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Three Essays in Macroeconomics of Fiscal and Monetary Policies

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Abstract

This dissertation studies the effects of inflation on long-term economic growth and economic inequality through the interactions of fiscal and monetary policy. Inflation is generated through the faster growth rate of nominal money supply. The fiscal policy in discussion includes different tax schedules, productive government spending, unproductive government spending, and transfer. Chapter one examines the effects of inflation on the distributions of income, earnings, consumption and wealth. We build a dynamic general equilibrium model in which consumers differ in terms of their earning abilities and time preference. Money is introduced via a generalized cash-in-advance constraint. In the quantitative analysis, we first calibrate the model to match the income and wealth distributions in the United States, and other key features of the U.S. economy. We then conduct a series of counterfactual experiments to quantify the distributional impacts of

inflation. Chapter two discusses the growth effects of inflation. We build a monetary search model with two subperiods. One subperiod captures the frictional, decentralized trading and bargaining between buyers and sellers. The other subperiod is the centralized, neoclassical growth model. The long-term economic growth is promoted by the productive government spending and the growth rate is endogenously determined. More money on the one hand generates inflation, and on the other, it facilitates stochastic trading and therefore enhances the assets accumulation. Chapter three investigates the question about how the property of the progressive tax affects the relationship between economic inequality and long-term economic growth. We find that heterogeneity has positive (negative) effects on capital accumulation and economic growth if and only if the marginal tax function of the progressive tax is concave (convex).

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1 Anticipated Inflation and Economic Inequality in a Cash-in-Advance Economy

1.1 Introduction

This paper examines the effects of anticipated inflation on economic inequality. To achieve this, we develop a dynamic general equilibrium model in which consumers differ in their earning abilities and subjective discount factors. Money is introduced through a cash-in-advance (CIA) constraint. A calibrated version of the model is able to match the extent of income and wealth inequality in the United States and replicate certain key features of the U.S. economy over the period 1960-2010. We then conduct a series of counterfactual experiments to quantify the impact of inflation on different groups of consumers.

The effects of inflation have long been a subject of interest to economists. There is now a large number of studies which explore the empirical and theoretical linkages between inflation and aggregate economic activities. Theoretical studies on the real effects of inflation include Tobin (1965), Sidrauski (1967), Stockman (1981), Wang and Yip (1992), and Suen and Yip (2005) among many others.[1] These existing studies, however, typically consider an environment in which consumers are identical and thus ignore the distributional effects of inflation. The present study departs from this literature by considering an environment in which consumers are *ex ante* heterogeneous. Our approach thus differs from İmrohoroğlu (1992), Erosa and Ventura (2002) and Camera and Chien (2014), which have examined the distributional effects of inflation in an environment in which consumers are *ex post*

[1] Empirical studies on the relationship between inflation and economic growth include Kormendi and Meguire (1985), De Gregorio (1993), Bullard and Keating (1995), Bruno and Easterly (1998) among many others.

heterogeneous (i.e., they experience different realizations of idiosyncratic shocks). Existing studies show that predetermined factors are at least as important as idiosyncratic shocks in explaining cross-sectional variation among consumers [see, for instance, Keane and Wolpin (1997) and Huggett *et al.* (2011)]. In the present study, we consider an environment in which consumers differ in their earning abilities and time preference. Using these two sources of consumer heterogeneity, our model is able to match the observed patterns of wealth and income inequality in the United States. This is not the first study that makes use of time preference heterogeneity to explain economic inequality.[2] In the literature of incomplete-market models, Krusell and Smith (1998) and Hendricks (2007) show that introducing this type of heterogeneity can generate a substantial concentration of wealth at the top end of the wealth distribution. In the neoclassical growth literature, Suen (2014) shows that time preference heterogeneity, together with a direct preferences for wealth, can lead to a high concentration of wealth. In Suen's (2014) model, the direct preferences for wealth prevent the wealth distribution from collapsing into a degenerate distribution.[3] In the present study, a progressive tax structure is used to serve this purpose. The same approach is also used in Sarte (1997) and Li and Sarte (2001).

Our main results show that an increase in money growth rate (and long-run inflation) will in general lower the inequality in real money holdings and consumption. This is the result of two effects. First, an increase in inflation raises the cost of holding real money balances. Holding other things constant, this will discourage the consumers from holding money, which will in turn suppress consumption through the CIA constraint. Second, a

[2] There is ample empirical evidence showing that consumers discount future values at difference rates. For a detailed review of these evidence, see Frederick *et al.* (2002) Section 6.

[3] Becker (1980) shows that in the standard neoclassical model where consumers have time-additive preferences and different subjective discount factors, all the wealth in the economy with eventually be owned by the most patient consumers.

faster money growth rate also means that more real money balances is transferred to the consumers. This will create a pure income effect which promotes consumption. For relatively poor consumers, the transfers represent a sizable portion of their income. Hence, an increase in inflation rate will induce them to have more consumption. But for the relatively affluent consumers, the negative effect of inflation dominates. These two forces together lead to a more equal distribution of consumption as inflation increases.

The present study is similar in spirit to Erosa and Ventura (2002), Heer and Süssmuth (2007), Camera and Chien (2014). Erosa and Ventura (2002) also show that when inflation increases, the distribution of wealth becomes more unequal, but their mechanism is different from ours. They emphasize the heterogeneity in transaction patterns and portfolio holdings across individuals through costly credit transactions as an alternative to monetary transactions. In their model, inflation is effectively a regressive consumption tax. They only discuss about the high-income and low-income families and therefore do not match the entire income and wealth distributions as we do in this paper. Heer and Süssmuth (2007) stress the heterogeneity in optimal portfolio holding across individuals and they explain the mechanism through Fredstein channel and stock market transaction costs. Higher inflation, due to Fredstein channel, increases real tax burden and due to the associated costs, widens the wealth disparity between the wealth-poor and the wealth-rich. But they didn't look at the distributional effects on income and earnings. Camera and Chien (2014) consider an incomplete-market environment in which money is used for self-insurance purpose. In the benchmark model where money is the only asset available for self-insurance against earning risks, an increase in the inflation rate will reduce wealth inequality and raise consumption inequality. When bond is introduced as a competing asset, an increase in inflation will lower consumption inequality and increase wealth inequality. Similar to the standard incomplete-market model, their model has difficulty in matching the actual extent of the wealth inequality observed in the data.

The rest of this paper is organized as follows. Section 2 describes the model environment and define the competitive equilibrium. Section 3 describes the benchmark parameter values that we use in the quantitative exercise. Section 4 presents the results from the benchmark model and the counterfactual experiments. Section 5 concludes.

1.2 The Model

1.2.1 Consumers

Consider an economy inhabited by a continuum of infinitely-lived consumers. These consumers are different in terms of their innate characteristics. Specifically, there are $S > 1$ different types of consumers. Each type $i \in \{1, 2, ..., S\}$ is identified by a pair of fixed predetermined characteristics (β_i, ε_i), where $\beta_i \in (0, 1)$ is the subjective discount factor and $\varepsilon_i > 0$ is labor productivity. Consumers within the same group are identical in all aspects. The share of type-i consumers in the population is given by $\lambda_i \in (0, 1)$. The size of total population is constant over time and is normalized to one so that $\sum_{i=1}^{S} \lambda_i = 1$.

There is a single commodity in this economy which can be used for consumption and investment. In each period, each consumer is endowed with one unit of time which can be divided between labor and leisure. The consumer's preferences are given by

$$\sum_{t=0}^{\infty} \beta_i^t u(c_{i,t}, l_{i,t}),$$

where $c_{i,t}$ is the consumption of a type-i consumer at time t, and $l_{i,t}$ denotes his labor hours.

The period utility function is assumed to be additively separable in its arguments, i.e.,

$$u(c,l) = \ln c - A\frac{l^{1+\varphi}}{1+\varphi},$$

where $\varphi > 0$ is the inverse of the Frisch elasticity of labor supply, and $A > 0$ is a constant.[4]

In each period, each consumer receives two types of taxable income: labor income from work and interest income from non-monetary assets. Let w_t be the wage rate for an effective unit of labor at time t. Then the labor income for a type-i consumer at time t is given by $w_t \varepsilon_i l_{i,t}$. Let $a_{i,t}$ be the non-monetary assets owned by a type-i consumer at the beginning of time t, and let r_t be the rate of return. Then he receives an interest income of $r_t a_{i,t}$ at time t. The sum of these two sources of incomes, denoted by $y_{i,t} \equiv w_t \varepsilon_i l_{i,t} + r_t a_{i,t}$, is subject to a progressive income tax. The total amount of income tax that a type-i consumer has to pay at time t is $\tau_t(y_{i,t})$, where $\tau_t : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuously differentiable, strictly increasing and strictly convex function that satisfies $\tau_t(0) = 0$ for all $t \geq 0$. In each period, each consumer can accumulate wealth in the form of non-monetary assets and money. Let $M_{i,t}$ denote the nominal money holding of a type-i consumer at the beginning of time t, and let P_t be the general price level. The consumer's budget constraint at time t is then given by

$$c_{i,t} + a_{i,t+1} - a_{i,t} + \pi_{t+1} m_{i,t+1} - m_{i,t} = y_{i,t} - \tau_t(y_{i,t}) + \xi_{i,t},$$

where $m_{i,t} \equiv M_{i,t}/P_t$ is the real money balances, $\pi_{t+1} \equiv P_{t+1}/P_t$ is the growth factor of the general price level, and $\xi_{i,t}$ is a lump-sum real money transfer from the government. Similar to Dotsey and Sarte (2000), we assume that money is required for consumption purchases as

[4] It is well known that if the period utility function is additively separable in consumption and labor, then the utility function of consumption must be logarithmic in order to be consistent with balanced growth.

well as investment. In particular, the consumers are subject to the following cash-in-advance (CIA) constraint in every period,

$$c_{i,t} + \phi(a_{i,t+1} - a_{i,t}) \leq m_{i,t},$$

where $\phi \in [0,1]$. If $\phi = 0$, then money is only required for consumption purchases. This specification is often referred to as the Clower (1967) CIA constraint. If $\phi = 1$, then money is required for both consumption purchases and investment. This specification is first introduced by Stockman (1981). In the present study, we consider the general case in which ϕ can take any value between zero and one.

Given prices and government policies, the consumers' problem is to choose a sequence of consumption, labor hours, non-monetary assets and real money balances so as to maximize his lifetime utility, subject to the sequential budget constraints and the CIA constraints. Formally, this is given by

$$\max_{\{c_{i,t} l_{i,t}, a_{i,t+1}, m_{i,t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta_i^t u(c_{i,t}, l_{i,t})$$

subject to

$$c_{i,t} + a_{i,t+1} - a_{i,t} + \pi_{t+1} m_{i,t+1} - m_{i,t} = y_{i,t} - \tau_t(y_{i,t}) + \xi_{i,t}, \tag{1}$$

$$c_{i,t} + \phi(a_{i,t+1} - a_{i,t}) \leq m_{i,t}, \tag{2}$$

$$a_{i,t+1} \geq 0, \qquad m_{i,t+1} \geq 0, \qquad l_{i,t} \in [0,1],$$

and the initial conditions: $a_{i,0} \geq 0$ and $m_{i,0} \geq 0$.

Set up the Lagrangian for the consumer's problem

$$\begin{aligned}\mathcal{L} \;=\; & \sum_{t=0}^{\infty} \beta_i^t \left\{ u(c_{i,t}, l_{i,t}) + \zeta_{i,t}[y_{i,t} - \tau_t(y_{i,t}) + \xi_{i,t} - c_{i,t} - a_{i,t+1} + a_{i,t} - \pi_{t+1} m_{i,t+1} + m_{i,t}] \right. \\ & \left. + \theta_{i,t}[m_{i,t} - c_{i,t} - \phi(a_{i,t+1} - a_{i,t})] \right\},\end{aligned}$$

where $\zeta_{i,t}$ and $\theta_{i,t}$ are the Lagrangian multipliers for (68) and (69), respectively. The first-order conditions for an interior solution are given by

$$c_{i,t}^{-1} = \zeta_{i,t} + \theta_{i,t}, \tag{3}$$

$$A l_{i,t}^{\varphi} = \zeta_{i,t} w_t \varepsilon_i [1 - \tau_t'(y_{i,t})], \tag{4}$$

$$\zeta_{i,t} + \phi\theta_{i,t} = \beta_i \left\{ \zeta_{i,t+1} \left[\left(1 - \tau_{t+1}'(y_{i,t+1})\right) r_{t+1} + 1 \right] + \phi\theta_{i,t+1} \right\}, \tag{5}$$

$$\pi_{t+1}\zeta_{i,t} = \beta_i \left(\zeta_{i,t+1} + \theta_{i,t+1} \right). \tag{6}$$

Using (70) and (71), we can obtain

$$\zeta_{i,t} = \beta_i \left(\pi_{t+1} c_{i,t+1} \right)^{-1}, \tag{7}$$

$$\theta_{i,t} = c_{i,t}^{-1} - \beta_i \left(\pi_{t+1} c_{i,t+1} \right)^{-1}. \tag{8}$$

Note that the CIA constraint is binding at time t if $\theta_{i,t} > 0$. As we will see later on, this condition is always satisfied in any balanced-growth equilibrium. Substituting (74) into (72) gives the optimality condition for labor hours

$$A l_{i,t}^{\varphi} = \frac{\beta_i w_t \varepsilon_i [1 - \tau_t'(y_{i,t})]}{\pi_{t+1} c_{i,t+1}}. \tag{9}$$

The intuition behind this condition is as follows. Suppose a type-i consumer wants to increase his labor supply at time t from $l_{i,t}$ to $l_{i,t} + \zeta$, where $\zeta > 0$ is infinitesimal. On one hand, such an increase will lower his current utility by $Al_{i,t}^{\varphi}\zeta$ units. On the other hand, this will raise the after-tax labor income by $w_t \varepsilon_i \left[1 - \tau_t'(y_{i,t})\right] \zeta$. This allows the consumer to hold $w_t \varepsilon_i \left[1 - \tau_t'(y_{i,t})\right] \zeta / \pi_{t+1}$ more units of $m_{i,t+1}$, which can then be used to purchase consumption at time $t+1$. Thus, the discounted marginal benefits of increasing labor supply at time t is given by $\beta_i w_t \varepsilon_i \left[1 - \tau_t'(y_{i,t})\right] \zeta / \pi_{t+1} c_{i,t+1}$. In the optimal situation, these marginal benefits are equated to the marginal costs.

Using (73)-(75), we can obtain

$$\frac{\phi}{c_{i,t}} = \frac{\beta_i}{c_{i,t+1}} \left(\phi - \frac{1-\phi}{\pi_{t+1}} \right) + \frac{(\beta_i)^2}{\pi_{t+2} c_{i,t+2}} \left\{ \left[1 - \tau_{t+1}'(y_{i,t+1})\right] r_{t+1} + 1 - \phi \right\}, \tag{10}$$

which is the Euler equation for consumption. A detailed derivation of this equation can be found in the Appendix. Equation (78) equates the marginal costs of having an additional unit of non-monetary asset at time t to its discounted marginal benefits. The intuition of this equation can best be explained by considering two special cases: $\phi = 0$ and $\phi = 1$. When $\phi = 0$, the Euler equation becomes

$$\frac{1}{\pi_{t+1} c_{i,t+1}} = \frac{\beta_i}{\pi_{t+2} c_{i,t+2}} \left\{ \left[1 - \tau_{t+1}'(y_{i,t+1})\right] r_{t+1} + 1 \right\}. \tag{11}$$

Suppose a type-i consumer wants to increase his future asset holdings $a_{i,t+1}$ by $\zeta > 0$ units. In order to balance his budget, the consumer will have to simultaneously lower $m_{i,t+1}$ by (ζ/π_{t+1}). [5] Through the CIA constraint at time $t+1$, this will lower $c_{i,t+1}$

[5] Under the Clower CIA constraint, current consumption $c_{i,t}$ is determined by the current real money holdings $m_{i,t}$, which is predetermined in the previous period. Hence, the consumer cannot adjust his current consumption in this scenario.

by the same amount, which will in turn lower future utility by $(\pi_{t+1} c_{i,t+1})^{-1} \zeta$. On the other hand, the increase in future asset holdings will generate an after-tax gross return of $\left\{ \left[1 - \tau'_{t+1} \left(y_{i,t+1} \right) \right] r_{t+1} + 1 \right\} \zeta$. These additional resources will allow the consumer to increase $m_{i,t+2}$ and $c_{i,t+2}$ by $\left\{ \left[1 - \tau'_{t+1} \left(y_{i,t+1} \right) \right] r_{t+1} + 1 \right\} \zeta / \pi_{t+2}$. The expression on the right-hand side of (11) is the discounted marginal benefits generated by the increase in $a_{i,t+1}$.

When $\phi = 1$, the Euler equation becomes

$$\frac{1}{c_{i,t}} = \frac{\beta_i}{c_{i,t+1}} + \frac{(\beta_i)^2}{\pi_{t+2} c_{i,t+2}} \left\{ \left[1 - \tau'_{t+1} \left(y_{i,t+1} \right) \right] r_{t+1} \right\},$$

In this case, the sequential budget constraint and the CIA constraint at time t can be rewritten as

$$\pi_{t+1} m_{i,t+1} = y_{i,t} - \tau_t(y_{i,t}) + \xi_{i,t},$$

and

$$c_{i,t} + a_{i,t+1} - a_{i,t} = m_{i,t}.$$

Suppose a type-i consumer wants to increase his future asset holdings $a_{i,t+1}$ by $\zeta > 0$ units. In order to maintain the CIA constraint, the consumer has to lower $c_{i,t}$ by the same amount, which in turn lowers his current utility by $(\zeta / c_{i,t})$. The increase in $a_{i,t+1}$ has two positive effects. First, by the CIA constraint at time $t+1$, an increase in $a_{i,t+1}$ allows the consumer to have more $c_{i,t+1}$. This will raise his lifetime utility by $(\beta_i \zeta / c_{i,t+1})$. Second, the increase in $a_{i,t+1}$ will raise his future after-tax incomes by $\left[1 - \tau'_{t+1} \left(y_{i,t+1} \right) \right] r_{t+1} \zeta$, which allows him to increase $m_{i,t+2}$ by $\left[1 - \tau'_{t+1} \left(y_{i,t+1} \right) \right] r_{t+1} \zeta / \pi_{t+2}$. This can be used to support more consumption at time $t+2$, which will then increase lifetime utility by $(\beta_i)^2 \left\{ \left[1 - \tau'_{t+1} \left(y_{i,t+1} \right) \right] r_{t+1} \right\} \zeta / \left(\pi_{t+2} c_{i,t+2} \right)$. In the optimal situation, the discounted marginal benefits from these two effects must be balanced by the corresponding marginal costs.

1.2.2 Production Function

On the supply side of the economy, there is a large number of identical firms. In each period, each firm hires labor and rents physical capital from the competitive factor markets, and produces output according to a production technology

$$Y_t = K_t^{\alpha} \left(X_t N_t \right)^{1-\alpha} H_t^{\varrho}, \qquad \alpha \in (0,1) \text{ and } 1-\alpha > \varrho > 0,$$

where Y_t denotes output at time t, K_t denotes capital input, N_t denotes labor input, X_t is a labor-augmenting technological factor, and H_t is the infrastructure capital provided by the government. The technological factor is assumed to grow at a constant exogenous rate so that $X_t \equiv \rho^t$ for all t, where $\rho \geq 1$ and X_0 is normalized to one. Both X_t and H_t are taken as exogenously given by the firms. Since the production function exhibits constant returns to scale in private inputs (i.e., K_t and N_t), we can focus on the choices made by a single representative firm. Let R_t be the rental price of physical capital at time t. Then the representative firm solves the following problem

$$\max_{K_t, N_t} \left\{ K_t^{\alpha} \left(X_t N_t \right)^{1-\alpha} H_t^{\varrho} - w_t N_t - R_t K_t \right\},$$

and the first-order conditions are

$$R_t = \alpha K_t^{\alpha-1} \left(X_t N_t \right)^{1-\alpha} H_t^{\varrho}, \qquad \text{and} \qquad w_t = (1-\alpha) X_t K_t^{\alpha} \left(X_t N_t \right)^{-\alpha} H_t^{\varrho}. \tag{12}$$

Let $\delta \in (0,1)$ be the depreciation rate of physical capital. Then the rate of return from non-monetary assets r_t is given by $r_t = R_t - \delta$.

1.2.3 Government

The government in this economy implements both fiscal and monetary policies. In terms of monetary policies, the nominal supply of money $\left(\overline{M}_t^s\right)$ is assumed to grow at a deterministic time-invariant rate $\mu > 0$ in every period, so that $\overline{M}_{t+1}^s = (1+\mu)\overline{M}_t^s$, for all $t \geq 0$. A fraction $\varpi \in (0, 1]$ of the newly printed money (i.e., seigniorage) is distributed to the consumers through the lump-sum transfer, so that

$$P_t \sum_{i=1}^{S} \lambda_i \xi_{i,t} = \varpi \left(\overline{M}_{t+1}^s - \overline{M}_t^s\right) = \varpi \mu \overline{M}_t^s. \tag{13}$$

To simplify the analysis, we assume that each type of consumer receives a fixed proportion of the seigniorage revenue in each period, so that $P_t \xi_{i,t} = \overline{\xi}_i \varpi \mu \overline{M}_t^s$ and $\sum_{i=1}^{S} \lambda_i \overline{\xi}_i = 1$. The rest of the seigniorage revenue is used to support government expenditures.

As for fiscal policies, following Guo and Lansing (1998) and Li and Sarte (2004), we assume that the progressive tax function is given by

$$\tau_t(y_t) = \widetilde{\eta}_t (y_t)^{1+\varkappa}, \tag{14}$$

where $\varkappa$ is a positive constant, and $\widetilde{\eta}_t > 0$ is a time-varying parameter. Under this specification, the average tax rate (ATR) and the marginal tax rate (MTR) faced by a consumer with total taxable income y_t are given by

$$\text{ATR} \equiv \frac{\tau_t(y_t)}{y_t} = \widetilde{\eta}_t (y_t)^{\varkappa}, \qquad \text{and} \qquad \text{MTR} \equiv \frac{d\tau_t(y_t)}{dy_t} = (1+\varkappa)\widetilde{\eta}_t (y_t)^{\varkappa}.$$

Hence, the degree of progressivity, defined as the ratio between the marginal tax rate and the average tax rate is given by $(1+\varkappa)$, i.e.,

$$\text{Degree of Progressivity} = \frac{\text{MTR}}{\text{ATR}} = 1+\varkappa.$$

The government is required to balance its budget in each period. In particular, all the tax revenues collected by the government, together with a fraction $(1-\varpi)$ of the seigniorage revenue, are spent on infrastructure investment (I_t) and unproductive government spending (G_t). The government's budget constraint at time t is given by

$$I_t + G_t = \sum_{i=1}^{S} \lambda_i \tau_t(y_{i,t}) + (1-\varpi)\left(\frac{\overline{M}_{t+1}^s - \overline{M}_t^s}{P_t}\right), \qquad \text{for all } t \geq 0. \tag{15}$$

The stock of infrastructure capital then evolves according to

$$H_{t+1} = I_t + (1-\delta)\, H_t, \qquad \text{for all } t \geq 0. \tag{16}$$

Finally, we assume that a fraction $\overline{v} \in (0,1]$ of aggregate output is used as unproductive government spending in every period, i.e., $G_t = \overline{v} Y_t$, for all $t \geq 0$.

1.2.4 Competitive Equilibrium

To define a competitive equilibrium, we first define $\mathbf{c}_t = (c_{1,t}, c_{2,t}, ..., c_{S,t})$, $\mathbf{l}_t = (l_{1,t}, l_{2,t}, ..., l_{S,t})$, $\mathbf{a}_t = (a_{1,t}, a_{2,t}, ..., a_{S,t})$, $\mathbf{m}_t = (m_{1,t}, m_{2,t}, ..., m_{S,t})$ and $\boldsymbol{\xi}_t = \left(\xi_{1,t}, \xi_{2,t}, ..., \xi_{S,t}\right)$ as the cross-sectional distributions of consumption, labor hours, non-monetary assets, real money holdings and lump-sum transfers at time t. The exogenous policy instruments in this economy include a sequence of progressive tax functions $\{\tau_t(\cdot)\}_{t=0}^{\infty}$, and the parameters $\left\{\mu, \varpi, \overline{v}, \overline{\xi}_1, ..., \overline{\xi}_S\right\}$. Given these policy instruments, a competitive equilibrium of this economy consists of sequences of distributions $\{\mathbf{c}_t, \mathbf{l}_t, \mathbf{a}_t, \mathbf{m}_t\}_{t=0}^{\infty}$, aggregate inputs $\{K_t, N_t\}_{t=0}^{\infty}$, policy variables $\{\overline{M}_t^s, I_t, \boldsymbol{\xi}_t\}_{t=0}^{\infty}$, and prices $\{w_t, r_t, R_t, P_t\}_{t=0}^{\infty}$, such that

(i) Given prices and government policies, the allocation $\{c_{i,t}, l_{i,t}, a_{i,t+1}, m_{i,t+1}\}_{t=0}^{\infty}$ solves a type-i consumer's problem.

(ii) Given prices and infrastructure capital, the aggregate inputs $\{K_t, N_t\}_{t=0}^{\infty}$ solve the representative firm's problem in every period.

(iii) The government's budget is balanced in every period, i.e., (43) holds. Nominal money supply is determined by $\overline{M}_{t+1}^s = (1+\mu)\overline{M}_t^s$.

(iv) All markets cleared in every period, i.e.,

$$K_t = \sum_{i=1}^{S} \lambda_i a_{i,t}, \qquad N_t = \sum_{i=1}^{S} \lambda_i \varepsilon_i l_{i,t} \qquad \text{and} \qquad \overline{M}_t^s = P_t \sum_{i=1}^{S} \lambda_i m_{i,t}, \quad \text{for all } t.$$

Balanced Growth Equilibrium In the following analysis, we confine our attention to balanced-growth equilibria. A balanced-growth equilibrium is a competitive equilibrium that satisfies the following additional conditions:

(i) The real rate of return from capital is constant over time, i.e., $r_t = r^* > 0$ for all t.

(ii) The supply of labor by each consumer is constant over time, i.e., $l_{i,t} = l_i^* \in (0,1)$ for all t and for all i.

(iii) The average tax rate and the marginal tax rate faced by the consumers are constant over time.

(iv) All other variables are growing at the same constant factor γ.

The common growth factor γ can be derived as follows. Suppose condition (i) is satisfied, i.e., $r_t = r^* > 0$ for all t. Using (79), we can get

$$\alpha K_t^{\alpha-1} \left(X_t N_t\right)^{1-\alpha} H_t^{\varrho} = r^* + \delta.$$

Since N_t is constant in any balanced-growth equilibrium, the above condition holds if $K_t^{\alpha-1} X_t{}^{1-\alpha} H_t^{\varrho}$ is constant over time, i.e.,

$$K_t^{\alpha-1} X_t^{1-\alpha} H_t^{\varrho} = K_{t+1}^{\alpha-1} X_{t+1}^{1-\alpha} H_{t+1}^{\varrho}$$

$$\Rightarrow \left(\frac{X_{t+1}}{X_t}\right)^{1-\alpha} = \rho^{1-\alpha} = \left(\frac{K_{t+1}}{K_t}\right)^{1-\alpha} \left(\frac{H_t}{H_{t+1}}\right)^{\varrho} = \gamma^{1-\alpha-\varrho}$$

$$\Rightarrow \gamma = \rho^{\frac{1-\alpha}{1-\alpha-\varrho}} \geq 1.$$

Conditions (i) and (iii) in the above definition require that the after-tax return from capital, i.e., $\left[1 - \tau'_{t+1}\left(y_{i,t+1}\right)\right] r_{t+1}$, is time-invariant in a balanced-growth equilibrium.[6] This can be achieved by setting $\widetilde{\eta}_t \equiv \eta \left(\gamma^t\right)^{-\varkappa}$ for all $t \geq 0$. Under this specification, the average tax rate and the marginal tax rate faced by a consumer with total taxable income y_t are now given by

$$\text{ATR} = \eta \left(\frac{y_t}{\gamma^t}\right)^{\varkappa}, \qquad \text{and} \qquad \text{MTR} = (1+\varkappa)\eta \left(\frac{y_t}{\gamma^t}\right)^{\varkappa}.$$

In these expressions, we have essentially removed the growth trend γ^t from the income variable. Hence, the resulting average tax rate and marginal tax rate are time-invariant in any balanced-growth equilibrium. Condition (iv) in the above definition implies that all the aggregate variables, including w_t, K_t, H_t and $\left(\overline{M}_t^s / P_t\right)$, will share the same growth trend γ^t

[6] Note that, cross-sectionally, different types of consumers with different levels of taxable income will face a different *after-tax* return from capital in the balanced-growth equilibrium.

in any balanced-growth equilibria. Hence, we can write each of these variables in terms of the common trend and their detrended counterpart, i.e., $w_t \equiv \gamma^t \widehat{w}^*$, $K_t \equiv \gamma^t \widehat{k}^*$, $G_t \equiv \gamma^t \widehat{g}^*$, and $\left(\overline{M}_t^s / P_t\right) \equiv \gamma^t \widehat{m}^*$, where $\left\{\widehat{w}^*, \widehat{k}^*, \widehat{g}^*, \widehat{m}^*\right\}$ is a set of stationary values that needs to be determined. Similarly, all the individual variables, including $c_{i,t}$, $a_{i,t}$, $m_{i,t}$, $\xi_{i,t}$ and $y_{i,t}$, will share the same growth trend γ^t in any balanced-growth equilibria, so that $c_{i,t} \equiv \gamma^t \widehat{c}_i^*$, $a_{i,t} \equiv \gamma^t \widehat{a}_i^*$, $m_{i,t} \equiv \gamma^t \widehat{m}_i^*$, $\xi_{i,t} \equiv \gamma^t \widehat{\xi}_i^*$, and $y_{i,t} \equiv \gamma^t \widehat{y}_i^*$ for all i. The set of stationary values $\left\{\widehat{c}_i^*, \widehat{a}_i^*, \widehat{m}_i^*, \widehat{\xi}_i^*, \widehat{y}_i^*\right\}_{i=1}^S$ are also endogenously determined in the balanced-growth equilibrium.

We now outline the steps for computing a balanced-growth equilibrium.[7] For given value of r^*, we can solve for the value of $\widehat{\mathbf{y}}^* \equiv \{\widehat{y}_1^*, \widehat{y}_2^*, ..., \widehat{y}_S^*\}$ using

$$\phi = \frac{\beta_i}{\gamma}\left(\phi - \frac{1-\phi}{\pi^*}\right) + \frac{1}{\pi^*}\left(\frac{\beta_i}{\gamma}\right)^2 \left\{\left[1 - (1+\chi)\,\eta(\widehat{y}_i^*)^{\varkappa}\right] r^* + 1 - \phi\right\}, \tag{17}$$

where $\pi^* \equiv (1+\mu)/\gamma$ is the growth factor of the general price level in the balanced-growth equilibrium. Equation (17) can be obtained from the Euler equation in (78). Once $\widehat{\mathbf{y}}^*$ is determined, the value of $\left\{\widehat{h}^*, \widehat{w}^*, \widehat{k}^*, N^*, \widehat{m}^*\right\}$ can be obtained by solving

$$(\gamma + \delta - 1)\,\widehat{h}^* + \frac{\overline{v}\,(r^* + \delta)}{\alpha}\widehat{k}^* = \sum_{i=1}^S \lambda_i \eta (\widehat{y}_i^*)^{1+\varkappa} + (1 - \varpi)\,\mu \widehat{m}^*. \tag{18}$$

$$\widehat{w}^* = (1-\alpha)\left(\frac{\alpha}{r^* + \delta}\right)^{\frac{\alpha}{1-\alpha}} \left(\widehat{h}^*\right)^{\frac{\varrho}{1-\alpha}}, \tag{19}$$

$$\widehat{k}^* = \frac{\alpha\left(\sum_{i=1}^S \lambda_i \widehat{y}_i^*\right)}{r^* + (1-\alpha)\,\delta}, \tag{20}$$

$$N^* = \left[\left(\frac{r^* + \delta}{\alpha}\right)\left(\widehat{h}^*\right)^{-\varrho}\right]^{\frac{1}{1-\alpha}} \widehat{k}^*, \tag{21}$$

[7] A detailed derivation of the equations mentioned below can be found in the Appendix.

$$\widehat{m}^* = \frac{1}{1+(1-\varpi)\,\mu}\left\{\sum_{i=1}^{S}\lambda_i \widehat{y}_i^*[1-\eta(\widehat{y}_i^*)^{\varkappa}] - (1-\phi)(\gamma-1)\widehat{k}^*\right\}. \tag{22}$$

Equation (46) follows from the government's budget constraint and the law of motion for infrastructure capital. Equations (47)-(49) can be obtained by combining the first-order conditions in (79). For each $i \in \{1, 2, ..., S\}$, the value of $\{\widehat{c}_i^*, \widehat{a}_i^*, \widehat{m}_i, l_i^*\}$ can be obtained by solving

$$\widehat{y}_i^* = \varepsilon_i \widehat{w}^* l_i^* + r^* \widehat{a}_i^*,$$

$$\widehat{c}_i + \phi\,(\gamma-1)\,\widehat{a}_i^* = \widehat{m}_i,$$

$$A(l_i^*)^{\varphi} = \frac{1}{\pi^*}\left(\frac{\beta_i}{\gamma}\right)\frac{\widehat{w}^*\varepsilon_i}{\widehat{c}_i^*}[1-(1+\varkappa)\eta(\widehat{y}_i^*)^{\varkappa}], \tag{23}$$

$$\widehat{c}_i^* + (\gamma-1)\widehat{a}_i^* + \mu\widehat{m}_i = \widehat{y}_i^*[1-\eta(\widehat{y}_i^*)^{\varkappa}] + \overline{\xi}_i\varpi\mu\widehat{m}^*. \tag{24}$$

Equation (51) follows immediately from the first-order condition in (76). Equation (24) is the consumer's budget constraint along the balanced growth path. Finally, we need to make sure that the distribution of individual asset holdings $\{\widehat{a}_1^*, \widehat{a}_2^*, ..., \widehat{a}_S^*\}$ is consistent with the aggregate variable $\widehat{k}^*$ obtained from (48). To achieve this, we will need to solve for the value of r^* that clears the market for physical capital, i.e.,

$$\widehat{k}^* = \sum_{i=1}^{S}\lambda_i\widehat{a}_i^*.$$

1.3 Calibration

The main purpose of the numerical analysis is to quantify the effects of monetary policy based on the model developed above. To this end, we first construct a parameterized version of the model and show that it is able to replicate certain features of the U.S. economy. The benchmark parameter values are summarized in Table 1. Some of these values are chosen based on empirical evidence. Others are chosen to match certain real-world statistics. The details of this procedure are explained below.

One period in the model is one year. The share of capital income in total output (α) is 0.36. The value of ϱ is set to 0.20, which matches the empirical estimates reported in Glomm and Ravikumar (1997). The Frisch elasticity of labor supply ($1/\varphi$) is set to 0.40, which is consistent with the empirical estimates obtained by MaCurdy (1981) and Altonji (1986). As explained earlier, the degree of progressivity ($1 + \varkappa$) can be measured by the ratio between the average income tax rate and the marginal income tax rate. Using TAXSIM data over the period 1960-2010, we obtain an estimate of 1.877 for the degree of progressivity, which means $\varkappa = 0.877$.[8] As for the parameter ϕ in the CIA constraint, Dotsey and Sarte (2000) suggest that for industrialized countries with well-developed financial markets (such as the U.S. and other OECD countries), the fraction of investment that requires cash is likely to be close to zero. Hence, we set $\phi = 0$ in our benchmark model. We also set $\varpi = 1$ and $\overline{\xi}_i = 1$ for all i in the benchmark model so that all the seigniorage is distributed evenly to the consumers.

The remaining parameters are calibrated to match some targeted statistics. First, the growth factor of the technological factor (ρ) is chosen so that the common growth rate ($\gamma - 1$) is 2.06%, , which matches the average annual growth rate of real per-capita U.S. GDP over the period 1960-2010. Given $\alpha = 0.36$ and $\eta = 0.20$, the required value of ρ is 1.0141. Second, the benchmark money growth rate (μ) is chosen so that the inflation

[8] Data on the average federal income tax rates and the marginal federal income tax rates in the United States over the period 1960-2010 are available from the website: http://users.nber.org/~taxsim/allyup/fixed-ally.html. We choose to use the data based on a fixed 1984 sample of taxpayers. To get the value of $\varkappa$, we first compute the ratio between the average tax rate and the marginal tax rate for each year and then take the average over the entire sample period.

rate $(\pi^* - 1)$ is 4.07%, which matches the average annual growth rate of the Consumer Price Index over the period 1960-2010. Given $\gamma = 1.0206$, the required value of μ is 6.21%. Third, individual's subjective discount factors and labor productivity are chosen to match the distributions of wealth and income in the United States. We use the data from the 2007 Survey of Consumer Finance (SCF) as reported in Díaz-Giménez *et al.* (2011). In the quantitative analysis, we partition the population in the model economy into ten groups, i.e., $S = 10$. This partition is intended to mimic the income groups reported in Table 5 of Díaz-Giménez *et al.* (2011). The only difference is that we have discarded the bottom 1% of the income distribution, which has a negative average income in the SCF data.[9] Hence, the first group in our model represents the bottom 1-5% of the income distribution and the tenth group represents the top 1%.[10] The average income and average wealth for group $i \in \{1, 2, ..., S\}$ in the balanced-growth equilibrium are given by $\widehat{y}_i^*$ and $(\widehat{a}_i^* + \widehat{m}_i^*)$, respectively. In the calibration procedure, we compute the value of $\{\beta_1, \beta_2, ..., \beta_{S-1}\}$ so that the income ratios $\{\widehat{y}_i^* / \widehat{y}_S^*\}_{i=1}^{S-1}$ match the values reported in Table 5 of Díaz-Giménez *et al.* (2011). Similarly, the labor productivity parameters $\{\varepsilon_2, \varepsilon_3, ..., \varepsilon_S\}$ are computed so that wealth ratios $\{(\widehat{a}_i^* + \widehat{m}_i^*) / (\widehat{a}_1^* + \widehat{m}_1^*)\}_{i=2}^{S}$ match their real-world counterparts. The value of ε_1 is set to 0.1.[11]

Five parameters remain undetermined up to this point, these include the preference parameter A, the depreciation rate of physical capital δ, the subjective discount factor for the top 1% of the income distribution β_S, a scale parameter in the progressive tax function η and the share of aggregate output used as unproductive government spending $\overline{v}$. These parameters

[9] The consumers in our model economy will never have a negative before-tax income.

[10] The other groups represent the 5-10%, 10-20%, 20-40%, 40-60%, 60-80%, 80-90%, 90-95%, 95-99% of the income distribution.

[11] This normalization is innocuous. The same results can be obtained if we set $\varepsilon_S = 1$ and choose the value of $\{\varepsilon_1, \varepsilon_2, ..., \varepsilon_{S-1}\}$ so as to match the observed value of $\{(\widehat{a}_i^* + \widehat{m}_i^*) / (\widehat{a}_1^* + \widehat{m}_1^*)\}_{i=1}^{S-1}$.

are chosen so that the benchmark balanced-growth equilibrium has the following features: First, the average time spent on working is about one-third, i.e., $\sum_{i=1}^{S} \lambda_i l_i^* = 0.33$. Second, the capital-output ratio is 3.0. Third, the ratio of total tax receipts to aggregate output is 7.8%, which matches the U.S. data over the period 1960-2010.[12] Fourth, unproductive government spending accounts for about 67.5% of all the government spending.[13] Fifth, the long-run rate of return from physical capital (r^*) is 6%.

1.4 Findings

This section reports the main findings obtained from the numerical analysis. In Section 4.1, we summarize the main properties of the benchmark equilibrium. In Section 4.2, we present the results obtained from a series of counterfactual policy experiments. The purpose of these experiments is to quantify the effects of inflation on economic inequality.

1.4.1 Benchmark Economy

Major Economic Variables Table 2 summarizes the key statistics in our benchmark economy and compares them to their empirical counterparts. Apart from the five targeted statistics, our model is also able to match quite well the average value of the consumption-output ratio in the United States over the same time period.

[12] To compute this value, we first obtain data on the personal current taxes received by the U.S. federal government over the period 1960-2010. These data are available from the National Income and Product Accounts. We then compute the ratio between these tax revenues and US GDP. The average value over the sample period is 7.8%.

[13] According to the NIPA data, consumption expenditures made by the U.S. federal government accounted for about 67.5% of all its nondefense expenditures over the period 1960-2010. The rest was used as gross investment in structures and equipments.

Economic Inequality Table 3 summarizes the characteristics of the earnings, income and wealth distributions obtained under the benchmark parameter values. The first column reports the Gini coefficient of the three variables and compares them to the actual data. The rest of the table shows the share of earnings, income and wealth held by consumers in different percentiles of the corresponding distribution. Since we have discarded the bottom 1% of the income distribution, it is not surprising that the extent of inequality generated by the model tends to be lower than that observed in the data. Nonetheless, our model is able to replicate the high concentration of wealth and income observed in the U.S. economy. For instance, in our benchmark economy, the top 1% of the wealth distribution owns about 26% of total wealth, and the top 1% of the income distribution owns about 21% of total income. These values are identical to those in the actual data.

Table 4 shows that share of wealth held in the form of money across different groups of consumers. Overall, income-poor and wealth-poor consumers tend to hold a larger fraction of their wealth as real money balances. This happens because poor consumers tend to spend a larger fraction of their income on consumption. Since consumption purchases require the use of money under the CIA constraint, this in turn generates a strong demand for money among the poor.

1.4.2 Policy Experiments

Distributional Effect of Inflation: Case I (Benchmark Economy) To gauge the effects of anticipated inflation on economic inequality, we conduct a series of counterfactual experiments by changing the growth factor of nominal money supply (i.e., μ) in the benchmark economy. In particular, a higher money growth rate would imply a higher inflation rate in the long-run equilibrium. Since we calibrate our model to the U.S. economy, we only consider episodes of relatively mild inflation, namely 2%, 4.07% (benchmark), 5%, and 10%. The associated value of μ are 4.10%, 6.21%, 7.16% and 12.27%,respectively. To analyze other countries with hyperinflation, we need to recalibrate the model before we can conduct these experiments. In the policy experiments for our benchmark model, we compute the long-run equilibrium for each value of μ, holding all other parameters constant. The resulting distributions of wealth, earnings, consumption (or equivalently real money balances), after-tax income, non-monetary assets holding, labor supply along with the Gini coefficients for the first five distributions are reported in Tables 5-10. Our results indicate that, as inflation rate increases, the distributions of (after-tax) income, earnings and non-monetary assets holding are becoming more equal and distributions of wealth, consumption and real money balances are becoming more unequal.

In the following discussion, we will refer to those in the bottom 1-5% of the (before-tax) income distribution as the poorest, and those who are in the 5-50% of the same distribution as the poor. Similarly, we will refer to the top 1% of the income distribution as the richest and those who are in the 50-99% of the income distribution as the rich.

In our experiments, we find that when inflation increases, wage rate increases while the interest rate is almost unchanged[14] for the benchmark model and non-monetary assets holding drops for all groups. Although the levels of non-monetary assets holding are dramatically different across groups under certain inflation rate, the decreases across groups are not that dramatic as inflation increases. Even when inflation goes up from 4.07% to 10%, the Gini coefficient just changes from 0.7083 to 0.7106. Since the interest rate doesn't change much, the change in interest income across groups largely reflects the change in consumer's non-monetary assets holding.

[14] Equilibrium interest rate decreases from 6.2000% under 4.07% inflation to 6.1970% under 10% inflation. And this causes all groups hold less non-monetary assets when inflation increases.

Table 10 shows that an increase in the inflation rate will generate substantial distortions in the consumer's labor supply. In general, such an increase will discourage labor supply for all types of consumers. The reduction among the poor, however, is much larger. According to equation (51), a higher inflation rate will create a direct negative effect on labor supply. But an increase in inflation also tends to lower individual income (hence the marginal tax rate) and raise the wage rate, which will promote labor supply. Our results in Table 10 thus suggest that these indirect effects are dominated by the direct effect under the benchmark parameter values. Since consumption are lower for the rich and the richest and higher for the poorest and the poor under high inflation, and again labor supply decreases for all groups, so effect of consumption is not the main reason for the shorter hours either. Notice that the reduction in labor hours is larger for the lower half of the income distribution. This can be attributed to two factors: the opposite changes in consumption for the poor and the rich when inflation increases, and the magnitude of wage rate effect is related to consumer's predetermined characteristics.

As for earnings, the increase in the effective wage, $\widehat{w}^{*}\varepsilon_i$, for the poorest, the poor, and even the rich is dominated by the decrease in individual labor supply, so their earnings decrease as inflation increases. For the richest, due to their high labor productivity (Table 1b), the effective wage effect dominates, and thus their earnings increases. The decrease in the earnings for all below-top-1% consumers reaches the peak around the middle of the distribution, but the lower half has even more decrease in their earnings than the upper half, so the distribution of earnings becomes more unequal as inflation increases, as shown in Table 6. Similar patterns, but in opposite direction, are observed when there is a decline in the inflation rate (from 4.07% to 2%).

When inflation increases, the before-tax income[15] as well as the tax payment decrease for all groups. Since the former decreases more than the latter, the after-tax income decreases for all groups. The changes in after-tax income across groups are very close. Therefore the distribution becomes a little unequal as shown in Table 8.

With the parameter values in our benchmark model, the budget constraint in (24) can be rewritten as $(1+\mu)c_i^* = y_i^* - \tau(y_i^*) - (\gamma - 1)a_i^* + \mu m^*$. As inflation increases, for most of the groups (except the top 1% group), the effect of less saving on non-monetary assets is dominated by the effect of decreasing after-tax income. Across groups, the change in $y_i^* - \tau(y_i^*) - (\gamma - 1)a_i^*$ reaches its peak around the middle of the distribution. This change across groups is then adjusted by the faster growth rate of money supply, $\frac{y_i^* - \tau(y_i^*) - (\gamma-1)a_i^*}{1+\mu}$.The effect of decreasing value of real money is so powerful that all groups tend to decrease their consumption. Since the relatively affluent consumers have more resources left from after-tax income minus saving, the richer they are, the more their consumption is affected by the monetary policy, i.e. how much the negative effect of higher μ affect the group's consumption depends on the level of their after-tax, after-saving resources, rather than the change of it. Meanwhile, all groups will receive an equal lump-sum transfer from seigniorage. This creates a pure income effect which promotes consumption. The overall effects on consumption depend on the consumer's position in the income distribution. For the relatively poor consumers, especially those in the lower half of the income distribution, the lump-sum transfers represent a sizable portion of their income. Consequently, an increase in transfer will induce them to have more consumption and hold more real money. But for those in the upper half of the income distribution, the negative effect dominates. As a result, the relatively affluent

[15] When $\phi = 0$, equation (17) becomes $1 = (\frac{\beta_i}{\gamma})\{[1-(1+\chi)\eta(y_i^*)^{\chi}]r^* + 1\}$. Before-tax income is determined by parameters, γ and equilibrium interest rate. In the experiment, r^* doesn't change much here under high inflation, so is y_i^*.

consumers will choose to have fewer consumption and hold less money. These two forces together lead to a more equal distribution of consumption and real money balance, as is shown in Table 7.

Since both non-monetary assets holding and real money balances for the rich and the richest decrease as inflation increases, their wealth shrinks a lot. For the lower half of the income distribution, the two assets adjust in the opposite directions, and their wealth doesn't change too much. Therefore, the wealth distribution becomes equalizing. We can tell by Table 4, money holding counts as rather an important share of the wealth for the poor, so the increase in their money holding dominates the decrease in their non-monetary assets holding and their wealth is increasing. For the poorest, money is not as important component in their portfolio as it's for the poor, so the poorest has less wealth as inflation increases.

The results for distributions of wealth, earnings, consumption, after-tax income, non-monetary assets holding, and labor hours under inflation rates of 4.07% and 10% in the benchmark model are presented as Case I in table 11 in order to facilitate comparison to other cases.

Case II and Case III The results in Tables 7 suggest that the lump-sum transfers generated by the creation of money might be an important factor in determining the effects of inflation on consumption. In the second group of experiments, Case II and Case III, we consider an alternative setting in which only part of the seigniorage is transferred to the consumers and the rest of seigniorage is invested in infrastructure. In particular, we set $\varpi = 0.6$ in Case II and $\varpi = 0$ in Case III. To facilitate comparison with the benchmark economy, we recalibrate the value of η and A so that the long-run rate of return from physical capital is 6% and the average time spent on working is about one-third. All other parameter values are the same as in the benchmark model (i.e., Case I). In particular, the common growth rate and inflation rate are again equal to 2.06% and 4.07%, respectively. Since $\phi = 0$, CIA constraint implies that individual consumption is equivalent to their real money balances. Lump-sum transfer is still *equally* distributed across all consumers. Equation (24) can be rewritten as $(1+\mu)c_i^* = y_i^* - \tau(y_i^*) - (\gamma - 1)a_i^* + \varpi\mu m^*$. Both ϖ and μ have direct effects on consumption. We then conduct the policy experiment for 10% inflation rate to show the distributional effects of inflation in both cases. The results are shown in column 4-7 of Table 11. To conserve space, we only report the Gini coefficients of the individual-level variables.

When inflation rate increases from 4.07% to 10%, we find that equilibrium wage rate and interest rate go up in both cases. Due to the higher interest rate, the lower half of distribution now save more in terms of non-monetary assets as inflation increases, while the upper half still save less as they do in Case I. As a result, the dispersion of non-monetary assets across groups decreases. We still observe a distortion of labor supply across groups, and the ones towards the lower end of the distribution decrease their labor supply more as is observed in Case I. Now both the before-tax income and after-tax income for all groups increase as inflation increases. For the lower half of the distribution, their interest income increases more than the decrease in their earnings, while for the upper half, both their earnings and interest income increase. Income distribution becomes more equal under higher inflation. All groups consume more and carry more real money balances. The wealth for the lower half increases, and for the upper half decreases. The higher price harms the capital accumulation on the aggregate level. Similar results are observed in Case III.

Case IV and Case V In Case IV we generalize the CIA constraint and assume that a small fraction (5%) of aggregate investment is subject to the CIA constraint. The model is recalibrated by adjusting parameters A and η. Transfer is again equally distributed across all consumers. By comparing with Case II, the findings are almost the same, so the pattern in the above results are not quite affected by the value of ϕ. In Case V, we change the transfer allocation across groups. When we let the two groups at the lower end of the income distribution get no transfer at all and the part of the seigniorage is equally distributed among the rest eight groups, the same pattern of change is still observed as in Case II. Especially, the change of consumption across groups. So, there is real effects of inflation on the change of inequality and other aggregate variables and the numerical findings are robust under certain changes of the model.

1.5 Conclusion

In this paper, a dynamic general equilibrium model with heterogeneous consumers is used to study the distributional effects of monetary policy. Money is introduced through a generalized cash-in-advance constraint. The benchmark model is calibrated to match the wealth and income distributions in the United States and other key features of the U.S. economy. The effects of inflation on economic inequality are quantified through a series of counterfactual experiments. Our results suggest that, as inflation rate increases, inequality in wealth, income (both before-tax and after-tax) and earnings also increase. But the distributions of real money balances and consumption may actually become more equal when inflation increases.

Table 1a Benchmark Parameters

Parameter	Description	Value
A	Parameter in utility function	32.2
φ	Inverse of Frisch elasticity of labor supply	2.50
α	Share of capital income in total output	0.360
δ	Depreciation rate of capital	0.0600
$\varkappa$	Degree of progressivity	0.877
ϕ	Fraction of investment required cash	0
μ	Growth rate of nominal money supply	0.0621
β_{10}	Subjective discount factor of top income group	0.997
ϖ	Fraction of newly printed money used as transfer	1.00
η	Parameter in the progressive tax function	0.0551
ϱ	Contribution of H_t in production function	0.200
ρ	Growth factor of technological factor	1.01
ν	Fraction of G_t in aggregate output	0.0520

Table 1b Consumer Characteristics

Income Brackets (%)	β_i	ε_i
$1-5$	0.9613	0.1000
$5-10$	0.9614	0.1911
$10-20$	0.9616	0.2712
$20-40$	0.9620	0.4791
$40-60$	0.9625	0.7992
$60-80$	0.9633	1.264
$80-90$	0.9643	1.904
$90-95$	0.9655	2.542
$95-99$	0.9692	4.393
$99-100$	0.9973	27.50

Table 2 Benchmark Results

	Model	Data
Annual Growth Rate (%)	2.06	2.06*
Inflation Rate (%)	4.07	4.07
Ratios to GDP		
Physical Capital	2.95	3.00*
Private Consumption	0.683	0.630
Receipts from Personal Income Tax	0.079	0.078*

Note: Data marked with an asterisk are the targeted statistics.

Table 3 Inequality in the Benchmark Economy

		Shares in Each Group of Total Sample									
	Gini	Bottom (%)		Quintiles					Top (%)		
	Coeff.	$1-5$	$5-10$	$1st$	$2nd$	$3rd$	$4th$	$5th$	$90-95$	$95-99$	$99-100$
Earnings:											
Model	0.555	0.26	0.70	2.99	7.44	12.65	20.07	56.85	9.90	13.19	18.62
Data	0.636	0.10	0.20	1.30	5.70	11.40	20.30	61.30	11.30	16.60	17.40
Income:											
Model	0.560	0.33	0.63	2.89	6.74	11.26	18.31	60.81	10.13	15.81	20.95
Data	0.575	0.30	0.60	2.80	6.70	11.30	18.30	60.90	10.20	15.90	21.00
Wealth:											
Model	0.670	0.60	0.48	2.84	5.06	7.68	13.71	70.70	10.85	23.06	26.17
Data	0.816	0.60	0.50	3.70	5.00	7.60	13.60	70.10	10.80	22.90	25.90

Data Source: Díaz-Giménez *et al.* (2011), Table 5.

Table 4 Importance of Money Holdings

Income Brackets (%)	Share of Wealth held as Money (%)
$1 - 5$	17.80
$5 - 10$	47.87
$10 - 20$	28.00
$20 - 40$	31.18
$40 - 60$	32.41
$60 - 80$	28.10
$80 - 90$	26.71
$90 - 95$	17.98
$95 - 99$	11.98
$99 - 100$	9.70

Table 5 Wealth by Groups

	Model	Experiments		
	$\mu = 6.21\%$	4.10%	7.16%	12.27%
Income Brackets (%)	$\pi = 1.0407$	1.02	1.05	1.10
$1 - 5$	0.2430	0.2452	0.2415	0.2301
$5 - 10$	0.1576	0.1546	0.1585	0.1592
$10 - 20$	0.2880	0.2860	0.2886	0.2892
$20 - 40$	0.4131	0.4121	0.4135	0.4144
$40 - 60$	0.6261	0.6282	0.6253	0.6216
$60 - 80$	1.118	1.125	1.115	1.101
$80 - 90$	1.731	1.745	1.725	1.698
$90 - 95$	3.541	3.563	3.532	3.487
$95 - 99$	9.406	9.445	9.389	9.304
$99 - 100$	42.69	42.84	42.63	42.30
Gini Coeff.	0.6701	0.6707	0.6699	0.6689

Table 6 Earnings by Groups

	Benchmark	Experiments		
Income Brackets (%)	$\pi = 1.0407$	1.02	1.05	1.10
$1-5$	0.0185	0.0194	0.0182	0.0166
$5-10$	0.0403	0.0416	0.0397	0.0372
$10-20$	0.0582	0.0596	0.0576	0.0549
$20-40$	0.1069	0.1083	0.1062	0.1031
$40-60$	0.1817	0.1832	0.1811	0.1778
$60-80$	0.2883	0.2897	0.2878	0.2847
$80-90$	0.4352	0.4364	0.4347	0.4319
$90-95$	0.5687	0.5696	0.5683	0.5660
$95-99$	0.9472	0.9476	0.9470	0.9460
$99-100$	5.349	5.345	5.351	5.360
Gini Coeff.	0.5216	0.5190	0.5228	0.5283

Table 7 Consumption by Groups

	Benchmark	Experiments		
Income Brackets (%)	$\pi = 1.0407$	1.02	1.05	1.10
$1 - 5$	0.0433	0.0390	0.0451	0.0546
$5 - 10$	0.0597	0.0560	0.0613	0.0697
$10 - 20$	0.0807	0.0774	0.0821	0.0893
$20 - 40$	0.1288	0.1266	0.1297	0.1346
$40 - 60$	0.2030	0.2023	0.2032	0.2047
$60 - 80$	0.3141	0.3157	0.3134	0.3099
$80 - 90$	0.4625	0.4670	0.4605	0.4503
$90 - 95$	0.6366	0.6446	0.6331	0.6153
$95 - 99$	1.127	1.145	1.119	1.080
$99 - 100$	4.140	4.216	4.106	3.935
Gini Coeff.	0.4770	0.4844	0.4737	0.4569

Table 8 After-tax Income by Groups

	Benchmark	Experiments		
Income Brackets (%)	$\pi = 1.0407$	1.02	1.05	1.10
$1-5$	0.0308	0.0321	0.0303	0.0274
$5-10$	0.0462	0.0475	0.0456	0.0426
$10-20$	0.0707	0.0721	0.0701	0.0669
$20-40$	0.1234	0.1249	0.1227	0.1194
$40-60$	0.2051	0.2066	0.2044	0.2009
$60-80$	0.3310	0.3326	0.3303	0.3267
$80-90$	0.4981	0.4997	0.4974	0.4937
$90-95$	0.7168	0.7184	0.7160	0.7124
$95-99$	1.348	1.350	1.348	1.344
$99-100$	5.172	5.172	5.171	5.171
Gini Ceoff.	0.5218	0.5195	0.5228	0.5279

Table 9 Non-monetary Assets Holding by Groups

	Benchmark	Experiments		
Income Brackets (%)	$\pi = 1.0407$	1.02	1.05	1.10
$1-5$	0.1997	0.2061	0.1964	0.1755
$5-10$	0.0980	0.0986	0.0972	0.0895
$10-20$	0.2074	0.2086	0.2066	0.1999
$20-40$	0.2843	0.2855	0.2837	0.2798
$40-60$	0.4232	0.4259	0.4221	0.4169
$60-80$	0.8039	0.8096	0.8016	0.7910
$80-90$	1.269	1.278	1.265	1.248
$90-95$	2.905	2.918	2.899	2.872
$95-99$	8.279	8.300	8.269	8.224
$99-100$	38.55	38.63	38.52	38.36
Gini Coeff.	0.7088	0.7083	0.7090	0.7106

Table 10 Individual Labor Supply by Groups

	Benchmark	Experiments		
Income Brackets (%)	$\pi = 1.0407$	1.02	1.05	1.10
$1-5$	0.2824	0.2965	0.2768	0.2520
$5-10$	0.3215	0.3324	0.3170	0.2959
$10-20$	0.3274	0.3354	0.3241	0.3078
$20-40$	0.3402	0.3451	0.3380	0.3272
$40-60$	0.3467	0.3498	0.3453	0.3384
$60-80$	0.3478	0.3498	0.3470	0.3425
$80-90$	0.3486	0.3498	0.3480	0.3450
$90-95$	0.3412	0.3421	0.3408	0.3387
$95-99$	0.3288	0.3292	0.3286	0.3275
$99-100$	0.2966	0.2967	0.2966	0.2964

Table 11 Sensitivity Analysis

	Case I (Benchmark)		Case II		Case III	
Parameters	$\phi = 0, \xi_i = 1,$ $\varpi = 1$		$\phi = 0, \xi_i = 1,$ $\varpi = 0.6$		$\phi = 0, \xi_i = 1,$ $\varpi = 0$	
Inflation Rate	4.07%	10%	4.07%	10%	4.07%	10%
Wage Rate	0.6558	0.6575	0.8306	0.9204	1.0603	1.2468
Interest Rate	0.0620	0.06197	0.0610	0.0613	0.0605	0.0609
K/Y Ratio	2.9508	2.9515	2.9742	2.9680	2.9873	2.9765
Gini Coefficients						
Wealth	0.6701	0.6689	0.6701	0.4968	0.6701	0.3979
Earnings	0.5216	0.5283	0.5231	0.5370	0.5248	0.5415
Consumption	0.4770	0.4569	0.4895	0.4474	0.5079	0.4589
After-tax Income	0.5218	0.5279	0.5222	0.4745	0.5224	0.4434
Non-monetary Assets	0.7088	0.7106	0.7054	0.5002	0.7011	0.3790

Table 11 Sensitivity Analysis (cont'd)

	Case IV		Case V	
Parameters	$\phi = 0.05, \xi_i = 1,$		$\phi = 0, \xi_1 = 0,$	
	$\varpi = 0.6$		$\xi_2 = 0, \varpi = 0.6$	
Inflation Rate	4.07%	10%	4.07%	10%
Wage Rate	0.8304	0.9197	0.8778	0.9726
Interest Rate	0.0612	0.0617	0.0605	0.0607
K/Y Ratio	2.9696	2.9586	2.9877	2.9815
Gini Coefficients				
Wealth	0.6701	0.4976	0.6701	0.4940
Earnings	0.5230	0.5369	0.5235	0.5373
Consumption	0.4894	0.4473	0.4929	0.4536
After-tax				
Income	0.5222	0.4747	0.5224	0.4745
Non-monetary				
Assets	0.7055	0.5012	0.7057	0.4977

1.6 Appendix

1.6.1 Derivation of Euler Equation

Using (74) and (75), we can get

$$\begin{aligned}\zeta_{i,t}+\phi\theta_{i,t} &= \phi(\zeta_{i,t}+\theta_{i,t})+(1-\phi)\zeta_{i,t}\\ &= \phi c_{i,t}^{-1}+(1-\phi)\beta_i\left(\pi_{t+1}c_{i,t+1}\right)^{-1}.\end{aligned}$$

Substituting these into (73) gives the Euler equation

$$\frac{\phi}{c_{i,t}}+\beta_i\left(\frac{1-\phi}{\pi_{t+1}c_{i,t+1}}\right)=(\beta_i)^2\frac{\left[1-\tau'_{t+1}\left(y_{i,t+1}\right)\right]r_{t+1}}{\pi_{t+2}c_{i,t+2}}+\beta_i\left[\frac{\phi}{c_{i,t+1}}+\beta_i\left(\frac{1-\phi}{\pi_{t+2}c_{i,t+2}}\right)\right]$$

$$\Rightarrow\frac{\phi}{c_{i,t}}=\frac{\beta_i}{c_{i,t+1}}\left(\phi-\frac{1-\phi}{\pi_{t+1}}\right)+\frac{(\beta_i)^2}{\pi_{t+2}c_{i,t+2}}\left\{\left[1-\tau'_{t+1}\left(y_{i,t+1}\right)\right]r_{t+1}+1-\phi\right\}.$$

1.6.2 Balanced Growth Equilibrium

We now provide the formal derivation of the equations in (17)-(24). In any balanced-growth equilibrium, the real money supply is growing at the common growth rate $\gamma-1$, so that

$$\frac{\overline{M}^s_{t+1}}{P_{t+1}}=\gamma\frac{\overline{M}^s_t}{P_t}\Rightarrow\frac{\overline{M}^s_{t+1}}{\overline{M}^s_t}=1+\mu=\gamma\frac{P_{t+1}}{P_t}.$$

Hence, the growth factor of the general price level in any balanced-growth equilibrium is given by

$$\frac{P_{t+1}}{P_t}=\frac{1+\mu}{\gamma}=\pi^*. \tag{25}$$

Next, substituting $c_{i,t} \equiv \gamma^t \widehat{c}_i^*$, $\pi_t = \pi^*$ and $r_t = r^*$ into the Euler equation in (78) gives

$$\phi = \frac{\beta_i}{\gamma}\left(\phi - \frac{1-\phi}{\pi^*}\right) + \frac{1}{\pi^*}\left(\frac{\beta_i}{\gamma}\right)^2 \left\{\left[1 - (1+\chi)\,\eta(\widehat{y}_i^*)^{\varkappa}\right] r^* + 1 - \phi\right\}.$$

which is (17) in the text. Next, using the first-order conditions in (79), we can get

$$\frac{K_t}{N_t} = \frac{\alpha w_t}{(1-\alpha)\,(r_t+\delta)} \tag{26}$$

$$\Rightarrow w_t = (1-\alpha)\, X_t H_t^{\frac{\varrho}{1-\alpha}} \left(\frac{\alpha}{r_t+\delta}\right)^{\frac{\alpha}{1-\alpha}}.$$

Dividing both sides by γ^t gives

$$\widehat{w}^* = (1-\alpha)\left(\frac{\alpha}{r^*+\delta}\right)^{\frac{\alpha}{1-\alpha}} \left(\widehat{h}^*\right)^{\frac{\varrho}{1-\alpha}}, \tag{27}$$

which is (47) in the text. By the definition of $y_{i,t}$, we can get

$$\widehat{y}_i^* = \varepsilon_i \widehat{w}^* l_i^* + r^* \widehat{a}_i^*.$$

Aggregating across all groups of consumers gives

$$\begin{aligned}
\sum_{i=1}^{S} \lambda_i \widehat{y}_i^* &= \widehat{w}^* \sum_{i=1}^{S} \lambda_i \varepsilon_i l_i^* + r^* \sum_{i=1}^{S} \lambda_i \widehat{a}_i^* \\
&= \widehat{w}^* N^* + r^* \widehat{k}^* \\
&= \left(\frac{\widehat{w}^* N^*}{\widehat{k}^*} + r^*\right) \widehat{k}^* \\
&= \frac{1}{\alpha}\left[r^* + (1-\alpha)\,\delta\right] \widehat{k}^*. \qquad (28)
\end{aligned}$$

The second line uses the labor market clearing condition and the capital market clearing condition. The fourth line uses (26). Rearranging terms in the above equation gives (48).

Next, combining (43), (16) and $G_t = \overline{v} Y_t$ gives

$$H_{t+1} - (1-\delta) H_t + \overline{v} Y_t = \sum_{i=1}^{S} \lambda_i \widetilde{\eta}_t y_{i,t}^{1+\varkappa} + (1-\varpi) \left(\frac{\overline{M}_{t+1}^s - \overline{M}_t^s}{P_t} \right).$$

In a balanced-growth equilibrium, this becomes

$$(\gamma + \delta - 1) \widehat{h}^* + \overline{v} \left(\frac{Y_t}{K_t} \right) \widehat{k}^* = \eta \sum_{i=1}^{S} \lambda_i (\widehat{y}_i^*)^{1+\varkappa} + \varpi \mu \widehat{m}^*.$$

Substituting $(r^* + \delta) = \alpha Y_t / K_t$ into the above equation gives (46).

Combining the individual budget constraint and the CIA constraint gives

$$(1-\phi)(a_{i,t+1} - a_{i,t}) + \pi_{t+1} m_{i,t+1} = y_{i,t} \left[1 - \widetilde{\eta}_t (y_{i,t})^{\varkappa} \right] + \overline{\xi}_i \varpi \mu \overline{m}_t^s.$$

Summing across all types of consumers gives

$$(1-\phi)(K_{t+1} - K_t) + \pi_{t+1} \overline{m}_{t+1}^s = \sum_{i=1}^{S} \lambda_i y_{i,t} \left[1 - \widetilde{\eta}_t (y_{i,t})^{\varkappa} \right] + \varpi \mu \overline{m}_t^s.$$

In a balanced-growth equilibrium, this becomes

$$(1-\phi)(\gamma - 1) \widehat{k}^* + (1 + \mu - \varpi \mu) \widehat{m}^* = \sum_{i=1}^{S} \lambda_i \widehat{y}_i^* [1 - \eta (\widehat{y}_i^*)^{\varkappa}].$$

Equation (50) can be obtained by rearranging terms in the above expression.

2 Inflation and Economic Growth in a Search-Theoretic Model of Money

2.1 Introduction

The relationship between inflation and long-term economic growth is a classic question in macroeconomics. It belongs to a more general discussion on the property of non-superneutrality of money. Increasing anticipated inflation, which due to a faster growth rate of nominal money supply, has growth and welfare effects. Empirical evidence shows that whether inflation is beneficial or detrimental to growth depending upon the countries and episodes considered in the study (Fischer, 1990; Barro 2013; Mallik and Chowdhury, 2001). For high inflation rates, which is often above 10 percent, it's clear that the effects on long-term growth is detrimental, while for lower inflation episodes, the evidence is mixed.

It's not the first time that this question has been addressed, but when introducing money, previous work implicitly presumes there is certain role of money, and builds frictionless model based on that, i.e. either by putting money in the utility function or imposing cash-in-advance constraint. Comparing with these reduced-form approaches, we use a new monetarist approach which explicitly specifying money in a frictional search and matching setting to reexamine the relationship between inflation and growth. New monetarist approach can go back at least to Kiyotaki and Wright (1989, 1993), Shi (1995, 1999), Kocherlakota (1998) and others. Then, new monetarist approach has been integrated in a tractable way to study mainstream macro issues (Lagos and Wright, 2005). These applied works include Aruoba, Waller, and Wright (2012) which examine the effects of inflation on investment, Aruoba and Chugh (2010) which study the optimal fiscal and monetary policy, Lagos and Rocheteau

(2008) which examine issues related to liquidity in asset markets, Williamson (2012) which applies this framework to analyze the effects of financial crises, etc. Williamson and Wright (2010a,b) provide more comprehensive surveys about this literature.

The other feature of this study is endogenous growth, introduced through productive government spending (Barro 1990). The intuition for this ingredient is that an important use of government spending is to provide infrastructures that can facilitate private production, such as building highways, etc. With productive government spending, economy grows perpetually in the long run, and the growth rate is endogenously determined in the model. Our contribution is to reexamine the effects of inflation on this growth rate when we explicitly consider the role of money as a medium of exchange. We emphasize the growth effects of inflation through those extra benefits of holding money, which are the increase in the trade surplus in stochastic trading process and the increase in the efficiency of the whole economy. Put it in another way, without explicitly consider this aspect, inflation is a tax on investment, and economy reaches a steady state with lower capital per worker in the long run than it's in the efficient case, while with explicitly consider this aspect, by holding either more money or more capital, the consumers might become better off in the stochastic trading process, which is novel in the endogenous growth literature. The goal of this chapter is to quantify this effect.

The rest of this chapter is organized as follows. Section 2 presents the model which modifies the model in Aruoba, Waller and Wright (2011) to allow for endogenous economic growth. Section 3 describes the calibration procedure and presents the numerical results.

2.2 The Model

2.2.1 The Environment

Time is discrete and is denoted by $t \in \{0, 1, 2, ...\}$. The economy under study is inhabited by a continuum of *ex ante* identical, infinitely-lived consumers. The size of population is constant over time and is normalized by one. Each time period is divided into two subperiods. In the first subperiod, a single commodity is produced and traded in a decentralized market (DM). Transactions in the DM are not monitored and occur anonymously between pairs of randomly matched buyers and sellers. This creates a need for using money as medium of exchange.[16] In the second subperiod, a different commodity is produced and traded in a frictionless centralized market environment (CM). This part of the model economy is essentially identical to the standard neoclassical economy. In each period, firms hire labor and physical capital from competitive factor markets in order to produce the CM good, while consumers supply labor and decide how much to save and consume. Since there is no trading friction in the CM, money is not needed in the second subperiod. The government in this economy collects taxes from the consumers and provides a constantly growing supply of money. It also finances and maintains infrastructure capital which is conducive to the production of goods.

2.2.2 Centralized Markets

Consumer's Problem All consumers have preferences over consumption and labor hours in the CM in each period. Let c_t and l_t denote consumption and labor hours in the second

[16] In order to rule out the use of physical capital as medium of exchange in the DM, it is assumed that physical capital is unportable and that claims to physical capital can be costlessly counterfeited by anyone in the economy.

subperiod at time t. Consumer's preferences over (c_t, l_t) are represented by

$$\mathcal{U}(c_t, l_t) = \ln c_t - Al_t,$$

where A is a positive constant. As is well-known in the growth theory literature, if the period utility function is additively separable in consumption and labor hours, then the utility function of consumption must be logarithmic in order to be consistent with balanced growth.

In each period, each consumer receives two types of taxable income, namely labor income from work and interest income from physical capital. Let w_t be the market wage rate at time t. Then a typical consumer's labor income at time t is given by $w_t l_t$. This type of income is taxed at a constant tax rate of $\tau_l \in (0, 1)$. Let $\widehat{k}_t$ be the quantity of physical capital owned by a typical consumer at the beginning of the second subperiod, and r_t be the net rate of return from physical capital at time t. Then the amount of interest income is $r_t \widehat{k}_t$, which is taxed at a constant rate of $\tau_k \in (0, 1)$. Consumers in this economy can save by holding physical capital and money. Let $\widehat{m}_t$ denote nominal money holdings at the beginning of the second subperiod at time t, and let p_t be the general price level in the CM. The consumer's budget constraint in this subperiod is given by

$$(1 + \tau_c) c_t + k_{t+1} - \widehat{k}_t + \frac{m_{t+1} - \widehat{m}_t}{p_t} = (1 - \tau_l) w_t l_t + (1 - \tau_k) r_t \widehat{k}_t + \phi_t, \tag{29}$$

where $\tau_c > 0$ is a constant tax rate on consumption and ϕ_t is a lump-sum transfer of real money from the government. In the above equation, k_{t+1} and m_{t+1} represent the stock of physical capital and nominal money that the consumer brings into the DM at time $t + 1$.[17]

[17] The notations (m_t, k_t) and $\left(\widehat{m}_t, \widehat{k}_t\right)$ are used to highlight the fact that the consumers enter the DM and

Let $W_t\left(\widehat{m},\widehat{k}\right)$ be the expected lifetime utility for a consumer who enters the CM at time t with assets $\left(\widehat{m},\widehat{k}\right)$, and let $V_t(m,k)$ be the expected lifetime utility for a consumer who enters the DM at time t with assets (m,k). All consumers use the same subjective discount factor $\beta \in (0,1)$ to discount utilities from future time periods, but there is no discounting between the DM and CM of the same time period. Given a set of government policies $\{\tau_c, \tau_l, \tau_k, \phi_t\}$ and a set of prices $\{w_t, r_t, p_t\}$, the consumer's problem in the CM is to choose an allocation $\{c_t, l_t, k_{t+1}, m_{t+1}\}$ so as to maximize his expected lifetime utility. Formally, this is given by

$$W_t\left(\widehat{m}_t,\widehat{k}_t\right) = \max_{c_t,l_t,k_{t+1},m_{t+1}} \left\{\ln c_t - Al_t + \beta V_{t+1}\left(m_{t+1},k_{t+1}\right)\right\} \tag{30}$$

subject to the sequential budget constraint in (68). After substituting l_t in (69) with the budget constraint, we can obtain

$$W_t\left(\widehat{m}_t,\widehat{k}_t\right) = \Omega_t\left(\widehat{m}_t,\widehat{k}_t\right) + \max_{c_t,m_{t+1},k_{t+1}} \left\{\ln c_t - \frac{A}{(1-\tau_l)w_t}\left[(1+\tau_c)c_t + k_{t+1} + \frac{m_{t+1}}{p_t}\right] + \beta V_{t+1}\left(m_{t+1},k_{t+1}\right)\right\} \tag{31}$$

where

$$\Omega_t\left(\widehat{m}_t,\widehat{k}_t\right) \equiv \frac{A}{(1-\tau_l)w_t}\left\{\left[1+(1-\tau_k)r_t\right]\widehat{k}_t + \frac{\widehat{m}_t}{p_t} + \phi_t\right\}.$$

The expression in (70) makes clear that the optimal choices of (c_t, m_{t+1}, k_{t+1}) are independent of $\left(\widehat{m}_t,\widehat{k}_t\right)$. This is due to the quasi-linearity of the utility function $\mathcal{U}(c_t, l_t)$. If $V_{t+1}(\cdot)$ is strictly concave, then the consumer's problem has a unique solution in (m_{t+1}, k_{t+1}) and this implies that all consumers will enter the DM at time $t+1$ with the same amount of assets. Formally, let $\mathcal{F}_{t+1}(m,k)$ be the cross-sectional distribution of assets at the beginning of the DM at time $t+1$. Then this distribution is degenerate if $V_{t+1}(\cdot)$ is strictly concave.[18] We

the CM of the same time period with different amount of assets.

[18] Since the consumers are assumed to be *ex ante* identical, they all have the same amount of initial assets at

will formally establish the strict concavity of $V_{t+1}(\cdot)$ in the next subsection.

The first-order conditions with respect to c_t, m_{t+1} and k_{t+1} are given by

$$\frac{1}{(1+\tau_c)\,c_t} = \frac{A}{(1-\tau_l)\,w_t}, \tag{32}$$

$$\frac{A}{(1-\tau_l)\,p_t w_t} = \beta \frac{\partial V_{t+1}(m_{t+1}, k_{t+1})}{\partial m_{t+1}}, \tag{33}$$

$$\frac{A}{(1-\tau_l)\,w_t} = \beta \frac{\partial V_{t+1}(m_{t+1}, k_{t+1})}{\partial k_{t+1}}. \tag{34}$$

The expression in (70) also makes clear that $W_t\left(\widehat{m}_t, \widehat{k}_t\right)$ is linear in $\left(\widehat{m}_t, \widehat{k}_t\right)$ with partial derivatives

$$\overline{W}_{m,t} \equiv \frac{\partial W_t\left(\widehat{m}_t, \widehat{k}_t\right)}{\partial \widehat{m}_t} = \frac{A}{(1-\tau_l)\,p_t w_t}, \tag{35}$$

$$\overline{W}_{k,t} \equiv \frac{\partial W_t\left(\widehat{m}_t, \widehat{k}_t\right)}{\partial \widehat{k}_t} = \frac{[1+(1-\tau_k)\,r_t]\,A}{(1-\tau_l)\,w_t}. \tag{36}$$

Production of CM Goods On the supply side of the centralized environment, there is a large number of identical firms which produce the CM good. The production technology is given by

$$Y_t = \widehat{K}_t^{\varrho}\,(H_t L_t)^{1-\varrho}, \qquad \text{with } \varrho \in (0,1),$$

where Y_t denotes output produced at time t, $\widehat{K}_t$ is capital input, L_t is labor input and H_t is the stock of infrastructure capital available at the beginning of time t. The value of H_t is taken as exogenously given by individual firms. Since the production function exhibits constant returns to scale in the private inputs (i.e., $\widehat{K}_t$ and L_t), we can focus on the choices

the beginning of the first subperiod at time 0. In other words, the initial distribution $\mathcal{F}_0(m,k)$ is degenerate by assumption.

made by a single, price-taking firm. Let R_t be the rental price of physical capital at time t. Then the representative firm solves the following problem

$$\max_{\widehat{K}_t, L_t} \left\{ \widehat{K}_t^{\varrho} \left(H_t L_t\right)^{1-\varrho} - w_t L_t - R_t \widehat{K}_t \right\},$$

and the first-order conditions are given by

$$w_t = (1-\varrho) \widehat{K}_t^{\varrho} L_t^{-\varrho} H_t^{1-\varrho}, \qquad \text{and} \qquad R_t = \varrho \widehat{K}_t^{\varrho - 1} \left(H_t L_t\right)^{1-\varrho}.$$

Let $\delta \in (0,1)$ be the depreciation rate of physical capital in the second subperiod. Then the rate of return r_t is determined by $r_t = R_t - \delta$.

2.2.3 Decentralized Market

Similar to Aruoba, Waller and Wright (2011), we assume that consumers face idiosyncractic uncertainty regarding their identity in the DM. Specifically, with equal probability $\alpha \in (0, 1/2)$, a consumer is either a buyer or a seller in the DM. In the current model, a buyer is someone who has access to an investment opportunity which converts one unit of DM good into $\overline{z} > 0$ units of physical capital within the same subperiod. A buyer, however, does not possess the knowledge or technology to produce the DM goods. Thus, he has to purchase these goods from the decentralized market. A seller, on the other hand, is someone who can produce the DM goods but does not have access to the investment opportunity. Hence, the sole purpose of producing the DM goods is to sell them for profit. The equal probability assumption ensures that there is an equal number of buyers and sellers in the DM. Finally, with probability $1 - 2\alpha$, a consumer is neither a buyer nor a seller, which means he does not have access to the investment opportunity nor the technology for producing the DM goods.

These consumers will be referred to as non-participants in the DM.

The sequence of events in the DM is as follows: At the beginning of the first subperiod, all consumers are randomly assigned to one of the following three groups: buyers, sellers and non-participants. The random identity is drawn independently across consumers and across time. A matching technology then assigns each buyer to exactly one seller. Each pair of trading partners will then determine the terms of trade through bargaining. Once the transaction is completed, the buyer will transform the purchased DM goods into physical capital.

Production of DM Goods First, consider a consumer who has been assigned as a seller in the DM at time t. The quantity of DM goods that can be produced is determined by three factors: (i) the seller's own capital k_t, (ii) the rate of utilization of the capital stock, denoted by $s_t \in [0,1]$, and (iii) the existing stock of infrastructure capital H_t, which is taken as exogenously given by the consumers. Formally, the technology for producing the DM good is given by

$$x_t = (s_t k_t)^{\epsilon} H_t^{1-\epsilon}, \qquad \text{with } \epsilon \in (0,1),$$

where x_t is the quantity of DM goods produced. The variable s_t captures the intensity with which the seller's own capital is used in the production process. In particular, a higher value of s_t means that it is being used more intensively, which will result in a faster depreciation rate during the first subperiod. Formally, the depreciation rate in the first subperiod is endogenously determined by

$$\widetilde{\delta}(s_t) = \eta s_t^{\sigma}, \qquad \text{with } \eta \in (0,1) \text{ and } \sigma > 1.$$

Since s_t is bounded above by one, this means for any given $(k_t, H_t) \in \mathbb{R}^2_{++}$ there is a limit on how much a seller can produce at time t. This limit is denoted by $\overline{x}_t \equiv k_t^{\epsilon} H_t^{1-\epsilon}$. For any $x_t \in [0, \overline{x}_t]$, the depreciation rate of physical capital in the first subperiod can be rewritten as

$$\widetilde{\delta}(x_t, k_t; H_t) \equiv \eta \left[\frac{\left(x_t H_t^{\epsilon - 1} \right)^{\frac{1}{\epsilon}}}{k_t} \right]^{\sigma}. \tag{37}$$

The restrictions $\sigma > 1 > \epsilon$ imply that $\widetilde{\delta}(x_t, k_t; H_t)$ is strictly convex in x_t for any $(k_t, H_t) \in \mathbb{R}^2_{++}$.

Bargaining Process Consider a random encounter in the DM that involves a buyer with assets $\left(m_t^b, k_t^b\right)$ and a seller with assets $\left(m_t^s, k_t^s\right)$. Let d_t denote the payment (in units of money) for x_t units of DM goods. By acquiring this amount of DM goods, the buyer can generate $\overline{z} x_t$ units of physical capital which he can bring forward to the second subperiod. Thus, the buyer's gain from trade is

$$\begin{aligned} & W_t\left(m_t^b - d_t, k_t^b + \overline{z} x_t\right) - W_t\left(m_t^b, k_t^b\right) \\ = \quad & \frac{A}{(1-\tau_l) w_t} \left\{ \left[1 + (1-\tau_k) r_t\right] \overline{z} x_t - \frac{d_t}{p_t} \right\}. \end{aligned}$$

The second line follows from (74) and (75). As for the seller, after producing x_t units of DM goods, a fraction $\widetilde{\delta}(x_t, k_t^s; H_t)$ of his capital will be lost through depreciation. This is compensated by a gain of d_t units of money. The seller's gain from trade is thus

$$\begin{aligned} & W_t\left(m_t^s + d_t, \left(1 - \widetilde{\delta}(x_t, k_t^s; H_t)\right) k_t^s\right) - W_t\left(m_t^s, k_t^s\right) \\ = \quad & \frac{A}{(1-\tau_l) w_t} \left\{ \frac{d_t}{p_t} - \left[1 + (1-\tau_k) r_t\right] \widetilde{\delta}(x_t, k_t^s; H_t) k_t^s \right\}. \end{aligned}$$

Summing these two expressions gives the total trade surplus,

$$\frac{A}{(1-\tau_l)\,w_t}\left[1+(1-\tau_k)\,r_t\right]\left[\overline{z}x_t-\widetilde{\delta}\left(x_t,k_t^s;H_t\right)k_t^s\right].$$

Since $\widetilde{\delta}\left(x_t,k_t;H_t\right)$ is strictly convex in x_t, there exists a unique value $\widehat{x}_t>0$ that maximizes the total trade surplus for any given $(k_t^s,H_t)\in\mathbb{R}_{++}^2$ (see Figure 1).

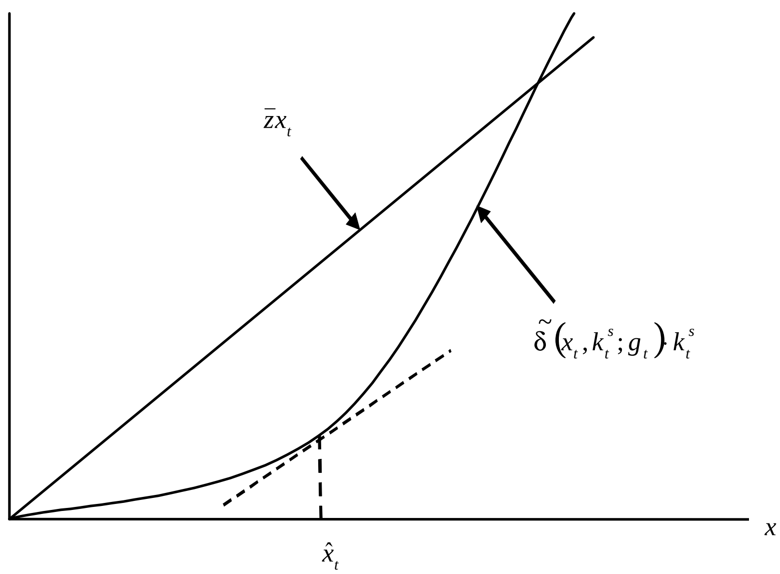

Figure 1: Total Trade Surplus in the DM.

In the current study, we contemplate a bargaining process in which the total trade surplus is divided proportionally between the buyer and the seller. Kalai (1977) provides an axiomatic foundation for this type of bargaining outcome.[19] In the monetary search liter-

[19]In particular, Kalai (1977) shows that the solution of a n-person bargaining game is proportional if and only if it satisfies the axiom of monotonicity. In words, this axiom requires that no player will be made worse off when additional options are made available to them.

ature, proportional bargaining solution has also been considered in Aruoba *et al.* (2007), Craig and Rocheteau (2008) and Waller (2011). Under this type of bargaining process, the buyer's gain from trade is proportional to the seller's gain from trade, so that

$$[1+(1-\tau_k)\,r_t]\,\overline{z}x_t - \frac{d_t}{p_t} = \frac{\theta}{1-\theta}\left\{\frac{d_t}{p_t} - [1+(1-\tau_k)\,r_t]\,\widetilde{\delta}\,(x_t, k_t^s; H_t)\,k_t^s\right\}, \tag{38}$$

where $\theta \in (0,1)$ is an exogenous parameter that indicates the share of total trade surplus received by the buyer. Given the sharing rule in (78), the values of d_t and x_t are chosen so as to maximize the total trade surplus, subject to the buyer's cash constraint and the seller's production capacity constraint. Formally, the bargaining solution can be obtained by solving

$$\max_{d_t, x_t}\left[\overline{z}x_t - \widetilde{\delta}\,(x_t, k_t^s; H_t)\,k_t^s\right] \tag{P1}$$

subject to (78), $d_t \leq m_t^b$ and $x_t \in [0, \overline{x}_t]$. To simplify the analysis, we focus on equilibria in which $\widehat{x}_t > \overline{x}_t$ for all $t \geq 0$. [20] In this type of equilibria, total trade surplus is always maximized at $x_t = \overline{x}_t$. But the actual quantity of DM goods being traded may be different from $\overline{x}_t$ as it depends on the quantity of money held by the buyer. Formally, after substituting $x_t = \overline{x}_t$ and $\widetilde{\delta}\,(\overline{x}_t, k_t^s; H_t) = \eta$ into (78), we can obtain

$$\overline{d}_t \equiv p_t\,[1+(1-\tau_k)\,r_t]\left[(1-\theta)\,\overline{z}\,(k_t^s)^{\epsilon}\,H_t^{1-\epsilon} + \theta\eta k_t^s\right], \tag{39}$$

[20] This condition holds if and only if

$$\overline{z} > \widetilde{\delta}_x\,(\overline{x}_t, k_t^s; H_t)\,k_t^s = \frac{\eta\sigma}{\epsilon}\left(\frac{k_t^s}{H_t}\right)^{1-\epsilon}$$

for all $t \geq 0$. In the following analysis, we will first characterize the long-run balanced-growth equilibria under this assumption, and then check that it is satisfied in these equilibria.

which is the required payment for $\overline{x}_t$ units of DM goods under proportional bargaining. If the buyer has enough money to make this payment, i.e., $m_t^b \geq \overline{d}_t$, then the bargaining outcome is $\left(\overline{d}_t, \overline{x}_t\right)$. Note that both $\overline{d}_t$ and $\overline{x}_t$ are independent of m_t^b. If the buyer does not have enough cash for $\overline{x}_t$ units of DM goods, then the quantity being traded is uniquely and endogenously determined by

$$\frac{m_t^b}{p_t} = \left[1 + (1 - \tau_k) r_t\right] \left[(1 - \theta) \overline{z} x_t + \theta \widetilde{\delta} \left(x_t, k_t^s; H_t\right) k_t^s\right]. \tag{40}$$

Equation (40) implicitly defines a strictly increasing function $x_t = \Lambda_t \left(m_t^b, k_t^s\right)$, which indicates the volume of trade in this case. By comparing (39) to (40), it is obvious to see that $\overline{x}_t = \Lambda_t \left(\overline{d}_t, k_t^s\right)$. The partial derivatives of $\Lambda_t \left(m_t^b, k_t^s\right)$ are given by

$$\frac{\partial \Lambda_t \left(m_t^b, k_t^s\right)}{\partial m_t^b} = \frac{1}{p_t \left[1 + (1 - \tau_k) r_t\right] \left[(1 - \theta) \overline{z} + \theta \widetilde{\delta}_x \left(x_t, k_t^s; H_t\right) k_t^s\right]} > 0,$$

$$\frac{\partial \Lambda_t \left(m_t^b, k_t^s\right)}{\partial k_t^s} = \frac{\theta (\sigma - 1) \widetilde{\delta} \left(x_t, k_t^s; H_t\right)}{(1 - \theta) \overline{z} + \theta \widetilde{\delta}_x \left(x_t, k_t^s; H_t\right) k_t^s} > 0. \tag{41}$$

Holding other things constant, an increase in the amount of money held by the buyer (say from m_t^b to $m_t^b + \varepsilon$, with $\varepsilon > 0$ and $m_t^b + \varepsilon < \overline{d}_t$) will relax the buyer's cash constraint and allow more goods to be traded. Hence, $\Lambda_t \left(m_t^b, k_t^s\right)$ is strictly increasing in m_t^b. On the other hand, an increase in k_t^s means that more DM goods can be produced by the seller. Hence, $\Lambda_t \left(m_t^b, k_t^s\right)$ is also strictly increasing in k_t^s.

The outcome of this bilateral trading process can be summarized by a pair of functions, $D_t \left(m_t^b, k_t^s\right)$ and $X_t \left(m_t^b, k_t^s\right)$, which specify the payment and the quantity of goods being traded, respectively. These functions are given by $D_t\left(m_t^b, k_t^s\right) = \min\left\{\bar{d}_t, m_t^b\right\}$ and

$$X_t\left(m_t^b, k_t^s\right) = \begin{cases} \left(k_t^s\right)^{\epsilon} H_t^{1-\epsilon} & \text{if } m_t^b \geq \bar{d}_t, \\ \Lambda_t\left(m_t^b, k_t^s\right) & \text{if } m_t^b < \bar{d}_t. \end{cases}$$

The relationship between $D_t\left(m_t^b, k_t^s\right)$ and $X_t\left(m_t^b, k_t^s\right)$ is depicted in Figure 2.

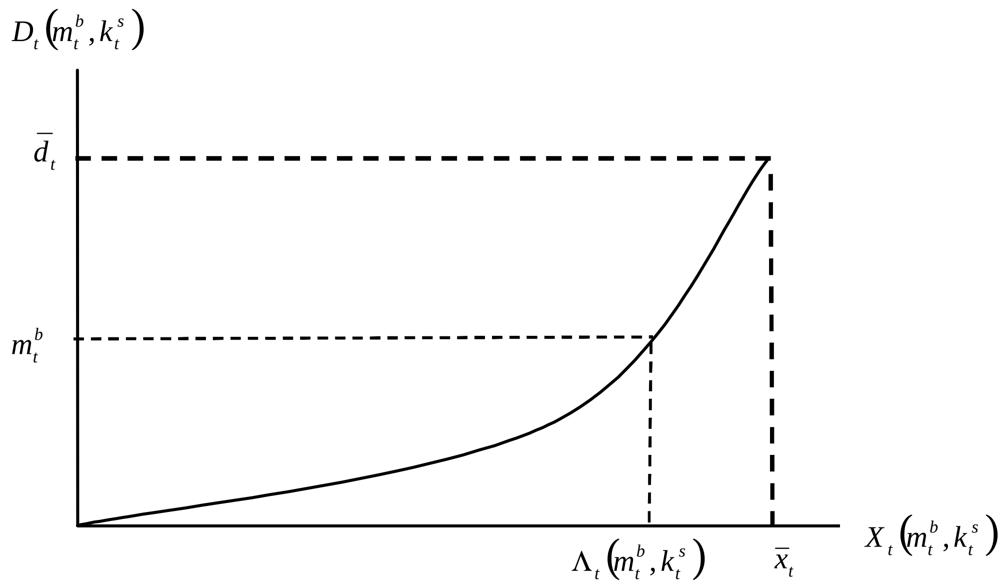

Figure 2: Relationship between $D_t\left(m_t^b, k_t^s\right)$ and $X_t\left(m_t^b, k_t^s\right)$.

Expected Value in the DM We now characterize the value function $V_t\left(m_t, k_t\right)$, which indicates the expected value of consumer in the DM at time t *before* the random identity is revealed. The main result of this subsection is Proposition 3 which establishes the strict concavity of $V_t\left(\cdot\right)$.

Suppose a consumer with assets $\left(m_t, k_t\right)$ has been assigned as a buyer in the DM. Then his expected lifetime utility is given by

$$V_t^b(m_t, k_t) = \int W_t\left(m_t - D_t\left(m_t, \widetilde{k}_t\right), k_t + \overline{z} X_t\left(m_t, \widetilde{k}_t\right)\right) d\mathcal{F}_t\left(\widetilde{m}_t, \widetilde{k}_t\right).$$

In the above expression, $\widetilde{k}_t$ denotes the quantity of physical capital owned by a potential trading partner (a seller). This quantity is randomly drawn according to the distribution $\mathcal{F}_t\left(\widetilde{m}_t, \widetilde{k}_t\right)$. If the same consumer has been assigned as a seller in the DM, then his expected lifetime utility is

$$V_t^s(m_t, k_t) = \int W_t\left(m_t + D_t\left(\widetilde{m}_t, k_t\right), k_t - \Phi_t\left(\widetilde{m}_t, k_t\right)\right) d\mathcal{F}_t\left(\widetilde{m}_t, \widetilde{k}_t\right),$$

where $\Phi_t\left(\widetilde{m}_t, k_t\right) \equiv \widetilde{\delta}\left(X_t\left(\widetilde{m}_t, k_t\right), k_t; H_t\right) k_t$, and $\widetilde{m}_t$ is the quantity of money held by a potential trading partner (a buyer). If the consumer is a non-participant in the DM, then his expected value is simply $W_t(m_t, k_t)$. Thus, before his identity in the DM is revealed, we have

$$V_t(m_t, k_t) = \alpha V_t^b(m_t, k_t) + \alpha V_t^s(m_t, k_t) + (1 - 2\alpha) W_t(m_t, k_t). \tag{42}$$

Proposition 3 provides a sufficient condition under which $V_t(\cdot)$ is strictly concave in (m_t, k_t) for all $t \geq 0$. The proof can be found in the Appendix.

Proposition 1 *Suppose $\varepsilon \leq \sigma(2 - \sigma)$. Then the value function $V_t(m_t, k_t)$ defined in (42) is strictly concave in (m_t, k_t) for all $t \geq 0$.*

From this point onward, we will assume that the condition in Proposition 3 is satisfied so that all the consumers will carry the same amount of assets when they enter the DM in every period. In terms of notations, this means there is no need to distinguish between (m_t, k_t) and $\left(\widetilde{m}_t, \widetilde{k}_t\right)$. This also means that all the bilateral trade in the DM will lead to the same outcome.

Implications for the CM Using the decentralized trading outcome described above, we can now provide a sharper prediction for the consumer's choices in the CM. In the CM at time t, all consumers have to decide on the value of (m_{t+1}, k_{t+1}), taking into account the bilateral trading outcome in the DM at time $t+1$. The main consideration here is whether it is optimal for the consumer to carry an excess amount of money into the DM, i.e., $m_{t+1} > \overline{d}_{t+1}$. Since all the sellers in the DM can at most produce $\overline{x}_{t+1}$ units of DM goods (and sell at a price of $\overline{d}_{t+1}$), carrying an excess amount of money will have no effect on the bargaining outcome. Thus, conditional on $m_{t+1} > \overline{d}_{t+1}$, the marginal benefit of increasing m_{t+1} is given by

$$\frac{\partial V_{t+1}\left(m_{t+1}, k_{t+1}\right)}{\partial m_{t+1}} = \overline{W}_{m,t+1} = \frac{A}{\left(1-\tau_l\right) p_{t+1} w_{t+1}},$$

which comes from the store-of-value function of money. Substituting this into (73) gives the condition under which $m_{t+1} > \overline{d}_{t+1}$ is optimal,

$$\frac{A}{\left(1-\tau_l\right) p_t w_t} = \beta \frac{A}{\left(1-\tau_l\right) p_{t+1} w_{t+1}} \Leftrightarrow \frac{p_{t+1} w_{t+1}}{p_t w_t} = \beta < 1.$$

Thus, it is optimal to have $m_{t+1} > \overline{d}_{t+1}$ only when the nominal wage is *decreasing* at a rate of $(1-\beta)$ between time t and $t+1$. If instead we have $p_{t+1} w_{t+1} > \beta p_t w_t$, then the marginal benefit of reducing m_{t+1} will outweigh its marginal cost of doing so, i.e.,

$$\frac{A}{\left(1-\tau_l\right) p_t w_t} > \beta \frac{A}{\left(1-\tau_l\right) p_{t+1} w_{t+1}},$$

so that it is not optimal to have $m_{t+1} > \overline{d}_{t+1}$. This result is summarized in the following lemma.

Lemma 2 *Suppose $p_{t+1}w_{t+1} > \beta p_t w_t$, for some $t \geq 0$. Then the optimal choice of m_{t+1} in the CM at time t must be bounded above by $\overline{d}_{t+1}$, i.e., $m_{t+1} \leq \overline{d}_{t+1}$.*

Suppose now the consumer chooses to have $m_{t+1} < \overline{d}_{t+1}$. In this case, any small change in m_{t+1} would affect the outcome in the DM at time $t+1$. In particular, the partial derivative of $V_{t+1}(m_{t+1}, k_{t+1})$ with respect to m_{t+1} is now given by

$$\frac{\partial V_{t+1}(m_{t+1}, k_{t+1})}{\partial m_{t+1}} = \frac{A}{(1-\tau_l)\, p_{t+1}w_{t+1}} \left\{ 1 + \alpha \left[p_{t+1}\rho_{t+1}\overline{z}\frac{\partial \Lambda_{t+1}(m_{t+1}, k_{t+1})}{\partial m_{t+1}} - 1 \right] \right\}, \quad (43)$$

where $\rho_{t+1} \equiv [1 + (1-\tau_k)\, r_{t+1}]$. Equation (43) summarizes the marginal benefits of carrying more money into period $t+1$. These benefits are twofold. The first type of benefit comes from the store-of-value function of money. Specifically, an increase in m_{t+1} means that the consumer will enter the CM at time $t+1$ with more assets, which raises the value of $W_{t+1}\left(\widehat{m}_{t+1}, \widehat{k}_{t+1}\right)$. The marginal gain in $W_{t+1}(\cdot)$ is $A/[(1-\tau_l)\, p_{t+1}w_{t+1}]$. The second type of benefit comes from the medium-of-exchange function of money. In particular, an increase in m_{t+1} relaxes the buyer's cash constraint in the DM and promotes trade. The marginal gain in lifetime utility is given by

$$\frac{A}{(1-\tau_l)\, p_{t+1}w_{t+1}} \left[p_{t+1}\rho_{t+1}\overline{z}\frac{\partial \Lambda_{t+1}(m_{t+1}, k_{t+1})}{\partial m_{t+1}} - 1 \right].$$

Note that the second type of benefit arises only when the consumer appears as a buyer in the DM, which happens with probability α. Equation (73) states that the optimal quantity of m_{t+1} is determined by equating the discounted marginal benefits of holding more money to its marginal cost. Substituting (43) into (73) gives

$$\frac{c_{t+1}}{c_t} = \frac{w_{t+1}}{w_t} = \frac{\beta p_t}{p_{t+1}} \left\{ 1 + \alpha \left[p_{t+1} \rho_{t+1} \overline{z} \frac{\partial \Lambda_{t+1} \left(m_{t+1}, k_{t+1} \right)}{\partial m_{t+1}} - 1 \right] \right\}. \tag{44}$$

Next, we turn to the marginal benefits of holding more capital in period $t+1$. These benefits are given by

$$\frac{\partial V_{t+1} \left(m_{t+1}, k_{t+1} \right)}{\partial k_{t+1}} = \frac{\rho_{t+1} A}{\left(1 - \tau_l \right) w_{t+1}} \left[1 - \alpha \frac{\partial \Phi_{t+1} \left(m_{t+1}, k_{t+1} \right)}{\partial k_{t+1}} \right].$$

Holding other things constant, an increase in k_{t+1} not only yields a gross after-tax return ρ_{t+1}, it also allows the consumer to produce more DM goods if he appears as a seller in the DM at time $t+1$. The second type of benefit is captured by the expression $-\alpha \partial \Phi_{t+1} \left(m_{t+1}, k_{t+1} \right) / \partial k_{t+1}$, which is strictly positive. Substituting the above expression into (71) gives

$$\frac{c_{t+1}}{c_t} = \frac{w_{t+1}}{w_t} = \beta \rho_{t+1} \left[1 - \alpha \frac{\partial \Phi_{t+1} \left(m_{t+1}, k_{t+1} \right)}{\partial k_{t+1}} \right]. \tag{45}$$

2.2.4 Government

The government in this economy implements both fiscal and monetary policies. In terms of monetary policies, the nominal supply of money (M_t) is assumed to grow at a deterministic constant rate $\mu > 0$ in every period, so that $M_{t+1} = (1+\mu) M_t$, for all $t \geq 0$. The seigniorage is distributed evenly to the consumers through the lump-sum transfer, so that $p_t \phi_t = M_{t+1} - M_t = \mu M_t$. In terms of fiscal policies, all the tax revenues are spent on infrastructure investment (I_t) and unproductive government spending (G_t). The latter is assumed to consume a fraction $\overline{v} \in (0,1)$ of aggregate output in every period, so that

$G_t = \overline{v} Y_t$ for all t. The government's budget constraint at time t is given by

$$\tau_c c_t + \tau_l w_t l_t + \tau_k r_t \widehat{k}_t = I_t + G_t, \tag{46}$$

The accumulation of infrastructure capital is governed by

$$H_{t+1} = I_t + (1 - \delta_h) H_t, \tag{47}$$

where $\delta_h \in (0, 1)$ is the depreciation rate of infrastructure capital and $H_0 > 0$ is given.

2.2.5 Equilibrium

In equilibrium, the factor markets in the CM clear in every period, so that $\widehat{K}_t = \widehat{k}_t$ and $L_t = l_t$ for all $t \geq 0$. All the nominal money issued by the government is held by the consumers, so that $M_t = m_t$ for all $t \geq 0$. The goods-market-clearing condition in the CM and the equilibrium dynamics of physical capital can be derived as follows: Let K_t denote the quantity of per-capita capital that is available in the DM at time t. The initial value $K_0 > 0$ is exogenously given. In equilibrium, all consumers enter the DM at time $t \geq 0$ with the same amount of assets (M_t, K_t). Those who have been assigned as a seller will enter the ensuing CM with physical capital $[K_t - \Phi_t (M_t, K_t)]$. Those who have been assigned as a buyer will enter the CM with physical capital $[K_t + \overline{z} X_t (M_t, K_t)]$. Finally, those who are inactive in the DM will enter the CM at time t with K_t. Thus, the quantity of physical capital available in the CM at time t is

$$\widehat{K}_t = K_t + \alpha \left[\overline{z} X_t (M_t, K_t) - \Phi_t (M_t, K_t) \right]. \tag{48}$$

Using the consumer's budget constraint in (68), the government's budget constraint in (46) and the other market clearing conditions, we can obtain the goods-market-clearing condition in the CM, which is given by

$$c_t + K_{t+1} - (1-\delta)\widehat{K}_t + I_t = (1-\overline{v})\widehat{K}_t^{\varrho}(H_t L_t)^{1-\varrho}. \tag{49}$$

Equations (48) and (49) together describe the equilibrium dynamics of $\widehat{K}_t$ and K_t.

Given a set of policy instruments $\{\tau_c, \tau_l, \tau_k, \mu\}$ and a set of initial conditions $\{M_0, K_0, H_0\}$, an equilibrium of this economy consists of sequences of allocations $\left\{c_t, l_t, k_t, m_t, \widehat{k}_t, \widehat{m}_t s_t, x_t, d_t\right\}_{t=0}^{\infty}$, aggregate inputs $\left\{K_t, \widehat{K}_t, L_t\right\}_{t=0}^{\infty}$, fiscal and monetary policy variables $\{M_{t+1}, H_{t+1}, G_t, \phi_t\}_{t=0}^{\infty}$, and prices $\{p_t, w_t, r_t, R_t\}_{t=0}^{\infty}$ such that

(i) Given prices and government policies, $\left\{c_t, l_t, k_t, m_t, \widehat{k}_t, \widehat{m}_t s_t, x_t, d_t\right\}_{t=0}^{\infty}$ solves the consumer's problem in the DM and CM in every period.

(ii) Given prices and the sequence of infrastructure capital, $\left\{\widehat{K}_t, L_t\right\}_{t=0}^{\infty}$ solves the representative firm's problem in the CM in every period.

(iii) The sequences $\left\{K_t, \widehat{K}_t\right\}_{t=0}^{\infty}$ satisfy (48) in every period.

(iv) The government's budget is balanced in every period, so that (46) holds for all $t \geq 0$. The stock of infrastructure capital accumulates according to (47). Nominal money supply is determined by $M_{t+1} = (1+\mu) M_t$ and $G_t = \overline{v} Y_t$ for all $t \geq 0$.

(v) All markets clear in every period.

In the following analysis, we confine our attention to balanced-growth equilibria in which all variables are growing at some positive constant rates. Specifically, we focus on equilibria which satisfy the following additional conditions:

(vi) The rental price of physical capital and the supply of labor in the CM are both constant over time, i.e., $R_t = R^*$ and $l_t = l^*$ for all $t \geq 0$.

(vii) The rate of capital utilization in the DM is constant over time, i.e., $s_t = s^*$ for all $t \geq 0$.

(viii) All other real variables, including $\left\{c_t, K_t, \widehat{K}_t, H_t, x_t, w_t, \phi_t\right\}$, are growing at the same constant rate. The common growth rate γ^* is non-negative.

(ix) All nominal variables are growing at the same rate as the nominal money supply and the growth rate is non-negative, i.e., $\mu \geq 0$.

Since $\mu \geq 0$, it follows from Lemma 2 that it is never optimal for the consumers to have $m_{t+1} > \overline{d}_{t+1}$. Thus, the bilateral trading outcome in the DM is given by $X_t(M_t, K_t) = \Lambda_t(M_t, K_t)$ and $D_t(M_t, K_t) = M_t$ for all t.

Define the transformed variables $\nu_t \equiv K_t/H_t$ and $\widehat{\nu}_t \equiv \widehat{K}_t/H_t$. In any balanced growth equilibrium, both variables are constant over time, so that $\nu_t = \nu^*$ and $\widehat{\nu}_t = \widehat{\nu}^*$. The set of stationary values $\{\gamma^*, R^*, s^*, \nu^*, \widehat{\nu}^*\}$ is determined by the following system of equations:

$$(s^*)^{\sigma-\epsilon}(\nu^*)^{1-\epsilon} = \frac{\overline{z}\epsilon}{\theta\eta\sigma}\left[\frac{\alpha\beta}{1+\mu-\beta(1-\alpha)} - (1-\theta)\right] \equiv \overline{z}\Omega(\mu), \tag{50}$$

$$1+\gamma^* = \beta\left[1+(1-\tau_k)(R^*-\delta)\right]\left\{1+(1-\theta)(\sigma-1)\left[\frac{1+\mu-\beta(1-\alpha)}{\beta}\right]\eta(s^*)^{\sigma}\right\}, \tag{51}$$

$$\tau_c B\left(\frac{\varrho}{R^*}\right)^{\frac{\varrho}{1-\varrho}} + \left[\widetilde{\tau}\left(\frac{R^*}{\varrho}\right) - \tau_k\delta\right]\widehat{\nu}^* = \gamma^* + \delta_h, \tag{52}$$

$$\widehat{\nu}^* = \nu^* + \left[1-\eta\Omega(\mu)\right]\alpha\overline{z}(s^*\nu^*)^{\epsilon}, \tag{53}$$

$$B\left(\frac{\varrho}{R^*}\right)^{\frac{\varrho}{1-\varrho}} + \nu^*(1+\gamma^*) + \gamma^* + \delta_h = \left[(1-\overline{v})\left(\frac{R^*}{\varrho}\right) + 1 - \delta\right]\widehat{\nu}^*, \tag{54}$$

where B and $\widetilde{\tau}$ are defined as

$$B \equiv \frac{(1-\tau_l)(1-\varrho)}{(1+\tau_c)A} \qquad \text{and} \qquad \widetilde{\tau} \equiv \tau_l(1-\varrho) + \tau_k \varrho - \overline{v}.$$

A formal derivation of these equations can be found in the Appendix. Once the values of $\{\gamma^*, R^*, s^*, \nu^*, \widehat{\nu}^*\}$ are determined, all other variables in the balanced-growth equilibrium can be uniquely determined.

Next, we provide the expression of some "great ratios" which are of interest in the numerical analysis. In each period t, aggregate output ($\boldsymbol{Y}_t$) is defined as the sum of output produced in the CM and the DM, i.e., $\boldsymbol{Y}_t \equiv \widehat{K}_t^{\varrho}(H_t L_t)^{1-\varrho} + \alpha\overline{z}(s_t K_t)^{\epsilon} H_t^{1-\epsilon}$. In any balanced-growth equilibrium, the ratio between $\widehat{K}_t$ and aggregate output is given by

$$\frac{\widehat{K}_t}{\boldsymbol{Y}_t} = \frac{\widehat{\nu}^*}{(\widehat{\nu}^*)^{\varrho}(L^*)^{1-\varrho} + \alpha\overline{z}(s^*\nu^*)^{\epsilon}}. \tag{55}$$

Aggregate investment, on the other hand, is given by $\boldsymbol{I}_t \equiv \alpha\overline{z}(s_t K_t)^{\epsilon} H_t^{1-\epsilon} + K_{t+1} - (1-\delta)\widehat{K}_t$. Using this, we can derive an expression for the investment-capital ratio in any balanced-growth equilibrium, which is

$$\frac{\boldsymbol{I}_t}{\widehat{K}_t} = \frac{\alpha\overline{z}(s^*\nu^*)^{\epsilon} + (1+\gamma^*)\nu^*}{\widehat{\nu}^*} - (1-\delta). \tag{56}$$

The velocity of money in any period t is defined as $\boldsymbol{V}_t = p_t\boldsymbol{Y}_t/M_t$. Since $d_t = m_t = M_t$ in the DM, we can compute the value of $\boldsymbol{V}_t$ based on $\boldsymbol{Y}_t$ and (40).

2.3 Quantitative Analysis

We now explore the quantitative implications of the above model. Our goal is to quantify the effects of inflation on long-term economic growth. To achieve this, we first construct a benchmark model which is parameterized to match certain key features of the U.S. economy over the period 1960-2010. We then construct a series of policy experiments to gauge to the effects of inflation on economic growth. The benchmark parameter values are summarized in Table 1. Most of these values are chosen based on empirical evidence. Others are chosen to match certain real-world statistics. The details of this procedure are explained below.

2.3.1 Calibration

One period in the model is one year. The parameter ϱ in the production function of the CM good is chosen to match the share of labor income in US GDP, which is 0.60 over the period 1960-2010.[21] The required value is $\varrho = 0.40$. We use the same value for ϵ so that physical capital has the same importance in the production of the CM good and the DM good. The tax rate on labor income and interest income are chosen based on the time series of marginal income tax rate reported in Barro and Redlick (2011, Table 1). In particular, they report data on both the federal and state income tax rates over the period 1912-2006. We take the sum of the two and compute the average value over the period 1960-2006. The resulting value is 27.6%. Hence, we set $\tau_l = \tau_k = 0.276$. The tax rate on private consumption expenditures (τ_c) is set to 4.6%, which is based on the data from the National Income

[21] This value is computed using data on compensation of employees and proprietor's income over the period 1960-2010. We assume that the share of labor income in proprietor's income is identical to that of the entire economy, so that

$$\text{Labor's share of income} = \frac{\text{Compensation of Employees}}{\text{GDP - Proprietor's income}}.$$

See Gomme and Rupert (2007, Section 4.2) for more details on this. Cooley and Prescott (1995) use a different approach to compute this share, but they also obtain a value of 0.60.

and Product Accounts.[22] We set $\overline{v} = 0.16$, which matches the percentage of government consumption expenditures in US GDP over the period 1960-2010. The depreciation rate of infrastructure capital (δ_h) is chosen to match the ratio between government gross investment and government fixed assets over the same time period. The implied value is $\delta_h = 0.067$. We set $\alpha = 0.5$ so that the population in the model economy is equally divided into buyers and sellers in the DM in each period. The value of η is normalized to one.

Seven parameters remain undetermined up to this point. These include the subjective discount factor (β), a preference parameter (A), the depreciation rate of physical capital in the CM (δ), the growth rate of nominal money supply (μ), the share of total trade surplus claimed by the buyer (θ), and two parameters related to the activities in the DM ($\overline{z}$ and σ). These parameters are chosen so that the benchmark balanced-growth equilibrium has the following properties: First, the time spent on working is one-third. Second, the common growth rate for real variables is 2%, which matches the average annual growth rate of real per-capita GDP in the United States over the period 1960-2010. Third, the capital-output ratio as defined in (55) is 3.0. Fourth, the inflation rate in the long-run equilibrium is 4.1%, which is the average annual growth rate of the Consumer Price Index over this time period. Fifth, the rate of return from physical capital in the CM is 5%. Sixth, the ratio between investment and physical capital as defined in (56) is 8.2%. Finally, the velocity of money is 5.381. The same target is also used in Aruoba, Waller and Wright (2011).

[22] Specifically, we first obtain annual data on total sales taxes collected by state and local governments over the period 1960-2010. We then compute the ratio between these tax revenues and total private consumption expenditures for each year. This ratio is used as proxies for the consumption tax rate in these years. The average value over the sample period is 4.6%.

2.3.2 Results

Table 2 summarizes the main properties of the benchmark equilibrium and compares them to their empirical counterparts. Besides the seven targeted statistics, the model is also able to generate reasonable values for the share of consumption in aggregate output. In our benchmark model, about 86% of trade surplus in the DM is claimed by the buyers. In Aruoba, Waller and Wright (2011), trade surplus is divided through generalized Nash bargaining. In their parameterization, the bargaining power for the buyer is 92%. Another thing worth mentioning is that, in their model the decentralized market only accounts for about 3% of total output. In our benchmark model, the decentralized market accounts for about 30% of total output.

Finally, we conduct a couple of counterfactual experiments by increasing the value of μ. All other parameters are fixed as in the benchmark scenario. In the first experiment, we set $\mu = 0.0863$ which gives a long-run inflation rate of 6.5%. In the second experiment, we set $\mu = 0.1222$ which generates a long-run inflation rate of 10%. The results are summarized in Table 3. In general, our results show that an increase in inflation is accompanied by an increase in the long-term growth rate but the magnitude of the effect is very small. It is also accompanied by an increase in working hours in the CM and an increase in capital accumulation (as indicated by the capital-output ratio).

Table 1 Benchmark Parameters

Parameter	Description	Value
β	Subjective discount factor	0.9820
A	Preference parameter	1.5696
ϱ	Parameter in CM production	0.4000
δ	Depreciation rate of capital in the CM	0.0446
τ_l	Tax rate on labor income	0.2760
τ_k	Tax rate on interest income	0.2760
τ_c	Tax rate on consumption expenditures	0.0460
μ	Growth rate of nominal money supply	0.0618
$\overline{v}$	Share of unproductive government spending	0.1600
δ_h	Depreciation rate of infrastructure	0.0670
α	Probability of being a buyer in the DM	0.5000
θ	Share of total surplus claimed by buyers	0.8595
σ	Parameter in the DM	1.7124
η	Parameter in the DM	1.0000
ϵ	Parameter in DM production	0.4000
$\overline{z}$	Parameter in the DM	0.9164

Table 2 Properties of Benchmark Model

	Data/Target	Model
Labor hours (L^*)	0.330*	0.3299
Long-term growth rate (γ^*)	0.020*	0.0200
Inflation rate (π^*)	0.041*	0.0410
Real interest rate (r^*)	0.050*	0.0500
Utilization rate of capital in DM (s^*)	—	0.1550
Output in the CM	—	0.8626
Output in the DM	—	0.3530
Velocity of money	5.385*	5.3834
Capital-output ratio	3.000*	2.9996
Consumption-output ratio	0.630	0.5690
Investment-Capital ratio	0.082*	0.0820

Note: Figures marked with an asterisk are the targeted statistics.

Table 3 Policy Experiments

	Benchmark	Experiments	
	$\mu = 0.0618$	$\mu = 0.0863$	$\mu = 0.1222$
Labor hours (L^*)	0.3299	0.3304	0.3313
Long-term growth rate (γ^*)	0.0200	0.0201	0.0202
Inflation rate (π^*)	0.0410	0.0650	0.1000
Real interest rate (r^*)	0.0500	0.0502	0.0504
Utilization rate of capital in DM (s^*)	0.1550	0.1492	0.1415
Output in the CM	0.8626	0.8629	0.8636
Output in the DM	0.3531	0.3476	0.3400
Velocity of money	5.3834	5.5950	5.9004
Capital-output ratio	2.9996	3.0084	3.0196
Consumption-output ratio	0.5690	0.5708	0.5730
Investment-Capital ratio	0.0820	0.0808	0.0794

2.4 Appendix

2.4.1 Proof of Proposition 3

Since $W_t(m_t, k_t)$ is separable in its argument and linear in both k_t and m_t, it suffice to show that $V_t^b(m_t, k_t)$ and $V_t^s(m_t, k_t)$ are jointly strictly concave in (m_t, k_t). Consider a buyer in the DM at time t with assets (m_t, k_t). Depending on whom he trade with, there are two possible bargaining outcomes: (i) $m_t \geq \overline{d}_t$ and (ii) $m_t < \overline{d}_t$, where

$$\overline{d}_t \equiv p_t\left[1 + (1 - \tau_k) r_t\right]\left[(1 - \theta)\overline{z}\left(\widetilde{k}_t\right)^{\epsilon} H_t^{1-\epsilon} + \theta\eta\widetilde{k}_t\right].$$

If $m_t \geq \overline{d}_t$, then the buyer's payoff is

$$W_t\left(m_t - \overline{d}_t, k_t + \overline{z}\left(\widetilde{k}_t\right)^{\epsilon} H_t^{1-\epsilon}\right) = W_t(m_t, k_t) - \overline{W}_{m,t}\overline{d}_t + \overline{W}_{k,t}\overline{z}\left(\widetilde{k}_t\right)^{\epsilon} H_t^{1-\epsilon},$$

which is a linear function in m_t and k_t. If $m_t < \overline{d}_t$, then the buyer's payoff is

$$W_t\left(0, k_t + \overline{z}\Lambda_t\left(m_t, \widetilde{k}_t\right)\right) = W_t(m_t, k_t) - \overline{W}_{m,t}m_t + \overline{W}_{k,t}\overline{z}\Lambda_t\left(m_t, \widetilde{k}_t\right), \tag{57}$$

which is linear in k_t. This payoff function is strictly concave in m_t if and only if $\Lambda_t\left(m_t, \widetilde{k}_t\right)$ is strictly concave in m_t. As shown in the main text, we have

$$\frac{\partial\Lambda_t\left(m_t, \widetilde{k}_t\right)}{\partial m_t} = \frac{1}{p_t\left[1 + (1 - \tau_k) r_t\right]\left[(1 - \theta)\overline{z} + \theta\widetilde{\delta}_x\left(x_t, \widetilde{k}_t; H_t\right)\widetilde{k}_t\right]} > 0, \tag{58}$$

where

$$\widetilde{\delta}_x\left(x_t,\widetilde{k}_t;H_t\right)\widetilde{k}_t=\frac{\eta\sigma}{\epsilon}\left[x_t^{\frac{\sigma}{\epsilon}-1}H_t^{\frac{(\epsilon-1)\sigma}{\epsilon}}\left(\widetilde{k}_t\right)^{1-\sigma}\right]\qquad\text{and}\qquad x_t=\Lambda_t\left(m_t,\widetilde{k}_t\right).$$

Differentiating (58) with respect to m_t gives

$$\begin{aligned}&\frac{\partial^2\Lambda_t\left(m_t,\widetilde{k}_t\right)}{\partial m_t^2}\\ =&\ -\frac{\eta\theta\sigma}{\epsilon}\left(\frac{\sigma}{\epsilon}-1\right)\frac{\left[\Lambda_t\left(m_t,\widetilde{k}_t\right)\right]^{\frac{\sigma}{\epsilon}-2}H_t^{\frac{(\epsilon-1)\sigma}{\epsilon}}\left(\widetilde{k}_t\right)^{1-\sigma}}{p_t\left[1+(1-\tau_k)r_t\right]\left[(1-\theta)\overline{z}+\theta\widetilde{\delta}_x\left(x_t,\widetilde{k}_t;H_t\right)\widetilde{k}_t\right]^2}\left[\frac{\partial\Lambda_t\left(m_t,\widetilde{k}_t\right)}{\partial m_t}\right]<0,\end{aligned}$$

as $\sigma>1>\epsilon$. Hence, the payoff function in (57) is strictly concave in m_t. Let $\mathcal{S}_{1,t}$ be the set of $\left(\widetilde{m}_t,\widetilde{k}_t\right)$ under which $m_t\geq\overline{d}_t$, and let $\mathcal{S}_{2,t}$ be the set of $\left(\widetilde{m}_t,\widetilde{k}_t\right)$ under which $m_t<\overline{d}_t$. Then, the buyer's expected value can be expressed as

$$\begin{aligned}V_t^b\left(m_t,k_t\right)\ =&\ \int_{\mathcal{S}_{1,t}}W_t\left(m_t-\overline{d}_t,k_t+\overline{z}\left(\widetilde{k}_t\right)^{\epsilon}H_t^{1-\epsilon}\right)d\mathcal{F}_t\left(\widetilde{m}_t,\widetilde{k}_t\right)\\ &+\int_{\mathcal{S}_{2,t}}W_t\left(0,k_t+\overline{z}\Lambda_t\left(m_t,\widetilde{k}_t\right)\right)d\mathcal{F}_t\left(\widetilde{m}_t,\widetilde{k}_t\right).\end{aligned}$$

The above results imply that $V_t^b\left(m_t,k_t\right)$ is separable in its arguments, linear in k_t and strictly concave in m_t. Hence, it is jointly strictly concave in $\left(m_t,k_t\right)$.

Next, consider a seller in the DM at time t with assets $\left(m_t,k_t\right)$. Again, depending on whom he trade with, there are two possible bargaining outcomes: (i) $\widetilde{m}_t\geq\overline{d}_t$ and (ii) $\widetilde{m}_t<\overline{d}_t$, where $\overline{d}_t$ is now given by

$$\overline{d}_t\left(k_t\right)\equiv p_t\left[1+(1-\tau_k)r_t\right]\left[(1-\theta)\overline{z}k_t^{\epsilon}H_t^{1-\epsilon}+\theta\eta k_t\right],$$

where we highlight its dependence on k_t. Since $\epsilon \in (0,1)$, $\overline{d}_t(k_t)$ is strictly concave in k_t. If $\widetilde{m}_t \geq \overline{d}_t(k_t)$, then the seller's payoff is

$$W_t\left(m_t + \overline{d}_t, (1-\eta)k_t\right) = W_t(m_t, k_t) + \overline{W}_{m,t}\overline{d}_t(k_t) - \overline{W}_{k,t}\eta k_t.$$

This payoff function is linear in m_t and strictly concave in k_t. If $\widetilde{m}_t < \overline{d}_t(k_t)$, then the seller's payoff is

$$W_t(m_t + \widetilde{m}_t, k_t - \Phi_t(\widetilde{m}_t, k_t)) = W(m_t, k_t) + \overline{W}_{m,t}\widetilde{m}_t - \overline{W}_{k,t}\Phi_t(\widetilde{m}_t, k_t), \tag{59}$$

where

$$\Phi_t(\widetilde{m}_t, k_t) = \widetilde{\delta}(\Lambda_t(\widetilde{m}_t, k_t), k_t; H_t)k_t = \eta[\Lambda_t(\widetilde{m}_t, k_t)]^{\frac{\sigma}{\epsilon}} H_t^{\frac{(\epsilon-1)\sigma}{\epsilon}} k_t^{1-\sigma}. \tag{60}$$

The payoff function in (59) is separable in (m_t, k_t) and linear in m_t. It is strictly concave in k_t if and only if $\Phi_t(\widetilde{m}_t, k_t)$ is strictly *convex* in k_t for any $\widetilde{m}_t > 0$. Since $\Phi_t(\widetilde{m}_t, k_t)$ is a strictly positive real-valued function, it is strictly convex in k_t if $\ln[\Phi_t(\widetilde{m}_t, k_t)]$ is strictly convex in k_t. Using (60), we can get

$$\ln[\Phi_t(\widetilde{m}_t, k_t)] = \ln\left[\eta H_t^{\frac{(\epsilon-1)\sigma}{\epsilon}}\right] + \frac{\sigma}{\epsilon}\ln[\Lambda_t(\widetilde{m}_t, k_t)] - (\sigma - 1)\ln k_t.$$

Since $-(\sigma-1)\ln k_t$ is strictly convex, it suffice to show that $\ln[\Lambda_t(\widetilde{m}_t, k_t)]$ is strictly convex in k_t. For ease of exposition, we will adopt the following simplified notation in the remaining part of the proof,

$$\widetilde{\delta}_t \equiv \widetilde{\delta}(\Lambda_t(\widetilde{m}_t, k_t), k_t; H_t),$$

$$\widetilde{\delta}_{x,t} \equiv \widetilde{\delta}_x(\Lambda_t(\widetilde{m}_t, k_t), k_t; H_t) = \frac{\sigma}{\epsilon}\frac{\widetilde{\delta}_t}{\Lambda_t(\widetilde{m}_t, k_t)} > 0, \tag{61}$$

$$\widetilde{\delta}_{k,t} \equiv \widetilde{\delta}_k \left(\Lambda_t \left(\widetilde{m}_t, k_t\right), k_t; H_t\right) = -\frac{\sigma \widetilde{\delta}_t}{k_t} < 0.$$

It is important to note the difference between the partial derivative $\widetilde{\delta}_{k,t}$ and the following total derivative

$$\frac{d\widetilde{\delta}_t}{dk_t} = \widetilde{\delta}_{x,t} \left[\frac{\partial \Lambda_t \left(\widetilde{m}_t, k_t\right)}{\partial k_t}\right] + \widetilde{\delta}_{k,t}.$$

Using these notations, we can write

$$\frac{\partial \Phi_t \left(\widetilde{m}_t, k_t\right)}{\partial k_t} = k_t \frac{d\widetilde{\delta}_t}{dk_t} + \widetilde{\delta}_t.$$

Using (77), we can obtain

$$\frac{\partial \ln \left[\Lambda_t \left(\widetilde{m}_t, k_t\right)\right]}{\partial k_t} = \frac{1}{\Lambda_t \left(\widetilde{m}_t, k_t\right)} \frac{\partial \Lambda_t \left(\widetilde{m}_t, k_t\right)}{\partial k_t} = \frac{\theta \left(\sigma - 1\right) \widetilde{\delta}_t}{\Lambda_t \left(\widetilde{m}_t, k_t\right) \left[\left(1 - \theta\right) \overline{z} + \theta \widetilde{\delta}_{x,t} k_t\right]}. \tag{62}$$

Using (61), we can express the denominator of the above expression as

$$\begin{aligned} \Delta_t &= \left(1 - \theta\right) \overline{z} \Lambda_t \left(\widetilde{m}_t, k_t\right) + \theta \left(\frac{\sigma}{\epsilon}\right) \widetilde{\delta}_t k_t \\ &= \left(1 - \theta\right) \overline{z} \Lambda_t \left(\widetilde{m}_t, k_t\right) + \theta \widetilde{\delta}_t k_t + \theta \left(\frac{\sigma}{\epsilon} - 1\right) \widetilde{\delta}_t k_t \\ &= \frac{\widetilde{m}_t}{p_t \left[1 + \left(1 - \tau_k\right) r_t\right]} + \theta \left(\frac{\sigma}{\epsilon} - 1\right) \widetilde{\delta}_t k_t. \end{aligned}$$

The third line follows from (40). Differentiating the expression in (62) with respect to k_t again gives

$$\frac{\partial^2 \ln \left[\Lambda_t \left(\widetilde{m}_t, k_t\right)\right]}{\partial k_t^2} = \frac{\theta \left(\sigma - 1\right)}{\Delta_t^2} \left[\left(\frac{d\widetilde{\delta}_t}{dk_t}\right) \Delta_t - \theta \left(\frac{\sigma}{\epsilon} - 1\right) \widetilde{\delta}_t \left(k_t \frac{d\widetilde{\delta}_t}{dk_t} + \widetilde{\delta}_t\right)\right].$$

The expression inside the square brackets can be simplified to become

$$\begin{aligned}
& \frac{\widetilde{m}_t}{p_t\left[1+\left(1-\tau_k\right) r_t\right]}\left(\frac{d \widetilde{\delta}_t}{d k_t}\right)-\theta\left(\frac{\sigma}{\epsilon}-1\right) \widetilde{\delta}_t^2 \\
= & \frac{\widetilde{m}_t}{p_t\left[1+\left(1-\tau_k\right) r_t\right]}\left\{\widetilde{\delta}_{x, t}\left[\frac{\partial \Lambda_t\left(\widetilde{m}_t, k_t\right)}{\partial k_t}\right]+\widetilde{\delta}_{k, t}\right\}-\theta\left(\frac{\sigma}{\epsilon}-1\right) \widetilde{\delta}_t^2 .
\end{aligned}$$

Since $\widetilde{\delta}_{k,t} < 0$, it suffice to show that

$$\frac{\widetilde{m}_t}{p_t\left[1+\left(1-\tau_k\right) r_t\right]} \widetilde{\delta}_{x, t}\left[\frac{\partial \Lambda_t\left(\widetilde{m}_t, k_t\right)}{\partial k_t}\right]-\theta\left(\frac{\sigma}{\epsilon}-1\right) \widetilde{\delta}_t^2<0 .$$

Using

$$\frac{\widetilde{m}_t}{p_t\left[1+\left(1-\tau_k\right) r_t\right]}=\Delta_t-\theta\left(\frac{\sigma}{\epsilon}-1\right) \widetilde{\delta}_t k_t \quad \text { and } \quad \widetilde{\delta}_{x, t}=\frac{\sigma}{\epsilon} \frac{\widetilde{\delta}_t}{\Lambda_t\left(\widetilde{m}_t, k_t\right)},$$

we can get

$$\begin{aligned}
& \frac{\widetilde{m}_t}{p_t\left[1+\left(1-\tau_k\right) r_t\right]} \widetilde{\delta}_{x, t}\left[\frac{\partial \Lambda_t\left(\widetilde{m}_t, k_t\right)}{\partial k_t}\right]-\theta\left(\frac{\sigma}{\epsilon}-1\right) \widetilde{\delta}_t^2 \\
= & \left[\Delta_t-\theta\left(\frac{\sigma}{\epsilon}-1\right) \widetilde{\delta}_t k_t\right] \frac{\sigma}{\epsilon}\left[\frac{\theta(\sigma-1) \widetilde{\delta}_t^2}{\Delta_t}\right]-\theta\left(\frac{\sigma}{\epsilon}-1\right) \widetilde{\delta}_t^2 \\
= & \frac{\theta \widetilde{\delta}_t^2}{\Delta_t}\left\{\left[\Delta_t-\theta\left(\frac{\sigma}{\epsilon}-1\right) \widetilde{\delta}_t k_t\right] \frac{\sigma(\sigma-1)}{\epsilon}-\left(\frac{\sigma}{\epsilon}-1\right) \Delta_t\right\} \\
= & \frac{\theta \widetilde{\delta}_t^2}{\Delta_t}\left\{\left[\frac{\sigma(\sigma-1)}{\epsilon}-\left(\frac{\sigma}{\epsilon}-1\right)\right] \Delta_t-\theta\left(\frac{\sigma}{\epsilon}-1\right) \frac{\sigma(\sigma-1)}{\epsilon} \widetilde{\delta}_t k_t\right\},
\end{aligned}$$

which is strictly negative if $\frac{\sigma(\sigma-1)}{\epsilon} \leq \left(\frac{\sigma}{\epsilon}-1\right)$. The latter is equivalent to $\epsilon \leq \sigma(2-\sigma)$. This proves that the payoff function in (59) is strictly concave in k_t.

Let $\widehat{\mathcal{S}}_{1,t}$ be the set of $\left(\widetilde{m}_t, \widetilde{k}_t\right)$ under which $\widetilde{m}_t \geq \overline{d}_t$, and let $\widehat{\mathcal{S}}_{2,t}$ be the set of $\left(\widetilde{m}_t, \widetilde{k}_t\right)$

under which $\widetilde{m}_t < \overline{d}_t$. Then, the seller's expected value can be expressed as

$$\begin{aligned} V_t^s(m_t, k_t) &= \int_{\widehat{S}_{1,t}} W_t\left(m_t + \overline{d}_t, (1-\eta)\,k_t\right) d\mathcal{F}_t\left(\widetilde{m}_t, \widetilde{k}_t\right) \\ &\quad + \int_{\widehat{S}_{2,t}} W_t\left(m_t + \widetilde{m}_t, k_t - \Phi_t\left(\widetilde{m}_t, k_t\right)\right) d\mathcal{F}_t\left(\widetilde{m}_t, \widetilde{k}_t\right). \end{aligned}$$

The above results imply that $V_t^s(m_t, k_t)$ is separable in its arguments, linear in m_t and strictly concave in k_t. Hence, it is also jointly strictly concave in (m_t, k_t). This completes the proof of Proposition 3.

2.4.2 Derivations of (50)-(54)

Since $K_t = k_t$, $\widehat{K}_t = \widehat{k}_t$, $M_t = m_t$ and $L_t = l_t$ for all $t \geq 0$ in equilibrium, we will use the lowercase variables and uppercase variables interchangeably in this section. Define the transformed variable $\nu_t \equiv k_t / H_t$. Then we can write

$$\frac{x_t}{H_t} = \left(s_t \cdot \frac{k_t}{H_t}\right)^{\epsilon} \Rightarrow \frac{x_t}{H_t} = \left(s_t \nu_t\right)^{\epsilon},$$

$$\widetilde{\delta}_{x,t} k_t = \frac{\sigma}{\epsilon}\left(\frac{\widetilde{\delta}_t k_t}{x_t}\right) = \frac{\eta\sigma}{\epsilon}\left[\frac{s_t^{\sigma}\nu_t}{\left(s_t\nu_t\right)^{\epsilon}}\right] = \widetilde{\eta}\left(s_t^{\sigma-\epsilon}\nu_t^{1-\epsilon}\right),$$

where $\widetilde{\eta} \equiv \eta\sigma/\epsilon > 0$. Using these expressions, we can express the partial derivatives of $\Lambda_t(m_t, k_t)$ as

$$\frac{\partial \Lambda_t(m_t, k_t)}{\partial m_t} = \frac{1}{p_t\left[1 + (1-\tau_k)\,r_t\right]\left[(1-\theta)\,\overline{z} + \theta\widetilde{\eta} s_t^{\sigma-\epsilon}\nu_t^{1-\epsilon}\right]},$$

$$\frac{\partial \Lambda_t(m_t, k_t)}{\partial k_t} = \frac{\theta(\sigma-1)\,\eta s_t^{\sigma}}{(1-\theta)\,\overline{z} + \theta\widetilde{\eta} s_t^{\sigma-\epsilon}\nu_t^{1-\epsilon}}.$$

Next, we need to derive an expression for $\partial\Phi_t(m_t, k_t)/\partial k_t$. This is given by

$$\begin{aligned}
\frac{\partial\Phi_t(m_t, k_t)}{\partial k_t} &= k_t\left[\widetilde{\delta}_{x,t}\frac{\partial\Lambda_t(m_t, k_t)}{\partial k_t} + \widetilde{\delta}_{k,t}\right] + \widetilde{\delta}_t \\
&= \widetilde{\delta}_t\frac{\theta(\sigma-1)\widetilde{\delta}_{x,t}k_t}{(1-\theta)\overline{z} + \theta\widetilde{\delta}_{x,t}k_t} - (\sigma-1)\widetilde{\delta}_t \\
&= -\frac{(1-\theta)\overline{z}(\sigma-1)\widetilde{\delta}_t}{(1-\theta)\overline{z} + \theta\widetilde{\delta}_{x,t}k_t} = -\frac{(1-\theta)\overline{z}(\sigma-1)\eta s_t^{\sigma}}{(1-\theta)\overline{z} + \theta\widetilde{\eta}s_t^{\sigma-\epsilon}\nu_t^{1-\epsilon}} < 0.
\end{aligned}$$

The Euler equations in (44) and (45) can now be rewritten as

$$\frac{p_{t+1}c_{t+1}}{p_t c_t} = \beta\left[1-\alpha + \frac{\alpha\overline{z}}{(1-\theta)\overline{z} + \theta\widetilde{\eta}s_{t+1}^{\sigma-\epsilon}\nu_{t+1}^{1-\epsilon}}\right], \tag{63}$$

$$\frac{c_{t+1}}{c_t} = \beta\left[1 + (1-\tau_k)r_{t+1}\right]\left\{1 + (1-\theta)(\sigma-1)\eta s_{t+1}^{\sigma}\left[\frac{\alpha\overline{z}}{(1-\theta)\overline{z} + \theta\widetilde{\eta}s_{t+1}^{\sigma-\epsilon}\nu_{t+1}^{1-\epsilon}}\right]\right\}. \tag{64}$$

In a balanced-growth equilibrium, we have $s_t = s^*$ and $\nu_t = \nu^*$ for all t. In addition, all nominal variables must be growing at the same rate as μ. Hence, (63) can be rewritten as

$$1 + \mu = \beta\left[1-\alpha + \frac{\alpha\overline{z}}{(1-\theta)\overline{z} + \theta\widetilde{\eta}(s^*)^{\sigma-\epsilon}(\nu^*)^{1-\epsilon}}\right]$$

$$\Rightarrow \frac{1+\mu-\beta(1-\alpha)}{\beta} = \frac{\alpha\overline{z}}{(1-\theta)\overline{z} + \theta\widetilde{\eta}(s^*)^{\sigma-\epsilon}(\nu^*)^{1-\epsilon}} \tag{65}$$

$$\Rightarrow (s^*)^{\sigma-\epsilon}(\nu^*)^{1-\epsilon} = \frac{\overline{z}}{\theta\widetilde{\eta}}\left[\frac{\alpha\beta}{1+\mu-\beta(1-\alpha)} - (1-\theta)\right] \equiv \overline{z}\Omega(\mu),$$

which is equation (50) in the main text. This equation also implies

$$(s^*)^{\sigma}\nu^* = \overline{z}\Omega(\mu)(s^*\nu^*)^{\epsilon}. \tag{66}$$

Similarly, after imposing the conditions for a balanced-growth equilibrium, (64) can be rewritten as

$$\begin{aligned} 1+\gamma^* &= \beta\left[1+(1-\tau_k)(R^*-\delta)\right]\left\{1+(1-\theta)(\sigma-1)\eta(s^*)^{\sigma}\left[\frac{\alpha\bar{z}}{(1-\theta)\bar{z}+\widetilde{\eta}(s^*)^{\sigma-\epsilon}(\nu^*)^{1-\epsilon}}\right]\right\} \\ &= \beta\left[1+(1-\tau_k)(R^*-\delta)\right]\left\{1+(1-\theta)(\sigma-1)\eta(s^*)^{\sigma}\left[\frac{1+\mu-\beta(1-\alpha)}{\beta}\right]\right\}, \end{aligned}$$

which is equation (51) in the main text. Note that we have used (65) in order to derive the second equality.

Define the transformed variable $\widehat{\nu}_t \equiv \widehat{K}_t/H_t$. From the representative firm's first-order conditions, we can get

$$R_t = \varrho\left(\frac{\widehat{\nu}_t}{l_t}\right)^{\varrho-1} \qquad \text{and} \qquad \frac{w_t}{H_t} = (1-\varrho)\left(\frac{\widehat{\nu}_t}{l_t}\right)^{\varrho},$$

which imply

$$\frac{\widehat{\nu}_t}{l_t} = \left(\frac{\varrho}{R_t}\right)^{\frac{1}{1-\varrho}} \Rightarrow \frac{w_t}{H_t} = (1-\varrho)\left(\frac{\varrho}{R_t}\right)^{\frac{\varrho}{1-\varrho}}, \tag{67}$$

$$l_t = \left(\frac{R_t}{\varrho}\right)^{\frac{1}{1-\varrho}}\widehat{\nu}_t,$$

$$\frac{w_t l_t}{H_t} = (1-\varrho)(\widehat{\nu}_t)^{\varrho}(l_t)^{1-\varrho} \qquad \text{and} \qquad \frac{R_t\widehat{K}_t}{H_t} = \varrho(\widehat{\nu}_t)^{\varrho}(l_t)^{1-\varrho}.$$

From the first-order condition of the consumer's problem, we have

$$c_t = \frac{(1-\tau_l)w_t}{(1+\tau_c)A} \Rightarrow \frac{c_t}{H_t} = \underbrace{\frac{(1-\tau_l)(1-\varrho)}{(1+\tau_c)A}}_{B}\left(\frac{\varrho}{R_t}\right)^{\frac{\varrho}{1-\varrho}}.$$

The second equality follows from the expression in (67). Next, consider the government's budget constraint in (46). Dividing both sides by H_t gives

$$\tau_c\left(\frac{c_t}{H_t}\right)+\left[\tau_l\left(1-\varrho\right)+\tau_k\varrho-\overline{v}\right]\left(\widehat{\nu}_t\right)^{\varrho}\left(l_t\right)^{1-\varrho}-\tau_k\delta\widehat{\nu}_t=\frac{H_{t+1}}{H_t}-\left(1-\delta_h\right)$$

$$\Rightarrow\tau_c\left(\frac{c_t}{H_t}\right)+\left[\tau_l\left(1-\varrho\right)+\tau_k\varrho-\overline{v}\right]\left(\frac{R_t}{\varrho}\right)\widehat{\nu}_t-\tau_k\delta\widehat{\nu}_t=\frac{H_{t+1}}{H_t}-\left(1-\delta_h\right)$$

$$\Rightarrow\tau_c\left(\frac{c_t}{H_t}\right)+\left[\widetilde{\tau}\left(\frac{R_t}{\varrho}\right)-\tau_k\delta\right]\widehat{\nu}_t=\frac{H_{t+1}}{H_t}-\left(1-\delta_h\right),$$

where $\widetilde{\tau}\equiv\tau_l\left(1-\varrho\right)+\tau_k\varrho-\overline{v}$. In a balanced-growth equilibrium, the above expression becomes

$$\tau_c B\left(\frac{\varrho}{R^*}\right)^{\frac{\varrho}{1-\varrho}}+\left[\widetilde{\tau}\left(\frac{R^*}{\varrho}\right)-\tau_k\delta\right]\widehat{\nu}^*=\gamma^*+\delta_h,$$

which is equation (52).

Finally, consider the equilibrium dynamics of K_t and $\widehat{K}_t$. Dividing both sides of (48) by H_t gives

$$\widehat{\nu}_t=\frac{K_t}{H_t}+\alpha\left[\overline{z}\left(\frac{x_t}{H_t}\right)-\eta s_t^{\sigma}\nu_t\right]=\upsilon_t+\alpha\left[\overline{z}\left(s_t\nu_t\right)^{\epsilon}-\eta s_t^{\sigma}\nu_t\right].$$

As shown in (66), in a balanced-growth equilibrium, we have $\left(s^*\right)^{\sigma}\nu^*=\overline{z}\Omega\left(\mu\right)\left(s^*\nu^*\right)^{\epsilon}$. Hence, we can obtain

$$\widehat{\nu}^*=\nu^*+\left[1-\eta\Omega\left(\mu\right)\right]\alpha\overline{z}\left(s^*\nu^*\right)^{\epsilon},$$

which is equation (53) in the text. Dividing both sides of (49) by H_t gives

$$\frac{c_t}{H_t}+\frac{K_{t+1}}{H_{t+1}}\frac{H_{t+1}}{H_t}-\left(1-\delta\right)\widehat{\nu}_t+\frac{H_{t+1}}{H_t}-\left(1-\delta_h\right)=\left(1-\overline{v}\right)\left(\widehat{\nu}_t\right)^{\varrho}\left(l_t\right)^{1-\varrho}.$$

Equation (54) can be obtained after imposing the conditions for a balanced-growth equilibrium.

3 Chapter 3 Progressive Taxation, Inequality and Growth

3.1 Introduction

In this chapter we examine theoretically the long-run macroeconomic effects of inequality, and the role of progressive taxation in determining these effects. The same topics have been previously studied by Li and Sarte (2004) and Carroll and Young (2011) among others. Most of the existing studies, however, focus on specific forms of the progressive tax function. The present study departs from this literature by considering a general tax function and examine how the general properties of this function will affect the relationship between inequality and economic development. We show that in a variety of model environments the concavity of the marginal tax function (i.e., the third derivative of the progressive tax function) plays a crucial role in determining the relationship between inequality and economic growth.

In Section 1 of this chapter, we present our baseline analytical framework, which is a dynamic general equilibrium model in which consumers differ in terms of their time preferences and labor productivity. Both labor income and interest income are subject to a progressive income tax. In the baseline model, there is no long-term economic growth. The main purpose of this section is to examine the effects of consumer heterogeneity on the steady-state value of per-capita capital. To this end, we compare the steady state in the heterogeneous-agent (HA) economy to that of an identical-agent (IA) economy. Inequality is said to be beneficial (or harmful) to long-run capital accumulation if the HA steady state has more (or less) physical capital than its IA counterpart. Our main result shows that inequality is beneficial (or harmful) to long-run capital accumulation if and only if the marginal tax function is concave (or convex).

In Section 2, we generalize this result in two ways: First, we show that the main result holds in an environment in which long-term economic growth exists and is driven by an exogenously growing productivity factor. Second, we extend our analysis to a class of endogenous growth model and show that inequality is beneficial (or harmful) to long-term economic growth in and only if the marginal tax function is concave (or convex).

3.2 The Baseline Model

3.2.1 Consumers

Time is discrete and is denoted by $t \in \{0, 1, 2, ...\}$. The economy under study is inhabited by a continuum of infinitely-lived consumers which differ in terms of their innate characteristics. Specifically, there are $S > 1$ different types of consumers. Each type $i \in \{1, 2, ..., S\}$ is identified by a pair of fixed predetermined characteristics (β_i, ε_i), where $\beta_i \in (0, 1)$ is the subjective discount factor and $\varepsilon_i > 0$ is labor productivity.[23] Consumers within the same group are identical in all aspects. The share of type-i consumers in the population is given by $\lambda_i \in (0, 1)$. The size of total population is constant over time and is normalized to one so that $\sum_{i=1}^{S} \lambda_i = 1$. For future reference, we also define $\overline{\varepsilon} \equiv \sum_{i=1}^{S} \lambda_i \varepsilon_i$ as the average labor productivity and $\rho_i \equiv 1/\beta_i - 1$ as the rate of time preference for a type-i consumer.

There is a single commodity in this economy which can be used for consumption and investment. All consumers have preferences over consumption sequences which can be represented by

$$\sum_{t=0}^{\infty} \beta_i^t u(c_{i,t}), \tag{68}$$

where $c_{i,t}$ is the consumption of a type-i consumer at time t. The utility function $u : \mathbb{R}_+ \to \mathbb{R}$ is assumed to be twice continuously differentiable, strictly increasing, strictly concave and

[23] Heterogeneity in labor productivity is not essential for our main results. We include this feature just to make our analysis more general.

satisfies the Inada condition: $\lim_{c \to 0} u'(c) = +\infty$.

In each period, each consumer is endowed with one unit of time which they supply inelastically to work. The labor income for a type-i consumer is given by $w_t \varepsilon_i$, where w_t is the wage rate for an effective unit of labor at time t. The consumers can save by holding a single risk-free asset. Let $a_{i,t}$ be the amount of assets owned by the consumer at the beginning of time t. The interest income generated by these assets is $r_t a_{i,t}$, where r_t is the rate of return. The sum of these two types of income, denoted by $y_{i,t} \equiv w_t \varepsilon_i + r_t a_{i,t}$, is subject to a progressive income tax. Specifically, the amount of tax that the consumer has to pay is determined by a function $\tau : \mathbb{R}_+ \to \mathbb{R}_+$, which is thrice continuously differentiable, strictly increasing, strictly convex and satisfies $\tau(0) = 0$. The consumer also receives a lump-sum transfer $\theta_{i,t}$ from the government which is not subject to tax. Thus, the net taxes that the consumer has to pay is given by $\widetilde{\tau}_t(y_{i,t}) \equiv \tau(y_{i,t}) + \theta_{i,t}$, which can be either positive or negative. The consumer's budget constraint at time t is then given by

$$c_{i,t} + a_{i,t+1} - a_{i,t} = y_{i,t} - \tau(y_{i,t}) + \theta_{i,t}. \tag{69}$$

Taking prices and the income tax schedule as given, the consumers' problem is to choose a sequence of consumption and asset holdings so as to maximize his lifetime utility in (68), subject to the sequential budget constraint in (69). The Euler equation for this problem is given by

$$u'(c_{i,t}) = \beta_i u'(c_{i,t+1}) \{1 + r_{t+1} [1 - \tau'(y_{i,t+1})]\}. \tag{70}$$

3.2.2 Production

On the supply side of the economy, there is a large number of identical firms. In each period, each firm hires labor and rents physical capital from the competitive factor markets, and produces output using a neoclassical production technology

$$Y_t = F(K_t, N_t),$$

where Y_t denotes output at time t, K_t and N_t denote capital input and labor input, respectively. The production function $F : \mathbb{R}_+^2 \to \mathbb{R}_+$ is twice continuously differentiable, strictly increasing and strictly concave in (K_t, N_t), exhibits constant returns to scale in the two inputs and satisfies the Inada conditions. Let R_t be the rental price of physical capital at time t. Then the representative firm solves the following problem

$$\max_{K_t, N_t} \{F(K_t, N_t) - w_t N_t - R_t K_t\},$$

and the first-order conditions are

$$R_t = F_K(K_t, N_t), \qquad \text{and} \qquad w_t = F_N(K_t, N_t).$$

3.2.3 Government

The tax revenues collected by the government are either spent on unproductive government spending (G_t) or distributed as transfers to the consumers.[24] The government's budget is

[24]Government spending G_t is called "unproductive" because it has no direct effect on consumers' utility and the production of goods.

balanced in every period, so that

$$\sum_{i=1}^{S} \lambda_i \tau(y_{i,t}) = G_t + \sum_{i=1}^{S} \lambda_i \theta_{i,t}, \qquad \text{for all } t. \tag{71}$$

3.2.4 Competitive Equilibrium

To define a competitive equilibrium, we first define $\mathbf{c}_t = (c_{1,t}, c_{2,t}, ..., c_{S,t})$ and $\mathbf{a}_t = (a_{1,t}, a_{2,t}, ..., a_{S,t})$ as the cross-sectional distributions of consumption and asset holdings at time t. The exogenous policy variables in this economy include the progressive tax function $\tau(\cdot)$ and a sequence of unproductive government spending $\{G_t\}_{t=0}^{\infty}$. Given these policy variables, a competitive equilibrium of this economy consists of sequences of distributions $\{\mathbf{c}_t, \mathbf{a}_t\}_{t=0}^{\infty}$, aggregate inputs $\{K_t, N_t\}_{t=0}^{\infty}$, prices $\{w_t, r_t, R_t\}_{t=0}^{\infty}$ and transfers $\{\theta_{i,t}\}_{t=0}^{\infty}$ such that

(i) Given prices and government policies, $\{c_{i,t}, a_{i,t}\}_{t=0}^{\infty}$ solves a type-i consumer's problem.

(ii) Given prices, $\{K_t, N_t\}_{t=0}^{\infty}$ solves the representative firm's problem in every period.

(iii) The government's budget is balanced in every period.

(iv) All markets clear in every period, so that $K_t = \sum_{i=1}^{S} \lambda_i a_{i,t}$, and $N_t = \sum_{i=1}^{S} \lambda_i \varepsilon_i \equiv \bar{\varepsilon}$, for all t.

We focus on the stationary equilibria or steady states of this economy. In this type of equilibria, both the unproductive government spending and lump-sum transfers are time-invariant. Define $k_t = K_t/N_t$. In any steady state, the equilibrium prices are given by $r^* = F_K(k^*, 1) - \delta$ and $w^* = F_N(k^*, 1)$ and the Euler equation becomes

$$1 = \beta_i \{1 + r^* [1 - \tau'(y_i^*)]\} \Rightarrow \tau'(y_i^*) = 1 - \frac{\rho_i}{r^*}, \tag{72}$$

where $\rho_i \equiv 1/\beta_i - 1$. Let $\phi(\cdot)$ be the inverse of the marginal tax function, i.e., $\phi[\tau'(y)] = y$ for all $y \geq 0$. Since $\tau'(\cdot)$ is strictly monotone, its inverse is a well-defined single-valued function. Straightforward differentiation yields $\phi'[\tau'(y)] = [\tau''(y)]^{-1} > 0$ and

$$\phi''[\tau'(y)] = -\frac{\phi'[\tau'(y)]\,\tau'''(y)}{[\tau''(y)]^2} \gtrless 0 \quad \text{if and only if } \tau'''(y) \lessgtr 0.$$

The above expression shows a close connection between the third derivative of the progressive tax function $\tau(\cdot)$ and the concavity of $\phi(\cdot)$. Specifically, a positive (or negative) third derivative of $\tau(\cdot)$ means that $\phi(\cdot)$ is a concave (or convex) function. Equation (72) can be used to obtain

$$y_i^* = w^*\varepsilon_i + r^*a_i^* = \phi\left(1 - \frac{\rho_i}{r^*}\right).$$

Summing the above expression across all types of consumers gives

$$\sum_{i=1}^{S} \lambda_i \phi\left(1 - \frac{\rho_i}{r^*}\right) = \overline{\varepsilon}\left[F(k^*, 1) - \delta k^*\right], \tag{73}$$

where $r^* = F_K(k^*, 1) - \delta$. Equation (73) can be used to solve for a unique value of k^*.[25] Note that this steady-state value is independent of the heterogeneity in labor productivity. Once k^* is known, all other variables including w^*, r^* and per-capita output $F(k^*, 1)$ can be uniquely determined. It follows that all these variables are independent of the heterogeneity in labor productivity.

[25] The uniqueness of k^* is formally established in the proof of Proposition 3.

3.2.5 Main Result

We now examine the effects of consumer heterogeneity on the steady-state value k^*, and the role of progressive taxation in determining these effects. In what follows, we will refer to the economy with consumer heterogeneity as the heterogeneous-agent (HA) economy and the solution of (73) as the HA steady state. Our goal is to compare this to the steady-state value implied by an economy with identical agents (IA). The IA economy is a special case of the above economy with $\rho_i = \overline{\rho} \equiv \sum_{j=1}^{S} \lambda_i \rho_j > 0$, $\varepsilon_i = \overline{\varepsilon}$ and $\theta_{i,t} = \overline{\theta}_t \equiv \sum_{j=1}^{S} \lambda_i \theta_{j,t}$ for all i and for all t. All other aspects of the IA economy are identical to its heterogeneous-agent counterpart. In particular, the two economies share the same production technology, progressive tax function and unproductive government spending.

Let $\overline{k}^*$ be the steady-state value of capital-labor ratio in the IA economy. This value is uniquely determined by

$$\phi \left(1 - \frac{\overline{\rho}}{\overline{r}^*}\right) = \overline{\varepsilon} \left[F\left(\overline{k}^*, 1\right) - \delta \overline{k}^*\right], \tag{74}$$

where $\overline{r}^* = F_K\left(\overline{k}^*, 1\right) - \delta$. Since the HA economy and IA economy are identical except for the presence or absence of consumer heterogeneity, the difference between k^* and $\overline{k}^*$ thus indicates the long-run effects of consumer inequality. Specifically, inequality is said to be harmful (or beneficial) to long-term capital accumulation if $k^* < \overline{k}^*$ (or $k^* > \overline{k}^*$). The main result of this section (Proposition 3) shows that the long-run effects of inequality depend crucially on the third derivative of the progressive tax function. The proof of this and other results can be found in the Appendix.

Proposition 3 *Consumer heterogeneity is beneficial to long-term capital accumulation if and only if the progressive tax function has negative third derivative, i.e.,* $k^* > \overline{k}^*$ *if and only if* $\tau''' (\cdot) < 0$.

Discussions In the existing studies on progressive taxation, two specific forms of tax function are commonly used. The first one is an isoelastic function that has been used in Guo and Lansing (1998), Li and Sarte (2004) and many subsequent studies. This type of function can be represented by

$$\tau(y) = \zeta y^{1+\chi},$$

with $\zeta > 0$ and $\chi > 0$. One distinctive feature of this function is that the ratio between the marginal tax rate $\tau'(y)$ and average tax rate $\tau(y)/y$ is captured by $(1+\chi)$. The ratio is often used as a measure of progressivity of the tax schedule.[26] Under this specification, the marginal tax function is given by

$$\tau'(y) = \zeta(1+\chi)\, y^{\chi},$$

which is strictly concave (or strictly convex) if $\chi < 1$ (or $\chi > 1$). Using tax returns data in the United States, Li and Sarte (2011) estimate that the value of χ in 1985 and 1991 are 0.88 and 0.75, respectively, and they use these values in their computations. Thus, in their quantitative analysis, the marginal tax function is strictly concave, i.e., $\tau'''(\cdot) < 0$.

Another commonly used tax function is the one proposed and estimated by Gouveia and Strauss (1994),

$$\tau(y) = a_0\left[y - \left(y^{-a_1} + a_2\right)^{-\frac{1}{a_1}}\right].$$

This functional form has been used in Sarte (1997), Conesa and Krueger (2006), Erosa and Koreshkova (2007), Carroll and Young (2011) among others. The first, second and third-

[26] Technically, $(1+\chi)$ is the elasticity of the tax function with respect to y, i.e., $y\tau'(y)/\tau(y) = 1+\chi$ for all $y \geq 0$.

order derivatives of this tax function are

$$\tau'(y) = a_0 \left[1 - (1 + a_2 y^{a_1})^{-\left(1+\frac{1}{a_1}\right)}\right],$$

$$\tau''(y) = a_0 a_2 (1 + a_1)(1 + a_2 y^{a_1})^{-\left(2+\frac{1}{a_1}\right)} y^{a_1 - 1},$$

$$\tau'''(y) = \frac{\tau''(y)}{y}\left[a_1 - 1 - (2a_1 + 1)\left(\frac{a_2 y^{a_1}}{1 + a_2 y^{a_1}}\right)\right]. \tag{75}$$

In all existing applications, the parameters a_0, a_1 and a_2 are taken to be strictly positive so as to ensure $\tau''(y) > 0$. Gouveia and Strauss (1994) estimate that the value of in the U.S. system a_1 is about 0.768. Similar values are also used in Sarte (1997) and Conesa and Krueger (2006). From (75), it is obvious that $0 < a_1 \leq 1$ implies $\tau'''(\cdot) < 0$. In their quantitative analysis, Carroll and Young (2011) have also considered counterfactual experiments in which $a_1 > 1$. In this case, there exists a unique threshold value of income below which $\tau'''(y) > 0$ and above which $\tau'''(y) < 0$.

3.3 Extensions

3.3.1 Exogenous Growth

In this section, we show that the results in Proposition 3 can be easily extended to an economy with exogenous productivity growth. To achieve this, we make three changes to the economy described above. First, the production technology is now given by

$$Y_t = F(K_t, X_t N_t),$$

where X_t is a labor-augmenting technological factor. This factor is assumed to grow by a constant factor $\gamma > 1$ in every period, so that $X_t = \gamma^t$ for all t. The production function $F(\cdot)$ is assumed to have the same properties as before. Second, in order to be consistent with balanced growth, the period utility function is now assumed to take the CRRA form, i.e.,

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma},$$

where $\sigma > 0$ is the inverse of the intertemporal elasticity of substitution. Finally, we need to ensure that the marginal tax rate is constant along the balanced growth path. To achieve this, we assume that the progressive tax function is now changing over time and is denoted by $T_t(y_{i,t})$, and the marginal tax function exhibits the following property:

$$T_t'(y_{i,t}) = \tau'\left(\frac{y_{i,t}}{\gamma^t}\right), \qquad \text{for all } t \geq 0.$$

As before, the function $\tau'(\cdot) : \mathbb{R}_+ \to [0,1]$ is twice continuously differentiable and strictly increasing. Define the transformed variables: $k_t \equiv K_t/(X_t N_t)$, and $\widehat{y}_{i,t} \equiv y_{i,t}/\gamma^t$. The Euler equation for consumption is now given by

$$\left(\frac{c_{i,t+1}}{c_{i,t}}\right)^{\sigma} = \beta_i \left\{1 + r_{t+1}\left[1 - \tau'(\widehat{y}_{i,t+1})\right]\right\},$$

where $r_{t+1} = F_K(k_{t+1}, 1) - \delta$. In any balanced-growth equilibrium, we have $k_t = k^*$, $c_{i,t+1} = \gamma c_{i,t}$ and $\widehat{y}_{i,t} = \widehat{y}_i^*$ for all t. Thus, (72) is now modified to become

$$\gamma^{\sigma} = \beta_i \left\{1 + r^*\left[1 - \tau'(\widehat{y}_i^*)\right]\right\} \Rightarrow \tau'(\widehat{y}_i^*) = 1 - \frac{1}{r^*}\left[\gamma^{\sigma}(1+\rho_i) - 1\right].$$

Defining $\phi(\cdot)$ as the inverse of $\tau'(\cdot)$ and summing across all types of consumers give

$$\sum_{i=1}^{S} \lambda_i \phi \left(1 - \frac{1}{r^*}\left[\gamma^\sigma (1+\rho_i) - 1\right]\right) = \overline{\varepsilon}\left[F(k^*, 1) - \delta k^*\right].$$

We will again refer to the solution of this equation as the HA steady state. The IA steady state $\left(\overline{k}^*\right)$ is characterized by

$$\phi \left(1 - \frac{1}{\overline{r}^*}\left[\gamma^\sigma (1+\overline{\rho}) - 1\right]\right) = \overline{\varepsilon}\left[F\left(\overline{k}^*, 1\right) - \delta \overline{k}^*\right],$$

where $\overline{r}^* = F_K\left(\overline{k}^*, 1\right) - \delta$.

The main result of this section is Proposition 4 which extends the result in Proposition 3 to this environment. The main ideas of the two proofs are essentially identical, hence the proof of Proposition 4 is omitted.

Proposition 4 *In the model with exogenous productivity growth, Consumer heterogeneity is beneficial to long-term capital accumulation if and only if the progressive tax function has negative third derivative, i.e.,* $k^* > \overline{k}^*$ *if and only if* $\tau'''(\cdot) < 0$.

3.3.2 Endogenous Growth

We now generalize our main result to an economy with endogenous growth. The model is essentially identical to the one considered in Li and Sarte (2004, Section II). There are two types of commodities in this economy: a consumption good (C_t) and an investment good (I_t). As in Li and Sarte (2004), the consumption good is produced by a Cobb-Douglas production function

$$C_t = BK_{c,t}^{\alpha} N_{c,t}^{1-\alpha}, \qquad \text{with } B > 0 \text{ and } \alpha \in (0,1), \tag{76}$$

where $K_{c,t}$ and $N_{c,t}$ denote capital input and labor input, respectively. Investment good is produced by a linear technology that only uses physical capital as input, i.e.,

$$I_t = AK_{I,t}, \qquad \text{with } A > 0,$$

where $K_{I,t}$ denote the amount of capital input in the investment-good sector. Firms in both sectors rent the inputs from the competitive factor markets. Let R_t be the rental price of physical capital and w_t be the market wage rate. Then the first-order conditions from the firms' problem are given by

$$R_t = A = \alpha B K_{c,t}^{\alpha-1} N_{c,t}^{1-\alpha} \qquad \text{and} \qquad w_t = (1-\alpha) B K_{c,t}^{\alpha} N_{c,t}^{-\alpha}.$$

The consumer's problem is essentially identical to the one in the baseline model. Specifically, a type-i consumer solves the following problem

$$\max_{\{c_{i,t}, a_{i,t+1}\}_{t=0}^{\infty}} \left[\sum_{t=0}^{\infty} \beta_i^t \left(\frac{c_{i,t}^{1-\sigma}}{1-\sigma} \right) \right]$$

subject to the sequential budget constraint

$$q_t c_{i,t} + a_{i,t+1} - a_{i,t} = y_{i,t} - T_t(y_{i,t}) + \theta_{i,t},$$

where q_t is the price of consumption good expressed in units of investment good. Unlike Li and Sarte (2004), we do not impose a specific functional form on $T_t(\cdot)$. However, in order to ensure that the marginal tax function is constant along the balanced growth path, we assume that the marginal tax function satisfies the following property:

$$T_t'(y_{i,t}) = \tau'\left(\frac{y_{i,t}}{Y_t}\right), \qquad \text{for all } t \geq 0,$$

where $Y_t = \sum_{i=1}^{S} \lambda_i y_{i,t}$. The Euler equation for consumption is now given by

$$\frac{q_{t+1}}{q_t}\left(\frac{c_{i,t+1}}{c_{i,t}}\right)^{\sigma} = \beta_i \left\{1 + r_{t+1}\left[1 - \tau'\left(\frac{y_{i,t}}{Y_t}\right)\right]\right\},$$

for all i and for all $t \geq 0$.

In equilibrium, the markets for physical capital and labor are cleared in every period so that

$$K_{c,t} + K_{I,t} = \sum_{i=1}^{S} \lambda_i a_{i,t} \qquad \text{and} \qquad N_{c,t} = \sum_{i=1}^{S} \lambda_i \varepsilon_i.$$

In any balanced-growth equilibria, $K_{c,t}$, $K_{I,t}$ and Y_t grow by the same factor in every period. The common growth factor is endogenously determined and is denoted by γ^*. Since $N_{c,t}$ is a fixed factor, it follows from (76) that the growth factor of C_t is given by $(\gamma^*)^{\alpha}$. Since total consumption expenditures $(q_t C_t)$ must be growing at the same rate as aggregate income, the growth factor of q_t is $(\gamma^*)^{1-\alpha}$. Along any balanced-growth path, the net rate of return from asset holdings is given by $r^* = A - \delta > 0$. Using these, we can express the Euler equation as

$$(\gamma^*)^{1-\alpha(1-\sigma)} = \beta_i \left\{1 + (A - \delta)\left[1 - \tau'(\psi_i^*)\right]\right\}, \qquad \text{for all } i, \tag{77}$$

where $\psi_i^* \equiv y_{i,t}/Y_t$. Define $\phi(\cdot)$ as the inverse of $\tau'(\cdot)$. Then we can rewrite the above equation as

$$\psi_i^* = \phi\left(1 - \frac{1}{A - \delta}\left[\frac{(\gamma^*)^{1-\alpha(1-\sigma)}}{\beta_i} - 1\right]\right).$$

Summing across all types of consumers gives

$$\sum_{i=1}^{S} \lambda_i \psi_i^* = \sum_{i=1}^{S} \lambda_i \phi \left(1 - \frac{1}{A-\delta}\left[\frac{(\gamma^*)^{1-\alpha(1-\sigma)}}{\beta_i} - 1\right]\right) = 1. \tag{78}$$

This provides a single equation that relates the endogenous growth factor γ^* and the consumer heterogeneity $\{\beta_1, ..., \beta_S\}$. In the following analysis, we will refer to γ^* as the HA growth factor. The growth factor in the IA world is denoted by $\overline{\gamma}^*$ and is completely characterized by

$$\phi \left(1 - \frac{1}{A-\delta}\left[\frac{(\overline{\gamma}^*)^{1-\alpha(1-\sigma)}}{\overline{\beta}} - 1\right]\right) = 1.$$

In order to ensure equation (78) has a unique solution, we need to impose some additional conditions. To start, we assume $\widetilde{\sigma} \equiv 1 - \alpha(1-\sigma) > 0$, which is satisfied when $\sigma \geq 1$. Next, we assume that

$$(1 + A - \delta)\beta_{\min} > \beta_{\max}, \tag{79}$$

where $\beta_{\min}$ and $\beta_{\max}$ are the minimum and maximum values of $\{\beta_1, ..., \beta_S\}$, respectively. Then define $\overline{\gamma}$ and $\underline{\gamma}$ according to

$$\overline{\gamma} \equiv [(1 + A - \delta)\beta_{\min}]^{\frac{1}{\widetilde{\sigma}}} \qquad \text{and} \qquad \underline{\gamma} \equiv (\beta_{\max})^{\frac{1}{\widetilde{\sigma}}}.$$

Since $\widetilde{\sigma} > 0$, the condition in (79) ensures that $(\underline{\gamma}, \overline{\gamma})$ is a nonempty interval. As we will show in the proof of Proposition 5, any solution of (78) must be contained in this interval. Thus, condition (79) is essential for the existence of balanced-growth equilibria. Finally, define the function $\Phi : \left(\underline{\gamma}, \overline{\gamma}\right) \rightarrow \mathbb{R}_+$ according to

$$\Phi\left(\gamma\right) \equiv \sum_{i=1}^{S} \lambda_i \phi \left[1 - \frac{1}{A - \delta}\left(\frac{\gamma^{\widetilde{\sigma}}}{\beta_i} - 1\right)\right].$$

Proposition 5 states that consumer heterogeneity is beneficial to long-term economic growth if and only if $\tau'''\left(\cdot\right) < 0$.

Proposition 5 *Suppose* $\widetilde{\sigma} \equiv 1 - \alpha\left(1 - \sigma\right) > 0$ *and* $\left(1 + A - \delta\right)\beta_{\min} > \beta_{\max}$. *In addition, suppose* $\Phi\left(\underline{\gamma}\right) > 1 > \Phi\left(\overline{\gamma}\right)$. *Then equation (78) has a unique solution* γ^*. *Furthermore, consumer heterogeneity is beneficial to long-term economic growth if and only if the progressive tax function has negative third derivative, i.e.,* $\gamma^* > \overline{\gamma}^*$ *if and only if* $\tau'''\left(\cdot\right) < 0$.

3.4 Appendix

3.4.1 Proof of Proposition 1

Define the reduced-form production function $f(k) \equiv F(k,1)$ for all $k \geq 0$. Define the function $\Phi(k) \equiv \overline{\varepsilon}\left[f(k) - \delta k\right],$ which is the left side of (73). Since $f(\cdot)$ is strictly increasing and strictly concave, there exists a unique value $k_{GR} > 0$ such that $\Phi'(k) \gtreqless 0$ if and only if $k \lesseqgtr k_{GR}$. Next, consider (72) and (73). Since $\tau'(y) \geq 0$ for all $y \geq 0$, equation (72) essentially imposes a restriction on the steady-state value r^*, which is $r^* \geq \rho_{\max} \equiv \max\left\{\rho_{1,}\rho_2, ..., \rho_S\right\}.$ By the strict concavity of $f(\cdot)$ and the Inada conditions $\lim_{k\to 0} f'(k) = \infty$ and $\lim_{k\to\infty} f'(k) = 0$, there exists a unique value $k_{\max} \in (0, k_{GR})$ such that

$$f'(k_{\max}) = \delta + \rho_{\max}.$$

Note that $r^* \geq \rho_{\max}$ if and only if $k^* \leq k_{\max}$. Thus, any solution of (73) must be contained in the range $(0, k_{\max})$. Define the function $\Gamma : (0, k_{\max}) \to \mathbb{R}_+$ according to

$$\Gamma(k) \equiv \sum_{i=1}^{S} \lambda_i \phi \left[1 - \frac{\rho_i}{f'(k) - \delta}\right].$$

Straightforward differentiation shows that $\Gamma(\cdot)$ is strictly decreasing over $(0, k_{\max})$. In addition, as k approaches zero, $\Gamma(k)$ tends to $\phi(1) > 0 = \Phi(0)$. As k approaches $k_{\max}$, $\Gamma(k)$ becomes $\sum_{i=1}^{S} \lambda_i \phi(1 - \rho_i/\rho_{\max}) > 0$. A solution of (73) exists if and only if

$$\Phi(k_{\max}) > \sum_{i=1}^{S} \lambda_i \phi \left(1 - \frac{\rho_i}{\rho_{\max}}\right).$$

In addition, a solution if exists must be unique. A graphical illustration of this is shown in Figure A1.

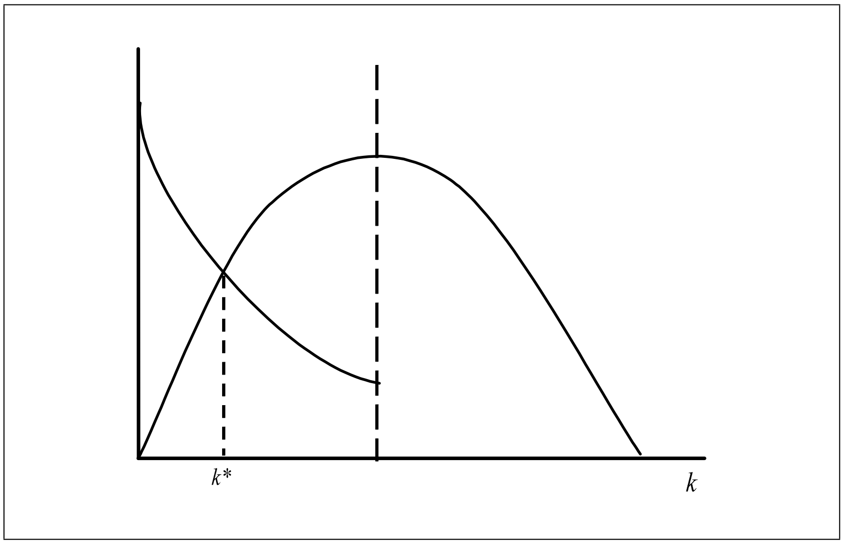

Figure A1: Existence and Uniqueness of Steady State.

We now turn to the comparison between k^* and $\overline{k}^*$. The latter is the solution of (74) and its uniqueness can be established by using a similar argument as above. If $\phi(\cdot)$ is a strictly concave function, which is equivalent to $\tau'''(\cdot) > 0$, then we have

$$\sum_{i=1}^{S} \lambda_i \phi \left(1 - \frac{\rho_i}{\overline{r}^*}\right) < \phi \left(1 - \frac{\overline{\rho}}{\overline{r}^*}\right) = \Phi\left(\overline{k}^*\right).$$

Using Figure A1, it is immediate to see this condition is valid if and only if $\overline{k}^* > k^*$. If $\phi(\cdot)$ is strictly convex, then we have

$$\sum_{i=1}^{S} \lambda_i \phi \left(1 - \frac{\rho_i}{\overline{r}^*}\right) > \phi \left(1 - \frac{\overline{\rho}}{\overline{r}^*}\right) = \Phi\left(\overline{k}^*\right)$$

which hold if and only if $\overline{k}^* < k^*$. This completes the proof of Proposition 3.

3.4.2 Proof of Proposition 5

Since the marginal tax rate $\tau'(\cdot)$ is restricted between zero and one, this essentially imposes an upper bound and a lower bound on the equilibrium growth factor γ^*. To see this, first rewrite (77) as

$$\tau'(\psi_i^*) = 1 - \frac{1}{A-\delta}\left[\frac{(\gamma^*)^{\widetilde{\sigma}}}{\beta_i} - 1\right],$$

where $\widetilde{\sigma} \equiv 1 - \alpha(1-\sigma) > 0$. Then $\tau'(\cdot) > 0$ implies

$$\gamma^* < \left[(1 + A - \delta)\beta_i\right]^{\frac{1}{\widetilde{\sigma}}}, \qquad \text{for all } i.$$

Likewise, $\tau'(\cdot) < 1$ implies $\gamma^* > (\beta_i)^{\frac{1}{\widetilde{\sigma}}}$ for all i. Thus, any solution of (78) must be contained in the range $\left(\underline{\gamma}, \overline{\gamma}\right)$.

Define the function $\Phi : \left(\underline{\gamma}, \overline{\gamma}\right) \to \mathbb{R}_+$ according to

$$\Phi(\gamma) \equiv \sum_{i=1}^{S} \lambda_i \phi\left[1 - \frac{1}{A-\delta}\left(\frac{\gamma^{\widetilde{\sigma}}}{\beta_i} - 1\right)\right].$$

Straightforward differentiation yields

$$\Phi'(\gamma) \equiv -\left(\frac{\widetilde{\sigma}\gamma^{\widetilde{\sigma}-1}}{A-\delta}\right)\sum_{i=1}^{S} \frac{\lambda_i}{\beta_i}\phi'\left[1 - \frac{1}{A-\delta}\left(\frac{\gamma^{\widetilde{\sigma}}}{\beta_i} - 1\right)\right] < 0.$$

Thus, $\Phi(\cdot)$ is strictly decreasing over the range $\left(\underline{\gamma}, \overline{\gamma}\right)$. If $\Phi\left(\underline{\gamma}\right) > 1 > \Phi(\overline{\gamma})$, then equation (78) has a unique solution.

If $\phi(\cdot)$ is a strictly convex function, i.e., $\tau'''(\cdot) < 0$, then we have

$$\begin{aligned}\Phi(\overline{\gamma}^*) &= \sum_{i=1}^{S} \lambda_i \phi\left(1 - \frac{1}{A-\delta}\left[\frac{(\overline{\gamma}^*)^{1-\alpha(1-\sigma)}}{\beta_i} - 1\right]\right) \\ &> \phi\left(1 - \frac{1}{A-\delta}\left[\frac{(\overline{\gamma}^*)^{1-\alpha(1-\sigma)}}{\overline{\beta}} - 1\right]\right) = 1.\end{aligned}$$

The second line is obtained by Jensen's inequality. Since $\Phi(\cdot)$ is strictly decreasing, this means $\overline{\gamma}^* < \gamma^*$. This completes the proof of Proposition 5.

References

[1] Altonji, J. "Intertemporal Substitution in Labor Supply: Evidence from Micro Data." *Journal of Political Economy*, vol. 94, 1986, pp.S176-S215.

[2] Angyridis, C. "Endogenous Growth with Public Capital and Progressive Taxation." *Macroeconomic Dynamics*, 2014, forthcoming.

[3] Aruoba, S. B., G. Rocheteau, and C. Waller. "Bargaining and the Value of Money." *Journal of Monetary Economics*, vol. 54, 2007, pp.2636-2655.

[4] Aruoba, S. B., C. J. Waller, and R. Wright. "Money and Capital." *Journal of Monetary Economics*, vol. 58, 2011, pp.98-116.

[5] Barro, R. J. "Government Spending in a Simple Model of Endogenous Growth." *Journal of Political Economy*, vol. 98 (S5), 1990, pp.103-125.

[6] Barro, R. J. "Inflation and Economic Growth." *Annals of Economics and Finance*, vol. 14, 2013, pp.121-144.

[7] Becker, R. A. "On the Long-Run Steady State in a Simple Dynamic Model of Equilibrium with Heterogeneous Households." *Quarterly Journal of Economics*, vol. 95, 1980, pp.375–82.

[8] Bruno, M., and W. Easterly. "Inflation Crises and Long-run Growth." *Journal of Monetary Economics*, vol. 41, 1998, pp.3-26.

[9] Bullard, J., and J.W. Keating. "The long-run Relationship between Inflation and Output in Postwar Economies." *Journal of Monetary Economics*, vol. 36, 1995, pp.477-496.

[10] Camera, G., and Y. Chien. "Understanding the Distributional Impact of Long-Run Inflation." *Journal of Money, Credit and Banking*, vol. 46, 2014, p. 1137-1170.

[11] Carroll, D., and E. Young. "The Long Run Effects of Changes in Tax Progressivity." *Journal of Economic Dynamics and Control*, vol. 35, 2011, pp. 1451-1473.

[12] Clower, R. "A Reconsideration of the Microfoundations of Monetary Theory." *Economic Inquiry,* vol. 6, 1967, pp.1-8.

[13] Craig, B., and G. Rocheteau. "Inflation and Welfare: A Search Approach." *Journal of Money, Credit and Banking*, vol. 40, 2008, pp.89-119.

[14] Greenwood, J., Z. Hercowitz, and G. W. Huffman. "Investment, Capacity Utilization, and the Real Business Cycle." *American Economic Review*, vol. 78, 1988, pp.402-417.

[15] Conesa, J., and D. Krueger. "On the Optimal Progressivity of the Income Tax Code." *Journal of Monetary Economics*, vol. 53, 2006, pp.1425-1450.

[16] Cooley, T., and E. Prescott. "Economic Growth and Business Cycles." in *Frontiers of Business Cycle Research*, edited by T. Cooley. Princeton, New Jersey: Princeton University Press, 1994, pp.1-38.

[17] De Gregorio, J. "Inflation, Taxation, and Long-run Growth." *Journal of Monetary Economics*, vol. 31, 1993, pp.271-298.

[18] Díaz-Giménez, J., A. Glover, and J. Ríos-Rull. "Facts on the Distributions of Earnings, Income, and Wealth in the United States: 2007 Update." *Federal Reserve Bank of Minneapolis Quarterly Review,* vol. 34, 2011, pp.2-31.

[19] Dotsey, M., and P. Sarte. "Inflation Uncertainty and Growth in a Cash-in-advance Economy." *Journal of Monetary Economics*, vol. 45, 2000, pp.631-655.

[20] Erosa, A., and T. Koreshkova. "Progressive Taxation in a Dynastic Model of Human Capital." *Journal of Monetary Economics*, vol. 54, 2007, pp.667-685.

[21] Erosa, A., and G. Ventura. "On Inflation as a Regressive Consumption Tax." *Journal of Monetary Economics*, vol. 49, 2002, pp.761-795.

[22] Frederick, S., G. Loewenstein, and T. O'Donoghue. "Time Discounting and Time Preference: A Critical Review." *Journal of Economic Literature*, vol. XL, 2002, pp.351-401.

[23] Glomm, G., and B. Ravikumar. "Productive Government Expenditures and Long-run Growth." *Journal of Economic Dynamics & Control*, vol. 21, 1997, pp.183-204.

[24] Guo, J.T., and K. Lansing. "Indeterminacy and Stabilization Policy." *Journal of Economic Theory*, vol. 82, 1998, pp.481-490.

[25] Heer, B., and B. Süssmuth. "Effects of Inflation on Wealth Distribution: Do Stock Market Participation Fees and Capital Income Taxation Matter?" *Journal of Economic Dynamics & Control*, vol. 31, 2007, pp.277-303.

[26] Hendricks, L. "How Important is Discount Rate Heterogeneity for Wealth Inequality?" *Journal of Economic Dynamics & Control*, vol. 31, 2007, pp.3042–68.

[27] Huggett, M., G. Ventura, and A. Yaron. "Sources of Lifetime Inequality." *American Economic Review,* vol. 101, 2011, pp.2923-2954.

[28] İmrohoroğlu, A. "The Welfare Cost of Inflation under Imperfect Insurance." *Journal of Economic Dynamics and Control*, vol. 16, 1992, pp.79-91.

[29] Jaimovich, N., and S. Rebelo. “News about the Future Drive the Business Cycle?” *American Economic Review*, vol. 99, 2009, pp.1097-1118.

[30] Kalai, E. “Proportional Solutions to Bargaining Situations: Interpersonal Utility Comparisons.” *Econometrica*, vol. 45, 1977, pp.1623-1630.

[31] Keane, M. P., and K. I. Wolpin. “The Career Decisions of Young Men.” *Journal of Political Economy,* vol. 105, 1997, pp.473-522.

[32] Kiyotaki, N., and R. Wright. “On Money as a Medium of Exchange.” *Journal of Political Economy*, vol. 97, 1989, pp.927–954.

[33] Kiyotaki, N., and R. Wright. “A Search-Theoretic Approach to Monetary Economics.” *American Economic Review,* vol. 83, 1993, pp.63–77.

[34] Kocherlakota, N. “Money is Memory.” *Journal of Economic Theory,* vol. 81, 1998, pp.232–251.

[35] Kormendi, R., and P. Meguire. "Macroeconomic Determinants of Growth: Cross-country evidence." *Journal of Monetary Economics*, vol. 16, 1985, pp.141-163.

[36] Krusell, P., and A. Smith, Jr. "Income and Wealth Heterogeneity in the Macroeconomy." *Journal of Political Economics*, vol. 106, 1998, pp.867-896.

[37] Lagos, R., and R. Wright. “A Unified Framework for Monetary Theory and Policy Analysis.” *Journal of Political Economy*, vol. 113, 2005, pp.463-484.

[38] Li, W., and P. Sarte. "Progressive Taxation and Long-Run Growth." *American Economic Review*, vol. 94, 2004, pp.1705-1716.

[39] MaCurdy, T.E. "An Empirical Model of Labor Supply in a Life Cycle Setting." *Journal of Political Economics*, vol. 89, 1981, pp.1059-1085.

[40] Mallik, G., and A. Chowdhury. "Inflation and Economic Growth: Evidence from Four South Asian Countries." *Asia-Pacific Development Journal* vol. 8, 2001, pp.123-135.

[41] Sarte, P. "Progressive Taxation and Income Inequality in Dynamic Competitive Equilibrium." *Journal of Public Economics*, vol. 66, 1997, pp.145-171.

[42] Shi S. "Money and Prices: A Model of Search and Bargaining." *Journal of Economic Theory,* vol. 67, 1995, pp.467-496.

[43] Shi, S. "Money, Capital, and Redistributive Effects of Monetary Policies." *Journal of Economic Dynamics and Control*, vol. 23, 1999, pp.565-590.

[44] Sidrauski, M. "Inflation and Economic Growth." *Journal of Political Economy*, vol. 75, 1967, pp.796-810.

[45] Stockman, A. "Anticipated Inflation and the Capital Stock in a Cash-in-advance Economy." *Journal of Monetary Economics*, vol. 8, 1981, pp.387-393.

[46] Suen, R. "Time Preference and the Distributions of Wealth and Income." *Economic Inquiry*, vol. 52, 2014, pp.364-381.

[47] Suen, R., and C.K. Yip. "Superneutrality, Indeterminacy and Endogenous Growth." *Journal of Macroeconomics*, vol. 27, 2005, pp.579-595.

[48] Tobin, J. "Money and Economic Growth." *Econometrica*, 1965, volume 33, pp.671-684.

[49] Waller, C. "Random Matching and Money in the Neoclassical Growth Model: Some Analytical Results." *Macroeconomic Dynamics*, vol. 15, 2011, pp.293-312.

[50] Wang, P., and C.K. Yip. "Alternative Approaches to Money and Growth." *Journal of Money, Credit, and Banking,* volume 24, 1992, pp.553-562.

[51] Wen, Y. “Capacity Utilization under Increasing Returns to Scale.” *Journal of Economic Theory*, vol. 81, 1998, pp.7-36.

[52] Williamson, S. D., and R. Wright. “Monetarist Economics: Methods.” *Federal Reserve Bank of St. Louis, Review* 92, 2010a, pp.265–302.

[53] Williamson, S. D., and R. Wright. “New Monetarist Economics: Models.” *Handbook of Monetary Economics*, vol. 3, B. Friedman and M. Woodford (ed.), 2010b, pp.25–96.