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Confinement in 3+1 Dimensions

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In this thesis we focus on two important problems of modern physics: the phenomenon of confinement in non-Abelian field theories and the unitarity of theories with higher derivatives.

In the first part we describe an effective theory of a scalar field, motivated by some features expected in the low energy theory of gluodynamics in 3+1 dimensions. The theory describes two propagating massless particles in a certain limit, which we identify with the Abelian QED limit, and has classical string solutions in the general case. The string solutions are somewhat unusual as they are multiply degenerate due to the spontaneous breaking of diffeomorphism invariance. Nevertheless, all solutions yield an identical electric field and have the same string tension. We conclude the first part by further investigating the Abelian limit of the model presented and constructing a Lagrangian with a four-derivative kinetic term and demonstrate that, despite the seeming nonlinearity of the theory, it is equivalent to a theory of a free photon.

In the second part we start by giving a simple discussion of ghosts, unitarity violation, negative norm states, and quantum vs classical behavior in the simplest model with a four-derivative action - the Pais-Uhlenbeck oscillator. We also point out that the normalizable "vacuum state" (in the sense defined below) of this model can be understood as a spontaneous breaking of the emergent conformal symmetry. We provide an example of an interacting system that couples the "particle" and "ghost" degrees of freedom and nevertheless remains unitary on both the classical and quantum levels. The rest of the second part focuses on the analysis of conformal gravity in translationally invariant approximation, where the metric is taken to depend on time but not on spatial coordinates. We find that the field mode, which in perturbation theory has a ghostlike kinetic term, turns into a tachyon when nonlinear interaction is accounted for. The kinetic term and potential for this mode have opposite signs. Solutions of nonlinear classical equations of motion develop a singularity in finite time determined by the initial conditions.

Confinement in 3+1 Dimensions

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I dedicate this work to my Family.

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Everybody, thank you.

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Chapter 1

Introduction

QCD is a theory of quarks and gluons, but its particle spectrum is composed of hadrons and mesons. This behavior is attributed to confinement phenomenon. However, there is no direct analytical explanation for confinement starting from the basic principles. On the other end of the Standard Model spectrum (QED), we have the massless photon and the quantized electric charge. There is no a priori reason for the electric charge to be quantized, except for the experimental fact that it is. Massless nature of the photon is usually attributed to gauge symmetry, but this is not really a satisfactory explanation as there are different mechanisms (Higgs mechanism, topological mass etc) for the gauge particles to acquire mass.

In the first part of this thesis (Chapter 2), we discuss our attempt in explaining the mechanism responsible for confinement. In the first section of chapter 2, we introduce the problem at hand in more details, and present a theory that works very well in 2+1 dimensions. Sections 2.2 is our first attempt to a generalization of the ideas in 2+1 dimensions to 3+1. Section 2.3 aims to improve some of the properties of the model presented in section 2.2. A detailed outline of Chapter 2 is presented at the begining of the chapter. Sections 2.2 and 2.3 are published in [1] and [2] respectively.

In Chapter 3, we turn our attention to a different problem in physics: theories with higher time derivatives, their importance in physics and their consistency. Such theories are commonly used in modified theories of gravity. We start that chapter by presenting a more detailed introduction to such theories. In section 3.1, we discuss some rather less known concepts of theories with ghosts, and show that there exists stable interacting such quantum mechanical systems. Section 3.2 focuses on conformal gravity, which is a very promising candidate for a quantum theory of gravity, as explained in that section. The material presented in chapter 3 is published in [3] and [4]. A more detailed outline is presented at the beginning of that chapter 3.

Chapter 2

Confinement

As discussed in the introduction, QCD is a theory of quarks and gluons. It is defined by the Lagrangian:

$$\mathcal{L} = -\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu a} + \Psi_I (i \not{D} - m_I) \Psi_I \tag{2.1}$$

The first term represents the Yang-Mills part or gluodynamics, and the second term represents quarks and their interactions with gluons. This Lagrangian is well understood in the short distance or high energy regime, but it is hard to say the same thing about the long distance behavior or low energies. The problem is very simple: the fundamental degrees of freedom are quarks and gluons, but the asymptotic states are composed of mesons and baryons.

This behavior is explained through confinement: the force between a quark antiquark pair does not decrease by separation, the potential between them increases linearly. This behavior is well established by lattice calculations [5] [6]. Also, there is evidence coming directly from the experiments [7], such as the

"Regge behavior". Figure 2.1 [8] is a plot of spins of some mesons against their mass squared, which appear to be proportional to each other. This is what one expects if the meson can be defined as a "spinning stick": a straight line with constant mass per unit length. Such a description of a meson not only gives the above proportionality, but also provides a constant force thus a linearly increasing potential. So, if we can find a mechanism that squeezes the field between a quark-antiquark pair in a similar way, into a color electric flux tube, the energy stored in such a configuration will be linear with separation. But the question is how such a flux tube forms in QCD?

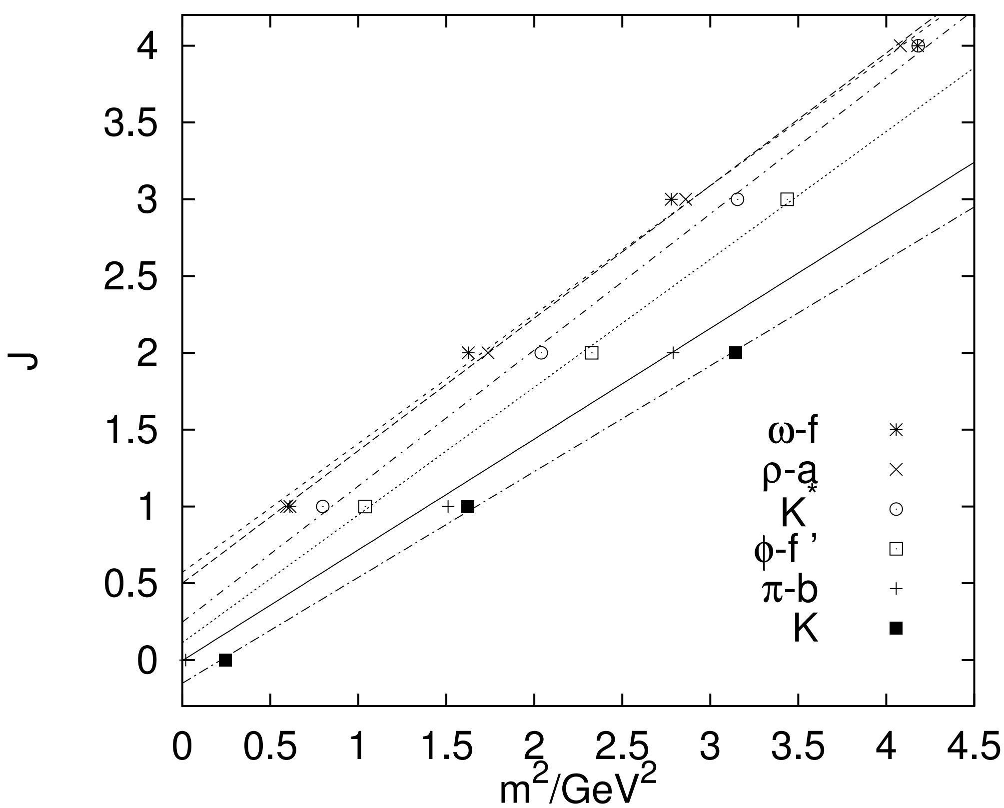


Fig. 2.1: Regge trajectories [8]

One simple and probably the most well known idea in explaining this behavior comes from superconductivity. Consider a type-II superconductor. Imagine the situation where a monopole pair is placed in such a medium. Magnetic flux needs to be conserved, but on the other hand the superconductor cannot tolerate the magnetic field. This "paradox" is resolved by the formation of a flux tube - called the Abrikosov vortices, to which the magnetic field is squeezed into. This mechanism is suggested as a prototype for quark confinement by 't Hooft and Mandelstam [9] [10]. All one has to do in order to get a proper definition in this sense is to replace monopoles by quarks, and instead of a magnetic flux tube, we need a "color electric flux" tube formed.

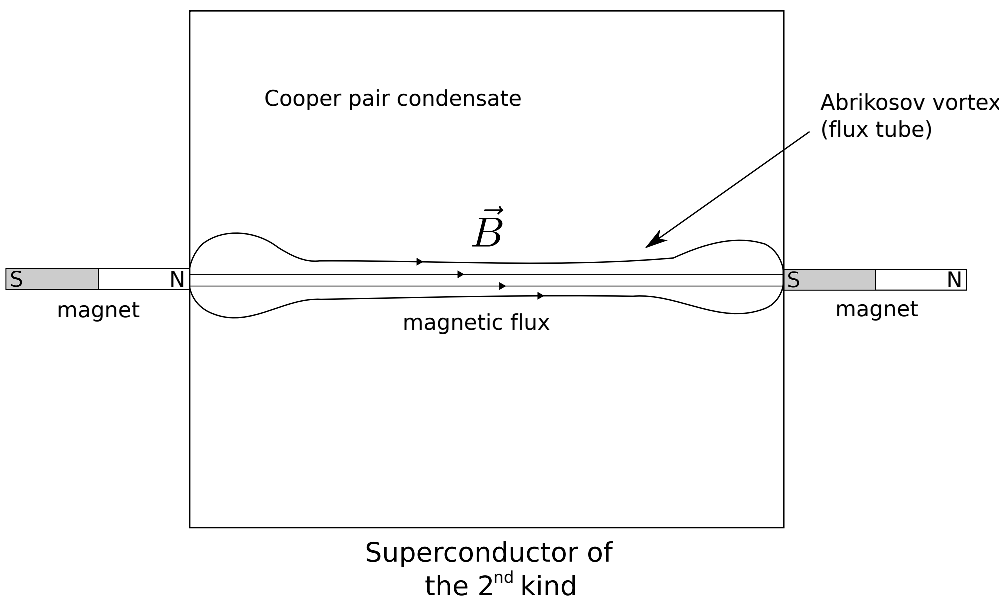


Fig. 2.2: The Meissner effect in QED [11]

In quantum field theory, above ideas can be formalized by the Landau-

Ginzburg description of the superconductor. It is given by the Abelian Higgs model:

$$\mathcal{L} = \frac{1}{4}F^2 + |D_\mu \phi|^2 + \lambda(\phi^* \phi - v^2)^2 \tag{2.2}$$

Here, in a similar way, Nielsen-Olsen vortices are formed. The behavior of these vortices and the spectrum of the above model is well known.

Even though this picture is very simple and appealing, it does not really capture the problem in its entirety. One of the main problems with this description comes from the lack of a gauge invariant description of a monopole for a Non-Abelian theory. One can define a similar object by fixing the gauge, but the results, in particular the properties of the strings will be gauge dependent [12]. In relation to this problem, the description above is essentially a mechanism for an Abelian theory, and generalization to non-Abelian theories are usually follow some procedure which is based on singling out an Abelian subgroup.

One can make another objection to the dual superconductivity based on its spectrum. If this model is a good effective description of the low energy dynamics, the lowest mass excitation of the Abelian Higgs model should correspond to those of QCD. The lowest lying spectrum of QCD is now well known from lattice calculations, and the first two are a scalar glueball with mass 1.7 GEV and a spin 2 tension glueball of mass 2.4 GEV. On the other hand, for the Abelian Higgs model, the lowest excitation are the well known scalar and vector fields. In particular the vector field, which is absent in the QCD picture is critical for the description of the superconductor. [13].

In this chapter our aim is to describe confinement by “guessing” the low energy effective theory of pure Yang-Mills using the symmetries of the theory. It is largely accepted that confinement is a property of pure gluodynamics [14], so we are interested in this part of the QCD Lagrangian only. Our approach is based on very simple principles: in a quantum field theory in order to write an effective theory that describes the long distance behavior, we do not need to know every detail of the full theory. One can construct an effective theory based on a “bottom up” approach, in which we impose constraints on our candidate Lagrangian based on the expected symmetries of the theory at low energies. In addition to that, at large separations, only the particles with no or small mass will be important, and there exists a natural candidates for such particles if the theory has spontaneously broken continuous global symmetries: the Goldstone bosons.

Even though it is true that we do not really know the low energy description of QCD, It was argued by t 'Hooft that non-Abelian gauge theories without fundamental fields exhibit a discrete magnetic $Z(N)$ symmetry, and it is also discussed that the spontaneous breaking of this symmetry is a direct indication of confinement in such theories [15]. Details of this formulation is given in section 2.1.1. If it is true that this symmetry governs the low energy dynamics of QCD, we should be able to obtain an effective theory following a symmetry based approach, as discussed above. The magnetic $Z(N)$ symmetry here is not a continuous symmetry, which means one cannot directly employ Goldstone's theorem to get the low energy excitations, but as we discuss in the next section, it can still be useful in describing the low energy dynamics.

2.1 An Effective Model of Confinement

2.1.1 Magnetic $Z(N)$ Symmetry in $2+1$ Dimensions

Here we present the arguments of 't Hooft regarding the existence of a $Z(N)$ symmetry, and its relation to confinement [15], [16]: our initial setup is a $SU(N)$ theory with Adjoint Higgs fields, so all the fields present in the theory are invariant under the center $Z(N)$ of $SU(N)$. The potential is not important for our purposes, as long as it is chosen such that Higgs mechanism takes place: gauge symmetry is broken, and the spectrum consists of the usual massive gluons and the Higgs particles; but there also exists extended soliton solutions: heavy stable magnetic vortices. These particles are heavy compared to the rest of the spectrum, but they are stable by a topological argument: the nearby field configuration is nontrivial but far away from the core, it is in pure gauge:

$$\Phi_i(x) = U(x)\phi_i, \; A_\mu = iU(x)\partial_\mu U(x) \tag{2.3}$$

.

Here ϕ_i is the vacuum expectation value of the Higgs field Φ_i.

In the presence of a vortex, the space is no more simply connected. Starting with a group element U, as we go around the vortex, it does not return to its original value, but instead goes to another element differing by a center element of the gauge group: $e^{2\pi i N}U$. Normally, such multivalued configurations are not well defined, but Adjoint Higgs fields do not feel this discontinuity by the center of the gauge group, and the energy of such a configuration is finite. In fact, given the potential one can calculate its mass. Furthermore, such a configuration cannot be smoothly deformed into the vacuum, but it is possible for N such vortex configuration to annihilate each other since for an $SU(N)$ gauge group, that configuration is equivalent to the vacuum. Explicit forms of the operators that creates these vortices are called "singular gauge transformation" can be derived using their properties.

The stability of this configuration can be explained by the properties of the vacuum manifold: there is a nontrivial mapping between the vacuum manifold and the boundary of the space, or in other words, the first homotopy group is nontrivial: $\Pi_1(SU(N)/Z(N)) = Z(N)$.

Existence of such heavy stable particles can be possible if they carry conserved quantum number, that is the magnetic $Z(N)$ - (since vortex number is preserved mod N). This is the basis of the reasoning that lead to the existence of a "magnetic $Z(N)$ symmetry" in the completely Higgled phase. Note that this symmetry is associated with the charge of the vortex and it is not the same $Z(N)$

as the center of the gauge group.

So we have established that the theory in completely Higgsed phase has a $Z(N)$ symmetry. We can keep changing the parameters smoothly such that the theory goes to the confining phase from the completely Higgs phase. Further changing the parameter we can decouple the Higgs fields completely from the gluons. We end up with a pure Yang-Mills theory, through a smooth limiting procedure which does not change the topology of the gauge group so the $Z(N)$ symmetry will also be present in this limit, even though it will be realized in a different way. The argument is that magnetic $Z(N)$ will be spontaneously broken in the confining phase. 't Hooft argues that spontaneous breaking of this discrete symmetry will lead to N different vacua characterized by the value of the topological charge $Z(N)$. These vacua will be separated by Bloch walls with energy, which are closed strings in 2+1 dimensions. If we introduce heavy quarks in the fundamental representation, they will be confined by such a string, providing a linear confinement.

This formulation is the basis of our ideas in constructing in an effective theory of confinement. The vortices discussed here are analogs of Nielsen-Olesen vortices. As we will see below, these ideas and the effective theories are studied in detail $2+1$ dimension, where things are under control and the symmetry is well understood. Even though 't Hooft's arguments can also be generalized to 3+1 dimensions, rest of the construction cannot be generalized in a straightforward manner.

2.1.2 The Effective Theory in 2+1 Dimensions

In 2+1 dimensions one has a very simple and straightforward relation between confinement and spontaneous breaking of a discrete magnetic symmetry as discussed above [15], [17]. Additionally, in 2+1 dimensions, non-Abelian gauge theories exhibiting the phenomenon of confinement are related to Abelian theories on the effective field theory level by a simple symmetry breaking deformation. This deformation breaks the continuous $U(1)$ magnetic symmetry of an Abelian theory down to a discrete group Z_N for $SU(N)$ gauge theories with adjoint matter. The mere fact of the presence of this deformation, coupled with the spontaneous breaking of the residual discreet group leads with certainty to a confining long distance behavior [17], [18].

Before discussing the model itself, we will recall briefly the story of 2+1 dimensional gauge theories. As a prototypical Abelian gauge theory consider scalar QED. It posesses a continuous $U_\mu(1)$ global symmetry generated by the total magnetic flux through the plane of the system, $\Phi = \int d^2xB(x)$. The order parameter for this symmetry is one complex field V, which creates point-like magnetic vortices. In the Coulomb phase $\langle V\rangle = v \neq 0$ and $U_\mu(1)$ is spontaneously broken. The low energy dynamics is qualitatively described by the effective "dual"

Lagrangian

$$L = -\partial^\mu V \partial_\mu V^* - \lambda(V^*V - \frac{e^2}{8\pi})^2 \qquad (2.4)$$

The Goldstone boson of the $U_\mu(1)$ symmetry breaking is identified with the massless photon, while the electric charge in the dual formulation is the topological charge of the field V

$$J_\mu = \frac{1}{e}\epsilon_{\mu\nu\lambda}\partial_\nu V^* \partial_\lambda V \qquad (2.5)$$

A charged state of QED in the effective description appears as a hedgehog - like soliton of V: $V(x) = ve^{i\theta(x)}$, with $\theta = \tan^{-1} y/x$.

This effective formulation is also a good basis for description of confinement in non-Abelian theories. In particular the effective theory of a weakly interacting $SU(N)$ model is essentially the same as eq.(2.4), except with a potential which breaks the magnetic $U_\mu(1)$ symmetry down to Z_N

$$L = -\partial^\mu V \partial_\mu V^* - \lambda(V^*V - \frac{e^2}{8\pi})^2 + \mu(V^N + V^{*N}) \qquad (2.6)$$

The perturbation reduces the infinite degeneracy of vacua of the Abelian theory to a finite number of degenerate vacuum states connected by the Z_N transformations. As a result, a charged state does not have a rotational symmetry anymore, but the winding is concentrated within a quasi one dimensional “flux tube” [17].

This is a very simple picture, and a very appealing one inasmuch as it identifies charged objects with topological defects which inherently have long range interactions due to their topological nature. It also identified photons with Goldstone bosons, providing a natural symmetry based explanation for masslessness of the photon.

It is natural to ask, whether in 3+1 dimensions one can have a similar description, which encompasses the massless nature of photons in QED as well as topological mechanism of confinement in non-Abelian theories. The situation here of course, is much more complicated. First of all, in 3+1 dimensions photons are vector particles and so it is not clear at all whether they can be understood as Goldstone bosons. Even if such a case can be made for photons, it is not easy to identify the relevant conserved current that breaks spontaneously. It is clear that the current has to be related to the dual field strength $\tilde{F}_{\mu\nu}$ consistent with the fact that photons have spin one [19]. The dual field strength, however has no local order parameter, and thus is an object of a very different kind than ordinary vector currents, which we are used to deal with. Another complication is, that classical effective description assumes weakly interacting theory, while QCD is of course strongly interacting.

All these are difficult questions, to which we do not attempt to provide answers here. Instead we will be content to construct a model that encompasses the following basic features:

1. The model should describe dynamics of scalar fields, and contain no fundamental gauge fields.

2. The model should have the limit (putative "Abelian regime") in which

it has two massless degrees of freedom, which are identified as Goldstone bosons. These massless Goldstone bosons in our model are intended to play the role of photons.

3. In the Abelian regime the model must provide for existence of classical topological solitons, which play the role of electrically charged particles. We require the topological charge that is carried by these solitons to reflect the mapping of the spatial infinity onto the manifold of vacua, and thus be given by $\Pi_2(M)$. The energy of these solitons has to be finite in the infrared. The energy density of a soliton solution should decrease as $1/r^4$ far from the soliton core. This is nontrivial in 3+1 dimensions, since our model has no gauge fields, while scalar fields that contribute to Π_2 have to be long range.

4. Soliton must become confined in the "Non-Abelian regime", when a symmetry breaking perturbation is added. This same perturbation must eliminate massless Goldstones by explicitly breaking the (previously) spontaneously broken symmetry group down to a discrete subgroup. Confinement should be accompanied by formation of string between the solitons.

In this chapter we will present our attempt in constructing such a model in 3+1 dimensions. We will start section 2.2 by constructing a very simple theory that satisfies above requirements. In section 2.2.1 we will analyze the Abelian limit and the symmetries of the model. The model has a very large and rather puzzling global symmetry, that will be further discussed in later sections. We will show both by constructing photon states and discussing the equations of motion that even though the model has similarities to the electrodynamics, it is not an exact description. In section 2.2.2, following the guidelines from 2+1 dimensions, we will introduce a perturbation that will get us to the non-Abelian limit of the model and look for confining string solutions, which will also have rather unusual properties. A further perturbation that breaks the symmetry down to $Z(N)$ is presented in section 2.2.3. We will conclude section 2.2.4 by discussing our results.

Section 2.3 is an attempt to improve the Abelian limit of the model presented in section 2.2. We will identify the source of the problem and fix it directly by modifying the action in section 2.3.1. This modification will give us a model that is exactly equivalent to a theory of a free photon at the Hamiltonian level. Lorentz transformation properties of the electromagnetic fields discussed in section 2.3.2 shows that the fields no more transform as usual scalar field, but have "anomalous" terms in their transformations. This modified model comes with a new gauge symmetry that has no analog in 2+1 dimension, which makes the understanding of the non-Abelian limit more challenging. We discuss some ideas in this direction in the section 2.3.3.

2.2 The Curious Case of an Effective Theory

In this section we discuss a 3+1 dimensional model which has all the above features and discuss its properties, which are somewhat unusual. In particular, the requirement of the finiteness of the energy of a topological soliton in the Abelian regime is very restrictive. It leads to rather unusual properties of the confining strings in the Non-Abelian regime such as existence of an infinite number of zero modes. This degeneracy can be lifted, however that requires the addition of another perturbation which is not clearly related with breaking of symmetries of the theory. In the Abelian regime, the model contains classical solutions with magnetic charge density, and thus the effective dual field strength tensor is not conserved. Related to that, although we are able to construct solutions of equations of motion that behave as single photons, the model has no solutions that correspond to a two photon state with arbitrary polarization vectors. Even though it is clear that this model cannot be taken literally as the effective theory of QCD, it does have some similarity with Fadeev-Niemi model, that has been proposed as an effective theory of glueballs from a completely different perspective [20].

2.2.1 The Abelian Model

The Field Space and the Lagrangian.

As explained in the previous section, we wish to construct a model of scalar fields which contains two massless degrees of freedom and solitons of finite energy. The simplest option that we adopt is a theory with two scalar degrees of freedom endowed with $SU(2)$ symmetry. Spontaneous breaking of this symmetry must lead to two massless modes. Thus we choose as the configuration space the $O(3)$

nonlinear σ-model.

$$\phi^a,\ a = 1, 2, 3; \quad \phi^2 = 1 \tag{2.7}$$

The moduli space allows for the requist topology $\Pi_2(S^2)$. The topological charge associated with it, is identified with the electric charge of QED

$$Q = \frac{e}{4\pi^2} \int d^3x \epsilon_{abc} \epsilon_{ijk} \partial^i \phi^a \partial_j \phi^b \partial_k \phi^c \tag{2.8}$$

As the first task, we have to contend with the following potential problem. In a theory with the standard kinetic term, the energy of a state with non-vanishing topological charge diverges in the infrared. A typical topologically nontrivial field configuration is a rotationally symmetric hedgehog

$$\phi_h^a(x) = \frac{r^a}{|r|} f(|r|); \qquad f(|r|) \to_{r\to\infty} 1 \tag{2.9}$$

The standard two derivative kinetic energy diverges quadratically on such a configuration. In order to make the energy of the soliton finite, we need to introduce a kinetic term with more than two derivatives.

In fact, there exists a unique four derivative term which is a natural choice for a kinetic term for our model. The identification of the electric charge with the topological charge eq.(2.8) also naturally leads to the identification of the electric current as

$$J^\mu = \frac{e}{4\pi^2} \epsilon_{abc} \epsilon^{\mu\nu\lambda\sigma} \partial_\nu \phi^a \partial_\lambda \phi^b \partial_\sigma \phi^c \tag{2.10}$$

and therefore electromagnetic field tensor as

$$F^{\mu\nu} = \epsilon^{abc} \epsilon^{\mu\nu\lambda\sigma} \phi^a \partial_\lambda \phi^b \partial_\sigma \phi^c \tag{2.11}$$

Since our goal is to construct a model that resembles QED as close as possible, the natural choice for the kinetic term is the square of the field strength tensor, which is just the well known Skyrme term.

Hence we consider the model of a triplet of scalar fields defined by the following Lagrangian:

$$L = \frac{1}{16e^2} F^{\mu\nu} F_{\mu\nu} + \lambda(\phi^2 - 1)^2 \tag{2.12}$$

We note, that the sign of the F^2 term in the Lagrangian is opposite to that in QED. In the framework of eq.(2.12) the sign is determined so that the Hamiltonian is positive, rather than negative definite. This feature is common to models related by duality. For example the same is true in the 2+1 dimensional models described in the introduction, where the kinetic term in the Lagrangian of the effective theory when written in terms of the field strength tensor has the opposite sign to that of Electrodynamics. The reason for this inversion, is that while in QED the electric field is proportional to the time derivative of the basic field (in this case A_μ), in the effective dual description it is the magnetic field that contains time derivative of the vortex field V. Thus in order for the Hamiltonian of the two models in terms of E and B to be the same, the Lagrangians have to have opposite sign. This inversion of sign also takes place in our model, and is the natural consequence of the relation between the field strength tensor and the basic scalar degrees of freedom eq.(2.11).

In the strongly coupled limit $\lambda \to \infty$, the isovector ϕ has unit length, and the field strength is trivially conserved

$$\partial_\nu F^{\mu\nu} = 0 \tag{2.13}$$

This limit therefore corresponds to QED without charges. In this limit the energy of the soliton eq.(2.9) diverges linearly in the ultraviolet. At finite coupling λ the variation of the radial component of the field ϕ^a softens the UV behavior, and the soliton energy is UV finite. It is also IR finite thanks to our choice of the four derivative action. In fact on the hedgehog configuration eq.(2.9) the "electric field" decreases as $E_i(x) \propto \frac{r^1}{|r|^3}$, and the energy density away from the soliton core decreases as $1/r^4$, just like the Coulomb energy of a static electric charge in the electrodynamics.

The Equations of Motion

We now derive the equations of motion for the model. For convenience we define in the strong coupling limit

$$\phi_3 = z, \qquad \psi = \phi_1 + i\phi_2 = \sqrt{1-z^2}e^{i\chi} \tag{2.14}$$

With this parametrization one has

$$F^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta}\epsilon_{abc}\phi^a\partial_\alpha\phi^b\partial_\beta\phi^c = -2\epsilon^{\mu\nu\alpha\beta}\partial_\alpha z\partial_\beta\chi \tag{2.15}$$

The Lagrangian can be written as

$$L = \frac{1}{4e^2}(\partial_\mu z\partial_\nu\chi - \partial_\mu\chi\partial_\nu z)^2 \tag{2.16}$$

The equations of motion read

$$\partial^\mu \left[\frac{1}{e^2} \partial^\nu \chi \left(\partial_\mu z \partial_\nu \chi - \partial_\nu z \partial_\mu \chi \right) \right] = 0$$
$$\partial^\mu \left[\frac{1}{e^2} \partial^\nu z \left(\partial_\mu z \partial_\nu \chi - \partial_\nu z \partial_\mu \chi \right) \right] = 0 \tag{2.17}$$

This can be combined into

$$\frac{1}{e^2} \partial_\nu G(z,\chi) \partial_\mu \left(\partial_\mu z \partial_\nu \chi - \partial_\nu z \partial_\mu \chi \right) = \frac{1}{e^2} \partial_\nu \left[G(z,\chi) \partial_\mu \left(\partial_\mu z \partial_\nu \chi - \partial_\nu z \partial_\mu \chi \right) \right] = 0 \tag{2.18}$$

where $G(z,\chi)$ is an arbitrary function of two variables. These equations have a form of conservation equations for currents defined as

$$J^G_\nu = G(z,\chi) \partial^\mu \left(\partial_\mu z \partial_\nu \chi - \partial_\nu z \partial_\mu \chi \right) \tag{2.19}$$

The Symmetries and the Correspondence to Electrodynamics.

The conserved currents of eq.(2.19) can indeed be identified with conserved Noether currents. An unexpected consequence of the choice of the Skyrme term as the kinetic term in the Lagrangian, is that the global symmetry group of the model is much larger than the $SO(3)$ group we have started with.

To see this note that the field strength as defined in eq.(2.11) is related to an infinitesimal area on a configuration space. Let us be more precise here. A given field configuration $\phi^a(x)$ defines a map from space-time to a sphere S^2. Consider a given component the field strength tensor, say F_{12} at some point x. To calculate it in terms of ϕ we consider three infinitesimally close points $A \equiv x^\mu$,

$B \equiv x^\mu + \delta^{\mu 1} a$ and $C \equiv x^\mu + \delta^{\mu 2} a$. These three points in space-time map into three infinitesimally close points on the sphere $\phi^a(A)$, $\phi^a(B)$, $\phi^a(C)$. The field strength F_{12} is proportional (up to the factor a^{-2}) to the area of the infinitesmal triangle on S^2 defined by these three points. Since the action of our toy model depends only on $F^{\mu\nu}$, it is clear that any reparametrization of the sphere which preserves area is an invariance of our action.

Thus the $SO(3)$ global symmetry we started with, is a *small* subgroup of the area preserving diffeomorphisms of S^2, which we denote $Sdiff(2)$ [21]. This is the group of canonical transformations of a classical mechanics of one degree of freedom. The infinitesimal symmetry transformation in terms of z and χ is

$$z \to z + \frac{\partial G}{\partial \chi}; \quad \chi \to \chi - \frac{\partial G}{\partial z} \tag{2.20}$$

with arbitrary $G(z, \chi)$. The appropriate Noether currents are precisely those of eq.(2.19) and the equations of motion are indeed equivalent to conservation equations of these currents.

It is amusing to note that this symmetry is similar to the world sheet diffeomorphism invariance of the Nambu-Goto string. Indeed, if one thinks of the fields z and χ as the world sheet string coordinates, the world sheet diffeomorphism invariance is precisely eq.(2.20) [1]. Although our setup looks very different from a string theory, there may be more to this analogy than meets the eye, as the basic "order parameters" of the magnetic symmetry in QED_4 are indeed magnetic

[1] We thank Michael Lublinsky for pointing this out to us.

vortex strings [19]. The S^2 topology of the world sheet then implies closed string loops. We will not develop this analogy any further here, and instead will return to the field theoretical approach.

The enhanced symmetry means that the moduli space is much larger than S^2 as would be naively the case for symmetry breaking pattern $SO(3) \to SO(2)$. Any configuration $\phi^a(x)$ that maps the configuration space into an arbitrary **one dimensional** curve on S^2 has vanishing action and is thus a point on the moduli space. The moduli space is therefore the union of maps $\phi^a(x)$ that map R^4 to L, where L is an arbitrary point or a one dimensional curve on S^2.

Nevertheless, even though the moduli space is not a simple sphere, the topological charge Q is quantized for any smooth classical configuration of fields $\phi(x)$. A twist in the tale is that there are many more degenerate soliton configurations than just the rotationally invariant hedgehog of eq.(2.9). Any $Sdiff(2)$ transformation corresponding to an arbitrary regular function G of eq.(2.20) applied to the configuration eq.(2.9) generates a soliton configuration $\phi_h^{aG}(x)$ which is degenerate in energy with $\phi_h^a(x)$. Note, that although these are different field configurations, they all correspond to the same electric field $E_i = \epsilon_{ijk}\epsilon^{abc}\phi^a\partial_j\phi^b\partial_k\phi^c$, since the electric (as well as magnetic) field is invariant under the action of $Sdiff(2)$ transformations.

Plane waves - photon states.

Returning to the Lagrangian eq.(2.12), the natural question to ask is how much of a relation does it have with electrodynamics. With the identification eq.(2.11), we know that the field strength $F^{\mu\nu}$ satisfies half of Maxwell's equations. The equations of motion eq.(2.18) are quite reminisicent of the other half of Maxwell's equations. They can be rewritten in terms of $F^{\mu\nu}$ as

$$[\partial_\nu G(z,\chi)]\partial_\mu \tilde{F}^{\mu\nu} = 0 \qquad (2.21)$$

Thus, any configuration of the fields $z,\ \chi$ that satisfies $\partial_\mu \tilde{F}^{\mu\nu} = 0$, also satisfies the equations of motion of our model. The converse is not true: there are solutions of the equations of motion eq.(2.18) which do not satisfy the equations of motion of electrodynamics. We give an example of such a solution in Appendix 4.

The model eq.(2.12) is therefore not equivalent to electrodynamics. Nevertheless, it is interesting to ask whether the spectrum of solutions of eq.(2.12) contains basic excitations of QED, in particular the photons. This is a slight abuse of language, since we are dealing with a classical theory. We will nevertheless refer to plane wave configurations of $F^{\mu\nu}$ with light-like momentum as photons.

Our aim in this section is to show that the free wave excitations are indeed solutions of equations eq.(2.18). To this end consider the configuration

$$\chi(x) = A\epsilon^\mu x_\mu; \qquad z(x) = \sin k^\mu x_\mu \qquad (2.22)$$

where the vector ϵ^μ is normalized as $\epsilon^\mu \epsilon_\mu = -1$. On this configuration

$$\tilde{F}^{\mu\nu} = A(\epsilon^\mu k^\nu - \epsilon^\nu k^\mu) \cos k \cdot x \tag{2.23}$$

Thus

$$\partial_\mu \tilde{F}^{\mu\nu} = -A\Big[(\epsilon \cdot k) k^\nu - k^2 \epsilon^\nu\Big] \sin k \cdot x \tag{2.24}$$

This vanishes, provided the momentum vector is light-like and the polarization vector ϵ is perpendicular to k:

$$k^2 = 0; \quad \epsilon \cdot k = 0 \tag{2.25}$$

For a given light-like momentum k_μ this equation has three independent solutions for ϵ^μ. One of them, however is proportional to k_μ itself. With this polarization vector, the field strength tensor vanishes. Thus there are two independent polarization vectors $\epsilon^\mu_\lambda, \quad \lambda = 1, 2$ that correspond to plane wave solutions for $F^{\mu\nu}$. Just like in QED, it is convenient to choose the polarization vectors so that their zeroth component vanishes $\epsilon^\mu_\lambda = (0, \epsilon^i_\lambda)$. The constant A is the overall amplitude of the electromagnetic wave, whose square is proportional to the number of photons with a given momentum and a given polarization in the wave.

The arbitrariness in the choice of the polarization vectors is precisely the same as in the choice of polarization vectors in electrodynamics

$$\epsilon^\mu \to \epsilon^\mu + a k^\mu \tag{2.26}$$

Note that this shift of polarization vector is affected by the transformation

$$\chi \to \chi + a \arcsin z \tag{2.27}$$

which is one of the $Sdiff(2)$ transformations eq.(2.20). In fact the two field configurations eq.(2.22) can be transformed by any element of $Sdiff(2)$ without changing $F^{\mu\nu}$.

The solution (equations 2.22, 2.23, 2.25) describes a state in all respects equivalent to the freely propagating photon, and we will refer to it as such. The setup here is dual to the usual free QED. Normally one introduces the vector potential A_μ via $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. This relation potentiates the homogeneous Maxwell's equation $\partial_\mu \tilde{F}^{\mu\nu} = 0$. However, in the free chargeless QED entirely analogously one can potentiate the other Maxwell equation by introducing the dual vector potential via $\tilde{F}_{\mu\nu} = \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu$. The dynamics of the dual vector potential $\tilde{A}_\mu$ is identical to that of A_μ, and it can be expanded in exactly the same polarization basis as A_μ. In this formulation QED poccesses a dual gauge symmetry $\tilde{A}_\mu \to \tilde{A}_\mu + \partial_\mu \lambda(x)$.

To make the correspondence between our model and the Electrodynamics more transparent, we can define a dual vector potential

$$\tilde{A}_\mu = z\partial_\mu \chi \tag{2.28}$$

Under the $Sdiff(2)$ transformation eq.(2.20) it transforms as

$$\tilde{A}_\mu \to \tilde{A}_\mu + \partial_\mu [G - z\frac{\partial G}{\partial z}] \tag{2.29}$$

which has a form reminiscent of the dual gauge transformation in Electrodynamics with the gauge function $\lambda(x) = G - z\frac{\partial G}{\partial z}$.

The analogy of eq.(2.28) with the dual vector potential of QED is suggestive, but one has to be aware that this is only an analogy rather than equivalence. First, the transformation eq.(2.29) is not a gauge transformation, but rather the action of a global symmetry transformation of the Lagrangian on $\tilde{A}_\mu$ of eq.(2.28). More importantly, the vector field defined in eq.(2.28) in terms of two scalar fields is not the most general form of a vector field, even allowing for a possible gauge ambiguity. For that reason the variation of the Lagrangian with respect to such a constrained vector potential does not lead to the homogeneous Maxwell's equation directly, but instead to eq.(2.21), which allows additional solutions.

Finally we note, that the solution eq.(2.22) gives a nice and simple interpretation for the properties of the photon states in terms of the effective theory. The photon momentum is the momentum associated with the variation of the third component of the isovector ϕ^a, while the direction of the photon polarization vector is determined by the spatial variation of the phase χ.

Although a plane wave $\tilde{F}_{\mu\nu}$ solves the equations of motion, the equations eq.(2.21) are not linear in the basic field variables, and thus an arbitrary superposition of two such solutions, itself is not a solution. We may try to construct a two photon state by slightly extending the ansatz eq.(2.22).

$$\chi = \lambda_\mu x_\mu; \quad z = a \sin k^\mu x_\mu + b \sin p^\mu x_\mu \tag{2.30}$$

with k^μ and p^μ - both lightlike vectors, $\lambda^\mu k_\mu = \lambda^\mu p_\mu = 0$ and $\lambda^\mu \lambda_\mu = -1$. The latter two conditions can be satisfied by taking

$$\lambda^\mu = \alpha\Big[\epsilon^\mu - \frac{\epsilon\cdot k}{k\cdot p}p_\mu - \frac{\epsilon\cdot p}{k\cdot p}k_\mu\Big] \tag{2.31}$$

with an arbitrary vector ϵ^μ and an appropriate normalization constant α.

The dual field strength tensor can be written as:

$$\tilde{F}_{\mu\nu} = a(k_\mu\epsilon^k_\nu - k_\nu\epsilon^k_\mu)\cos k\cdot x + b(p_\mu\epsilon^p_\nu - p_\nu\epsilon^p_\mu)\cos p\cdot x \tag{2.32}$$

with the polarization vectors

$$\epsilon^k_\mu = \lambda_\mu - \frac{\lambda_0}{k_0}k_\mu; \quad \epsilon^p_\mu = \lambda_\mu - \frac{\lambda_0}{p_0}p_\mu; \tag{2.33}$$

This is a bona fide two photon state. Unfortunately by varying λ_μ at fixed p and k one cannot obtain two most general polarization vectors for photons with momenta k and p. This is obvious since both polarization vectors ϵ^k and ϵ^p have equal component in the direction perpendicular to the plane spanned by p^i; k^i. Thus we are lacking one degree of freedom to be able to construct a two photon state with arbitrary polarizations of both photons. In Appendix 4 we show that this is problem is not restricted to our ansatz eq.(2.30), but is a genuine limitation of our Lagrangian.

2.2.2 The Non-Abelian perturbation and the string solution.

In analogy with 2+1 dimensions we now perturb the theory with the simplest perturbation which breaks the $O(3)$ global symmetry. This perturbation should eliminate the vacuum degeneracy associated with the spontaneous symmetry breaking.

We find it convenient to choose a potential that fixes the vacuum expectation value of the field z to be equal to unity. We thus consider the Lagrangian

$$L = \frac{1}{16e^2} F^{\mu\nu} F_{\mu\nu} - \frac{2}{e^2} \Lambda^2 (z-1)^2 \tag{2.34}$$

The perturbation breaks not only the $SO(3)$ symmetry, but also generic $Sdiff(2)$ transformations. Nevertheless, the subgroup generated by

$$\chi \to \chi - \frac{dG(z)}{dz} \tag{2.35}$$

remains unbroken. We keep this in mind throughout the discussion of this section.

The equations of motion now are

$$\begin{aligned} \partial^\mu \left[\frac{1}{e^2} \partial^\nu \chi \left(\partial_\mu z \partial_\nu \chi - \partial_\nu z \partial_\mu \chi \right) \right] &= \frac{4}{e^2} \Lambda^2 (z-1) \\ \partial^\mu \left[\frac{1}{e^2} \partial^\nu z \left(\partial_\mu z \partial_\nu \chi - \partial_\nu z \partial_\mu \chi \right) \right] &= 0 \end{aligned} \tag{2.36}$$

With this perturbation there are no finite energy solutions with non-vanishing topological charge Q. Instead, we expect to find translationally invariant string-like solution. In the presence of a soliton and anti-soliton such strings will connect the two and will provide for a linear confining potential. To find such a solution consider a static field configuration translationally invariant in the third direction. For such a configuration the only non-vanishing components of $F_{\mu\nu}$ are:

$$F^{03} = 2\epsilon^{ij} \partial_i z \partial_j \chi \tag{2.37}$$

We look for a solution invariant under rotations in the plane perpendicular to the string

$$\chi(x) = \theta(x); \qquad z(x) = z(r) \tag{2.38}$$

where r and θ are the polar coordinates in the x_1, x_2 plane. Such a configuration has a unit winding in the x_1, x_2 plane which is precisely what one expects from the string connecting the soliton and anti-soliton. The soliton resides at some very large negative value of x_3. Far to the left of the soliton the field configuration must be vacuum $\phi^1 = \phi^2 = 0;\ z = 1$. Thus the topological charge calculated on a surface enclosing such a soliton is equal to the two dimensional topological charge - the winding of the phase χ on any plane pierced by the string. An identical argument applies for the anti-soliton, which resides at large positive value of x_3. Thus indeed our ansatz is appropriate for the string connecting a soliton and an anti-soliton residing far apart. For our ansatz the equation of motion for the field χ is trivially satisfied. The equation for z becomes

$$4z'' = 4\Lambda^2(z-1) \tag{2.39}$$

where $z' \equiv \frac{dz}{dr^2}$

For the solution to be well defined in the middle of the string and approach vacuum far away from it, z must have the asymptotic behavior:

$$z(0) = -1,\ \ z(\infty) = 1 \tag{2.40}$$

The solution is easy to find

$$z(r^2) = 1 - 2\exp\{-\Lambda r^2\} \tag{2.41}$$

The solution has some intuitively expected properties: it has a finite width, outside which the fields approach vacuum, while inside the string the field values are different from the vacuum and thus it carries a finite energy density. The string tension can be calculated exactly

$$\sigma = 8\pi\frac{\Lambda}{e^2} \tag{2.42}$$

Nevertheless, the solution is rather peculiar, since it does not approach the vacuum exponentially, but rather as a Gaussian. the string therefore has a very sharply defined width, outside of which the vacuum is reached very quickly. In a theory with a finite mass gap and a finite number of massive excitations such behavior is impossible. This strange behavior can be traced back to our non-canonical kinetic term. Indeed, for simple dimensional reasons, the kinetic energy for a rotationally invariant configuration is a second derivative with respect to r^2 rather than r, which results in a Gaussian rather than exponential decay of the solution.

2.2.3 The Z_N preserving perturbation.

The perturbation considered in the last section was the simplest potential that breaks the $SO(3)$ as well as the $Sdiff(2)$ symmetries but leaves an $O(2)$ subgroup of $SO(3)$ and large subgroup $Sdiff(2)$, eq.(2.35) intact. Following the parallel with the 2+1 dimensional discussion, we expect the global symmetries to be broken down to Z_N if our effective theory has a chance of mimicking some properties of $SU(N)$ gauge theories. In this section therefore we consider an additional perturbation, which breaks the remaining $O(2)$ symmetry down to the Z_N subgroup. We modify the Lagrangian to

$$\begin{aligned} L =& \frac{1}{16e^2}F^2 - \frac{2}{e^2}\Lambda^2(z-1)^2\Big[1 - \mu(\psi^N + \psi^{\star N})\Big] \\ =& \frac{1}{16e^2}F^2 - \frac{2}{e^2}\Lambda^2(z-1)^2\Big[1 - 2\mu(1-z^2)^{N/2}\cos N\chi\Big] \end{aligned} \tag{2.43}$$

We will only consider regime where the minimum of the potential is at $z = 1$. It is easy to see that this is the case as long as

$$\mu < \frac{1}{2} \tag{2.44}$$

For field configurations which do not depend on the longitudinal coordinate x_3, the energy per unit length is given by

$$E = \int d^2x \frac{1}{2e^2}(\epsilon_{ij}\partial_i z \partial_j \chi)^2 + \frac{2}{e^2}\Lambda^2(z-1)^2\Big[1 - 2\mu(1-z^2)^{N/2}\cos N\chi\Big] \tag{2.45}$$

Perturbative solution

Let us first consider the perturbation to be small, $\mu \ll 1$, and find the first order corrections to the string solution of the previous section. We take the following ansatz for the perturbative solution:

$$z(r,\theta) = z(r); \chi = \theta + \chi_1(r,\theta) = \theta + f(r)\sin N\theta \tag{2.46}$$

where $z(r)$ is given by eq.(2.41). This is not the most general form of the perturbation, but which nevertheless yields a solution to first order in μ, as we now show.

To first order in μ the equation for χ_1 is

$$\frac{1}{e^2}8N^2(z')^2 f\sin N\theta = \frac{1}{e^2}N\mu\Lambda^2(z-1)^2(1-z^2)^{N/2}\sin N\theta \tag{2.47}$$

solved by

$$f(r^2) = \frac{\mu}{N}\left[2e^{-\Lambda r^2}(1-e^{-\Lambda r^2})\right]^{N/2} \tag{2.48}$$

The second minimization equation reads

$$\frac{1}{e^2}8N\left[2z''f+z'f'\right] = \frac{1}{e^2}4\mu\Lambda\left[2(z-1)(1-z^2)^{N/2}-Nz(z-1)^2(1-z^2)^{N/2-1}\right] \tag{2.49}$$

It is straightforward to check that this equation is indeed satisfied by the perturbative solution eq.(2.48) and $z(r)$ given by eq.(2.41).

Calculating the longitudinal electric field corresponding to this solution we find

$$F^{03} = -4\Lambda e^{-\Lambda r^2}\left[1+\mu\Big(2e^{-\Lambda r^2}(1-e^{-\Lambda r^2})\Big)^{N/2}\cos N\theta\right] \tag{2.50}$$

The electric field is concentrated within the radius $\Lambda^{1/2}$ in the transverse plane, with the Z_N invariant perturbation providing a slight angular modulation, as expected.

The General Solution

Let us now consider the minimization equations beyond perturbation theory and beyond the simple ansatz of the previos subsection. Minimization of the energy functional eq.(2.45) gives the following equations:

$$\begin{aligned} \frac{1}{e^2}\epsilon_{ij}\partial_j\chi\partial_i F &= \frac{\partial U}{\partial z} \\ \frac{1}{e^2}\epsilon_{ij}\partial_j z\partial_i F &= -\frac{\partial U}{\partial \chi} \end{aligned} \tag{2.51}$$

where

$$F \equiv \frac{1}{2}F^{03} = \epsilon_{ij}\partial_i z\partial_j\chi \tag{2.52}$$

and U is the potential energy in eq.(2.45).

Multiplying the first equation by $\partial_k z$, the second by $\partial_k\chi$ and subtracting, we find:

$$\frac{1}{2e^2}\partial_k(F^2) = \partial_k U \tag{2.53}$$

For any finite energy configuration the electric field vanishes at infinity. Since the potential U appearing in eq.(2.45) also vanishes at infinity, the integration constant needed to integrate eq.(2.53) is trivial and we find

$$F^2 = 2e^2U; \qquad F = \sqrt{2e^2U} \tag{2.54}$$

To solve the equation it is convenient to use the coordinates ($\tau = r^2, \theta$):

$$\partial_\tau z\partial_\theta\chi - \partial_\theta z\partial_\tau\chi = \sqrt{\frac{1}{2}e^2U} \tag{2.55}$$

This equation obviously has many solutions. The infinite degeneracy follows from a symmetry of the energy functional eq.(2.45). Consider a transformation

$$(z(x), \chi(x)) \to (z(x'), \chi(x')); \qquad \frac{\partial(x'^1, x'^2)}{\partial(x^1, x^2)} = 1 \tag{2.56}$$

Such transformations form the group of area preserving diffeomorphisms on a plane $SDiff(R^2)$. Note that it is a diffeomorphism transformation on the co-ordinate space rather than on the field space, and thus is very different from $Sdiff(2)$, which we discussed in the previous section. This transformation is clearly a symmetry of the energy functional eq.(2.45). Thus, starting from any string solution one can generate an infinite number of degenerate solutions with the help of $SDiff(R^2)$ transformations. Note that since the longitudinal electric field is itself invariant under eq.(2.56), all these solutions have the same electric field profile.

We will discuss here two such solutions. Let us look for solution with a prescribed and simple dependence of χ on the angle: $\chi = \theta$. Eq.(2.55) then becomes an equation for z:

$$\partial_\tau z = \sqrt{\frac{1}{2}e^2 U} = \sqrt{\Lambda^2 (z-1)^2 \Big[1 - 2\mu(1-z^2)^{N/2}\cos N\theta\Big]} \tag{2.57}$$

The dependence on θ here is parametric, and so for every value of θ it is solved by

$$\tau = \int_{-1}^{z(\tau)} dz \frac{1}{\sqrt{\Lambda^2 (z-1)^2 \Big[1 - 2\mu(1-z^2)^{N/2}\cos N\theta\Big]}} \tag{2.58}$$

The solution has the correct asymptotics, since as $\tau \to \infty$ the function z has to approach unity for the RHS to diverge. In fact it is easy to find the large distance asymptotics of the solution. When z is close to unity, we can neglect the term proportional to μ in the denominator, and for the IR asymptotics we have

$$\tau = \int_{-1}^{z(\tau)} dz \frac{1}{\sqrt{\Lambda^2 (z-1)^2}} \tag{2.59}$$

which is solved by

$$z(\tau \to \infty) = 1 - 2e^{-\Lambda\tau} \tag{2.60}$$

This is the same as eq.(2.41), showing that the IR asymptotics of the string solution is unaffected by the Z_N perturbation.

Let us now consider a solution where z only has radial dependence. In this case, we have:

$$\partial_\tau z \partial_\theta \chi = \sqrt{\Lambda^2 (z-1)^2 \Big[1 - 2\mu(1-z^2)^{N/2} \cos N\chi\Big]} \tag{2.61}$$

This can be formally solved for θ at fixed r:

$$\theta = \int_0^{\chi(r,\theta)} \frac{z' d\chi}{\sqrt{\Lambda^2 (z-1)^2 \Big[1 - 2\mu(1-z^2)^{N/2} \cos N\chi\Big]}} \tag{2.62}$$

The right hand side can be expressed in terms of elliptic integrals:

$$\theta = \frac{2}{N} \frac{z'}{\sqrt{\Lambda^2 (z-1)^2 (1 - 2\mu(1-z^2)^{N/2})}} F(\frac{N\chi}{2}, \frac{4\mu(1-z^2)^{N/2}}{2\mu(1-z^2)^{N/2} - 1}) \tag{2.63}$$

where $F(\phi, m)$ is the incomplete elliptic integral of the first kind:

$$F(\phi, m) = \int_0^{\phi} (1 - m \sin \theta^2)^{-1/2} d\theta \tag{2.64}$$

The solution has to satisfy the boundary condition

$$\chi(\theta + 2\pi) = \chi(\theta) + 2\pi \tag{2.65}$$

Imposing this condition gives the equation for the radial dependence of z. Using the relation $F(\frac{k\pi}{2}, m) = kK(m)$, where $K(m)$ is the complete elliptic integral of the first kind, we have:

$$2\pi = \frac{4z'}{\sqrt{\Lambda^2(z-1)^2(1-2\mu(1-z^2)^{N/2})}} K(\frac{4\mu(1-z^2)^{N/2}}{2\mu(1-z^2)^{N/2}-1}) \tag{2.66}$$

One can easily check, that in the infrared for $z \to 1$ the equation reduces to

$$z' = \Lambda(1 - z) \tag{2.67}$$

and thus has the same asymptotics as in eq.(2.41).

2.2.4 Discussion

In this section we tried to follow the template of 2+1 dimensions and, based on a couple of simple requirements "guess" a scalar theory which could be a candidate of the effective theory of 3+1 dimensional gauge theories. The theory we ended up with is not entirely satisfactory, but it does have several interesting and intriguing features.

In the Abelian limit it has a large global symmetry group, which is spontaneously broken by lowest energy classical solutions. This symmetry is not reflected in the observables which we tentatively identified with the observables of QED. This is similar to 2+1 dimensions, where the electromagnetic field was invariant under the magnetic $U(1)$ symmetry which acted nontrivially on the vortex field. In our 3+1 dimensional model the electromagnetic field is also invariant under the action of the (large) global symmetry group $Sdiff(2)$ which is nontrivially represented on the effective scalar fields.

Just like in 2+1 dimensions, this global symmetry is broken by the lowest energy configurations. However, the situation here is more complicated. Whereas in 2+1 dimensions the symmetry breaking pattern is the standard one, in our 3+1 dimensional model the space of vacuum configurations is very large. It includes field configurations that have nontrivial spatial dependence, and thus breaks translational invariance in addition to the global $Sdiff(2)$ symmetry. In fact, it could well be that classical analysis fails in this model quite badly. Many of the classical vacua differ from each other only in finite region of space. Generically in such a case one expects that upon quantization these configurations become connected by tunneling transitions of finite probability. Thus it is natural to expect that the quantum portrait of moduli space is significantly different from the classical one.

We will address some of the issues of the Abelian limit in the next section. There, it will be shown that is it possible to modify the action of this section in such a way that, the new action will be equivalent to free electrodynamics without vectors. Curiously enough, this modification promotes the large global symmetry group of this section into a global gauge symmetry which is a new feature compared to 2+1 dimensional models.

Upon introduction of the symmetry breaking perturbation, the model provides a simple classical description of confinement similarly to the 2+1 dimensional case. However here also there are some peculiarity. In particular, string solutions are infinitely degenerate, as static energy for configurations which depend only on two coordinates has an additional diffeomorphism invariance. This is a different invariance than in the Abelian limit, as it involves diffeomorphism transformations in coordinate space rather than in field space. Nevertheless it results in degeneracy of the solutions, although all such solutions yield the same electric field. In the sense of electric field distributions, the solution seems to be unique. This again begs the question about the behavior of such a string in a quantum theory, since it carries a large entropy associated with large degeneracy.

We note that the string degeneracy is lifted if one adds the standard kinetic term for the $O(3)$ sigma model fields, $\partial^\mu \phi^a \partial_\mu \phi^a$. Addition of such a term also makes our model identical with the one proposed by Faddeev and Niemi in [20] as an effective theory of QCD. Interestingly, the picture we suggest is quite distinct from and complementary to that of [20]. Whereas the authors of [20] concentrated on closed string solutions supposedly representing glueballs, our picture is that of open strings, with the endpoints corresponding to "constituent gluons" like in 2+1 dimensions [17], [22]. The stability of closed strings in the Faddeev-Niemi model is achieved due to nontrivial twisting of the phase of the scalar field along the string. Open strings on the other hand, do not require any twist and in principle can break into shorter strings similarly to QCD. The approximate stability of relatively long strings must be due to dynamical properties of the theory which should make the endpoints sufficiently heavy [23].

Finally we note that with the standard kinetic term our model becomes very similar to CP^1 model, which has been recently discussed in the literature in relation to effective models of confinement [24].

2.3 Photons without Vector Fields

In the previous section we constructed an effective theory of scalar fields in 3+1 dimensions with certain features that could potentially mimic the low energy limit of gluodynamics. Although the main aim of this construction was to explore possible low energy representation of non-Abelian theories, a certain limit of the construction should have encompassed an Abelian gauge theory, and its simplest limit - the theory of a free photon.

The construction of section 2.2, despite having some useful features failed to describe exactly this limit. In this section we modify action and demonstrate that this new model is exactly equivalent to a theory of the free photon.

Clearly, the Abelian limit of section 2.2 failed to describe a free photon primarily because the magnetic field is not required to be divergenceless $\partial_i B_i \neq 0$. To rectify this problem, we now consider the following setup

$$\mathcal{L} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{2}(\vec{E}^2 - \vec{B}^2) \tag{2.68}$$

$$F^{\mu\nu} = g\epsilon^{\mu\nu\alpha\beta}[\epsilon^{abc}\phi_a \partial_\alpha \phi_b \partial_\beta \phi_c + (n \cdot \partial) n_\alpha \partial_\beta \Phi] \tag{2.69}$$

where $n = (1, 0, 0, 0)$ is a timelike unit vector and Φ is an additional scalar field [2].

The presence of an explicit timelike vector seems to render the model non Lorentz invariant. However, as we will see below this is not quite the case. The model does possess a Lorentz invariant superselection sector, and it is this sector that is equivalent to QED.

In this section we discuss how this modification changes the model in the Abelian limit. We will show that the theory has the same canonical structure as free electrodynamics. This includes the commutation relations between "electric" and "magnetic" fields as well as the Hamiltonian. The model is therefore equivalent to a theory of a free photon, even though it is not formulated in terms of the vector potential. We also discuss the action of Lorentz transformations on the basic degrees of freedom of the model. We show explicitly that the fields ϕ^i are not covariant scalar fields, but rather have an "anomalous" term in their Lorentz

[2] A somewhat similar model, but with a dual interpretation was considered in [26]

transformation law. With this modification we show that the model is indeed Lorentz invariant.

2.3.1 Equation of Motion and Canonical Structure

Equations of Motion

In terms of the fields of previous sections (equation 2.14) and Φ, the electromagnetic field is:

$$F^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta}[-2\partial_\beta\chi\partial_\alpha z + n_\alpha\partial_\beta\partial_0\Phi] \tag{2.70}$$

The modification does not affect the electric field:

$$E_i = 2\epsilon_{ijk}\partial_j z\partial_k\chi, \tag{2.71}$$

but now the magnetic field is:

$$B_k = [2(\partial_k\chi\partial_0 z - \partial_0\chi\partial_k z) - \partial_k\partial_0\Phi] \tag{2.72}$$

The Lagrangian equations of motion that follow from the Lagrangian eq.(2.68) are

$$\partial_0\partial_k\left[F_{ij}\epsilon^{ij0k}\right] = 0 = \partial_0\partial_k B_k \tag{2.73}$$

$$\partial_\beta\chi\partial_\alpha(F_{\mu\nu}\epsilon^{\mu\nu\alpha\beta}) = 0 = \partial_k\chi\partial_\alpha(F_{\mu\nu}\epsilon^{\mu\nu\alpha k}) = \partial_k\chi(\partial_0 B_k + (\partial\times E)_k) \tag{2.74}$$

$$\partial_\beta z \partial_\alpha (F_{\mu\nu}\epsilon^{\mu\nu\alpha\beta}) = 0 = \partial_k z \partial_\alpha (F_{\mu\nu}\epsilon^{\mu\nu\alpha k}) = \partial_k z (\partial_0 B_k + (\partial \times E)_k) \tag{2.75}$$

Eq.(2.73) is a local conservation equation of a "magnetic charge density" $\partial_k B_k$. It ensures that the Hilbert space of the theory is divided into "super-selection sectors" with fixed value of the magnetic charge density. In order to preserve translational invariance we limit ourselves to the sector with $\partial_k B_k = 0$. Our considerations in the rest of this chapter pertain to this super-selection sector alone.

Using this constraint on the magnetic field, equations eq.(2.74) and eq.(2.75) can be inverted, with the result[3]

$$\partial_0 B_k + (\partial \times E)_k = 0 \tag{2.76}$$

Recall that with the field strength components given by eq.(3.102), the equation

$$\partial_\mu F^{\mu\nu} = 0 \tag{2.77}$$

is satisfied identically. We thus have the full set of Maxwell's equations.

[3] Strictly speaking there is an ambiguity in the inversion of eqs.(2.74,2.75). The general solution is $\partial_0 B_k + (\partial \times E)_k = \alpha E_k$ with an arbitrary constant α. However, given that B and $\partial \times E$ are pseudo vectors while E is a vector, a non vanishing value of α would violate parity. Requiring parity invariance of the equations resolves the ambiguity and sets $\alpha = 0$.

The Hamiltonian.

We now demonstrate that the Hamiltonian and the canonical commutation relations of the electromagnetic fields in our model are identical to those in pure QED.

The canonical momenta can be calculated from equation (3.102) as :

$$p_z = \frac{\delta L}{\delta \partial_0 z} = F_{ij}\epsilon^{ij0k}\partial_k \chi = 2B_k \partial_k \chi = 2\partial_k \chi[2(\partial_k \chi \partial_0 z - \partial_0 \chi \partial_k z) - \partial_k \partial_0 \Phi] \quad (2.78)$$

$$p_\chi = \frac{\delta L}{\delta \partial_0 \chi} = F_{ij}\epsilon^{ijk0}\partial_k z = -2B_k \partial_k z = -2\partial_k z[2(\partial_k \chi \partial_0 z - \partial_0 \chi \partial_k z) - \partial_k \partial_0 \Phi] \quad (2.79)$$

$$p_\Phi = \frac{\delta L}{\delta \partial_0 \Phi} = \frac{1}{2}\partial_k (F_{ij}\epsilon^{ij0k}) = \partial_k B_k = \partial_k[2(\partial_k \chi \partial_0 z - \partial_0 \chi \partial_k z) - \partial_k \partial_0 \Phi] \quad (2.80)$$

It is a straightforward matter to express the time derivative of χ and z as:

$$\dot{\chi} = \frac{1}{E^2}[p_z(z\chi) + p_\chi \chi^2 + \epsilon_{ijk}\dot{\Phi}_i E_j \chi_k] \quad (2.81)$$

$$\dot{z} = \frac{1}{E^2}[p_z z^2 + p_\chi (z\chi) + \epsilon_{ijk}\dot{\Phi}_i E_j z_k] \quad (2.82)$$

The time derivative of Φ is related to canonical momenta via:

$$p_\Phi = \partial_k \left[\frac{1}{E^2}\epsilon_{klm}E_l \left(p_z z_m + p_\chi \chi_m\right) - \frac{1}{E^2}E_k E_i \dot{\Phi}_i\right] \quad (2.83)$$

or in terms of a "vector potential"

$$A_k = \frac{1}{E^2}\epsilon_{klm}E_l\left(p_z z_m + p_\chi \chi_m\right) \tag{2.84}$$

as

$$p_\Phi = \partial_k\left(A_k - \hat{E}_k\hat{E}_i\dot{\Phi}_i\right) \tag{2.85}$$

The Hamiltonian is then calculated as:

$$H = \int d^3x\left[p_z\dot{z} + p_\chi\dot{\chi} + p_\Phi\dot{\Phi} - L\right] = \int d^3x\frac{1}{2}\left(E^2 + B^2\right) \tag{2.86}$$

where we have neglected a boundary term $\int d^3x\partial_k\left(B_k\dot{\Phi}\right)$.

Canonical structure

To show that our model is equivalent to QED we need to make sure that the canonical commutation relations of E_i and B_i are identical in the two theories.

First off all, since all components of the electric field in our model are functions only of coordinates and not canonical momenta, they commute with each other

$$[E_i(x), E_j(y)] = 0 \tag{2.87}$$

Our next goal is to calculate the commutator between electric and magnetic field. In order to do that, we set $p_\Phi = 0$, as we are only interested in this super selection sector of the theory. Then, eq.(2.85) becomes

$$\frac{\partial_k A_k}{E} = \hat{E}_k \partial_k \left(\frac{\hat{E}_i \dot{\Phi}_i}{E} \right) \tag{2.88}$$

where we have used $\partial_k E_k = 0$.

The formal solution of this equation can be obtained as

$$\hat{E}_i \dot{\Phi}_i = E(x) \int_{-\infty}^{x} dl_C \frac{\partial_k A_k}{E} \tag{2.89}$$

where the integral is along the contour C which starts at x and goes to infinity (boundary of space). The contour is everywhere parallel to the direction of the electric field.

Using the definition, we have:

$$B_k = A_k - E_k \int_{-\infty}^{x} dl_C \frac{\partial_m A_m}{E} \tag{2.90}$$

As an intermediate step for the calculation of the commutator $[E, B]$ we consider

$$\begin{aligned}
[E_i(x), A_k(y)] &= 2i \frac{E_l(y)}{E^2(y)} \epsilon_{iab} \epsilon_{klm} \left[\partial_a^x \delta(x-y) \chi_b(x) z_m(y) + \partial_b^x \delta(x-y) z_a(x) \chi_m(y) \right] \\
&= 2i \frac{E_l(y)}{E^2(y)} \epsilon_{iab} \epsilon_{klm} \partial_a^x \delta(x-y) [\chi_b(y) z_m(y) - z_b(y) \chi_m(y)] \\
&= i \hat{E}_l(y) \hat{E}_c(y) \epsilon_{iab} \epsilon_{klm} \epsilon_{cmb} \partial_a^x \delta(x-y) \\
&= i \left[\epsilon_{iak} - \hat{E}_b(y) \hat{E}_k(y) \epsilon_{iab} \right] \partial_a^x \delta(x-y)
\end{aligned} \tag{2.91}$$

Using this, we can calculate

$$
\begin{aligned}
[E_i(x), B_k(y)] &= [E_i(x), A_k(y)] - E_k(y) \int_{-\infty}^{y} dl_C \frac{\partial_m^t \left[E_i(x), A_m(t)\right]}{E(t)} \\
&= [E_i(x), A_k(y)] \\
&- E_k(y) \int_{-\infty}^{y} dl_C \frac{1}{E(t)} \partial_m^t \left[\left(\epsilon_{iam} - \hat{E}_b(t)\hat{E}_m(t)\epsilon_{iab} \right) \partial_a^x \delta(x-t) \right] \\
&= [E_i(x), A_k(y)] + E_k(y) \int_{\infty}^{y} dl_C \hat{E}_m(t) \partial_m^t \left(\frac{\hat{E}_b(t)}{E(t)} \epsilon_{iab} \partial_a^x \delta(x-t) \right) \\
&= i\epsilon_{iak} \partial_a^x \delta(x-y)
\end{aligned}
\tag{2.92}
$$

where we have used the fact that the integration contour C is defined to run in the direction of electric field, and have assumed that the fields decrease fast enough at the boundary.

The commutator eq.(2.92) coincides with the corresponding commutator in QED.

We now turn to the commutator of components of magnetic fields.

It is straightforward to show that $[B_i(x), B_a(y)] = 0$ as long as the curve C_x that defines $B_i(x)$ in eq.(2.90) does not contain the point y, and C_y does not contain x. When this condition is not met, the direct calculation of the commutator is not straightforward. Instead of attempting it, we take an indirect way. A set of relations that involve the commutator in question are easily obtained.

Consider for instance

$$[B_i(x)\partial_i\chi(x), B_j(y)\partial_j z(y)] = [p_z(x), p_\chi(y)] = 0 \tag{2.93}$$

Trivially:

$$\begin{aligned}
&B_i(x)\partial_j z(y)[\partial_i\chi(x), B_j(y)] + B_j(y)\partial_i\chi(x)[B_i(x), \partial_j z(y)] \\
&+ \partial_i\chi(x)\partial_j z(y)[B_i(x), B_j(y)] \\
&= \left(B_i(x)\partial_j z(y)\partial_i^{(x)}\frac{\partial A_j(y)}{\partial p_\chi(x)} - B_j(y)\partial_i\chi(x)\partial_j^{(y)}\frac{\partial A_i(x)}{\partial p_z(y)}\right) \\
&+ \partial_i\chi(x)\partial_j z(y)[B_i(x), B_j(y)] \\
&= (B_i(y)\partial_i^{(x)}\delta(x-y) + B_i(x)\partial_i^{(y)}\delta(x-y)) + \partial_i\chi(x)\partial_j z(y)[B_i(x), B_j(y)] \\
&= \partial_i\chi(x)\partial_j z(y)[B_i(x), B_j(y)] = 0
\end{aligned} \tag{2.94}$$

Here we used the fact that $E_k z_k = E_k\chi_k = 0$ and the constraint $\partial_i B_i = 0$. Similarly

$$\partial_i z(x)\partial_j z(y)[B_i(x), B_j(y)] = \partial_i\chi(x)\partial_j\chi(y)[B_i(x), B_j(y)] = 0 \tag{2.95}$$

And by $\partial_k B_k = 0$, we have:

$$\partial_i z(x)\partial_j^y[B_i(x), B_j(y)] = \partial_i\chi(x)\partial_j^y[B_i(x), B_j(y)] = \partial_i^x\partial_j^y[B_i(x), B_j(y)] = 0 \tag{2.96}$$

Thus the commutator matrix $M_{ij}(x,y) \equiv [B_i(x), B_j(y)]$, antisymmetric under the exchange $(i,x) \leftrightarrow (j,y)$ satisfies the set of equations eqs.(2.94-2.96). The general solution for these equations is given by

$$M_{ij}(x,y) = E_i(x)F_j(y) - E_j(y)F_i(x) \tag{2.97}$$

where $F_i(x)$ is an arbitrary function. However, we have already established that if x does not belong to C_y and y does not belong to C_x, then $M_{ij}(x,y) = 0$. This unambiguously fixes $F_i(x) = 0$, so that we have:

$$[B_i(x), B_j(y)] = 0 \tag{2.98}$$

for all x, y.

2.3.2 Lorentz Transformations of the Fields

The final point we address is the Lorentz transformation properties of the fields z and χ. Since the electric and magnetic field are covariant components of the Lorentz tensor, it is clear that z and χ cannot be covariant scalar fields. The transformations of z and χ under rotations are the same as those of covariant fields, and we will not deal with those here.

Let us parametrize the infinitesimal Lorentz transformation properties of these fields in the following way:

$$\begin{aligned} z(x) &\to z(\Lambda^{-1}x) = (1+\beta\Delta)z(x) + a \\ \chi(x) &\to \chi(\Lambda^{-1}x) = (1+\beta\Delta)\chi(x) + b \\ \Theta(x) \equiv \partial_0\Phi &\to \Theta(\Lambda^{-1}x) = \Theta(x) + c \end{aligned} \tag{2.99}$$

Here β is the boost parameter and $\Delta \equiv \omega^{\mu}{}_{\nu}x^{\nu}\partial_{\mu}$ with ω^{μ}_{ν} - an antisymmetric generator of Lorentz transformation. In particular for a boost in the direction of a unit vector $\hat{n}$, $\omega_0^i = \hat{n}_i$. The noncanonical terms a, b and c are to be determined such that $F^{\mu\nu}$ transforms as a tensor.

For simplicity, let us consider explicitly a boost transformation in the first direction, $\hat{n} = (1, 0, 0)$. The transformation of the components of the field strength tensor are

$$E_2(x) \to E_2(\Lambda^{-1}x) - \beta B_3(\Lambda^{-1}x) \tag{2.100}$$

on the other hand, writing this in terms of z, χ and Θ we have:

$$\begin{aligned} &E_2(x) = 2[\partial_3 z(x)\partial_1\chi(x) - \partial_1 z(x)\partial_3(x)] \\ &\to 2[\partial_3 z(\Lambda^{-1}x)\partial_1\chi(\Lambda^{-1}x) - \partial_1 z(\Lambda^{-1}x)\partial_3(\Lambda^{-1}x)] \end{aligned} \tag{2.101}$$

Equating the two we obtain:

$$-\beta\partial_3\Theta + 2[\partial_3 z\partial_1 b + \partial_3 a\partial_1\chi - \partial_1 z\partial_3 b - \partial_1 a\partial_3\chi] = 0 \tag{2.102}$$

Similarly by considering the transformation of E_1 we obtain

$$2(\partial_2 z\partial_3 b + \partial_2 a\partial_3\chi - \partial_3 z\partial_2 b - \partial_3 a\partial_2\chi) = 0 \tag{2.103}$$

and for E_3:

$$\beta\partial_2\Theta + 2[\partial_1 z\partial_2 b + \partial_1 a\partial_2\chi - \partial_2 z\partial_1 b - \partial_2 a\partial_1\chi] = 0 \tag{2.104}$$

Defining for convenience $f_i = 2(a\partial_i\chi - b\partial_i z)$, and $u_i = (0, \beta\partial_3\Theta, -\beta\partial_2\Theta)$ the above equations can be written as

$$\epsilon_{ijk}\partial_j f_k = u_i \tag{2.105}$$

The general solution for f is:

$$f_i = -\frac{\epsilon_{ijk}\partial_j u_k}{\partial^2} + \partial_i\tilde{\lambda}$$
$$= \beta\hat{n}_i\Theta + \partial_i\lambda \tag{2.106}$$

where

$$\tilde{\lambda} - \beta\frac{\hat{n}_i\partial_i}{\partial^2}\Theta = \lambda \tag{2.107}$$

where the function λ still has to be determined.

We can now solve eq.(2.106) for a and b by noting that eq.(2.106) is identical to eq.(2.72) with the substitution

$$\begin{aligned} \partial_0 z &\to a \\ \partial_0\chi &\to b \\ \partial_0\Phi &\to \lambda \\ B_k &\to \beta\Theta\hat{n}_k \end{aligned} \tag{2.108}$$

Using eqs (2.81, 2.82) we find:

$$\begin{aligned} a &= \frac{1}{E^2}(\beta\Theta\hat{n}_i + \lambda_i)\epsilon_{ijk}E_j z_k \\ b &= \frac{1}{E^2}(\beta\Theta\hat{n}_i + \lambda_i)\epsilon_{ijk}E_j \chi_k \end{aligned} \tag{2.109}$$

With this, eq. (2.106) yields the equation for λ:

$$E_i(\beta\Theta\hat{n}_i + \partial_i\lambda) = 0 \tag{2.110}$$

from which we get:

$$\lambda(x) = -\beta\int_{\infty}^{x} dl_C \hat{E}_i\hat{n}_i\Theta \tag{2.111}$$

where again, C is a curve in the direction of E

To determine the remaining function c we consider the transformation of the magnetic field For the transformation of the magnetic field, we have:

$$\begin{aligned} B_1(x) &\to B_1(\Lambda^{-1}x) \\ &= 2[\partial_1\chi(\Lambda^{-1}x)\partial_0(\Lambda^{-1}x) - \partial_0\chi(\Lambda^{-1}x)\partial_1 z(\Lambda^{-1}x)] - \partial_1\Theta(\Lambda^{-1}x) \end{aligned} \tag{2.112}$$

which yields

$$2[\partial_1\chi\partial_0 a + \partial_1 b\partial_0 z - \partial_0\chi\partial_1 a - \partial_0 b\partial_1 z] - \partial_1 c + \beta\Delta\partial_1\Theta = 0 \tag{2.113}$$

Similarly the transformation of B_2 and B_3 yields

$$\begin{aligned} &2[\partial_2\chi\partial_0 a + \partial_0 z\partial_2 b - \partial_0\chi\partial_2 a - \partial_2 z\partial_0 b] - \partial_2 c + \beta\Delta\partial_2\Theta = 0 \\ &2[\partial_3\chi\partial_0 a + \partial_0 z\partial_3 b - \partial_0\chi\partial_3 b - \partial_3 z\partial_0 b] - \partial_3 c + \beta\Delta\partial_3\Theta = 0 \end{aligned} \tag{2.114}$$

These can be written a single vector equation

$$\partial_0 f_i - \partial_i f_0 - \partial_i c + \beta\Delta\partial_i\Theta = 0 \tag{2.115}$$

Using eq. (2.106) this can be written as:

$$\partial_i[\partial_0\lambda - f_0 - c + \beta\Delta\Theta] = 0 \tag{2.116}$$

yielding

$$\begin{aligned} c &= 2(a\partial_0\chi - b\partial_0 z) - \partial_0\lambda - \beta\Delta\Theta \\ &= \beta\left[\frac{2}{E^2}\epsilon_{ijk}E_j(\partial_0\chi z_k - \partial_0 z\chi_k)\left[\Theta\hat{n}_i - \partial_i\int_\infty^x dl_C\hat{E}_l\hat{n}_l\Theta\right] + \partial_0\int_\infty^x dl_C\hat{E}_i\hat{n}_i\Theta - \Delta\Theta\right] \end{aligned} \tag{2.117}$$

Thus we find that the fields z, χ and Φ under Lorentz boost transform according to eq.(2.99) with a,b,c given in equations (2.109), (3.107) and (3.112).

2.3.3 Discussion

In this section, we have amended the model suggested in section 2.2 as a candidate for effective description of a gauge theory. We have considered only the Abelian limit in which the model is equivalent to a theory of a free massless photon. We have proven this equivalence by considering the canonical structure of the theory. We have also shown that the basic fields of our model have interesting Lorentz transformation properties.

We note that the modification discussed here also solves a certain puzzle posed in section 2.2. Namely, the global symmetry generated by

$$C_F = \int d^3x \left[p_z \frac{\partial G[z,\chi]}{\partial \chi} - p_\chi \frac{\partial G[z,\chi]}{\partial z} \right] \tag{2.118}$$

for an arbitrary function of two variables $G[z,\chi]$. These transformations constitute a group of area preserving diffeomorphisms on a sphere. The electric and magnetic fields were invariant under the action of this group. That in fact suggested that the theory had some degrees of freedom in addition to the electrodynamic ones, since the action of the transformation generated by eq.(2.118) on the full phase space of the theory was nontrivial. However now the direct consequence of eqs. (2.78),(2.79) and (2.80), is that the generator of this transformation vanishes,

$$C_F = 0 \tag{2.119}$$

and thus there are no physical degrees of freedom that transform nontrivially under eq.(2.118). Thus in the present model the group of (global) diffeomorphisms $Sdiff(S^2)$ is in fact a global gauge symmetry, that is the states are invariant under the action of C_F, and what used to be extra degrees of freedom in section 2.2 now becomes unphysical “gauge” coordinate. This ensures that electric and magnetic fields are the only physical degrees of freedom.

Since here we are dealing with the theory of a free photon, the charged states are not present. It should be however straightforward to extend this discussion to include electrically charged states. Just like in section 2.2 we should lift the constraint of constant length of the field ϕ^a, and instead allow dynamics of the modulus ϕ^2. This will regulate the energy of the charged states in the UV and will make it finite. Since the configuration space of the model is $SO(3) \times R$, and the $SO(3)$ symmetry is broken to $O(2)$, the moduli space should have nontrivial homotopy group $\Pi_2(M) = Z$, and the relevant topological charge should be identifiable with the electric charge[4].

The next set of questions to be addressed is how to move in this theory to the non-Abelian regime. According to the logic of section 2.2 we need to find a perturbation that breaks the global symmetries of the model and through this breaking generates linear potential between the charges. The question one has to

[4] There may be some subtlety in this argument related to the fact that the global gauge group $Sdiff(S^2)$ has to be modded out. However, since the gauge transformation is global, we do not anticipate any problems.

address, does this perturbation have to preserve the $Sdiff(S^2)$ gauge symmetry, or should it break it explicitly. This global gauge symmetry is a new element compared to 2+1 dimensions [17], and we do not have any guidance from the 2+1 dimensional models. Perhaps one should deal directly with the breaking of the generalized magnetic symmetry - the symmetry generated by the magnetic flux [19], [27] in terms of its order parameter - the 't Hooft loop [15]. These question will be addressed in future work.

Chapter 3

Unitarity

The physics of systems with ghosts has recently attracted renewed attention [28]. The most interest in these systems is in connection with the theories of gravity.

One strong motivation for this is the discovery of cosmic acceleration [29] and the associated need for a non-vanishing cosmological constant, which has no natural explanation within general relativity. One can hope that modifying gravitational interactions at large distance scales might bring a natural understanding of this problem. Another problem where modifying gravity can potentially bring dividends is dark matter. Dark matter has not been observed directly, although within the standard cosmological model it is necessary to account for the energy balance of the universe, as well as explaining rotation curves of galaxies.

Conformal gravity is an example of a modified theory of gravity which is potentially interesting in both these contexts [30] [31] [32] [33]. An important aspect of conformal gravity that singles it our from other higher derivative extensions of GR is that it is renormalizable by power counting in the ultraviolet [34], and on this basis has been considered as a candidate for a consistent quantum theory of gravity.

It is however not clear whether conformal gravity - or any other modified theory of gravity - is consistent [35]. The problem, like with many higher derivative theories, is that in perturbation theory it has ghost modes - the modes whose kinetic energy is negative [36]. This is usually considered to be a hindrance for a physical theory. Indeed an absence of a stable vacuum (lowest energy) state is disconcerting and is likely to lead to an instability, whereby the evolution extracts energy from the negative energy modes and pumps it into the positive energy modes producing a runaway instability.

As long as interactions between the field modes are neglected, the wrong sign of kinetic energy is not a problem as such. Since in a free approximation any field theory has infinite number of conserved quantities, the classical motion of such a system is bounded. All the oscillators simply oscillate independently of each other, and the sign of the energy for each one is a matter of convention. However, once interactions between the modes are turned on, one generally expects that the classical motion becomes ergodic, and samples all available phase space. If the total energy is not bounded from below, this is expected to lead to classical instability with positive and negative contributions to energy growing without bound. Sometimes the ghosts are said to violate unitarity of a quantum theory. As we will explain in section 3.1, this is simply another way of stating the same problem. In such a quantum system time evolution evolves a normalizable quantum state into a state which has support only for “infinite” values of the field, thereby “violating unitarity”. A classical theory with such behavior cannot yield a consistent quantum field theory upon quantization.

Despite all the drawbacks of a theory with ghosts, the attractive features of the modified theories of gravity prompted attempts to solve this problem. One approach in conformal gravity attempts to separate the ghost modes from the positive norm gravitons and ban their propagation “by hand” [28]. Another attempt is to quantize the theory using a nonstandard definition of a quantum mechanical norm [37], [38] following a more general program of quantizing PT invariant but non hermitian Hamiltonians [39]. In a free limit this is essentially equivalent to treating the ghost modes as purely imaginary, which flips the sign of the ghost part of the Hamiltonian.

Such attempts were not made only for conformal gravity, but also for the massive gravity: Much effort has also been spent to understand whether the ghosts in this theory can be consistently decoupled [40]. On the other hand it has been also convincingly argued recently that one does not need to decouple the ghost, since nonperturbatively the theory cures itself and the full nonlinear Hamiltonian of spontaneously broken gravity is bounded from below [41].

On the other hand, the instability in question may not be necessarily a fatal flaw. This is especially so in a theory of gravity, which governs the evolution of the universe and thus never actually relaxes to its ground state. Thus the nonexistence of a ground state in gravity may be just a way of life. In particular it has been suggested that a negative pressure due to ghosts may be a cause of the cosmological acceleration [42]. It has been argued that the time scale in which instability develops is way too short in theories which contain ghosts in the matter sector [43]. We are unaware however of a similar analysis of gravitational ghosts themselves, that is the ghost partners of the gravitons that arise in conformal gravity: the rapid decay of the vacuum discussed in [43] may be preventable if the ghost coupling to gravitons is nonlocal [44]. It is not obvious therefore that the last word on viability of theories with ghosts has been uttered yet.

Our aim in this chapter is first to discuss and explain important properties of systems with ghost by using the simplest such theory in section 3.1: The Pais-Uhlenbeck oscillator. We will start section 3.1.1 by discussing the Hamiltonian formalism. In section 3.1.2, we will investigate a limit of the model which is most relevant to conformal gravity. In section 3.1.3 we discusss the classical and quantum stability of the model and finally in section 3.1.4 we argue by analytical and numerical methodsthat presence of an interaction does not necessarily lead to an instability. Section 3.1.5 consists of the lessons learned from this simple model.

Section 3.2 is an investigation of the stability of conformal gravity. We aim to see if the ghosts degrees of freedom could potentially "fix" themselves in the interacting theory. Since the full theory is extremely complicated, we introduce a simplified approximation in section 3.2.1. Section 3.2.2 is the analysis of the equations of motion. section 3.2.3 is the discussion of rather unexpected fate of ghost degrees of freedom in our approximation.

3.1 Some Comments on Ghosts and Unitarity: The Pais-Uhlenbeck Oscillator Revisited.

The purpose of this section is rather modest and pedagogical. The potential interest notwithstanding, theories with ghosts are still considered somewhat esoteric and are not frequently discussed in particle physics literature. We aim to discuss pedagogically the simplest example of a theory with ghosts - the Pais-Uhlenbeck oscillator [45]. Our goal is to explicitly demonstrate in this simple framework the meaning of some rather paradoxical notions that are sometimes used in the context of theories with ghosts, like negative norm states and violation of unitarity in theories with ostensibly perfectly hermitian Hamiltonian. We also demonstrate explicitly by solving the time dependent evolution in this theory how the soft UV behavior arises in dynamical context.

We stress that the Pais-Uhlenbeck oscillator is in fact a unitary theory even though it possesses a ghost mode, and also give an example of a theory of interacting "particle" and "ghost" modes which is nevertheless unitary on the quantum level. All the above statements apply to quantum mechanical systems with the standard Dirac norm, as we do not recourse to a non standard quantization approach *a la* [38].

The Pais - Uhlenbeck oscillator was the subject of several papers in recent years, and its solution is well known [37,46,47]. Nevertheless we feel that our simple and straightforward approach to the problem is illuminating and is worth recording.

3.1.1 Pais-Uhlenbeck Oscillator

The Pais-Uhlenbeck system is the theory of a single degree of freedom which satisfies a fourth order equation of motion. It is defined by the Lagrangian

$$L = (\frac{d^2}{dt^2}z + \omega_1^2 z)(\frac{d^2}{dt^2}z + \omega_2^2 z) \tag{3.1}$$

For definiteness we assume $\omega_1 > \omega_2$. Our aim is to study the Hamiltonian dynamics with the view of quantum mechanical system, since a discussion of evolution of a wave function is most convenient in the Hamiltonian formalism. Although there exist a general formalism for calculating a Hamiltonian of four derivative systems, developed by Ostragradsky [48], we find it more straightforward in the context of this particular model to introduce a pair of variables

$$X = \frac{d^2}{dt^2}z + \omega_1^2 z; \qquad Y = \frac{d^2}{dt^2}z + \omega_2^2 z \tag{3.2}$$

and consider them as independent coordinates. The rationale of this choice is, that the fourth order equation for the variable z

$$\left(\frac{d^2}{dt^2} + \omega_1^2\right)\left(\frac{d^2}{dt^2} + \omega_2^2\right) z = 0 \tag{3.3}$$

can be written as a pair of second order equations for X and Y

$$\frac{d^2}{dt^2}X+\omega_2^2X=0;\qquad \frac{d^2}{dt^2}Y+\omega_1^2Y=0 \tag{3.4}$$

To find the Hamiltonian we introduce the Lagrange multipliers for the constraints eq.(3.2)

$$L=XY+\alpha(\frac{d^2}{dt^2}z+\omega_1z-X)+\beta(\frac{d^2}{dt^2}z+\omega_2z-Y) \tag{3.5}$$

Canonical momenta are calculated in the standard fashion $p_i=\partial L/\partial\dot{x}_i$. This definition leads to the following constraints

$$P_X=P_Y=0;\quad p_\alpha=p_\beta=-\dot{z};\qquad p_z=-\dot{\alpha}-\dot{\beta} \tag{3.6}$$

The Hamiltonian, calculated in the standard way as the Legendre transform of the Lagrangian is

$$H=-\frac{1}{2}(p_\alpha+p_\beta)p_z-XY-\alpha(\omega_1^2z-X)-\beta(\omega_2^2z-Y) \tag{3.7}$$

Commuting (calculating the Poisson bracket of) H with the primary constraints, eq.(3.6) we obtain secondary constraints

$$[H,P_X]=\alpha-Y=0;\;[H,P_Y]=\beta-X=0;\;[H,p_\alpha-p_\beta]=(\omega_1^2-\omega_2^2)z-(X-Y)=0 \tag{3.8}$$

These can be used to express α, β and z in terms of X and Y,

$$\alpha=Y;\quad\beta=X;\quad z=\frac{X-Y}{\omega_1^2-\omega_2^2} \tag{3.9}$$

The Dirac procedure for constraint systems requires that we use the Dirac brackets instead of the Poisson brackets to derive equations of motion. The net result of switching to the Dirac brackets is clear without a detailed calculation. The new "commutation relations" are such that the dynamical variables "commute" with all the constraints. Also, the modification is present only for those variables whose Poisson bracket with the original constraints does not vanish. Without any calculation the result in the present case is obvious

$$p_\alpha = \pi_Y; \quad p_\beta = \pi_X; \quad p_z = \frac{1}{2}(\omega_1^2 - \omega_2^2)(\pi_X - \pi_Y) \tag{3.10}$$

with the Dirac brackets

$$[\pi_i, X_j]_D = -\delta_{ij} \tag{3.11}$$

The Hamiltonian then becomes

$$H = \frac{1}{2}\Omega\Delta[\pi_Y^2 - \pi_X^2] + \frac{\omega_1^2}{2\Omega\Delta}Y^2 - \frac{\omega_2^2}{2\Omega\Delta}X^2 \tag{3.12}$$

where we have defined $\Omega = \frac{\omega_1+\omega_2}{2}; \quad \Delta = \omega_1 - \omega_2$. Finally rescaling the variables $\pi_x = (\Omega\Delta)^{1/2}\pi_X; \quad x = (\Omega\Delta)^{-1/2}X$, and similarly for y we obtain

$$H = \frac{1}{2}\pi_y^2 + \frac{1}{2}\omega_1^2 y^2 - \frac{1}{2}\pi_x^2 - \frac{1}{2}\omega_2^2 x^2 \tag{3.13}$$

In terms of the new variables, the original coordinate z is expressed as

$$z = \frac{1}{\sqrt{4\Omega\Delta}}(x - y) \tag{3.14}$$

This is a very simple Hamiltonian. It is not bounded either from below nor from above, but nevertheless it generates a perfectly acceptable evolution. The two degrees of freedom x and y are decoupled, and classically each one simply satisfies a harmonic oscillator equation of motion. Classically there are no runaway solutions for these equations of motion for an arbitrary initial condition. Quantum mechanically the system also possesses finite positive norm states which evolve unitarily with time.

Before turning to the quantum problem, we wish to stress again that we are not going to discuss nonstandard quantization in the spirit of the one proposed in Ref. [37]. Our view is that the quantum problem is not defined solely by the abstract axioms of quantum mechanics, and thus any quantization that preserves the basic mathematical structure is allowed. On the contrary, we take the view that the correspondence principle is no less important than the abstract axioms. In other words, given that the classical problem is defined by the Lagrangian (eq.3.1) and classically one is interested in the real time evolution of the real degree of freedom z, the quantum problem must have the same classical limit. This requires the quantum variable z and all its time derivatives, as well as the variables x and y defined above, to be Hermitian operators. The standard Dirac norm, where the matrix elements are calculated by integration over the real values of x and y, is eminently appropriate, and we will use it throughout this paper. The quantum norm used in the approach of Ref [37], on the other hand, requires the time derivative of z to be anti-Hermitian, as explained for example in Ref. [49], and is not appropriate for our purposes. Thus our aim in this paper is not to explore alternative options for the quantization of this theory, but rather to explain the issues related to ghosts and unitarity in the theory with a standard Dirac norm.

As we will see below, quantum mechanically the Pais-Uhlenbeck system possesses finite positive (Dirac) norm states which evolve unitarily with time. Nevertheless it is commonly said that this quantum theory has negative norm states. In the next subsection we will clarify what this statement technically means, and stress that it is not a hindrance for the peaceful existence of a unitary evolution in this model.

The Negative Face of a Divergent Integral

The Hamiltonian of the Pais-Uhlenbeck system is not bounded from below. This is unusual and somewhat disturbing, since we normally expect that any open system will interact with some external degrees of freedom and generally relax to its ground state by loosing any excess energy to those degrees of freedom. However if the system is closed, no such loss of energy is possible and unboundedness of energy from below does not have to be a problem. In particular, in the present case the two harmonic oscillators do not interact with each other, no energy transfer from one to another occurs and the evolution is perfectly unitary, provided at the initial moment in time we start with a state which is localized at finite values of x and y.

On the other hand, for states with large negative energy (which at finite volume are close to the lowest-energy state) the evolution becomes non-unitary. This is simply due to the fact that these states are localized at very large values of x, close to the spatial boundary at a finite IR regulator. The probability stored in a state like this simply "leaks" through the spatial boundary during the evolution. In the infinite-volume limit these states are non-normalizable, which is the manifestation of the fact that they are localized at un-physically large values of the coordinates.

To see this explicitly, let us define creation and annihilation operators in the standard way

$$a = \sqrt{\frac{\omega}{2}}x + i\sqrt{\frac{1}{2\omega}}\pi_x \tag{3.15}$$

The Fock vacuum of a is the normalized Gaussian state

$$a|0\rangle = 0; \qquad \langle x|0\rangle = Ne^{-\frac{\omega}{2}x^2} \tag{3.16}$$

This is the state with highest energy in the x-sector.

One can also formally define a state which corresponds to lowest energy eigenvalue, as the vacuum of $a^\dagger$ [46]

$$a^\dagger|\Phi\rangle = 0; \qquad \langle x|\Phi\rangle = N_- e^{\frac{\omega}{2}x^2} \tag{3.17}$$

This state is non-normalizable and not physical, since a particle in this state is localized exclusively at infinity. The probability to find the particle at finite value of coordinate vanishes, since in the infinite volume limit the normalization constant N_- vanishes faster than exponentially.

Nevertheless in a certain formal way it corresponds to the lowest energy state. To see this, write the Hamiltonian for x mode in the standard form

$$H_x = -\omega a a^\dagger + E_0 \tag{3.18}$$

Consider a tower of states above $|\Phi\rangle$ generated by the action of operator a.

$$H|\Phi\rangle = E_0|\Phi\rangle; \quad H|1\rangle \equiv Ha|\Phi\rangle = E_0 a|\Phi\rangle - \omega a a^\dagger a|\Phi\rangle = (E_0+\omega)|1\rangle; \quad ... \tag{3.19}$$

Thus applying operator a increases the energy of the state by ω, and the spectrum seems to be bounded from below. Another formal argument suggests that at least some of these states have negative norm. Let us calculate the norm of the "one particle state"

$$\langle 1|1\rangle = \langle\Phi|a^\dagger a|\Phi\rangle = \langle\Phi|aa^\dagger - 1|\Phi\rangle = -\langle\Phi|\Phi\rangle \tag{3.20}$$

Taken literally, this argument suggests that either the "one particle" state or the "vacuum" state has a negative norm. This is the origin of the usual statement that the theory has negative norm states.

In fact, of course the norm of both of these states is positive once we regulate the system by putting it into a finite volume [1]. The "vacuum" state is just a Gaussian which grows at large values of x. Its norm is positive in finite volume,

[1] If one assumes that both $|\Phi\rangle$ and $|1\rangle$ in infinite volume belong to some Hilbert space, this space necessarily contains negative norm states [49]. However, this is certainly not the Hilbert space with Dirac norm and is not the subject of our discussion

and diverges (while remaining positive) as the infrared cutoff is removed. The one particle wave function can be found explicitly

$$a|\Phi\rangle = \left(\sqrt{\frac{\omega}{2}}x + \sqrt{\frac{1}{2\omega}}\frac{d}{dx}\right)e^{\frac{\omega}{2}x^2} = \sqrt{2\omega}xe^{\frac{\omega}{2}x^2} \tag{3.21}$$

The norm of this state obviously is also positive, and is even more divergent than that of the vacuum in large volume. None of the norms is negative. The flaw in the formal eq.(3.21) is of course precisely the fact that the states in question are not normalizable. To interpret the expectation value of $a^\dagger a$ as the norm of a one particle state, one needs to act with $a^\dagger$ on the bra, which amounts to integration by parts of the derivative in $a^\dagger$. The integration by parts however is not allowed, since the wave function grows at infinity. In particular

$$\langle 1|1\rangle \neq |a|\Phi\rangle|^2 \tag{3.22}$$

as one can easily verify by an explicit calculation. In fact the difference between the two sides of the inequality is infinite. Thus "negative norm" is merely a jargon which refers to the fact that neither the norm nor matrix element of any reasonable operator like x^n or p^n is defined in the states of the form eq.(3.19) due to strong infrared divergence.

Sometimes the procedure described above is referred to as a "quantization scheme", in the sense that the states of the tower eq.(3.19) do not belong to the Hilbert space of normalizable states. The unitarity in this quantization scheme is broken exactly for the reason explained above. All the wave functions with finite number of "excitations" above the "vacuum" $|\Phi\rangle$ live on the edge of space. Once an infrared regulator (which makes the norm finite) is removed the wave functions vanish everywhere in the bulk. Such states run great risk of disappearing through the boundary under time evolution.

On the other hand it is clear, that states which are created by the action of $a^\dagger$ on $|0\rangle$ are normalizable and their evolution is perfectly unitary. One is normally interested in the situation when a particle can be detected in the bulk with finite probability. This physical condition makes the non-normalizable states physically irrelevant and devoid of interest.

3.1.2 The degenerate case $\Delta = 0$

A special case of the Pais-Uhlenbeck system is when the two oscillators have the same frequency, $\Delta = 0$. In terms of analogy with the W^2 gravity, this case is the most interesting. In this section we discuss some interesting features of the equal frequency limit.

The Fate of the Normalized Wave Functions

The limit $\Delta = 0$ of the previous expressions is a little tricky, since the transformation between the original variable z and x, y becomes singular. It is therefore not straightforward to take the limit directly on the level of the Hamiltonian. One cannot simply drop the terms in the Hamiltonian which naively vanish in the limit

$\Delta \to 0$, since the operators that multiply Δ may have divergent matrix elements. To illustrate this, let us first rewrite the Hamiltonian in terms of variables X and z, avoiding any singular redefinition of variables (here the variable x is defined as originally: $X = \frac{d^2}{dt^2}z + \omega^2 z$).

$$H = -\frac{1}{2}\pi_X\pi_z + X^2 - 2(\Omega + \frac{1}{2}\Delta)^2 zX + 2\Delta\Omega(\Omega + \frac{1}{2}\Delta)^2 z^2 - \frac{1}{2}\Delta\Omega\pi_X^2 \tag{3.23}$$

Suppose we naively drop the last two terms in eq.(3.23), which formally vanish in the limit $\Delta \to 0$.

$$H_0 = -\frac{1}{2}\pi_X\pi_z + X^2 - 2(\Omega + \frac{1}{2}\Delta)^2 zX \tag{3.24}$$

Let us now look for Gaussian eigenstates of the resulting Hamiltonian. Recall that at nonzero Δ we had four Gaussian eigenstates

$$\exp \pm \left\{ \frac{\omega_2}{2\Omega\Delta}X^2 \pm \frac{\omega_1}{2\Omega\Delta}Y^2 \right\} = \exp \pm \left\{ \frac{1}{\omega_1 \pm \omega_2}X^2 \pm 2\omega_1\Omega\Delta z^2 \mp 2\omega_1 Xz \right\} \tag{3.25}$$

Three of these were non-normalizable and only one was the well behaved normalizable state peaked at $x, z = 0$: The normalizable state is

$$\Psi = \exp - \left\{ \frac{1}{\Delta}X^2 + 2\omega_1\Omega\Delta z^2 - 2\omega_1 Xz \right\} \tag{3.26}$$

However if we seek *all* Gaussian eigenstates of the truncated Hamiltonian eq.(3.24), we find only two states

$$\exp \pm \left\{ -\frac{1}{2\Omega}(X - 2\Omega^2 z)^2 + 2\Omega^3 z^2 \right\} \tag{3.27}$$

Evidently none of these two states is normalizable. These two Gaussian states are indeed obtained in the limit $\Delta \to 0$ from two of the states eq.(3.25). Thus we seem to find no normalizable Gaussian eigenstates of a quadratic Hamiltonian eq.(3.24), even though for any finite Δ a normalizable Gaussian eigenstate exists. This means that the Hamiltonian eq.(3.24) is not diagonalizable, which indeed can be formally proven [38], [46].

This conclusion is however a little hasty, as it is based on neglecting the last two terms in eq.(3.23). However, even though these terms are multiplied by Δ, in order to be able to neglect them, we need to be sure that they have vanishing matrix elements in the limit $\Delta \to 0$. It is easy to see that this is not the case here. Indeed, in the normalizable state eq.(3.26) we have

$$\langle z^2 \rangle \sim \langle \pi_X^2 \rangle \sim \frac{1}{\Delta} \tag{3.28}$$

so that in fact the last two terms in eq.(3.23) are finite in the limit $\Delta \to 0$ and therefore cannot be simply discarded.

The normalizable state eq.(3.26) does not disappear without a trace in the degenerate limit, but rather tends to a delta function of X

$$\Psi^2(X) \to \delta(X) \tag{3.29}$$

The action of the Hamiltonian eq.(3.24) on this state is ambiguous due to the first term in the Hamiltonian. One does obtain this state unambiguously, however as the equal frequency limit of eq.(3.26)

Thus on the normalizable states the auxiliary variable X is frozen at zero, while the original variable z fluctuates freely with infinite amplitude.

Interestingly, this suggests that in a sense the oscillator looses half of its degrees of freedom and also becomes "classical". Recall that the variable X is essentially the classical equation of motion for half the original modes of z, since $X = \frac{d^2}{dt^2}z + \omega^2 z$. In the limit $\Delta \to 0$, this quantity is fixed at zero without fluctuations. On the other hand the coordinate z itself fluctuates without restriction. Thus essentially the quantum system becomes a classical oscillator which can oscillate with arbitrary amplitude.

Dynamical conformal symmetry

As an interesting aside, we note that at $\Delta = 0$ the theory dynamically develops a conformal symmetry, which is spontaneously broken by normalizable states. For the purpose of this discussion, it is convenient to revert to normalization in which the Hamiltonian is simplest in the limit $\Delta \to 0$, eq.(3.13). In the equal frequency limit the Hamiltonian eq.(3.13) is invariant under the following transformation

$$x \to x \cosh t + y \sinh t; \quad y \to y \cosh t + x \sinh t; \qquad z \to e^{-t} z \tag{3.30}$$

It is natural to refer to this symmetry as conformal. This symmetry is not obviously present in the Lagrangian eq.(3.1). In fact the Lagrangian is multiplied by a constant under the transformation eq.(3.30). However, as we have seen in the previous subsection, in the equal frequency limit the dynamics of z is such that on normalizable states it is pinned to satisfy $X = \frac{d^2}{dt^2}z + \omega^2 z = 0$. As a result the Lagrangian vanishes for all physically interesting configurations. Scaling of the Lagrangian by a finite factor therefore is indeed a "dynamical" symmetry in this limit.

Interestingly this symmetry is spontaneously broken, in the sense that the normalizable "vacuum", or in fact any of the normalizable physical states, is not invariant under it. The wave function of the "lowest energy", the non-normalizable eigenstate of the operator $a^\dagger$ is indeed invariant under the conformal transformation:

$$\exp\left\{-\frac{1}{2\Omega}\left[x^2 - y^2\right]\right\} \tag{3.31}$$

However for the normalizable Gaussian

$$\exp\left\{-\frac{1}{2\Omega}\left[x^2 + y^2\right]\right\} \to \exp\left\{-\frac{1}{2\Omega}\left[\cosh(2t)\left[x^2 + y^2\right] + 2\sinh(2t)xy\right]\right\} \tag{3.32}$$

It is clear that any state whose wave function is localized at finite values of x and y is necessarily not invariant under the transformation eq.(3.30). Thus the conformal symmetry is "spontaneously broken" on normalizable states. Since the representations of conformal group eq.(3.30) are infinitely dimensional, the finite energy spectrum is infinitely degenerate. This is of course well known and obvious since adding any number of excitations of the x oscillator and the same number of excitations of the y oscillator does not change the energy in the degenerate limit [46,47]. It is nevertheless amusing, that this degeneracy can be understood as a spontaneous breaking of conformal symmetry.

Here we wish to add a comment regarding the nature of the spectrum at $\Delta = 0$. Naively taking $\Delta = 0$ we have two degenerate harmonic oscillators. Taking the nth excited state of the y oscillator, $\psi_n(y)$, and mth state of the x oscillator, $\psi_m(x)$, we conclude that the energy eigenvalue is $E_{n-m} = \Omega_{n-m}$ and the degeneracy of every level is infinite, corresponding to arbitrary n and fixed $n-m$. There is however a subtlety in this argument [47] [2]. To find the spectrum at $\Delta \to 0$ we need to calculate the density of states at small but finite Δ and then take the limit $\Delta \to 0$. Keeping $\epsilon \gg \Delta$ and counting the number of states in an arbitrarily small interval E to $E + \epsilon$, we find that it is infinite. To regulate the calculation we need to cut off in some way the spectrum of each harmonic oscillator, n, $m \leq \kappa$. For finite κ we find $N(E \neq \Omega k, \epsilon) \propto \kappa\Delta$. We now have to take the limit $\Delta \to 0$ and $\kappa \to \infty$. If one takes $\Delta \to 0$ at finite κ first, the density of states for $E \neq \Omega k$ vanishes and one recovers the infinitely degenerate but discrete spectrum. On the other hand, if one takes κ to scale as $\kappa = \frac{L}{\Delta}$ with $L \to \infty$ last, the spectrum becomes continuous and infinitely degenerate for any E [44], [50]. What is important to realize, however, is that all the states with $E \neq \Omega k$ are of the form $\Psi_n(x)\Psi_m(y)$ with m,$n \propto \frac{1}{\Delta}$. Thus in the limit $\Delta \to 0$, only infinitely high excitations of both harmonic oscillators contribute to the continuous part of the spectrum. These states are localized at “infinite” values of the variables x and y: $\langle x^2 \rangle \sim \langle y^2 \rangle \sim \frac{1}{\Delta}$. Thus any state initially localized in

[2] We thank A. Smilga for pointing this out to us

a finite volume has zero overlap with these states and will not feel their presence during the evolution. The situation is even more extreme for the original variable z, since $\langle z^2 \rangle \sim \frac{1}{\Delta}\langle x^2 \rangle \sim \frac{1}{\Delta^2}$. We therefore conclude that for the evolution of physically interesting states, the existence of eigenstates with $E \neq \Omega k$ is not important. In this sense only the discrete (but infinitely degenerate) part of the spectrum is physical.

3.1.3 Dynamics: Classical vs Quantum

The dynamics of the classical Pais-Uhlenbeck oscillator is identical to that of two decoupled harmonic oscillators. The variables x and y satisfy the harmonic oscillator equations of motion, and the fact that the energy of the x-oscillator is negative is irrelevant, since the energies of each oscillator are separately conserved.

Quantum mechanically, however the situation is very different. Here the overall sign of energy is reflected in the sign of the phase of the wave function. For the evolution of states which are initially product wave functions $\Psi_1(x)\Psi_2(y)$ this is again unimportant, however it affects strongly the time evolution of "entangled" states. The simplest calculation where the quantum mechanical importance of the sign flip for the x-oscillator manifests itself, is the propagator of the z. It is of course well known, that the UV behavior of the propagator in four derivative theories is much softer than in theories with ordinary kinetic term. The Pais-Uhlenbeck oscillator is the simplest example of this kind. Although this is a trivial calculation, we present it here for completeness.

The propagator

To calculate the propagator of z we need to calculate the propagator of x and y separately. For y this is the usual harmonic oscillator calculation.

The y propagator:

The Hamiltonian for the y mode is

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega_1 y^2 \tag{3.33}$$

The annihilation operator a

$$a = \sqrt{\frac{\omega_1}{2}}(y + \frac{ip}{\omega_1}) \tag{3.34}$$

evolves in time according to

$$a(t) = a(0)e^{-i\omega_1 t} \tag{3.35}$$

For the Feynman propagator:

$$\begin{aligned} G_y(t) &= < T\{y(t)y(0)\} > \\ &= \frac{1}{2\omega_1} < \Theta(t)[a(t)a^\dagger(0) + a^\dagger(t)a(0)] + \Theta(-t)[a^\dagger(0)a(t) + a(0)a^\dagger(t)] > \end{aligned} \tag{3.36}$$

we have

$$G_y(t) = \frac{1}{2\omega_1}[\Theta(t)e^{-i\omega_1 t} + \Theta(-t)e^{i\omega_1 t}] \tag{3.37}$$

To perform the Fourier transform, as usual we introduce the regulator which makes the integral converent for large times

$$G_y(p) = \frac{1}{2\omega_1}\int dte^{ipt}[\Theta(t)e^{-i\omega_1 t}e^{-\epsilon t} + \Theta(-t)e^{i\omega_1 t}e^{\epsilon t}] = \frac{i}{p^2 - \omega_1^2 + i\epsilon} \tag{3.38}$$

This is the standard result, which upon integration over the frequency p gives the equal time expectation value in the vacuum

$$\langle y^2 \rangle = \int \frac{dp}{2\pi} G_y(p) = \frac{1}{2\omega_1} \tag{3.39}$$

The x propagator:

The propagator of x is equally easy to calculate in the physically relevant "vacuum"- the highest energy state. The Hamiltonian now is

$$H = -\frac{1}{2}p^2 - \frac{1}{2}\omega_2 x^2 \tag{3.40}$$

and

$$a = \sqrt{\frac{\omega_2}{2}}(x + \frac{ip}{\omega_2}); \qquad a(t) = a(0)e^{i\omega_2 t} \tag{3.41}$$

The same calculation as before now gives

$$G_x(t) \equiv \langle 0|T\{x(t)x(0)\}|0\rangle = \frac{1}{2\omega_2}[\Theta(t)e^{i\omega_2 t} + \Theta(-t)e^{-i\omega_2 t}] \tag{3.42}$$

and

$$G_x(p) = \frac{-i}{p^2 - \omega_2^2 - i\epsilon} \tag{3.43}$$

This differs from eq.(3.38) by the overall sign and also by the sign of the regulator ϵ. As is easily seen, these two sign changes cancel each other in the calculation of equal time quantities. For example

$$\langle 0|x^2|0\rangle = \int \frac{dp}{2\pi} G_x(p) = \frac{1}{2\omega_2} \tag{3.44}$$

which is the correct result for the normalizable Gaussian eigenstate of the x oscillator.

The z propagator:

Finally combining the results for x and y, and noting that due to the symmetries of the system the mixed propagator vanishes $\langle x(t)y(0)\rangle = 0$, we obtain

$$G_z(p) = \frac{1}{4\Omega\Delta}[G_y(p) + G_x(p)] = \frac{i}{2(p^2-\omega_1^2)(p^2-\omega_2^2)+i\epsilon} \tag{3.45}$$

Again, this is the standitard result, showing a softened UV behavior, since the propagator of z vanishes much faster for high frequencies than that of a harmonic oscillator. This indicates of course, that the time evolution of z is very smooth and has a very small high frequency component.

The "propagator" in the unbounded state:

What happens if we try to calculate the propagator of the x oscillator in the unbounded Gaussian state? Of course, as explained above this calculation is purely formal, since the integrals over this wave function are divergent. Still, formally proceeding as before we can define

$$G_x^-(t) = \langle \Phi|T\{x(t)x(0)\}|\Phi\rangle \tag{3.46}$$

We still use eq.(3.41), but this time it is $a^\dagger$ that annihilates the state Φ. We then formally obtain:

$$G_x^-(t) = -\frac{1}{2\omega_2}[\Theta(t)e^{-i\omega_2 t} + \Theta(-t)e^{i\omega_2 t}] \tag{3.47}$$

and

$$G_x^-(p) = \frac{-i}{p^2 - \omega^2 + i\epsilon} \tag{3.48}$$

The sign of the regulator ϵ is now the same as for the positive energy harmonic oscillator, which is simply the reflection of the fact that the state $|\Phi\rangle$ is formally the lowest energy state of the system. However this propagator leads to the same paradox of negative norm states as discussed in the previous section. Calculating the equal time expectation value, which should be by definition positive, we find

$$\langle\Phi|x^2|\Phi\rangle = -\frac{1}{2\omega_2} \tag{3.49}$$

This again underscores the point, that non-normalizable states, if manipulated formally, can be mistaken to have negative norm.

Time evolution: the wave function

It is instructive to see explicitly how the wave function of the system evolves in time. In particular we would like to see the origin of the smooth UV behavior of the Pais-Uhlenbeck system in terms of the time evolution of wave functions.

We are mostly interested in the degenerate case $\Delta \to 0$, and will therefore study time evolution generated by the Hamiltonian

$$H = -\frac{1}{2}\frac{\partial^2}{\partial y^2} + \frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{2}\Omega^2 y^2 - \frac{1}{2}\Omega^2 x^2. \tag{3.50}$$

We want to follow the time dependence of simple quantum averages, like $\langle z^2(t)\rangle$ and $\langle (X(t)+Y(t))^2\rangle$. The first observable is the obvious choice, since it is the fluctuation of the coordinate of the original oscillator, while the second one is the fluctuation of the second order equation of motion. We will choose an initial state such that both these operators have sensible (finite) averages.

We are not interested in states which are simple product states of the form $\psi_1(x)\psi_2(y)$. As far as the expectation values of all Hermitian operators go, the evolution of such a product state is identical to that of a state $\psi_1(x)\psi_2^*(y)$ evolved with the positive energy harmonic oscillator. We will thus be interested in states which are not trivial product states in the variables x and y. A simple initial wave function that satisfies these requirements is

$$\psi(0) = N\exp\left\{-\frac{1}{2}\left[\frac{\Delta\Omega}{\xi^2}(x+y)^2 + \frac{1}{4\Omega\tau^2\Delta}(x-y)^2\right]\right\} =$$
$$N\exp\left\{-\frac{1}{2}\left[(\frac{\Delta\Omega}{\xi^2}+\frac{1}{4\Omega\tau^2\Delta})x^2 + (\frac{\Delta\Omega}{\xi^2}+\frac{1}{4\Omega\tau^2\Delta})y^2 + 2(\frac{\Delta\Omega}{\xi^2}-\frac{1}{4\Omega\tau^2\Delta})xy\right]\right\}. \tag{3.51}$$

Note that we have scaled out the dependence on the frequency difference Δ explicitly. Strictly speaking for nonvanishing Δ we also have to keep the frequencies of the two oscillators in the Hamiltonian different. However the Hamiltonian itself is smooth in the degenerate limit, and it is only the relation between x, y and z that involves divergent coefficients. Thus with the appropriate choice of the wave function we can make z finite also at $\Delta \to 0$. Specifically, for the state eq.(3.51) we have

$$\langle z^2 \rangle = \tau^2; \qquad \langle (X+Y)^2 \rangle = \xi^2 \tag{3.52}$$

Since the evolution is free, a Gaussian wave function preserves its Gaussian shape at any later time. Thus at any time t we have

$$\psi(t) = N(t) \exp\left[-\frac{1}{2}A(t)x^2 - \frac{1}{2}B(t)y^2 - C(t)xy\right]. \tag{3.53}$$

Acting on this wave function with the Hamiltonian we obtain the evolution of the coefficients

$$\dot{A} = i[A^2 - C^2 - \Omega^2]; \quad \dot{B} = i[C^2 - B^2 - \Omega^2]; \quad \dot{C} = iC[A - B]. \tag{3.54}$$

After some algebra this leads to

$$\dot{C} = C\frac{\dot{A} + \dot{B}}{A + B} \tag{3.55}$$

which is solved by

$$C(t) = \alpha[A(t) + B(t)] \tag{3.56}$$

with

$$\alpha = \frac{C(0)}{A(0) + B(0)} \tag{3.57}$$

Using this result for $C(t)$ in eq.(3.54), and defining $A(t) + B(t) \equiv u(t)$, $A(t) - B(t) = v(t)$. we have:

$$\dot{u} = iuv \tag{3.58}$$

$$\dot{v} = i[(\frac{1}{2} - 2\alpha^2)u^2 + \frac{1}{2}v^2 - 2\Omega^2] \tag{3.59}$$

with the initial conditions:

$$u(0) = 2\left(\frac{\Delta\Omega}{\xi^2} + \frac{1}{4\Omega\tau^2\Delta}\right),\ v(0) = 0 \tag{3.60}$$

It is easy to see that the solution has the form

$$u(t) = \frac{1}{f_+ + f_- \cos 2\Omega t}, \qquad v(t) = \frac{-i2f_-\Omega \sin 2\Omega t}{f_+ + f_- \cos 2\Omega t} \tag{3.61}$$

where $f_\pm$ are constants determined by the equations of motion and the initial conditions. After some algebra, for the initial conditions eq.(3.60) we obtain

$$f_\pm = \frac{\Delta}{\Omega}(\pm 1 + \Omega^2\xi^2\tau^2)\frac{1}{\xi^2 + 4\Omega^2\Delta^2\tau^2} \tag{3.62}$$

and

$$\begin{aligned} A(t) &= \frac{\Omega}{2\Delta}\frac{(\xi^2 + 4\Omega^2\Delta^2\tau^2) - i2\Delta(1 - \Omega^2\xi^2\tau^2)\sin 2\Omega t}{(1 - \cos 2\Omega t) + \Omega^2\xi^2\tau^2(1 + \cos 2\Omega t)} \\ B(t) &= \frac{\Omega}{2\Delta}\frac{(\xi^2 + 4\Omega^2\Delta^2\tau^2) + i2\Delta(1 - \Omega^2\xi^2\tau^2)\sin 2\Omega t}{(1 - \cos 2\Omega t) + \Omega^2\xi^2\tau^2(1 + \cos 2\Omega t)} \\ C(t) &= -\frac{\Omega}{2\Delta}\frac{\xi^2 - 4\Omega^2\Delta^2\tau^2}{(1 - \cos 2\Omega t) + \Omega^2\xi^2\tau^2(1 + \cos 2\Omega t)} \end{aligned} \tag{3.63}$$

The time dependent probability density can be written as:

$$\begin{aligned} \psi^\dagger\psi =& N^2 \exp\left[-\frac{\Omega}{2\Delta}(x+y)^2\frac{4\Omega^2\Delta^2\tau^2}{(1 - \cos 2\Omega t) + \Omega^2\xi^2\tau^2(1 + \cos 2\Omega t)}\right. \\ &\left. -\frac{\Omega}{2\Delta}(x-y)^2\frac{\xi^2}{(1 - \cos 2\Omega t) + \Omega^2\xi^2\tau^2(1 + \cos 2\Omega t)}\right] \end{aligned} \tag{3.64}$$

Thus we find

$$\begin{aligned} < z^2(t) > &= \frac{1}{2}\left[\frac{1}{\Omega^2\xi^2}(1 - \cos 2\Omega t) + \tau^2(1 + \cos 2\Omega t)\right] \\ \langle (X(t) + Y(t))^2 \rangle &= \frac{1}{2}\left[\frac{1}{\Omega^2\tau^2}(1 - \cos 2\Omega t) + \xi^2(1 + \cos 2\Omega t)\right] \end{aligned} \tag{3.65}$$

These expressions are notable for their absence of features. Normally one expects that if the initial state is very far from the vacuum, the evolution should delocalize it in a short time, so that the amplitude of the fluctuation of the coordinates should become very large. This is exactly what happens in the standard positive Hamiltonian harmonic oscillator, as we will demonstrate in the next subsection. However eq.(3.65) shows that in the Pais-Uhlenbeck system both interesting averages evolve smoothly in time on the scale determined by the initial state averages. Clearly, if both τ and ξ are finite, the averages stay finite throughout the evolution. This is despite the fact, that the "vacuum" of the system is such that $\tau^2 \propto 1/\Delta \to \infty$, $\xi^2 \propto \Delta \to 0$, as discussed in the previous section. If we start the system "close" to its vacuum state, that is with $\xi^2 \propto 1/\tau^2 \propto \Delta$, it is still true that at all times parametrically the averages are the same, fluctuation with the amplitude proportional to the initial average. Thus it does not matter, if the system starts off far from the vacuum, or close to it, the evolution is smooth and the averages at all times are proportional to those in the initial state.

To underscore that this is very different from the standard harmonic oscillator, we perform the same exercise as above for the two decoupled oscillator systems.

The baseline: oscillators with positive energy

We now consider time evolution generated by

$$H = -\frac{1}{2}\frac{\partial^2}{\partial y^2} - \frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{2}\Omega^2 y^2 + \frac{1}{2}\Omega^2 x^2 \tag{3.66}$$

For a Gaussian wave function eq.(3.53) the evolution of the parameters $A(t)$, $B(t)$ and $C(t)$ is given by:

$$\dot{A} = i[-C^2 - A^2 + \Omega^2]; \quad \dot{B} = i[-C^2 - B^2 + \Omega^2]; \quad \dot{C} = -iC(A + B) \tag{3.67}$$

This is simplified for our initial state where $A(t)$ and $B(t)$ stay equal for all times:

$$\dot{A} = i[-C^2 - A^2 + \Omega^2]; \qquad \dot{C} = -2iAC, \tag{3.68}$$

with initial conditions given as in 3.51. This is solved by

$$C = \frac{1}{f + g\cos(2\Omega t + \phi)}; \qquad A = \frac{ig\Omega \sin(2\Omega t + \phi)}{f + g\cos(2\Omega t + \phi)} \tag{3.69}$$

provided

$$f^2 - g^2 = \frac{1}{\Omega^2} \tag{3.70}$$

Imposing the initial conditions,

$$A(0) = \frac{\Delta\Omega}{\xi^2} + \frac{1}{4\Omega\tau^2\Delta} = \frac{ig\Omega\sin\phi}{2(f + g\cos\phi)}; \qquad C(0) = \frac{\Delta\Omega}{\xi^2} - \frac{1}{4\Omega\tau^2\Delta} = \frac{1}{f + g\cos\phi}. \tag{3.71}$$

we find

$$f = \frac{2\Delta}{\Omega}\frac{\Omega^2\tau^2\xi^2 - 1}{4\Omega^2\Delta^2\tau^2 - \xi^2}; \quad g\sinh\Phi = -\frac{1}{\Omega}\frac{4\Omega^2\Delta^2\tau^2 + \xi^2}{4\Omega^2\Delta^2\tau^2 - \xi^2};$$
$$g\cosh\Phi = 2\Delta\frac{\Omega^2\tau^2\xi^2 + 1}{4\Omega^2\Delta^2\tau^2 - \xi^2} \tag{3.72}$$

where $\Phi = i\phi$. Finally, the solution for our initial conditions is

$$\begin{aligned} A(t) = B(t) =& \frac{\Omega}{2\Delta}\frac{(4\Omega^2\Delta^2\tau^2 + \xi^2)\cos 2\Omega t + i2\Delta(\Omega^2\tau^2\xi^2 + 1)\sin 2\Omega t}{\Omega^2\tau^2\xi^2(1+\cos 2\Omega t) - (1-\cos 2\Omega t) + i(2\Omega^2\Delta\tau^2 + \frac{\xi^2}{2\Delta})\sin 2\Omega t} \\ C(t) =& \frac{\Omega}{2\Delta}\frac{4\Omega^2\Delta^2\tau^2 - \xi^2}{\Omega^2\tau^2\xi^2(1+\cos 2\Omega t) - (1-\cos 2\Omega t) + i(2\Omega^2\Delta\tau^2 + \frac{\xi^2}{2\Delta})\sin 2\Omega t} \end{aligned} \tag{3.73}$$

For small Δ we expand these expressions to second nontrivial order

$$\begin{aligned} A(t) = B(t) &= i\Omega\tan 2\Omega t + \frac{2\Omega\Delta}{\xi^2}\frac{(\Omega^2\tau^2\xi^2 - 1)\cos 2\Omega t + (\Omega^2\tau^2\xi^2 + 1)\cos 4\Omega t}{\sin^2 2\Omega t} \\ C(t) &= -i\frac{\Omega}{\sin 2\Omega t} - \frac{2\Omega\Delta}{\xi^2}\frac{\Omega^2\tau^2\xi^2(1+\cos 2\Omega t) - (1-\cos 2\Omega t)}{\sin^2 2\Omega t} \end{aligned} \tag{3.74}$$

Generically at arbitrary time we have

$$Re[A+C] \propto Re[A-C] \propto \frac{\Delta}{\sin^2 2\Omega t} \tag{3.75}$$

and thus

$$\langle (x-y)^2 \rangle \propto \langle (x+y)^2 \rangle \propto \frac{\sin^2 2\Omega t}{\Delta} \tag{3.76}$$

This is precisely what one normally expects. Our initial state is very far away from the ground state. It was chosen in such a way that the center of mass coordinate $x + y$ had large fluctuations, $O(1/\Delta)$, whereas the relative coordinate $x - y$ had small fluctuations $O(\Delta)$. One expects a state like this to expand very quickly and become delocalized in all coordinates. Indeed eq.(3.76) displays precisely this feature: the relative coordinate fluctuates with amplitude of order $1/\Delta$ almost all the time, except for a very short time interval $\delta t \propto \Delta$ within every period of evolution.

Thus indeed, we see that the time evolution of the Pais - Uhlenbeck oscillator is smoother than that of a system of decoupled harmonic oscillators, in the sense that the averages in the Pais-Uhlenbeck case fluctuate on the scale given by the initial state and do not develop additional large variations throughout the evolution.

3.1.4 A simple unitary interaction

We have seen that the quantum evolution of the Pais-Uhlenbeck oscillator is unitary. This is not very surprising, nor very exciting since the two second order degrees of freedom in this case are decoupled, and each one follows a Harmonic oscillator evolution. In fact the system has two conserved quantum numbers - not just the total energy, but also the energy of each individual oscillator is conserved. For this reason the classical motion in the X, Y plane is bounded and the quantum evolution is unitary.

A more interesting and general question is whether interacting systems with ghosts can be unitary. The worry is clear. We have a Hamiltonian which is unbounded neither from above nor from below, and once the two modes x and

y are allowed to interact, there is a real and present danger that the system can develop an instability, where both x and y run away to infinity even though the total energy stays conserved.

In the quantum mechanical context one can pose the following question: does a system of coupled "particle" and "ghost" degrees of freedom possess normalizable eigenstates. If the answer is affirmative, such system enjoys unitary quantum evolution, since the probability to find the system in finite volume does not decrease with time [3]. If this is not the case, such systems would not allow for unitary quantum mechanical evolution and probability would leak out completely through the boundaries in a finite amount of time.

The aim of this section is to present a simple example of a model, which remains unitary even though it contains interacting particle and ghost degrees of freedom [4]. Let us add to our Hamiltonian a quartic interaction of the form

$$H = \frac{1}{2}\pi_y^2 + \frac{1}{2}\omega_1^2 y^2 + \lambda_1 y^4 - \frac{1}{2}\pi_x^2 - \frac{1}{2}\omega_2^2 x^2 - \lambda_2 x^4 + \mu x^2 y^2 \tag{3.77}$$

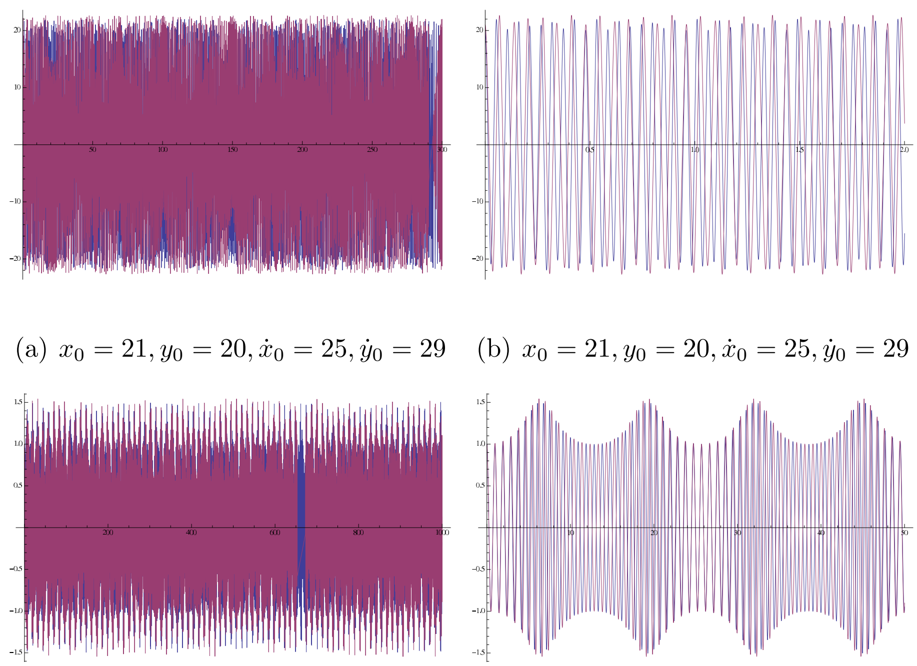
[3] One should qualify this statement slightly. An initial state that has a finite but nonunit projection onto a subspace spanned by normalizable eigenstates will leak probability initially. This leakage will stop after a while and the rest of the evolution will be unitary, preserving the part of the total probability associated with the normalizable subspace. Such behavior is physically perfectly admissible and we will refer to it as unitary disregarding any initial transient leakage of probability

[4] We note that an example of a stable supersymmetric system with ghosts was discussed in [51].

For definiteness we choose $\mu > 0$. At $\mu = 0$ the theory is clearly unitary, as the particle and ghost degrees of freedom are decoupled, and evolution of each one separately is unitary in exactly the same sense as for the Pais - Uhlenbeck oscillator.

The question about stability can be asked already on the classical level. It was noted in [52] and also [47], that some systems of this kind allow for classically stable solutions, namely oscillatory solutions for which the amplitude does not grow without bound as a function of time. Specifically ref. [52] studied numerically the evolution of eq.(3.77) for $\lambda_{1,2} = 0$ and found that the classical behavior of the system is stable as long as the initial energy stored in the oscillators is not too large. Denoting the initial displacement of the oscillators from the equilibrium by M, ref. [52] found that for $M^2 < M_c^2 = \frac{1}{\mu}\omega_2^2$ the behavior is oscillatory, while for $M^2 > M_c^2$ the amplitude of oscillations grows without bound. The addition of the quartic self interaction $\lambda_{1,2}$ further stabilizes the system. We have repeated the numerical exercise of [52] for the system eq.(3.77), and have found a similar behavior in a wider range of parameters. In fact as long as the coupling μ remains small $\mu \ll \lambda_{1,2}$ we did not see classical instability for any initial conditions that we have tried. Examples of evolution for several initial conditions are given in Fig.1. This suggests that when the interaction is weak enough, the classical system is absolutely stable, although it is not possible to prove such a statement by numerical methods.

Reference [47] performed a similar classical study of a relative of the Hamiltonian (eq. 3.77) for which the interaction potential is antisymmetric under $x \to y$. All classical trajectories explored in this paper also did not exhibit any instability. Thus, at least classically, it is certainly possible to find systems with ghosts which are stable.



(a) $x_0 = 21, y_0 = 20, \dot{x}_0 = 25, \dot{y}_0 = 29$ (b) $x_0 = 21, y_0 = 20, \dot{x}_0 = 25, \dot{y}_0 = 29$

(c) $x_0 = 1, y_0 = 1, \dot{x}_0 = 0, \dot{y}_0 = 0$ (d) $x_0 = 1, y_0 = 1, \dot{x}_0 = 0, \dot{y}_0 = 0$

Fig. 3.1: Typical time evolution of x (red) and y (blue) for different initial conditions. The parameters are chosen as $\omega_1 = 3, \lambda_1 = 10, \omega_2 = 5, \lambda_2 = 7, \mu = 3$. The evolution is plotted over two time intervals to show the detailed structure of time dependence and to demonstrate the absense of instability over very long times.

Note, that in order for the quantum system to be unitary, its classical counterpart has to have stable evolution for arbitrary initial conditions. Otherwise quantum tunneling will connect stable and unstable regions of the phase space and will inevitably lead to violation of unitarity. This is the situation, for example in the upside down Mexican hat potential $U(x) = -\lambda(x^2 - x_0^2)^2$. Classical solutions with total energy $-\lambda x_0^4 < E < 0$ and initial displacement $|x| < x_0$ are regular. However quantum mechanically the system is non-unitary due to finite probability of tunneling into the unbounded region $|x| > x_0$.

In the present case, one can give an argument that the theory remains stable, at least in a limited range of parameters. Let us consider the limit $\omega_1 \ll \omega_2$. In this case one can use the classical Born-Oppenheimer approximation. Since y oscillates much faster than x, one can consider the motion of y in the background of fixed x. Thus for given x the dynamics of y is given simply by an anharmonic oscillator with the frequency, which for large x behaves as $\omega^2 = \mu x^2$. This is clearly a well defined bounded motion. The dynamics of x is affected by the average value of y^2 for a given trajectory. Given the initial energy E stored in the mode y, we have (for large x, which is the interesting and potentially dangerous regime) $\bar{y}^2 \propto E/\mu x^2$. The dynamics of x then is governed by the effective potential

$$\frac{1}{2}\omega_2^2 x^2 + \lambda_2 x^4 - \mu \bar{y}^2 x^2 = \frac{1}{2}\omega_2^2 x^2 + \lambda_2 x^4 - E \tag{3.78}$$

Thus the dynamics of x in this approximation is unaffected by y and is bounded and stable. A similar argument can be given for the opposite case $\omega_1 \gg \omega_2$.

Thus at least when the two frequencies are very different there is no instability for arbitrary initial conditions. In this case one expects that the quantum theory is well defined and unitary in the sense explained above.

In the next subsection we present another line of reasoning supporting the same conclusion for small μ.

Asymptotics of Eigenfunctions for Small μ

One way to establish that a quantum theory has normalizable eigenstate is to find asymptotics of eigenfunctions for large values of coordinates x and y.

As usual, we introduce an eikonal S via

$$\Psi = Ne^{-S(x,y)} \tag{3.79}$$

If the eikonal is positive and divergent for large values of the coordinates, the wave function is normalizable. For large values of S, $|x|$ and $|y|$ it satisfies the following "semiclassical" equation:

$$-\frac{1}{2}\left(\frac{\partial S}{\partial y}\right)^2 + \frac{1}{2}\left(\frac{\partial S}{\partial x}\right)^2 + \lambda_1 y^4 - \lambda_2 x^4 + \mu x^2 y^2 = 0 \tag{3.80}$$

We will not attempt to solve this equation in full generality, but rather explore the behavior of S for small values of μ. For $\mu = 0$ the solution is simply a sum of the solutions for two decoupled degrees of freedom:

$$S_0(x,y) = \frac{\sqrt{2\lambda_1}}{3}|y|^3 + \frac{\sqrt{2\lambda_2}}{3}|x|^3 \tag{3.81}$$

The crucial point is that the structure of the potential is such that for $\mu \ll \lambda_i$, the perturbation is smaller than the leading order potential for generic large values of x and y. This is of course very different from the standard perturbation theory around a harmonic oscillator potential, where a perturbation is usually bigger than the unperturbed potential for large values of the coordinate. Thus although the standard perturbation theory around a Harmonic potential is asymptotic, we expect the perturbation theory in μ to have a finite radius of convergence.

Let us therefore solve eq.(3.79) perturbatively. Let $S = S_0 + S_1$, where $S_1 \propto \mu$. We first solve the equation for $x, y > 0$. To first order in μ we have:

$$-\sqrt{2\lambda_1}y^2\frac{\partial S_1}{\partial y} + \sqrt{2\lambda_2}x^2\frac{\partial S_1}{\partial x} + \mu x^2 y^2 = 0 \tag{3.82}$$

Changing variables $\bar{x} = \frac{1}{\sqrt{2\lambda_2}x}$, $\bar{y} = \frac{1}{\sqrt{2\lambda_1}y}$ and defining $x^{\pm} = \bar{x} \pm \bar{y}$ the equations becomes simple

$$\frac{\partial S_1}{\partial x^-} = \frac{4\mu}{\lambda_1\lambda_2}\frac{1}{(x^{+2} - x^{-2})^2} \tag{3.83}$$

A well behaved solution to this equation is:

$$S_1(x^+, x^-) = \frac{4\mu}{\lambda_1\lambda_2}\frac{1}{2x^{+3}}\left[-\frac{x^+x^-}{x^{-2} - x^{+2}} + arctanh\left(\frac{x^-}{x^+}\right)\right] \tag{3.84}$$

In terms of the original variables, the solution can be written as:

$$\begin{aligned} S_1(x > 0, y > 0) =& \sqrt{2}\mu x^2 y^2 \frac{\sqrt{\lambda_1}y - \sqrt{\lambda_2}x}{(\sqrt{\lambda_1}y + \sqrt{\lambda_2}x)^2} \\ &+ 2\sqrt{2}\mu\sqrt{\lambda_1\lambda_2}\frac{x^3y^3}{(\sqrt{\lambda_1}y + \sqrt{\lambda_2}x)^3}\log(\sqrt{\frac{\lambda_1}{\lambda_2}}\frac{y}{x}) \end{aligned} \tag{3.85}$$

Extending the solution to other regions of the plane we find

$$\begin{aligned} S_1(x,y) =& \sqrt{2}\mu x^2 y^2 \frac{\sqrt{\lambda_1}|y| - \sqrt{\lambda_2}|x|}{(\sqrt{\lambda_1}|y| + \sqrt{\lambda_2}|x|)^2} \\ & + 2\sqrt{2}\mu\sqrt{\lambda_1\lambda_2}\frac{|x|^3|y|^3}{(\sqrt{\lambda_1}|y| + \sqrt{\lambda_2}|x|)^3}\log(\sqrt{\frac{\lambda_1}{\lambda_2}}\frac{|y|}{|x|}) \end{aligned} \tag{3.86}$$

As expected, the correction S_1 is smaller than S_0 at large values of the arguments, and thus the asymptotics of the wave function is determined by S_0. Thus we find that for small μ our model quantum mechanically has normalizable eigenstates, and therefore unitary evolution.

3.1.5 Discussion

Although Pais-Uhlenbeck quantum mechanics is just a toy model, the existence of unitary ghost theories could potentially have very interesting implications, especially if the mechanism of stabilization by self-interaction can be extended to the realm of field theory. In particular, one can ask whether the perturbative divergence of the rate of decay discussed in [43] is always an indication of instability. In principle it is possible that this perturbative result only means that the perturbative state gets modified at all momenta, including the ultraviolet; however, modification does not necessarily mean instability. In the quantum-mechanical example discussed in this section, perturbation theory would certainly give a finite decay rate of the "perturbative vacuum" (the Gaussian eigenstate of the noninteracting system). However, since the theory is in fact stable, it still has a normalizable eigenstate which evolves into the perturbative vacuum when all couplings vanish. It is conceivable that the fate of the vacuum in the model discussed in [42] [43] is similar; the perturbative vacuum is not unstable, but simply evolves into another normalizable state. Since the theory has Lorentz invariance, the new state presumably will also be Lorentz invariant, and will have a relativistic excitation spectrum, which could be identified with physical particles. This mechanism would require a strong enough self interaction of all the modes, including the deep ultraviolet ones, and it may not be possible to achieve this with the standard model interactions (in the particle and ghost sectors), or indeed in any renormalizable matter theory. There is however an interesting possibility that, if the gravitational sector is described by conformal gravity - which is much softer in the ultraviolet than general relativity-the "decay rate" does not diverge and the stability of the "vacuum" can be achieved with a renormalizable self-interaction. Such a scenario would of course necessitate the stability of the conformal gravity itself.

3.2 Conformal Gravity Redux: Ghost Turned Tachyon

As explicitly demonstrated in section 3.1, in simple quantum mechanical systems presence of ghosts does not immediately signal instability even if the theory is interacting. This is by no means an isolated example, other discussions of consistent simple models with interacting ghost and normal modes can be found in [47] [53]. But in a quantum field theory such stability must be much harder to achieve due to many excitation channels available [43]. Nevertheless it is an interesting open question, whether the ghost modes in conformal gravity do indeed render the full interacting theory unstable, or perhaps the theory is consistent "as is"[5]. In fact it has been shown that the number of local conserved quantities in conformal gravity is equal to the number of perturbative ghost modes [54]. This can give hope that the dynamics is constrained enough and not ergodic to an extent that instabilities do not appear even in the interacting theory.

Complete analysis of an interacting theory of gravity is a very complicated proposition. Our aim in this section is much more modest. We ask if the theory has instabilities when the number of degrees of freedom is restricted to translationally invariant modes. The requirement of translational invariance is very severe and reduces the field theory to a theory of a finite albeit relatively large number of classical degrees of freedom. We derive the Hamiltonian for this system and study classical behavior of its solutions. Our result is somewhat unexpected. We find that the theory is unstable on the classical level. The instability is of a somewhat different nature than what we may have expected from the previous argument. It is not due to transfer of large amount of energy from ghost modes to normal modes. Instead the nonlinearity of the interaction induces a potential for the

[5] We note that it may be possible to make sense of conformal gravity even if ghosts do cause instability by either restricting oneself to a subset of solutions [28], or invoking quantization with nonstandard inner product [37]. However we are not aware at the moment of a fully consistent way of implementing either of these suggestions in conformal gravity.

ghost modes which is positive. Thus the ghost becomes also a tachyon - it's kinetic term is negative, while its potential is positive. Thus the ghost sector becomes unstable by itself. We find simple classical solutions for which normal modes are vanishing, and ghost modes diverge within a finite amount of time, set by the initial conditions.

3.2.1 The Hamiltonian of the Reduced Theory

Conformal gravity is defined by the action

$$S = -\int d^4x\sqrt{-g}(3R_{\mu\nu}R^{\mu\nu} - R^2) \tag{3.87}$$

with the usual definitions of the Riemann and Ricci tensors $R^{\rho}{}_{\mu\sigma\lambda} = -\partial_\sigma \Gamma^{\rho}{}_{\mu\lambda} + \ldots$ and $R_{\mu\lambda} = R^{\sigma}{}_{\mu\sigma\lambda}$. We use the metric convention $(+,-,-,-)$. Since our interest is in the classical theory, we set the dimensionless coupling constant to unity, as its value does not affect solutions of equation of motion.

We treat this Lagrangian as a Lagrangian of an ordinary field theory. We will derive the Hamiltonian which generates classical time evolution by Legendre transforming it rather than using the ADM procedure [55]. Since the Lagrangian possesses gauge invariance, this is of course a constrained system, and constraints have to be properly taken into account. The Lagrangian, as is well known is gauge invariant under the general linear transformation

$$g_{\rho\sigma}(x) \to g'_{\rho\sigma}(x') = g_{\mu\nu}(x)\frac{\partial x^{\mu}}{\partial x^{\rho\prime}}\frac{x^{\nu}}{\partial x^{\sigma\prime}} \tag{3.88}$$

and, in addition the local conformal transformation:

$$g_{\mu\nu}(x) \to \tilde{g}_{\mu\nu}(x) = \Omega^2(x) g_{\mu\nu}(x) \tag{3.89}$$

We choose to impose a simple gauge fixing condition:

$$g_{00} = 1, \qquad g_{i0} = 0. \tag{3.90}$$

This gauge condition does not fix one combination of conformal and general linear transformations (see Appendix 5), and we will deal with this remaining gauge symmetry later.

We truncate the theory by taking the metric to be space independent $g_{\mu\nu} = g_{\mu\nu}(t)$. The non vanishing components of the Christoffel symbol and Ricci tensor, in the gauge eq.(3.90) for metric that does not depend on spatial coordinates, are:

$$\Gamma^0{}_{ij} = -\frac{1}{2}\partial g_{ij}, \Gamma^i{}_{0j} = \frac{1}{2} g^{ik} \partial g_{jk} \tag{3.91}$$

$$R_{00} = \partial \Gamma^i{}_{0i} + \Gamma^i{}_{j0}\Gamma^j{}_{i0} = \frac{1}{2}\partial(g^{ij}\partial g_{ij}) + \frac{1}{4} g^{ik}\partial g_{kj} g^{jm} \partial g_{mi} = \frac{1}{2}\partial\alpha - \frac{1}{4}\beta \tag{3.92}$$

$$R_{ij} = -(\partial\Gamma^0{}_{ij} + \Gamma^k{}_{k0}\Gamma^0{}_{ij}) + (\Gamma^0{}_{kj}\Gamma^k{}_{0i} + \Gamma^k{}_{0j}\Gamma^0{}_{ki}) = \frac{1}{2}\partial^2 g_{ij} + \frac{1}{4}\alpha\partial g_{ij} - \frac{1}{2}\alpha^k{}_j \partial g_{ki} \tag{3.93}$$

$$R = \frac{1}{4}\partial\alpha - \frac{1}{4}\beta + \frac{1}{4}\alpha^2 \tag{3.94}$$

where, we have defined:

$$\alpha^i{}_j = g^{ik}\partial g_{kj}; \qquad \alpha = \alpha^i{}_i \tag{3.95}$$

$$\beta^i{}_j = \partial g^{ik} \partial g_{kj}; \qquad \beta = \beta^i{}_i \tag{3.96}$$

The action can be written as:

$$\begin{aligned} S = & - \int dt \sqrt{-g} \left[3 \left((\frac{1}{2}\partial\alpha - \frac{1}{4}\beta)^2 + (\frac{1}{2}\partial\alpha^a{}_j + \frac{1}{4}\alpha\alpha^a{}_j)(\frac{1}{2}\partial\alpha^j{}_a + \frac{1}{4}\alpha\alpha^j{}_a) \right) \right. \\ & - \left. \left((\frac{1}{2}\partial\alpha - \frac{1}{4}\beta) + (\frac{1}{2}\partial\alpha + \frac{1}{4}\alpha^2) \right)^2 \right] \\ = & - \int dt \sqrt{-g} \left[-\frac{1}{2}(\beta + \alpha^2)(\frac{1}{2}\partial\alpha - \frac{1}{4}\beta) + 3(\frac{1}{4}\partial\tilde{\alpha}^a_b \partial\tilde{\alpha}^b_a + \frac{1}{4}\alpha\tilde{\alpha}^a_b\partial\tilde{\alpha}^b_a \right. \\ + & \left. \frac{1}{16}\alpha^2 \tilde{\alpha}^a_b \tilde{\alpha}^b_a) \right] \end{aligned} \tag{3.97}$$

Where, $\tilde{\alpha}^a_b$ is the traceless part of $\alpha^a{}_b$

$$\tilde{\alpha}^a_b = \alpha^a{}_b + \frac{1}{3}\alpha g^a_b \tag{3.98}$$

After some simple manipulations, involving integration by parts, this can be written as

$$S = -\int dt \sqrt{-g} \left[\frac{3}{4}\partial\tilde{\alpha}^a_b \partial\tilde{\alpha}^b_a - \frac{1}{8}\partial\alpha \mathrm{tr}(\tilde{\alpha}^2) - \frac{1}{24}\alpha^2 \mathrm{tr}(\tilde{\alpha}^2) + \frac{1}{8}\left[\mathrm{tr}(\tilde{\alpha}^2)\right]^2 \right] \tag{3.99}$$

Or using the identity

$$\partial[\sqrt{-g}\left[\mathrm{tr}(\tilde{\alpha}^2)\right]] = \frac{1}{2}\sqrt{-g}\alpha^2\left[\mathrm{tr}(\tilde{\alpha}^2)\right] + \sqrt{-g}\partial\alpha\tilde{\alpha}^a_b\partial\tilde{\alpha}^b_a + 2\sqrt{-g}\alpha\tilde{\alpha}^a_b\partial\tilde{\alpha}^b_a \tag{3.100}$$

and integrating by parts

$$S = -\int dt \sqrt{-g} \left[3\partial\tilde{\alpha}^a_b\partial\tilde{\alpha}^b_a + \alpha\tilde{\alpha}^a_b\partial\tilde{\alpha}^b_a + \frac{1}{2}\mathrm{tr}(\tilde{\alpha}^2)\left(\mathrm{tr}(\tilde{\alpha}^2) + \frac{\alpha^2}{6}\right) \right] \tag{3.101}$$

The latter form is more convenient for applications since it makes it obvious that no time derivatives of α appear in the action.

Since we imposed gauge conditions in the action, we must in principle separately keep track of constraints that would be generated by variation of the action with respect to g_{00} and g_{i0}. However in our reduced theory this turns out not to be necessary. The variation of the action with respect to $g_{\mu 0}$ results in the equations

$$B_{\mu 0} = 0 \tag{3.102}$$

where $B_{\mu\nu}$ is the so called Bach tensor:

$$B_{\mu\nu} \equiv \nabla^{\alpha}\nabla^{\beta}C_{\mu\alpha\nu\beta} - \frac{1}{2}R^{\alpha\beta}C_{\mu\alpha\nu\beta} = 0. \tag{3.103}$$

Here $C_{\mu\alpha\nu\beta}$ is the conformal tensor - the traceless part of the Riemann tensor:

$$C_{\mu\nu\alpha\beta} = R_{\mu\nu\alpha\beta} - (g_{\mu[\alpha}R_{\beta]\nu} - g_{\nu[\alpha}R_{\beta]\mu}) + \frac{1}{3}Rg_{\mu[\alpha}g_{\beta]\nu}. \tag{3.104}$$

However, in the gauge g_{i0} in the reduced theory (no x_i dependence) it is obvious that $B_{i0} = 0$ identically. The Bach tensor is by definition traceless, thus identically

$$B_{00} = g^{ij}B_{ij} \tag{3.105}$$

Therefore B_{00} vanishes automatically when the spatial components vanish. These are required to vanish by equations of motion that follow from the action eq.(3.99). Thus in the translationally invariant approximation, constraints eq.(3.102) do not add any new information, and we can forget about their existence.

The Hamiltonian

Our aim now is to derive the Hamiltonian for the system described by the action eq.(3.99). Since the fields α are related to the time derivative of g_{ij}, we introduce this relation into the action with the help of the Lagrange multiplier

$$S = -\int dt\sqrt{-g}\left[3\partial\tilde{\alpha}^a_b\partial\tilde{\alpha}^b_a + \alpha\tilde{\alpha}^a_b\partial\tilde{\alpha}^b_a + \frac{1}{2}\mathrm{tr}(\tilde{\alpha}^2)\left(\mathrm{tr}(\tilde{\alpha}^2) + \frac{\alpha^2}{6}\right)\right. \\ \left. - \lambda^a{}_b(\alpha^b{}_a - g^{bc}\partial g_{ca})\right] \tag{3.106}$$

The conjugate momenta are:

$$p^{ij} = \frac{\partial L}{\partial(\partial g_{ij})} = -\frac{\sqrt{-g}}{2}[\lambda^i_b g^{jb} + \lambda^j_b g^{ib}] \tag{3.107}$$

$$\beta^i{}_j = \frac{\partial L}{\partial(\partial\tilde{\alpha}^j_i)} = -\sqrt{-g}(6\partial\tilde{\alpha}^i_j + \alpha\tilde{\alpha}^i_j) \tag{3.108}$$

and

$$p_\alpha = p_\lambda = 0 \tag{3.109}$$

To find the Hamiltonian, we take the Legendre transform of the action and use eq.(3.107) to express λ^a_b in terms of the momenta p^{ij}. The resulting Hamiltonian is

$$H = p^{ij}\partial g_{ij} - L = -\frac{1}{\sqrt{-g}}\frac{1}{12}\beta^i{}_j\beta^j{}_i + \frac{1}{6}\alpha\tilde{\alpha}^i_j\beta^j{}_i + \frac{1}{2}\sqrt{-g}\left[\mathrm{tr}(\tilde{\alpha}^2)\right]^2 - \alpha^b{}_a p^{an} g_{nb} \tag{3.110}$$

The is complemented by a primary constraint

$$\beta = 0 \tag{3.111}$$

Commuting (calculating the Poisson brackets) the constraint with the Hamiltonian, we obtain the secondary constraint

$$\{H,\beta\} = C_1 = \frac{1}{6}\tilde{\alpha}^i_j\beta^j{}_i - \frac{1}{3}p^{ac}g_{ac} = 0 \tag{3.112}$$

In turn, commuting C_1 with the Hamiltonian, we obtain another secondary constraint

$$\{H,C_1\} = \frac{1}{12}\frac{\beta^j{}_i\beta^i{}_j}{\sqrt{-g}} - \frac{1}{2}\sqrt{-g}\left[\mathrm{tr}(\tilde{\alpha}^2)\right]^2 + \tilde{\alpha}^i_j p^{jk}g_{ki} = C_2 \tag{3.113}$$

Commuting this with the Hamiltonian no new constraints are generated.

Note that

$$H = -C_2 + \alpha C_1 \tag{3.114}$$

and thus the Hamiltonian vanishes on the constraint surface. This is natural in a conformal theory. Classically however, it only means that we should consider such solutions of equations of motion which have zero energy. The Hamiltonian is still an important quantity, as it generates the equations of motion, even though the energy vanishes on intersting classical trajectories.

Following the standard Dirac procedure, the first order constraint eq.(3.111) can be supplemented by another condition which turns the constraints into second order. A convenient choice is

$$\alpha = 0 \tag{3.115}$$

With this choice the Hamiltonian simplifies and we will adopt it in the following.

General Linear Transformations

Before analyzing equations of motion and their solutions, we note that our model has a large number of symmetries. We have already discussed gauge symmetry, which was inherited from the complete theory where original gauge transformations were taken to be independent of spatial coordinates. However there is a larger subgroup of the original space-time dependent gauge group, which preserves the independence of the metric on x_i. These transformations appear in the reduced model not as gauge symmetries with associated constraints, but rather as global symmetries. The reason there are no constraints associated with these symmetries in the reduced model, is that they are automatically satisfied when the fields are taken to be x_i-independent.

Consider a general linear transformation that does not induce space dependence in the metric, and preserves the gauge conditions eq.(3.90). It's infinitesimal form is:

$$x'^{\alpha} = (\delta^{\alpha}{}_{\beta} + \omega^{\alpha}{}_{\beta})x^{\beta} \tag{3.116}$$

with $\omega^0{}_0 = 0; \quad \omega^0{}_k = 0$

The transformation of the metric is:

$$\begin{aligned} g_{ij} &\rightarrow g_{ij} - g_{ik}\omega^k{}_j - g_{kj}\omega^k{}_i \\ g^{ij} &\rightarrow g^{ij} + g^{ik}\omega^j{}_k + g^{kj}\omega^i{}_k \end{aligned} \tag{3.117}$$

For this to be a canonical transformation, the momenta have to transform as

$$\delta p^{ij} = \omega^i{}_b p^{bj} + \omega^j{}_b p^{ib} \tag{3.118}$$

The transformation of α and β can be found using the expression of α in terms of time derivative of g, and again requiring that the transformation is canonical

$$\begin{aligned} \alpha^i{}_j &\rightarrow \alpha^i{}_j(1+\frac{1}{3}\omega) - \omega^k{}_j\alpha^i{}_k + \omega^i{}_k\alpha^k{}_j \\ \beta^i{}_j &\rightarrow \beta^i{}_j(1-\frac{1}{3}\omega) - \omega^k{}_j\beta^i{}_k + \omega^i{}_k\beta^k{}_j \end{aligned} \tag{3.119}$$

where, $\omega \equiv \omega^i_i$.

It is indeed easy to check that this transformation leaves the Hamiltonian invariant. One has

$$\delta H = \frac{\omega}{3} H \tag{3.120}$$

which vanishes on the constraint surface.

The matrix ω_{ij} is an arbitrary real matrix, thus providing us with 9 symmetries. One of them, corresponding to $\omega_{ij} \propto \delta_{ij}$ however coincides with the conformal transformation. We should therefore strictly speaking consider only the traceless part ω_{ij} as generators of global symmetry transformations. The theory thus has 8 symmetries. With such large number of conserved quantities, as discussed in the introduction, one might hope that the dynamics of the model is stable. We will see however, that this is not the case. Nevertheless this large number of conserved quantity is handy to be able to find solutions of equations of motion.

3.2.2 Solving the equations of motion

Before directly tackling the solution of equations of motion it is useful to introduce a different set of coordinates, which simplifies this problem somewhat. At the moment our Hamiltonian is written in terms of basic variables g_{ij} and $\tilde{\alpha}^i_j$. However not all of them are independent. The metric g_{ij} is symmetric and contains 6 degrees of freedom, while $\tilde{\alpha}^i_j$ is not symmetric, but is nevertheless constrained sine $g_{ij}\tilde{\alpha}^j_k$ is by definition a symmetric matrix. Additionally, we set $\alpha = 0$. Also the constraint eq.(3.112) can be used to eliminate one more degree of freedom. We can use it for example to fix $g = -1$. Thus in total we have 10 degrees of freedom. We will use the parametrization that makes these independent degrees of freedom more accessible.

We introduce the general real matrix Λ by

$$g_{ij} = -\left[\Lambda\Lambda^T\right]_{ij} \tag{3.121}$$

This relation defines Λ only up to a rotation, as Λ and ΛO give the same matrix g. To define it completely we take

$$\tilde{\alpha}_{ij} = \left[\Lambda^{T^{-1}}\gamma\Lambda^T\right]_{ij} \tag{3.122}$$

with γ - a diagonal traceless matrix

$$\gamma = \begin{vmatrix} \gamma_1 & 0 & 0 \\ 0 & \gamma_2 & 0 \\ 0 & 0 & -(\gamma_1+\gamma_2) \end{vmatrix} .$$

With general γ eq.(3.122) is just a similarity transformation, but requiring γ to be diagonal fixes the freedom in Λ left undetermined by eq.(3.121). Tracelessnes of γ follows from the tracelessnes of $\tilde{\alpha}$. The general matrix Λ has 9 degrees of freedom, which we will reduce to 8 by requiring $|\Lambda| = 1$. Together with two components of diagonal, traceless γ this constitutes the original 10 degrees of freedom present in $\{g, \tilde{\alpha}\}$.

In terms of the new variables we have

$$\begin{aligned} \dot{g} &= -(\dot{\Lambda}\Lambda^T + \Lambda\dot{\Lambda}^T) \\ \dot{\alpha} &= \Lambda^{T^{-1}}(\dot{\gamma} + \gamma\dot{\Lambda}^T\Lambda^{T^{-1}} - \dot{\Lambda}^T\Lambda^{T^{-1}}\gamma) = \Lambda^{T^{-1}}(D_0\gamma)\Lambda^T \end{aligned} \tag{3.123}$$

where

$$D_0\gamma \equiv \dot{\gamma} + [\gamma, M]; \qquad M \equiv \dot{\Lambda}^T\Lambda^{T^{-1}} \tag{3.124}$$

The action eq.(3.99) can now be written as :

$$\begin{aligned} S = -|\Lambda| \int dt \Big\{ & 3\mathrm{tr}\left(\dot{\gamma}^2 + [\gamma, M]^2\right) + \frac{1}{2}[\mathrm{tr}[\gamma^2]]^2 - \mathrm{tr}\ \tilde{\mu}\left[\gamma - \left(M + M^T\right)\right] \\ & + \alpha\mathrm{tr}[\gamma\dot{\gamma}] + \frac{1}{3}\mu[\alpha - 2\mathrm{tr}M] + \frac{1}{2}\alpha^2\mathrm{tr}[\gamma^2] \Big\} \end{aligned} \tag{3.125}$$

The Lagrange multiplier (symmetric) matrix $\tilde{\mu}$ enforces the constraint relating $\tilde{\alpha}$ to time derivative of g. Just like in the previous section, we can set $\alpha = 0$, since there is no time derivative of α in eq.(3.125). This can be done, but only after requiring that the variation of S with respect to α vanishes. This variation leads to a constraint

$$\frac{\partial S}{\partial \alpha}\Big|_{\alpha=0} = |\Lambda|(\mathrm{tr}\gamma\dot{\gamma} + \frac{1}{3}\mu) = 0; \tag{3.126}$$

This is the generator of the conformal gauge transformation expressed in the new variables.

Calculating momenta conjugate to Λ, we find

$$p_{ij} = \frac{\partial L}{\partial \dot{\Lambda}_{ij}} = -|\Lambda| \left[\Lambda^{T^{-1}} \left(6[[\gamma, M], \gamma] + 2(\tilde{\mu} - \frac{1}{3}I\mu)\right)\right]_{ij} \tag{3.127}$$

Note that on the constraint surface the symmetric part of matrix M is proportional to γ. Thus only the antisymmetric part of M contributes to the commutator in eqs.(3.125,3.127). Using this, we find

$$\begin{aligned} \frac{1}{2}(\Lambda^T p - p\Lambda^T) &= -6|\Lambda|[[\gamma, \frac{1}{2}(M - M^T)], \gamma] \\ \frac{1}{2}(\Lambda^T p + p\Lambda^T) &= -2|\Lambda|(\tilde{\mu} - \frac{1}{3}I\mu) \end{aligned} \tag{3.128}$$

Conjugates to γ are found as

$$p_1 = \frac{\partial L}{\partial \dot{\gamma}_1} = -6|\Lambda|(2\dot{\gamma}_1 + \dot{\gamma}_2), \qquad p_2 = \frac{\partial L}{\partial \dot{\gamma}_2} = -6|\Lambda|(2\dot{\gamma}_2 + \dot{\gamma}_1) \tag{3.129}$$

The Hamiltonian is:

$$H = \frac{1}{2}\Lambda^T p\gamma - 3|\Lambda|[\gamma, \frac{1}{2}(M - M^T)]^2 + \frac{1}{18|\Lambda|}(-p_1^2 - p_2^2 + p_1 p_2) + |\Lambda|(\gamma_1^2 + \gamma_2^2 + \gamma_1\gamma_2)^2 \tag{3.130}$$

It is now possible to express the second term in terms of conjugate momenta using eq.(3.128). It is most simply done by expanding both sides of eq.(3.128) in terms of the complete basis of 3×3 matrices. After some straightforward algebra, we find:

$$[\gamma, \frac{1}{2}(M - M^T]^2$$
$$= \frac{1}{18|\Lambda|^2}\left[\left(\frac{(\Lambda^T p - p\Lambda^T)_{12}}{\gamma_2 - \gamma_1}\right)^2 + \left(\frac{(\Lambda^T p - p\Lambda^T)_{13}}{\gamma_2 + 2\gamma_1}\right)^2 + \left(\frac{(\Lambda^T p - p\Lambda^T)_{23}}{2\gamma_2 + \gamma_1}\right)^2\right] \tag{3.131}$$

Finally, diagonalizing the quadratic term in the Hamiltonian, we obtain:

$$H = -\frac{1}{18|\Lambda|}[\tilde{p_1}^2 + \tilde{p_2}^2] + \frac{9}{16}|\Lambda|[\tilde{\gamma}_1^2 + \tilde{\gamma}_2^2]^2 + \frac{1}{2}\text{tr}\left(\Lambda^T p\gamma\right)$$
$$-\frac{1}{6|\Lambda|}\left[\left(\frac{(\Lambda^T p - p\Lambda^T)_{12}}{\gamma_2 - \gamma_1}\right)^2 + \left(\frac{(\Lambda^T p - p\Lambda^T)_{13}}{\gamma_2 + 2\gamma_1}\right)^2 + \left(\frac{(\Lambda^T p - p\Lambda^T)_{23}}{2\gamma_2 + \gamma_1}\right)^2\right] \tag{3.132}$$

Where,

$$\tilde{p_1} = \frac{1}{2}(p_1 + p_2), \qquad \tilde{p_2} = \frac{\sqrt{3}}{2}(-p_1 + p_2) \tag{3.133}$$

and

$$\tilde{\gamma_1} = (\gamma_1 + \gamma_2), \ \tilde{\gamma_2} = \frac{1}{\sqrt{3}}(-\gamma_1 + \gamma_2) \tag{3.134}$$

The canonical form of the constraint eq.(3.126), which supplements this Hamiltonian is:

$$\frac{1}{3}(p_1\gamma_1 + p_2\gamma_2) + \text{tr}(\Lambda^T p) = 0 \tag{3.135}$$

As noted above, we fix the gauge freedom associated with this constraint by setting

$|\Lambda| = 1$[6].

Our goal here is to see whether the Hamiltonian has unstable solutions. We will not look for a general solution of equations of motion, but instead will analyze a simple subset of those. The simplification is possible due to the following observation. Let us define for convenience traceless matrices

$$\tau_1 = \mathrm{diag}(1,0,-1); \quad \tau_2 = \mathrm{diag}(0,1,-1); \quad \sigma^a_{ij} = \epsilon_{aij} \tag{3.136}$$

$$\lambda^1 = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}; \ \lambda^2 = \begin{vmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix}; \ \lambda^3 = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \tag{3.137}$$

and associated generators of the general linear transformations

$$G^i = \mathrm{tr}(\Lambda^T p \tau^i); \quad G^a_A = \mathrm{tr}(\Lambda^T p \sigma^a); \quad G^a_S = \mathrm{tr}(\Lambda^T p \lambda^a) \tag{3.138}$$

In terms of these, the Hamiltonian is written

$$\begin{aligned} H = & -\frac{1}{18|\Lambda|}[\tilde{p_1}^2 + \tilde{p_2}^2] + \frac{9}{16}|\Lambda|[\tilde{\gamma}_1^2 + \tilde{\gamma}_2^2]^2 + \frac{1}{2}\Sigma_i \left(G^i \gamma_i\right) \\ & -\frac{1}{6|\Lambda|}\left[\left(\frac{G^3_A}{\gamma_2 - \gamma_1}\right)^2 + \left(\frac{G^2_A}{\gamma_2 + 2\gamma_1}\right)^2 + \left(\frac{G^1_A}{2\gamma_2 + \gamma_1}\right)^2\right] \end{aligned} \tag{3.139}$$

Note that for all of these generators, we have $[|\Lambda|, G^\alpha] = 0$. Consider a solution, which at initial time has $G^i = G^a_A = G^a_S = 0$. Since commutator of any of the generators G^α with the Hamiltonian eq.(3.139) is proportional to, at least the first

[6] We do this only after deriving equations of motion to avoid the necessity to introduce Dirac brackets.

power of G^β, this condition is preserved in time, and all the generators G^α vanish at all times. We can think of this initial condition, as an initial condition imposed on p_{ij} for arbitrary initial Λ_{ij}. For this set of initial conditions, the equations of motion therefore simplify considerably. The equation of motion for Λ becomes

$$\dot{\Lambda}_{ij} = \frac{1}{2}(\gamma\Lambda)_{ij} \tag{3.140}$$

This determines Λ once the solution for γ is known as

$$\Lambda = A\left(\exp\int_0^t \frac{\gamma}{2}dt\right) \tag{3.141}$$

where A is the initial condition.

The equations of motion for γ then are derived from he reduced Hamiltonian

$$H = -\frac{1}{18}[\tilde{p_1}^2 + \tilde{p_2}^2] + \frac{9}{16}[\tilde{\gamma}_1^2 + \tilde{\gamma}_2^2]^2 \tag{3.142}$$

where we have set $|\Lambda| = 1$ with accordance to previous discussion.

The reduced Hamiltonian is a simple upside-down unharmonic oscillator. The kinetic term is negative, in accordance with the fact that γ_i appear as ghost modes in the linearized theory, where the unharmonic potential is absent. Interestingly, the sign of the potential is positive, and therefore it is clear that the dynamics of the reduced model is unstable. To see this explicitly, consider a simple solution of equations of motion, corresponding to vanishing "'angular momentum" in the $\tilde{\gamma}_1 - \tilde{\gamma}_2$ plane. We also have to impose the constraint of zero energy, which is an easy task in the reduced model. Solutions under these conditions are very simple

$$\tilde{\gamma}_1 = \gamma_r \ \cos\theta; \quad \tilde{\gamma}_2 = \gamma_r \ \sin\theta \tag{3.143}$$

with

$$\theta = const; \quad \gamma_r = \frac{\gamma_0}{1 \pm \frac{\gamma_0}{2^{3/2}} t} \tag{3.144}$$

The two solutions correspond to the sign of the initial radial velocity. For negative initial velocity (sign + in eq.(3.144)), the “particle” initially moves towards the origin. This is a stable solution, since at infinite time the particle simply climbs to the top of the potential, and ends up there with zero velocity. For positive initial relative velocity (sign − in eq.(3.144)) the particle moves away from the origin. This solution is unstable. The instability is in fact much worse than would be for an upside down harmonic oscillator. The particle reaches infinite distance within a finite time $t_c = 2^{3/2}/\gamma_0$.

Transforming to the original variables we find

$$\gamma_{1,2} = \frac{1}{2}(\cos\theta \mp \sqrt{3}\sin\theta)\gamma_r = \frac{1}{2}(\cos\theta \mp \sqrt{3}\sin\theta)\frac{\gamma_0}{1 \pm \frac{\gamma_0}{2^{3/2}} t} \tag{3.145}$$

The metric g is found to be

$$g_{ij} = -[A\Gamma A^T]_{ij} \tag{3.146}$$

where Γ is the diagonal matrix with the following non-vanishing matrix elements

$$\begin{aligned} \Gamma_{11} =& |1 \pm \frac{\gamma_0}{2^{3/2}} t|^{2^{3/2}\left(\cos\theta - \sqrt{3}\sin\theta\right)}; \\ \Gamma_{22} =& |1 \pm \frac{\gamma_0}{2^{3/2}} t|^{2^{3/2}\left(\cos\theta + \sqrt{3}\sin\theta\right)}; \quad \Gamma_{33} = [\Gamma_{11}\Gamma_{22}]^{-1} \end{aligned} \tag{3.147}$$

Either one or two eigenvalues of the metric g diverge at the terminal time t_c, while the rest of the eigenvalues (two or one) vanish.

3.2.3 Discussion

In this section we have considered conformal gravity in translationally invariant approximation. Our main finding is that the nonlinear interactions lead to instability in the dynamics of zero momentum modes. Specifically we displayed a simple solution of equations of motion which diverges within a finite time. The reason for such a severe divergence is that the dynamical modes γ, which in the perturbative regime have ghostlike kinetic term, acquire in addition a positive potential. Thus this sector of the reduced theory is equivalent to two dimensional upside down anharmonic oscillator. Close to the minimum of the potential γ behaves as a pertubative ghost with zero mass. However at any non-vanishing distance from the minimum, the signs of kinetic and potential energies are opposite and γ behaves as a tachyon.

Thus the perturbative ghost problem is not cured, but is rather exacerbated by nonlinear gravitational interactions. Thinking about quantization, it is clear that the theory does not allow sensible quantization via standard methods, i.e. using standard Dirac norm. The possibility that the use of a nonstandard norm, like in [37] could lead to a unitary theory may be worth exploring, although such a procedure is rather non intuitive.

Finally we note that another way to view the present calculation is as a study of possible homogeneous cosmologies in conformal gravity. The universe described by eqs.(3.146,3.147) is certainly very far from reality, since it is not isotropic. In fact the only isotropic and homogeneous space allowed by conformal gauge symmetry is Minkowski space, since any isotropic metric is conformally equivalent to Minkowski one. Nevertheless, an interesting property of this metric, is that it describes accelerated dynamics. As we indicated above, some dimensions in this space undergo accelerated expansion, while others accelerated contraction. Perhaps, when supplemented by conformal anomaly in the matter part [56], which we have not considered here, it could acquire more realistic features while still retaining the property of acceleration. This would be interesting to study.

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Appendices

Chapter 4

Appendix A

In this appendix we show that the model considered in this section 2.2 does not admit two photon solutions with arbitrary polarizations. We are looking for two photon solutions for which the electromagnetic tensor is of the form:

$$\tilde{F}_{\mu\nu} = \partial_{[\mu} z \partial_{\nu]} \chi = A(k_\mu \epsilon^1_\nu - k_\nu \epsilon^1_\mu) \cos kx + B(p_\mu \epsilon^2_\nu - p_\nu \epsilon^2_\mu) \cos px \tag{4.1}$$

Foer simplicity we choose the case when the first photon has momentum k in x-direction and polarization a in y-direction, while the second photon has momentum p in y-direction and polarization b in z direction. Note that this case is not covered by our construction of two photon states in the body of the paper.

Now, for components of $\tilde{F}_{\mu\nu}$, we have:

$$\partial_{[0} z \partial_{1]} \chi = 0 = \partial_{[1} z \partial_{3]} \chi = 0 \tag{4.2}$$

$$\partial_{[0} z \partial_{2]} \chi = ka \cos kx = -\partial_{[1} z \partial_{2]} \chi \tag{4.3}$$

$$\partial_{[0} z \partial_{3]} \chi = pb \cos px = -\partial_{[2} z \partial_{3]} \chi \tag{4.4}$$

Introducing new coordinates $(x, y, z, t) \rightarrow (\bar{x} = t - x, \bar{y} = t - y, \bar{t} = t, \bar{z} = z)$, and using unbarred symbols for notational simplicity, we have:

$$\partial_{[t} z \partial_{y]} \chi = \partial_{[t} z \partial_{z]} \chi = \partial_{[x} z \partial_{z]} \chi = 0 \tag{4.5}$$

$$\partial_{[t} z \partial_{x]} \chi = \partial_{[x} z \partial_{y]} \chi = -ka \cos kx \tag{4.6}$$

$$\partial_{[y} z \partial_{z]} \chi = pb \cos py \tag{4.7}$$

These equations have no solutions. Assuming $\partial_t z \neq 0$, the first two equations in eq.(4.5) imply $\partial_y z \partial_z \chi - \partial_z z \partial_y \chi = 0$, which contradicts eq.(4.7)the last equation. Alternatively, assuming $\partial_t z = 0$, implies vanishing of either $\partial_t \chi$, or two other partial derivatives of z . It is then easy to see that both these options are in conflict with the rest of the equations. The result is that a two photon state with this polarization pattern cannot be constructed in this model.

The model also contains solutions which do not satisfy the homogeneous Maxwell equation. As an example of such a solution consider the configuration

$$\chi = \sin p \cdot x; \quad z = \sin k \cdot x \tag{4.8}$$

It is easy to see that this configuration satisfies equations of motion, provided

$$(p \cdot k)^2 - p^2 k^2 = 0 \tag{4.9}$$

A simple example is a lightlike momentum k^μ and a spacelike momentum p^μ

satisfying $p \cdot k = 0$. This yields the dual field strength

$$\tilde{F}_{\mu\nu} \propto (k_\mu p_\nu - k_\nu p_\mu)[\cos(p+k) \cdot x + \cos(p-k) \cdot x] \tag{4.10}$$

which is not conserved

$$\partial^\mu \tilde{F}_{\mu\nu} \propto p^2 k_\nu [\sin(p+k) \cdot x + \sin(p-k) \cdot x] \tag{4.11}$$

In fact, both momenta $k+p$ and $k-p$ are spacelike, and thus $\tilde{F}_{\mu\nu}$ looks tachyonic. However, as mentioned in the Discussion, since the model classically has many degenerate vacua with broken translational invariance, interpretaton of classical solutions as excitations is not so clear.

Chapter 5

Appendix B

Residual gauge symmetry of the action

In this appendix we show that the action eq.(3.99) of section 4 after gauge fixing is still invariant under a combination of a general linear and conformal transformation which has not been gauge fixed by eq.(3.90).

Under a combined transformation the metric transforms as

$$g_{\rho\sigma}(x) \to g'_{\rho\sigma}(x') = \Omega^2(x) g_{\mu\nu}(x) \frac{\partial x^\mu}{\partial x^{\rho\prime}} \frac{\partial x^\nu}{\partial x^{\sigma\prime}} \tag{5.1}$$

In order for the metric to remain a function of time only, we must only consider the transformation of the type

$$x^i = x^{i'}, \quad x^0 = f(x^{0'}), \quad \Omega = \Omega(t) \tag{5.2}$$

With this restriction we get $g'_{i0}(x') = 0$ if $g_{i0}(x) = 0$, thus this gauge fixing condition is preserved. In order to maintain the condition $g_{00}(x') = 1$, we need to take $\Omega^2(t) = \frac{1}{f'^2}$. The spatial components of the metric transform under this transformation as

$$g_{ij}(t) \to g'_{ij}(t') = \frac{1}{f'^2} g_{ij}(t(t')) \tag{5.3}$$

. Denoting $\frac{1}{f'} = F$, we can write

$$g'_{ij}(t) = F^2 g_{ij}(f(t)), \qquad g'^{ij}(t) = \frac{1}{F^2} g^{ij}(f(t)) \tag{5.4}$$

Then, using

$$\frac{\partial}{\partial t} = \frac{1}{F}\frac{\partial}{\partial f} \tag{5.5}$$

we obtain

$$\partial_t g_{ij}(t) \to \partial_t g'_{ij}(t) = \partial_t F^2 g_{ij}(f) + F^2 \partial_t g_{ij}(f) = \partial_t(F^2) g_{ij} + F \partial_f g_{ij} \tag{5.6}$$

and

$$\alpha^k{}_i(t) \to \alpha'^k{}_i(t) = g'^{kj}\partial_t g' ij = \frac{1}{F^2} g^{kj}[\partial_t F^2 g_{ij} + F\partial_f g_{ij}] = \frac{\partial_t F^2}{F^2}\delta^k{}_i + \frac{1}{F}\alpha^k{}_i(f) \tag{5.7}$$

, Or

$$\tilde{\alpha}^k_j(t) \to \frac{1}{F}\tilde{\alpha}^k_j(f); \qquad \alpha(t) \to 3\frac{\partial_t(F^2)}{F^2} + \frac{1}{F}\alpha(f) \tag{5.8}$$

Similarly, it follows that:

$$\partial_t \alpha^k{}_j(t) \to \partial_t \alpha'^k{}_j(t) = \partial_t(\frac{\partial_t(F^2)}{F^2})\delta^k_j + \partial_t(\frac{1}{F})\alpha^k{}_j(f) + \frac{1}{F^2}\partial_f \alpha^k{}_j(f) \tag{5.9}$$

Or

$$\begin{aligned} \partial\tilde{\alpha}^k_j(t) \to& \partial_t(\frac{1}{F})\tilde{\alpha}^k_j(f) + \frac{1}{F^2}\partial_f \tilde{\alpha}^k_j(f); \\ \partial_t \alpha(t) \to& 3\partial_t(\frac{\partial_t(F^2)}{F^2}) + \partial_t(\frac{1}{F})\alpha(f) + \frac{1}{F^2}\partial_f \alpha(f) \end{aligned} \tag{5.10}$$

It is now straightforward to substitute these transformed fields in the expression for the action eq.(3.99). Upon discarding total derivative terms and changing the integration variables $t \to f$ it is then easy to see that the action is indeed invariant.