}_C(\tau_C^{-1}\Omega_C^{-1} N, \mathrm{Gen}\, N) = 0$. We note that $\tau_B N = \begin{matrix}3\\44\\1\,5\end{matrix} \oplus \begin{matrix}3\\4\\1\,5\end{matrix} \oplus \begin{matrix}4\\1\end{matrix}$ and $\mathrm{Hom}_B(N, \tau_B N) = 0$ in accordance with Theorem 4.2.1.

The next two examples will illustrate Corollary 4.2.4.

Example 5.3.3. Consider the tilting C-module

$$M = 4 \oplus \begin{matrix}4\\5\end{matrix} \oplus \begin{matrix}3\\4\\5\end{matrix} \oplus \begin{matrix}1\\2\\3\end{matrix} \oplus \begin{matrix}2\\3\\4\\5\end{matrix}.$$

Recall that $\tau_C\Omega_C(DC) = \begin{matrix}3\\4\end{matrix} \oplus 4$. Since $\mathrm{Hom}_C(M, \tau_C\Omega_C(DC)) \neq 0$, Lemma 2.1.1 says $\mathrm{id}_C\, M = 2$. Note that $\tau_B M = 1 \oplus \begin{matrix}4\\1\end{matrix} \oplus \begin{matrix}3\\4\\1\,5\end{matrix}$ and we have $\mathrm{Hom}_B(M, \tau_B M) \neq 0$ in accordance with Corollary 4.2.4.

Example 5.3.4. Consider the tilting C-module

$$T = 2 \oplus \begin{matrix}2\\3\end{matrix} \oplus \begin{matrix}2\\3\\4\end{matrix} \oplus \begin{matrix}1\\2\\3\end{matrix} \oplus \begin{matrix}2\\3\\4\\5\end{matrix}.$$

Since $\operatorname{Hom}_C(T, \tau_C \Omega_C(DC)) = 0$, Lemma 2.1.1 says $\operatorname{id}_C T \leq 1$. We note that

$$\tau_C T \cong \tau_B T = \; 3 \;\oplus\; \begin{matrix} 3 \\ 4 \end{matrix} \;\oplus\; \begin{matrix} 3 \\ 4 \\ 5 \end{matrix}$$

and $\operatorname{Hom}_B(T, \tau_B T) = 0$ in accordance with Corollary 4.2.4.

Bibliography

[1] T. Adachi, O. Iyama and I. Reiten, τ-tilting theory, *Compos. Math.* **150** (2014), no. 3, 415–452.

[2] C. Amoit, Cluster categories for algebras of global dimension 2 and quivers with potential, *Ann. Inst. Fourier* **59**, (2009), no. 6, 2525–2590.

[3] I. Assem, T. Brüstle and R. Schiffler, Cluster-tilted algebras as trivial extensions, *Bull. Lond. Math. Soc.* **40** (2008), 151–162.

[4] I. Assem, T. Brüstle and R. Schiffler, Cluster-tilted algebras and slices, *J. of Algebra* **319** (2008), 3464–3479.

[5] I. Assem, T. Brüstle and R. Schiffler, On the Galois covering of a cluster-tilted algebra, *J. Pure Appl. Alg.* **213** (7) (2009), 1450–1463.

[6] I. Assem, T. Brüstle and R. Schiffler, Cluster-tilted algebras without clusters *J. Algebra* **324**, (2010), 2475–2502.

[7] I. Assem and N. Marmaridis, Tilting modules and split-by-nilpotent extensions, *Comm. Algebra* **26** (1998), 1547–1555.

[8] I. Assem, D. Simson and A. Skowronski, *Elements of the Representation Theory of Associative Algebras, 1: Techniques of Representation Theory*, London Mathematical Society Student Texts 65, Cambridge University Press, 2006.

[9] I. Assem and D. Zacharia, Full embeddings of almost split sequences over split-by-nilpotent extensions, *Coll. Math.* **81**, (1) (1999), 21–31.

[10] M. Auslander and S. O. Smalø, Almost split sequences in subcategories, *J. Algebra* **69** (1981), no. 2, 426–454.

[11] M. Barot, E. Fernandez, I. Pratti, M. I. Platzeck and S. Trepode, From iterated tilted to cluster-tilted algebras, *Adv. Math.* **223** (2010), no. 4, 1468–1494.

[12] L. Beaudet, T. Brustle and G. Todorov, Projective dimension of modules over cluster-tilted algebras, *Algebr. and Represent. Theory* **17** (2014), no. 6, 1797–1807.

[13] M. A. Bertani-Økland, S. Oppermann and A. Wrålsen, Constructing tilted algebras from cluster-tilted algebras, *J. Algebra* **323** (2010), no. 9, 2408–2428.

[14] A. B. Buan, R. Marsh, M. Reineke, I. Reiten and G. Todorov, Tilting theory and cluster combinatorics, *Adv. Math.* **204** (2006), no. 2, 572–618.

[15] A. B. Buan, R. Marsh and I. Reiten, Cluster-tilted algebras, *Trans. Amer. Math. Soc.* **359** (2007), no. 1, 323-332.

[16] A. B. Buan, R. Marsh and I. Reiten, Cluster-tilted algebras of finite representation type, *J. Algebra* **306** (2006), no. 2, 412–431.

[17] A. B. Buan, R. Marsh and I. Reiten, Cluster mutation via quiver representations, *Comment. Math. Helv.* **83** (2008), no. 1, 143–177.

[18] P. Caldero, F. Chapoton and R. Schiffler, Quivers with relations arising from clusters (A_n case), *Trans. Amer. Math. Soc.* **358** (2006), no. 4, 359–376.

[19] P. Caldero, F. Chapoton and R. Schiffler, Quivers with relations and cluster tilted algebras, *Algebr. and Represent. Theory* **9**, (2006), no. 4, 359–376.

[20] S. Fomin and A. Zelevinsky, Cluster algebras I: Foundations, *J. Amer. Math. Soc.* **15** (2002), 497–529.

[21] B. Keller, On triangulated orbit categories, *Documenta Math.* **10** (2005), 551–581.

[22] B. Keller and I. Reiten, Cluster-tilted algebras are Gorenstein and stably Calabi-Yau, *Adv. Math.* **211** (2007), no. 1, 123–151.

[23] P. G. Plamondon, Cluster algebras via cluster categories with infinite-dimensional morphism spaces, *Compos. Math.* **147** (2011), no. 6, 1921–1954.

[24] R. Schiffler, *Quiver Representations*, CMS Books in Mathematics, Springer International Publishing, 2014.

[25] R. Schiffler and K. Serhiyenko, Induced and coinduced modules in cluster-tilted algebras, preprint, **arXiv** : **1410**.**1732**.