5-4-2016

Bernstein-Sato Polynomials for Quivers

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University of Connecticut, 2016

In this thesis we investigate an important singularity invariant, Bernstein-Sato polynomials, also called b-functions. Together with the classical notion of the b-function (of one variable) of a single polynomial, we also consider more general ones, such as the b-function of several variables (of several polynomials) and Bernstein-Sato polynomials (of one variable) of arbitrary varieties. We propose several techniques for their computation in various equivariant settings. The main applications will belong to the quiver setting, namely (semi-)invariant polynomials (resp. nullcones, orbit closures) of quivers.

We give a formula relating b-functions of semi-invariants corresponding to each other under castling transforms (or reflection functors). This, in particular, allows the computation of the b-functions for all Dynkin quivers, and also extended Dynkin quivers with prehomogeneous dimension vectors.

We give another computational technique using slices, that is efficient for semi-invariants with "small weights". Among other uses of slices, we give a way of finding locally semi-simple representations and an easy rule for the determination of the canonical decomposition for type $\mathbb{D}$ quivers.

We compute the Bernstein-Sato polynomial of the ideal generated by maximal minors (which is a type $\mathbb{A}_2$ orbit closure) and sub-maximal Pfaffians. We settle the Strong Monodromy Conjecture in these cases.

Using our computational results, we give various results on the geometry of orbit closures of quivers. In particular, we prove that codimension 1 orbit closures of Dynkin quivers have rational singularities. We prove the same result for extended Dynkin quivers for dimension vectors that are not "too small". We give results on the reduced property of the nullcone of quivers and establish a connection between b-functions of several variables and the Bernstein-Sato polynomial of varieties. With these we give some criteria for rational singularities in higher codimensions.

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B.S., Babeş-Bolyai University, Romania, 2010

M.S., Northeastern University, 2012

A Dissertation

Submitted in Partial Fulfillment of the

Requirements for the Degree of

Doctor of Philosophy

at the

University of Connecticut

2016

APPROVAL PAGE

Doctor of Philosophy Dissertation

Bernstein–Sato Polynomials for Quivers

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2016

ACKNOWLEDGEMENTS

I would like to thank my advisor, Jerzy Weyman, for suggesting this project in the first place. I am deeply indebted to him for his continuing patience and support throughout my graduate years at both Northeastern University and the University of Connecticut.

I would also like to thank my friend, José Eduardo Simental Rodríguez, for helpful comments and proofreading some of the results.

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Chapter 1

Introduction

The study of Bernstein-Sato polynomials (or b-functions, see Definition 1.1.1) originated by the work of Sato and Shintani [59] and Bernstein [5]. The former used them in the study of zeta functions in equivariant settings, while the latter proved their existence using holonomic D-modules, and used them in order to analytically continue powers of functions f^s. In general the b-function gives a measure of the singularities of the scheme defined by $f = 0$. Recently b-functions have been extended to more general schemes in [14].

Since their introduction numerous connections of b-functions to different subjects were discovered. Its zeros are closely related to the eigenvalues of the monodromy on the cohomology of the Milnor fiber. Others include: zeta functions, log-canonical resolutions, multiplier ideals, jumping coefficients etc.

Despite much research, the calculation of b-functions remains notoriously difficult: several algorithms have been implemented to compute b-functions, and a number of examples have been worked out in the literature, but basic instances are still not understood well.

A lot of effort has been made to calculate the b-functions in various cases. An important class is that of semi-invariants of prehomogeneous vector spaces, that is, representations with a dense orbit (for example, see [35, 58, 68, 36]). In this setting, there is a natural consideration of the more general b-functions of several variables (for several semi-invariants).

In this thesis we will focus on various b-functions coming from quivers. The theory of quivers originate from the study of representation theory of algebras, and it has lead to a rich area of mathematics with a number of connections with other fields. Semi-invariants of quivers and the geometry of their zero sets (nullcones) have received considerable attention in the past decades (for example, [1, 61, 21, 63, 17, 49, 50]).

In [66] the b-functions for quivers of type $\mathbb{A}$ are computed. We will generalize these computations in several ways (Chapters 3, 4) and provide some geometric consequences (Chapter 5). For the computation of b-functions of semi-invariants we consider two independent reduction techniques: in Chapter 3 using castling transforms (or reflection functors) and in Chapter 4 using slices. In the case of quivers, these techniques give algorithms similar to those in [66]. In Chapter 6, we compute the b-functions of the ideal of maximal minors and sub-maximal Pfaffians. As this is a b-function of a variety of higher codimension, we use various tools of different kind.

Most of the results from Chapters 3 and 4 can be found in the preprint [40]. The results from Chapter 5 are from the preprint [41]. The work in Chapter 6 is joint with C. Raicu, J. Weyman and U. Walther [42].

Take $X = (x_{ij})$ an $n \times n$ generic matrix of variables, and ∂X the matrix formed by the partial derivatives $\dfrac{\partial}{\partial x_{ij}}$. Its determinant is a differential operator. The formulas we get for quivers can be understood as generalizations of *Cayley's classical identity* (see [16, (1.1)]):

$$\det(\partial X) \cdot \det(X)^{s+1} = (s+1)(s+2)\cdots(s+n)\det(X)^s.$$

This gives the b-function of the determinant $b_{\det}(s) = (s+1)(s+2)\cdots(s+n)$.

A simple, yet non-trivial example of interest is the following semi-invariant, coming from the quiver $\mathbb{D}_4$:

$$\det \begin{pmatrix} X & Y & 0 \\ 0 & Y & Z \end{pmatrix}.$$

Here X, Y, Z are generic matrices of variables, with $X \in M_{\beta_4,\beta_1}, Y \in M_{\beta_4,\beta_2}, Z \in M_{\beta_4,\beta_3}$ and $\beta_1+\beta_2+\beta_3 = 2\beta_4$. We compute its b-function in Section 4.3, but the technique from Section 3.2 is also applicable.

In the first chapter we give a general background on b-functions and their properties. In Section 1.2 we introduce the main equivariant setting of consideration, namely the *multiplicity-free property* for semi-invariants. The main examples of this are semi-invariants of prehomogeneous vector spaces. In the last section of this chapter we give the definition and main properties for b-functions of arbitrary varieties.

1.1 Bernstein-Sato polynomials or b-functions

First we define and briefly recall some basic properties of b-functions. For more details we refer the reader to [13, 34].

As usual, $\mathbb{N}$ will denote the set of all non-negative integers. Throughout this chapter we work over the complex field $\mathbb{C}$. Let V be an N-dimensional vector space, $N \in \mathbb{N}$.

Consider a polynomial ring $S = \mathbb{C}[V] = \mathbb{C}[x_1, \ldots, x_N]$ and let $\mathcal{D}_V = S[\partial_1, \ldots, \partial_N]$ denote the associated Weyl algebra of differential operators with polynomial coefficients ($\partial_i = \frac{\partial}{\partial x_i}$).

Definition 1.1.1. Take a non-constant polynomial $f \in S$. Then the set of polynomials $b(s) \in \mathbb{C}[s]$ for which there exists a differential operator $P \in \mathcal{D}_V[s] := \mathcal{D}_V \otimes \mathbb{C}[s]$ such that

$$P \cdot f^{s+1} = b(s) \cdot f^s \tag{1.1}$$

forms a non-zero ideal. The monic generator of this ideal is called the *Bernstein–Sato polynomial* (or the *b-function*) of f), and is denoted $b_f(s)$.

Our main example is for the determinant, in which case we have the Capelli identity, as mentioned in the Introduction. Another example:

Example 1.1.2. $f = x^2 + y^3$, then applying the operator

$$P = \frac{1}{12} y \partial_x^2 \partial_y + \frac{1}{27} \partial_y^3 + s \frac{1}{4} \partial_x + \frac{3}{8} \partial_x^2$$

we obtain the equation

$$P \cdot f^{s+1} = b_f(s) \cdot f^s,$$

where the b-function $b_f(s)$ is

$$b_f(s) = (s+1)(s + \frac{5}{6})(s + \frac{7}{6}).$$

As its existence, the other fundamental result about b-functions is highly non-trivial, and uses Hironaka's resolution of singularities (see [34]):

Theorem 1.1.3. *All roots of $b_f(s)$ are negative rational numbers.*

Note that -1 is always a root of $b_f(s)$, as can be seen by taking $s = -1$ in the defining equation.

Similarly, for an element $v \in V$, we can define *local* b-functions $b_{f,v}(s)$ by allowing differential operators P with coefficients that are rational functions regular at v. Clearly, $b_{f,v}$ divides $b_f(s)$. Also, if $f(v) \neq 0$, then $\dfrac{1}{f}$ is a regular differential operator at v, hence $b_{f,v} = 1$, since $\dfrac{1}{f} \cdot f^{s+1} = f^s$. One can recover the (global) b-function by the following result :

Lemma 1.1.4 ([26, Lemma 2.5.2])**.** *The b-function $b_f(s)$ is the least common multiple of $\{b_{f,v}(s) | v \in V\}$.*

Among the numerous connections of roots of b-functions with other singularity invariants, we will consider applications to rational singularities.

For a variety X, we call a morphism $h : Z \to X$ a *resolution of singularities*, if Z is a smooth variety and h is a proper birational map.

Definition 1.1.5 ([72]). We say a variety X has *rational singularities*, if for a (hence any) resolution of singularities $h : Z \to X$, we have

(a) The natural map $\mathcal{O}_X \to h_*\mathcal{O}_Z$ is an isomorphism of sheaves,

(b) The higher direct images $R^i h_*\mathcal{O}_Z = 0$ vanish for $i > 0$.

It is known that if X has rational singularities, it is normal and Cohen-Macaulay [72].

Theorem 1.1.6 ([54]). *Assume f is a reduced polynomial. Then the zero-set $Z(f)$ has rational singularities if and only if -1 is the largest root of $b(s)$ and its multiplicity is* 1*.*

1.2 Multiplicity-free property and prehomogeneous vector spaces

In this section we introduce the basic equivariant settings that we will be working with.

Let G be a connected reductive algebraic group, acting rationally on V. We denote the orbit of an element $v \in V$ by $\mathcal{O}_v \subset V$. We have an action of G on $\mathbb{C}[V]$ by $(g \cdot f)(v) = f(g^{-1}v)$, for $g \in G$, $f \in \mathbb{C}[V]$. We call a polynomial $f \in \mathbb{C}[V]$ a *semi-invariant*, if there is a character $\sigma \in \operatorname{Hom}(G, \mathbb{C}^\times)$ such that $g \cdot f = \sigma(g)f$, that is, $f(gv) = \sigma(g)^{-1}f(v)$. In this case we say the weight of f is σ. In case σ is trivial, f is called an invariant.

Now we form the ring of semi-invariants

$$\operatorname{SI}(G, V) = \bigoplus_{\sigma} \operatorname{SI}(G, V)_\sigma,$$

where the sum runs over all characters σ of G and the weight spaces are

$$\mathrm{SI}(G,V)_\sigma = \{f \in \mathbb{C}[V] | f \text{ is a semi-invariant of weight } \sigma\}.$$

Note that by this definition not all elements in $\mathrm{SI}(G,V)$ are G-semi-invariants. But they are all $[G,G]$-invariants, moreover, it is well-known that the reductivity of G implies the equality

$$\mathrm{SI}(G,V) = \mathbb{C}[V]^{[G,G]}. \tag{1.2}$$

Throughout we deal with representations (G,V) that have a *unique closed orbit*. For a rational representation (G,V), this means $\{0\}$ is the unique closed orbit of the action. This happens iff there are no non-constant polynomial invariants. In particular, a prehomogeneous vector space (see below) has a unique closed orbit. One can always induce the unique closed orbit property by enlarging the group action to $G' = G \times \mathbb{C}^\times$, where $\mathbb{C}^\times$ acts via the action induced by the vector space structure of V.

The connection between orbits and (local) b-functions is made clear by the following lemma:

Lemma 1.2.1. *Let $f \in \mathbb{C}[V]$ be a semi-invariant for an action (G,V). Then*

(a) If $\mathcal{O}$ is an orbit, and $v, w \in \mathcal{O}$, then $b_{f,v} = b_{f,w} =: b_{f,\mathcal{O}}$.

(b) If $\mathcal{O}_1 \subset \overline{\mathcal{O}_2}$, then $b_{f,\mathcal{O}_2}$ divides $b_{f,\mathcal{O}_1}$.

Proof. Suppose the equation for the local b-function at v is given by

$$P_v(s) \cdot f^{s+1}(x) = b_v(s) f^s(x)$$

where $P_v(s) \in \mathcal{D}_{V,v}[s]$. If $w = gv$, then applying g to the equation, we get the equation for the local b-function at w, with $P_w(s) = \dfrac{1}{\sigma(g)}(g \cdot P(s)) \in \mathcal{D}_{V,w}[s]$. This finished part (a).

Now let $v_1 \in \mathcal{O}_1$, and suppose we have the equation for the local b-function b_{v_1} defined in an open neighborhood U, with $v_1 \in U$. Then $U \cap \mathcal{O}_2 \neq \emptyset$, so pick $v_2 \in U \cap \mathcal{O}_2$. Then $b_{v_2} | b_{v_1}$, which, together with part (a), gives the conclusion. $\square$

We have the following corollary (see also [26, Lemma 2.5.2]):

Corollary 1.2.2. *If (G, V) has a unique closed orbit, and f is a semi-invariant, then $b_{f,0} = b_f$. In particular, this holds if f is any homogeneous polynomial on V.*

Proof. This follows immediately from Lemma 1.1.4 and Lemma 1.2.1. □

The *multiplicity* of σ is $\dim \mathrm{SI}(G, V)_\sigma$.

Definition 1.2.3. We say that σ is *multiplicity-free*, if the multiplicity of σ^k is 1, for any $k \in \mathbb{N}$. We call a non-zero semi-invariant f multiplicity-free if it has a multiplicity-free weight.

A multiplicity-free semi-invariant must be homogeneous [26]. If $f \in \mathbb{C}[V]$ is a non-zero semi-invariant of multiplicity-free weight σ, there exists (canonical up to a constant) a non-zero dual semi-invariant $f^* \in \mathbb{C}[V^*]$ of weight σ^{-1}, which we view as a constant coefficient differential operator. We can construct such dual semi-invariants explicitly in the following way [60]:

Take the map of algebraic groups $\rho : G \to \mathrm{GL}(V)$ defining the action of G on V. Since G is reductive, it is known (see [36]) that one can choose a basis in $V \cong \mathbb{C}^n$ such that $\rho(G) = \rho(G)^t \subset \mathrm{GL}_n(\mathbb{C})$, where $\rho(G)^t$ is the set consisting of the transposes of the elements in $\rho(G)$. Using this basis, we can define $f^* \in \mathbb{C}[V^*]$ by replacing the variables in f with the dual variables (see Introduction for the case of the determinant). It is a multiplicity-free semi-invariant with weight σ^{-1}. In the cases we consider, the dual semi-invariant will always take this simple form.

The following result follows by [26, Corollary 2.5.10]:

Theorem 1.2.4. *Let f be multiplicity-free and f^* its dual as above. Then we have the following equation*

$$f^*(\partial x) \cdot f(x)^{s+1} = b(s) f(x)^s, \tag{1.3}$$

and $b(s)$ coincides with the b-function $b_f(s)$ of f up to a non-zero constant factor. Moreover, $\deg b(s) = \deg f$.

Hence we call the polynomial $b(s) = b_f(s)$ above the b-function of f, which is unambiguous up to a factor.

Definition 1.2.5. We call (G, V) a prehomogeneous vector space, if V has a dense G-orbit $\mathcal{O}$, i.e. $\overline{\mathcal{O}} = V$.

In this case, the dense orbit $\mathcal{O}$ is open in V. By a standard result (cf. [36]), (G, V) is prehomogeneous iff all weight multiplicities of the ring of semi-invariants are at most 1. Moreover, we have [60]:

Theorem 1.2.6. *Let S_i be the hypersurface irreducible components of $V \backslash \mathcal{O}$ for $i = 1, \ldots, m$. Write $S_i = \{x \in V \mid f_i(x) = 0\}$ for some irreducible polynomials f_i (called fundamental semi-invariants). Then $\mathrm{SI}(G, V) = \mathbb{C}[f_1, \ldots, f_m]$ is a polynomial ring.*

Next we introduce the more general notion of b-functions of several variables (cf. [57]). For $f_1, \ldots, f_l$ semi-invariants of weights $\sigma_1, \ldots, \sigma_l$, we say that the tuple $\underline{f} = (f_1, \ldots, f_l)$ is multiplicity-free, if the multiplicity of $\sigma_1^{k_1} \cdots \sigma_l^{k_l}$ is 1, for any $(k_1, \ldots, k_l) \in \mathbb{N}^l$. This is equivalent to the product $f_1 \cdots f_l$ being a multiplicity-free semi-invariant. Take respective duals $f_1^*, \ldots, f_l^*$, and put $\underline{f}^* = (f_1^*, \ldots, f_l^*)$. For a multi-variable $\underline{s} = (s_1, \ldots, s_l)$, we define the powers by $\underline{f}^{\underline{s}} = \displaystyle\prod_{i=1}^{l} f_i^{s_i}$, and $\underline{f}^{*\underline{s}} = \displaystyle\prod_{i=1}^{l} f_i^{*s_i}$.

Lemma 1.2.7. *If $\underline{f}$ is a multiplicity-free tuple, then for any l-tuple $\underline{m} = (m_1, \ldots, m_l) \in \mathbb{N}^l$ there is a polynomial $b_{\underline{m}}(\underline{s})$ of l variables such that*

$$\underline{f}^{*\underline{m}}(\partial x) \cdot \underline{f}^{\underline{s}+\underline{m}}(x) = b_{\underline{m}}(\underline{s}) \underline{f}^{\underline{m}}(x). \tag{1.4}$$

If the tuple $\underline{f}$ is multiplicity-free, then all weights f_i are multiplicity-free semi-invariants, and one can easily recover the b-function b_{f_i} of one variable from $b_{\underline{m}}$. Again, if (G, V) is prehomogeneous, then any tuple of semi-invariants multiplicity-free.

Although we won't consider it, the b-function of several variables (1.4) has been generalized to the case of arbitrary (not necessarily semi-invariant) polynomials by [27, 53].

It is known that $b_{\underline{m}}(\underline{s})$ is a product of linear polynomials ([57, 68]) and has an expression (up to a constant) of the form:

$$b_{\underline{m}}(\underline{s}) = \prod_{j=1}^{N} \prod_{k=1}^{\mu_j} \prod_{i=0}^{\gamma^j \cdot \underline{m} - 1} (\gamma^j \cdot \underline{s} + \alpha_{j,k} + i), \tag{1.5}$$

where $N \in \mathbb{N}$, $\mu_j \in \mathbb{N}$, $\gamma^j \in \mathbb{N}^r$ and $\alpha_{j,k} \in \mathbb{Q}_+$.

1.3 Bernstein-Sato polynomials of varieties

In this section we review the results and definitions from [14] that are most relevant for our considerations. For a collection $\underline{f} = (f_1, \ldots, f_r)$ of arbitrary non-zero polynomials in S, we consider a set of independent commuting variables $s_1, \ldots, s_r$, one for each f_i. We form the $\mathcal{D}_V[s_1, \ldots, s_r]$-module

$$B_{\underline{f}}^{\underline{s}} = S_{f_1 \cdots f_r}[s_1, \ldots, s_r] \cdot \underline{f}^{\underline{s}}, \tag{1.6}$$

where $S_{f_1 \cdots f_r}$ denotes the localization of S at the product of the f_i's, and $\underline{f}^{\underline{s}}$ stands for the formal product $f_1^{s_1} \cdots f_r^{s_r}$. $B_{\underline{f}}^{\underline{s}}$ is a free rank one $S_{f_1 \cdots f_r}[s_1, \ldots, s_r]$-module with generator $\underline{f}^{\underline{s}}$, which admits a natural action of $\mathcal{D}_V$: the partial derivatives ∂_i act on the generator $\underline{f}^{\underline{s}}$ via

$$\partial_i \cdot \underline{f}^{\underline{s}} = \sum_{j=1}^{r} \frac{s_j \cdot (\partial_i \cdot f_j)}{f_j} \cdot \underline{f}^{\underline{s}}.$$

For a tuple $\hat{c} \in \mathbb{Z}^r$, denote

$$\hat{c}_{\underline{s}} = \prod_{c_i < 0} s_i \cdot (s_i - 1) \cdots (s_i + c_i + 1). \tag{1.7}$$

and write $s = s_1 + \cdots + s_r$. Then the following generalizes Definition 1.1.1:

Definition 1.3.1. The Bernstein–Sato polynomial or b-function $b_{\underline{f}}(s)$ (of one variable) of $\underline{f}$ is the monic polynomial of the lowest degree in s for which $b_{\underline{f}}(s) \cdot \underline{f}^{\underline{s}}$ belongs to the $\mathcal{D}_V[s_1, \ldots, s_r]$-submodule of $B_{\underline{f}}^{\underline{s}}$ generated by all expressions

$$\hat{c}_{\underline{s}} \cdot \prod_{i=1}^{r} f_i^{s_i + c_i},$$

where $\hat{c} = (c_1, \ldots, c_r)$ runs over the r-tuples in $\mathbb{Z}^r$ with $c_1 + \cdots + c_r = 1$ (for short $|\hat{c}| = 1$).

Equivalently, $b_{\underline{f}}(s)$ is the monic polynomial of lowest degree for which there exist a finite set of tuples $\hat{c} \in \mathbb{Z}^r$ with $|\hat{c}| = 1$, and corresponding operators $P_{\hat{c}} \in \mathcal{D}_V[s_1, \ldots, s_r]$ such that

$$\sum_{\hat{c}} P_{\hat{c}} \cdot \hat{c}_{\underline{s}} \cdot \prod_{i=1}^{r} f_i^{s_i + c_i} = b_{\underline{f}}(s) \cdot \underline{f}^{\underline{s}}. \tag{1.8}$$

Just as in the case $r = 1$ (of a single hypersurface), $b_{\underline{f}}(s)$ exists and is a polynomial whose roots are negative rational numbers. Moreover, $b_{\underline{f}}(s)$ only depends on the ideal I generated by $f_1, \ldots, f_r$, which is why we'll often write $b_I(s)$ instead of $b_{\underline{f}}(s)$. Furthermore, let $Z \subset V$ denote the subscheme defined by $f_1, \ldots, f_r$.

The following generalizes Theorem 1.1.6:

Theorem 1.3.2 ([14, Theorem 4])**.** *Assume Z is a reduced complete intersection of codimension r. Then Z has rational singularities if and only if $-r$ is the largest root of $b_I(s)$ and its multiplicity is* 1.

If we set

$$b_Z(s) = b_I(s - \operatorname{codim}_V(Z)) \tag{1.9}$$

then $b_Z(s)$ only depends on the affine scheme Z and not on its embedding in an affine space. The polynomial $b_Z(s)$ is called the *Bernstein–Sato polynomial* *of* Z (or the *b-function of* Z), and, as in the hypersurface case, is meant as a measure of the singularities of Z: the higher the degree of $b_Z(s)$, the worse are the singularities of Z. For instance, one has that T is smooth if and only if $b_T(s) = s$. Moreover, it follows from [14, Theorem 5] that for any Z and any smooth T we have

$$b_{Z\times T}(s) = b_Z(s). \tag{1.10}$$

Analogous to Lemma 1.1.4 Z is irreducible and $Z = Z_1 \cup \cdots \cup Z_k$ is an open cover of Z then

$$b_Z(s) = \operatorname{lcm}\{b_{Z_i}(s) : i = 1, \ldots, k\}. \tag{1.11}$$

Chapter 2

Quivers

In this chapter we give the necessary background on quivers that will be used in the following parts. In Section 2.1 we deal with the basic definitions. In Section 2.2 we introduce the important notions of reflection functors and the Auslander-Reiten transformation. In Section 2.3, we consider the geometric approach to studying quiver representation spaces and discuss semi-invariants of quivers. For background material, we refer the reader to [3, 11, 21].

2.1 Representations of quivers

Throughout this chapter we work over an algebraically closed field k.

A quiver Q is an oriented graph, i.e. a pair $Q = (Q_0, Q_1)$ formed by the set of vertices Q_0 and the set of arrows Q_1. An arrow a has a head ha, and tail ta, that are elements in Q_0:

$$ta \xrightarrow{a} ha$$

A representation V of Q is a family of finite dimensional vector spaces $\{V(x) \,|\, x \in Q_0\}$ together with linear maps $\{V(a) : V(ta) \to V(ha) \,|\, a \in Q_1\}$. The *dimension vector* $\underline{d}(V) \in \mathbb{N}^{Q_0}$ of a representation V is the tuple $\underline{d}(V) := (d_x)_{x \in Q_0}$, with $d_x = \dim V(x)$. A morphism $\phi : V \to W$ of two representations is a collection of linear maps $\phi = \{\phi(x) : V(x) \to W(x) \,|\, x \in Q_0\}$, with the property that for each $a \in Q_1$ we have $\phi(ha)V(a) = W(a)\phi(ta)$. Denote by $\mathrm{Hom}_Q(V, W)$ the vector space of morphisms of representations from V to W.

For two vectors $\alpha, \beta \in \mathbb{Z}^{Q_0}$, we define the Euler form

$$\langle \alpha, \beta \rangle = \sum_{x \in Q_0} \alpha_x \beta_x - \sum_{a \in Q_1} \alpha_{ta} \beta_{ha}.$$

For any two representations V and W, we have the following exact sequence:

$$\begin{aligned} 0 \to \mathrm{Hom}_Q(V, W) \xrightarrow{i} \bigoplus_{x \in Q_0} \mathrm{Hom}(V(x), W(x)) \\ \xrightarrow{d_W^V} \bigoplus_{a \in Q_1} \mathrm{Hom}(V(ta), W(ha)) \xrightarrow{p} \mathrm{Ext}_Q(V, W) \to 0 \end{aligned} \tag{2.1}$$

Here, the map i is the inclusion, d_W^V is given by

$$\{\phi(x)\}_{x \in Q_0} \mapsto \{\phi(ha)V(a) - W(a)\phi(ta)\}_{a \in Q_1}$$

and the map p builds an extension of V and W by adding the maps $V(ta) \to W(ha)$ to the direct sum $V \oplus W$.

From the exact sequence (2.1) we have that $\langle \underline{d}(V), \underline{d}(W) \rangle = \dim \mathrm{Hom}(V, W) - \dim \mathrm{Ext}(V, W)$.

Throughout we assume that Q is a quiver *without oriented cycles* .

We recall (cf. [3]) the notions of a *Dynkin quiver* (of type $\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8$) and of an *extended Dynkin quiver* (of type $\tilde{\mathbb{A}}_n, \tilde{\mathbb{D}}_n, \tilde{\mathbb{E}}_6, \tilde{\mathbb{E}}_7, \tilde{\mathbb{E}}_8$). Since we are dealing with quivers without oriented cycles, we will exclude the oriented cycle quiver of type $\tilde{\mathbb{A}}_n$.

Simple, projective and injective representations have an easy description [3]. The main goal in the representation theory of quivers is to classify all indecomposable representations. A quiver is called of *finite type* if it has only finitely many indecomposables. The following result, called Gabriel's Theorem, is fundamental ([3]:

Theorem 2.1.1. *A quiver Q is of finite representation type if and only if it is a Dynkin quiver.*

There is a notion of *tame type*, in which there are infinitely many indecomposables, but there is still a classification as they form finitely many one-parameter families [3]. The tame quivers are precisely the extended Dynkin quivers. The rest of the quivers are called of *wild type*.

2.2 Reflection functors and Auslander-Reiten theory

A vertex $x \in Q_0$ is called a sink (resp. source) if there is no arrow in Q starting (resp. ending) in x.

First, we introduce some terminology for reflection functors, that is, castling transforms in the quiver setting (for details, see [3, Section VII.5.]). Throughout, let Q be a quiver without oriented cycles. Given any vertex $i \in Q_0$, we form a new quiver $c_i Q$ by reversing all arrows that start or end in i. An ordering of $i_1, \ldots, i_n$ of the vertices of Q is called admissible if for each k the vertex i_k is a sink for $c_{i_{k-1}} \ldots c_{i_1} Q$. In such case, it is easy to see that $c_{i_n} \ldots c_{i_1} Q = Q$. Since Q has no oriented cycles, Q has admissible orderings, and we fix one. For $x \in Q_0$, we take the following the linear map of dimension vectors that we denote by the same letter

$$c_x : \mathbb{Z}^n \to \mathbb{Z}^n$$

$$c_x(\beta)_y = \begin{cases} \beta_y & \text{if } x \neq y, \\ -\beta_x + \displaystyle\sum_{\text{edges } x\text{—}z} \beta_z & \text{if } x = y. \end{cases}$$

Also, let $c = c_{i_n} \cdots c_{i_1}$ be the Coxeter transformation. It is independent on the choice of the admissible ordering. As a matrix, we have that $c = -E^{-1}E^t$, where E denotes the Euler matrix of Q corresponding to the Euler product.

We have the reflection functors on the representation level $C_x : \mathrm{rep}(Q) \to \mathrm{rep}(c_x Q)$ such that $C_x(S_x) = 0$, and for all other indecomposables X, $C_x(X)$ is non-zero indecomposable representation with dimension vector $c_x(\underline{d}(X))$ (see [4]).

Now denote by C the Coxeter functor defined by $C = C_{i_n} \cdots C_{i_1} : \mathrm{rep}(Q) \to \mathrm{rep}(Q)$.

Then $C(P_y) = 0$, for the projective module P_y corresponding to any vertex $y \in Q_0$, and for all other indecomposables X, $C(X)$ is a non-zero indecomposable representation with dimension vector $c(\underline{d}(X))$. In fact, in the latter case, the Coxeter functor coincides with the *Auslander-Reiten translation* τ, namely, $C(X) \cong \tau X$. For a quiver Q, we can associate an Auslander-Reiten quiver whose vertices are all the indecomposable representations of Q and arrows the irreducible morphisms between them. For more on Auslander-Reiten theory, we direct the reader to [3, Chapter IV.].

We say an indecomposable X is preprojective, if $C^k(X) = 0$, for some $k \in \mathbb{N}$. Preprojective representations are generic and we also call their dimension vectors preprojective. Dually, we can define these notions with x a source, we get the Coxeter transformation c^{-1} and preinjective representations. An indecomposable representation is called regular is it is neither preprojective nor preinjective.

A quiver is of Dynkin type if and only if all the indecomposables are preprojective (and preinjective), see [3].

2.3 Geometric considerations and semi-invariants of quivers

We define the vector space of representations with dimension vector $\alpha \in \mathbb{N}^{Q_0}$ by

$$\mathrm{Rep}(Q, \alpha) := \bigoplus_{a \in Q_1} \mathrm{Hom}(k^{\alpha_{ta}}, k^{\alpha_{ha}}).$$

The group

$$\mathrm{GL}(\alpha) := \prod_{x \in Q_0} \mathrm{GL}(\alpha_x)$$

acts on $\mathrm{Rep}(Q, \alpha)$ in a natural way. This action corresponds to changing bases and under this action, two representations lie in the same orbit iff they are isomorphic representations. It is well known that if Q has no oriented cycles, then $0 \in$

$\operatorname{Rep}(Q,\alpha)$ is the only semi-simple representation, and $(\mathrm{GL}(\alpha), \operatorname{Rep}(Q,\alpha))$ satisfies the unique closed orbit property.

There is a useful geometric interpretation of the sequence (2.1), when $V = W$. Then $\bigoplus_{x\in Q_0} \operatorname{Hom}(V(x), V(x)) \cong \mathfrak{gl}(\underline{d}(V))$, the Lie algebra of $\mathrm{GL}(\underline{d}(V))$. The map d_V^V is the differential at the identity of the orbit map

$$g \mapsto g \cdot V \in \operatorname{Rep}(Q, \underline{d}(V)).$$

Also, $\operatorname{Hom}_Q(V,V) \cong \mathfrak{gl}_V(\underline{d}(V))$, the isotropy subalgebra of $\mathfrak{gl}(\underline{d}(V))$ at V, and we have a natural $\operatorname{Aut}_Q(V)$-equivariant identification of the normal space

$$\operatorname{Ext}_Q(V,V) \cong \operatorname{Rep}(Q,\underline{d}(V))/T_V(\mathcal{O}_V),$$

where $\mathcal{O}_V$ is the orbit of V. In particular, $\mathcal{O}_V$ is dense iff $\operatorname{Ext}_Q(V,V) = 0$, in which case we say that V is a *generic representation*, and that $\underline{d}(V)$ is a *prehomogeneous dimension vector*.

For two representations $M, N \in \operatorname{Rep}(Q,\alpha)$, we say N is a degeneration of M if $N \in \overline{\mathcal{O}(M)}$, where $\overline{\mathcal{O}(M)}$ is the closure of the orbit of M. We say that N is a minimal degeneration of M if for any representation Q such that N is a degeneration of Q and Q is a degeneration of M we have either $Q \cong N$ or $Q \cong M$. By semi-continuity, if N is a degeneration of M, then $\dim \operatorname{Hom}_Q(M,X) \leq \dim \operatorname{Hom}_Q(N,X)$ and $\dim \operatorname{Ext}_Q(M,X) \leq \dim \operatorname{Ext}_Q(N,X)$ for any $X \in \operatorname{Rep}(Q)$.

It is an easy fact that if we have an exact sequence

$$0 \to A \to B \to C \to 0$$

then $A \oplus C$ is a degeneration of B. In fact for Dynkin quivers we have the following converse by [10]:

Lemma 2.3.1. *Let Q be a Dynkin quiver and $M, N \in \operatorname{Rep}(Q)$ such that N be a minimal degeneration of M. Then there exists indecomposables U, V such that $N = U \oplus V \oplus X$, $M = Z \oplus X$ and we have an exact sequence*

$$0 \to U \to Z \to V \to 0.$$

Now we turn to semi-invariants of a quiver representation space $\mathrm{Rep}(Q, \beta)$. We form the ring of semi-invariants $\mathrm{SI}(Q, \beta) \subset k[\mathrm{Rep}(Q, \beta)]$ by

$$\mathrm{SI}(Q, \beta) = \bigoplus_{\sigma} \mathrm{SI}(Q, \beta)_{\sigma} = k[\mathrm{Rep}(Q, \alpha)]^{\mathrm{SL}(\alpha)}$$

Here σ runs through all the characters of $\mathrm{GL}(\beta)$. Each character σ of $\mathrm{GL}(\beta)$ is a product of determinants, that is, of the form

$$\prod_{x \in Q_0} \det_x^{\sigma(x)}$$

where $\det_x$ is the determinant function on $\mathrm{GL}(\beta_x)$. In this way, we will view a character σ as a function $\sigma : Q_0 \to \mathbb{Z}$, or equivalently, as an element $\sigma \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{Q_0}, \mathbb{Z})$. With this convention, we view characters as duals to dimension vectors, namely:

$$\sigma(\beta) = \sum_{x \in Q_0} \sigma(x)\beta_x.$$

We recall the definition of an important class of determinantal semi-invariants, first constructed by Schofield [61]. Fix two dimension vectors α, β, such that $\langle \alpha, \beta \rangle = 0$. The latter condition says that for every $V \in \mathrm{Rep}(Q, \alpha)$ and $W \in \mathrm{Rep}(Q, \beta)$ the matrix of the map d^V_W in (2.1) will be a square matrix. We define the semi-invariant c of the action of $\mathrm{GL}(\alpha) \times \mathrm{GL}(\beta)$ on $\mathrm{Rep}(Q, \alpha) \times \mathrm{Rep}(Q, \beta)$ by $c(V, W) := \det d^V_W$. Note that we have

$$c(V, W) = 0 \Leftrightarrow \mathrm{Hom}(V, W) \neq 0 \Leftrightarrow \mathrm{Ext}(V, W) \neq 0.$$

Next, for a fixed V, restricting c to $\{V\} \times \mathrm{Rep}(Q, \beta)$ defines a semi-invariant $c^V \in \mathrm{SI}(Q, \beta)$. Similarly, for a fixed W, restricting c to $\mathrm{Rep}(Q, \alpha) \times \{W\}$, we get a semi-invariant $c_W \in \mathrm{SI}(Q, \alpha)$. The weight of c^V is $\langle \alpha, \cdot \rangle \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{Q_0}, \mathbb{Z})$, and the weight of c_W is $-\langle \cdot, \beta \rangle$. The semi-invariants c^V and c_W are well-defined up to scalar, that is, if V is isomorphic to V', then c^V and $c^{V'}$ are equal up to a scalar. Furthermore, taking any projective (resp. injective) resolution (not the canonical one that gives (2.1) doesn't affect the outcome up to a scalar. We will always ignore such scalars. We note that, out of convenience, throughout we work with both types of semi-invariants $c^V \in \mathrm{Rep}(Q,\beta)$ and $c_W \in \mathrm{Rep}(Q,\alpha)$.

Theorem 2.3.2 ([21, 63]). *For a fixed vector β, the ring of semi-invariants $\mathrm{SI}(Q,\beta)$ is spanned by the semi-invariants c^V, with $\langle \underline{d}(V), \beta\rangle = 0$. Analogously, the semi-invariants c_W span $\mathrm{SI}(Q,\alpha)$.*

In fact, the algebra of semi-invariants $\mathrm{SI}(Q,\beta)$ is generated by semi-invariants c^V, with $\langle \underline{d}(V), \cdot\rangle = 0$ and V a Schur representation (that is, $\mathrm{End}_Q(V) = k$). We call a dimension vector α a *real Schur root* , if there is a Schur representation V with $\underline{d}(V) = \alpha$ and the orbit of V dense in $\mathrm{Rep}(Q,\alpha)$.

For prehomogeneous dimension vectors, we have the a more precise statement of the theorem above as follows. Let β be prehomogeneous, and T the generic representation. Denote by ${}^{\perp}T$ the *left perpendicular category* of T, that is, the full subcategory of $\mathrm{Rep}(Q)$ consisting of objects Y that satisfy

$$\mathrm{Hom}(Y,T) = \mathrm{Ext}(Y,T) = 0.$$

Let $n = |Q_0|$ and let m denote the number of pairwise non-isomorphic indecomposable summands of T. By [61, Theorem 2.5], ${}^{\perp}T$ is equivalent to the category of representations of a quiver ${}^{\perp}Q$ without oriented cycles and with $n-m$ vertices

$${}^{\perp}T \xrightarrow{\sim} \mathrm{Rep}({}^{\perp}Q).$$

We denote the simple objects in ${}^{\perp}T$ by $S_{m+1},\ldots,S_n$. We have the following (compare Theorem 1.2.6):

Theorem 2.3.3 ([61, Theorem 4.3]). *The semi-invariants c^{S_j}, $j = m+1,\ldots,n$, are algebraically independent generators (fundamental semi-invariants) of the ring $\mathrm{SI}(Q,\beta)$.*

Again, the analogous result holds for using the right perpendicular category $T^{\perp}$.

To find multiplicity-free weights for semi-invariants on $\mathrm{Rep}(Q,\beta)$ with β not necessarily prehomogeneous, the following reciprocity result is useful:

Lemma 2.3.4 ([21, Corollary 1])**.** *Let α and β be two dimension vectors, with $\langle\alpha,\beta\rangle = 0$. Then*

$$\dim \mathrm{SI}(Q,\beta)_{\langle\alpha,\cdot\rangle} = \dim \mathrm{SI}(Q,\alpha)_{-\langle\cdot,\beta\rangle}.$$

In particular, if f is a non-zero semi-invariant of weight $\langle\alpha,\cdot\rangle$ with α prehomogeneous, then any multiple of α is also prehomogeneous, hence we see that $f = c^V$ has multiplicity-free weight with V generic in $\mathrm{Rep}(Q,\alpha)$.

Remark 2.3.5. For any $V \in \mathrm{Rep}(Q,\alpha)$, it is easy to write down the semi-invariants $c^V(W)$ explicitly, as determinants of suitable block matrices. Namely, label the rows formed by the blocks with the arrows $a \in Q_1$, and label the columns with the vertices in Q_0. Then, for an arrow a, we put two block entries in the row of a: $I_{\dim V(ta)} \otimes W(a)$ in the column ta, and $-V(a)^t \otimes I_{\dim W(ha)}$ in the column ha.

Example 2.3.6. Let Q be the following $\mathbb{D}_4$ quiver:

$$\begin{tikzcd} & 2 \arrow[d] & \\ 1 \arrow[r] & 4 & 3 \arrow[l] \end{tikzcd}$$

Let V be the indecomposable $V = \begin{matrix} & 1 & \\ 1 & 1 & 1 \end{matrix}$. Then $\langle\alpha,\beta\rangle = 0$ gives $\beta = (\beta_1,\beta_2,\beta_3,\beta_4)$ with $\beta_1+\beta_2+\beta_3 = 2\beta_4$. Let X, Y, Z be generic matrices of variables, with $X \in M_{\beta_4,\beta_1}, Y \in M_{\beta_4,\beta_2}, Z \in M_{\beta_4,\beta_3}$. Then c^V is the determinant of the following square matrix of variables:

$$\det\begin{pmatrix} X & 0 & 0 & I_{\beta_4} \\ 0 & Y & 0 & I_{\beta_4} \\ 0 & 0 & Z & I_{\beta_4} \end{pmatrix} = \det\begin{pmatrix} X & Y & 0 \\ 0 & Y & Z \end{pmatrix}$$

Also, $c^V \neq 0$ iff $\beta_i \leq \beta_4$, for $i = 1, 2, 3$.

We give the following easy lemma:

Lemma 2.3.7. *Let Q be a quiver without oriented cycles, β a dimension vector and f a semi-invariant on $\operatorname{Rep}(Q, \beta)$ of weight $\sigma = \langle \alpha, \cdot \rangle$. Then we can view f as a semi-invariant on a new quiver with new weight according to the following simplification rules:*

(a) If $\alpha_1 = 0$, then we have (we put the values of α on top of β):

$$\begin{tikzcd}[row sep=small] & \overset{\alpha_{x_1}}{\beta_{x_1}} \cdots & & & \overset{\alpha_{x_1}}{\beta_{x_1}} \cdots \\ & \overset{\alpha_{x_2}}{\beta_{x_2}} \cdots & & & \overset{\alpha_{x_2}}{\beta_{x_2}} \cdots \\ \overset{0}{\beta_1} \arrow[uur] \arrow[ur] & \cdots & {} \arrow[r, squiggly] & {} & \cdots \\ & \overset{\alpha_{y_2}}{\beta_{y_2}} \cdots \arrow[ul] & & \overset{0}{\beta_1} & \overset{\alpha_{y_2}}{\beta_{y_2}} \cdots \arrow[l] \\ & \overset{\alpha_{y_1}}{\beta_{y_1}} \cdots \arrow[uul] & & \overset{0}{\beta_1} & \overset{\alpha_{y_1}}{\beta_{y_1}} \cdots \arrow[l] \end{tikzcd}$$

(b) Write $\sigma = -\langle \cdot, \alpha^* \rangle$. If 1 is a vertex with $\langle \alpha, \underline{d}(P_1) \rangle = \alpha_1^* = 0$, then the same simplification rule holds as in part (a) by replacing α with α^*, with the arrows reversed.

Proof. (a) We can assume $f = c^V$, with $\underline{d}(V) = \alpha$. Then we see explicitly that f doesn't depend on the arrows from 1 to x_i, hence we can drop them. Finally, we can split vertex 1 so that the arrows from y_i have different heads, not changing f.

(b) We can assume $f = c_W$, with $\underline{d}(W) = \gamma$. Then the we see the simplifications explicitly as in part (a).

□

Chapter 3

b-functions via reflection functors

In this chapter, we give an efficient method for computing b-functions of semi-invariants of quivers. In Section 3.1 we give a relation of b-functions of semi-invariants corresponding to each other under castling transforms (or reflection functors). In Section 3.2 we show how this allows us to compute all b-functions (of one- and several variables) for Dynkin quivers, and extended Dynkin quivers with prehomogeneous dimension vectors. We provide several examples for calculations of b-functions of one and several variables.

Notation. For $a, b, d \in \mathbb{N}, a \leq b$, we use the following notation in $\mathbb{C}[s]$:

$$[s]^d_{a,b} := \prod_{i=a+1}^{b} \prod_{j=0}^{d-1} (ds + i + j).$$

In the case $d = 1$, we sometimes write $[s]_{a,b} := [s]^1_{a,b}$. Also, if $a = 0$, we sometimes write $[s]^d_b := [s]^d_{0,b}$. Hence $[s]^d_{a,b}[s]^d_a = [s]^d_b$.

Now fix an l-tuple $\underline{m} = (m_1, \ldots, m_l) \in \mathbb{N}^l$. Then for any l-tuple $(d_1, \ldots, d_l)$, we use the following notation in $\mathbb{C}[s_1, \ldots, s_l]$:

$$[s]^{d_1,\ldots,d_l}_{a,b} = \prod_{i=a+1}^{b} \prod_{j=0}^{d-1} (d_1 s_1 + \cdots + d_l s_l + i + j),$$

where $d = m_1 d_1 + \cdots + m_l d_l$.

3.1 b-functions under castling transforms

Let G be a connected reductive algebraic group, let (π, V) and (ρ, W) be two finite-dimensional rational representations of G and fix $\dim V = n$. Denote by Λ_1 the standard representation of GL. Take two numbers $r_1, r_2 \in \mathbb{N}$ such that $r_1 + r_2 = n$. Following [32, Section 2.3], we form two representations:

$$R_1 = (G \times \mathrm{GL}_{r_1}, (\pi^* \otimes \Lambda_1) \oplus (\rho \otimes 1), V^{r_1} \oplus W),$$

$$R_2 = (G \times \mathrm{GL}_{r_2}, (\pi \otimes \Lambda_1^*) \oplus (\rho \otimes 1), V^{r_2} \oplus W).$$

In [60] such representations R_1, R_2 are said to be *castling transforms* of each other, while in representation theory of quivers the functors relating the representation spaces are called *reflection functors* . By [32], there are canonical isomorphisms of rings of invariants

$$\begin{aligned} \mathbb{C}[R_1]^{G\times \mathrm{SL}_{r_1}} &\cong \mathbb{C}[R_2]^{G\times \mathrm{SL}_{r_2}}, && \text{when } r_1, r_2 > 0,\\ \mathbb{C}[R_1]^{G\times \mathrm{SL}_{r_1}} &\cong \mathbb{C}[R_2]^{G} \otimes \mathbb{C}[\det_{r_1}], && \text{when } r_2 = 0. \end{aligned} \tag{3.1}$$

The papers [35, 55] give relations between the b-functions of semi-invariants of prehomogeneous spaces related under castling transform with some extra hypothesis (so-called regularity condition). However, this condition is too restrictive for our purposes. We give an extended result, for the regularity condition turns out to be unnecessary. The proof we give is similar to the sketch of proof of [36, Theorem 7.52.].

Let $f \in \mathbb{C}[R_1]$ and $f' \in \mathbb{C}[R_2]$ be two semi-invariants (so $[G, G] \times \mathrm{SL}_{r_i}$-invariants) corresponding under the isomorphisms above. Let d be the absolute value of their GL_{r_i}-weights (they are equal).

Theorem 3.1.1. *Assume $f \in \mathbb{C}[R_1]$ and $f' \in \mathbb{C}[R_2]$ are $G \times \mathrm{GL}_{r_i}$-semi-invariants with multiplicity-free weights corresponding under the isomorphisms (3.1). Then* *their b-functions satisfy*

$$b_f(s)\cdot[s]^d_{r_2}=b_{f'}(s)\cdot[s]^d_{r_1}.$$

Proof. The case $r_2=0$ is easy to check directly, since the second isomorphism in (3.1) gives a separation of variables. So we can assume $r_1,r_2>0$. Let x_{ij} ($1\le i\le n, 1\le j\le r_1$) and y_{ij} ($1\le i\le n, 1\le j\le r_2$) be indeterminates in $\mathbb{C}[V^{r_1}]$ and $\mathbb{C}[V^{r_2}]$ respectively. Also, put

$$\Lambda := \{\lambda=(i_1,\dots,i_{r_1})\mid 0<i_1<\cdots<i_{r_1}\le n\},$$

$$\Lambda' := \{\lambda'=(j_1,\dots,j_{r_2})\mid 0<j_1<\cdots<j_{r_2}\le n\}.$$

For $\lambda=(i_1,\dots,i_{r_1})\in\Lambda$ (resp. for $\lambda'=(j_1,\dots,j_{r_2})\in\Lambda'$), put $|\lambda|=i_1+\cdots+i_{r_1}$ (resp. $|\lambda'|=j_1+\cdots+j_{r_2}$) and

$$x_\lambda=\det\begin{pmatrix} x_{i_1,1} & \dots & x_{i_1,r_1}\\ \vdots & \ddots & \vdots\\ x_{i_{r_1},1} & \dots & x_{i_{r_1},r_1}\end{pmatrix},\ \text{resp. } y_{\lambda'}=\det\begin{pmatrix} y_{j_1,1} & \dots & y_{j_1,r_2}\\ \vdots & \ddots & \vdots\\ y_{j_{r_2},1} & \dots & y_{j_{r_2},r_2}\end{pmatrix}.$$

Let A (resp. A') be the subring of $\mathbb{C}[R_1]$ (resp. $\mathbb{C}[R_2]$) generated by the polynomials x_λ, where $\lambda\in\Lambda$ (resp. $y_{\lambda'}$, where $\lambda'\in\Lambda'$). Let A_k (resp. A'_k) denote its homogeneous part of degree r_1k (resp. r_2k). Similarly, we define the ring of differential operators D and D' generated by ∂x_λ (resp. $\partial y_{\lambda'}$), where

$$\partial x_\lambda=\det\begin{pmatrix} \partial x_{i_1,1} & \dots & \partial x_{i_1,r_1}\\ \vdots & \ddots & \vdots\\ \partial x_{i_{r_1},1} & \dots & \partial x_{i_{r_1},r_1}\end{pmatrix},\ \text{resp. } \partial y_{\lambda'}=\det\begin{pmatrix} \partial y_{j_1,1} & \dots & \partial y_{j_1,r_2}\\ \vdots & \ddots & \vdots\\ \partial y_{j_{r_2},1} & \dots & \partial y_{j_{r_2},r_2}\end{pmatrix}.$$

Let D_k (resp. D'_k) denote the homogeneous part of degree r_1k (resp. r_2k).

Now we endow A with the natural action of GL_n. In fact, A can be viewed as the coordinate algebra of the affine Grassmannian $\widetilde{\mathrm{Gr}}(r_1,V)$, that is, the affine cone of the usual Grassmannian variety. Similarly, we equip A' with the dual action of GL_n, viewing it as the coordinate algebra of $\widetilde{\mathrm{Gr}}(r_2, V^*)$. Due to the natural isomorphism $\widetilde{\mathrm{Gr}}(r_1, V) \cong \widetilde{\mathrm{Gr}}(r_2, V^*)$, we have a GL_n-equivariant isomorphism of graded algebras $\tau : A \to A'$. Similarly, we naturally equip D_k (resp. D'_k) with the GL_n-structure dual to A_k (resp. A'_k) via the pairing $\langle Q(\partial x_\lambda), P(x_\lambda)\rangle = Q(\partial x_\lambda)P(x_\lambda)$.

More explicitly, for $\lambda \in \Lambda$ let $\lambda' \in \Lambda'$ be the complementary set to λ, namely, $\{\lambda, \lambda'\} = \{1, \ldots, n\}$. Then τ is given by $x_\lambda \mapsto (-1)^{|\lambda'|} y_{\lambda'}$, and τ' is given by $\partial x_\lambda \mapsto (-1)^{|\lambda'|} \partial y_{\lambda'}$.

For any $k, l \in \mathbb{N}, k \leq l$, we have a GL_n-equivariant map $\phi_{k,l} : D_k \otimes A_l \to A_{l-k}$ given by $Q(\partial x_\lambda) \otimes P(x_\lambda) \mapsto Q(\partial x_\lambda)P(x_\lambda)$, and similarly a map $\phi'_{k,l} : D'_k \otimes A'_l \to A'_{l-k}$. So $\phi_{k,l}$ and $\tau^{-1} \circ \phi'_{k,l} \circ (\tau' \otimes \tau)$ are two GL_n-module morphisms $D_k \otimes A_l \to A_{l-k}$:

$$\begin{tikzcd}
D_k \otimes A_l \arrow[r, "\phi_{k,l}"] \arrow[d, "\tau'\otimes\tau"'] & A_{l-k} \\
D'_k \otimes A'_l \arrow[r, "\phi'_{k,l}"] & A'_{l-k} \arrow[u, "\tau^{-1}"']
\end{tikzcd}$$

We claim that the diagram commutes up to a constant $c_{k,l} \in \mathbb{C}$. It is well-known that A_k is an irreducible GL_n-representation corresponding to the Young tableaux of rectangular shape having r_1 rows and l columns (see for instance [72, Proposition 3.1.4]). Using this and the Littlewood-Richardson rule (cf. [72]), one easily sees that in the decomposition of $D_k \otimes A_l$ into irreducible GL_n-modules, A_{l-k} appears with multiplicity 1. This in turn implies using Schur's lemma that there is a constant $c_{k,l}$ such that $\phi = c_{k,l} \cdot \tau^{-1} \circ \phi'_{k,l} \circ (\tau' \otimes \tau)$.

To determine $c_{k,l}$, we look on the value of $\phi_{k,l}$ and $\tau^{-1} \circ \phi'_{k,l} \circ (\tau' \otimes \tau)$ on

$$\partial x^k_{(1,\ldots,r_1)} \otimes x^l_{(1,\ldots,r_1)}.$$

Using the classical Cayley identity for the determinant, we get

$$\phi_{k,l}(\partial x^k_{(1,\ldots,r_1)} \otimes x^l_{(1,\ldots,r_1)}) = \prod_{i=0}^{k-1}\prod_{j=0}^{r_1-1}(l-i+j)x^{l-k}_{(1,\ldots,r_1)},$$

$$\left(\tau^{-1}\circ\phi'_{k,l}\circ(\tau'\otimes\tau)\right)(\partial x^k_{(1,\ldots,r_1)} \otimes x^l_{(1,\ldots,r_1)}) = \prod_{i=0}^{k-1}\prod_{j=0}^{r_2-1}(l-i+j)x^{l-k}_{(1,\ldots,r_1)}.$$

From this, we get $c_{k,l} = \dfrac{\prod_{i=0}^{k-1}\prod_{j=0}^{r_2-1}(l-i+j)}{\prod_{i=0}^{k-1}\prod_{j=0}^{r_1-1}(l-i+j)}$.

Now, by the First Fundamental Theorem for SL (cf. [46]), we know that $f \in A_d \otimes \mathbb{C}[W]$ and $f' \in A'_d \otimes \mathbb{C}[W]$, moreover, $(\tau\otimes 1)(f) = f'$. From these facts we obtain that

$$b_f(s) = c_{d,d(s+1)} \cdot b_{f'}(s),$$

for any $s \in \mathbb{N}$, hence the conclusion. □

By the same argument, we give the version for the b-function of several variables.

Let $f_i \in \mathbb{C}[R_1]$ and $f'_i \in \mathbb{C}[R_2]$, where $i = 1,\ldots,l$ be semi-invariants corresponding respectively under the isomorphisms (3.1), such that the l-tuples $\underline{f} = (f_1,\ldots,f_l)$ and $\underline{f'} = (f'_1,\ldots,f'_l)$ have multiplicity-free weights. Denote by d_i the GL-weight of f_i and f'_i.

Theorem 3.1.2. *Using the notation above, the b-functions of $\underline{f}$ and $\underline{f'}$ satisfy*

$$b_{\underline{f}}(s)\cdot[s]^{d_1,\ldots,d_l}_{r_2} = b_{\underline{f'}}(s)\cdot[s]^{d_1,\ldots,d_l}_{r_1}.$$

3.2 Calculation of b-functions of quivers using reflections

Suppose x is a sink, $\beta_x \neq 0$. The isomorphisms from (3.1) translate into the quiver setting as:

$$\begin{aligned} c_x &: \mathrm{SI}(Q,\beta) \cong \mathrm{SI}(c_xQ, c_x(\beta)), && \text{when } c_x(\beta)_x > 0, \\ c_x &: \mathrm{SI}(Q,\beta) \cong \mathrm{SI}(c_xQ, c_x(\beta)) \otimes \mathbb{C}[\det_{\beta_x}], && \text{when } c_x(\beta)_x = 0. \end{aligned} \tag{3.2}$$

Note that $\mathrm{SI}(Q,\beta) \cong \mathrm{SI}(Q, \beta - \beta_x \epsilon_x)$, when $c_x(\beta)_x < 0$, where $\epsilon_x = \underline{d}(S_x)$, since in this case a semi-invariant doesn't depend on the arrows ending in x, hence we can simply drop the vertex.

All these isomorphisms respect weight spaces: when $c_x(\beta)_x \geq 0$, we have

$$\mathrm{SI}(Q,\beta)_{\langle \alpha, \cdot \rangle} \cong \mathrm{SI}(c_xQ, c_x(\beta))_{\langle c_x(\alpha), \cdot \rangle}.$$

We have the reflection functors on the representation level $C_x : \mathrm{rep}(Q) \to \mathrm{rep}(c_xQ)$ such that $C_x(S_x) = 0$, and for all other indecomposables X, $C_x(X)$ is a non-zero indecomposable representation with dimension vector $c_x(\underline{d}(X))$ (see [4]).

Theorem 3.2.1. *Let Q be a quiver without oriented cycles and $f_i \in \mathrm{SI}(Q,\beta)_{\langle \alpha^i, \cdot \rangle}$ be semi-invariants, where $i = 1, \ldots, k$. Assume $\underline{f} = (f_1, \ldots, f_k)$ has multiplicity-free weight and the coordinates of $c(\beta)$ are non-negative. Then the b-function satisfies the formula*

$$b_{\underline{f}}(s) = b_{c(\underline{f})}(s) \prod_{x \in Q_0} \frac{[s]_{\beta_x}^{c(\alpha^1)_x, \ldots, c(\alpha^k)_x}}{[s]_{c(\beta)_x}^{c(\alpha^1)_x, \ldots, c(\alpha^k)_x}}.$$

Proof. Fix a sink x. First, note that the case $c_x(\beta)_x < 0$ implies just that none of the semi-invariants depend on x (and all of them have weight 0 at x), so we can drop the vertex.

Since x is a source in c_xQ, the absolute value of the GL_{β_x}-weight of f and $c_x(f)$ is $c_x(\alpha^i)_x$. Now applying Theorem 3.1.2, we get that

$$b_{\underline{f}}(s) = b_{c_x(\underline{f})}(s) \cdot \frac{[s]_{\beta_x}^{c_x(\alpha^1)_x, \ldots, c_x(\alpha^k)_x}}{[s]_{c_x(\beta)_x}^{c_x(\alpha^1)_x, \ldots, c_x(\alpha^k)_x}}.$$

Applying this to an admissible sequence we get the desired formula. $\square$

We call a semi-invariant $f \in \mathrm{SI}(Q,\beta)$ *reducible by reflections* if, after applying reflection functors finitely many times, we can reduce it to a constant function via the isomorphisms in (3.2). In this case, we can compute the b-function of f by Theorem 3.2.1. Similarly, we say the tuple $\underline{f} = f_1, \dots, f_k \in \mathrm{SI}(Q,\beta)$ is reducible by reflections, if we can compute their b-function of several variables using reflections. Note that we will use simplifications according to Lemma 2.3.7 whenever possible.

Theorem 3.2.2. *Let Q be a quiver without oriented cycles, $f \in \mathrm{SI}(Q,\beta)$ a semi-invariant of weight $\langle \alpha, \cdot \rangle$. Assume one of the following cases holds:*

(a) The dimension vector α is preprojective or preinjective, or

(b) The dimension vector β is prehomogeneous and any indecomposable in the canonical decomposition of β is preprojective or preinjective.

Then f is reducible by reflections.

Proof. (a) We show this for α preprojective. By Lemma 2.3.4, f has multiplicity-free weight. Applying the Coxeter transformation sufficiently many times, we arrive at a semi-invariant whose weight corresponds to a projective dimension vector, which implies that it is constant. The only thing one has to deal with is that after applying a reflection transformation c_x, one might end up with a vector β' with $\beta'_x < 0$. But in this case the function doesn't depend on x, so after replacing β'_x with 0, we can carry on with the procedure.

(b) If the canonical decomposition of β doesn't have the simple S_x as summand, for x a sink, then applying c_x to each indecomposable in the canonical decomposition gives us the canonical decomposition of $c_x(\beta)$. We apply the reflection functor in the order given by the admissible sequence repeatedly until we reach a simple, that is, by applying $c_{x_{i-1}}$ we reach the simple S_{x_i} as a summand of the canonical decomposition. Write $f = c_W$. Since f

doesn't vanish on the generic element, we have that $W_x = \mathrm{Hom}(S_x, W) = 0$. By Lemma 2.3.7, this implies that f doesn't depend on the arrows of the sink x_i. So we can drop all S_{x_i} from the canonical decomposition and then continue by applying c_{x_i}. After we get rid of all preprojectives this way, we start working dually (with sources) to get rid of all preinjectives.

□

The following result is immediate by either case in Theorem 3.2.2:

Theorem 3.2.3. *All (tuples of) semi-invariants of Dynkin quivers are reducible by reflections.*

Remark 3.2.4. The fact that the b-function is truly a polynomial in the end is not obvious from the formula given in Theorem 3.2.1. Also, in case of Dynkin quivers, we can apply the procedure in both directions, either with sinks or with sources. The b-function holds interesting combinatorial information.

Theorem 3.2.5. *Let Q be an extended Dynkin quiver, and β a prehomogeneous dimension vector. Then (tuples of) semi-invariants in* $\mathrm{SI}(Q, \beta)$ *are reducible by reflections.*

Proof. Using the procedure described in Proposition 3.2.2, we can reduce to the case when all the indecomposables in the canonical decomposition are regular. By [50, Lemma 5.1], the left orthogonal category of the generic representation contains a preprojective representation. Hence applying the Coxeter functor sufficiently many times, we arrive at a dimension vector β' such that $0 = \langle \underline{d}(P_x), \beta' \rangle = \beta'_x$, where P_x is the projective corresponding to a vertex x. Hence we can drop vertex x, reducing to the Dynkin case. □

Now we consider some examples:

Example 3.2.6. We take the following quiver of type $\mathbb{E}_6$, f the semi-invariant of weight $\langle \alpha, \cdot \rangle$, we write the values α above the values β, and $\beta_1 + \beta_2 + \beta_3 + \beta_4 + 2\beta_5 =$

$3\beta_6$, together with the necessary inequalities that assure f is non-zero (see below). The Coxeter transformation is given by

$$c = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & -1 \\ 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 1 & -1 \end{pmatrix}$$

$$\begin{tikzcd} & & \overset{2}{\beta_5} \arrow[d] & & \\ \overset{1}{\beta_1} \arrow[r] & \overset{2}{\beta_2} \arrow[r] & \overset{3}{\beta_6} & \overset{2}{\beta_4} \arrow[l] & \overset{1}{\beta_3} \arrow[l] \end{tikzcd} \quad \overset{\text{Apply } c}{\rightsquigarrow}$$

$$\begin{tikzcd} & & \underset{\beta_2+\beta_4-\beta_6}{1} \arrow[d] & & \\ \underset{\beta_4+\beta_5-\beta_6}{1} \arrow[r] & \underset{\beta_1+\beta_4+\beta_5-\beta_6}{2} \arrow[r] & \underset{\beta_2+\beta_4+\beta_5-\beta_6}{3} & \underset{\beta_2+\beta_3+\beta_5-\beta_6}{2} \arrow[l] & \underset{\beta_2+\beta_5-\beta_6}{1} \arrow[l] \end{tikzcd}$$

$$\begin{tikzcd} & & \underset{\beta_1+\beta_3+\beta_5-\beta_6}{1} \arrow[d] & & \\ \underset{\beta_2+\beta_3-\beta_6}{0} \arrow[r] & \underset{\beta_6-\beta_5-\beta_1}{1} \arrow[r] & \underset{\beta_6-\beta_5}{2} & \underset{\beta_6-\beta_5-\beta_3}{1} \arrow[l] & \underset{\beta_1+\beta_4-\beta_6}{0} \arrow[l] \end{tikzcd}$$

Here, using the Lemma 2.3.7 (a), we can simplify by dropping vertex 1 and 3, and as the last step we apply only c_6:

$$\begin{tikzcd} & \underset{\beta_1+\beta_3+\beta_5-\beta_6}{1} & \\ \underset{\beta_6-\beta_5-\beta_1}{1} & \underset{0}{1} \arrow[l] \arrow[u] \arrow[r] & \underset{\beta_6-\beta_5-\beta_3}{1} \end{tikzcd}$$

Here the semi-invariant becomes constant, so we stop. Hence the b-function is

$$b(s) = \left(\frac{[s]_{\beta_1} \cdot [s]^2_{\beta_2} \cdot [s]_{\beta_3} \cdot [s]^2_{\beta_4} \cdot [s]_{\beta_5} \cdot [s]^3_{\beta_6}}{[s]_{\beta_4+\beta_5-\beta_6} [s]^2_{\beta_1+\beta_4+\beta_5-\beta_6} [s]_{\beta_2+\beta_5-\beta_6} [s]^2_{\beta_2+\beta_3+\beta_5-\beta_6} [s]_{\beta_2+\beta_4-\beta_6} [s]^3_{\beta_2+\beta_4+\beta_5-\beta_6}} \right).$$

$$\cdot\left(\frac{[s]_{\beta_1+\beta_4+\beta_5-\beta_6}[s]_{\beta_2+\beta_3+\beta_5-\beta_6}[s]_{\beta_2+\beta_4-\beta_6}[s]^2_{\beta_2+\beta_4+\beta_5-\beta_6}}{[s]_{\beta_6-\beta_5-\beta_1}[s]_{\beta_6-\beta_5-\beta_3}[s]_{\beta_1+\beta_3+\beta_5-\beta_6}[s]^2_{\beta_6-\beta_5}}\right)\cdot[s]_{\beta_6-\beta_5}$$

Note that using the inequalities between β_i, we can reduce this expression to a polynomial. We can read the inequalities looking at the dimension vectors in the Coxeter transformations:

$$\beta_6 \leq \beta_4+\beta_5, \beta_2+\beta_5, \beta_2+\beta_4, \beta_1+\beta_3+\beta_5$$

$$\beta_6 \geq \beta_1+\beta_5, \beta_3+\beta_5$$

Using these, one way to write the b-function as a polynomial is:

$$b(s) = [s]_{\beta_4+\beta_5-\beta_6,\beta_1+\beta_4+\beta_5-\beta_6}[s]_{\beta_2+\beta_5-\beta_6,\beta_2+\beta_3+\beta_5-\beta_6}[s]^2_{\beta_2+\beta_3+\beta_5-\beta_6,\beta_2}[s]^2_{\beta_1+\beta_4+\beta_5-\beta_6,\beta_4}\cdot$$

$$\cdot[s]^3_{\beta_2+\beta_4+\beta_5-\beta_6,\beta_6}[s]_{\beta_1+\beta_3+\beta_5-\beta_6,\beta_1}[s]_{\beta_6-\beta_5-\beta_1,\beta_3}[s]_{\beta_6-\beta_5-\beta_3,\beta_6-\beta_5}[s]^2_{\beta_6-\beta_5,\beta_2+\beta_4+\beta_5-\beta_6}[s]_{\beta_5}$$

In the next example, we compute the b-function of several variables of 4 semi-invariants:

Example 3.2.7. Take the following $\mathbb{D}_5$ quiver

$$\begin{tikzcd} & 5 & & \\ 1 \arrow[r] & 4 \arrow[u] & 3 \arrow[l] & 2 \arrow[l] \end{tikzcd}$$

with dimension vector $\beta = (n, n, 2n, 2n, n)$, where $n \in \mathbb{N}$. There are 4 fundamental invariants f_i, $i = 1, \dots, 4$, with weights $\langle \alpha^i, \cdot \rangle$, where $\alpha^1 = (0,0,1,0,0), \alpha^2 = (1,0,0,1,0), \alpha^3 = (0,1,1,1,0), \alpha^4 = (1,1,1,1,1)$. Explicitly, if we label the generic matrices as

$$\begin{tikzcd} & n & & \\ n \arrow[r, "A"] & 2n \arrow[u, "D"] & 2n \arrow[l, "C"'] & n \arrow[l, "B"'] \end{tikzcd}$$

then the semi-invariants are $f_1 = \det C$, $f_2 = \det(DA)$, $f_3 = \det(DCB)$, $f_4 = \det(A, CB)$. The Coxeter transformation is given by

$$c = \begin{pmatrix} 0 & 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 & -1 \\ 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

Again, we write the (ordered) values of α on top of β, and apply c:

$$\begin{tikzcd} & \overset{0,0,0,1}{n} & & & \\ \overset{0,1,0,1}{n} \arrow[r] & \overset{0,1,1,1}{2n} \arrow[u] & \overset{1,0,1,1}{2n} \arrow[l] & \overset{0,0,1,1}{n} \arrow[l] & {} \arrow[r, rightsquigarrow, "\text{Apply } c"] & {} \end{tikzcd}$$

$$\begin{tikzcd} & \overset{0,1,1,0}{n} & & \\ \overset{1,0,1,0}{n} \arrow[r] & \overset{1,1,1,1}{2n} \arrow[u] & \overset{0,1,1,1}{n} \arrow[l] & \overset{0,1,0,0}{0} \arrow[l] \end{tikzcd}$$

Here we can drop vertex 2, and the second semi-invariant, since it became constant. So we are left with

$$\begin{tikzcd} & \overset{0,1,0}{n} & & & & \overset{1,0,1}{n} & \\ \overset{1,1,0}{n} \arrow[r] & \overset{1,1,1}{2n} \arrow[u] & \overset{0,1,1}{n} \arrow[l] & {} \arrow[r, rightsquigarrow] & \overset{0,0,1}{0} \arrow[r] & \overset{1,1,1}{n} \arrow[u] & \overset{1,0,0}{0} \arrow[l] \end{tikzcd}$$

Here the first and last semi-invariants became constant, and the middle is the determinant of size n. So the b-function is:

$$b_{\underline{f}}(s) = [s]_n^{0,1,0,0} \cdot [s]_{n,2n}^{0,1,1,1} \cdot [s]_n^{0,0,0,1} \cdot [s]_n^{1,0,0,0} \cdot [s]_{n,2n}^{1,0,1,1} \cdot [s]_n^{0,0,1,0}.$$

Example 3.2.8. Let Q be the Kronecker quiver

$$1 \rightrightarrows 2$$

with the prehomogeneous dimension vector $\beta = (n \cdot k, (n+1)k)$. We have the following semi-invariant f_n with weight $\langle \alpha, \cdot \rangle$ corresponding to $\alpha = (n+1, n+2)$:

$$f_n = \det \begin{pmatrix} X & Y & 0 & \dots & 0 & 0 \\ 0 & X & Y & \dots & 0 & 0 \\ & \vdots & & \ddots & & \\ 0 & 0 & 0 & \dots & Y & 0 \\ 0 & 0 & 0 & \dots & X & Y \end{pmatrix}.$$

Here there are $n+1$ block columns and n block rows. Applying c_2 we get the same quiver (after renumbering) with dimension vector $c_2(\beta) = ((n-1)k, nk)$ and $c_2(\alpha) = (n, n+1)$. Hence

$$b_{f_n}(s) = b_{f_{n-1}} \cdot [s]^n_{(n-1)k,(n+1)k} = \prod_{i=1}^{n} [s]^i_{(i-1)k,(i+1)k}.$$

In the following example (compare with example in [50]), we see that the method by reflection functors does not give the full b-function of several variables. We outline a remedy for this:

Example 3.2.9. Let (Q, β) be the following quiver

$$\begin{tikzcd} 8n \arrow[drr] & 3n \arrow[dr] & 3n \arrow[d] & 3n \arrow[dl] & 3n \arrow[dll] \\ & & 9n & & \end{tikzcd}$$

where $n \in \mathbb{N}$. Then β is a prehomogeneous dimension vector, and is a multiple of the regular indecomposable $\begin{smallmatrix} 8\,3\,3\,3\,3 \\ 9 \end{smallmatrix}$. We have 5 fundamental semi-invariants $f_i = c^{V_i} \in \mathrm{SI}(Q, \beta), i = 1, \dots, 5$, where the $\underline{d}(V_i) = \alpha_i$ correspond to the following simples in the left orthogonal category:

$$\begin{matrix} 0\,0\,1\,1\,1 \\ 2 \end{matrix}, \quad \begin{matrix} 0\,1\,0\,1\,1 \\ 2 \end{matrix}, \quad \begin{matrix} 0\,1\,1\,0\,1 \\ 2 \end{matrix}, \quad \begin{matrix} 0\,1\,1\,1\,0 \\ 2 \end{matrix}, \quad \begin{matrix} 3\,1\,1\,1\,1 \\ 3 \end{matrix}.$$

These are also regular representations. So we see that the method of reflections won't work directly if we want to compute their b-function of several variables. However, if we consider only the first 4 semi-invariants f_1, f_2, f_3, f_4, then we can drop the first source vertex (since the semi-invariants have weight 0), and the resulting quiver being extended Dynkin, the b-function b_{f_1,f_2,f_3,f_4} (of 4 variables) can be computed. The b-function (of one variable) of f_5 can be computed by applying the inverse Coxeter transformation c^{-1} first, when we get

$$c^{-1}(\alpha_5) = \begin{array}{ccccc} 0 & 2 & 2 & 2 & 2 \\ & & 5 & & \end{array}.$$

Then we can again drop the first source vertex, and we are left with an extended Dynkin quiver.

Hence, we can compute b_{f_1,f_2,f_3,f_4} and b_{f_5}. In order to obtain information about the b-function of 5 variables, one should compute the a-function first, and then employ the structure theorem on b-functions ([57, Theorem 2] or [68, Theorem 1.3.5], see also [66]).

Chapter 4

b-functions and slices

In this chapter, we discuss slices with some applications including another method of computing b-functions. The technique is similar to the localization methods used in [68, 71, 56]. In Section 4.1, we describe the slice method and work in a more general setting of reductive groups. Under some technical assumptions, the roots of the b-function turn out to be invariants of the root system. Using Theorem 4.1.4 we show how to compute the b-functions of classical semi-invariants like the determinant, Pfaffian, and symmetric determinant. We also give the analogous result for b-functions of several variables (Theorem 4.1.11). In Section 4.2, we apply slices in the quiver setting. This gives useful algebra maps between rings of semi-invariants of two quivers. In Section 4.3, the slice technique is applied to arrows of quivers, which gives a practical reduction method for computing b-functions of many determinantal quiver semi-invariants (in which case we call them sliceable), including those of quivers of type $\mathbb{A}, \mathbb{D}$. The reduction provides other useful information as well. We work out several examples and theorems on b-functions of one variable, finishing with a couple of examples of b-functions of several variables. Using Proposition 4.3.10, we give an example of a semi-invariant of E_6 that is not sliceable. In Section 4.4, we give a method for the determination of the canonical decomposition for type $\mathbb{D}$ quivers.

In the formulas for b-functions, we continue using the Notation 3 from Chapter 3.

4.1 Reductions using slices

Let G be a connected reductive algebraic group acting rationally and non-trivially on V, and let f be a non-zero semi-invariant of weight σ. Let $\mathfrak{g}$ be the Lie algebra of G. Fix $v \in V$ an arbitrary point. Then we can take the tangent space to the orbit $\mathcal{O} = Gv$ of v, $T_v(\mathcal{O}) = \mathfrak{g} \cdot v$. The stabilizer G_v acts on $T_v(\mathcal{O})$, however, it might fail to be reductive - Matsushima's criterion says that G_v is reductive iff $\mathcal{O}_v$ is affine. Nevertheless, we can still write $G_v = L'_v \ltimes U_v$, where U_v is the unipotent radical of G_v, and $L'_v \cong G_v/U_v$ is reductive. Let L_v be the connected component of L'_v. Take the L_v-complement W in $V = T_v(\mathcal{O}) \oplus W$, and call (L_v, W) the *slice representation* at v.

Define a new semi-invariant f_v on (L_v, W) by $f_v(w) := f(v+w)$, for $w \in W$. As in [68], we consider the map

$$\mu : G \times W \to V$$

$$\mu(g, w) = g(v + w).$$

Computing the differential at the identity, we see that the map is smooth. In particular, the algebra map μ^* is injective. The map separates variables, for

$$\mu^*(f) = \sigma^{-1} \otimes f_v. \tag{4.1}$$

We have the following lemma by [68, p. 57]:

Lemma 4.1.1. *Using the notation above, $f_v \neq 0$, and $b_{f_v,0} = b_{f,\mathcal{O}}$. That is, the local b-function of f_v at 0 coincides with the local b-function of f at v. In particular, if the slice representation (L_v, W) has a unique closed orbit, then b_{f_v} divides b_f.*

Proof. Consider the map

$$\mu : G \times W \to V$$

$$\mu(g, w) = g(v + w).$$

Computing the differential at the identity $(1, 0)$, we see that the map is smooth. In particular, the algebra map μ^* is injective. The map separates variables, for $\mu^*(f) = \sigma^{-1} \otimes f_v$ by

$$\mu^*(f)(g, w) = f(g(v + w)) = \sigma(g)^{-1} f_v(w).$$

Hence $f_v \neq 0$. Using the holomorphic Constant Rank Theorem, together with [26, Lemma 2.5.4], one gets $b_{f_v,0} = b_{f,\mathcal{O}}$. □

We want the explicit equation giving the local b-function at v, and using these to construct the equation for the global b-function. We illustrate this in the case when v is a highest weight vector.

From now on, we assume G is a connected reductive algebraic group. Let us recall some standard structure theory of reductive groups (for example, see [65]). Pick T a maximal torus in G, B a Borel subgroup containing T. Let $(X(T), \Phi, X(T)^\vee, \Phi^\vee)$ be the root datum with pairing $\langle\ ,\ \rangle : X(T) \times X(T)^\vee \to \mathbb{Z}$. Let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{t} \oplus \mathfrak{n}^+$ the root space decomposition for the Lie algebra of G, with $\mathfrak{n}^\pm = \bigoplus_{\alpha \in \Phi^\pm} \mathfrak{g}_\alpha$, where Φ^+ (resp. Φ^-) is the set of positive (resp. negative) roots, $\Phi = \Phi^- \cup \Phi^+$. Let Δ be a fixed set of simple roots α_i, and $\lambda_i \in \mathbb{Q} \otimes_{\mathbb{Z}} X(T)$ be the fundamental dominant weight corresponding to α_i, i.e. $\langle \lambda_i, \alpha_j^\vee \rangle = \delta_{ij}$. Also denote by $X(T)_0 = \{x \in X(T) | \langle x, \alpha^\vee \rangle = 0, \text{ for all } \alpha \in \Phi\}$. For a dominant weight $\lambda = a_1\lambda_1 + \cdots + a_k\lambda_k$, where $a_i \in \mathbb{N}$, we denote by $a_1\Lambda_1 + \cdots + a_k\Lambda_k$ the irreducible representation corresponding to λ.

Assume $v \in V$ is a highest weight vector of dominant weight $\lambda \neq 0$. Put $I = \{\alpha_i \in \Delta | \langle \lambda, \alpha_i^\vee \rangle = 0\}$, and let Φ_I be the corresponding root subsystem of Φ spanned by the simple roots in I. Denote $\mathfrak{n}_I^\pm = \bigoplus_{\alpha \in \Phi_I^\pm} \mathfrak{g}_\alpha$. Let $\mathfrak{p}_I = \mathfrak{n}_I^- \oplus \mathfrak{t} \oplus \mathfrak{n}^+$ the parabolic subalgebra, P_I the corresponding parabolic subgroup. Also, let

$\mathfrak{l}_I = \mathfrak{n}_I^- \oplus \mathfrak{t} \oplus \mathfrak{n}_I^+$ be the Levi subalgebra, L_I the corresponding Levi subgroup, $\mathfrak{u}^\pm = \bigoplus_{\alpha \in \Phi^\pm \setminus \Phi_I^\pm} \mathfrak{g}_\alpha$, $U^\pm$ the corresponding unipotent subgroups. We have the usual Levi decomposition $P_I = L_I \ltimes U^+$. It is well known that P_I is the stabilizer of the line $\mathbb{C}v$. According to our previous notation, $G_v \subset P_I$, $L_v \subset L_I$ are subgroups of codimension 1. As before, take the L_I (or L_v) complement W to $\mathfrak{g}v$, that is $V = \mathfrak{g}v \oplus W$. Note that $\mathfrak{g}v = \mathbb{C}v \oplus \mathfrak{u}^- v$. We assume that under the choice $V \cong \mathbb{C}^n$ we have a weight basis for T, moreover, v and the weight vectors in $\mathfrak{u}^- v$ will be basis elements.

We will compute recursively under the following *main technical assumption:*

$$\text{There exists } z \in \mathfrak{t} \text{ such that } z \cdot W = 0 \text{ and } z \cdot v \neq 0. \tag{4.2}$$

Denote $\mathfrak{t}_v := \mathbb{C}z \subset \mathfrak{t}$, and let $p : \mathbb{C}^\times \to T$ be the corresponding 1-parameter subgroup (or co-weight). We have $\mathfrak{l}_I = \mathfrak{t}_v \oplus \mathfrak{l}_v$.

Let $\Pi(W)$ be the $\mathbb{Z}$-span of the T-weights of W. Then the following is equivalent to condition (4.2) above:

$$\mathbb{Z}\lambda \cap \Pi(W) = \{0\}. \tag{4.3}$$

Example 4.1.2. Condition (4.3) holds for the representation $(\mathrm{GL}_n \times \mathrm{GL}_n, M_n(\mathbb{C}))$ (see Example 4.1.7). However, it is easy to see that the condition is not satisfied for $(\mathrm{GL}_6, \Lambda^3\mathbb{C}^6)$.

We derive a result on how restrictive assumption (4.3) is:

Lemma 4.1.3. *Let λ be a dominant weight as above. If condition (4.3) is satisfied, then $\langle \lambda, \alpha^\vee \rangle \leq 2$, for all $\alpha \in \Phi^+$, and λ is necessarily of the following form:*

$$\lambda = \epsilon_1 \lambda_{j_1} + \epsilon_2 \lambda_{j_2} + \lambda_0,$$

where $\epsilon_1, \epsilon_2 \in \{0, 1\}$, $\lambda_{j_1}, \lambda_{j_2}$ are fundamental dominant weights, $\lambda_0 \in X(T)_0$. If

λ_{j_1} and λ_{j_2} are distinct, then they correspond to different irreducible root systems (that is, they are nodes in different connected components of the associated graph).

Proof. The irreducible representation $V(\lambda)$ of highest weight λ is a direct summand of V. If there is an $\alpha \in \Phi^+ \backslash \Phi_I^+$ such that $\langle \lambda, \alpha^\vee \rangle \geq 3$, then $\lambda - 2\alpha \in \Pi(W)$ and $\lambda - 3\alpha \in \Pi(W)$, hence $\lambda \in \Pi(W)$, contradiction. Hence $\langle \lambda, \alpha^\vee \rangle \leq 2$, for all $\alpha \in \Phi^+$. In particular, all coefficients of fundamental weights are ≤ 2.

Now assume $\lambda = \lambda_{j_1} + \lambda_{j_2} + \lambda' + \lambda_0$, where λ_{j_1} and λ_{j_2} are distinct fundamental weights, and λ' is a sum of fundamental weights. We want to show that the corresponding simple roots $\alpha_{j_1}, \alpha_{j_2}$ lie in different irreducible root systems. Suppose α_{j_1} and α_{j_2} are in the same connected Dynkin diagram, then there is a root $\alpha_{j_1} + \sum_{i=1}^{l} \alpha^i + \alpha_{j_2}$ corresponding to a path. Denote $\beta = \alpha_{j_1} + \sum_{i=1}^{l} \alpha^i$. Since $\langle \lambda, \alpha^\vee \rangle \leq 2$ for all $\alpha \in \Phi^+$, $2\beta + \alpha_{j_2}$ and $\beta + 2\alpha_{j_2}$ are not roots. Then $\lambda - 2\beta - \alpha_{j_2}, \lambda - \beta - 2\alpha_{j_2}, \lambda - 2\beta - 2\alpha_{j_2}$ are all weights of W, hence $\lambda \in \Pi(W)$, a contradiction. So α_{j_1} and α_{j_2} lie in different irreducible root systems.

If $\lambda = 2\lambda_{j_1} + \lambda_{j_2} + \lambda_0$, then by the previous discussion, α_{j_1} and α_{j_2} are in particular orthogonal. Hence $\lambda - \alpha_1 - \alpha_2, \lambda - 2\alpha_1, \lambda - 2\alpha_1 - \lambda_2 \in \Pi(W)$, hence $\lambda \in \Pi(W)$, a contradiction.

If $\lambda = \lambda_{j_1} + \lambda_{j_2} + \lambda_{j_3} + \lambda_0$, with all λ_{j_i} distinct, for $i = 1, 2, 3$, then the λ_{j_i} lie in different irreducible root systems, and we have $\lambda - \alpha_{j_1} - \alpha_{j_2}, \lambda - \alpha_{j_1} - \alpha_{j_3}, \lambda - \alpha_{j_2} - \alpha_{j_3}, \lambda - \alpha_{j_1} - \alpha_{j_2} - \alpha_{j_3} \in \Pi(W)$. This again contradicts (4.3). □

As before, let $f \in \mathbb{C}[V]$ be a G-semi-invariant of weight σ, and $f_v \in \mathbb{C}[W]$ the associated L_v-semi-invariant of weight $\sigma|_{L_v}$. Let $f^* \in \mathbb{C}[V^*]_{\sigma^{-1}}$ be a dual semi-invariant of f, and let v^* be the dual of v with $v^*(v) = 1$, a lowest weight vector in V^*. Due to condition (4.3), v^* is canonical.

From the L_v-decomposition $V = \mathfrak{g}v \oplus W$, we have an L_v-decomposition $V^* = \mathfrak{g}v^* \oplus W^*$. Let $f_v^* \in \mathbb{C}[W^*]$ be $f_v^*(w^*) := (f^*)_{v^*}(w^*) = f^*(v^* + w^*)$, for every

$w^* \in W^*$. It is also an L_v-semi-invariant, of weight $\sigma^{-1}|_{L_v}$.

For any element $a \in V$, denote by $\deg_a f$ the degree of f at a, namely

$$\deg_a f = \max\left\{k \in \mathbb{N} \mid \frac{\partial^k f}{\partial a^k} \neq 0\right\}$$

Since f is a semi-invariant, the degree doesn't depend on the orbit, that is, $\deg_a f = \deg_{g\cdot a} f$, for any $g \in G$.

We will assume that f depends on v, i.e. $\deg_v f > 0$. The following is the main result of this section:

Theorem 4.1.4. *Using the notations above, assume the weight $\sigma|_{L_v}$ of f_v is multiplicity-free, and let $b_v(s)$ be the b-function of f_v. Furthermore, assume that $\mathbb{Z}\lambda \cap \Pi(W) = \{0\}$.*

(a) In the neighborhood $v^* \neq 0$, we have the following equation for the local b-function of f at v:

$$\left[\frac{1}{v^*(x)^d} f_v^*(\partial w)\right] f(x)^{s+1} = b_v(s) f(x)^s.$$

(b) σ is multiplicity-free, and global b-function of f is

$$b(s) = b_v(s) \prod_{i=0}^{d-1} (ds + r + i)$$

where $d = \deg_v f$, and $r \in \mathbb{Q}_+$ is given by $r = 1 + \sum_{\alpha \in \Phi^+ \setminus \Phi_I^+} \frac{\langle \lambda - \alpha, p\rangle}{\langle \lambda, p\rangle}$, for any 1-parameter subgroup p as in (4.2).

Proof. (a) Since $\sigma|_{L_v}$ is multiplicity-free, we have the following equation on W:

$$f_v^*(\partial w) f_v^{s+1}(w^*) = b_v(s) f_v^s(w^*) \tag{4.4}$$

Here $\partial w = \left(\frac{\partial}{\partial w_1}, \dots, \frac{\partial}{\partial w_k}\right)$, and $w^* = (w_1^*, \dots, w_k^*)$ with respect to our basis elements $\{w_1, \dots, w_k\}$ from W.

By Lemma 4.1.1, $b_v(s)$ is also the local b-function of f at v, and σ is multiplicity-free too.

Consider the étale map:

$$\mu : \mathbb{C}^\times \times U^- \times W \to V$$

$$(t, u, w) \mapsto u(t \cdot v + w)$$

As before, the differential at identity induces an isomorphism of tangent spaces. Also, the differential at a point (t, u, w) evaluated at $\dfrac{\partial}{\partial w}$ is:

$$d_{(t,u,w)}\mu : \mathbb{C} \oplus \mathfrak{u}^- \oplus W \to V$$

$$d_{(t,u,w)}\mu(0, 0, \frac{\partial}{\partial w}) = u \cdot \frac{\partial}{\partial w} \tag{4.5}$$

Since condition (4.2) is satisfied, we have the one-parameter subgroup $p : \mathbb{C}^\times \to T$, such that $\langle \lambda, p \rangle = K$, where we can assume K is a positive integer, and $\langle \Pi(W), p \rangle = 0$. Also, let $\langle \sigma, p \rangle = -K' \in \mathbb{Z}$. Hence for any $t \in \mathbb{C}^\times, w \in W$ we have:

$$f(t^K v + w) = f(p(t)v + w) = f(p(t)(v + w)) = \sigma(p(t))^{-1} f(v + w) = t^{K'} f_v(w)$$

Since $f_v \neq 0$, choose $w_0 \in W$ such that $f_v(w_0) \neq 0$, so we have that $f(t^K v + w_0) = t^{K'} f_v(w_0) \neq 0$. Since f is a polynomial, we have $K|K'$, and $K' \geq 0$. Denote $d' = K'/K$. We have $f(t \cdot v + w) = t^{d'} f_v(w)$. Obviously, $d' \leq d = \deg_v f$. We want to show that $d' = d$.

For each root α, take the 1-dimensional unipotent subgroup U_α with Lie algebra $\mathfrak{g}_\alpha$. We have an additive isomorphism $u_\alpha : \mathbb{C} \to U_\alpha$. By Lemma 4.1.3, we have $\langle \lambda, \alpha^\vee \rangle \leq 2$ for $\alpha \in \Phi^+ \backslash \Phi_I^+$, hence there exist elements $0 \neq v_\alpha \in \mathfrak{g}_{-\alpha} v$ and $w_\alpha \in \mathfrak{g}_{-\alpha} v_\alpha \subset W$, such that $u_{-\alpha}(y) \cdot v = v + y \cdot v_\alpha + y^2 \cdot w_\alpha$. We fix a weight basis such that the v_α are basis elements, for $\alpha \in \Phi^+ \backslash \Phi_I^+$. In particular, they form a basis for $\mathfrak{u}^- v$.

Assume $d' < d$. Since $f(t \cdot v + w) = t^{d'} f(v + w), w \in W$, and $d = \deg_v f$, the polynomial f must have a non-zero term of the form $v^{*d} \prod v_\alpha^{*a_\alpha} \cdot q^*$, where q^* doesn't depend on v^* and v_α^*, for $\alpha \in \Phi^+ \backslash \Phi_I^+$, and the numbers $a_\alpha \in \mathbb{N}$ are not all zero. First, assume there is an element $\alpha_0 \in \Phi^+ \backslash \Phi_I^+$ with $a_{\alpha_0} > 0$, and $\langle \lambda, \alpha_0 \rangle = 1$. Hence we have that $w_{\alpha_0} = 0$. Take $a + b$ maximal with the property such that f has a non-zero term $v^{*a} \cdot v_{\alpha_0}^{*b} \cdot q^*$, where q^* doesn't depend on v^*, and $v_{\alpha_0}^*$, with $a, b \in \mathbb{N}$, and $a + b > d$. We look at the following expression, which is polynomial in y after expanding:

$$\left(\frac{\partial}{\partial(v + y v_{\alpha_0})} \right)^{a+b} \cdot f$$

The coefficient of y^b in is $a! \cdot b! \cdot q^* \neq 0$. But v and $v + y \cdot v_\alpha$ lie in the same orbit, hence the degree of f at $v + y \cdot v_\alpha$ is also d, in contradiction with $d < a + b$.

So if $a_\alpha > 0$, then $\langle \lambda, \alpha \rangle = 2$, by Lemma 4.1.3. In this case, $\lambda - 2\alpha$ is a weight of W, we have $\langle \lambda - 2\alpha, p \rangle = 0$, hence $\langle \lambda - \alpha, p \rangle = K/2 > 0$. Hence the weight of $\prod v_\alpha^{*a_\alpha}$ under the action of p is strictly positive. But then the monomial term $v^{*d} \prod v_\alpha^{*a_\alpha} \cdot q^*$ has weight strictly greater than K', which is a contradiction.

Hence $d' = d$. With similar reasoning, we get that $\deg_{v^*} f^* = d$.

Now we pull back via μ:

$$\mu^* \left(\frac{1}{v^{*d}} f \right) (t, u, w) = \frac{1}{t^d} f(u(t \cdot v + w)) = \frac{1}{t^d} f(t \cdot v + w) = f_v(w),$$

where we have used the fact that $\sigma(u) = 1$. Hence we can rewrite (4.4) as

$$f_v^*(\partial w) \mu^* \left(v^{*-d} f \right)^{s+1} = b_v(s) \mu^* \left(v^{*-d} f \right)^s$$

By change of variables, the following holds in a neighborhood of v:

$$d\mu_*(f_v^*)(v^*(x)^{-d} f(x))^{s+1} = b_v(s)(v^*(x)^{-d} f(x))^s$$

By (4.5) and $u{\cdot}f^*(v^*{+}w^*) = f^*(v^*{+}w^*)$, we have that $d\mu_*(f_v^*)(\partial x) = f_v^*(\partial w)$ (here we view f_v^* as a function on V^*). After canceling $v^{*-d\cdot s}$ from both sides, we get (a).

(b) First, note that viewing f_v as a function on V (i.e. it is 0 on $\mathfrak{g}v$), we have

$$f_v = \frac{1}{d!}\left(\frac{\partial}{\partial v}\right)^d f$$

We rewrite equation (a) in the following form:

$$\frac{\partial^d f^*}{\partial v^{*d}}(\partial x) f^{s+1}(x) = c_0(s) v^*(x)^d f^s(x),$$

where $c_0(s) = d! \cdot b(s)$.

For the sake of notation, denote $v_0 := v$ and number the elements $\alpha \in \Phi^+ \backslash \Phi_I^+$ by $1, 2, \ldots m$. If v_i corresponds to v_α, write $r_i = \dfrac{\langle \lambda - \alpha, p\rangle}{\langle \lambda, p\rangle}$ and let $r_0 = 1$. Let

$$r = \sum_{i=0}^{m} r_i = m - \frac{\langle \sum \alpha, p\rangle}{\langle \lambda, p\rangle},$$

where the latter sum is over $\alpha \in \Phi^+ \backslash \Phi_I^+$. We will see that r is independent of the choice of p.

Differentiating the action of $p(t)$ on f, and evaluating at $t = 1$, we get that the Euler operator

$$E = \sum_{i=0}^{m} r_i v_i^* \frac{\partial}{\partial v_i}$$

satisfies $E \cdot f = d \cdot f$, and also $E^* \cdot f^* = d \cdot f^*$, where E^* is the dual operator.

Now we build up the global b-function by induction. We want to prove that there exist functions $c_{l-1}(s)$ with

$$\frac{\partial^{d-l+1} f^*}{\partial v_0^{*d-l+1}}(\partial x) f^{s+1}(x) = c_{l-1}(s) v_1^*(x)^{d-l+1} f^s(x) \tag{4.6}$$

for any integer l with $0 < l \leq d+1$. We proceed by induction on l. For $l = 1$, this is equation is satisfied if we take $c_0(s) = d! \cdot b_v(s)$.

Assume (4.6) is satisfied for a fixed integer $0 < l \leq d$. Applying the 1-dimensional unipotent subgroups $U_{-i}(y)$ to both sides of (4.6), we get the following equations for $i = 1, \ldots, m$:

$$\frac{\partial^{d-l+1} f^*}{\partial(v_0 + yv_i + y^2 w_i)^{*d-l+1}}(\partial x) \cdot f^{s+1} = c_{l-1}(s)(v_0^* + yv_i^* + y^2 w_i^*)^{d-l+1} f^s$$

Selecting the coefficients of y in the equation above, and then applying $r_i \frac{\partial}{\partial v_i}$ to both sides, we get:

$$\left[\left(\frac{\partial}{\partial v_0^*}\right)^{d-l} r_i \frac{\partial}{\partial v_i} \frac{\partial}{\partial v_i^*} f^*\right](\partial x) \cdot f^{s+1} = c_{l-1} v_0^{*(d-l)} \left(r_i v_i^* \frac{\partial}{\partial v_i} + r_i\right) f^s \quad (4.7)$$

Next, using the commutation formulas

$$x\left(\frac{\partial}{\partial x}\right)^{d-l} = \left(\frac{\partial}{\partial x}\right)^{d-l} x - (d-l)\left(\frac{\partial}{\partial x}\right)^{d-l-1},$$

$$\frac{\partial}{\partial x} x^{d-l+1} = x^{d-l+1}\frac{\partial}{\partial x} + (d-l+1)x^{d-l}.$$

we get after multiplying both sides of (4.6) by $\dfrac{\partial}{\partial v_0}$:

$$\left[\left(\frac{\partial}{\partial v_0^*}\right)^{d-l} \frac{\partial}{\partial v_0} \frac{\partial}{\partial v_0^*} f^* - (d-l)\left(\frac{\partial}{\partial v_0^*}\right)^{d-l} f^*\right] \cdot f^{s+1} =$$

$$= c_{l-1} v_0^{*(d-l)} \left(v_0^* \frac{\partial}{\partial v_0} + d - l + 1\right) f^s \quad (4.8)$$

Summing equation (4.8) with all the equations (4.7) for $i = 1, \ldots, m$, we get:

$$\frac{\partial^{d-l} f^*}{\partial v_1^{*d-l}}(\partial x) f^{s+1} = \frac{1}{l}(ds + r + d - l) c_{l-1}(s) v_1^{*(d-l)} f^s$$

On the LHS we have used the identity $E^* \cdot f^* = d \cdot f^*$, and on the RHS $E \cdot f^s = d \cdot s \cdot f^s$.

Hence we get equation (4.6) with $l + 1$, and

$$c_l(s) = b_v(s) \cdot \frac{d!}{l!} \prod_{i=1}^{l} (ds + r + d - i).$$

This finishes the induction.

Now putting $l = d+1$ in (4.6), we get the required equation:

$$f^*(\partial x) f(x)^{s+1} = b_v(s) \prod_{i=0}^{d-1} (ds + r + i) \cdot f(x)^s \tag{4.9}$$

This equation gives the (non-monic) b-function of f, which finishes part (b). $\square$

Remark 4.1.5. The only place where we used the multiplicity-free requirement is that an equation of the form (4.4) exists. Hence, instead of this requirement it is enough to assume that such an equation exists.

Remark 4.1.6. The slice technique will be applied to compute b-functions recursively. Generally, we can make further improvements in the following ways.

Firstly, we can always make the action to be faithful. We can also work on a space potentially smaller than V by considering only the support of f, that is, we take the complement of the vector space consisting of elements $a \in V$ such that $\deg_a f = 0$. Since f is a semi-invariant, this space is G-stable.

Moreover, after slicing, say, at a highest weight vector v and obtaining the slice representation (L_v, W), it can happen that there is a natural candidate for a reductive group H larger than L_v acting on W, for which f_v is still a semi-invariant. By abuse of terminology, *we also call such action* (H, W) *the slice representation.*

Now we provide some examples, all of which are irreducible prehomogeneous spaces. These can be found in [35, Appendix], hence we follow similar notation:

Example 4.1.7. $(\mathrm{GL}_n \times \mathrm{GL}_n, \Lambda_1 \otimes \Lambda_1^*)$, the determinant:

We have $G = \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$, $V = M_n(\mathbb{C})$ is the space of $n \times n$ matrices, and $(g_1, g_2) \in G$ acts by $(g_1, g_2) \cdot M = g_2 M g_1^{-1}$. Then $f = \det$ is a semi-invariant of weight $\sigma = (1, -1)$, that is, $\sigma(g_1, g_2) = \det g_1 \cdot \det^{-1} g_2$. We show how to find the b-function $b_n(s)$ of f.

For an appropriate choice of Borel subgroups, we can view the element $v = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$ as a highest weight vector. Note that the degree of f at v is $d = 1$. The Levi subgroup $L_I \subset G$ is given by

$$L_I = \begin{pmatrix} * & 0 & \dots & 0 \\ 0 & * & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \dots & * \end{pmatrix} \times \begin{pmatrix} * & 0 & \dots & 0 \\ 0 & * & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \dots & * \end{pmatrix}$$

Hence $L_I \cong \mathrm{GL}_{n-1}(\mathbb{C}) \times \mathrm{GL}_{n-1}(\mathbb{C}) \times \mathbb{C}^\times \times \mathbb{C}^\times$. Note that $L_v \subset L_I$ is the subgroup of codimension 1 of the elements with equal entries in the top-left corners. We have $V = \mathfrak{g}v \oplus W$, where tangent space $\mathfrak{g}v$ at the orbit of v and the L_I-complement W are given by

$$\mathfrak{g}v = \begin{pmatrix} * & * & \dots & * \\ * & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & 0 & \dots & 0 \end{pmatrix}, \text{ and } W = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & * & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \dots & * \end{pmatrix}$$

Hence W can be identified with $M_{n-1}(\mathbb{C})$, $f_v = \det$ on $M_{n-1}(\mathbb{C})$, and ignoring the trivial action we can take $G_1 = \mathrm{GL}_{n-1} \times \mathrm{GL}_{n-1} \subset L_I$. Then $f_v = \det$ on $M_{n-1}(\mathbb{C})$, and $(G_1, M_{n-1}(\mathbb{C}))$ is the slice representation for the next step.

Condition (4.2) is satisfied: we can take our one-parameter subgroup $p : \mathbb{C} \to G$ acting trivially on W to be

$$t \longmapsto \left(I_n, \begin{pmatrix} t & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} \right)$$

where I_n is the identity matrix. Summing the coefficients of the corresponding Euler operator of f, it is immediate that $r = n$. Hence, by Theorem 4.1.4, we can write $b_n = (s+n)b_{n-1} = \cdots = (s+n)\cdots(s+1)$.

Example 4.1.8. $(\mathrm{GL}_{2k}, \Lambda_2)$, the Pfaffian:

We have $V = \bigwedge^2 \mathbb{C}^{2k}$, where we can also think of elements $A \in V$ as $2k \times 2k$ skew-symmetric matrices, $A + A^t = 0$, and the action is given by $g \cdot A = gAg^t$. The Pfaffian $f = \mathrm{Pf}$ is a semi-invariant. We can take $v = e_1 \wedge e_2$ as a highest weight vector, and the degree of f at v is $d = 1$. Condition (4.2) is satisfied, the slice will be $(\mathrm{GL}_{2k-2}, \lambda_2)$, and f_v is the $2k-2$ Pfaffian. The number r will be $r = 2k-1$, hence $b_{2k} = (s+2k-1)b_{2k-2} = \cdots = (s+1)(s+3)\cdots(s+2k-1)$.

Example 4.1.9. $(\mathrm{GL}_n, 2\Lambda_1)$, the symmetric determinant:

We can think of elements $M \in V = \mathrm{Sym}^2\,\mathbb{C}^n$ as symmetric matrices $M = M^t$, then the action is given by $g \cdot M = gMg^t$. The semi-invariant is $f = \det$, and we can take $v = e_1^2$ as a highest weight vector, with $d = 1$. The slice will give $(\mathrm{GL}_{n-1}, 2\lambda_1)$, and condition (4.2) is satisfied. We have $r = \frac{n+1}{2}$. Hence the b-function is $b_n(s) = (s + \frac{n+1}{2})b_{n-1}(s) = (s+1)(s+\frac{3}{2})\cdots(s+\frac{n+1}{2})$.

Remark 4.1.10. The technique used in Theorem 4.1.4 is also applicable sometimes for elements v that are not highest weight vectors. Indeed, we can get the explicit equations giving the local b-functions for other elements, as taking slices at $v_k' := v_1 + v_2 + \cdots + v_k$ (see also Remark 4.1.6). In the example of the determinant, picking $v_k' = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$, it is easy to modify μ in the proof by an appropriate étale map μ_k. With the change of variables, we will get the equation in the neighborhood of v'_k:

$$\frac{1}{\det X_J} \det \partial X_{J^c} \det X^{s+1} = (s+1)(s+2)\cdots(s+k)\det X^s,$$

where $J = \{1, \ldots, k\}$, J^c is the complement, and $X_J, \partial X_{J^c}$ are the corresponding principal minors. Due to equivariance, we get such formulas for any minor. This equation is sometimes called the Capelli identity. We will not pursue further in writing the equations giving the local b-functions.

Now let (G, V) be a representation and (L_v, W) be the slice representation. Assume $f_1, \ldots, f_l$ are semi-invariants on V of weights $\sigma_1, \ldots, \sigma_l$, with duals $f_1^*, \ldots, f_l^*$, and let $d_i = \deg_v f_i$. As before, define the L_v-semi-invariants $f_{i,v}$ on W, and put $\underline{f}_v = (f_{1,v}, f_{2,v}, \ldots, f_{l,v})$ and $\underline{f}_v^* = (f_{1,v}^*, f_{2,v}^*, \ldots, f_{l,v}^*)$. We have the following result for the b-function of several variables, the proof of which goes along the proof of Theorem 4.1.4, mutatis mutandis:

Theorem 4.1.11. *Using the notations above, assume the weight $\underline{\sigma}|_{L_v}$ of $\underline{f}$ is multiplicity-free and the b-function of $\underline{f}_v$ is $b_{v,\underline{m}}(\underline{s})$. Furthermore, assume that $\mathbb{Z}\lambda \cap \Pi(W) = \{0\}$. Then b-function of $\underline{f}$ is*

$$b_{\underline{m}}(\underline{s}) = b_{v,\underline{m}}(\underline{s}) \prod_{i=0}^{d-1} (d_1 s_1 + \cdots + d_l s_l + r + i)$$

where $d = \sum_{i=1}^{l} d_i m_i$.

4.2 Slices of quivers

The identification of the normal space gives an explicit description of the slice at any element $V \in \operatorname{Rep}(Q, \alpha)$. Decompose V as a sum of indecomposables $V = \bigoplus_{i=1}^t V_i^{m_i}$, where $m_i \in \mathbb{N}\backslash\{0\}$ are the multiplicities, and $V_i \not\cong V_j$, for $i \neq j$.

By [11], the isotropy subgroup of V is

$$\mathrm{Aut}_Q(V) \cong U \rtimes \prod_{i=1}^{t} \mathrm{GL}(m_i),$$

where U is a closed normal unipotent subgroup. Let $\beta_V := \underline{m}$ and consider the reductive part $\mathrm{GL}(\beta_V) = \prod_{i=1}^{t} \mathrm{GL}(m_i)$. Then the slice representation is in fact isomorphic to the quiver representation space

$$(\mathrm{GL}(\underline{m}), \bigoplus_{i,j=1}^{t} \mathrm{Ext}_Q(V_i, V_j) \otimes \mathrm{Hom}(\mathbb{C}^{m_i}, \mathbb{C}^{m_j})) = (\mathrm{GL}(\beta_V), \mathrm{Rep}(Q_V, \beta_V)), \quad (4.10)$$

where Q_V is the quiver with t vertices numbered $1, 2, \ldots, t$ such that the number of arrows from i to j is $\dim \mathrm{Ext}_Q(V_i, V_j)$. In particular, if the indecomposables V_i don't have "cyclic" extensions between them, the quiver doesn't have oriented cycles, hence the slice has the unique closed orbit property.

Lemma 4.2.1. *Consider a representation $V \in \mathrm{Rep}(Q, \beta)$ with $V = \oplus_{i=1}^{t} V_i^{m_i}$, where V_i are indecomposable, and take the slice $\mathrm{Ext}_Q(V, V)$ at V as in (4.10). If $f \in \mathrm{Rep}(Q, \beta)$ is a semi-invariant of weight σ, then the induced semi-invariant on the slice $f_V \in \mathrm{Rep}(Q_V, \beta_V)$ has weight σ_V defined by*

$$\sigma_V = \sigma \cdot T,$$

where the transformation $T \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^t, \mathbb{Z}^{Q_0})$ is given by $T_{ij} = \underline{d}(V_j)_i$. Moreover, we have a weight-preserving map of $\mathbb{C}$-algebras

$$\phi_V : \mathrm{SI}(Q, \beta) \to \mathrm{SI}(Q_V, \beta_V)$$

injective on the level of weight spaces.

Proof. The weight of f_V is just the restriction of σ to $\prod_{i=1}^{t} \mathrm{GL}(m_i)$. We put the matrices $V(a)$, $a \in Q_1$ in block diagonal forms, and make the inclusion $\prod_{i=1}^{t} \mathrm{GL}(m_i) \subset \mathrm{GL}(Q, \beta)$ explicit. It is immediate that $\sigma'(i) = \sigma(\underline{d}(V_i))$. We have the algebra map

$$\mathbb{C}[\mathrm{Rep}(Q, \beta)] \ni P \longmapsto P_v \in \mathbb{C}[\mathrm{Rep}(Q_V, \beta_V)]$$

and this map restricts to an injective linear map by (4.1)

$$\mathrm{SI}(Q,\beta)_\sigma \ni f \longmapsto f_v \in \mathrm{SI}(Q_V,\beta_V)_{\sigma_V}.$$

□

Consider Q' a full subquiver of Q, and the dimension vector β' induced from β. Assume β' is a sincere prehomogeneous dimension vector, and let $V' \in \mathrm{Rep}(Q',\beta')$ be a generic representation. By [33, 61], we have a decomposition

$$V' = \oplus_{i=1}^t V_i'^{n_i},\ n_i > 0,$$

where $\mathrm{Ext}_Q(V_i',V_j') = 0$, $\mathrm{End}_Q(V_i') = \mathbb{C}$ and we can number the indecomposables V_i' such that $i > j$ implies $\mathrm{Hom}_{Q'}(V_i',V_j') = 0$. It is easy to see that the dimension vectors $\underline{d}(V_i')$, are linearly independent, hence $t \leq \#Q_0'$, where $\#Q_0'$ is the number of vertices of Q_0'. (see also [23, Section 2.7.]). Moreover, the matrix T has a left inverse over $\mathbb{Z}$, where $T_{ij} = \underline{d}(V_j')_i$.

Extending V' by zero matrices, we view it as an element $V \in \mathrm{Rep}(Q,\beta)$, so that

$$V = \bigoplus_{i=1}^t V_i^{n_i} \oplus \bigoplus_{x\in Q_0\setminus Q_0'} S_x^{\beta_x}.$$

We say that such an element V is dense in its support Q'.

Slicing at such an element (which can be thought of slicing at a full subquiver) gives the following:

Proposition 4.2.2. *Assume $V \in \mathrm{Rep}(Q,\beta)$ is dense in its support Q' as above. Consider the decomposition $\mathrm{Aut}_Q(V) = U \rtimes L$, with unipotent $U = \oplus_{i<j} \mathrm{Hom}_Q(V_i,V_j) \otimes \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{n_i},\mathbb{C}^{n_j})$ and Levi factor $L = \prod_i \mathrm{GL}(n_i) \times \prod_x \mathrm{GL}(\beta_x) =: \mathrm{GL}(\beta_V)$. Considering the natural action of U on the slice $\mathrm{Ext}_Q(V,V)$, we have the surjective weight-preserving map of $\mathbb{C}$-algebras*

$$\phi_V : \mathrm{SI}(Q,\beta) \twoheadrightarrow \mathbb{C}[\mathrm{Rep}(Q_V,\beta_V)]^{U \rtimes \mathrm{SL}(\beta_V)} \subset \mathrm{SI}(Q_V,\beta_V).$$

Moreover, ϕ_V is an isomorphism if and only if $t = \#Q_0'$.

Proof. We have the $G := \mathrm{GL}(\beta)$-equivariant decomposition

$$\mathrm{Rep}(Q,\beta) = \mathrm{Rep}(Q',\beta') \oplus W\ ,\ W = \bigoplus_{a\in Q_1\setminus Q_1'} \mathrm{Hom}_{\mathbb{C}}(\beta_{ta},\beta_{ha}).$$

So the slice $W = \mathrm{Ext}_Q(V,V)$ here is in fact G-stable, in particular it is $G_V = \mathrm{Aut}_Q(V)$-stable. But f_V is then G_V-semi-invariant. The ring of G_V-semi-invariants on $\mathrm{Rep}(Q_V,\beta_V)$ is equal to $\mathbb{C}[\mathrm{Rep}(Q_V,\beta_V)]^{U\rtimes \mathrm{SL}(\beta_V)}$. Hence we have a map

$$\phi_V : \mathrm{SI}(Q,\beta) \longrightarrow \mathbb{C}[\mathrm{Rep}(Q_V,\beta_V)]^{U\rtimes \mathrm{SL}(\beta_V)} = (\mathbb{C}[\mathrm{Rep}(Q_V,\beta_V)]^{U})^{\mathrm{SL}(\beta_V)}.$$

Pick a semi-invariant $f' \in \mathbb{C}[\mathrm{Rep}(Q_V,\beta_V)]^{U\rtimes \mathrm{SL}(\beta_V)}$ of weight σ'. Since T has a left inverse, we can lift the weight σ' to a weight $\sigma \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{Q_0},\mathbb{Z})$. Consider the function F defined on the open set $\mathcal{O}_V \times W \subset \mathrm{Rep}(Q,\beta)$ by $F(gV,w) := \sigma(g)f'(g^{-1}w)$. Then F is a well-defined rational semi-invariant of weight σ and we can write $F = \dfrac{f}{h}$, with $f,h \in \mathrm{SI}(Q,\beta)$ relatively prime.

By [33], we have fundamental invariants $f_1,\ldots,f_m \in \mathrm{SI}(Q',\beta') \subset \mathrm{SI}(Q,\beta)$, where $m = \#Q_0' - t$. If $\mathcal{O}_V$ is the dense orbit of V, we have [60]:

$$\mathrm{Rep}(Q,\beta)\setminus(\mathcal{O}_V \times W) = S \cup \bigcup_{i=1}^{m} Z(f_i),$$

where S has codimension ≥ 2 and $Z(f_i)$ is the zero-set of f_i.

Now take h' an irreducible factor of h. Then $Z(h') \subset \mathrm{Rep}(Q,\beta)\setminus(\mathcal{O}_V \times W)$, hence $h' = f_i$ (up to constant), for some i. Since $\phi_V(f_i) = (f_i)_V$ is constant, h_V is constant as well. From $F_V = f'$ we get $f_V = f'$ (up to constant), hence ϕ_V is surjective.

If $\#Q_0' > t$, then there is a non-constant semi-invariant f_1. But $\phi_V(f_1)$ is constant, so ϕ_V is not injective.

If $\#Q_0' = t$, then there are only constant $G' = [G,G]$-invariants, hence V has a dense G'-orbit. Hence applying the slice with G' now, we see that ϕ_V is injective, since (in general) it is injective on weight spaces. Also, in this case T is an invertible matrix over $\mathbb{Z}$. $\square$

Remark 4.2.3. The proof uses little about quivers. Surjectivity needs only a condition on lifting weights. Also, the case of isomorphism appears in [46, Proposition 3.15.].

Following [64], for a semi-invariant $f \in \mathrm{SI}(Q, \beta)$, we say that a representation $A \in \mathrm{Rep}(Q, \beta)$ is *locally semi-simple* (corresponding to f), if the orbit $\mathcal{O}_A$ of A is closed in the principal open set defined by $f \neq 0$. Such an orbit is unique if and only if the weight of f is multiplicity-free. Taking the slice as in (4.10) at such a representation gives:

Lemma 4.2.4. *Let $f \in \mathrm{SI}(Q, \beta)$ be a non-zero semi-invariant of weight $\langle \alpha, \cdot \rangle$, with α prehomogeneous. Consider the locally semi-simple representation $A \in \mathrm{Rep}(Q, \beta)$ corresponding to f. Then the induced function f_A on the slice $\mathrm{Ext}_Q(A, A)$ is a non-zero constant function.*

Proof. Since α is a prehomogeneous dimension vector, $f = c^V$, with V generic and the weight $\langle \alpha, \cdot \rangle$ is multiplicity-free. Hence, we have a unique locally semi-simple representation up to isomorphism. By [64, Theorem 11], A can be written as a finite direct sum $A = \oplus S_i^{m_i}$, $m_i \in \mathbb{N} \backslash \{0\}$, where S_i are simple objects from the right perpendicular category $V^{\perp}$, that is, the full subcategory of representations B with $\mathrm{Hom}_Q(V, B) = 0 = \mathrm{Ext}_Q(V, B)$. This subcategory is closed under extensions and direct sums. By Schofield [61, Theorem 2.5], since the orbit of V is dense, this subcategory is actually equivalent to a category of representations of a quiver Q' without oriented cycles.

By (4.10), the slice at A corresponds to the quiver representation space

$$(\mathrm{GL}(\underline{m}), \bigoplus \mathrm{Ext}_Q(S_i, S_j) \otimes \mathrm{Hom}(\mathbb{C}^{m_i}, \mathbb{C}^{m_j})).$$

Clearly, this quiver Q_A is in fact a subquiver of Q', hence it doesn't have oriented cycles either.

The induced polynomial f_A on the slice is $\mathrm{GL}(\underline{m})$-semi-invariant. However, since $f_A(0) \neq 0$, f_A has to be invariant. Since Q_A has no oriented cycles, the only invariants are the constant functions. $\square$

The lemma above is stated more generally in [66, Lemma 7.6]. However, we believe the condition on the weight is necessary, as the following example shows:

Example 4.2.5. Take Q to be the Kronecker quiver $1 \rightrightarrows 2$. Take the dimension vectors $\beta = (n, n)$, $n \in \mathbb{N}$, $\alpha = (1, 1)$. Take $V = (I_1, -I_1) \in \mathrm{Rep}(Q, \alpha)$ and $f = c^V \in \mathrm{SI}(Q, \beta)$, which is just the semi-invariant $f(X, Y) = \det(X + Y)$, for $(X, Y) \in \mathrm{Rep}(Q, \beta)$. Take $A = (I_n, 0) = V_1^n \in \mathrm{Rep}(Q, \beta)$, where $V_1 = (I_1, 0)$ is indecomposable. Clearly, A is a locally semi-simple representation corresponding to f. The slice representation is isomorphic to $(\mathrm{GL}(n), \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n))$ with the conjugation action (this is the 1-loop quiver). We see that the induced semi-invariant is $f_A(Y) = \det(I_n + Y)$ which is an invariant function, but not constant.

4.3 b-functions via slices

In this section we apply the results from the previous sections to compute roots of the b-functions of quiver semi-invariants.

At first, Q will be an arbitrary quiver. We start with the following lemma, which is just a restatement of Lemma 4.1.1 in the quiver setting:

Lemma 4.3.1. *Let $A \in \mathrm{Rep}(Q, \beta)$ be any representation, and $f \in \mathrm{SI}(Q, \beta)$ a non-zero semi-invariant. If f_A is the semi-invariant induced on the slice $\mathrm{Ext}_Q(A, A)$ of A, then $b_{f_A,0} = b_{f,A}$. If Q_A has no oriented cycles, then b_{f_A} divides b_f. If f_A has multiplicity-free weight, the same is true for f.*

For a vertex $x \in Q_0$, denote by ϵ_x the function on Q_0 defined by $\epsilon_x(x) = 1$, and $\epsilon_x(y) = 0$, for $x \neq y$. Denote by S_x the simple representation of Q

corresponding to a vertex $x \in Q_0$, that is, the representation with all linear maps 0 and $S_x(y) = \mathbb{C}^{\epsilon_x(y)}$.

For an arrow $a \in Q_1$, denote by S_a the representation of Q obtained from the non-split exact sequence

$$0 \to S_{ha} \to S_a \to S_{ta} \to 0,$$

with the matrix $S_a(a)$ of rank 1.

Fix a dimension vector β. Then the element $0 \in \operatorname{Rep}(Q, \beta)$ corresponds to the semi-simple representation $SS = \bigoplus_{x \in Q_0} S_x^{\beta_x}$. For an integer k with $0 \leq k \leq \min\{\beta_{ta}, \beta_{ha}\}$ we introduce the following representation of Q:

$$SS_a^k := S_a^k \oplus \bigoplus_{x \in Q_0} S_x^{\beta_x - k(\epsilon_{ta}(x) + \epsilon_{ha}(x))}.$$

In other words, the matrix $SS_a^k(a')$ will be zero, if $a' \in Q_1 \backslash \{a\}$, and the matrix $SS_a^k(a)$ will be of rank k. We will always choose the latter to be in elementary form, with the values of the first k diagonal entries 1, and the other entries 0. Note that $SS_a^0 = SS$. Also, let $SS_a := SS_a^1$. The representations SS_a are "almost semi-simple" representations of Q. Indeed, the decomposition of the representation space

$$\operatorname{Rep}(Q, \beta) := \bigoplus_{a \in Q_1} \operatorname{Hom}(\mathbb{C}^{\beta_{ta}}, \mathbb{C}^{\beta_{ha}}).$$

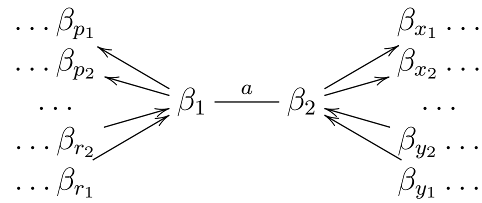
under the action of $G = \operatorname{GL}(\beta)$ is in fact an irreducible decomposition (if Q has no loops), and the corresponding highest weight vectors are SS_a, for a suitable choice of the Borel subgroup. This will put us in the setting of the Theorem 4.1.4.

We say that an that arrow a is a *1-source* (resp. *1-sink*) if ta (resp. ha) is not a vertex of any arrow other than a.

We will focus on slicing at single arrows a (i.e. we assume a is the only arrow between ta and ha):

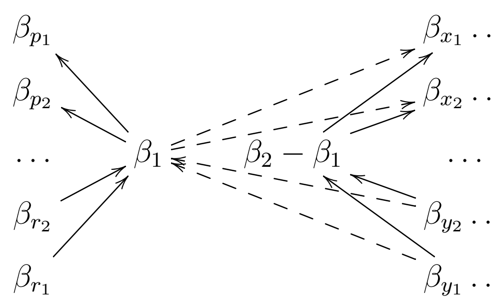
Proposition 4.3.2. *Let Q be a quiver without oriented cycles, β be a dimension vector. Take $\vec{a} \in Q_1$ an arrow, and number its vertices by $1, 2$, such that $\beta_1 \leq \beta_2$. So we have the following general picture (where the orientation of $\vec{a}$ is arbitrary):*

$Q :$



(a) The slice at the arrow $\vec{a}$ (that is, at the element $SS_a^{\beta_1}$) is a representation space (Q_a, β_a) corresponding to the following quiver:

$Q_a :$



Let f be a semi-invariant of weight $\sigma = \langle \alpha_a, \cdot \rangle$ and f_a be the induced semi-invariant on Q_a with induced weight $\sigma_a = \langle \alpha_a, \cdot \rangle$. Under the obvious correspondence of vertices between Q and Q_a, σ_a differs from σ only at vertex 1, with $\sigma_a(1) = \sigma(1) + \sigma(2)$.

Assume $0 < \beta_1 < \beta_2$. If $1 \xrightarrow{a} 2$, take $U = \mathrm{Hom}(\mathbb{C}^{\beta_2 - \beta_1}, \mathbb{C}^{\beta_1})$ (resp. if $1 \xleftarrow{a} 2$, take $U = \mathrm{Hom}(\mathbb{C}^{\beta_1}, \mathbb{C}^{\beta_2 - \beta_1})$). Then U acts on $\mathrm{Rep}(Q_a, \beta_a)$ naturally, and f_a is also U-invariant. Moreover, we have an isomorphism

$$\phi_a : \mathrm{SI}(Q, \beta) \cong \mathbb{C}[\mathrm{Rep}(Q_a, \beta_a)]^{U \rtimes \mathrm{SL}(\beta_a)}.$$

If $\beta_1 = \beta_2$ (we can drop vertex 2 from Q_A) and $\vec{a}$ is a 1-source or 1-sink, we have an isomorphism

$$\phi_a : \mathrm{SI}(Q, \beta)/(\det X_a - 1) \cong \mathrm{SI}(Q_a, \beta_a).$$

(b) Let $\vec{a}$ be a 1-source or 1-sink (so there are no vertices p_i and r_i). Then Q_a has no oriented cycles. Moreover, if σ_a is a multiplicity-free weight, then

$$b_f(s) = b_{f_a}(s) \cdot [s]^{|\sigma_1|}_{\beta_2-\beta_1,\beta_2},$$

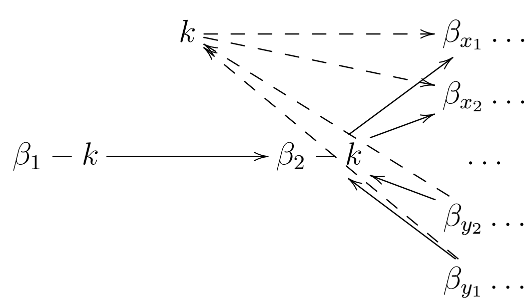
and if $Z' \in \mathrm{Rep}(Q_a, \beta_a) \subset \mathrm{Rep}(Q, \beta)$ is the locally semi-simple representation corresponding to f_a, then (assuming $\sigma_1 \neq 0$) the locally semi-simple representation corresponding to f is $Z = Z' + SS_a^{\beta_1}$.

Proof. (a) This part follows immediately from Proposition 4.2.2 and Lemma 4.3.1. The case $\beta_1 = \beta_2$ follows from the fact that if a semi-invariant depends on the arrow a, then we can factor out the semi-invariant $\det X_a$.

(b) We fix the standard basis in $\mathrm{Rep}(Q, \beta)$ consisting of elements E^b_{ij}, where E^b_{ij} is the element with all matrices 0 except for $E_{ij}(b)$, for which the (i, j) entry is 1 and is the only non-zero entry. Under this choice, it is clear that the action of $\mathrm{GL}(\beta)$ is self-adjoint, and to construct the dual f^* of a determinantal semi-invariant, we can just replace the variables by partial derivatives.

We discuss only the case when a is a 1-source. One sees that for $k = 0, \ldots, \beta_1$ the slice at the element $v_k = SS_a^k$ is given by the following quiver representation space (Q_k, β^k).

$Q_k :$



(4.11)

Note that $Q_0 = Q$ and $Q_{\beta_1} = Q_a$ (disregarding the vertices with 0). Let f_k be the induced semi-invariant. We claim that if we slice Q_k at the element

$SS_a \in \mathrm{Rep}(Q_k, \beta^k)$, we get $\mathrm{Rep}(Q_{k+1}, \beta^{k+1})$. This is straightforward, taking into account Remark 4.1.6: instead of taking the Levi factor of the stabilizer of $SS_a \in \mathrm{Rep}(Q_k, \beta^k)$ acting on the slice, we can take the bigger Levi factor of the stabilizer of $SS_a^{k+1} \in \mathrm{Rep}(Q, \beta)$. Note that it might happen that the weight of f_k is not multiplicity-free.

Slicing at SS_a, Condition (4.2) is satisfied, hence we can apply Theorem 4.1.4. Clearly, $d := \deg_{SS_a} f = |\sigma_1|$, and as always, we consider $d > 0$. Applying Theorem 4.1.4 repeatedly (keeping in mind Remark 4.1.5) we get that the b-function of f_k is:

$$b_{f_k}(s) = b_{f_{k-1}}(s) \cdot \prod_{j=0}^{d-1}(ds + j + k) = b_{f_a}(s) \cdot \prod_{i=\beta_2-\beta_1+1}^{k} \prod_{j=0}^{d-1}(ds + i + j)$$

Putting $k = 0$, we get the desired formula.

Now take any representation $B \in \mathrm{Rep}(Q, \beta)$ such that $f(B) \neq 0$. If $\sigma_1 \neq 0$, we can assume B is of the form $B = SS_a^{\beta_1} + B'$, with B' a representation in the slice $\mathrm{Rep}(Q_a, \beta_a)$. Since $f_a(B') \neq 0$, we have that the closure of $\mathrm{GL}(\beta_a) \cdot B'$ contains Z'. Since $\mathrm{GL}(\beta_a)$ is contained in the stabilizer of $SS_a^{\beta_1}$, the element $Z = SS_a^{\beta_1} + Z'$ is in the closure of the orbit of B. Since B was arbitrary, Z is the unique closed orbit in the principal open $f \neq 0$.

□

Remark 4.3.3. We see from part b) that we also get formulas for the b-functions of the induced semi-invariants f_k of the quivers Q_k (4.11).

Remark 4.3.4. Since we know the weight $\sigma_a = \langle \alpha_a, \cdot \rangle$ on the slice, we implicitly also know α_a. In examples, we prefer working with α rather than σ. Let P_i be the indecomposable projective module of Q_a at vertex i and S_i the simple module of Q_a at vertex i. The formulas are:

(a) If a is a 1-source, then $\alpha_a = \alpha + (\alpha_2 - \alpha_1)\underline{d}(P_1) - \alpha_1 \cdot \underline{d}(S_2)$,

(b) If a is a 1-sink, then $\alpha_a = \alpha + \alpha_1 \cdot \underline{d}(P_1) - \alpha_1 \cdot \underline{d}(S_1)$.

Moreover, in these cases we can see by direct computation that if $f = c^V$, then $f_a = c^{V'}$, where $V' \in \operatorname{Rep}(Q_a, \alpha_a)$ can be written down explicitly. Since we will be working with Schur representations V, we will write only the corresponding real Schur roots.

For a quiver Q and a semi-invariant f, we say the pair (Q, f) is *sliceable* if, after slicing repeatedly at 1-sinks and 1-arrows as described in Proposition 4.3.2 and simplifying as in Lemma 2.3.7, we can reach the empty quiver (equivalently, a non-zero constant function). In this case we can compute the entire b-function of f using the slice technique. Due to Remark 4.1.5, if (Q, f) is sliceable, then for the calculation of the b-function we can argue by reverse induction and ignore the multiplicity-free requirement completely.

Remark 4.3.5. The isomorphism $\operatorname{SI}(Q, \beta) \cong \mathbb{C}[\operatorname{Rep}(Q_a, \beta_a)]^{U \rtimes \operatorname{SL}(\beta_a)}$ also gives inductively the homogeneous inequalities between the values β_i of the dimension vector that are needed for the semi-invariant to be non-zero. These will be encoded in the positivity of the roots of the b-function. Unless otherwise specified, we will work with "general" dimension vectors β, which means that these inequalities are strict.

We now show how to use Proposition 4.3.2 in examples. We will put the values of α on top of the values of the dimension vector β, and use a dashed line at the arrow at which we are slicing. We will indicate (below the curly arrow) the simplification law used from Lemma 2.3.7 and retain the part of the b-function given by the recursion in Proposition 4.3.2 part b).

Example 4.3.6. We compute the b-function of the semi-invariant from Example

2.3.6. Recall $\beta_1+\beta_2+\beta_3=2\beta_4$.

$$\begin{tikzcd} & \overset{1}{\beta_2} \arrow[d] & \\ \overset{1}{\beta_1} \arrow[r, dashed] & \overset{1}{\beta_4} & \overset{1}{\beta_3} \arrow[l] \end{tikzcd} \overset{[s]_{\beta_4-\beta_1,\beta_4}}{\rightsquigarrow} \begin{tikzcd} \overset{1}{\beta_2} \arrow[r] \arrow[d] & \overset{0}{\beta_4-\beta_1} \\ \overset{1}{\beta_1} & \overset{1}{\beta_3} \arrow[l] \arrow[u] \end{tikzcd} \underset{a)}{\rightsquigarrow}$$

$$\begin{tikzcd} \overset{1}{\beta_2} \arrow[r, dashed] \arrow[d] & \overset{0}{\beta_4-\beta_1} & \\ \overset{1}{\beta_1} & \overset{1}{\beta_3} \arrow[l] \arrow[r] & \overset{0}{\beta_4-\beta_1} \end{tikzcd} \underset{a)}{\overset{[s]_{\beta_1+\beta_2-\beta_4,\beta_2}}{\rightsquigarrow}} \begin{tikzcd} \overset{1}{\beta_4-\beta_3} \arrow[d, dashed] & & \\ \overset{1}{\beta_1} & \overset{1}{\beta_3} \arrow[l] \arrow[r] & \overset{0}{\beta_4-\beta_1} \end{tikzcd} \underset{b)}{\overset{[s]_{\beta_1+\beta_3-\beta_4,\beta_1}}{\rightsquigarrow}}$$

$$\begin{tikzcd} \overset{0}{\beta_4-\beta_2} & \overset{1}{\beta_3} \arrow[l, dashed] \arrow[r] & \overset{0}{\beta_4-\beta_1} \end{tikzcd} \underset{a)}{\overset{[s]_{\beta_2+\beta_3-\beta_4,\beta_3}}{\rightsquigarrow}} \begin{tikzcd} \overset{1}{\beta_4-\beta_1} \arrow[r, dashed] & \overset{0}{\beta_4-\beta_1} \end{tikzcd} \overset{[s]_{\beta_4-\beta_1}}{\rightsquigarrow} \emptyset$$

Hence the b-function is

$$b(s)=[s]_{\beta_4-\beta_1,\beta_4}\cdot[s]_{\beta_1+\beta_2-\beta_4,\beta_2}\cdot[s]_{\beta_1+\beta_3-\beta_4,\beta_1}\cdot[s]_{\beta_2+\beta_3-\beta_4,\beta_3}\cdot[s]_{\beta_4-\beta_1}=$$

$$=[s]_{\beta_4}\cdot[s]_{\beta_1+\beta_2-\beta_4,\beta_2}\cdot[s]_{\beta_2+\beta_3-\beta_4,\beta_3}\cdot[s]_{\beta_1+\beta_3-\beta_4,\beta_1}.$$

We get that the locally semi-simple representation is

$$A=V_1^{\beta_4-\beta_1}\oplus V_2^{\beta_4-\beta_2}\oplus V_3^{\beta_4-\beta_3},$$

where the indecomposables are $V_1=\begin{array}{ccc} & 1 & \\ 0 & 1 & 1\end{array}$, $V_2=\begin{array}{ccc} & 0 & \\ 1 & 1 & 1\end{array}$, $V_3=\begin{array}{ccc} & 1 & \\ 1 & 1 & 0\end{array}$. Note that this is also the generic element.

We also get the b-functions of the semi-invariants of the quivers that are used in the steps, including the extra ones as in Remark 4.3.3. For example, here we also get the b-function of $f_k=c^{V_k'}$, where $\underline{d}(V_k')=\alpha'$ all maps of V_k' are 1, and the quiver Q_k is the following:

$$\begin{tikzcd} & \overset{1}{\beta_2} \arrow[r] \arrow[d] & \overset{1}{k} \\ \overset{1}{\beta_1-k} \arrow[r] & \overset{1}{\beta_4} & \overset{1}{\beta_3} \arrow[l] \arrow[u] \end{tikzcd}$$

Note that Q_k is not a tree, and the weight of f_k is not multiplicity-free. The b-function is

$$b_{f_k} = [s]_{\beta_4 - k} \cdot [s]_{\beta_1+\beta_2-\beta_4,\beta_2} \cdot [s]_{\beta_2+\beta_3-\beta_4,\beta_3} \cdot [s]_{\beta_1+\beta_3-\beta_4,\beta_1}.$$

In fact, we have the following more general result for any tree quiver, that is, for a quiver whose underlying graph has no cycles. This generalizes the A_n case:

Theorem 4.3.7. *Let Q be a tree quiver, and f a non-zero semi-invariant of weight $\langle \alpha, \cdot \rangle = -\langle \cdot, \alpha^* \rangle$. If $\alpha_x \leq 1$ for any $x \in Q_0$ (or $\alpha^* \leq 1$ for any $x \in Q_0$), then (Q, f) is sliceable.*

Proof. By duality, it is enough to consider the case $\alpha_x \leq 1$ for all $x \in Q_0$. It is immediate that α is a prehomogeneous dimension vector, hence the weight $\langle \alpha, \cdot \rangle$ is multiplicity-free. As usual, we work with the support of f, that is, we can drop the arrows which f doesn't depend on. Since Q is a tree, we can take an arrow $\vec{a} \in Q_1$ that is a 1-source or 1-sink. We use the notation as in Proposition 4.3.2.

Assume $\vec{a}$ is 1-source. If f depends on $\vec{a}$, we must have $\alpha_1 = 1$. Let A be the generic matrix of variables corresponding to $\vec{a}$. If $\alpha_2 = 0$, then by Lemma 2.3.7 part a) we can disconnect the quiver, A has to be a square matrix, and we can separate variables $f = f' \cdot \det A$, where f' is a semi-invariant on the smaller quiver without the arrow $\vec{a}$. Hence we can assume $\alpha_2 = 1$.

Similarly, if $\vec{a}$ is a 1-sink, we can assume $\alpha_1 = 0$ and $\alpha_2 = 1$.

In any case, slicing at $\vec{a}$ simplifies due to Lemma 2.3.7, so we get a quiver Q_a which is still a tree quiver, and the weight α_a of the induced semi-invariant f_a on Q_a still satisfies $(\alpha_a)_x \leq 1$, for any $x \in (Q_a)_0$. In particular, the weight of f_a is multiplicity-free, and by Proposition 4.3.2 we get

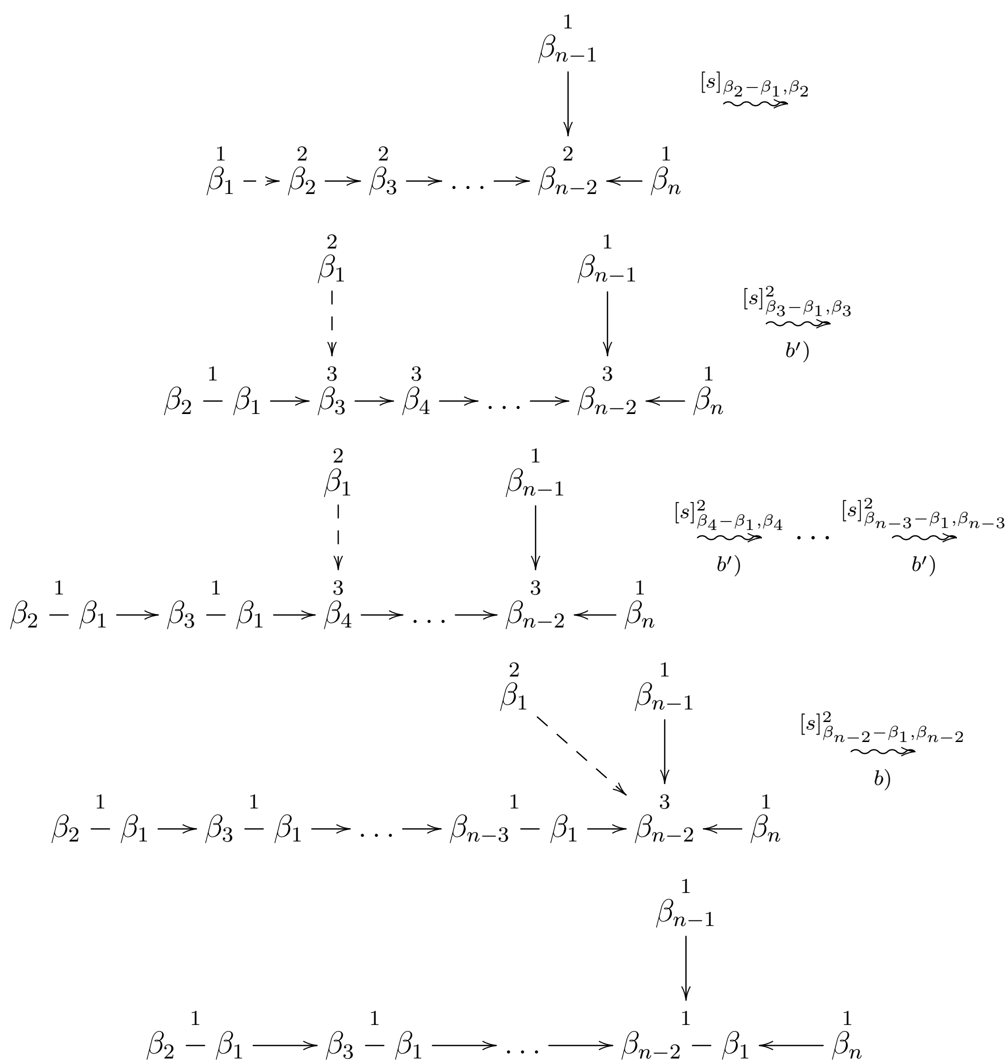
$$b_f(s) = b_{f_a}(s) \cdot [s]_{\beta_2 - \beta_1, \beta_2}.$$

Since the dimension of the representation space strictly decreases by slicing, this procedure is finite and stops when we arrive at the empty quiver (or a constant function). $\square$

We consider the next family of Dynkin quivers:

Theorem 4.3.8. *All fundamental semi-invariants for quivers of type $\mathbb{D}$ are sliceable.*

Proof. First, we prove this with the orientation of $\mathbb{D}_n$ chosen so that all arrows point to the joint vertex. Also, using Lemma 2.3.7, we can reduce the proof to the case when α is the longest root.



At this stage we know that the quiver above is sliceable, by Theorem 4.3.7. Continuing

$$
\xrightsquigarrow[a)]{[s]_{\beta_3-\beta_2,\beta_3-\beta_1}}
\begin{tikzcd}
 & & & \overset{1}{\beta_{n-1}} \arrow[d] & \\
\overset{1}{\beta_2-\beta_1} \arrow[r, dashed] & \overset{1}{\beta_4-\beta_1} \arrow[r] & \ldots \arrow[r] & \overset{1}{\beta_{n-2}-\beta_1} & \overset{1}{\beta_n} \arrow[l]
\end{tikzcd}
\xrightsquigarrow[a)]{[s]_{\beta_4-\beta_2,\beta_4-\beta_1}} \ldots
$$

$$
\ldots \xrightsquigarrow[a)]{[s]_{\beta_{n-3}-\beta_2,\beta_{n-3}-\beta_1}}
\begin{tikzcd}
 & \overset{1}{\beta_{n-1}} \arrow[d] & \\
\overset{1}{\beta_2-\beta_1} \arrow[r] & \overset{1}{\beta_{n-2}-\beta_1} & \overset{1}{\beta_n} \arrow[l]
\end{tikzcd}
\xrightsquigarrow[\text{Example 4.3.6}]{} \emptyset
$$

Hence the b-function is:

$$
b(s) = [s]_{\beta_2-\beta_1,\beta_2} \prod_{i=3}^{n-2} \left([s]^2_{\beta_i-\beta_1,\beta_i} [s]_{\beta_i-\beta_2,\beta_i-\beta_1} \right) \cdot
$$

$$
\cdot [s]_{\beta_{n-2}-\beta_{n-1}-\beta_1,\beta_{n-2}-\beta_1} [s]_{\beta_{n-2}-\beta_n-\beta_1,\beta_{n-2}-\beta_1} [s]_{\beta_{n-2}-\beta_1}.
$$

Accordingly, the homogeneous inequalities that are necessary and sufficient for the semi-invariant to be non-zero are:

$$
\beta_1 \leq \beta_2 \leq \beta_i, i = 3, \ldots, n-2,
$$

$$
\beta_{n-1}, \beta_n \leq \beta_{n-2} - \beta_1
$$

Also, we can write down the corresponding locally semi-simple representation explicitly.

We now consider the other cases. Note that by duality, if every semi-invariant of a quiver is sliceable, the same is true for the opposite quiver. Hence, we can assume that the arrow between $n-3$ and $n-2$ goes from $n-3$ to $n-2$. One can always reduce the long arm of the quiver. In the end, we arrive at a $\tilde{\mathbb{D}}_4$ quiver

$$
\begin{tikzcd}
 & 3 \arrow[d, dash] & \\
2 \arrow[r] & 5 \arrow[r, dash] & 4 \\
 & 1 \arrow[u] &
\end{tikzcd}
$$

Excluding the semi-invariants covered by Theorem 4.3.7, we have three main cases according to the orientation of the arrows $3-5$ and $4-5$. The case with all arrows pointing towards the joint vertex has already been discussed, where we used simplification (b).

We only write the values of α at vertices. The next case is

$$\begin{tikzcd}
& 1 \arrow[d] & & 1 \arrow[d] & & \\
2 \arrow[r, dashed] & 3 \arrow[r] & 2 \arrow[r, squiggly, "b')"'] & 1 \arrow[r] & 1 & 1 \arrow[l] \\
& 1 \arrow[u] & & 1 \arrow[u] & &
\end{tikzcd}$$

We stop since we know that the RHS quiver is sliceable.

The last main case is

$$\begin{tikzcd}
& 2 & & 2 & 2 \arrow[l] \arrow[d] & & \\
2 \arrow[r, dashed] & 3 \arrow[u] \arrow[r] & 2 \arrow[r, squiggly, "b')"'] & 1 \arrow[u] \arrow[r] & 2 \arrow[r, squiggly, "a)"'] & 2 & 2 \arrow[l] \arrow[d] \\
& 1 \arrow[u] & & 1 \arrow[u, dashed] & & 1 \arrow[u] \arrow[r] & 2
\end{tikzcd}$$

$$\begin{tikzcd}
& 1 & 1 \arrow[l] & \\
{} \arrow[r, squiggly, "\text{split source with value 2}", "b')"'] & 1 \arrow[u] \arrow[r] & 1 & 1 \arrow[l]
\end{tikzcd}$$

We stop since we know that the last quiver is sliceable. □

We give an example of extended Dynkin type:

Example 4.3.9. We take $\tilde{\mathbb{D}}_4$ with the dimension vector β, with $2\beta_1 + \beta_2 + \beta_3 + \beta_4 = 3\beta_5$, semi-invariant (unique up to constant) $f = c^V$, where $\underline{d}(V) = \alpha =$

$(2,1,1,1,2)$ is a real Schur root:

$$
\begin{tikzcd}
 & \overset{1}{\beta_2} \arrow[d] & & & \overset{1}{\beta_2} \arrow[d] \arrow[r, dashed] & \overset{0}{\beta_5 - \beta_1} & & \\
\overset{2}{\beta_1} \arrow[r, dashed] & \overset{2}{\beta_5} & \overset{1}{\beta_3} \arrow[l] \arrow[r, squiggly, "{[s]^2_{\beta_5-\beta_1,\beta_5}}", "a)"'] & {} & \overset{2}{\beta_1} & \overset{1}{\beta_3} \arrow[l] \arrow[r, dashed] & \overset{0}{\beta_5 - \beta_1} \arrow[r, squiggly] & {} \\
 & \overset{1}{\beta_4} \arrow[u] & & & \overset{1}{\beta_4} \arrow[u] \arrow[r, dashed] & \overset{0}{\beta_5 - \beta_1} & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & & \beta_1 + \overset{1}{\beta_3} - \beta_5 \arrow[d] & & & \\
{} \arrow[r, squiggly, "{[s]_{\beta_1+\beta_2-\beta_5,\beta_2} \cdot [s]_{\beta_1+\beta_3-\beta_5,\beta_3} \cdot [s]_{\beta_1+\beta_4-\beta_5,\beta_4}}", "a)"'] & {} & \beta_1 + \overset{1}{\beta_2} - \beta_5 \arrow[r] & \overset{2}{\beta_1} & \beta_1 + \overset{1}{\beta_4} - \beta_5 \arrow[l] & {} \arrow[r, squiggly, "{[s]_{\beta_1}}"] & \emptyset
\end{tikzcd}
$$

At the last step we noticed the shortcut that the semi-invariant is just the square determinant of size β_1.

So the b-function of f is

$$b_f(s) = [s]^2_{\beta_5-\beta_1,\beta_5} \cdot [s]_{\beta_1+\beta_2-\beta_5,\beta_2} \cdot [s]_{\beta_1+\beta_3-\beta_5,\beta_3} \cdot [s]_{\beta_1+\beta_4-\beta_5,\beta_4} \cdot [s]_{\beta_1}.$$

The following proposition gives a clearer picture of sliceable semi-invariants:

Proposition 4.3.10. *Let $f = c^V \in \mathrm{SI}(Q,\beta)$ be an irreducible semi-invariant of weight $\langle \alpha, \cdot \rangle = -\langle \cdot, \alpha^* \rangle$ and assume f depends on all arrows of Q. If α (and α^*) is not a real Schur root, then f is not sliceable.*

Furthermore, take an arrow $\vec{a}$ that is a 1*-source or* 1*-sink between* 1 *and* 2 *such that $\beta_1 \leq \beta_2$, and assume α is a real Schur root. Let $\langle \alpha_a, \cdot \rangle$ be the weight of the induced semi-invariant f_a on the slice (Q_a, β_a), and let $\langle \alpha'_a, \cdot \rangle$ be the weight on (Q'_a, β'_a) after possible simplifications as in Lemma 2.3.7. Then the following are equivalent:*

(a) α_a is a real Schur root;

(b) α'_a is a real Schur root;

(c) $\vec{a}$ *is a* 1*-source with* $\alpha_1 = \alpha_2$ *or* $\alpha_1^* = 0$*, or* $\vec{a}$ *is a* 1*-sink with* $\alpha_1 = 0$ *or* $\alpha_1^* = \alpha_2^*$*.*

Proof. We will assume a is a 1-source (the case with 1-sink is similar). Since f depends on all arrows of Q and is irreducible, we have by Proposition 4.3.2 part a) that β and β_a are sincere dimension vectors. Due to the isomorphism $\mathrm{SI}(Q, \beta) \cong \mathbb{C}[\mathrm{Rep}(Q_a, \beta_a)]^{U \rtimes \mathrm{SL}(\beta_a)}$, we also have that $f_a = c^{V'}$ is irreducible. Since β and β_a are sincere, V and V' are Schur representations by [21, Lemma 1].

Note that $\langle \alpha, \alpha \rangle = -\langle \alpha, \alpha^* \rangle = \langle \alpha^*, \alpha^* \rangle$. The following formula is straightforward

$$\langle \alpha_a, \alpha_a \rangle_a = \langle \alpha, \alpha \rangle - (\alpha_2 - \alpha_1)\alpha_1^*,$$

where $\langle \cdot, \cdot \rangle_a$ is the Euler form on Q_a. This implies that this value decreases by slicing (at least before simplifications), and it remains the same iff $\alpha_2 = \alpha_1$ or $\alpha_1^* = 0$. However, we can simplify according to Lemma 2.3.7 precisely under these conditions, when we get a reduced quiver Q_a' with α_a'. But an easy computation yields that the value $\langle \alpha_a', \alpha_a' \rangle_a' = \langle \alpha_a, \alpha_a \rangle_a$ still remains the same. Since V (resp. V') are Schur representations, α (resp. α_a) is a real Schur root if and only if $\langle \alpha, \alpha \rangle = 1$ (resp. $\langle \alpha_a, \alpha_a \rangle = 1$). This proves the second part.

Now assume f is sliceable. Since V is a Schur representation, we have $\langle \alpha, \alpha \rangle \leq 1$. Since this value can only decrease by slicing, and the last value is trivially 1, we must have that all values are 1, and all the encountered dimension vectors are real Schur roots. □

Using the proposition above, we find a Dynkin quiver with a semi-invariant that is not sliceable:

Example 4.3.11. Take the following quiver of type $\mathbb{E}_6$ with semi-invariant of weight $\langle \alpha, \cdot \rangle = -\langle \cdot, \alpha^* \rangle$, with α being the highest root:

$$\begin{tikzcd}
 & & 2 \arrow[d] & & & & & & & 1 \arrow[d] & & \\
1 \arrow[r] & 2 \arrow[r] & 3 & 2 \arrow[l] & 1 \arrow[l] & {} \arrow[r, rightsquigarrow, "\text{Rewrite with } \alpha^*"] & {} & 1 \arrow[r] & 2 \arrow[r] & 3 & 2 \arrow[l] & 1 \arrow[l]
\end{tikzcd}$$

There are no 1-sinks and there is no 1-source a with $\alpha_{ta} = \alpha_{ha}$ nor with $\alpha^*_{ta} = 0$. By Theorem 4.3.10 the semi-invariant is not sliceable.

We show in the next example how to apply Proposition 4.3.2 together with Theorem 4.1.11 to compute b-functions of several variables. The main difference in the process is that we can make only simultaneous simplifications for the semi-invariants. For the $\mathbb{A}$ type this process is always applicable (however, it is not always applicable directly for type $\mathbb{D}$), and it can be also understood as superimposing the separate slice quivers, similar to [66].

Example 4.3.12. (b-function of several variables) Take the following $\mathbb{D}_5$ quiver with non-zero semi-invariants $f_i = c^{V_i}$, for $i = 1, 2$, $\alpha^1 = \underline{d}(V_1) = (0, 1, 1, 0, 1)$, $\alpha^2 = \underline{d}(V_2) = (1, 1, 0, 0, 0)$ and $\beta_1 + \beta_4 = \beta_3$, $\beta_2 = \beta_5$. We put the values of α^1 and α^2 on top of β:

$$\begin{tikzcd}
 & & & & & & \overset{1,0}{\beta_3} \arrow[d] & & \\
 & & \overset{1,0}{\beta_3} \arrow[d] & & {} \arrow[r, rightsquigarrow, "{[s]^{1,0}_{\beta_2-\beta_1,\beta_2}}"] & \overset{1,1}{\beta_2 - \beta_1} \arrow[r, dashed] & \overset{1,1}{\beta_5} \arrow[r] & \overset{0,1}{\beta_4} & {} \arrow[r, rightsquigarrow, "{[s]^{1,1}_{\beta_1,\beta_5}}", "a)"'] & {} \\
\overset{0,1}{\beta_1} & \overset{1,1}{\beta_2} \arrow[l, dashed] \arrow[r] & \overset{1,0}{\beta_5} \arrow[r] & \overset{0,0}{\beta_4} & & & \overset{0,1}{\beta_1} \arrow[u] & &
\end{tikzcd}$$

$$\begin{tikzcd}
\overset{1,0}{\beta_3} \arrow[r] \arrow[d] & \overset{0,0}{\beta_1} & & & & & & \\
\overset{1,1}{\beta_2 - \beta_1} \arrow[r] & \overset{0,1}{\beta_4} & {} \arrow[r, rightsquigarrow, "{[s]^{0,1}_{\beta_1}}"] & \overset{1,0}{\beta_3} \arrow[r, dashed] \arrow[d] & \overset{0,0}{\beta_1} & {} \arrow[r, rightsquigarrow, "{[s]^{1,0}_{\beta_4,\beta_3}}", "a)"'] & \overset{1,0}{\beta_4} \arrow[d, dashed] & \\
\overset{0,1}{\beta_1} \arrow[u] \arrow[r, dashed] & \overset{0,0}{\beta_1} & & \overset{1,0}{\beta_2 - \beta_1} \arrow[r] & \overset{0,0}{\beta_4} & & \overset{1,0}{\beta_2 - \beta_1} \arrow[r] & \overset{0,0}{\beta_4}
\end{tikzcd}$$

$$\overset{[s]^{1,0}_{\beta_2-\beta_3,\beta_2-\beta_1}}{\rightsquigarrow} \quad \overset{1,0}{\beta_4} \dashrightarrow \overset{1,0}{\beta_4} \quad \overset{[s]^{1,0}_{\beta_4}}{\rightsquigarrow} \quad \emptyset$$

Hence we have

$$b_{\underline{m}}(s_1, s_2) = [s]^{1,1}_{\beta_1,\beta_2}[s]^{0,1}_{\beta_1}[s]^{1,0}_{\beta_3}[s]^{1,0}_{\beta_2-\beta_3,\beta_2}.$$

In the following example, we will combine the slice technique with the method of a-functions from [57] to compute the b-function of several variables:

Example 4.3.13. Take again the $\mathbb{D}_4$ quiver

$$\begin{tikzcd} & 2 \arrow[d] & \\ 1 \arrow[r] & 4 & 3 \arrow[l] \end{tikzcd}$$

with the choice of $\beta = (n, n, n, 2n)$, $n \in \mathbb{N}$. There are 3 fundamental semi-invariants f_1, f_2, f_3 with $\alpha_1 = (0, 1, 1, 1), \alpha_2 = (1, 0, 1, 1), \alpha_3 = (1, 1, 0, 1)$. Explicitly, if we label the matrices as

$$\begin{tikzcd} & n \arrow[d, "Y"] & \\ n \arrow[r, "X"] & 2n & n \arrow[l, "Z"'] \end{tikzcd}$$

then $f_1 = \det(YZ), f_2 = \det(ZX), f_3 = \det(XY)$. We see that the slice method cannot be applied simultaneously for all 3 semi-invariants, but it can be applied to any 2 of them. For example, we pick f_1 and f_2:

$$\begin{tikzcd} & \overset{1,0}{n} \arrow[d] & \\ \overset{0,1}{n} \arrow[r] & \overset{1,1}{2n} & \overset{1,1}{n} \arrow[l, dashed] \end{tikzcd} \quad \underset{a)\ b)}{\overset{[s]^{1,1}_{n,2n}}{\rightsquigarrow}} \quad \begin{tikzcd} \overset{1,0}{n} \arrow[r, dashed] & \overset{0,0}{n} \\ \overset{0,1}{n} \arrow[r, dashed] & \overset{0,0}{n} \end{tikzcd} \quad \overset{[s]^{1,0}_{n}[s]^{0,1}_{n}}{\rightsquigarrow} \quad \emptyset$$

Hence the b-function of two variables of f_1 and f_2 is

$$b'_{m_1,m_2}(s_1, s_2) = \prod_{i=0}^{m_1+m_2-1} \prod_{j=n+1}^{2n} (s_1+s_2+i+j) \prod_{i=0}^{m_1-1} \prod_{j=1}^{n} (s_1+i+j) \prod_{i=0}^{m_2-1} \prod_{j=1}^{n} (s_2+i+j).$$

By symmetry, we have the b-function of two variables for any pair.

To compute the b-function of three variables of f_1, f_2 and f_3, we employ the structure theorem on b-functions [57, Theorem 2] or [68, Theorem 1.3.5]. This method is also used for the $\mathbb{A}_n$ case in [66]. We refer the reader to these papers for details on the method.

First, we take an explicit generic element: $A_0 = \left[\begin{pmatrix} I_n \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ I_n \end{pmatrix}, \begin{pmatrix} I_n \\ I_n \end{pmatrix}\right]$. Next, we compute

$$\operatorname{grad}\log f_k(A_0) = \frac{1}{f_k(A_0)}\left[\Big(\frac{\partial f_k}{x_{i,j}}(A_0)\Big)_{i,j}, \Big(\frac{\partial f_k}{y_{i,j}}(A_0)\Big)_{i,j}, \Big(\frac{\partial f_k}{z_{i,j}}(A_0)\Big)_{i,j}\right].$$

We get

$$\operatorname{grad}\log f_1(A_0) = \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -I_n \\ I_n \end{pmatrix}, \begin{pmatrix} I_n \\ 0 \end{pmatrix}\right],$$

$$\operatorname{grad}\log f_2(A_0) = \left[\begin{pmatrix} I_n \\ -I_n \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ I_n \end{pmatrix}\right],$$

$$\operatorname{grad}\log f_3(A_0) = \left[\begin{pmatrix} I_n \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ I_n \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right].$$

The next step is computing

$$a_k(\underline{s}) = f_k(A_0)\cdot f_k\Big(\sum_{i=1}^{3} s_i \operatorname{grad}\log f_i(A_0)\Big) = s_k^n \cdot (s_1+s_2+s_3)^n.$$

Hence the a-function is

$$a_{\underline{m}}(\underline{s}) = a_1(\underline{s})^{m_1} a_2(\underline{s})^{m_2} a_3(\underline{s})^{m_3} = s_1^{nm_1} s_2^{nm_2} s_3^{nm_3} (s_1+s_2+s_3)^{n(m_1+m_2+m_3)}.$$

By the structure theorem, the b-function is of the form $b_{\underline{m}}(\underline{s}) =$

$$\prod_{j=1}^{n}\left[\prod_{i=0}^{m_1-1}(s_1+\alpha_{1,j}+i)\prod_{i=0}^{m_2-1}(s_2+\alpha_{2,j}+i)\prod_{i=0}^{m_3-1}(s_3+\alpha_{3,j}+i)\prod_{i=0}^{m_1+m_2+m_3-1}(s_1+s_2+s_3+\alpha_{4,j}+i)\right]$$

Since $b_{m_1,m_2,0}(s_1,s_2,0) = b'_{m_1,m_2}(s_1,s_2)$, we get that $\alpha_{1,j} = \alpha_{2,j} = \alpha_{3,j} = j$ and $\alpha_{4,j} = n+j$.

4.4 Canonical decomposition for quivers of type $\mathbb{D}$

In this sections we give a rule to determine the canonical decomposition for type $\mathbb{D}$ quivers.

Let Q be a quiver, and α a prehomogeneous dimension vector. Following [33], we call a decomposition

$$\alpha = \alpha_1 \oplus \alpha_2 \oplus \cdots \oplus \alpha_t$$

the *canonical decomposition* if the generic representation of dimension α decomposes into indecomposable representations of dimensions $\alpha_1, \alpha_2, \ldots \alpha_t$. As already discussed in Chapter 2, in this case α_i are real Schur roots, with $\mathrm{Ext}_Q(\alpha_i, \alpha_j) = 0$ (that is, the corresponding generic representations have no self-extensions). Moreover, rewriting

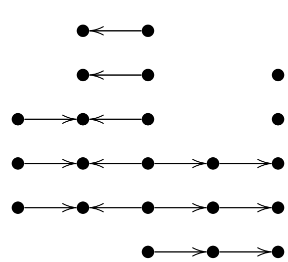
$$\alpha = \alpha_1^{\oplus r_1} \oplus \alpha_2^{\oplus r_2} \oplus \cdots \oplus \alpha_t^{\oplus r_t}.$$

with α_i distinct, we may assume, after a suitable rearrangement, that $\mathrm{Hom}_Q(\alpha_i, \alpha_j) = 0$, for $i < j$ (again, this means that there are no morphisms between the corresponding generic representations). For more details , see [23, 33].

Though there exist algorithms to determine the canonical decomposition for a dimension vector (see [22]), it is of interest to give clear-cut procedures for simpler cases. There is such a rule for quivers of type $\mathbb{A}$, and this is described in [1, Proposition 3.1]. We illustrate this construction by the following example:

$$3 \longrightarrow 5 \longleftarrow 6 \longrightarrow 3 \longrightarrow 5$$

The canonical decomposition is given by the following diagram (the connected horizontal components are the indecomposables):



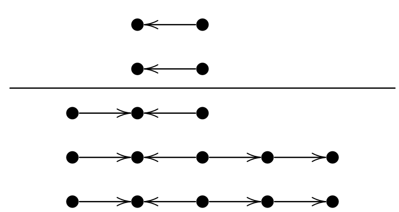
Based on the $\mathbb{A}$ case, we extend the rule for quivers of type $\mathbb{D}$. Take a quiver with underlying graph $\mathbb{D}_n$ and the following labeling:

$$\begin{tikzcd}
 & n & & & \\
1 \arrow[r, dash] & 2 \arrow[u] \arrow[r, dash] & 3 \arrow[r, dash] & \dots \arrow[r, dash] & n-1
\end{tikzcd}$$

Since the canonical decomposition of a quiver and its opposite quiver coincide, we will fix without loss of generality the orientation of the arrow $2 \to n$. We illustrate the procedure by examples first. Take the following $\mathbb{D}_n$ quiver with $n = 6$ and $\alpha = (3, 5, 6, 3, 5, 4)$:

$$\begin{tikzcd}
 & 4 & & & \\
3 \arrow[r] & 5 \arrow[u] & 6 \arrow[l] \arrow[r] & 3 \arrow[r] & 5
\end{tikzcd}$$

First, take the canonical decomposition of the $\mathbb{A}_{n-1}$ quiver by dropping the n-th vertex. This was done in the example above. Then, the indecomposables that have 0 dimension at vertex 2 will also appear in the canonical decomposition for $\mathbb{D}_n$. Hence we drop them, and we are left with the following diagram:



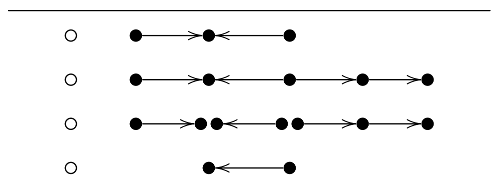
We separated by a horizontal line the two classes of indecomposables with dimension at vertex 1 equal to 0 or equal to 1. We call the indecomposables under this line of the first class and over the line of the second class. Now we place α_n symbols $\circ$ on the left of the diagram starting from the horizontal line and moving downwards ($\circ$ represents the simple representation S_n). When we stop, we put another horizontal line to the bottom. Then we move the indecomposables of the second class starting from the top of the diagram and add their dimension vectors starting from the bottom horizontal line and stop if either:

(a) We reach the top horizontal line or

(b) We run out of indecomposables of the second class or

(c) There exists a non-zero morphism from the indecomposable of the second class that we want to move to corresponding indecomposable of the first class.

In this example we stop due to part (b) and the diagram we get is:



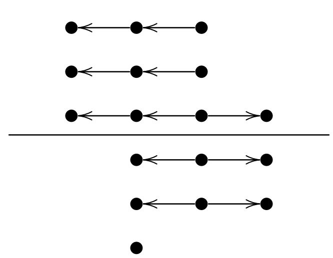
Now we are ready to read off the canonical decomposition. The indecomposables outside the horizontal lines will stay the same (there are none in this example). Finally, for each row between the two horizontal lines the dimension vector will have dimension 1 at vertex n. Hence we get in this case

$$(3,5,6,3,5,4) = (1,1,1,0,0,1)\oplus(1,1,1,1,1,1)\oplus(1,2,2,1,1,1)\oplus(0,1,0,0,0,1)\oplus \\ \oplus (0,0,0,0,1,0)^{\oplus 2} \oplus (0,0,1,1,1,0).$$

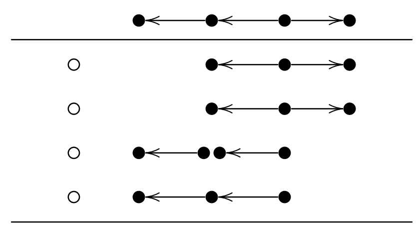
We give another example:

$$\begin{tikzcd} & 4 & & \\ 3 & 6 \arrow[l] \arrow[u] & 5 \arrow[l] \arrow[r] & 3 \end{tikzcd}$$

The canonical decomposition for the $\mathbb{A}_4$ part is



Note that all indecomposables have dimension 1 at vertex 2. The diagram joining the two classes of indecomposables is:



Here we stopped due to condition (c) since there is a non-zero map from the indecomposable $1 \leftarrow 1 \leftarrow 1 \rightarrow 1$ to the corresponding indecomposable $0 \leftarrow 1 \leftarrow 1 \rightarrow 1$. Hence the canonical decomposition is

$$(3,6,5,3,4) = (1,1,1,1,0) \oplus (0,1,1,1,1)^{\oplus 2} \oplus (1,2,1,0,1) \oplus (1,1,1,0,1).$$

Theorem 4.4.1. *The algorithm described above gives the canonical decomposition for $\mathbb{D}_n$ quivers.*

Proof. We give a proof using slices. First, write the canonical decomposition for a generic representation R of the $\mathbb{A}_{n-1}$ quiver in the form

$$R = \bigoplus_{i=1}^{m} V_i^{p_i} \oplus \bigoplus_{i=1}^{n} W_i^{q_i} \oplus \bigoplus_{i} Z_i$$

Here V_i and W_i are representations of the first and second class, respectively (separated by the horizontal line as in the examples) and Z_i are the representations with dimension 0 at vertex 2. We assume that the order is chosen such that:

(a) There is a map from V_i to V_j iff $j \leq i$;

(b) There is a map from W_i to V_j iff $j \leq i$;

(c) There are no maps from V_i to W_j for all i, j.

We note that this can be achieved immediately from the canonical decomposition algorithm (after dropping the representations Z_i): V_i are the representations below the horizontal line, ordered from top to bottom, and W_i are the representations above the horizontal line, ordered from top to bottom. With this in mind, we take the slice as in Proposition 4.2.2 (see also Remark 4.2.3). The generic element element V of $\mathbb{D}_n$ will be of the form $V = Z + R$, with $Z \in \mathrm{Hom}(\mathbb{C}^{\alpha_2}, \mathbb{C}^{\alpha_n})$ and Z having a dense orbit under the action of $\mathrm{GL}(\alpha_n) \times \mathrm{GL}(\underline{p}) \times \mathrm{GL}(\underline{q}) \times U \times U'$, where $U = \prod_{j<i} \mathrm{Hom}(\mathbb{C}^{p_i}, \mathbb{C}^{p_j}) \prod_{j<i} \mathrm{Hom}(\mathbb{C}^{q_i}, \mathbb{C}^{q_j})$ and $U' = \prod_{i,j} \mathrm{Hom}(W_i, V_j)^{p_j q_i}$. It can be easily seen that forgetting about the action of U', the following element is already generic:

$$
Z = \begin{array}{c}
\begin{array}{cccc|cccc} V_1 & V_1 & \cdots & V_m & W_1 & W_1 & \cdots & W_n \end{array} \\
\left[\begin{array}{cccc|cccc}
1 & 0 & \dots & 0 & & & & \\
0 & 1 & \dots & 0 & & & & \\
 & & \ddots & & & & \iddots & \\
 & & & & 0 & 1 & \dots & 0 \\
 & & & & 1 & 0 & \dots & 0
\end{array}\right]
\end{array}
$$

Here there are p_i (resp. q_i) columns corresponding to V_i (resp. W_i), and we put the ones diagonally in the first (resp. second) block starting from the top left (resp. bottom left) until we reach the bottom or right (resp. top or right) edge of the block. The arrangement of ones corresponds to stopping under condition (a) or (b). Now using the action of U', if two ones are in the same row corresponding to the columns of V_i and W_j, and $\mathrm{Hom}_Q(W_j, V_i) \neq 0$, then we can cancel the 1 in the column of W_j. This corresponds to stopping under condition (c). $\square$

Chapter 5

Singularities of zero sets of semi-invariants

The study of zero sets of semi-invariants for quivers has been initiated in [17], and has been intensively investigated later in several articles. In particular [49] shows that the nullcone for prehomogeneous dimension vectors is an irreducible complete intersection if the dimension vector is not "too small". Bounds have been given for tame quivers in [50]. In Section 5.1 we state analogous results concerning whether the nullcone is reduced (Theorem 5.1.7 and Theorem 5.1.10). Clearly, results about the nullcone are valid for the zero set of an arbitrary set of fundamental semi-invariants. We note that such questions have been investigated also outside the quiver setting (for example, see [38]).

In Section 5.2 we use b-functions and give some results on whether zero sets have rational singularities. This is based on the calculation of b-functions (of several variables) from our previous sections. We note that zero sets often turn out to be orbit closures (see [44, 45]) and the hypersurface case corresponds to codimension 1 orbits if the dimension vector is not "too small" (see [48]). In the latter case the results are sharper and we prove that for Dynkin quivers codimension 1 orbits have rational singularities (Theorems 5.2.4 for Dynkin and see 5.2.6 for extended Dynkin quivers). Such questions about the geometry of orbit closures have been thoroughly investigated before, for instance, it is known that for quivers of type $\mathbb{A}, \mathbb{D}$ all orbit closures have rational singularities by [7, 8].

For zero sets of more semi-invariants, we establish an elementary link between the b-function of several variables and the Bernstein-Sato polynomial of an ideal (Proposition 5.2.1). Then we state Lemma 5.2.10 and exhibit how it can be used in Example 5.2.11. Lastly, we state Theorem 5.2.12 involving some special semi-invariants for tree quivers.

5.1 On the reduced property of the nullcone

Let k be an algebraically closed field, Q a quiver without oriented cycles with $n = |Q_0|$ vertices, and α a dimension vector.

We investigate the geometry of the *nullcone* for the action of $\mathrm{SL}(\alpha)$, that is, the the set of common zeros of all semi-invariants of positive degree:

$$\mathcal{Z}(Q,\alpha) = \{X \in \mathrm{Rep}(Q,\alpha) \,:\, f(X) = 0, \text{ for all non-constant } f \in \mathrm{SI}(Q,\alpha)\}.$$

Throughout we assume that α is a *prehomogeneous dimension vector* . Without loss of generality, we assume α is a sincere dimension vector, that is, $\alpha_x > 0$ for all $x \in Q_0$. Denote by T the generic representation, and write $T = T_1^{\oplus\lambda_1} \oplus \cdots \oplus T_m^{\oplus\lambda_m}$, where the T_i are pairwise non-isomorphic direct summands.

Take the simple objects $S_{m+1}, \ldots, S_n$ in the right perpendicular category $T^{\perp}$. By Theorem 2.3.3 the nullcone can be described as follows:

$$\mathcal{Z}(Q,\alpha) = \{X \in \mathrm{Rep}(Q,\alpha) \mid \mathrm{Hom}(X, S_j) \neq 0, \text{ for all } j = m+1, \ldots, n\}.$$

It is shown in [49] that there is a large enough number $N = N(Q)$, such that if $c \geq N(Q)$ then $\mathcal{Z}(Q, c\cdot\alpha)$ is irreducible and a set-theoretic complete intersection. By [50], for tame quivers we have more precise control over N. Namely, the nullcone is a complete intersection for $N(Q)$ and irreducible for $N(Q)+1$ where

$$N(Q) = \begin{cases} 1 & \text{if } Q = \mathbb{A}_n \text{ or } \widetilde{A}_n, \\ 2 & \text{if } Q = \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7 \text{ or } \mathbb{E}_8, \\ 3 & \text{if } Q = \widetilde{\mathbb{D}}_n, \widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7 \text{ or } \widetilde{\mathbb{E}}_8. \end{cases} \tag{5.1}$$

To discuss geometric properties of the nullcone we need to first introduce some tools. We follow much of the notation introduced in [48, 44].

Let Y be a representation satisfying

$$\mathrm{Ext}(T, Y) = 0.$$

We denote by $\mathrm{tr}\, Y$ the trace of T in Y, that is, the sum of all the images of all maps from T to Y, and let $\overline{Y} = Y/\,\mathrm{tr}\, Y$. Then it is easy to see that $\overline{Y} \in T^{\perp}$.

Next, we recall a construction of Bongartz [9]. Let kQ be the path algebra of Q, viewed as a projective representation of Q. Let $\mu_i = \dim \mathrm{Ext}(T_i, kQ)$, $i = 1, \ldots, m$. Then there is an exact sequence

$$0 \to kQ \to \tilde{T} \to T \to 0$$

such that the induced map

$$\mathrm{Hom}(T_l, \oplus_{i=1}^{m} T^{\mu_i}) \to \mathrm{Ext}(T_l, kQ)$$

is surjective for all $l = 1, \ldots m$. This defines $\tilde{T}$ up to isomorphism, and $T \oplus \tilde{T}$ is a tilting module, so it has n pairwise distinct indecomposable summands and $\mathrm{Ext}(T \oplus \tilde{T}, T \oplus \tilde{T}) = 0$. There are $n - m$ non-isomorphic summands of $\tilde{T}$ and denote them by $T_{m+1}, \ldots, T_n$. We have the following

Proposition 5.1.1 ([48, 61])**.** *The representations $\overline{T}_{m+1}, \ldots, \overline{T}_n$ are representatives for the indecomposable projective objects in $T^{\perp}$.*

We order $\overline{T}_{m+1}, \ldots, \overline{T}_n$ so that they are the projective covers of $S_{m+1}, \ldots, S_n$, respectively.

For a representation $A = B_1^{b_1} \oplus \cdots \oplus B_t^{b_t}$, where B_i are pairwise non-isomorphic indecomposables, and the b_i are positive integers, we denote by $\mathrm{add}(A)$ the full subcategory of $\mathrm{Rep}(Q)$ whose objects are representations Y such that $Y \cong B_1^{c_1} \oplus \cdots \oplus B_t^{c_t}$, where the c_i are non-negative integers.

Now we recall a construction from [48]. For every $j = m+1, \ldots, n$ we have an exact sequence

$$0 \to T_j \to T_j^{++} \to Z_j \to 0, \tag{5.2}$$

where the first map is a source map, and T_j^{++} is a representation in $\mathrm{add}(T)$. Denote $Z = T_1 \oplus \cdots \oplus T_m \oplus Z_{m+1} \cdots \oplus Z_n$. We have the following lemma (see [44, 48]):

Lemma 5.1.2. *The following hold:*

(a) Z *is a tilting module, that is,* $\mathrm{Ext}(Z, Z) = 0$.

(b) $\dim \mathrm{Ext}(Z_j, S_k) = \dim \mathrm{Hom}(\overline{T}_j, S_k) = \dim \mathrm{Hom}(T_j, S_k) = \delta_{jk}$, *where* $j, k \in \{m+1, \ldots, n\}$.

(c) $\mathrm{Hom}(Z, S_k) = 0$, *where* $k \in \{m+1, \ldots, n\}$.

Definition 5.1.3. We say $X \in \mathcal{Z}(Q, \alpha)$ satisfies the *independent gradient conditions* if we have:

(a) $\dim \mathrm{Hom}(X, S_j) = 1$, for any $j = m+1, \ldots n$,

(b) For any $k = m+1, \ldots, n$ there exists an exact sequence

$$0 \to X \to Y_k \to X \to 0$$

such that $\dim \mathrm{Hom}(Y_k, S_j) = 2 - \delta_{jk}$ for $j = m+1, \ldots, n$, where δ is the Kronecker delta.

Using [51, Corollary 7.4] together with the Serre's Criterion (see for example [25, Theorem 18.15]) we have the following (see also [6, Proposition 4.6]):

Proposition 5.1.4. *Assume the nullcone $\mathcal{Z}(Q,\alpha)$ is a set-theoretic complete intersection. Then $\mathcal{Z}(Q,\alpha)$ is reduced iff each of its irreducible components contains a representation satisfying the independent gradient conditions (5.1.3).*

We define the following open subsets of $\mathcal{Z}(Q,\alpha)$:

$$\mathcal{Z}'(Q,\alpha) = \{X \in \mathcal{Z}(Q,\alpha) : \mathrm{Ext}(T,X) = \mathrm{Ext}(X,T) = 0\},$$

$$\mathcal{H}(Q,\alpha) = \{X \in \mathrm{Rep}(Q,\alpha) : \dim \mathrm{Hom}(X,S_j) = 1, \text{ for all } j = m+1,\dots n\}.$$

By the independent gradient conditions (5.1.3), if $\mathcal{H}(Q,\alpha) = \emptyset$, then $\mathcal{Z}(Q,\alpha)$ is not reduced. Now we are ready to prove our first result about reduced property of the nullcone:

Theorem 5.1.5. *Assume that $\mathcal{Z}'(Q,\alpha)$ is not empty. Then the nullcone $\mathcal{Z}(Q,\alpha)$ is reduced, irreducible and a complete intersection.*

Proof. By [49, Proposition 3.7], we know already that $\mathcal{Z}(Q,\alpha)$ is irreducible and a complete intersection. The set $\mathcal{H}(Q,\alpha) \cap E_T$ is open in $\mathcal{Z}(Q,\alpha)$ and non-empty. We prove that any element in $\mathcal{H}(Q,\alpha)\cap E_T$ also satisfies the independence condition (5.1.3 b). So take an arbitrary $X \in \mathcal{H}(Q,\alpha) \cap E_T$ and write

$$X \cong \widetilde{X} \oplus \bigoplus_{j=m+1}^{n} Z_j^{a_j},$$

where $\widetilde{X}$ and Z have no common indecomposable summands. By [44, Proposition 3.14], we have a minimal projective resolution of $\overline{X}$ in $T^{\perp}$ of the form

$$0 \to \bigoplus_{j\in J} \overline{T}_j \to \bigoplus_{j=m+1}^{n} \overline{T}_j \to \overline{X} \to 0,$$

where $J \subset \{m+1,\dots,n\}$. Moreover, $a_j = 0$ if $j \in J$, and $a_j = 1$, if $j \in J^c$, where J^c denotes the complement of J in $\{m+1,\dots,n\}$. We construct the exact sequences as in (5.1.3 b) by considering two cases, whether $k \in J$ or $k \in J^c$.

First, let $k \in J^c$. Consider the composite map $\phi : T_k \to \overline{T}_k \to \overline{X}$ where the first map is the projection $T_k \to T_k/\operatorname{tr} T_k$ and the second is from the minimal resolution of $\overline{X}$. Applying $\operatorname{Hom}(-, S_k)$ to the minimal resolution, together with Lemma 5.1.2 we have the induced isomorphisms of 1-dimensional spaces

$$\operatorname{Hom}(\overline{X}, S_k) \cong \operatorname{Hom}(\overline{T}_k, S_k) \cong \operatorname{Hom}(T_k, S_k) \cong \operatorname{Ext}(Z_k, S_k). \tag{5.3}$$

Consider the following diagram

$$\begin{tikzcd}
0 \arrow[r] & T_k \arrow[r] \arrow[d, "\phi"] & T_k^{++} \arrow[r] \arrow[d] & Z_k \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \overline{X} \arrow[r] & U_k \arrow[r] & Z_k \arrow[r] & 0
\end{tikzcd}$$

where the second row is the push-out of the first via ϕ. Take $j \in \{m+1, \ldots, n\}, j \neq k$. Applying $\operatorname{Hom}(-, S_j)$ to the second exact sequence, the induced long exact sequence together with Lemma 5.1.2 gives

$$\dim \operatorname{Hom}(U_k, S_j) = \dim \operatorname{Hom}(\overline{X}, S_j) = 1.$$

On the other hand, applying $\operatorname{Hom}(-, S_k)$ we get the exact sequence

$$0 \to \operatorname{Hom}(U_k, S_k) \to \operatorname{Hom}(\overline{X}, S_k) \to \operatorname{Ext}(Z_k, S_k),$$

where, by construction, the last map is the composition of isomorphisms in (5.3). Hence $\operatorname{Hom}(U_k, S_k) = 0$, so we have $\dim \operatorname{Hom}(U_k, S_j) = \delta_{jk}$, where $j \in \{m + 1, \ldots, n\}$.

Now applying $\operatorname{Hom}(Z_k, -)$ to the exact sequence

$$0 \to \operatorname{tr} X \to X \to \overline{X} \to 0$$

we get $\operatorname{Ext}(Z_k, X) \cong \operatorname{Ext}(Z_k, \overline{X})$, since $\operatorname{tr} X \in \operatorname{add}(Z)$ by [44, Proposition 3.9]. Hence can we lift (uniquely) the exact sequence with middle term U_k and get the following exact diagram:

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \operatorname{tr} X \arrow[r, equal] \arrow[d] & \operatorname{tr} X \arrow[d] & & \\
0 \arrow[r] & X \arrow[r] \arrow[d] & V_k \arrow[r] \arrow[d] & Z_k \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \overline{X} \arrow[r] \arrow[d] & U_k \arrow[r] \arrow[d] & Z_k \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}
$$

Applying $\operatorname{Hom}(-, S_j)$ for $j = m+1, \ldots, n$ to the middle column we get that $\dim \operatorname{Hom}(V_k, S_j) = \dim \operatorname{Hom}(U_k, S_j) = \delta_{jk}$. By Lemma 5.1.2 $\operatorname{Hom}(Z, S_j) = 0$, so $\dim \operatorname{Hom}(X, S_j) = \dim \operatorname{Hom}(\widetilde{X}, S_j) = 1$. Hence if we put

$$Y_k = V_k \oplus \widetilde{X} \oplus \bigoplus_{j \in J^c \setminus \{k\}} Z_j$$

we have an exact sequence

$$0 \to X \to Y_k \to X \to 0$$

satisfying the independence condition (5.1.3 b).

Now we consider the second case, when $k \in J$. Denote

$$Q_k = \bigoplus_{j \in J \setminus \{k\}} \overline{T}_j \text{ and } R_k = \bigoplus_{\substack{j=m+1 \\ j \neq k}}^{n} \overline{T}_j.$$

Let ψ denote the injective map of the minimal projective resolution of $\overline{X}$ in $T^{\perp}$, so $\psi : Q_k \oplus \overline{T}_k \to R_k \oplus \overline{T}_k$. Consider the following commutative diagram

$$
\begin{tikzcd}
0 \arrow[r] & Q_k \oplus \overline{T}_k \arrow[r, "\binom{I}{0}"] \arrow[d, "\psi"] & Q_k \oplus \overline{T}_k \oplus Q_k \oplus \overline{T}_k \arrow[r, "(0\ I)"] \arrow[d, "\psi_k"] & Q_k \oplus \overline{T}_k \arrow[r] \arrow[d, "\psi"] & 0 \\
0 \arrow[r] & R_k \oplus \overline{T}_k \arrow[r, "\binom{I}{0}"] & R_k \oplus \overline{T}_k \oplus R_k \oplus \overline{T}_k \arrow[r, "(0\ I)"] & R_k \oplus \overline{T}_k \arrow[r] & 0
\end{tikzcd}
$$

where the map $\psi_k : Q_k^2 \oplus \overline{T}_k^2 \to R_k^2 \oplus \overline{T}_k^2$ is obtained from $\psi \oplus \psi$ by adding the identity map from the second copy of $\overline{T}_k$ to the first copy of $\overline{T}_k$. Denote $W_k = \operatorname{coker} \psi_k$. By the snake lemma, ψ_k is injective and we have an exact sequence

$$0 \to \overline{X} \to W_k \to \overline{X} \to 0.$$

Moreover, applying $\operatorname{Hom}(-, S_j)$ to the (non-minimal) projective resolution $\psi_k : Q_k^2 \oplus \overline{T}_k^2 \to R_k^2 \oplus \overline{T}_k^2$, we have that $\dim \operatorname{Hom}(W_k, S_j) = 2 - \delta_{jk}$, for $j = m+1, \dots, n$. Now we pull-back the sequence above via the map $X \twoheadrightarrow \overline{X}$ to get the following diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \operatorname{tr} X \arrow[r, equal] \arrow[d] & \operatorname{tr} X \arrow[d] & \\
0 \arrow[r] & \overline{X} \arrow[r] \arrow[d, equal] & W_k' \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overline{X} \arrow[r] & W_k \arrow[r] \arrow[d] & \overline{X} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Applying $\operatorname{Hom}(-, S_j)$ to the middle column, we see that $\dim \operatorname{Hom}(W_k', S_j) = \dim \operatorname{Hom}(W_k, S_j) = 2 - \delta_{jk}$, where $j = m+1, \dots, n$.

Now the surjection $X \twoheadrightarrow \overline{X}$ gives a surjective map $\operatorname{Ext}(X, X) \twoheadrightarrow \operatorname{Ext}(X, \overline{X})$, hence we can lift the sequence $0 \to \overline{X} \to W_k' \to X \to 0$ to a sequence $0 \to X \to Y_k \to X \to 0$. As before, from the sequence $0 \to \operatorname{tr} X \to Y_k \to W_k' \to 0$ we see that $\dim \operatorname{Hom}(Y_k, S_j) = \dim \operatorname{Hom}(W_k', S_j) = 2 - \delta_{jk}$, where $j = m+1, \dots, n$, giving the desired property. $\square$

We can conclude a fortiori that for a reduced nullcone we have:

Corollary 5.1.6. *The set $\mathcal{H}(Q, \alpha) \cap \mathcal{Z}'(Q, \alpha)$ is contained in the smooth locus of $\mathcal{Z}(Q, \alpha)$, which in turn is contained in $\mathcal{H}(Q, \alpha)$.*

We also deduce the following result:

Theorem 5.1.7. *Let $T_1, \ldots, T_m$ be pairwise non-isomorphic indecomposables in* $\mathrm{Rep}(Q)$ *such that* $\mathrm{Ext}(T_i, T_j) = 0$*, for any* $i, j \leq m$*. Then there is a positive integer* N *such the nullcone* $\mathcal{Z}(Q, \alpha)$ *is reduced, irreducible and a complete intersection for any dimension vector* $\alpha = \lambda_1 \cdot \underline{d}(T_1) + \cdots + \lambda_m \cdot \underline{d}(T_m)$ *with* $\lambda_i \geq N$ *for* $i = 1, \ldots, m$*.*

Proof. If we pick N to be large enough, the set $\mathcal{Z}'(Q, \alpha)$ is not empty by [49, Corollary 3.4] or by [44, Proposition 4.7]. □

Remark 5.1.8. One can give the following short proof of the theorem above. If we allow N to be large enough so that $\alpha - \underline{d}(T^{++}_{m+1}) - \cdots - \underline{d}(T^{++}_n)$ is a dimension vector, then there is an element of the form $Z_{m+1} \oplus T_{m+1} \oplus \cdots \oplus Z_n \oplus T_n \oplus T'$ that lies in $\mathcal{H}(Q, \alpha) \cap \mathcal{Z}'(Q, \alpha)$, where $T' \in \mathrm{add}\, T$. Moreover, one can easily see that the independence condition (5.1.3 b) is also satisfied by using the just the sequences (5.2). However, this condition is cruder than the condition $\mathcal{Z}'(Q, \alpha) \neq \emptyset$.

Remark 5.1.9. Hence we can conclude that if α is not "too small", then the semi-invariants $c_{S_{m+1}}, \ldots, c_{S_n}$ form a regular sequence and generate a prime ideal in $k[\mathrm{Rep}(Q, \alpha)]$. In fact, these properties hold for an arbitrary field k (being geometrically reduced and irreducible). This is because the semi-invariants $c_{S_{m+1}}, \ldots, c_{S_n}$ are defined over any field k (not necessarily algebraically closed) by construction, since the representations S_i themselves are ($\dim S_k$ are real Schur roots), cf. [32, 62].

For finite/tame quivers, one can give more precise information on N. Bounds for a condition similar to $\mathcal{Z}'(Q, \alpha) \neq \emptyset$ have been investigated previously in [44]. So for Dynkin quivers N can be taken to be $N(Q) + 1$ as in (5.1). Also, for extended Dynkin quivers similar bounds have been announced in [45, Remark 6.7].

However, for Dynkin quivers we show by a different reasoning that for the nullcone to be reduced we only need $N = N(Q)$. We keep the usual notation.

Theorem 5.1.10. *Let Q we a Dynkin quiver and set $N(Q) = 1$, if Q is of type $\mathbb{A}$, and $N(Q) = 2$ otherwise. If $\lambda_i \geq N(Q)$ for all $i = 1, \dots, m$, then the nullcone $\mathcal{Z}(Q, \alpha)$ is reduced and a complete intersection.*

Proof. The bounds from (5.1) imply that the nullcone is a set-theoretic complete intersection. Hence it is enough to verify the independent gradient conditions (5.1.3). We are going to prove the result for arbitrary zero sets of semi-invariants $Z(c_{S_{i_1}}, \dots, c_{S_{i_k}})$, $i_j \in \{m+1, \dots, n\}$, for $j = 1, \dots, k$, by induction on the number of semi-invariants $k \leq n - m$. If $k = 0$ there is nothing to prove. For simplicity, we can assume $i_j = j$ and denote $f_j = c_{S_j}$, $j = 1, \dots, k$. Now take any irreducible component of $\mathcal{Z} := Z(f_1, \dots, f_k)$, which, since Q is of finite type, is the closure of the orbit of a representation, say X. Take $l = 1, \dots, k$ arbitrary and look at the zero-set $\mathcal{Z}_l = Z(f_1, \dots, f_{l-1}, f_{l+1}, \dots f_k)$. Since the zero-sets are complete intersections, $\dim \mathcal{Z}_l = \dim \mathcal{Z} + 1$. Hence there is an irreducible component of $\mathcal{Z}_l$ which is the closure of an orbit, say X_l, so that X is a minimal degeneration of X_l. By Lemma 2.3.1 we can write

$$X \cong A_l \oplus B_l \oplus Y_l \text{ and } X_l \cong C_l \oplus Y_l$$

such that A_l, B_l are indecomposables and there is an exact sequence $0 \to A_l \to C_l \to B_l \to 0$. By the induction hypothesis, X_l satisfies the independent gradient condition (5.1.3 a), hence $\dim \mathrm{Hom}(X_l, S_j) = \mathrm{Ext}(X_l, S_j) = 1 - \delta_{jl}$, for $j = 1, \dots, k$. This implies that $\mathrm{Hom}(Y_l, S_l) = \mathrm{Hom}(C_l, S_l) = \mathrm{Hom}(B_l, S_l) = \mathrm{Ext}(Y_l, S_l) = \mathrm{Ext}(C_l, S_l) = \mathrm{Ext}(A_l, S_l) = 0$, $\dim \mathrm{Hom}(X, S_l) = \dim \mathrm{Hom}(A_l, S_l) = \dim \mathrm{Ext}(B_l, S_l) > 0$.

Now assume that X does not satisfy the first independent gradient condition (5.1.3 a), hence we assume WLOG that $\dim \mathrm{Hom}(X, S_1) > 1$. Then $\dim \mathrm{Hom}(A_1, S_1) = \mathrm{Ext}(B_1, S_1) > 1$, hence A_1 and B_1 are not direct summands of X_l or Y_l, for any $l = 1, \dots, k$. Suppose $A_1 \cong B_l$, for some $l > 1$. Then the sequence $0 \to A_l \to C_l \to B_l \to 0$ gives $\dim \mathrm{Hom}(C_l, S_1) = \dim \mathrm{Hom}(A_1, S_1) > 1$, which again is a contradiction. Hence $A_l \cong A_1$, $B_l \cong B_1$, hence also $Y_l \cong Y_1$ for all $l = 1, \dots, k$. So put $A := A_1$, $B := B_1$, $Y = Y_1$. Summarizing, X must be of the form

$$X \cong A \oplus B \oplus Y,$$

with A, B indecomposables together with exact sequences $0 \to A \to C_l \to B \to 0$, for $l = 1, \dots, k$.

Now we also see that $\mathrm{Ext}(A, Y) = \mathrm{Ext}(Y, A) = \mathrm{Ext}(B, Y) = \mathrm{Ext}(Y, B) = \mathrm{Ext}(Y, Y) = \mathrm{Ext}(A, B) = 0$. Indeed, suppose, for example, that we have a non-trivial exact sequence $0 \to A \to U \to Y \to 0$. Then X is a degeneration of $U \oplus B$, hence $U \oplus B$ is not an element in $\mathcal{Z}$. But $\dim \mathrm{Ext}(U \oplus B, S_j) = \dim \mathrm{Ext}(B, S_j) > 0$, for any $j = 1, \dots, k$, which implies $U \oplus B \in \mathcal{Z}$, a contradiction. This proves $\mathrm{Ext}(Y, A) = 0$, and the other claims are analogous. Summarizing, we have $k = \dim \mathrm{Ext}(X, X) = \dim \mathrm{Ext}(B, A) > 0$.

Next, we claim that $Y \in \mathrm{add}(T)$. Since Q is Dynkin, using that an indecomposable has no self-extensions, we see from Lemma 2.3.1 that we can reach T from X by a sequence of minimal degenerations given by short exact sequences. However, since $\mathrm{Ext}(A, Y) = \mathrm{Ext}(Y, A) = \mathrm{Ext}(B, Y) = \mathrm{Ext}(Y, B) = \mathrm{Ext}(Y, Y) = 0$, Y is going to remain fixed in this sequence of short exact sequences. Hence when we reach T, we get that Y is a direct summand of T, so we can write $Y = T_1^{\beta_1} \oplus \cdots \oplus T_m^{\beta_m}$, with $0 \le \beta_i \le \lambda_i$, for $i = 1, \dots, m$. Also $\underline{d}(A) + \underline{d}(B) = (\lambda_1 - \beta_1)\underline{d}(T_1) + \cdots + (\lambda_m - \beta_m)\underline{d}(T_m)$.

If Q is of type $\mathbb{A}$, the dimension of the space of maps between any two indecomposables is at most 1, contradicting $\dim \mathrm{Hom}(A, S_1) > 1$. Hence we may assume $N(Q) = 2$. Then we claim that $\beta_i \ge 1$, for all $i = 1, \dots, m$. Assume the contrary, say $\beta_1 = 0$. Then $\langle \underline{d}(A) + \underline{d}(B), \underline{d}(A) + \underline{d}(B) \rangle \ge \lambda_1^2 \dim \mathrm{Hom}(T_1, T_1) \ge 4$. On the other hand, we also have $\langle \underline{d}(A) + \underline{d}(B), \underline{d}(A) + \underline{d}(B) \rangle = 2 + \langle \underline{d}(A), \underline{d}(B) \rangle + \langle \underline{d}(B), \underline{d}(A) \rangle$, hence the inequality $\langle \underline{d}(A), \underline{d}(B) \rangle + \langle \underline{d}(B), \underline{d}(A) \rangle \ge 2$. We claim that this is impossible for Dynkin quivers. Indeed, we can use reflection functors (see [3]) to reduce to the case when $A \cong S^x$ is a simple corresponding to a vertex x. Then B is not isomorphic to the simple S^x, and the value becomes $\langle \underline{d}(S^x), \underline{d}(B)\rangle + \langle \underline{d}(B), \underline{d}(S^x)\rangle = 2\dim B(x) - \dim B(y_1) - \cdots - \dim B(y_i)$, where y_i are all the neighbors of x. But for Dynkin quivers this value is known to be smaller than 2 (see [52, (4) on Page 4]). Hence $\beta_i \geq 1$, for all $i = 1, \ldots, m$. Since $\mathrm{Ext}(A, Y) = \mathrm{Ext}(Y, A) = \mathrm{Ext}(B, Y) = \mathrm{Ext}(Y, B) = 0$, this implies that $X \in \mathcal{Z}'(Q, \alpha)$, which is a contradiction, by Proposition 5.1.4. Hence X satisfies the first independent gradient condition (5.1.3 a), that is, $\dim(X, S_j) = 1$, for $j = 1, \ldots, k$.

Now we show that X satisfies the independence condition (5.1.3 b). Take the exact sequences $0 \to A_l \to C_l \to B_l \to 0$ as before. These induce sequences

$$0 \to X \to X \oplus X_l \to X \to 0$$

that satisfy the second indenpendent gradient condition, since $\dim \mathrm{Hom}(X \oplus X_l, S_j) = 2 - \delta_{jl}$, where $j = 1, \ldots, k$. Hence each irreducible component of $\mathcal{Z}$ is reduced, finishing the inductive step. $\square$

Remark 5.1.11. We note that for type $\mathbb{A}$ quivers the above result also follows from [37], as the fact that (in characteristic 0) the nullcone has rational singularities (see Section 5.2).

The following example shows that the nullcone of a Dynkin quiver is not always reduced.

Example 5.1.12. Let Q be $\mathbb{E}_8$ with the following orientation and dimension vector:

$$\begin{tikzcd}
 & & 3 \arrow[d] & & & & \\
2 \arrow[r] & 4 \arrow[r] & 7 & 4 \arrow[l] & 3 \arrow[l] & 2 \arrow[l] & 1 \arrow[l]
\end{tikzcd}$$

The decomposition of the generic representation is as follows:

$$\begin{smallmatrix} & & 3 & & & & \\ 2&4&7&4&3&2&1 \end{smallmatrix} = \begin{smallmatrix} & & 1 & & & & \\ 0&1&2&1&1&1&1 \end{smallmatrix} \oplus \begin{smallmatrix} & & 1 & & & & \\ 1&2&3&2&1&1&0 \end{smallmatrix} \oplus \begin{smallmatrix} & & 1 & & & & \\ 1&1&2&1&1&0&0 \end{smallmatrix}.$$

The simples in the right orthogonal category are the following:

$$\begin{smallmatrix} & & 0 & & & & \\ 0&0&1&1&1&1&1 \end{smallmatrix}, \begin{smallmatrix} & & 1 & & & & \\ 0&1&2&1&1&1&0 \end{smallmatrix}, \begin{smallmatrix} & & 0 & & & & \\ 1&1&1&0&0&0&0 \end{smallmatrix}, \begin{smallmatrix} & & 1 & & & & \\ 0&0&1&1&0&0&0 \end{smallmatrix}, \begin{smallmatrix} & & 0 & & & & \\ 0&1&1&1&1&0&0 \end{smallmatrix}.$$

A routine computation shows that the nullcone consists of 9 irreducible components, each the closure of one of the following representations:

$$N_1 = \begin{smallmatrix} & & 0 & & & & \\ 0&0&1&0&0&0&0 \end{smallmatrix} \oplus \begin{smallmatrix} & & 3 & & & & \\ 2&4&6&4&3&2&1 \end{smallmatrix}$$

$$N_2 = \begin{smallmatrix} & & 1 & & & & \\ 0&0&1&1&0&0&0 \end{smallmatrix} \oplus \begin{smallmatrix} & & 0 & & & & \\ 0&0&1&1&1&0&0 \end{smallmatrix} \oplus \begin{smallmatrix} & & 1 & & & & \\ 0&1&1&0&0&0&0 \end{smallmatrix} \oplus \begin{smallmatrix} & & 0 & & & & \\ 1&1&1&0&0&0&0 \end{smallmatrix} \oplus \begin{smallmatrix} & & 1 & & & & \\ 1&2&3&2&2&2&1 \end{smallmatrix}$$

$$N_3 = \begin{smallmatrix} & & 0 & & & & \\ 1&1&1&0&0&0&0 \end{smallmatrix} \oplus \begin{smallmatrix} & & 0 & & & & \\ 0&0&1&1&1&1&1 \end{smallmatrix} \oplus \begin{smallmatrix} & & 1 & & & & \\ 0&1&2&1&0&0&0 \end{smallmatrix} \oplus \begin{smallmatrix} & & 2 & & & & \\ 1&2&3&2&2&1&0 \end{smallmatrix}$$

$$N_4 = \begin{smallmatrix} & & 1 & & & & \\ 0&0&1&1&0&0&0 \end{smallmatrix} \oplus \begin{smallmatrix} & & 0 & & & & \\ 0&0&1&1&1&1&1 \end{smallmatrix} \oplus \begin{smallmatrix} & & 0 & & & & \\ 0&1&1&0&0&0&0 \end{smallmatrix} \oplus \begin{smallmatrix} & & 1 & & & & \\ 1&1&2&1&1&1&0 \end{smallmatrix} \oplus \begin{smallmatrix} & & 1 & & & & \\ 1&2&2&1&1&0&0 \end{smallmatrix}$$

$$N_5 = \begin{smallmatrix} & & 1 & & & & \\ 0&0&1&0&0&0&0 \end{smallmatrix} \oplus \begin{smallmatrix} & & 0 & & & & \\ 0&0&1&1&1&1&1 \end{smallmatrix} \oplus \begin{smallmatrix} & & 0 & & & & \\ 0&1&1&1&1&0&0 \end{smallmatrix} \oplus \begin{smallmatrix} & & 0 & & & & \\ 1&1&1&0&0&0&0 \end{smallmatrix} \oplus \begin{smallmatrix} & & 2 & & & & \\ 1&2&3&2&1&1&0 \end{smallmatrix}$$

$$N_6 = \begin{smallmatrix} & & 1 & & & & \\ 0&0&1&1&0&0&0 \end{smallmatrix} \oplus \begin{smallmatrix} & & 0 & & & & \\ 0&0&1&0&0&0&0 \end{smallmatrix} \oplus \begin{smallmatrix} & & 0 & & & & \\ 0&0&1&1&1&1&0 \end{smallmatrix} \oplus \begin{smallmatrix} & & 1 & & & & \\ 0&1&1&0&0&0&0 \end{smallmatrix} \oplus \begin{smallmatrix} & & 0 & & & & \\ 1&1&1&0&0&0&0 \end{smallmatrix} \oplus$$

$$\begin{smallmatrix} & & 1 & & & & \\ 1&1&2&1&1&1&1 \end{smallmatrix}$$

$$N_7 = \begin{smallmatrix} & & 1 & & & & \\ 0&0&1&1&0&0&0 \end{smallmatrix} \oplus \begin{smallmatrix} & & 0 & & & & \\ 0&0&1&1&1&1&1 \end{smallmatrix} \oplus \begin{smallmatrix} & & 0 & & & & \\ 0&1&1&1&1&0&0 \end{smallmatrix} \oplus \begin{smallmatrix} & & 1 & & & & \\ 0&1&2&1&1&1&0 \end{smallmatrix} \oplus \begin{smallmatrix} & & 0 & & & & \\ 1&1&1&0&0&0&0 \end{smallmatrix} \oplus$$

$$\begin{smallmatrix} & & 1 & & & & \\ 1&1&1&0&0&0&0 \end{smallmatrix}$$

$$N_8 = \begin{array}{c} 0 \\ 1\,1\,1\,0\,0\,0\,0 \end{array} \oplus \begin{array}{c} 0 \\ 0\,0\,1\,1\,0\,0\,0 \end{array} \oplus \begin{array}{c} 2 \\ 1\,2\,4\,3\,3\,2\,1 \end{array} \oplus \begin{array}{c} 1 \\ 0\,1\,1\,0\,0\,0\,0 \end{array}$$

$$N_9 = \begin{array}{c} 1 \\ 0\,0\,1\,1\,0\,0\,0 \end{array} \oplus \begin{array}{c} 0 \\ 0\,0\,1\,1\,1\,1\,1 \end{array} \oplus \begin{array}{c} 1 \\ 0\,1\,2\,1\,1\,0\,0 \end{array} \oplus \begin{array}{c} 0 \\ 1\,1\,1\,0\,0\,0\,0 \end{array} \oplus \begin{array}{c} 1 \\ 1\,2\,2\,1\,1\,1\,0 \end{array}$$

If we look at the first component N_1, we see that indeed the nullcone is not reduced, since

$$\dim \operatorname{Hom}\left(\begin{array}{c} 0 \\ 0\,0\,1\,0\,0\,0\,0 \end{array}, \begin{array}{c} 1 \\ 0\,1\,2\,1\,1\,1\,0 \end{array} \right) = 2,$$

contradicting the independent gradient condition (5.1.3 a). We also note that the nullcone is a set-theoretic complete intersection, since the codimension of each component is $\dim \operatorname{Ext}(N_i, N_i) = 5$, for all $i = 1, \ldots, 9$.

5.2 Bernstein-Sato polynomials and rational singularities of zero sets

In this section, we will work over the complex field $k = \mathbb{C}$. Let G be a connected reductive group acting rationally on a prehomogeneous vector space V, $k[V]$ the polynomial ring and $\mathcal{D}_V$ the ring of differential operators on V.

Consider a tuple of semi-invariants $\underline{f} = f_1, \ldots, f_r$ in $k[V]$. Denote by $\underline{e}^1, \ldots, \underline{e}^r$ the standard basis for $\mathbb{Z}^r$, and put $\underline{e} = \underline{e}^1 + \cdots + \underline{e}^r$. Let $s_1, \ldots, s_r$ be independent variables, and put $s = s_1 + \cdots + s_r$.

The first result of this section is a link between b-function of several variables $b_{\underline{m}}(\underline{s})$ (see (1.4)) and $b_{\underline{f}}(s)$ (Definition 1.3.1), beyond the case $r = 1$ when the two notions coincide. For $\hat{c} \in \mathbb{Z}^r$ with $|\hat{c}| = 1$, define $\hat{c}^+ \in \mathbb{N}^r$ by $\hat{c}_i^+ := \max\{c_i, 0\}$ and $\hat{c}^- := \hat{c} - \hat{c}^+$, and denote by $b_{\hat{c}} \in k[s_1, \ldots, s_r]$ the polynomial

$$b_{\hat{c}} := b_{\hat{c}^+}(\underline{s} + \hat{c}^-) \cdot \hat{c}_{\underline{s}}.$$

Let $\tilde{B}$ be the ideal generated by polynomials $b_{\hat{c}}$, where $\hat{c}$ runs over the elements in $\mathbb{Z}^r$ with $|\hat{c}| = \underline{e} \cdot \hat{c} = 1$.

Proposition 5.2.1. *There exists a polynomial in $\tilde{B}$ depending only in $s = s_1 + \cdots + s_r$; let $\tilde{b}(s)$ be such of lowest degree. We have $b_{\underline{f}}(s) \mid \tilde{b}(s)$.*

Proof. The second part of the proposition follows from

$$\prod_{i,c_i<0)^-}^{r} f_i^{-c_i} \prod_{i,c_i>0)^+}^{r} f_i^{*c_i} \cdot \hat{c}_{\underline{s}} \prod_{i=1}^{r} f_i^{s_i+c_i} = b_{\hat{c}} \cdot \prod_{i=1}^{r} f_i^{s_i}.$$

Put $L = \{\gamma^1, \dots, \gamma^N\} \cup \{\underline{e}^1, \dots, \underline{e}^r\}$. Choose arbitrary elements $\underline{l}^1, \dots, \underline{l}^k \in L$ and $u_1, \dots, u_k \in \mathbb{Q}$ and take the ideal $I = (\underline{l}^1 \cdot \underline{s} + u_1, \dots, \underline{l}^k \cdot \underline{s} + u_k)$, and assume I is a proper ideal. Since $\tilde{B}$ is finitely generated, in order to prove the first part of the proposition, it is enough to show that if $\tilde{B} \subset I$, then $\underline{e} \in \operatorname{span}_{\mathbb{Q}}\{\underline{l}^1, \dots, \underline{l}^k\}$. WLOG, we can assume that for any $\underline{l} \in L$ with $\underline{l} \in \operatorname{span}_{\mathbb{Q}}\{\underline{l}^1, \dots, \underline{l}^k\}$ we have $\underline{l} \in \{\underline{l}^1, \dots, \underline{l}^k\}$. Arguing by induction on r, we can further assume WLOG that there are no basis elements $\underline{e}^i$ among the vectors $\underline{l}^1, \dots, \underline{l}^k$. Then, we show that in fact $\underline{e} \in \operatorname{span}_{\mathbb{Q}_{\geq 0}}\{\underline{l}^1, \dots, \underline{l}^k\}$. Assuming the contrary, there exists by Farkas' lemma a vector $\hat{c} \in \mathbb{Z}^r$ such that $\underline{e} \cdot \hat{c} > 0$, and $\underline{l}^i \cdot \hat{c} < 0$, for all $i = 1, \dots, k$. Since $\underline{l}^i \in \mathbb{N}^r$, we can, by possibly scaling and decreasing the entries of $\hat{c}$, find a vector $\hat{c} \in \mathbb{Z}^r$ with $\underline{e} \cdot \hat{c} = 1$ and $\max_i\{\underline{l}^i \cdot \hat{c}\}$ arbitrary small. Hence, looking at largest constant terms of the factors in (1.5) we can find $\hat{c}$ such that none of the forms $\underline{l}^1 \cdot \underline{s} + u_1, \dots, \underline{l}^k \cdot \underline{s} + u_k$ is a factor of $b_{\hat{c}}$, which gives $b_{\hat{c}} \notin I$, a contradiction. $\square$

Remark 5.2.2. As we mentioned in Chapter 1, the b-function of several variables has been generalized to the case of arbitrary (not necessarily semi-invariant) polynomials. Proposition 5.2.1 can be adapted to this setting as well. In particular, this gives another proof for the existence of $b_{\underline{f}}(s)$ and rationality of its roots (see [14]). However, in general $\tilde{b}(s) \neq b_{\underline{f}}(s)$, moreover $\tilde{b}(s)$ may have positive roots (see Example 5.2.13).

As before, let Q be a quiver with n vertices, α a prehomogeneous dimension vector with generic representation $T = T_1^{\oplus \lambda_1} \oplus \cdots \oplus T_m^{\oplus \lambda_m}$. Let $r = n - m$, and

$S_{m+1}, \ldots, S_n$ the simple objects in $T^{\perp}$ with dimension vectors $\beta^1, \ldots, \beta^r$, respectively. For a fixed tuple $\underline{m} \in \mathbb{N}^r$, denote by b_α the b-function of several variables $b_{\underline{m}}(\underline{s})$ of the semi-invariants $c_{S_{m+1}}, \ldots, c_{S_n}$. Let c be the Coxeter transformation. We assume throughout that $c(\alpha) \in \mathbb{N}^n$. In this setting Theorem 3.2.1 gives

$$b_\alpha = b_{c(\alpha)} \prod_{x \in Q_0} \frac{[s]^{\beta^1_x, \ldots, \beta^r_x}_{\alpha_x}}{[s]^{\beta^1_x, \ldots, \beta^r_x}_{c(\alpha)_x}}. \tag{5.4}$$

Put $S := S_i$ for some $i \in \{m+1, \ldots, n\}$ and β its dimension vector. Following [48], we call α an *S-stable* dimension vector if T_i^{++} is a direct summand of T (see (5.2)). By [48], in such case the zero set $Z(c^S)$ is the closure of a codimension 1 orbit. Let τ be the Auslander-Reiten translation (see Section 2.2).

Proposition 5.2.3. *Assume α is S-stable and $c(\alpha)$ is τS-stable (assume S is not projective). Then $Z(c^S)$ has rational singularities iff $Z(c^{\tau S})$ has rational singularities.*

Proof. By (5.4) and Theorem 1.3.2, we see that it is enough to show that $\alpha \geq \beta$ and $c(\alpha) \geq \beta$. Since α is S-stable, we have the composite of injective maps $T_i \hookrightarrow T_i^{++} \hookrightarrow T$ and surjective maps $T_i \twoheadrightarrow \overline{T}_i \twoheadrightarrow S_i$, we get $\alpha \geq \beta$. Since $S \in {}^{\perp}(\tau T)$ and $c(\alpha)$ is τS-stable we get dually that $c(\alpha) \geq \beta$. $\square$

Note that if S is projective, we reduce as in Theorem 3.2.2 part (b) using reflection functors to the case when S is simple, and we obtain the root -1 of the b-function. Hence

Theorem 5.2.4. *If Q is a Dynkin quiver, then all codimension 1 orbits in $\mathrm{Rep}(Q, \alpha)$ have rational singularities.*

Proof. By [48], for Dynkin quivers all dimension vectors are stable. $\square$

Remark 5.2.5. As mentioned before, it is known that all orbit closures of Dynkin quiver of type $\mathbb{A}$ and $\mathbb{D}$ have rational singularities. For type $\mathbb{A}$ quivers, the roots of the b-functions corresponding to codimension 1 orbit closures are all integers (see the computations in [66] or Chapters 3, 4), while for type $\mathbb{D}$ they can be half-integers (Chapters 3, 4). We know that the global b-function is the same as the local b-function at 0, by Corollary 1.2.2. By [43, Theorem 1.4] orbit closures in these cases are hypersurfaces if and only if they are local hypersurfaces at 0, and this is a property preserved by smooth morphisms. We conclude that the types of singularities of orbit closures of type $\mathbb{A}$ and those of type $\mathbb{D}$ are not equivalent.

Theorem 5.2.6. *Let Q be an extended Dynkin quiver, $T_1, \dots, T_m$ be pairwise non-isomorphic indecomposables in* $\mathrm{Rep}(Q)$ *such that* $\mathrm{Ext}(T_i, T_j) = 0$*, for any* $i, j \leq m$*. Then there is a positive integer N such that all codimension* 1 *orbits in* $\mathrm{Rep}(Q, \alpha)$ *have rational singularities, for any dimension vector* $\alpha = \lambda_1 \cdot \underline{d}(T_1) + \cdots + \lambda_m \cdot \underline{d}(T_m)$ *with $\lambda_i \geq N$ for $i = 1, \dots, m$.*

Proof. By Theorem 3.2.5 we can compute the b-function using (5.4) in a finite number of steps, and for large enough N the dimension vectors are stable. $\square$

Based on this, we make the following

Conjecture 1. *Corollary 5.2.6 is true for any quiver Q.*

Now we illustrate how Proposition 5.2.1 can be used to determine the property of having rational singularities for nullcones. Assume the b-function of several variables $b(\underline{s})$ (we suppress the index $\underline{m}$) is of the form

$$b(\underline{s}) = \prod_{i=1}^{r} [s]_{d_i}^{\underline{e}^i} \prod_{j=1}^{N} [s]_{a_j, b_j}^{\gamma^j}, \tag{5.5}$$

where $d_i \in \mathbb{N}$ and for all $j = 1, \dots, N$ we have $a_j, b_j \in \mathbb{N}, \gamma^j \in \mathbb{N}^r$ with $\underline{e} \cdot \gamma^j \leq a_j$.

Remark 5.2.7. For example, if Q is Dynkin and the multiplicities $\lambda_i \geq N(Q) + 1$ for $i = 1, \dots, m$, then the b-function of several variables always looks as above (5.5). The only thing left to see is that $\underline{e} \cdot \gamma^j \leq a_j$, for every $j = 1, \dots, N$. This follows from the formula (5.4), since the multiplicity condition implies $\mathcal{Z}'(Q,\alpha)) \neq \emptyset$ by [44], while this in turn implies that $\dim\alpha \geq \beta_1 + \cdots + \beta_r$ by [49, Lemma 3.6].

Generalizing the case of 1 semi-invariant, we conjecture that the condition $\underline{e}\cdot\gamma^j \leq a_j$ on b-functions of several variables of semi-invariants of more general prehomogeneous vector spaces implies rational singularities of their zero sets.

Definition 5.2.8. We say an element $z \in Z(\tilde{B})$ is *good*, if either $z = -\underline{e}$ or $\underline{e}\cdot z < -r$.

Proposition 5.2.9. *Suppose $Z = Z(f_1,\ldots,f_r)$ is a reduced complete intersection and all elements in $Z(\tilde{B})$ are good. Then Z has rational singularities.*

Proof. Localizing $\tilde{B}$ at $z = -\underline{e}$ and looking at the elements $b_{\underline{e}^i}$ for $i = 1,\ldots,r$, we see that the conditions $\underline{e}\cdot\gamma^j \leq a_j$ imply that $-r$ is a root of $\tilde{b}(s)$ with multiplicity 1. Hence the largest root of the polynomial $\tilde{b}(s)/(s+r)$ is smaller than $-r$, so we conclude by combining Theorem 1.3.2 and Proposition 5.2.1. $\square$

Put $L := \{\gamma^1,\ldots,\gamma^N\}\cup\{\underline{e}^1,\ldots,\underline{e}^r\}$. Set $\Gamma = L\backslash\{\underline{e}^1,\ldots,\underline{e}^r\}$, and for each $i\in\{1,\ldots,r\}$, let $\Gamma_i := \{j\in\{1,\ldots,N\} : \gamma^j_i > 0, \gamma^j\in\Gamma\}$.

Lemma 5.2.10. *Using the notation above, we have the following reduction techniques:*

(a) Assume there exists $I = \{i_1,\ldots,i_k\}\subset\{1,\ldots,r\}$ with the property that for any $(j_1,\ldots,j_k)\in\Gamma_{i_1}\times\cdots\times\Gamma_{i_k}$ there exists $(u_1,\ldots,u_k)\in\mathbb{Q}_+^k$ such that $\displaystyle\sum_{i\in\{j_1,\ldots,j_k\}} u_i\gamma^{j_i} = \underline{e}$ and $\displaystyle\sum_{i\in\{j_1,\ldots,j_k\}} u_i(a_i+1) > r$. Assume further that for any $i\in I$, all $z\in Z(\tilde{B})$ with $z_i\leq -1$ are good. Then all the elements in $Z(\tilde{B})$ are good.

(b) Assume there exists $J = \{j_1,\ldots,j_k\}\subset\{1,\ldots,r\}$ such that $\underline{e}\notin E_J := \operatorname{span}_{\mathbb{Q}_+}\Gamma + \operatorname{span}_{\mathbb{Q}}\{\underline{e}^{j_1},\ldots,\underline{e}^{j_k}\}$, and suppose J is maximal with this property. Write $J^c = J_+\cup J_-$ with $J_+ = \{i\in J^c : \underline{e}\in E_J + \mathbb{Q}_+\cdot\underline{e}^i\}$ and $J_- = \{i \in$

$J^c : \underline{e} \in E_J + \mathbb{Q}_- \cdot \underline{e}^i\}$. *Then for any* $z \in Z(\tilde{B})$, *there is an element* $i \in J^c$ *such that if* $i \in J_+$ *then* $z_i \in \mathbb{Z}_{<0}$, *and if* $i \in J_-$ *then* $z_i \in \mathbb{N}$.

Proof. (a) Take an element $z \in Z(\tilde{B})$ such that $z_i > -1$, for all $i \in I$. We see that for $j = 1, \ldots, k$ the linear factors of $b_{\underline{e}^{i_j}}(\underline{s})$ involve the vectors from $\{\underline{e}^{i_j}\} \cup \Gamma_{i_j}$. Hence z is a root of a linear factor of $b_{\underline{e}^{i_j}}(\underline{s})$ involving a vector from Γ_{i_j}, for each $j = 1, \ldots, k$. But then the condition $\sum_{i \in \{j_1, \ldots, j_k\}} u_i(a_i+1) > r$ implies that $\underline{e} \cdot z < -r$, so z is good.

(b) Take an element $z \in Z(\tilde{B})$. As in the proof of Proposition 5.2.1, there exists $\underline{c} \in \mathbb{Z}^r$ such that $\underline{e} \cdot \underline{c} = 1$, $\underline{c}_{j_i} = 0$, for $i = 1, \ldots, k$, and $\max_{\gamma \in \Gamma}\{\gamma \cdot \underline{c}\}$ is arbitrary small. It is immediate that if $i \in J_+$, then $\underline{c}_i > 0$, and if $i \in J_-$, then $\underline{c}_i < 0$. Since z is a root of $b_{\underline{c}}$, it follows that there is an element $i \in J^c$ such that z is the root of a form involving a term $\underline{e}^i$, hence the conclusion. $\square$

Example 5.2.11. Take the following E_6 quiver and dimension vector:

$$\begin{array}{ccccccccc} & & & & n+m & & & & \\ & & & & \downarrow & & & & \\ n & \longrightarrow & 2n+m & \longrightarrow & 2n+m & \longleftarrow & 2n+m & \longleftarrow & n \end{array}$$

The generic representation is $T = T_1^{\oplus m} \oplus T_2^{\oplus n}$, where $T_1 = \begin{smallmatrix} & & 1 & & \\ 0 & 1 & 1 & 1 & 0 \end{smallmatrix}$, $T_2 = \begin{smallmatrix} & & 1 & & \\ 1 & 2 & 2 & 2 & 1 \end{smallmatrix}$. The simples in the right perpendicular category $T^{\perp}$ are the indecomposables with dimension vectors:

$$\begin{smallmatrix} & & 1 & & \\ 1 & 1 & 1 & 0 & 0 \end{smallmatrix}, \begin{smallmatrix} & & 1 & & \\ 0 & 0 & 1 & 1 & 1 \end{smallmatrix}, \begin{smallmatrix} & & 0 & & \\ 0 & 1 & 1 & 1 & 1 \end{smallmatrix}, \begin{smallmatrix} & & 0 & & \\ 1 & 1 & 1 & 1 & 0 \end{smallmatrix}.$$

By Theorem 5.1.10, if $n, m \geq 2$ the nullcone is reduced and a complete intersection. Using (5.4) repeatedly we obtain the b-function of several variables:

$$b(\underline{s}) = [s]^{1000}_{n+m} \cdot [s]^{0100}_{n+m} \cdot [s]^{0010}_{n} \cdot [s]^{0001}_{n} \cdot [s]^{0011}_{n,2n+m} \cdot [s]^{0110}_{n+m,2n+m} \cdot [s]^{1001}_{n+m,2n+m}.$$

We want to show that for $n, m \geq 2$, each element in $z \in Z(\tilde{B})$ is good. We see that Lemma 5.2.10 (a) applies with $I = \{1, 2\}$ and we have $(n+m+1)+(n+m+1) > 4$. Hence we can assume one of the 2 possibilities: $z_1 \leq -1$ or $z_2 \leq -1$. We consider the first case (the latter is analogous), put $z_1 = -k_1$, with $k_1 \geq 1$, (moreover, we can assume $k_1 \leq d_1 = n + m$). Then put $z' = (z_2, z_3, z_4)$ and evaluating $s_1 = -k_1$ we consider the b-function of several variables

$$b_1 = [s]^{100}_{n+m} \cdot [s]^{010}_{n} \cdot [s]^{001}_{n} \cdot [s]^{011}_{n,2n+m} \cdot [s]^{110}_{n+m,2n+m} \cdot [s]^{001}_{n+m-k_1,2n+m-k_1}.$$

If $\tilde{B}_1$ is the ideal associated to b_1, then $z' \in Z(\tilde{B}_1)$. Now we can apply Lemma 5.2.10 (b) with $J = \emptyset$. Then $J_+ = \{1, 3\}$ and $J_- = \{2\}$. Hence we have another 3 possibilities. Assume first that $z'_3 = -k_2$, with $k_2 \geq 1$ (the case of $1 \in J_+$ is analogous). This leads to

$$b_2 = [s]^{10}_{n+m} \cdot [s]^{01}_{n} \cdot [s]^{01}_{n-k_2,2n+m-k_2} \cdot [s]^{11}_{n+m,2n+m}.$$

Let $z'' = (z'_1, z'_2)$. Again, Lemma 5.2.10 (a) can be applied with $I = \{1\}$, since we have $n + m + 1 > 2$. Hence we can assume $z''_1 = -k_3$, $1 \leq k_3 \leq n + m$. Then we are left with $b_3 = [s]^{1}_{n} \cdot [s]^{1}_{n-k_2,2n+m-k_2} \cdot [s]^{1}_{n+m-k_3,2n+m-k_3}$. Hence for the last choice $k_4 \geq \min\{1, n - k_2 + 1\}$. Hence $\underline{e} \cdot z = -k_1 - k_2 - k_3 - k_4 \leq -4$, with equality only for $z = -\underline{e}$, hence z is good.

Now we return to b_1 and we are left with the case $2 \in J_-$, so put $z'_2 = k_2$, with $k_2 \geq 0$. Due to the previous discussion we can assume that $z'_1 > -1$ and $z'_3 > -1$. Hence we consider

$$b'_2 = [s]^{10}_{n+m} \cdot [s]^{01}_{n} \cdot [s]^{01}_{n+k_2,2n+m+k_2} \cdot [s]^{10}_{n+m+k_2,2n+m+k_2} \cdot [s]^{01}_{n+m-k_1,2n+m-k_1}.$$

Put $z'' = (z'_1, z'_3)$. Since $z''_1 > -1$, we conclude that z'' is not a root of $(b'_2)_{\underline{e}^1}$, finishing this last case.

In conclusion, if $n, m \geq 2$ the nullcone is a reduced complete intersection with rational singularities. Note that if $n, m \geq 3$, then it is the closure of an orbit.

Next, we state a result for tree quivers:

Theorem 5.2.12. *Let Q be a tree quiver, α a dimension vector, and c_{W_1}, c_{W_2} two irreducible semi-invariants on* $\operatorname{Rep}(Q, \alpha)$ *with $\underline{d}(W_i)_x \leq 1$, for all $x \in Q_0$. Then the hypersurface $Z(c_{W_i})$ has rational singularities. Moreover, if either α or $\underline{d}(W_1)+\underline{d}(W_2)$ is prehomogeneous,, then for large enough N the zero-set $Z(c_{W_1}, c_{W_2}) \subset$* $\operatorname{Rep}(Q, N \cdot \alpha)$ *has rational singularities.*

Proof. The first part follows from computation of the b-function by Theorem 4.3.7 and the fact that c_{W_i} are irreducible.

For the second part, For the second part, the assumptions imply multiplicity-freeness hence we can consider the b-function of 2 variables $b_{\underline{m}}(\underline{s})$. Specializing $b_{(1,0)}(s_1, 0)$ and $b_{(0,1)}(0, s_2)$, we see again by Theorem 4.3.7 that the set L of all linear forms appearing in $b_{\underline{m}}(\underline{s})$ are from $L \subset \{(1,0), (0,1), (1,1)\}$ and that the constants a_i, b_i from (5.5) increase linearly in N. Hence we can apply Lemma 5.2.10 (a) with $I = \{1\}$, say, and for large enough N we obtain that all elements in $Z(\tilde{B})$ are good. $\square$

Example 5.2.13. Here we give an example where Theorem 5.2.12 is not applicable and show that $\tilde{b}(s)$ can have positive roots already for 2 semi-invariants. We continue Example 5.1.12 with the dimension vector

$$\begin{tikzcd} & & 3n \arrow[d] & & & & \\ 2n \arrow[r] & 4n \arrow[r] & 7n & 4n \arrow[l] & 3n \arrow[l] & 2n \arrow[l] & n \arrow[l] \end{tikzcd}$$

Let S_1 and S_2 be the indecomposables $\begin{smallmatrix} & & 0 & & & & \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{smallmatrix}$, $\begin{smallmatrix} & & 1 & & & & \\ 0 & 1 & 2 & 1 & 1 & 1 & 0 \end{smallmatrix}$, respectively. The b-function of 2 variables of c_{S_1}, c_{S_2} is

$$b(\underline{s}) = [s]^{01}_{4n} \cdot ([s]^{01}_{n,3n})^2 \cdot [s]^{01}_{2n,4n} \cdot [s]^{10}_{n} \cdot [s]^{11}_{n,4n} \cdot [s]^{12}_{4n,7n}$$

It is easy to see that for any $\underline{c} \in \mathbb{Z}^2$ with $c_1 + c_2 = 1$, we have that either $s_1 + 2s_2 + 4n + 1$ or $s_1 + s_2 - 2n$ is a factor of $b_{\underline{c}}(\underline{s})$. Hence $(8n+1, -6n-1) \in Z(\tilde{B})$ and $2n$ is a root of $\tilde{b}(s)$.

Remark 5.2.14. We point out that most of the results in this section are valid over an arbitrary field k of characteristic 0. Consider the zero set $Z = Z(f_1, \dots, f_r)$ of some fundamental semi-invariants of a quiver Q with a prehomogeneous dimension vector α. Assume for simplicity that Z is reduced, irreducible with rational singularities over $\mathbb{C}$ (hence by Remark 5.1.9 Z is irreducible, reduced over k). We claim that Z has rational singularities also over k. One way we can see this as follows. Since fundamental semi-invariants have linearly independent weights, there exists a reductive group G, with $\mathrm{SL}(\alpha) \subset G \subset \mathrm{GL}(\alpha)$, such that Z is the nullcone for the action of G on $\mathrm{Rep}(Q, \alpha)$. By [28], there exists a desingularization $Z' \to Z$, where Z' is the total space of a vector bundle on a flag variety. We note that $Z' \to Z$ is defined already over $\mathbb{Q}$. Moreover, in this situation the property of rational singularities is equivalent to the vanishing of the cohomology of certain vector bundles on this flag variety (see [72]). This is independent of the field, as long as it is of characteristic 0.

Chapter 6

Bernstein-Sato polynomials for maximal minors and sub-maximal Pfaffians

In [12] and [13], N. Budur posed as a challenge and reviewed the progress on the problem of computing the b-function of the ideal of $p \times p$ minors of the generic $m \times n$ matrix. We solve the challenge for the case of maximal minors, and we also find the b-function for the ideal of $2n \times 2n$ Pfaffians of the generic skew-symmetric matrix of size $(2n+1) \times (2n+1)$.

When $m = n$, I is generated by a single equation – the determinant of the generic $n \times n$ matrix – and the formula for $b_I(s)$ is well-known.

The statement of the Strong Monodromy Conjecture of Denef and Loeser [19] extends naturally from the case of one hypersurface to arbitrary ideals, and it asserts that the poles of the topological zeta function of I are roots of $b_I(s)$. When $I = I_n$ is the ideal of maximal minors of (x_{ij}), the methods of [24] can be used to show that the poles of the topological zeta function of I are $-m, -m+1, \dots, -m+n-1$, so our theorem implies that the Strong Monodromy Conjecture holds in this case. If we replace I by the ideal I_p of $p \times p$ minors of (x_{ij}), $1 < p < n$, then it is no longer true that the roots of $b_{I_p}(s)$ coincide with the poles of the zeta function of I_p: as explained in [13, Example 2.12], a computer calculation of T. Oaku shows that for $m = n = 3$ one has $b_{I_2}(s) = (s+9/2)(s+4)(s+5)$, while [24, Thm. 6.5] shows that the only poles of the zeta function of I_2 are $-9/2$ and -4. Besides the Strong Monodromy Conjecture which predicts some of the roots of $b_{I_p}(s)$, we are not aware of any general conjectural formulas for $b_{I_p}(s)$ when $1 < p < n$.

In Section 6.1 we review some generalities on representation theory and $\mathcal{D}$-modules. In Section 6.2 we recall the necessary results on invariant differential operators and their eigenvalues. In Section 6.3 we illustrate some methods for bounding the b-function of an ideal: for upper bounds we use invariant differential operators, while for lower bounds we show how non-vanishing of local cohomology can be used to exhibit roots of the b-functions. These methods allow us to compute the b-function for sub-maximal Pfaffians, and to bound from above the b-function for maximal minors. In Section 6.5 we employ the SL_n-symmetry in the definition of the b-function of maximal minors in order to show that the upper bound obtained in Section 6.3 is in fact sharp.

Notation. We write $[N]$ for the set $\{1,\dots,N\}$, and for $k \le N$ we let $\binom{[N]}{k}$ denote the collection of k-element subsets of $[N]$. Throughout the chapter, $X = \mathbb{A}^N$ is an affine space, and $S = \mathbb{C}[x_1,\dots,x_N]$ denotes the coordinate ring of X. We write $\mathcal{D}_X$ or simply $\mathcal{D}$ for the Weyl algebra of differential operators on X.

6.1 Representation theory of the general linear group

We consider the group $\mathrm{GL}_N = \mathrm{GL}_N(\mathbb{C})$ of invertible $N \times N$ complex matrices, and denote by T_N the maximal torus of diagonal matrices. We will refer to N–tuples $\lambda = (\lambda_1,\dots,\lambda_N) \in \mathbb{Z}^N$ as *weights* of T_N and write $|\lambda|$ for the total size $\lambda_1 + \cdots + \lambda_N$ of λ. We say that λ is a *dominant weight* if $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_N$ and denote the collection of dominant weights by $\mathbb{Z}^N_{\mathrm{dom}}$. A dominant weight with $\lambda_N \ge 0$ is a *partition*, and we write $\mathcal{P}^N$ for the set of partitions in $\mathbb{Z}^N_{\mathrm{dom}}$. We will implicitly identify $\mathcal{P}^{N-1}$ with a subset of $\mathcal{P}^N$ by setting $\lambda_N = 0$ for any $\lambda \in \mathcal{P}^{N-1}$. For $0 \le k \le N$ and $a \ge 0$ we write (a^k) for the partition $\lambda \in \mathcal{P}^k \subset \mathcal{P}^N$ with $\lambda_1 = \cdots = \lambda_k = a$. Irreducible rational representations of $\mathrm{GL}_N(\mathbb{C})$ are in one-to-one correspondence with dominant weights λ. We denote by $S_\lambda\mathbb{C}^N$ the irreducible representation associated to λ, often referred to as a *Schur functor*, and note that $S_{(1^k)}\mathbb{C}^N = \bigwedge^k \mathbb{C}^N$ is the k-th exterior power of $\mathbb{C}^N$ for every $0 \le k \le N$. When $m \ge n$, we have Cauchy's formula [72, Cor. 2.3.3]:

$$\mathrm{Sym}(\mathbb{C}^m \otimes \mathbb{C}^n) = \bigoplus_{\lambda \in \mathcal{P}^n} S_\lambda\mathbb{C}^m \otimes S_\lambda\mathbb{C}^n. \tag{6.1}$$

If we identify $\mathbb{C}^m \otimes \mathbb{C}^n$ with the linear forms on the space $X = X_{m\times n}$ of $m \times n$ complex matrices, then (6.1) is precisely the decomposition into irreducible $\mathrm{GL}_m \times \mathrm{GL}_n$ representations of the coordinate ring S of X.

We consider the tuple of maximal minors $\underline{d} = (d_K)_{K\in\binom{[m]}{n}}$ of the generic matrix of indeterminates (x_{ij}), where

$$d_K = \det(x_{ij})_{i\in K, j\in[n]}, \tag{6.2}$$

The elements d_K form a basis for the irreducible representation $V = \bigwedge^n \mathbb{C}^m \otimes \bigwedge^n \mathbb{C}^n$ in (6.1), indexed by the partition $\lambda = (1^n)$. If we consider $\mathrm{SL}_n \subset \mathrm{GL}_n \subset \mathrm{GL}_m \times \mathrm{GL}_n$, the special linear group of $n \times n$ matrices, then

$$S^{\mathrm{SL}_n} = \mathbb{C}\left[d_K : K \in \binom{[m]}{n}\right]$$

is the $\mathbb{C}$-subalgebra of S generated by the maximal minors d_K. Moreover, S^{SL_n} can be identified with the homogeneous coordinate ring of the Grassmannian $\mathbb{G}(n,m)$ of n-planes in $\mathbb{C}^m$. We let

$$p_0 = d_{[n]}, \text{ and } p_{ij} = d_{[n]\setminus\{i\}\cup\{j\}}, \text{ for } i \in [n], j \in [m] \setminus [n], \tag{6.3}$$

and note that p_{ij}/p_0 give the coordinates on the open Schubert cell defined by $p_0 \neq 0$ inside $\mathbb{G}(n,m)$. It follows that if we take any $K \in \binom{[m]}{n}$, set $|[n] \setminus K| = k$, and enumerate the elements of the sets $[n]\setminus K = \{i_1, \dots, i_k\}$ and $K\setminus[n] = \{j_1, \dots, j_k\}$

in increasing order, then:

$$d_K = p_0^{1-k} \cdot \det \begin{pmatrix} p_{i_1 j_1} & \cdots & p_{i_1 j_k} \\ \vdots & \ddots & \vdots \\ p_{i_k j_1} & \cdots & p_{i_k j_k} \end{pmatrix}. \tag{6.4}$$

It will be important in Section 6.5 to note moreover that p_0, p_{ij} are algebraically independent and that

$$\left\{ p_0^{c_0} \cdot \prod_{i\in[n], j\in[m]\setminus[n]} p_{ij}^{c_{ij}} : c_0, c_{ij} \in \mathbb{Z} \right\} \text{ forms a } \mathbb{C}\text{-basis of } \left(S_{p_0 \cdot \prod_{i,j} p_{ij}} \right)^{\mathrm{SL}_n}. \tag{6.5}$$

For a partition λ we write $\lambda^{(2)} = (\lambda_1, \lambda_1, \lambda_2, \lambda_2, \dots)$ for the partition obtained by repeating each part of λ twice. The skew-symmetric version of Cauchy's formula [72, Prop. 2.3.8(b)] yields

$$\mathrm{Sym}\left(\bigwedge^2 \mathbb{C}^{2n+1} \right) = \bigoplus_{\lambda \in \mathcal{P}^n} S_{\lambda^{(2)}} \mathbb{C}^{2n+1}. \tag{6.6}$$

If we identify $\bigwedge^2 \mathbb{C}^{2n+1}$ with the linear forms on the space $X = X_n$ of $(2n+1) \times (2n+1)$ skew-symmetric matrices, then (6.6) describes the decomposition into irreducible GL_{2n+1}-representations of the coordinate ring of X.

Throughout this chapter, we will be studying various (left) $\mathcal{D}_X$-modules when X is a finite dimensional representation of some connected reductive linear algebraic group G. Differentiating the G-action on X yields a map from the Lie algebra $\mathfrak{g}$ into the vector fields on X, which in turn induces a map

$$\tau : \mathcal{U}(\mathfrak{g}) \longrightarrow \mathcal{D}_X, \tag{6.7}$$

where $\mathcal{U}(\mathfrak{g})$ denotes the universal enveloping algebra of $\mathfrak{g}$. In particular, any $\mathcal{D}_X$-module $\mathcal{M}$ inherits via τ the structure of a $\mathfrak{g}$-representation: if $g \in \mathfrak{g}$ and $m \in \mathcal{M}$ then $g \cdot m = \tau(g) \cdot m$. In order to make the action of $\mathcal{D}_X$ on $\mathcal{M}$ compatible with the $\mathfrak{g}$-action we need to consider the action of $\mathfrak{g}$ on $\mathcal{D}_X$ given by

$$g \cdot p = \tau(g) \cdot p - p \cdot \tau(g) \text{ for } g \in \mathfrak{g} \text{ and } p \in \mathcal{D}_X. \tag{6.8}$$

The induced Lie algebra action of $\mathfrak{g}$ on the tensor product $\mathcal{D}_X \otimes \mathcal{M}$ makes the multiplication $\mathcal{D}_X \otimes \mathcal{M} \to \mathcal{M}$ into a $\mathfrak{g}$-equivariant map: $g \cdot (p \cdot m) = (g \cdot p) \cdot m + p \cdot (g \cdot m)$ for $g \in \mathfrak{g}$, $p \in \mathcal{D}_X$, $m \in \mathcal{M}$.

6.2 Invariant operators and the Fourier transform

For a Lie subalgebra $\mathfrak{a} \subset \mathfrak{g}$, and a $\mathcal{D}_X$-module $\mathcal{M}$, we consider the collection $\mathcal{M}^{\mathfrak{a}}$ of $\mathfrak{a}$-*invariant sections in* $\mathcal{M}$:

$$\mathcal{M}^{\mathfrak{a}} = \{m \in \mathcal{M} : \tau(a) \cdot m = 0 \text{ for all } a \in \mathfrak{a}\}.$$

The main examples that we study arise from a tuple $\underline{f} = (f_1, \ldots, f_r) \in S^r$ of polynomial functions on X, where each f_i is $\mathfrak{a}$-invariant, and $\mathcal{M} = S_{f_1 \cdots f_r}$ is the localization of S at the product $f_1 \cdots f_r$. In this case we have that $\mathcal{M}^{\mathfrak{a}} = (S_{f_1 \cdots f_r})^{\mathfrak{a}}$ coincides with $(S^{\mathfrak{a}})_{f_1 \cdots f_r}$, the localization of $S^{\mathfrak{a}}$ at $f_1 \cdots f_r$.

The *ring of* $\mathfrak{a}$-*invariant differential operators on* X, denoted by $\mathcal{D}_X^{\mathfrak{a}}$ (not to be confused with $\mathcal{M}^{\mathfrak{a}}$ for $\mathcal{M} = \mathcal{D}_X$ as defined above), are defined via

$$\mathcal{D}_X^{\mathfrak{a}} = \{p \in \mathcal{D}_X : a \cdot p = 0 \text{ for all } a \in \mathfrak{a}\}, \tag{6.9}$$

and $\mathcal{M}^{\mathfrak{a}}$ is a $\mathcal{D}_X^{\mathfrak{a}}$-module whenever $\mathcal{M}$ is a $\mathcal{D}_X$-module. If we write $\mathcal{ZU}(\mathfrak{a})$ for the center of $\mathcal{U}(\mathfrak{a})$ then it follows from (6.8) and (6.9) that

$$\tau\left(\mathcal{ZU}(\mathfrak{a})\right) \subseteq \mathcal{D}_X^{\mathfrak{a}}. \tag{6.10}$$

An alternative way of producing $\mathfrak{a}$-invariant differential operators is as follows. Let $P = \mathbb{C}[\partial_1, \ldots, \partial_N]$ and write S_k (resp. P_k) for the subspace of S (resp. P) of homogeneous elements of degree k. The action of P on S by differentiation induces $\mathfrak{a}$-equivariant perfect pairings $\langle \ , \ \rangle : P_k \times S_k \to \mathbb{C}$ for each $k \geq 0$, namely $\langle w, v\rangle = w \cdot v$. If $V \subset S_k$, $W \subset P_k$ are dual $\mathfrak{a}$-subrepresentations, with (almost dual) bases $\underline{v} = (v_1, \ldots, v_t)$ and $\underline{w} = (w_1, \ldots, w_t)$, such that for some non-zero constant c

$$\langle w_i, v_j\rangle = 0 \text{ for } i \neq j, \quad \langle w_i, v_i\rangle = c \text{ for all } i, \tag{6.11}$$

then we can define elements of $\mathcal{D}_X^{\mathfrak{a}}$ via

$$D_{\underline{v},\underline{w}} = \sum_{i=1}^{t} v_i \cdot w_i, \quad D_{\underline{w},\underline{v}} = \sum_{i=1}^{t} w_i \cdot v_i. \tag{6.12}$$

In the examples that we consider, the basis w will have a very simple description in terms of v. For $p = p(x_1, \dots, x_N) \in S$, we define $p^* = p(\partial_1, \dots, \partial_N) \in P$. For the tuples of maximal minors and sub-maximal Pfaffians, it will suffice to take $w_i = v_i^*$ in order for (6.11) to be satisfied, in which case we'll simply write $D_{\underline{v}}$ instead of $D_{\underline{w},\underline{v}}$ and $D_{\underline{v}^*}$ or $D_{\underline{w}}$ instead of $D_{\underline{v},\underline{w}}$.

We specialize our discussion to the case when $X = X_{m,n}$ is the vector space of $m \times n$ matrices, $m \geq n$, and $G = \mathrm{GL}_m \times \mathrm{GL}_n$, $\mathfrak{g} = \mathfrak{gl}_m \oplus \mathfrak{gl}_n$. The coordinate ring of X is $S = \mathbb{C}[x_{ij}]$ with $i \in [m]$, $j \in [n]$. Recall the tuple of maximal minors $\underline{d} = (d_K)_{K \in \binom{[m]}{n}}$ of the generic matrix of indeterminates (x_{ij}), and take the tuple $\underline{\partial} = (\partial_K)_{K \in \binom{[m]}{n}}$ of maximal minors in the dual variables

$$\partial_K = d_K^* = \det(\partial_{ij})_{i \in K, j \in [n]}, \tag{6.13}$$

The elements d_K form a basis for the irreducible representation $V = \bigwedge^n \mathbb{C}^m \otimes \bigwedge^n \mathbb{C}^n$, while ∂_K form a basis for the dual representation W. If we let $c = n!$ then it follows from Cayley's identity (see Introduction) that (6.11) holds for the tuples d and ∂, so we get $\mathfrak{g}$-invariant operators

$$D_{\underline{\partial}} = \sum_{K \in \binom{[m]}{n}} d_K \cdot \partial_K, \quad D_{\underline{d}} = \sum_{K \in \binom{[m]}{n}} \partial_K \cdot d_K. \tag{6.14}$$

Throughout, by the determinant of a matrix $A = (a_{ij})_{i,j \in [r]}$ with non-commuting entries we mean the *column-determinant*: if $\mathcal{S}_r$ is the symmetric group of permutations of $[r]$, and sgn denotes the signature of a permutation, then

$$\text{col-det}(a_{ij}) = \sum_{\sigma \in \mathcal{S}_r} \mathrm{sgn}(\sigma) \cdot a_{\sigma(1)1} \cdot a_{\sigma(2)2} \cdots a_{\sigma(n)n}. \tag{6.15}$$

We consider the Lie algebra $\mathfrak{gl}_r$ and choose a basis $\{\mathcal{E}_{ij} : i, j \in [r]\}$ for it, where $\mathcal{E}_{ij}$ is the matrix whose only non-zero entry is in row i, column j, and it is equal to one. We think of $\mathcal{E}_{ij}$ as the entries of an $r \times r$ matrix $\mathcal{E}$ with entries in $\mathcal{U}(\mathfrak{gl}_r)$. We consider an auxiliary variable z, consider the diagonal matrix $\Delta = \operatorname{diag}(r-1-z, r-2-z, \cdots, 1-z, -z)$ and define the polynomial $\mathcal{C}(z) \in \mathcal{U}(\mathfrak{gl}_r)[z]$ using notation (6.15):

$$\mathcal{C}(z) = \text{col-det}(\mathcal{E} + \Delta). \tag{6.16}$$

For $a \geq 0$ we write

$$[z]_a = z(z-1)\cdots(z-a+1) \tag{6.17}$$

and define elements $\mathcal{C}_a \in \mathcal{U}(\mathfrak{gl}_r)$, $a = 0, \ldots, r$, by expanding the polynomial $\mathcal{C}(z)$ into a linear combination

$$\mathcal{C}(z) = \sum_{a=0}^{r} (-1)^{r-a} \mathcal{C}_a \cdot [z]_{r-a}. \tag{6.18}$$

In the case when $r = 2$ we obtain

$$\mathcal{C}(z) = \text{col-det} \begin{bmatrix} \mathcal{E}_{11} + 1 - z & \mathcal{E}_{12} \\ \mathcal{E}_{21} & \mathcal{E}_{22} - z \end{bmatrix} = [z]_2 - (\mathcal{E}_{11} + \mathcal{E}_{22}) \cdot [z] + ((\mathcal{E}_{11} + 1) \cdot \mathcal{E}_{22} - \mathcal{E}_{21} \cdot \mathcal{E}_{12}),$$

thus

$$\mathcal{C}_0 = 1, \quad \mathcal{C}_1 = \mathcal{E}_{11} + \mathcal{E}_{22}, \quad \mathcal{C}_2 = (\mathcal{E}_{11} + 1) \cdot \mathcal{E}_{22} - \mathcal{E}_{21} \cdot \mathcal{E}_{12}.$$

The elements $\mathcal{C}_a$, $a = 1, \ldots, r$ are called *the Capelli elements* of $\mathcal{U}(\mathfrak{gl}_r)$, and $\mathcal{ZU}(\mathfrak{gl}_r)$ is a polynomial algebra with generators $\mathcal{C}_1, \ldots, \mathcal{C}_r$. For $\lambda \in \mathbb{Z}^r_{\text{dom}}$, let V_λ denote an irreducible $\mathfrak{gl}_r$-representation of highest weight λ, and pick $v_\lambda \in V_\lambda$ to be a highest weight vector in V_λ, so that

$$\mathcal{E}_{ii} \cdot v_\lambda = \lambda_i \cdot v_\lambda, \quad \mathcal{E}_{ij} \cdot v_\lambda = 0 \text{ for } i < j. \tag{6.19}$$

Since $\mathcal{C}_a$ are central, their action on V_λ is by scalar multiplication, and the scalar (called the *eigenvalue of* $\mathcal{C}_a$ *on* V_λ) can be determined by just acting on v_λ. To record this action more compactly, we will consider how $\mathcal{C}(z)$ acts on v_λ. Expanding $\mathcal{C}(z)$ via (6.15), it follows from (6.19) that the only term that doesn't annihilate v_λ is the product of diagonal entries in the matrix $\mathcal{E}+\Delta$, hence

$$\mathcal{C}(z) \text{ acts on } V_\lambda[z] \text{ by multiplication by } \prod_{i=1}^{r}(\lambda_i + r - i - z). \tag{6.20}$$

We can think of $\mathcal{U}(\mathfrak{gl}_r)$ in terms of generators and relations as follows: it is generated as a $\mathbb{C}$-algebra by $\mathcal{E}_{ij}$, $i,j \in [r]$, subject to the relations

$$[\mathcal{E}_{ij}, \mathcal{E}_{kl}] = \delta_{jk} \cdot \mathcal{E}_{il} - \delta_{il} \cdot \mathcal{E}_{kj}, \tag{6.21}$$

where $[a,b] = ab - ba$ denotes the usual commutator, and δ is the Kronecker delta function. For every complex number $u \in \mathbb{C}$, the substitutions

$$\mathcal{E}_{ij} \longrightarrow -\mathcal{E}_{ji} \text{ for } i \neq j, \quad \mathcal{E}_{ii} \longrightarrow -\mathcal{E}_{ii} - u$$

preserve (6.21), so they define an involution $\mathcal{F}_u : \mathcal{U}(\mathfrak{gl}_r) \longrightarrow \mathcal{U}(\mathfrak{gl}_r)$ which we call *the Fourier transform with parameter u*. We can apply $\mathcal{F}_u$ to $\mathcal{C}(z)$ and obtain

$$\mathcal{F}_u\mathcal{C}(z) = \text{col-det}(-\mathcal{E}^t - u \cdot \mathrm{Id}_r + \Delta), \tag{6.22}$$

where $\mathcal{E}^t$ is the transpose of $\mathcal{E}$, and Id_r denotes the $r \times r$ identity matrix. The Fourier transforms $\mathcal{F}_u\mathcal{C}_1, \ldots, \mathcal{F}_u\mathcal{C}_r$ of the Capelli elements form another set of polynomial generators for $\mathcal{Z}\mathcal{U}(\mathfrak{gl}_r)$, hence they act by scalar multiplication on any irreducible $\mathfrak{gl}_r$-representation V_λ. To determine the scalars, we will consider the action on a lowest weight vector $w_\lambda \in V_\lambda$, so that

$$\mathcal{E}_{ii} \cdot w_\lambda = \lambda_{r+1-i} \cdot w_\lambda, \quad \mathcal{E}_{ji} \cdot w_\lambda = 0 \text{ for } i < j. \tag{6.23}$$

Expanding (6.22) via (6.15), it follows from (6.23) that the action of $\mathcal{F}_u\mathcal{C}_a$ on V_λ is encoded by the fact that

$$\mathcal{F}_u\mathcal{C}(z) \text{ acts on } V_\lambda[z] \text{ by multiplication by } \prod_{i=1}^{r}(-\lambda_{r+1-i} - u + r - i - z). \tag{6.24}$$

Lemma 6.2.1. *For $s \in \mathbb{Z}$, let $\lambda = (s^r)$ denote the dominant weight with all $\lambda_i = s$, and for $a = 1, \ldots, r$ let $P_a(s)$ (resp. $\mathcal{F}_uP_a(s)$) denote the eigenvalue of $\mathcal{C}_a$ (resp. $\mathcal{F}_u\mathcal{C}_a$) on V_λ. We have that $P_a(s)$ and $\mathcal{F}_uP_a(s)$ are polynomial functions in s and as such $\mathcal{F}_uP_a(s) = P_a(-s-u)$.*

Proof. If we let $P(s,z) = \sum_{a=0}^{r} (-1)^{r-a} P_a(s) \cdot [z]_{r-a}$ then it follows from (6.18) and (6.20) that

$$P(s,z) = \prod_{i=1}^{r} (s + r - i - z).$$

Expanding the right hand side as a linear combination of $[z]_0, [z]_1, \ldots, [z]_r$ shows that $P_a(s)$ is a polynomial in s. We define $\mathcal{F}_u P(s,z)$ by replacing $P_a(s)$ with $\mathcal{F}_u P_a(s)$ and obtain using (6.24) that

$$\mathcal{F}_u P(s,z) = \prod_{i=1}^{r} (-s - u + r - i - z).$$

Since $\mathcal{F}_u P(s,z) = P(-s-u, z)$, the conclusion follows. $\square$

Lemma 6.2.2. *For $s \in \mathbb{N}$, let $\lambda = (s^{r-1})$ denote the partition with $\lambda_1 = \cdots = \lambda_{r-1} = s$, $\lambda_r = 0$, and for $a = 1, \ldots, r$ let $Q_a(s)$ (resp. $\mathcal{F}_{r-1} Q_a(s)$) denote the eigenvalue of $\mathcal{C}_a$ (resp. $\mathcal{F}_{r-1}\mathcal{C}_a$) on V_λ. We have that $Q_a(s)$ and $\mathcal{F}_{r-1} Q_a(s)$ are polynomial functions in s and as such $\mathcal{F}_{r-1} Q_a(s) = Q_a(-s-r)$.*

Proof. We define $Q(s,z)$ and $\mathcal{F}_{r-1} Q(s,z)$ as in the proof of Lemma 6.2.1 and obtain using (6.20), (6.24) that

$$Q(s,z) = \left(\prod_{i=1}^{r-1} (s + r - i - z) \right) \cdot (0 - z) = (s+r-1-z) \cdot (s+r-2-z) \cdots (s+1-z) \cdot (-z),$$

$$\begin{aligned} \mathcal{F}_{r-1} Q(s,z) &= (-0 - (r-1) + r - 1 - z) \cdot \left(\prod_{i=2}^{r} (-s - (r-1) + r - i - z) \right) \\ &= (-z) \cdot (-s-1-z) \cdot (-s-2-z) \cdots (-s-r+1-z). \end{aligned}$$

It is immediate to check that $\mathcal{F}_{r-1} Q(s,z) = Q(-s-r, z)$, from which the conclusion follows. $\square$

We will need two easy lemmas for localization. Let $X_{m,n}$, $m \geq n$, denote the vector space of $m \times n$ matrices, write $Z_{m,n}$ for the subvariety of $X_{m,n}$ consisting of matrices of rank at most $n-1$, and let $U \subset X_{m,n}$ denote the open affine subset consisting of matrices $u = (u_{ij})$ with $u_{11} \neq 0$.

Lemma 6.2.3. *There exists an isomorphism of algebraic varieties*

$$\pi : U \cap Z_{m,n} \longrightarrow \mathbb{C}^* \times \mathbb{C}^{m-1} \times \mathbb{C}^{n-1} \times Z_{m-1,n-1}.$$

Proof. We define $\pi : U \to \mathbb{C}^* \times \mathbb{C}^{m-1} \times \mathbb{C}^{n-1} \times X_{m-1,n-1}$ via $\pi(u) = (t, \vec{c}, \vec{r}, M)$ where if $u = (u_{ij})$ then

$$t = u_{11},\ \vec{c} = (u_{21}, u_{31}, \ldots, u_{m1}),\ \vec{r} = (u_{12}, u_{13}, \ldots, u_{1n}),$$

$$M_{ij} = \det{}_{\{1,i+1\},\{1,j+1\}} \text{ for } i \in [m-1], j \in [n-1],$$

where $\det_{\{1,i+1\},\{1,j+1\}} = u_{11} \cdot u_{i+1,j+1} - u_{1,j+1} \cdot u_{i+1,1}$ is the determinant of the 2×2 submatrix of u obtained by selecting rows $1, i+1$ and columns $1, j+1$. It follows for instance from [30, Section 3.4] that the map π is an isomorphism, and that it sends $U \cap Z_{m,n}$ onto $\mathbb{C}^* \times \mathbb{C}^{m-1} \times \mathbb{C}^{n-1} \times Z_{m-1,n-1}$, which yields the desired conclusion. $\square$

We let X_n denote the vector space of $(2n+1) \times (2n+1)$ skew-symmetric matrices, and define $Z_n \subset X_n$ to be the subvariety of matrices of rank at most $(2n-2)$. We let $U \subset X_n$ denote the open affine subset defined by matrices (u_{ij}) with $u_{12} \neq 0$.

Lemma 6.2.4. *There exists an isomorphism of algebraic varieties*

$$\pi : U \cap Z_n \longrightarrow \mathbb{C}^* \times \mathbb{C}^{2n-1} \times \mathbb{C}^{2n-1} \times Z_{n-1}.$$

Proof. We define $\pi : U \to \mathbb{C}^* \times \mathbb{C}^{2n-1} \times \mathbb{C}^{2n-1} \times X_{n-1}$ via $\pi(u) = (t, \vec{c}, \vec{r}, M)$ where if $u = (u_{ij})$ then

$$t = u_{12},\ \vec{c} = (u_{13}, u_{14}, \ldots, u_{1,2n+1}),\ \vec{r} = (u_{23}, u_{24}, \ldots, u_{2,2n+1}),$$

$$M_{ij} = \frac{\mathrm{Pf}_{\{1,2,i+2,j+2\}}}{u_{12}} \text{ for } 1 \leq i, j \leq 2n-1,$$

where $\mathrm{Pf}_{\{1,2,i+2,j+2\}}$ is the Pfaffian of the 4×4 principal skew-symmetric submatrix of u obtained by selecting the rows and columns of u indexed by $1, 2, i+2$ and $j+2$. Since

$$M_{ij} = u_{i+2,j+2} - (u_{1,i+2}\cdot u_{2,j+2} - u_{1,j+2}\cdot u_{2,i+2})/u_{12}$$

one can solve for $u_{i+2,j+2}$ in terms of the entries of $M, \vec{r}, \vec{c}$ and u_{12} in order to define the inverse of π, which is therefore an isomorphism. We consider the $(2n+1)\times(2n+1)$ matrix

$$C = \left[\begin{array}{cc|cccc} 0 & 1 & u_{23}/u_{12} & u_{24}/u_{12} & \cdots & u_{2,2n+1}/u_{12} \\ 1/u_{12} & 0 & -u_{13}/u_{12} & -u_{14}/u_{12} & \cdots & -u_{1,2n+1}/u_{12} \\ \hline 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{array}\right]$$

Writing $\vec{0}$ for zero row/column vectors of size $(2n-1)$, we have (see also [31, Lemma 1.1])

$$C^t \cdot u \cdot C = \left[\begin{array}{cc|c} 0 & -1 & \vec{0} \\ 1 & 0 & \vec{0} \\ \hline \vec{0} & \vec{0} & M \end{array}\right]$$

Since $\mathrm{rank}(u) = \mathrm{rank}(C^t\cdot u\cdot C) = \mathrm{rank}(M)+2$, it follows that π sends $U\cap Z_n$ onto $\mathbb{C}^*\times\mathbb{C}^{2n-1}\times\mathbb{C}^{2n-1}\times Z_{n-1}$, so it restricts to the desired isomorphism. $\square$

6.3 Bounding the b-function

In this section we discuss some methods for bounding the b-function from above and below. As a consequence we obtain formulas for the b-function of the ideal of maximal minors of the generic $(n+1)\times n$ matrix, and for the b-function of the ideal of sub-maximal Pfaffians of a generic skew-symmetric matrix of odd size.

In order to obtain lower bounds for a b-function, it is important to be able to identify certain factors of the b-function which are easier to compute. One instance of this is given in equation (1.11): the b-function of Z is divisible by the b-function of any affine open subscheme. Combining (1.10) and (1.11) with the results from the previous section, we conclude that

$$b_{Z_{m-1,n-1}}(s) \text{ divides } b_{Z_{m,n}}(s), \text{ and } b_{Z_{n-1}}(s) \text{ divides } b_{Z_n}(s). \tag{6.25}$$

Sometimes it is possible to identify roots of the b-function (i.e. linear factors) by showing an appropriate inclusion of $\mathcal{D}$-modules. As before $\underline{f} = (f_1, \dots, f_r) \in S^r$, and $I \subset S$ is the ideal generated by the f_i's.

For $\alpha \in \mathbb{Z}$ we define F_α to be the $\mathcal{D}_X$-submodule of $S_{f_1 \cdots f_r}$ generated by

$$\underline{f}^{\underline{\alpha}} = \prod_{i=1}^{r} f_i^{\alpha_i}, \text{ where } \underline{\alpha} = (\alpha_1, \dots, \alpha_r) \in \mathbb{Z}^r, \ \alpha_1 + \cdots + \alpha_r = \alpha.$$

It is clear that $F_{\alpha+1} \subseteq F_\alpha$ for every $\alpha \in \mathbb{Z}$. We have moreover:

Proposition 6.3.1. *If $\alpha \in \mathbb{Z}$ and if there is a strict inclusion $F_{\alpha+1} \subsetneq F_\alpha$ then α is a root of $b_{\underline{f}}(s)$.*

Proof. By the definition of $b_{\underline{f}}(s)$, there exist tuples $\hat{c}$ and operators $P_{\hat{c}} \in \mathcal{D}_X[s_1, \dots, s_r]$ such that (1.8) holds. Assume now that $F_{\alpha+1} \subsetneq F_\alpha$ for some $\alpha \in \mathbb{Z}$, and consider any integers $\alpha_1, \dots, \alpha_r$ with $\alpha_1 + \cdots + \alpha_r = \alpha$. There is a natural $\mathcal{D}_X$-module homomorphism

$$\pi : S_{f_1 \cdots f_r}[s_1, \dots, s_r] \cdot \underline{f}^{\underline{s}} \longrightarrow S_{f_1 \cdots f_r}, \text{ defined by } \pi(s_i) = \alpha_i. \tag{6.26}$$

Applying π to (1.8) we find that $b_{\underline{f}}(\alpha) \cdot \underline{f}^{\underline{\alpha}} \in F_{\alpha+1}$. If $b_{\underline{f}}(\alpha) \neq 0$ then we can divide by $b_{\underline{f}}(\alpha)$ and obtain that $\underline{f}^{\underline{\alpha}} \in F_{\alpha+1}$ for all $\underline{\alpha}$ with $|\underline{\alpha}| = \alpha$. Since the elements $\underline{f}^{\underline{\alpha}}$ generate F_α it follows that $F_\alpha \subseteq F_{\alpha+1}$ which is a contradiction. We conclude that $b_f(\alpha) = 0$, i.e. that α is a root of $b_f(s)$. $\square$

We write $H^\bullet_I(S)$ for the local cohomology groups of S with support in the ideal I. Proposition 6.3.1 combined with non-vanishing results for local cohomology can sometimes be used to determine roots of the b-function as follows:

Corollary 6.3.2. *If $b_I(s)$ has no integral root α with $\alpha < -r$, and if $H^r_I(S) \neq 0$ then $b_I(-r) = 0$.*

Proof. For every $\alpha \in \mathbb{Z}$, $\alpha < -r$, and every $\underline{\alpha} = (\alpha_1, \ldots, \alpha_r)$ with $\alpha = \alpha_1 + \cdots + \alpha_r$, we can apply the specialization map (6.26) to the equation (1.8) to conclude that $b_I(\alpha) \cdot \underline{f}^{\underline{\alpha}} \in F_{\alpha+1}$. Since $b_I(\alpha) \neq 0$ by assumption, we conclude that $\underline{f}^{\underline{\alpha}} \in F_{\alpha+1}$ for all such α, and therefore $F_\alpha = F_{\alpha+1}$. It follows that

$$F_{-r} = F_{-r-1} = F_{-r-2} = \cdots = S_{f_1 \cdots f_r},$$

since the localization $S_{f_1 \cdots f_r}$ is the union of all F_α, $\alpha \leq -r$.

By Proposition 6.3.1, in order to show that $b_I(-r) = 0$, it is enough to show that $F_{-r+1} \subsetneq F_{-r}$, which by the above is equivalent to proving that F_{-r+1} does not coincide with the localization $S_{f_1 \cdots f_r}$. Consider any generator $\underline{f}^{\underline{\alpha}}$ of F_{-r+1}, corresponding to a tuple $\underline{\alpha} \in \mathbb{Z}^r$ with $\alpha_1 + \cdots + \alpha_r = -r + 1$. At least one of the α_i's has to be nonnegative, so that $\underline{f}^{\underline{\alpha}}$ belongs to $S_{f_1 \cdots \hat{f}_i \cdots f_r}$, the localization of S at a product of all but one of the generators f_i. This shows that

$$F_{-r+1} \subseteq \sum_{i=1}^r S_{f_1 \cdots \hat{f}_i \cdots f_r}. \tag{6.27}$$

Using the Čech complex description of local cohomology, and the assumption that $H^r_I(S) \neq 0$, we conclude that there is a strict inclusion

$$\sum_{i=1}^r S_{f_1 \cdots \hat{f}_i \cdots f_r} \subsetneq S_{f_1 \cdots f_r}.$$

Combining this with (6.27) we conclude that $F_{-r+1} \subsetneq F_{-r} = S_{f_1 \cdots f_r}$, as desired. $\square$

Obtaining upper bounds for b-functions is in general a difficult problem, since most of the time it involves determining the operators $P_{\hat{c}}$ in (1.8). In the presence of a large group of symmetries, invariant differential operators are natural candidates for such operators, and the problem becomes more tractable. As before, G is a connected reductive linear algebraic group, and $\mathfrak{g}$ is its Lie algebra.

Definition 6.3.3. A representation $V = V_\lambda \subset S$ of highest weight λ is said to be *multiplicity-free representation* if

(a) for every $\alpha \in \mathbb{N}$, the multiplicity of the representation $V_\alpha := V_{\alpha\cdot\lambda}$ of highest weight $\alpha \cdot \lambda$ inside S is equal to one,

(b) the algebra generated by the elements of V_λ in S decomposes as

$$\mathbb{C}[V_\lambda] = \bigoplus_{\alpha\in\mathbb{N}} V_{\alpha\cdot\lambda}.$$

A basis $\underline{f} = (f_1, \dots, f_r)$ of a multiplicity-free representation V_λ has the property that for every $\alpha \in \mathbb{N}$, the polynomials

$$\underline{f}^{\underline{\alpha}} = f_1^{\alpha_1} \cdots f_r^{\alpha_r}, \text{ for } \underline{\alpha} = (\alpha_1, \dots, \alpha_r) \in \mathbb{N}^r \text{ satisfying } \alpha_1 + \cdots + \alpha_r = \alpha,$$

span the irreducible G-subrepresentation $V_{\alpha\cdot\lambda} \subset S$.

A typical example of a multiplicity-free representation arises in the case $r = 1$ when $V = \mathbb{C}f$ and f is a multiplicity-free semi-invariant (in the sense of Definition 1.2.3). Our definition gives a natural generalization to tuples with $r > 1$ entries. We have the following:

Proposition 6.3.4. *Consider a basis $\underline{f} = (f_1, \dots, f_r)$ of a multiplicity-free representation V, and a G-invariant differential operator $D_{\underline{f}} = \sum_{i=1}^r g_i \cdot f_i$, where $g_i \in \mathcal{D}_X$. If we let $s = s_1 + \cdots + s_r$ then there exists a polynomial $b_V(s) \in \mathbb{C}[s]$ such that*

$$D_{\underline{f}} \cdot \underline{f}^{\underline{s}} = b_V(s) \cdot \underline{f}^{\underline{s}},$$

and moreover we have that $b_{\underline{f}}(s)$ divides $b_V(s)$.

Proof. Since the action of $D_{\underline{f}}$ preserves $B_{\underline{f}}^{\underline{s}}$, there exists an element $Q \in S_{f_1 \cdots f_r}[s_1, \ldots, s_r]$ with the property

$$D_{\underline{f}} \cdot \underline{f}^{\underline{s}} = Q \cdot \underline{f}^{\underline{s}}.$$

The goal is to show that, as a polynomial in $s_1, \ldots, s_r$, $Q = Q(s_1, \ldots, s_r)$ has coefficients in $\mathbb{C}$, and moreover that it can be expressed as a polynomial only in $s = s_1 + \cdots + s_r$. For this, it suffices to check that:

(a) $Q(\alpha_1, \ldots, \alpha_r) \in \mathbb{C}$ for every $\alpha_1, \ldots, \alpha_r \in \mathbb{N}$.

(b) For α_i as in (a), $Q(\alpha_1, \ldots, \alpha_r)$ only depends on $\alpha = \alpha_1 + \cdots + \alpha_r$.

Let $\alpha_1, \ldots, \alpha_r$ be arbitrary non-negative integers, and write $\alpha = \alpha_1 + \cdots + \alpha_r$. Since $V_{\alpha \cdot \lambda}$ is irreducible, the multiplicity of $V_{\alpha \cdot \lambda}$ in S is 1, and $D_{\underline{f}}$ is G-invariant, it follows from Schur's Lemma that $D_{\underline{f}}$ acts on $V_{\alpha \cdot \lambda}$ by multiplication by a scalar, i.e. $Q(\alpha_1, \ldots, \alpha_r) \in \mathbb{C}$ is a scalar that only depends on α, so conditions (a) and (b) are satisfied.

To see that $b_{\underline{f}}(s)$ divides $b_V(s)$, it suffices to note that $D_{\underline{f}} \cdot \underline{f}^{\underline{s}} = b_V(s) \cdot \underline{f}^{\underline{s}}$ can be rewritten in the form (1.8), where the sum is over tuples $\hat{c} = (0, \ldots, 0, 1, 0, \ldots, 0)$ with $c_i = 1$, $c_j = 0$ for $j \neq i$, with corresponding operator $P_{\hat{c}} = g_i$. Since $b_{\underline{f}}(s)$ is the lowest degree polynomial for which (1.8) holds, it follows that $b_{\underline{f}}(s)$ divides $b_V(s)$. □

Now let $X = X_{m,n}$ be the vector space of $m \times n$ matrices, $m \geq n$. The group $G = \mathrm{GL}_m \times \mathrm{GL}_n$ acts on X via row and column operations. The coordinate ring of X is $S = \mathbb{C}[x_{ij}]$, and we consider the representation $V = \bigwedge^n \mathbb{C}^m \otimes \bigwedge^n \mathbb{C}^n$ with basis the tuple $\underline{d} = (d_K)_{K \in \binom{[m]}{n}}$ of maximal minors defined in (6.2). Then V is multiplicity-free for the G-action, where for $\alpha \in \mathbb{N}$, the corresponding representation V_α in Definition 6.3.3 is $S_{(\alpha^n)}\mathbb{C}^m \otimes S_{(\alpha^n)}\mathbb{C}^n$ from (6.1) (see for instance [18, Thm. 6.1]). We associate to $\underline{d}$ the invariant differential operator $D_{\underline{d}}$ in (6.14) and by Proposition 6.3.4 there exists a polynomial $P_d(s)$ with

$$D_{\underline{d}} \cdot \underline{d}^{\underline{s}} = b_V(s) \cdot \underline{d}^{\underline{s}}. \tag{6.28}$$

Theorem 6.3.5. *With the notation above, we have that for maximal minors*

$$b_V(s) = \prod_{i=m-n+1}^{m} (s+i). \tag{6.29}$$

Proof. In order to compute $b_V(s)$, it suffices to understand the action of $D_{\underline{d}}$ on d_L^s for some fixed $L \in \binom{[m]}{n}$ (this corresponds to letting $s_K = 0$ for $K \neq L$ in (6.28)). We consider instead the action of the operator $D_{\underline{\partial}}$ in (6.14), and note that by Cayley's identity [16, (1.1)] one has

$$\partial_K \cdot d_L^s = 0 \text{ for } K \neq L, \quad \partial_L \cdot d_L^s = \left(\prod_{i=0}^{n-1}(s+i)\right) \cdot d_L^{s-1},$$

which implies

$$D_{\underline{\partial}} \cdot d_L^s = \left(\prod_{i=0}^{n-1}(s+i)\right) \cdot d_L^s. \tag{6.30}$$

Let $\mathcal{F} : \mathcal{D}_X \longrightarrow \mathcal{D}_X$ denote the (usual) Fourier transform, defined by $\mathcal{F}(x_{ij}) = \partial_{ij}$, $F(\partial_{ij}) = -x_{ij}$, and note that $D_{\underline{d}} = (-1)^n \cdot \mathcal{F}(D_{\underline{\partial}})$. We will obtain $b_V(s)$ by applying the Fourier transform to (6.30).

For $i, j \in [n]$, we consider the polarization operators

$$E_{ij} = \sum_{k=1}^{m} x_{ki} \cdot \partial_{kj}.$$

The action of the Lie algebra $\mathfrak{gl}_n \subset \mathfrak{gl}_m \oplus \mathfrak{gl}_n$ on X induces a map $\tau : \mathcal{U}(\mathfrak{gl}_n) \to \mathcal{D}_X$ as in (6.7), sending $\tau(\mathcal{E}_{ij}) = E_{ij}$ for all i, j. The Fourier transform sends

$$\mathcal{F}(E_{ij}) = -E_{ji} \text{ for } i \neq j, \quad \mathcal{F}(E_{ii}) = -E_{ii} - m,$$

so using the notation in Section 6.1 we obtain a commutative diagram

$$\begin{tikzcd} \mathcal{U}(\mathfrak{gl}_n) \arrow[r, "\mathcal{F}_m"] \arrow[d, "\tau"'] & \mathcal{U}(\mathfrak{gl}_n) \arrow[d, "\tau"] \\ \mathcal{D}_X \arrow[r, "\mathcal{F}"] & \mathcal{D}_X \end{tikzcd}$$

Since $D_{\underline{\partial}}$ is in $\tau(\mathcal{ZU}(\mathfrak{gl}_n))$ (it is in fact equal to $\tau(\mathcal{C}_n)$ by [29, (11.1.9)]), it follows from (6.30), from the commutativity of the above diagram and from Lemma 6.2.1 with $r = n$ and $u = m$ that

$$D_{\underline{d}} \cdot d_K^s = \left((-1)^n \prod_{i=0}^{n-1}(-s-m+i)\right) \cdot d_K^s = \left(\prod_{i=m-n+1}^{m}(s+i)\right) \cdot d_K^s,$$

which concludes the proof of our theorem. $\square$

Remark 6.3.6. A more direct way to prove (6.29) is to use for instance [15, Prop. 1.2] in order to obtain a determinantal representation for the operator $D_{\underline{d}}$, namely

$$D_{\underline{d}} = \text{col-det}\begin{pmatrix} E_{11}+m & E_{12} & \cdots & E_{1n} \\ E_{21} & E_{22}+m-1 & \cdots & E_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ E_{n1} & E_{n2} & \cdots & E_{nn}+m-n+1 \end{pmatrix},$$

from which the conclusion follows easily. The advantage of our proof of Theorem 6.3.5 is that it applies equally to the case of sub-maximal Pfaffians in Section 6.3, where we are not aware of a more direct approach.

6.4 Almost square matrices and sub-maximal Pfaffians

In the case of $(n+1) \times n$ matrices and sub-maximal Pfaffians, we can show that the lower and upper bounds obtained by the techniques described above agree, and we obtain the following special instance of the Theorem on Maximal Minors described in the Introduction:

Theorem 6.4.1. *If $\underline{d}$ is the tuple of maximal minors of the generic $(n+1) \times n$ matrix then its b-function is*

$$b_{\underline{d}}(s) = \prod_{i=2}^{n+1}(s+i).$$

Proof. We have by Proposition 6.3.4 and Theorem 6.3.5 that $b_{\underline{d}}(s)$ divides the product $(s+2)\cdots(s+n+1)$. If we write $Z_{n+1,n}$ for the variety of $(n+1)\times n$ matrices of rank smaller than n as before, then the defining ideal of $Z_{n+1,n}$ is generated by the entries of $\underline{d}$. Since $Z_{n+1,n}$ has codimension two inside $X_{n+1,n}$, $b_{Z_{n+1,n}}(s) = b_{\underline{d}}(s-2)$ by (1.9), and thus it suffices to show that

$$b_{Z_{n+1,n}}(s) \text{ is divisible by } \prod_{i=0}^{n-1}(s+i). \tag{6.31}$$

By induction on n, we may assume that $b_{Z_{n,n-1}} = \prod_{i=0}^{n-2}(s+i)$. Taking into account (6.25) we are left with proving that $(-n+1)$ is a root of $b_{Z_{n+1,n}}(s)$, or equivalently that $(-n-1)$ is a root of $b_{\underline{d}}(s)$. To do this we apply Corollary 6.3.2 with $r = n+1$, and I the defining ideal of $Z_{n+1,n}$. It follows from [73, Thm. 5.10] or [47, Thm. 4.5] that $H_I^{n+1}(S) \neq 0$, so the Corollary applies and concludes our proof. □

Remark 6.4.2. An alternative approach to proving Theorem 6.4.1 goes by first computing the b-function of several variables (1.4) associated to $d_1,\dots,d_{n+1}$. The space $X_{n+1,n}$ is prehomogeneous under the action of the smaller group $(\mathbb{C}^*)^{n+1}\times \mathrm{GL}_n(\mathbb{C})$. The maximal minors $d_1,\dots,d_{n+1}$ can be viewed as semi-invariants for the following quiver with $n+2$ vertices and dimension vector

$$\begin{tikzcd}
1 & 1 & \cdots & 1 & 1 \\
 & & n \arrow[ull] \arrow[ul] \arrow[ur] \arrow[urr] & &
\end{tikzcd}$$

The dimension vector is preinjective, hence by Theorem 3.2.2 we can compute the b-function of several variables using reflection functors to get:

$$b_{\underline{d}}(\underline{s}) = [s]_{n-1,n}^{1,1,\dots,1}\cdot[s]_1^{1,0,\dots,0}\cdot[s]_1^{0,1,\dots,0}\cdots[s]_1^{0,0,\dots,1}.$$

This means that we have formulas

$$d_i^*\cdot d_i\cdot \underline{d}^{\underline{s}} = (s_i+1)(s+2)(s+3)\cdots(s+n)\cdot\underline{d}^{\underline{s}},$$

which, together with Lemma 6.5.3 below gives readily the Bernstein-Sato polynomial of the ideal.

Now let $X = X_n$ be the vector space of $(2n+1)\times(2n+1)$ skew-symmetric matrices, with the natural action of $G = \mathrm{GL}_{2n+1}$. The coordinate ring of X is $S = \mathbb{C}[x_{ij}]$ with $1 \leq i < j \leq 2n+1$. We consider the tuple $\underline{d} = (d_1, d_2, \ldots, d_{2n+1})$, where d_i is the Pfaffian of the skew-symmetric matrix obtained by removing the i-th row and column of the generic skew-symmetric matrix $(x_{ij})_{i,j\in[2n+1]}$ (with the convention $x_{ji} = -x_{ij}$ and $x_{ii} = 0$). The tuple $\underline{d}$ is a basis for a multiplicity-free representation V, where for $\alpha \in \mathbb{N}$, the corresponding representation V_α in Definition 6.3.3 is $S_{(\alpha^{2n})}\mathbb{C}^{2n+1}$ from (6.6) (see for instance [2, Thm. 4.1]). We associate to $\underline{d}$ the invariant differential operator

$$D_{\underline{d}} = \sum_{i=1}^{2n+1} d_i^* \cdot d_i,$$

and by Proposition 6.3.4 there exists a polynomial $b_V(s)$ with

$$D_{\underline{d}} \cdot \underline{d}^{\underline{s}} = b_V(s) \cdot \underline{d}^{\underline{s}}. \tag{6.32}$$

Theorem 6.4.3. *If $\underline{d}$ is the tuple of sub-maximal Pfaffians of the generic $(2n+1)\times(2n+1)$ skew-symmetric matrix, then*

$$b_{\underline{d}}(s) = b_V(s) = \prod_{i=0}^{n-1}(s+2i+3). \tag{6.33}$$

Proof. We begin by showing, using the strategy from the proof of Theorem 6.3.5, that $b_V(s) = \prod_{i=0}^{n-1}(s+2i+3)$. We have a commutative diagram

$$\begin{tikzcd} \mathcal{U}(\mathfrak{gl}_{2n+1}) \arrow[r, "\mathcal{F}_{2n}"] \arrow[d, "\tau"'] & \mathcal{U}(\mathfrak{gl}_{2n+1}) \arrow[d, "\tau"] \\ \mathcal{D}_X \arrow[r, "\mathcal{F}"] & \mathcal{D}_X \end{tikzcd}$$

If we let $D_{\underline{d}^*} = \sum_{i=1}^{2n+1} d_i \cdot d_i^*$ then $D_{\underline{d}} = (-1)^n \cdot \mathcal{F}(D_{\underline{d}^*})$. It follows from [16, Thm. 2.3] that

$$d_i^* \cdot d_0^s = 0 \text{ for } i \neq 0, \quad d_0^* \cdot d_0^s = \left(\prod_{i=0}^{n-1}(s+2i)\right) \cdot d_0^{s-1},$$

from which we obtain

$$D_{\underline{d}^*} \cdot d_0^s = \left(\prod_{i=0}^{n-1}(s+2i)\right) \cdot d_0^s.$$

Since $D_{\underline{d}^*}$ is in $\tau(\mathcal{ZU}(\mathfrak{gl}_{2n+1}))$ by [29, Cor. 11.3.19], it follows from Lemma 6.2.2 with $r = 2n+1$ that

$$D_{\underline{d}} \cdot d_0^s = \left((-1)^n \cdot \prod_{i=0}^{n-1}(-s-2n-1+2i)\right) \cdot d_0^s,$$

from which we obtain

$$b_V(s) = \prod_{i=0}^{n-1}(s+2i+3). \tag{6.34}$$

We have that $b_{\underline{d}}(s) = b_{Z_n}(s+3)$ since Z_n has codimension three in X_n, so (6.33) is equivalent to $b_{Z_n}(s) = \prod_{i=0}^{n-1}(s+2i)$. By induction on n we have $b_{Z_{n-1}}(s) = \prod_{i=0}^{n-2}(s+2i)$, which divides $b_{Z_n}(s)$ by (6.25). This shows that $-3, -5, \dots, -2n+1$ are roots of $b_{\underline{d}}(s)$, and since $b_{\underline{d}}(s)$ divides $b_V(s)$, it follows from (6.34) that the only other possible root is $-2n-1$. Using [47, Thm. 5.5] and Corollary 6.3.2 with $r = 2n+1$ and I being the ideal generated by the d_i's, it follows that $-2n-1$ is indeed a root of $b_{\underline{d}}(s)$, hence (6.33) holds. $\square$

If we write Z_n for the zero locus of I, i.e. the variety of $(2n+1)\times(2n+1)$ skew-symmetric matrices of rank at most $(2n-2)$, then by (1.9) we get $b_{Z_n}(s) = \prod_{i=0}^{n-1}(s+2i)$. By [36, Appendix], this is the same as the b-function of the hypersurface of singular $2n \times 2n$ skew-symmetric matrices.

Remark 6.4.4. The method described in Remark 6.4.2 can be used in this case as well. Using the decomposition (6.6) and the Littlewood-Richardson rule, we see that $d_i^* \cdot S_{((\alpha+1)^{2n})}\mathbb{C}^{2n+1} \subset S_{(\alpha^{2n})}\mathbb{C}^{2n+1}$ for $\alpha \in \mathbb{N}$. Moreover, under the action of diagonal matrices the weights of $d_1, \dots, d_{2n+1}$ are linearly independent. Hence the tuple $\underline{d} = (d_1, d_2, \dots, d_{2n+1})$ has a b-function of several variables, and similarly in the proof of Theorem 3.1.1 we obtain the formulas

$$d_i^* \cdot d_i \cdot \underline{d}^{\underline{s}} = (s_i+1)(s+3)(s+5)\cdots(s+2n-1) \cdot \underline{d}^{\underline{s}}.$$

Together with the analogue of Lemma 6.5.3 below, this gives the Bernstein-Sato polynomial of the ideal.

6.5 Bernstein–Sato polynomials for maximal minors

In this section we generalize Theorem 6.4.1 to arbitrary $m \times n$ matrices. Recall that $\underline{d} = (d_K)_{K \in \binom{[m]}{n}}$ is the tuple of maximal minors as in (6.2).

Theorem 6.5.1. *The Bernstein–Sato polynomial of the tuple of maximal minors of the generic $m \times n$ matrix is*

$$b_{\underline{d}}(s) = \prod_{i=m-n+1}^{m} (s+i).$$

For general $m \geq n$, if we let $Z_{m,n}$ denote the zero locus of I, i.e. the variety of $m \times n$ matrices of rank at most $n-1$, then using the renormalization (1.9) our theorem states that the b-function of $Z_{m,n}$ is $\prod_{i=0}^{n-1}(s+i)$. It is interesting to note that this only depends on the value of n and not on m.

We know by Proposition 6.3.4 and Theorem 6.3.5 that $b_{\underline{d}}(s)$ divides $\prod_{i=m-n+1}^{m}(s+i)$. By induction, we also know from (6.25) that $b_{\underline{d}}(s)$ is divisible by $\prod_{i=m-n+1}^{m-1}(s+i)$, so we would be done if we can show that $-m$ is a root of $b_{\underline{d}}(s)$. This would follow from Proposition 6.3.1 if we could prove the following:

Conjecture 2. *If we associate as in Section 6.3 the $\mathcal{D}$-modules F_α, $\alpha \in \mathbb{Z}$, to the tuple d of maximal minors of the generic $m \times n$ matrix, then there exists a strict inclusion $F_{-m+1} \subsetneq F_{-m}$.*

We weren't able to verify this conjecture when $m > n+1$, so we take a different approach. We consider the $(1 + n \cdot (m-n))$-tuple

$$\underline{p} = (p_0, p_{ij}) \in S^{1+n\cdot(m-n)},$$

as in (6.3) and associate to p_0 a variable s_0, and to each p_{ij} a variable s_{ij}. We write $\underline{s} = (s_0, s_{ij})$ and consider $B_{\underline{p}}^{\underline{s}}$ as defined in (1.6). Inside $B_{\underline{p}}^{\underline{s}}$, we consider the

$\mathbb{C}[\underline{s}]$-submodule

$$A_{\underline{p}}^{\underline{s}} = \mathbb{C}[\underline{s}] \cdot \left\{ \underline{p}^{\underline{s}+\hat{c}} = p_0^{s_0+c_0} \cdot \prod_{i\in[n],j\in[m]\setminus[n]} p_{ij}^{s_{ij}+c_{ij}} : \hat{c} = (c_0, c_{ij}) \in \mathbb{Z}^{1+n\cdot(m-n)} \right\}. \tag{6.35}$$

A more invariant way of describing $A_{\underline{p}}^{\underline{s}}$ follows from the discussion in Section 6.2:

$$A_{\underline{p}}^{\underline{s}} \text{ consists precisely of the } \mathfrak{sl}_n\text{-invariants inside the } \mathcal{D}_X\text{-module } B_{\underline{p}}^{\underline{s}}. \tag{6.36}$$

It follows that $A_{\underline{p}}^{\underline{s}}$ is in fact a $\mathcal{D}_X^{\mathfrak{sl}_n}$-module. Since $\partial_K \in \mathcal{D}_X^{\mathfrak{sl}_n}$ for every $K \in \binom{[m]}{n}$, we can make the following:

Definition 6.5.2. We let $s = s_0 + \sum_{i,j} s_{ij}$ and define $a_{\underline{p}}(s)$ to be the monic polynomial of the lowest degree in s for which $a_{\underline{p}}(s) \cdot \underline{p}^{\underline{s}}$ belongs to

$$\mathbb{C}[\underline{s}] \cdot \left\{ \partial_K \cdot \underline{p}^{\underline{s}+\hat{c}} : K \in \binom{[m]}{n}, |\hat{c}| = 1 \right\}.$$

With $b_V(s)$ as computed in Theorem 6.3.5 we will prove that

$$a_{\underline{p}}(s) \text{ divides } b_{\underline{d}}(s), \text{ and} \tag{6.37}$$

$$b_V(s) \text{ divides } a_{\underline{p}}(s). \tag{6.38}$$

Combining (6.37) with (6.38), and with the fact that $b_{\underline{d}}(s)$ divides $b_V(s)$, concludes the proof of Theorem 6.5.1.

It follows from (6.5) that the elements $\underline{p}^{\underline{s}+\hat{c}}$ in (6.35) in fact give a basis of $A_{\underline{p}}^{\underline{s}}$ as a $\mathbb{C}[\underline{s}]$-module. We have

$$A_{\underline{p}}^{\underline{s}} = \bigoplus_{\alpha\in\mathbb{Z}} A_{\underline{p}}^{\underline{s}}(\alpha)$$

which we can think of as a weight space decomposition, where

$$A_{\underline{p}}^{\underline{s}}(\alpha) = \mathbb{C}[\underline{s}] \cdot \left\{ \underline{p}^{\underline{s}+\hat{c}} : |\hat{c}| = \alpha \right\} \tag{6.39}$$

is the set of elements in $A_{\underline{p}}^{\underline{s}}$ on which $g \in \mathfrak{gl}_n$ acts by multiplication by $\mathrm{tr}(g)\cdot(s+\alpha)$, and in particular each $A_{\underline{p}}^{\underline{s}}(\alpha)$ is preserved by $\mathcal{D}_X^{\mathfrak{gl}_n}$. Using (6.4) we obtain that multiplication by d_K sends $A^{\underline{s}}_{\underline{p}}(\alpha)$ into $A^{\underline{s}}_{\underline{p}}(\alpha+1)$. Since $d_K \cdot \partial_K \in \mathcal{D}_X^{\mathfrak{gl}_n}$, it then follows that multiplication by ∂_K sends $A^{\underline{s}}_{\underline{p}}(\alpha+1)$ into $A^{\underline{s}}_{\underline{p}}(\alpha)$. We obtain:

Lemma 6.5.3. *The polynomial $a_{\underline{p}}(s)$ is the monic polynomial of lowest degree for which there exist a finite collection of tuples $\hat{c} \in \mathbb{Z}^{1+n\cdot(m-n)}$ with $|\hat{c}| > 0$ and corresponding operators $Q_{\hat{c}} \in \mathcal{D}_X[\underline{s}]$ such that*

$$\sum_{\hat{c}} Q_{\hat{c}} \cdot \underline{p}^{\underline{s}+\hat{c}} = a_{\underline{p}}(s) \cdot \underline{p}^{\underline{s}}. \tag{6.40}$$

Proof. Using the fact that $\underline{p}^{\underline{s}+\hat{c}}$ and $a_{\underline{p}}(s)\cdot\underline{p}^{\underline{s}}$ are $\mathfrak{sl}_n$-invariants, we may assume that $Q_{\hat{c}} \in \mathcal{D}_X^{\mathfrak{sl}_n}[\underline{s}]$. Since every element in $\mathcal{D}_X^{\mathfrak{sl}_n}$ can be expressed as a linear combination of products $Q_1 \cdot Q_2 \cdot Q_3$, where Q_1 is a product of ∂_K's, Q_2 is a product of d_K's, and $Q_3 \in \mathcal{D}_X^{\mathfrak{gl}_n}$, the conclusion follows from the observation that $\mathcal{D}_X^{\mathfrak{gl}_n}$ preserves each weight space, d_K increases the weight by one, while ∂_K decreases the weight by one. $\square$

We are now ready to prove that $a_{\underline{p}}(s)$ divides $b_{\underline{d}}(s)$:

Proof of (6.37). Using (1.8) with $\underline{s} = (s_K)_{K\in\binom{[m]}{n}}$ we can find a finite collection of tuples $\hat{c} \in \mathbb{Z}^{\binom{[m]}{n}}$ with $|\hat{c}| = 1$, and corresponding operators $P_{\hat{c}} \in \mathcal{D}_X[\underline{s}]$ such that we have an equality inside $B^{\underline{s}}_{\underline{d}}$:

$$\sum_{\hat{c}} \hat{c}_{\underline{s}} \cdot P_{\hat{c}} \cdot \underline{d}^{\underline{s}+\hat{c}} = b_{\underline{d}}(s) \cdot \underline{d}^{\underline{s}}. \tag{6.41}$$

Note that by (1.7), setting $s_K = 0$ makes $\hat{c}_{\underline{s}} = 0$ whenever $\hat{c}$ is such that $c_K < 0$. We apply to (6.41) the specialization

$$\begin{aligned} &s_K = 0 \text{ whenever } |K \cap [n]| \leq n-2, \text{ and} \\ &s_{[n]} = s_0,\ s_{[n]\setminus\{i\}\cup\{j\}} = s_{ij} \text{ for } i \in [n], j \in [m]\setminus[n]. \end{aligned} \tag{6.42}$$

We then use the equalities $p_0 = d_{[n]}$, $p_{ij} = d_{[n]\setminus\{i\}\cup\{j\}}$ and (6.4), and regroup the terms to obtain (with an abuse of notation) a finite collection of tuples $\hat{c} = (c_0, c_{ij}) \in \mathbb{Z}^{1+n\cdot(m-n)}$ with $|\hat{c}| = 1$, and corresponding operators $Q_{\hat{c}} \in \mathcal{D}_X[\underline{s}]$, where

$\underline{s}$ denotes now the tuple of variables (s_0, s_{ij}), such that the following equality holds in $B_{\underline{p}}^{\underline{s}}$:

$$\sum_{\hat{c}} Q_{\hat{c}} \cdot \underline{p}^{\underline{s}+\hat{c}} = b_{\underline{d}}(s) \cdot \underline{p}^{\underline{s}}.$$

Using Lemma 6.5.3 it follows that $a_p(s)$ divides $b_d(s)$ as desired. ◻

We conclude by proving (6.38), but before we establish a preliminary result. For $|\hat{c}| = 1$ we observe that $\underline{p}^{\underline{s}+\hat{c}} \in A_{\underline{p}}^{\underline{s}}(1)$, thus $\partial_K \cdot \underline{p}^{\underline{s}+\hat{c}}$ can be expressed as a $\mathbb{C}[\underline{s}]$-linear combination of the basis elements of $A_{\underline{p}}^{\underline{s}}(0)$. We define $Q_{K,\hat{c}} \in \mathbb{C}[\underline{s}]$ to be the coefficient of $\underline{p}^{\underline{s}}$ in this expression, and write $\hat{e} = (1, 0^{n\cdot(m-n)})$.

Lemma 6.5.4. *Write $Q^0_{K,\hat{c}} \in \mathbb{C}[s_0]$ for the result of the specialization $s_{ij} = -1$ for all $i \in [n], j \in [m] \setminus [n]$, applied to $Q_{K,\hat{c}}$. We have that $Q^0_{K,\hat{c}} = 0$ unless $K = [n]$ and $\hat{c} = \hat{e}$.*

Proof. Since the specialization map commutes with the action of $\mathcal{D}_X$, we have that

$$Q^0_{K,\hat{c}} \text{ is the coefficient of } \frac{p_0^{s_0}}{\prod_{i,j} p_{ij}} \text{ inside } \partial_K \cdot \left(p_0^{s_0+c_0} \cdot \prod_{i,j} p_{ij}^{c_{ij}-1} \right).$$

Suppose first that $\hat{c}$ is a tuple with some entry $c_{i_0 j_0} \geq 1$: we show that for any K, $Q^0_{K,\hat{c}} = 0$. To see this, note that applying any sequence of partial derivatives to $p_0^{s_0+c_0} \cdot \prod_{i,j} p_{ij}^{c_{ij}-1}$ won't turn the exponent of $p_{i_0 j_0}$ negative. Since $\partial_K \in \mathcal{D}_X^{\mathfrak{sl}_n}$, we may then assume that

$$\partial_K \cdot \left(p_0^{s_0+c_0} \cdot \prod_{i,j} p_{ij}^{c_{ij}-1} \right) = p_0^{s_0+d_0} \cdot \prod_{i,j} p_{ij}^{d_{ij}} \cdot F, \tag{6.43}$$

where $d_0, d_{ij} \in \mathbb{Z}$, $d_{i_0 j_0} = 0$, and $F \in S^{\mathfrak{sl}_n}[s_0]$ is a polynomial in s_0 whose coefficients are $\mathfrak{sl}_n$-invariant. Since $S^{\mathfrak{sl}_n}$ is generated by the maximal minors d_K, we can apply (6.4) to rewrite the right hand side of (6.43) as a $\mathbb{C}[s_0]$-linear combination of $p_0^{s_0+e_0} \cdot \prod_{i,j} p_{ij}^{e_{ij}}$ where $e_0, e_{ij} \in \mathbb{Z}$ and $e_{i_0 j_0} \geq 0$. We conclude that $Q^0_{K,\hat{c}} = 0$.

From now on we assume that $\hat{c}$ is has all $c_{ij} \leq 0$. Since $|\hat{c}| = 1$, we must have $c_0 \geq 1$. We look at weights under the action of the subalgebra

$$\left\{ T_t = \begin{pmatrix} t \cdot I_n & 0 \\ 0 & 0 \end{pmatrix} : t \in \mathbb{C} \right\} \subset \mathfrak{gl}_m,$$

and note that

$$T_t \cdot \left(p_0^{s_0+c_0} \cdot \prod_{i,j} p_{ij}^{c_{ij}-1} \right) = t \cdot \left((s_0 + c_0) \cdot n + (n-1) \sum_{i,j} (c_{ij} - 1) \right) \cdot \left(p_0^{s_0+c_0} \cdot \prod_{i,j} p_{ij}^{c_{ij}-1} \right),$$

$$T_t \cdot \partial_K = -t \cdot |K \cap [n]| \cdot \partial_K, \text{ by(6.8), and}$$

$$T_t \cdot \left(\frac{p_0^{s_0}}{\prod_{i,j} p_{ij}} \right) = t \cdot \left(s_0 \cdot n + (n-1) \sum_{i,j} (-1) \right) \cdot \left(\frac{p_0^{s_0}}{\prod_{i,j} p_{ij}} \right).$$

It follows that $Q^0_{K,\hat{c}}$ can be non-zero only when

$$(s_0 + c_0) \cdot n + (n-1) \sum_{i,j} (c_{ij} - 1) - |K \cap [n]| = s_0 \cdot n + (n-1) \sum_{i,j} (-1),$$

which using the fact that $c_0 + \sum_{i,j} c_{ij} = 1$ is equivalent to $c_0 + (n-1) = |K \cap [n]|$. Since $c_0 \geq 1$ this equality can only hold when $c_0 = 1$ (which then forces all $c_{ij} = 0$), and $K = [n]$. $\square$

Proof of (6.38). Using Definition 6.5.2, we can find finitely many tuples $\hat{c} \in \mathbb{Z}^{1+n\cdot(m-n)}$ with $|\hat{c}| = 1$, and polynomials $P_{K,\hat{c}} \in \mathbb{C}[\underline{s}]$ for $K \in \binom{[m]}{n}$ such that

$$\sum_{K,\hat{c}} P_{K,\hat{c}} \cdot \partial_K \cdot \underline{p}^{\underline{s}+\hat{c}} = a_{\underline{p}}(s) \cdot \underline{p}^{\underline{s}}. \tag{6.44}$$

Using the definition of $Q_{K,\hat{c}}$, we obtain

$$\sum_{K,\hat{c}} P_{K,\hat{c}} \cdot Q_{K,\hat{c}} = a_{\underline{p}}(s).$$

Applying the specialization $s_{ij} = -1$ for all $i \in [n], j \in [m] \backslash [n]$, it follows from Lemma 6.5.4 that

$$P^0_{[n],\hat{e}} \cdot Q^0_{[n],\hat{e}} = \sum_{K,\hat{c}} P^0_{K,\hat{c}} \cdot Q^0_{K,\hat{c}} = a_{\underline{p}}(s_0 - n \cdot (m-n)),$$

where $P^0_{K,\hat{c}} \in \mathbb{C}[s_0]$ is (just as $Q^0_{K,\hat{c}}$) the specialization of $P_{K,\hat{c}}$. We will show that $Q^0_{[n],\hat{e}} = b_V(s_0 - n\cdot(m-n))$, from which it follows that $b_V(s_0 - n\cdot(m-n))$ divides $a_{\underline{p}}(s_0 - n\cdot(m-n))$. Making the change of variable $s = s_0 - n\cdot(m-n)$ proves that $b_V(s)$ divides $a_{\underline{p}}(s)$, as desired.

To see that $Q^0_{[n],\hat{e}} = b_V(s_0 - n\cdot(m-n))$, we consider the action of $D_{\underline{d}}$ on $\underline{p}^{\underline{s}}$: using (6.29), Theorem 6.3.5, and applying the specialization (6.42) as before, we obtain

$$\sum_{K\in\binom{[m]}{n}} \partial_K \cdot d_K \cdot \underline{p}^{\underline{s}} = D_{\underline{d}} \cdot \underline{p}^{\underline{s}} = b_V(s)\cdot \underline{p}^{\underline{s}}.$$

Using (6.4), we can rewrite the above equality as

$$\partial_{[n]} \cdot \underline{p}^{\underline{s}+\hat{e}} + \sum_{\substack{K\neq[n] \\ \hat{c} \text{ with } |\hat{c}|=1}} R_{K,\hat{c}} \cdot \partial_K \cdot \underline{p}^{\underline{s}+\hat{c}} = P_d(s)\cdot \underline{p}^{\underline{s}},$$

for some $R_{K,\hat{c}} \in \mathbb{C}[\underline{s}]$. We now apply the same argument as we did to (6.44): we consider the further specialization $s_{ij} = 0$ and use Lemma 6.5.4 to obtain $Q^0_{[n],\hat{e}} = P_d(s_0 - n\cdot(m-n))$, which concludes our proof. $\square$

6.6 On the Strong Monodromy Conjecture

Let $X = \mathbb{C}^N$ and $Y \subset X$ a closed subscheme with defining ideal I. Consider a *log resolution* $f : X' \to X$ of the ideal I (or of the pair (X,Y); see for instance [39, Sec. 9.1.B]), i.e. a proper birational morphism $f : X' \to X$ such that $I\mathcal{O}_{X'}$ defines an effective Cartier divisor E, f induces an isomorphism $f : X' \setminus E \to X \setminus Y$, and the divisor $K_{X'/X} + E$ has simple normal crossings support. Write E_j, $j \in \mathcal{J}$, for the irreducible components of the support of E, and express

$$E = \sum_{j\in\mathcal{J}} a_j E_j, \quad K_{X'/X} = \sum_{j\in\mathcal{J}} k_j \cdot E_j.$$

The *topological zeta function* of I (or of the pair (X,Y)) is defined as [19, 20, 70]

$$Z_I(s) = \sum_{\mathcal{I}\subseteq\mathcal{J}} \chi(E^\circ_{\mathcal{I}}) \cdot \prod_{i\in\mathcal{I}} \frac{1}{a_i\cdot s + k_i + 1}, \tag{6.45}$$

where χ denotes the Euler characteristic and $E_{\mathcal{I}}^{\circ} = (\bigcap_{i\in\mathcal{I}} E_i) \setminus (\bigcup_{i\notin\mathcal{I}} E_i)$. The topological zeta function is independent of the log resolution, and the Strong Monodromy Conjecture asserts that the poles of $Z_I(s)$ are roots of $b_I(s)$, and in an even stronger form that

$$b_I(s) \cdot Z_I(s) \text{ is a polynomial.} \tag{6.46}$$

We verify (6.46) for maximal minors and sub-maximal Pfaffians as a consequence of Theorems 6.4.3 and 6.5.1, by taking advantage of the well-studied resolutions given by *complete collineations* in the case of determinantal varieties, and *complete skew forms* in the case of Pfaffian varieties [69, 67, 30].

Let $m \geq n$ and $X = X_{m,n}$ denote the vector space of $m \times n$ matrices as before. Denote by Y the subvariety of matrices of rank at most $n-1$, and let I be the ideal of maximal minors defining Y. It follows from [30, Cor. 4.5 and Cor. 4.6] that I has a log resolution with $\mathcal{J} = \{0, \dots, n-1\}$ and

$$E = \sum_{i=0}^{n-1} (n-i) \cdot E_i, \quad K_{X'/X} = \sum_{i=0}^{n-1} ((m-i)(n-i) - 1) \cdot E_i.$$

It follows that $k_i + 1 = (m-i)(n-i)$, and $a_i = n - i$ for $i = 0, \dots, n-1$, and therefore by our Theorem 6.5.1 the denominator of every term in (6.45) divides $b_I(s)$. This is enough to conclude (6.46).

Let $X = X_n$ be the vector space of $(2n+1) \times (2n+1)$ skew-symmetric matrices. Denote by Y the subvariety of matrices of rank at most $2(n-1)$ and let I denote the ideal of sub-maximal Pfaffians defining Y. As shown below, there is a log resolution of I with $\mathcal{J} = \{0, \dots, n-1\}$ and

$$E = \sum_{i=0}^{n-1} (n-i) \cdot E_i, \quad K_{X'/X} = \sum_{i=0}^{n-1} (2(n-i)^2 + (n-i) - 1) \cdot E_i. \tag{6.47}$$

It follows that $(k_i + 1)/a_i = 2(n-i) + 1$ for $i = 0, \dots, n-1$, and thus our Theorem 6.4.3 implies (6.46).

We sketch the construction of the log resolution, based on the strategy in [30, Chapter 4]: this is perhaps well-known, but we weren't able to locate (6.47) explicitly in the literature. We write $Y_i \subset X$ for the subvariety of $(2n+1) \times (2n+1)$ skew-symmetric matrices of rank at most $2i$. We define the sequence of transformations $\pi_i : X^{i+1} \to X^i$, $f_i = \pi_0 \circ \pi_1 \circ \cdots \circ \pi_i : X^{i+1} \to X^0$, where $X^0 = X$, X^1 is the blow-up of X^0 at Y_0, and in general X^{i+1} is the blow-up of X^i at the strict transform Y^i of Y_i along f_{i-1}. The desired log resolution is obtained by letting $X' = X^n$ and $f = f_{n-1} : X' \to X$. Each Y^i is smooth (as we'll see shortly), so the same is true about the exceptional divisor E_i of the blow-up π_i. We abuse notation and write E_i also for each of its transforms along the blow-ups $\pi_{i+1}, \dots, \pi_{n-1}$. It follows from the construction below that the E_i's are defined locally by the vanishing of distinct coordinate functions, so $f : X' \to X$ is indeed a log resolution.

We show by induction on $i = n, n-1, \dots$ that X^{n-i} admits an affine open cover where each open set V in the cover has a filtration $V = V_i \supset V_{i-1} \supset \cdots \supset V_0$, isomorphic to

$$(Y_i^i \supset Y_{i-1}^i \supset \cdots \supset Y_0^i) \times \mathbb{C}^{4i+3} \times \cdots \times \mathbb{C}^{4(n-1)-1} \times \mathbb{C}^{4n-1}, \tag{6.48}$$

where $Y_i^n = Y_i$ and more generally

Y_j^i is the variety of $(2i+1) \times (2i+1)$ matrices of rank at most $2j$.

The key property of the filtration (6.48) is that for each $j = 0, \dots, i$, V_j is obtained by intersecting V with the strict transform of Y_{n-i+j} along f_{n-i-1}. In particular $V_0 = V \cap Y^{n-i}$ is (on the affine patch V) the center of blow-up for π_i. Since Y_0^0 is just a point, V_0 is an affine space and hence smooth.

When $i = n$, $X^{n-i} = X$, so we can take $V = X$ and (6.48) to be the filtration $X = Y_n \supset Y_{n-1} \supset \cdots \supset Y_0$. We discuss the first blow-up ($i = n-1$) and the associated filtration, while for $i < n-1$ the conclusion follows from an easy iteration of our argument. We write x_{ij} (resp. y_{ij}), $1 \leq i < j \leq 2n+1$ for the coordinate functions on X (resp. on $\mathbb{P}X$, the projectivization of X). X^1 is defined inside $X \times \mathbb{P}X$ by the equations $x_{ij}y_{kl} = x_{kl}y_{ij}$, and we choose $V \subset X^1$ to be the affine patch where $y_{12} \neq 0$ (similar reasoning applies on each of the affine patches $y_{ij} \neq 0$). The coordinate functions on V are $t_0 = x_{12}$ and $u_{ij} = y_{ij}/y_{12}$ for $(i,j) \neq (1,2)$. Setting $u_{12} = 1$, we get that the map $\pi_0 : V \to X^0$ corresponds to a ring homomorphism

$$\pi_0^* : \mathbb{C}[x_{ij}] \longrightarrow \mathbb{C}[t_0, u_{ij}] \text{ given by } x_{ij} \mapsto t_0 \cdot u_{ij},$$

and $E_0 \cap V$ is defined by the equation $t_0 = 0$. With the usual conventions $u_{ji} = -u_{ij}$, $u_{ii} = 0$, we write $M_{ij} = \mathrm{Pf}_{\{1,2,i+2,j+2\}}$ for the Pfaffian of the 4×4 principal skew-symmetric submatrix of (u_{ij}) obtained by selecting the rows and columns of u indexed by $1, 2, i+2$ and $j+2$, $1 \leq i, j \leq 2n-1$. Using the calculation in the proof of Lemma 6.2.4, we obtain that $\{M_{ij} : 1 \leq i < j \leq 2n-1\} \cup \{t_0\} \cup \{u_{1i}, u_{2i} : i = 3, \ldots, 2n+1\}$ is a system of coordinate functions on V, and moreover

$$\pi_0^*(I_{p+1}(x_{ij})) = t_0^{p+1} \cdot I_p(M_{ij}), \text{ for } p = 1, \ldots, n, \tag{6.49}$$

where $I_p(a_{ij})$ denotes the ideal generated by the $2p \times 2p$ Pfaffians of the skew-symmetric matrix (a_{ij}). Thinking of $\{t_0\} \cup \{u_{1i}, u_{2i} : i = 3, \ldots, 2n+1\}$, as the coordinate functions on $\mathbb{C}^{4n-1}$, and of $\{M_{ij}\}$ as the coordinate functions on $X_{n-1} = Y_{n-1}^{n-1}$, we identify Y_{p-1}^{n-1} with the zero locus of $I_p(M_{ij})$ for $p = 1, \ldots, n$, and note that by (6.49) it is the strict transform of Y_p which is the variety defined by $I_{p+1}(x_{ij})$. This yields the filtration (6.48) for $i = n-1$. By letting $p = n-1$ in (6.49) and noting that $I = I_n(x_{ij})$, we obtain that the inverse image $\pi_0^{-1}(I) = I\mathcal{O}_{X^1}$ vanishes with multiplicity n along E_0. Iterating this, we obtain the formula (6.47) for the exceptional divisor E. Pulling back the standard volume form $dx = dx_{12} \wedge \cdots \wedge dx_{n-1,n}$ on X along π_0, we obtain (on the affine patch V)

$$\pi_0^*(dx) = t_0^{2n^2+n-1} \cdot dt_0 \wedge du_{13} \wedge \cdots \wedge du_{n-1,n},$$

which vanishes with multiplicity $2n^2 + n - 1$ along E_0. Iterating this, we obtain formula (6.47) for $K_{X'/X}$.

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