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Traveling Wave Solutions To The Allen-Cahn Equations With Fractional Laplacians

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ABSTRACT

For any fixed $s \in (0,1)$, we consider the following problem:

$$\begin{cases} (-\Delta)^s u(x) - \mu u'(x) = f(u(x)), \quad \forall x \in \mathbb{R}, \\ |u(x)| \leq 1, \quad \forall x \in \mathbb{R}, \\ \lim\limits_{x \to \pm\infty} u(x) = \pm 1. \end{cases}$$

We proved that for any bistable nonlinearity f, then there exist a unique speed μ and unique function u up to translation such that (μ, u) is the solution to the previous problem. We use a continuation argument to show the existence of solution, in which a key ingredient is the estimation of the speed μ in terms of the potential function f. In the meantime, we prove some qualitative properties of the solution u: monotonicity, polynomial decays at infinity, Hamiltonian identity and Modica type estimate, and nondegeneracy. Moreover, we proved that for any balanced bistable nonlinearity f and any nonlinearity $g \in C^2(\mathbb{R})$, then for small $\epsilon > 0$, the traveling speed μ_ϵ corresponding to $f + \epsilon g$ linearly depends on the parameter ϵ.

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Contents

Chapter 1

Introduction

1.1 Traveling Wave with Standard Laplacian

Front propagation is a natural phenomenon which has appeared in phase transition, chemical reaction, combustion, biological spreading, etc. The mechanism of front propagation is often the competing effects of diffusion and reaction. Traveling wave solutions are typical profiles of physical states near the propagating front, and are therefore of great importance in the study of reaction diffusion processes. There has been a tremendous amount of literature on traveling wave solutions in mathematics as well as in various branches of applied sciences (see [1, 3, 7, 8, 6, 12, 28, 29, 35, 62] and references therein). Traveling wave solutions are essential building blocks in various phase field models and play an important role in pattern formation and phase separation (see [4, 22, 27] etc. for the classical model and [44, 55, 56] for nonlocal models with fractional Laplacians). Nonlocal phase transition models and related traveling wave solutions have been studied in [5, 23, 63] and references therein, where the kernels of convolution in the nonlocal operators are bounded, and in [31, 32, 47] where the kernels are periodic.

In the study of front propagation, traditionally the diffusion process is quite standard and normal, in the sense that the concerned particles or objects are engaged in a Brownian motion with a uniformly changed random variable. The resulting diffusion effect on the physical state, mathematically is represented by the Laplacian of this function. Therefore, the difference of various reaction diffusion systems relies on the nonlinear reaction effect which varies in combustion, chemical reaction, phase transition, biological pattern formation, etc. In general, a typical reaction diffusion system is in the form of

$$u_t(t,x) - \Delta_x u(t,x) = f(u(t,x)), \quad \forall t > 0,\ x \in \mathbb{R}^n, \tag{1.1.1}$$

where f is a nonlinear function. If the front of a solution u in large time propagates at a constant speed, the solution is typically close to a profile depending on the distance away from the traveling front. We, therefore, study traveling wave solutions of one spatial variable, although more complicated traveling waves solutions do exist (see, e.g., [7, 8, 37, 39, 40, 41, 42, 43, 52, 53, 59, 60, 65, 64] and references therein). We say that $u \in C^2(\mathbb{R}^2)$, is a traveling wave solution to (1.1.1) if u has the special form $u(t,x) = g(x - \mu t)$ for all $(x,t) \in \mathbb{R}^2$. The constant μ is called the speed of the traveling wave u, and the function g is called the profile of the traveling wave. We are interested in traveling front where g connects the two states, say -1 and 1. The sliding method implies that such g strictly increases. The study of traveling wave solution is reduced to the study of solution to the following system:

$$\begin{cases} -g''(x) - \mu g'(x) = f(g(x)), \quad \forall x \in \mathbb{R}, \\ g'(x) > 0, \quad \forall x \in \mathbb{R}, \\ \lim_{x \to \pm\infty} g(x) = \pm 1. \end{cases} \tag{1.1.2}$$

For the nonlinear function f, there are three interesting cases:

- Fisher-KPP or Monostable Model:

$$\begin{cases} f(t) > 0 = f(\pm 1), \quad \forall t \in (-1, 1), \\ f'(-1) > 0, \ f'(1) < 0. \end{cases}$$

- Combustion Model: There exists some $t_0 \in (-1, 1)$ such that

$$\begin{cases} f(t) = 0, \quad \forall t \in [-1, t_0], \\ f(t) > 0 = f(1), \quad \forall t \in (t_0, 1), \\ f'(1) < 0. \end{cases}$$

- Bistable or the Allen-Cahn Model: There exists some $t_0 \in (-1, 1)$ such that

$$\begin{cases} f(t) < 0 = f(\pm 1) = f(t_0), \quad \forall t \in (-1, t_0), \\ f(t) > 0, \quad \forall t \in (t_0, 1), \\ f'(\pm 1) < 0. \end{cases} \tag{1.1.3}$$

For the combustion and Fisher-KPP models, 0 and 1 are usually used as the limits with only 1 being stable (monostable model). The concerned states in the Allen-Cahn model are often represented by -1 and 1 with both states being stable (bistable model). In the Allen-Cahn model, there is also another nodal point t_0 of f in $(-1,1)$; this nodal point may represents an unstable state which is not the concerned state, since otherwise the equation may be regarded as a Fisher-KPP equation by restricting u in $(t_0,1)$.

Using phase plane analysis, one can show that the traveling wave solutions (g,μ) always exist for the above three types of non-linearities. More precisely, for the combustion and bistable models, there exists a unique pair (g,μ) as the traveling wave solution to (1.1.2). For the Fisher-KPP model, there exists a maximal speed μ_0 such that for any speed $\mu \leq \mu_0$, there exists a traveling wave solution (g,μ) to (1.1.2). Moreover, for the bistable non-linearity, the traveling wave solution to (1.1.2) decays exponentially at infinity, i.e., $g'(x) \sim e^{-\nu|x|}$ as $|x| \to \infty$, as a consequence, we get $1 - g(x) \sim e^{-\nu|x|}$ as $x \to \infty$ and $-1 + g(x) \sim e^{-\nu|x|}$ as $x \to -\infty$ for some positive constant $\nu > 0$. In addition to the method of phase analysis, one can also use the sub-super solution method and variational method to prove the same results (see [61, 66] and references therein).

Recently, there have been a fast increasing number of studies on front propagation of reaction diffusion systems with an anomalous diffusion such as super diffusion, which plays important roles in various physical, chemical, biological and geological processes. (See, e.g., [61] for a brief summary and references therein.) Mathematically, such a super diffusion is related to Lévy process and may be modeled by a fractional Laplace operator $(-\Delta)^s u$ with $0 < s < 1$, the definition of fractional Laplacians will be given in Section 1.2.

1.2 Traveling Waves with Fractional Laplacians

The fractional Laplacian is often defined by Fourier transformation, for any $0 < s < 1$ and $u \in C_c^\infty(\mathbb{R}^n)$, the fractional Laplacian $(-\Delta)^s u$ is defined as the inverse Fourier transform of $(2\pi|x|)^{2s}\hat{u}(x)$, i.e., $\widehat{(-\Delta)^s u}(x) = (2\pi|x|)^{2s}\hat{u}(x)$. It is well known (see [48]) that equivalently we have

$$(-\Delta)^s u(x) = C_{n,s} \text{ P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}}\, dz, \quad \forall x \in \mathbb{R}^n, \tag{1.2.1}$$

where $C_{n,s} = \frac{s 2^{2s} \Gamma\left(\frac{n+2s}{2}\right)}{\pi^{\frac{n}{2}}\Gamma(1-s)}$. The integral definition (1.2.1) of fractional Laplacian can be used for more general functions, in particular, for $u \in C^2(\mathbb{R}^n)$.

The fractional Laplacian can also be defined as a Dirichlet-to-Neumann map. First, the n-dimensional fractional Poisson kernel $P_{n,s}$ is defined as

$$P_{n,s}(x,y) = \frac{\Gamma\left(\frac{n+2s}{2}\right)}{\pi^{\frac{n}{2}}\Gamma(s)} \cdot \frac{y^{2s}}{\left[|x|^2 + y^2\right]^{\frac{n+2s}{2}}}, \quad \forall (x,y) \in \mathbb{R}_+^{n+1}.$$

For any $u \in C^2(\mathbb{R}^n) \bigcap L^\infty(\mathbb{R}^n)$, let $\overline{u}(x,y) = P_{n,s}(\cdot, y) * u(x)$ for all $(x,y) \in \mathbb{R}_+^{n+1}$ be the s-harmonic extension of u. A direct computation shows that $\overline{u}$ satisfies

$$\begin{cases} \text{div } \left[y^{1-2s}\nabla \overline{u}(x,y)\right] = 0, \quad \forall (x,y) \in \mathbb{R}_+^{n+1}, \\ \lim\limits_{y \searrow 0} \overline{u}(x,y) = u(y), \quad \forall x \in \mathbb{R}^n, \\ \lim\limits_{y \searrow 0} -y^{1-2s}\overline{u}_y(x,y) = d_s(-\Delta)^s u(x), \quad \forall x \in \mathbb{R}^n, \end{cases} \tag{1.2.2}$$

where $d_s = \frac{2^{1-2s}\Gamma(1-s)}{\Gamma(s)}$. This fact is well-known for $s = \frac{1}{2}$ and recently proved for all $s \in (0,1)$ by Caffarelli and Silvestre in [21]. The s-harmonic extension method allows us to use local techniques for the local differential operator L_s to study nonlocal integro-differential operator $(-\Delta)^s$, which will be used in Section 2.3 to get a Hamiltonian identity for traveling wave solution.

There have been a fast growing number of studies on the front propagation of the reaction diffusion system with an anomalous diffusion such as super diffusion, which plays an important role in various physical, chemical, biological and geological processes. Mathematically, such a super diffusion is related to the Lévy process and may be modeled by a fractional Laplacian operator $(-\Delta)^s u$ with $0 < s < 1$, and the reaction diffusion equation with respect to such super diffusion can be given by $u_t + (-\Delta)^s u = f(u)$. We are interested in traveling wave solutions in one dimensional spatial variable. Namely, we consider solutions $u(x,t)$ in the form of $u(x - \mu t)$ for some constant μ, equivalently, we will study the following problem:

$$\begin{cases} (-\Delta)^s u(x) - \mu u'(x) = f(u(x)), \quad \forall x \in \mathbb{R}, \\ u'(x) > 0, \quad \forall x \in \mathbb{R}, \\ \lim\limits_{x \to \pm\infty} u(x) = \pm 1. \end{cases} \tag{1.2.3}$$

Traveling wave solutions for reaction diffusion equations with the fractional Laplacians have been studied in the last few years (see [15, 20, 50, 51, 61]). It is very interesting to see that the study of (1.2.3) really depends on the nonlinearity f and parameter s.

When f is a Fisher-KPP non-linearity, it is known that the front propagation speed could be very fast depending on the initial values (see [9]). The Fisher-KPP equation with a fractional Laplacian displays a very different behavior, due to the super diffusion process involved. It was discovered numerically in [25, 24, 49] that the front propagation can accelerate exponentially in time. This phenomenon is rigorously studied and proved in [16] that for any $s \in (0,1)$, the position of all level sets of solution u to reaction diffusion equation $u_t + (-\Delta)^s u = f(u)$ with $|u| \leq 1$ moves exponentially fast in time t. Since a traveling wave front propagates linearly in t, it is an immediate consequence that there is no traveling wave solution for the Fisher-KPP equation with a fractional Laplacian.

For the combustion model, the results are much more interesting. Via the sub-super solution method, it is shown in [50] that when f is a combustive non-linearity, if $s \in (1/2, 1)$, there exists a unique pair (μ, u) as solution to (1.2.3), and the solution u decays algebraically at $-\infty$. On the other hand, by the s-harmonic extension, it is shown in [38] that if $s \in (0, 1/2]$, there does not exist a traveling wave solution to (1.2.3) for the combustion model. Moreover, they showed the nonexistence of traveling fonts for more general non-linearities including the Fisher-KPP case.

For the bistable model, there are two types of bistable non-linearities: balanced and unbalanced. When the bistable non-linearity is balanced, i.e., the associated double well potential $G(u) = -\int_{-1}^{u} f(t)\, dt$ has two wells with equal depths $G(1) = G(-1) = 0$, a traveling wave solution with one spatial variable for the balanced Allen-Cahn equation is indeed a standing wave, i.e., the speed μ must be zero. Such a solution is sometimes called a layer solution as it describes a transition layer structure near the interface between two physical states. The existence of the standing wave solution to (1.2.3) was proved in [17, 19] as a stationary point of the functional

$$\mathcal{E}_s(u) = \int_{\mathbb{R}} \left[\frac{1}{2} \left| (-\Delta)^{s/2} u \right|^2 + G(u) \right] dx,$$

Indeed, by using the s-harmonic extension and variational method, they proved the following result:

Theorem 1.2.1 (Cabré and Solà-Morales[19], Cabré and Sire [17]). *For any* $0 < s < 1$ *and balanced bistable nonlinearity* $f \in C^2(\mathbb{R})$*, i.e.,* f *satisfies (1.1.3) and* $\int_{-1}^{1} f(t)\ dt = 0$*, then there exists a unique function* $u \in C^2(\mathbb{R})$ *up to translation as the solution to (1.2.3) with* $\mu = 0$*. Moreover,* $u'(x) > 0$ *for all* $x \in \mathbb{R}$ *and there exists some constant* $C > 0$ *which only depends on* s *and* f *such that*

$$\frac{C^{-1}}{|x|^{1+2s}} \leq u'(x) \leq \frac{C}{|x|^{1+2s}}, \quad \forall\ |x| > 1.$$

It is shown that a multidimensional standing wave solution must be one dimensional (without counting the dimension of extension) for certain low dimensions and s (see also [13, 14]). These results are analogues of a well studied phenomenon for the classical Allen-Cahn equation and usually referred to as the De Giorgi conjecture (see [2, 26, 34, 33, 36, 54]).

For unbalanced bistable nonlinearity, it's natural to ask whether or not there is a traveling wave solution to the unbalanced fractional Allen-Cahn equation where $G(-1) \neq G(1)$. In [61], when f is a piecewise linear bistable nonlinearity, exact traveling wave solutions for fractional Allen-Cahn equations are computed for $s \in [\frac{1}{2}, 1)$.

Our main result, Theorem 4.2.1 in Chapter 4, proves the same statement as Theorem 1.2.1 for all Allen-Cahn equations with a fractional Laplacian, i.e., we can remove the restriction that $\int_{-1}^{1} f(t)\ dt = 0$. So the existence and nonexistence problems related to the traveling wave solution with a fractional Laplacian are completely solved. Several traveling waves problems similar to the fractional diffusion reaction models are investigated recently (see [15, 20]).

Chapter 2

Qualitative Properties of Traveling Wave Solutions

In this chapter, we will always assume that $0 < s < 1$, $f \in C^2(\mathbb{R})$ satisfies $f'(\pm 1) < 0$, and $G(t) = -\int_{-1}^{t} f(u)\, du$. Let's consider the following problem:

$$\begin{cases} (-\Delta)^s u(x) - \mu u'(x) = f(u(x)), \quad \forall x \in \mathbb{R}, \\ |u(x)| \leq 1, \quad \forall x \in \mathbb{R}, \\ \lim\limits_{x \to \pm\infty} u(x) = \pm 1. \end{cases} \tag{2.0.1}$$

In the following sections, we will prove some useful qualitative properties of traveling wave solution to the Fractional Allen-Cahn equation (2.0.1): monotonicity of profile u, uniqueness of speed μ and profile u, polynomial decays at infinity of u, Hamiltonian identity and Modica type estimate, and nondegeneracy. These properties will play important role in the proof of the existence of traveling wave solution, see more details in Chapter 4. Many of these properties are similar to those in [17, 19], where the standing wave solutions with $\mu = 0$ are considered.

2.1 Monotonicity of Profile, and Uniqueness of Speed and Profile

In order to study traveling wave solutions, first we state a slight variation of a maximum principle in [18] to include an advection term.

Lemma 2.1.1. *Let $c \in C(\mathbb{R})$, $d \in L^\infty(\mathbb{R})$ and $v \in C^2(\mathbb{R})$ satisfy*

$$\begin{cases} (-\Delta)^s v(x) + c(x)v'(x) + d(x)v(x) \geq 0, & \forall x \in \mathbb{R}, \\ \lim\limits_{|x|\to\infty} v(x) = 0. \end{cases}$$

Assume that there exists some subset $H \subset \mathbb{R}$ (may be empty set) such that

$$v(x) \geq 0,\ \forall x \in H \qquad \text{and} \qquad d(x) \geq 0,\ \forall x \notin H.$$

Then either $v(x) > 0$ in $\mathbb{R}$ or $v(x) \equiv 0$ in $\mathbb{R}$.

Proof. We consider the following two cases:

Case I: $v(x) \geq 0$ in $\mathbb{R}$. If $v(x) \equiv 0$ in $\mathbb{R}$, we are done. If $v(x) > 0$ in $\mathbb{R}$, we are done. If $v(x) \not\equiv 0$ in $\mathbb{R}$ and there exists some $x_0 \in \mathbb{R}$ such that $v(x_0) = 0$, that is, x_0 is a global minimum point of v, we get $v'(x_0) = 0$ and $(-\Delta)^s v(x_0) < 0$, which implies that $(-\Delta)^s v(x_0) + c(x_0)v'(x_0) + d(x_0)v(x_0) = (-\Delta)^s v(x_0) < 0$. We get a contradiction.

Case III: $v(x_1) < 0$ for some $x_1 \in \mathbb{R}$. Since $\lim\limits_{|x|\to\infty} v(x) = 0$, we know that v can not be a constant function in $\mathbb{R}$ and there exists some $x_2 \in \mathbb{R}$ such that

$v(x_2) = \inf_{x\in\mathbb{R}} v(x) < 0$. Hence we get $v'(x_2) = 0$ and $(-\Delta)^s v(x_2) < 0$. Since $v(x_2) < 0$, we know $x_2 \notin H$ and $d(x_2) \geq 0$, which implies that

$$(-\Delta)^s v(x_2) + c(x_2)v'(x_2) + d(x_2)v(x_2) = (-\Delta)^s v(x_2) + d(x_2)v(x_2) < 0,$$

which contradicts with the assumption.

In summary, we know that either $v(x) > 0$ in $\mathbb{R}$ or $v(x) \equiv 0$ in $\mathbb{R}$.

□

According to Lemma 2.1.1, we can use the sliding method which is introduced by Berestycki and Nirenberg in [10] to prove the monotonicity of profile, and the uniqueness of speed and profile.

Proposition 2.1.2 (Monotonicity)**.** *Let $\mu \in \mathbb{R}$ and $u \in C^2(\mathbb{R})$ be a solution to (2.0.1) with μ. Then $-1 < u(x) < 1$ and $u'(x) > 0$ for all $x \in \mathbb{R}$.*

Proof. First, we prove that $-1 < u(x) < 1$ in $\mathbb{R}$. Otherwise, since $|u(x)| \leq 1$ in $\mathbb{R}$, we know that there exists some $x_1 \in \mathbb{R}$ such that $u(x_1) = 1$ or -1. If $u(x_1) = 1 = \sup_{x\in\mathbb{R}} u(x)$, since $\lim_{y\to\pm\infty} u(x) = \pm 1$, we have $u(x) \not\equiv 1$ in $\mathbb{R}$, which implies that $u'(x_1) = 0$ and $(-\Delta)^s u(x_1) > 0$. Hence we have $(-\Delta)^s u(x_1) - \mu u'(x_1) - f(u(x_1) = (-\Delta)^s u(x_1) > 0$, contradiction. If $u(x_1) = -1 = \inf_{x\in\mathbb{R}} u(x)$, since $\lim_{x\to\pm\infty} u(x) = \pm 1$, we have $u(x) \not\equiv 1$ in $\mathbb{R}$, which implies that $u'(x_1) = 0$ and $(-\Delta)^s u(x_1) < 0$. Hence we have $(-\Delta)^s u(x_1) - \mu u'(x_1) - f(u(x_1) = (-\Delta)^s u(x_1) < 0$, contradiction. Hence we know that

$$-1 < u(x) < 1, \quad \forall x \in \mathbb{R}. \tag{2.1.1}$$

Now let's prove the monotonicity of u. For any $t > 0$, we define $u^t(x) = u(t + x)$,

$w^t(x) = u(t+x) - u(x)$ for all $x \in \mathbb{R}$, and $H^t = \{\{x \in \mathbb{R} : \ w^t(x) > 0\}$. It's easy to see that w^t satisfies

$$\begin{cases} (-\Delta)^s w^t(x) - \mu(w^t)'(x) + d^t(x)w^t(x) = 0, \quad \forall x \in \mathbb{R}, \\ |w^t(x)| \leq 2, \quad \forall x \in \mathbb{R}, \\ \lim\limits_{|x|\to\infty} w^t(x) = 0. \end{cases}$$

where

$$d^t(x) = \begin{cases} -\dfrac{f(u^t(x)) - f(u(x))}{u^t(x) - u(x)}, & \text{if } w^t(x) \neq 0, \\ 0, & \text{if } w^t(x) = 0. \end{cases}$$

By the Mean Value Theorem, we know that $|d^t(x)| \leq \|f'\|_{L^\infty([-1,1])}$ for all $t \geq 0$ and all $x \in \mathbb{R}$. Since $f'(\pm 1) < 0$, then there exists some $0 < \tau < 1$ such that

$$f'(t) < 0, \quad \forall \in [-1, -\tau] \bigcup [\tau, 1]. \tag{2.1.2}$$

Claim I: There exists some large $T_0 > 0$ such that for all $t \geq T_0$, we have $w^t(x) > 0$ in $\mathbb{R}$.

Since $\lim\limits_{x\to\pm\infty} u(x) = \pm 1$, there exists some large $Y_0 > 0$ such that $-1 < u(x) < -\tau$ for all $x \leq -Y_0$, and $\tau < u(x) < 1$ for all $x \geq Y_0$. By (2.1.1), we have

$$A_0 := \max_{x\in[-Y_0,Y_0]} u(x) < 1.$$

Since $\lim\limits_{x\to\pm\infty} u(x) = \pm 1$, there exists some large $Y_1 > 0$ such that $u(x) > A_0$ for all $x \geq Y_1$. Now let $T_0 = Y_1 + Y_0 > 0$, then for all $t \geq T_0$ and fix, and all $x \in [-Y_0, Y_0]$, we have $x + t \geq -Y_0 + Y_1 + Y_0 = Y_1$, which implies that $u^t(x) = u(t+x) > A_0 \geq u(x)$, in particular, we get

$$[-Y_0, Y_0] \subset H^t.$$

On the other hand, for all $x \notin H^t$, we have $u^t(x) \leq u(x)$, and $x \notin [-Y_0, Y_0]$. By the definition of Y_0, we know that $-1 < u(x) < -\tau$ if $x < -Y_0$, and $\tau < u(x) < 1$ if $x > Y_0$. If $x < Y_0$, since $u^t(x) \leq u(x)$, then $-1 < u^t(x) \leq u(x) < -\tau$. If $x > Y_0$, we have $t + x > x > y_0$, so $\tau < u^t(x) = u(t + x) \leq u(x) < 1$. In both cases, we know that both $u^t(x)$ and $u(x)$ lie in either $[-1, -\tau]$ or $[\tau, 1]$. Since $u^t(x) \leq u(x)$, by (2.1.2), we know that $d^t(x) \geq 0$ for all $x \notin H^t$.

Since $w^t(x) > 0$ in $[-Y_0, Y_0]$, by Lemma 2.1.1, we know that $w^t(x) > 0$ for all $x \in \mathbb{R}$.

Claim II: If for some fixed $t > 0$, $w^t(x) > 0$ in $\mathbb{R}$, then there exists some small $\epsilon_t \in (0, t)$ such that for all $|h| < \epsilon_t$, we have $w^{t+h}(x) > 0$ in $\mathbb{R}$.

Since $u^t(x), u(x) \to \pm 1$, as $x \to \pm\infty$, and $0 < \tau < 1$, then there exists some $Y_2 > Y_1$ such that for all $|x| \geq Y_2$, we have

$$|u^t(x)| \geq \frac{1+\tau}{2}, \quad \text{and} \quad |u(x)| \geq \frac{1+\tau}{2}.$$

In particular, we have

$$K^t := \left\{ x \in \mathbb{R} : \; |u^t(x)| < \frac{1+\tau}{2}, \text{ or } |u(x)| < \frac{1+\tau}{2} \right\} \subset [-Y_2, Y_2].$$

Since $\lim\limits_{x \to \pm\infty} u(x) = \pm 1$, then there exists $x_0 \in \mathbb{R}$ such that $u(x_0) = 0$, in particular, $x_0 \in K^t$ and $K^t \neq \emptyset$. Since $w^t(x) > 0$ in $\mathbb{R}$, we have

$$A_2 := \inf_{x \in K^t} w^t(x) > 0.$$

Since $u^t \in C^2(\mathbb{R})$, so u^t are Lipschitz continuous on $\mathbb{R}$, in particular, u^t is uniformly continuous on $\mathbb{R}$. Since $0 < \tau < 1$, then there exists some small $t > \epsilon_t > 0$ such that

$$|u^t(y) - u^t(z)| < \min\left\{\frac{A_2}{2}, \frac{1-\tau}{2}\right\}, \quad \text{for all } |y - z| < \epsilon_t.$$

For all $|h| < \epsilon_t$, we have

$$|w^{t+\mu}(x)| = |u^t(x+\mu) - u^t(x)| < \min\left\{\frac{A_2}{2}, \frac{1-\tau}{2}\right\}, \quad \forall x \in \mathbb{R}.$$

For all $x \in K^t$, by the definition of A_2, we get $w^t(x) \geq A_2$, which implies that

$$w^{t+\mu}(x) = w^{t+\mu}(y) + w^t(y) \geq -\frac{A_2}{2} + A_2 = \frac{A_2}{2} > 0,$$

which implies that $K^t \subset H^{t+h}$.

On the other hand, for any $x \notin H^{t+h}$, we have $x \notin K^t$, $|u(x)| \geq \frac{1+\tau}{2}$ and $u^{t+h}(x) \leq u(x)$. If $u(x) \leq -\frac{1+\tau}{2}$, by (2.1.1), we have $-1 < u^{t+h}(x) \leq u(x) \leq -\frac{1+\tau}{2} < -\tau$. If $u(x) \geq \frac{1+\tau}{2}$, since $w^t(x) > 0$, we have $u^t(x) > u(x) > \frac{1+\tau}{2}$, which implies that $u^{t+h}(x) = u^{t+h}(x) - u^t(x) + u^t(x) > -\frac{1-\tau}{2} + \frac{1+\tau}{2} = \tau$. In both cases, we know that both $u^{t+h}(x)$ and $u(x)$ lie in either $[-1, -\tau]$ or $[\tau, 1]$. Since $u^{t+h}(x) \leq u(x)$, by (2.1.2), we know that $d^{t+h}(x) \geq 0$ for all $x \notin H^{t+h}$.

Since $w^{t+h}(x) > 0$ in K^t, by Lemma 2.1.1, we know that $w^t(x) > 0$ for all $x \in \mathbb{R}$.
Claim III: For all $t > 0$, then $w^t(x) > 0$ in $\mathbb{R}$.

Let

$$S = \left\{t > 0 : \; w^t(x) > 0, \; \text{in } \mathbb{R}\right\}.$$

By Claim I, $[T_0, \infty) \subset S$, so $S \neq \emptyset$. By Claim II, S is open. If there exists some sequence $\{t_n\}_{n=1}^{\infty} \subset S$ such that $t_n \to t > 0$, as $n \to \infty$. For all $n \geq 1$, since $w^{t_n}(x) > 0$ in $\mathbb{R}$, by taking $n \to \infty$, we get $w^t(x) \geq 0$ in $\mathbb{R}$. If there exists some $x_0 \in \mathbb{R}$ such that $w^t(x_0) = 0$, by Lemma 2.1.1, we have $w^t(x) \equiv 0$ in $\mathbb{R}$, that is, $u(t+x) \equiv u(x)$ in $\mathbb{R}$, which implies that u is a periodic function in $\mathbb{R}$, which contradicts with $\lim_{x \to \pm\infty} u(x) = \pm 1$. So we must have $w^t(x) > 0$ in $\mathbb{R}$, that is, $t \in S$. Hence, S is closed with respect to $(0, \infty)$.

In summary, S is a nonempty open and closed subset of $(0, \infty)$. Since $(0, \infty)$ is connected, then $S = (0, \infty)$, that is, for all $t > 0$, then $w^t(x) > 0$ in $\mathbb{R}$.

Claim IV: $u'(x) > 0$ in $\mathbb{R}$.

By Claim III, we see that u is a strictly increasing function in $\mathbb{R}$, so $u'(x) \geq 0$ in $\mathbb{R}$. If $u'(x) > 0$ in $\mathbb{R}$, we are done. If there exists some $x_0 \in \mathbb{R}$ such that $u'(x_0) = 0$, let $g(x) = u'(x)$ in $\mathbb{R}$, then $(-\Delta)^s g(x) - \mu g'(x) = f'(u(x))g(x)$ and $g(x) \geq 0$ for all $x \in \mathbb{R}$. Since u is strictly increasing, then $g(x) \not\equiv 0$ in $\mathbb{R}$. Since $g(x_0) = u'(x_0) = \inf_{x \in \mathbb{R}} g(x)$, we have $g'(x_0) = 0$ and $(-\Delta)^s g(x_0) < 0$, which implies that $(-\Delta)^s g(x_0) - \mu g'(x_0) - f'(u(x_0))g(x_0) = (-\Delta)^s g(x_0) < 0$, contradiction. Therefore, we must have $u'(x) > 0$ in $\mathbb{R}$.

□

By using the same argument as Proposition 2.1.2 with small modifications, we can show the uniqueness of speed and profile. Since the proof is very similar, we omit it.

Proposition 2.1.3 (Uniqueness). *For $i = 1, 2$, let $\mu_i \in \mathbb{R}$ and $u_i \in C^2(\mathbb{R})$ be a solution to (2.0.1) with μ_i, respectively. Then $\mu_1 = \mu_2$ and there exists some $a \in \mathbb{R}$ such that $u_1(x) \equiv u_2(x + a)$ in $\mathbb{R}$.*

2.2 Polynomial Decays at Infinity

Let's quote three functions in [17]: for any $t > 0$ and all $x \in \mathbb{R}$, let

$$p_t(x) = \frac{1}{\pi}\int_0^\infty \cos(xr)e^{-tr^{2s}}\ dr,\ v_t(x) = -1 + 2\int_{-\infty}^x p_t(r)\ dr,\ \text{and}\ \varphi_t(x) = v_t'(x).$$

Theorem 3.1 in [17] says that $p_t, v_t \in C^\infty(\mathbb{R})$, and there exists some $f_t \in C^2([-1,1])$ which is an odd function in $[-1,1]$ and satisfies (1.1.3) and $f_t'(\pm 1) = -\frac{1}{t}$, such that v_t is a layer solution in $\mathbb{R}$ with nonlinearity f_t, that is, v_t is a solution to (2.0.1) with $f = f_t$ and $\mu = 0$. Moreover, we have

$$\lim_{|x|\to\infty} |x|^{1+2s}v_t'(x) = \frac{4ts\Gamma(2s)}{\pi}\cdot\sin(\pi s) > 0. \tag{2.2.1}$$

By (2.0.1), the equation which φ_t satisfies has φ_t', let's estimate $\varphi_t' = v_t''$.

Lemma 2.2.1. *There exists some constant $C_0 > 0$ such that*

$$|\varphi_t'(x)| \le \frac{tC_0}{|x|^{2+2s}}, \quad \forall x \neq 0.$$

Proof. First, let's estimate $\varphi_1'(x)$. For all $x \in \mathbb{R}\backslash\{0\}$, we have

$$\begin{aligned}
\varphi_1'(x) &= 2p_t'(x) \\
&= -\frac{2}{\pi}\int_0^\infty r\sin(xr)e^{-r^{2s}}\ dr \\
&= -\frac{2\text{sign } x}{\pi}\int_0^\infty r\sin(|x|r)e^{-r^{2s}}\ dr \\
&= -\frac{2\text{sign } x}{\pi}\int_0^\infty \frac{z}{|x|}\cdot\sin z e^{-\left(\frac{z}{|x|}\right)^{2s}}\cdot\frac{1}{|x|}\ dz, \quad \text{Let } z = |x|r \\
&= -\frac{2\text{sign } x}{\pi x^2}\int_0^\infty z\sin z e^{-\left(\frac{z}{|x|}\right)^{2s}}\ dz.
\end{aligned}$$

So we know that φ_1' is an odd function in $\mathbb{R}\backslash\{0\}$, and for any $x > 0$, using integration by parts, we have

$$\begin{aligned}
& -\int_0^\infty z \sin z e^{-\left(\frac{z}{x}\right)^{2s}}\, dz \\
= \;& z e^{-\left(\frac{z}{x}\right)^{2s}} \cos z\Big|_0^\infty - \int_0^\infty \cos z \left[e^{-\left(\frac{z}{x}\right)^{2s}} - z e^{-\left(\frac{z}{x}\right)^{2s}} 2s\left(\frac{z}{x}\right)^{2s-1}\frac{1}{x}\right] dz \\
= \;& -\int_0^\infty \cos z e^{-\left(\frac{z}{x}\right)^{2s}}\, dz + \frac{2s}{x^{2s}}\int_0^\infty z^{2s}\cos z e^{-\left(\frac{z}{x}\right)^{2s}}\, dz \\
= \;& -e^{-\left(\frac{z}{x}\right)^{2s}} \sin z\Big|_0^\infty - \int_0^\infty \sin z e^{-\left(\frac{z}{x}\right)^{2s}} 2s\left(\frac{z}{x}\right)^{2s-1}\frac{1}{x}\, dz + \frac{2s}{x^{2s}}\left\{ z^{2s} e^{-\left(\frac{z}{x}\right)^{2s}} \sin z\Big|_0^\infty \right. \\
& \left. -\int_0^\infty \sin z \left[2s z^{2s-1} e^{-\left(\frac{z}{x}\right)^{2s}} - z^{2s} e^{-\left(\frac{z}{x}\right)^{2s}} 2s\left(\frac{z}{x}\right)^{2s-1}\frac{1}{x}\right] dz\right\} \\
= \;& -\frac{2s+4s^2}{x^{2s}}\int_0^\infty z^{2s-1}\sin z e^{-\left(\frac{z}{x}\right)^{2s}}\, dz + \frac{4s^2}{x^{4s}}\int_0^\infty z^{4s-1}\sin z e^{-\left(\frac{z}{x}\right)^{2s}}\, dz.
\end{aligned}$$

So we get

$$\begin{aligned}
\varphi_1'(y) \;=\; & -\frac{4s(1+2s)}{\pi}\cdot\frac{1}{x^{2+2s}}\int_0^\infty z^{2s-1}\sin z e^{-\left(\frac{z}{x}\right)^{2s}}\, dz \\
& +\frac{8s^2}{\pi}\cdot\frac{1}{x^{2+4s}}\int_0^\infty z^{4s-1}\sin z e^{-\left(\frac{z}{x}\right)^{2s}}\, dz, \;\forall x > 0.
\end{aligned}$$

By taking $\kappa = 2, 4$ in Lemma 3.4 in [17], respectively, we can get

$$\begin{aligned}
\lim_{y\to\infty}\int_0^\infty z^{2s-1}\sin z e^{-\left(\frac{z}{y}\right)^{2s}}\, dz &= \Gamma(2s)\sin(s\pi), \\
\lim_{y\to\infty}\int_0^\infty z^{4s-1}\sin z e^{-\left(\frac{z}{y}\right)^{2s}}\, dz &= \Gamma(4s)\sin(2s\pi).
\end{aligned}$$

Hence, we have

$$\lim_{x\to\infty} x^{2+2s}\varphi_1'(x) = -\frac{4s(1+2s)}{\pi}\cdot\Gamma(2s)\sin(s\pi).$$

Since φ_1' is an odd function in $\mathbb{R}\backslash\{0\}$, from the above limit, we know that there exists some $C_0 > 0$ such that

$$|\varphi_1'(x)| \leq \frac{C_0}{|x|^{2+2s}}, \quad \forall x \neq 0.$$

By the formula (3.18) in [17], $v_t(x) = v_1\left(t^{-\frac{1}{2s}}x\right)$ in $\mathbb{R}$, then $\varphi_t'(x) = t^{-\frac{1}{2s}}\varphi_1\left(t^{-\frac{1}{2s}}x\right)$ in $\mathbb{R}$, which implies that

$$\begin{aligned} |\varphi_t'(x)| &= t^{-\frac{1}{s}}\left|\varphi_1'\left(t^{-\frac{1}{2s}}x\right)\right| \\ &\leq t^{-\frac{1}{s}} \cdot \frac{C_0}{\left|t^{-\frac{1}{2s}}x\right|^{2+2s}} \\ &= \frac{tC_0}{|x|^{2+2s}}, \quad \forall x \neq 0. \end{aligned}$$

□

Now we can prove the following asymptotic behaviors of traveling wave solutions.

Proposition 2.2.2. *Let $\mu \in \mathbb{R}$ and $u \in C^2(\mathbb{R})$ be a solution to (2.0.1) with μ. Then there exists some constant $C > 0$ which only depends on s, μ and f such that*

$$\frac{C^{-1}}{|y|^{1+2s}} \leq u'(y) \leq \frac{C}{|y|^{1+2s}}, \quad \forall |y| \geq 1.$$

As a consequence, we have $u' \in L^p(\mathbb{R})$ for any $1 \leq p \leq \infty$, and

$$\frac{C^{-1}}{y^{2s}} \leq 1 - u(y) \leq \frac{C}{y^{2s}}, \quad \forall y > 1 \qquad \text{and} \qquad \frac{C^{-1}}{|y|^{2s}} \leq 1 + u(y) \leq \frac{C}{|y|^{2s}}, \ \forall y < -1.$$

Proof. By the same compactness argument as Lemma 4.8 in [18], we know that

$$\lim_{|x|\to\infty} u'(x) = 0.$$

First, let's find the upper bound of u'. For any $\delta > 0$, let $w_{\delta,t}(x) = \delta\varphi_t(x) - u'(x)$ in $\mathbb{R}$, by (2.0.1), we get

$$\begin{aligned} & (-\Delta)^s w_{\delta,t}(x) - \mu w'_{\delta,t}(x) + \frac{3}{t} w_{\delta,t}(x) \\ = \; & \delta\varphi_t(x)\left[\frac{2}{t} + f'_t(v_t(x))\right] - u'(x)\left[\frac{3}{t} + f'(u(x))\right] + \delta\left[\frac{\varphi_t(x)}{t} - \mu\varphi'_t(x)\right]. \end{aligned}$$

Since $f'_t(\pm 1) = -\dfrac{1}{t} < 0$, $f'(\pm 1) < 0$ and $\lim\limits_{y\to\pm\infty} v_t(y) = \lim\limits_{y\to\pm\infty} u(y) = \pm 1$, then there exists some large $T_0 > 0$ and $R_1 > 0$ such that

$$\frac{2}{T_0} + f'_{T_0}(v_{T_0}(x)) > 0 \quad \text{and} \quad \frac{3}{T_0} + f'(u(x)) < 0, \quad \forall |x| \geq R_1.$$

By Proposition 2.1.2, we get

$$\delta\varphi_{T_0}(y)\left[\frac{2}{T_0} + f'_{T_0}(v_{T_0}(x))\right] - u'(x)\left[\frac{3}{T_0} + f'(u(x))\right] > 0, \quad \forall |x| \geq R_1. \tag{2.2.2}$$

By Lemma 2.2.1, we have

$$|\varphi'_{T_0}(x)| \leq \frac{T_0 C_0}{|x|^{2+2s}}, \quad \forall x \neq 0.$$

By (2.2.1), there exists some $C_1 > 0$ and $R_2 > R_1 > 0$ such that

$$\frac{1}{T_0}\varphi_{T_0}(x) \geq \frac{C_1}{|x|^{1+2s}}, \quad \forall |x| \geq R_2.$$

So we get

$$\begin{aligned}\frac{1}{T_0}\varphi_{T_0}(x)-\mu\varphi'_{T_0}(x) &\geq \frac{C_1}{|x|^{1+2s}}-\frac{|\mu|T_0C_0}{|x|^{2+2s}}\\ &= \frac{C_1}{|x|^{2+2s}}\left[|x|-\frac{|\mu|T_0C_0}{C_1}\right], \quad \forall |x|\geq R_2.\end{aligned}$$

Let $R_3=\max\left\{R_2,\frac{|\mu|T_0C_0}{C_1}+1\right\}$, we have

$$\frac{1}{T_0}\varphi_{T_0}(x)-\mu\varphi_{T_0})'(x)>0, \quad \forall |x|\geq R_3.$$

In summary, we know that for all $\delta>0$, we have

$$(-\Delta)^s w_{\delta,T_0}(x)-\mu w'_{\delta,T_0}(y)+\frac{3}{T_0}w_{\delta,T_0}(x)>0, \quad \forall |x|\geq R_3.$$

Since $\varphi_{T_0}(x)>0$ in $\mathbb{R}$, then there exists some large $\delta_0>0$ such that for all $\delta\geq\delta_0$, we have $w_{\delta,T_0}(x)=\delta\varphi_{T_0}(x)-u'(x)\geq 1$ for all $|x|\leq R_3+1$. Hence w_{δ,T_0} satisfies

$$\begin{cases}(-\Delta)^s w_{\delta,T_0}(x)-\mu w'_{\delta,T_0}(x)+\frac{3}{T_0}w_{\delta,T_0}(x)>0, \quad \forall |x|\geq R_3,\\ w_{\delta,T_0}(x)\geq 1>0, \quad \forall |x|\leq R_3+1,\\ \lim\limits_{|x|\to\infty} w_{\delta,T_0}(x)=0.\end{cases}$$

Claim I: $w_{\delta,T_0}(x)\geq 0$ in $\mathbb{R}$.

If Claim I is not true, since $\lim\limits_{|x|\to\infty} w_{\delta,T_0}(x)=0$, then there exists some $x_0\in\mathbb{R}$ such that $w_{\delta,T_0}(x_0)=\inf\limits_{x\in\mathbb{R}} w_{\delta,T_0}(x)<0$, in particular, $w'_{\delta,T_0}(x_0)=0$. Since $w_{\delta,T_0}(x)>0$ for all $|x|\leq R_3+1$, we get $|x_0|>R_3$, which implies that $(-\Delta)^s w_{\delta,T_0}(x_0)<0$. So we obtain

$$(-\Delta)^s w_{\delta,T_0}(x_0) - \mu w'_{\delta,T_0}(x_0) + \frac{3}{T_0} w_{\delta,T_0}(x_0) = (-\Delta)^s w_{\delta,T_0}(x_0) + \frac{3}{T_0} w_{\delta,T_0}(x_0) < 0,$$

we get a contradiction.

By Claim I, we get $u'(x) \le \delta\varphi_{T_0}(x) = \delta v'_{T_0}(x)$ in $\mathbb{R}$. By (2.2.1), we know that there exists some constant $A > 0$ such that

$$u'(x) \le \frac{A}{|x|^{1+2s}}, \quad \forall |x| \ge 1.$$

For lower bound of u', we follow the same idea as the upper bound. For any $\delta > 0$, we may define

$$\tilde{w}_{\delta,t}(x) = -w_{\delta,t}(x), \quad \forall x \in \mathbb{R}.$$

By (2.0.1), we have

$$\begin{aligned} & (-\Delta)^s \tilde{w}_{\delta,t}(x) - \mu \tilde{w}'_{\delta,t}(x) + \frac{1}{4t}\tilde{w}_{\delta,t}(x), \\ = \quad & -\delta\varphi_t(y)\left[\frac{1}{2t} + f'_t(v_t(x))\right] + u'(x)\left[\frac{1}{4t} + f'(u(x))\right] + \delta\left[\frac{1}{4t}\varphi_t(y) + \mu(\varphi_t)'(y)\right], \quad x \in \mathbb{R}. \end{aligned}$$

Since $u'(x) > 0$ and $\varphi_{T_1}(x) > 0$ in $\mathbb{R}$, $f'(\pm 1) < 0$, $f'_t(\pm 1) = -\frac{1}{t}$ and $\lim_{x\to\pm\infty} v_t(x) = \lim_{x\to\pm\infty} u(x) = \pm 1$, then there exists some small $T_1 > 0$ and large $R_4 > R_3 > 0$ such that for all $\delta > 0$, we have

$$\frac{1}{2T_1} + f'_{T_1}(v_{T_1}(x)) < 0 \quad \text{and} \quad \frac{1}{4T_1} + f'(u(x)) > 0, \quad \forall |x| \ge R_4.$$

Now look at $\frac{1}{4T_1}\varphi_{T_1}(x) + \mu\varphi'_{T_1}(x)$, by (2.2.1), Claim I and $v_t(x) = v_1\left(t^{-\frac{1}{2s}}x\right)$, there exist some constant $C_3 > 0$ and $C_4 > 0$ such that for all $|x| \geq R_4$, we have

$$\frac{1}{4T_1}\varphi_{T_1}(x) + \mu\varphi'_{T_1}(x) \geq \frac{C_4}{4|x|^{1+2s}} - \frac{|\mu|C_3}{|y|^{2+2s}} \geq \frac{C_4}{4|x|^{2+2s}}\left[|y| - \frac{4|\mu|C_3}{C_4}\right].$$

Taking $R_5 = \max\left\{R_4, \frac{4|\mu|C_3}{C_4} + 1\right\}$, then we have

$$\frac{1}{4T_1}\varphi_{T_1}(x) + \mu(\varphi_{T_1})'(x) > 0, \quad \forall |x| \geq R_5.$$

In summary, we know that for all $\delta > 0$, we have

$$(-\Delta)^s \tilde{w}_{\delta,T_1}(x) - \mu\tilde{w}'_{\delta,T_1}(x) + \frac{1}{4T_1}\tilde{w}_{\delta,T_1}(x) > 0, \quad \forall |x| \geq R_5.$$

Since $u'(x) > 0$ in $\mathbb{R}$, then there exists some small $\delta_1 > 0$ such that for all $0 < \delta \leq \delta_1$, we have

$$w_{\delta,T_1}(x) = \delta\varphi_{T_1}(x) - u'(x) < 0, \quad \forall |x| \leq R_5 + 1.$$

Then $\tilde{w}_{\delta,T_1}$ satisfies

$$\begin{cases} (-\Delta)^s \tilde{w}_{\delta,T_1}(x) - \mu\tilde{w}'_{\delta,T_1}(x) + \frac{1}{4T_1}\tilde{w}_{\delta,T_1}(x) > 0, \quad \forall |x| > R_5, \\ \tilde{w}_{\delta,T_1}(x) > 0, \quad \forall |x| < R_5 + 1, \\ \lim\limits_{|x|\to\infty} \tilde{w}_{\delta,T_1}(x) = 0. \end{cases}$$

By the same argument as Claim I, we can conclude that $\tilde{w}_{\delta,T_1}(x) \geq 0$ in $\mathbb{R}$, that is, $u'(x) \geq \delta\varphi_{T_1}(x) = \delta v'_{T_1}(x)$ in $\mathbb{R}$. By (2.2.1), we know that there exists some $B > 0$

such that

$$u'(x) \geq \frac{B}{|x|^{1+2s}}, \quad \forall |x| \geq 1.$$

□

Remark 2.2.3. *Let $1 \leq p \leq \infty$, by Proposition 2.2.2, we know that $u' \in L^p(\mathbb{R})$ for all $0 < s < 1$, but $f(u(\cdot)) \in L^p(\mathbb{R})$ if and only if $\frac{1}{2p} < s < 1$, that is, $(-\Delta)^s u \in L^p(\mathbb{R})$ if and only if $\frac{1}{2p} < s < 1$.*

2.3 Hamiltonian Identity and Modica Type Estimate

In this section, we always assume that $\mu \in \mathbb{R}$, $u \in C^2(\mathbb{R})$ is a solution to (2.0.1) with μ, and $\overline{u}$ is the s-harmonic extension of u. Since $\overline{u}$ is the s-harmonic extension in $\mathbb{R}^2_+$ of u, then $\overline{u} \in C^2(\mathbb{R}^2_+) \bigcap C\left(\overline{\mathbb{R}^2_+}\right)$ and satisfies

$$\begin{cases} \text{div}\left[y^{1-2s}\nabla\overline{u}(x,y)\right] = 0, \quad \forall (x,y) \in \mathbb{R}^2_+, \\ \lim\limits_{y\searrow 0} -y^{1-2s}\overline{u}_y(x,y) = d_s\left[\mu\overline{u}_x(x,0) + f(\overline{u}(x,0))\right], \quad \forall x \in \mathbb{R}, \\ \overline{u}(x,0) = u(x), \quad \forall x \in \mathbb{R}. \end{cases}$$

Following [18], we show similar Hamiltonian identity and Modica-type estimate for traveling wave solution. First, let's quote a Lemma from [18].

Lemma 2.3.1 (Lemma 5.1 in [18]). *The integral $\displaystyle\int_0^\infty t^{1-2s}|\nabla\overline{u}(x,t)|^2 t^{1-2s}\, dt$ is finite and differentiable with respect to $x \in \mathbb{R}$. Moreover, we have*

$$\lim_{|x|\to\infty} \int_0^\infty |\nabla\overline{u}(x,t)|^2 t^{1-2s}\, dt = 0.$$

Proposition 2.3.2 (Hamiltonian Identity). *For all $x \in \mathbb{R}$, we have*

$$\frac{1}{2}\int_0^\infty [\overline{u}_x^2(x,t) - \overline{u}_y^2(x,t)]t^{1-2s}\ dt = d_s\left[-\mu\int_{-\infty}^x [\overline{u}_x(r,0)]^2\ dr + G(\overline{u}(x,0))\right],$$

and

$$\mu\int_{-\infty}^\infty [\overline{u}_x(r,0)]^2\ dr = \mu\int_{\mathbb{R}} |u'(x)|^2\ dx = G(1).$$

Proof. By Lemma 2.3.1, we can define

$$v(x) = \frac{1}{2}\int_0^\infty [\overline{u}_x^2(x,t) - \overline{u}_y^2(x,t)]t^{1-2s}\ dt, \quad \forall x \in \mathbb{R}.$$

Then $\lim\limits_{|x|\to\infty} v(x) = 0$ and

$$v'(x) = \int_0^\infty [\overline{u}_x(x,t)\overline{u}_{xx}(x,t) - \overline{u}_y(x,t)\overline{u}_{xy}(x,t)]t^{1-2s}\ dt, \quad \forall x \in \mathbb{R}.$$

Since div $[y^{1-2s}\nabla\overline{u}(x,y)] = 0$ in $\mathbb{R}^2_+$, we get

$$\overline{u}_{xx}(x,y) = -y^{2s-1}D_y[y^{1-2s}\overline{u}_y(x,y)], \quad \forall (x,y) \in \mathbb{R}^2_+.$$

Use the integration by parts, we can get

$$\begin{aligned}
v'(x) &= \int_0^\infty \left[-\overline{u}_x(x,t)t^{2s-1}D_y[t^{1-2s}\overline{u}_y(x,t)] - \overline{u}_y(x,t)\overline{u}_{xy}(x,t)\right] t^{1-2s}\ dt \\
&= -\int_0^\infty \overline{u}_x(x,t)D_x[t^{1-2s}\overline{u}_y(x,t)]\ dt - \int_0^\infty \overline{u}_y(x,t)\overline{u}_{xy}(x,t)t^{1-2s}\ dt \\
&= \lim_{t\searrow 0} t^{1-2s}\overline{u}_y(x,t)\overline{u}_x(x,t) \\
&= -d_s\overline{u}_x(x,0)\left[\mu\overline{u}_x(x,0) + f(\overline{u}(x,0))\right] \\
&= d_s\left[-\mu\overline{u}_x^2(x,0) + \frac{d}{dx}G(\overline{u}(x,0))\right], \quad \forall x \in \mathbb{R}.
\end{aligned}$$

Then there exists some constant $C_0 > 0$ such that

$$v(x) = d_s \left[-\mu \int_{-\infty}^{x} \overline{u}_x^2(r, 0)\ dr + G(\overline{u}(x, 0)) + C_0 \right], \quad \forall x \in \mathbb{R}.$$

Take $x \to -\infty$ in the above identity, since $\lim_{x \to -\infty} v(x) = 0$, we know that $C_0 = -G(-1) = 0$. We complete the proof.

□

Remark 2.3.3. *A direct consequence of Proposition 2.3.2 is that μ has the same sign as $G(1) = -\int_{-1}^{1} f(t)\ dt$. In particular, u is standing wave (i.e., $\mu = 0$) if and only if $G(1) = G(-1) = 0$.*

For the second identity $\mu \int_{\mathbb{R}} |u'(x)|^2\ dx = G(1)$ in Propositoin 2.3.2, we have another direct approach. Multiply $u'(x)$ on the both sides of (2.0.1), we know that it suffices to show that

$$\int_{\mathbb{R}} (-\Delta)^s u(x) u'(x)\ dx = 0. \tag{2.3.1}$$

Proof of (2.3.1). By the Dominated Convergence Theorem and changing of variables, we have

$$\begin{aligned}
& \int_{\mathbb{R}} (-\Delta)^s u(x) u'(x)\ dx \\
= & -\frac{C_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{[u(x+y) + u(x-y) - 2u(x)]u'(x)}{|y|^{n+2s}}\ dydx \\
= & -\frac{C_{n,s}}{2} \lim_{\epsilon \searrow 0} \int_{\mathbb{R}^n} \int_{|y| > \epsilon} \frac{[u(x+y) + u(x-y) - 2u(x)]u'(x)}{|y|^{n+2s}}\ dydx \\
= & -\frac{C_{n,s}}{2} \lim_{\epsilon \searrow 0} \int_{\mathbb{R}^n} \int_{|y| > \epsilon} \frac{[u(x+y) - u(x)]u'(x)}{|y|^{n+2s}}\ dydx \\
& -\frac{C_{n,s}}{2} \lim_{\epsilon \searrow 0} \int_{\mathbb{R}^n} \int_{|y| > \epsilon} \frac{[u(x-y) - u(x)]u'(x)}{|y|^{n+2s}}\ dydx
\end{aligned}$$

$$\begin{aligned}
&= -\frac{C_{n,s}}{2} \lim_{\epsilon \searrow 0} \int_{\mathbb{R}^n} \int_{|y|>\epsilon} \frac{[u(y)-u(x)]u'(x)}{|x-y|^{n+2s}}\ dydx \\
&\quad -\frac{C_{n,s}}{2} \lim_{\epsilon \searrow 0} \int_{\mathbb{R}^n} \int_{|y|>\epsilon} \frac{[u(y)-u(x)]u'(x)}{|x-y|^{n+2s}}\ dydx \\
&= C_{1,s} \lim_{\epsilon \searrow 0} \int_{\mathbb{R}} \int_{|x-y|>\epsilon} \frac{[u(x)-u(y)]u'(x)}{|x-y|^{1+2s}}\ dydx \\
&= C_{1,s} \lim_{\epsilon \searrow 0} \left[\int_{\mathbb{R}} \int_{|x-y|>\epsilon} \frac{u(x)u'(x)}{|x-y|^{1+2s}}\ dydx - \int_{\mathbb{R}} \int_{|x-y|>\epsilon} \frac{u(y)u'(x)}{|x-y|^{1+2s}}\ dydx \right].
\end{aligned}$$

For $\int_{\mathbb{R}} \int_{|x-y|>\epsilon} \frac{u(x)u'(x)}{|x-y|^{1+2s}}\ dydx$, by Fubini-Tonelli's Theorem, and apply the integration by parts, we have

$$\begin{aligned}
\int_{\mathbb{R}} \int_{|x-y|>\epsilon} \frac{u(x)u'(x)}{|x-y|^{1+2s}}\ dydx &= \int_{\mathbb{R}} \int_{|z|>\epsilon} \frac{u(x)u'(x)}{|z|^{1+2s}}\ dydx \quad \text{Let } z = y - x \\
&= \int_{|z|>\epsilon} \frac{1}{|z|^{1+2s}} \left[\int_{\mathbb{R}} u(x)u'(x)\ dx \right]\ dz \\
&= \int_{|z|>\epsilon} \frac{1}{|z|^{1+2s}} \left[\frac{1}{2} |u(x)|^2 \Big|_{-\infty}^{\infty} \right]\ dz \\
&= 0, \quad \text{Since } \lim_{x\to\infty} u(x) = \pm 1.
\end{aligned}$$

For $\int_{\mathbb{R}} \int_{|x-y|>\epsilon} \frac{u(y)u'(x)}{|x-y|^{1+2s}}\ dydx$, let $J_\epsilon(z) = \frac{1}{|z|^{1+2s}} \cdot 1_{\mathbb{R}^n \setminus B_1(0)}(z)$ in $\mathbb{R}$, then J_ϵ is an even function in $\mathbb{R}$, and

$$\int_{\mathbb{R}} J_\epsilon(z)\ dz = \int_{|z|>\epsilon} \frac{1}{|z|^{1+2s}}\ dz = \frac{1}{s\epsilon^{2s}}.$$

Since $\lim_{x\to\pm\infty} u(x) = \pm 1$, by the Dominated Convergence Theorem, we have

$$\lim_{x\to\pm\infty} J_\epsilon * u(x) = \lim_{x\to\pm\infty} \int_{\mathbb{R}} J_\epsilon(y) u(x-y)\ dz = \pm \int_{\mathbb{R}} J_\epsilon(y)\ dz = \pm \frac{1}{s\epsilon^{2s}},$$

which implies that

$$\lim_{x\to\pm\infty} J_\epsilon * u(x)\cdot u(x) = \frac{1}{s\epsilon^{2s}}.$$

Use the integration by parts, we have

$$\begin{aligned}
\int_{\mathbb{R}}\int_{|x-y|>\epsilon} \frac{u(y)u'(x)}{|x-y|^{1+2s}}\ dydx &= \int_{\mathbb{R}} J_\epsilon * u(x)u'(x)dx \\
&= J_\epsilon * u(x)u(x)|_{-\infty}^{\infty} - \int_{\mathbb{R}}\int_{\mathbb{R}} u(x)J_\epsilon * u'(x)\ dx \\
&= -\int_{\mathbb{R}}\int_{\mathbb{R}} u(x)\left[\int_{\mathbb{R}} J_\epsilon(x-y)u'(y)\ dy\right]\ dx \\
&= -\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} J_\epsilon(y-x)u(x)u'(y)\ dydx \quad \text{Since } J \text{ is even} \\
&= -\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} J_\epsilon(x-y)u(y)u'(x)\ dxdy \\
&\qquad \text{By changing the positions of } x \text{ and } y \\
&= -\int_{\mathbb{R}}\int_{|x-y|>\epsilon} \frac{u(y)u'(x)}{|x-y|^{1+2s}}\ dydx.
\end{aligned}$$

So we know that

$$\int_{\mathbb{R}}\int_{|x-y|>\epsilon} \frac{u(y)u'(x)}{|x-y|^{1+2s}}\ dydx = 0.$$

In summary, we have

$$\int_{\mathbb{R}}\int_{|x-y|>\epsilon} \frac{u(x)u'(x)}{|x-y|^{1+2s}}\ dydx = \int_{\mathbb{R}}\int_{|x-y|>\epsilon} \frac{u(y)u'(x)}{|x-y|^{1+2s}}\ dydx = 0,$$

which implies that

$$\int_{\mathbb{R}} (-\Delta)^s u(x)u'(x)\ dx = 0.$$

□

Proposition 2.3.4 (Modica Type Estimate)**.** *For all* $(x,y) \in \overline{\mathbb{R}^2_+}$, *we have*

$$\frac{1}{2}\int_0^y [\overline{u}_x^2(x,t) - \overline{u}_y^2(x,t)]t^{1-2s}\ dt < d_s\left[-\mu\int_{-\infty}^x \overline{u}_x^2(r,0)\ dr + G(\overline{u}(x,0))\right].$$

As a consequence, we have

$$G(u(x)) > \mu\int_{-\infty}^x |u'(r)|^2\ dr, \quad \forall x \in \mathbb{R}. \tag{2.3.2}$$

Proof. Consider the function

$$v(x,y) = \frac{1}{2}\int_0^y [\overline{u}_x^2(x,t) - \overline{u}_y^2(x,t)]t^{1-2s}\ dt, \qquad \forall (x,y) \in \overline{\mathbb{R}^2_+}.$$

By Lemma 2.3.1, we know that

$$\lim_{|x|\to\infty} v(x,y) = 0, \quad \text{uniformly in } y \geq 0, \tag{2.3.3}$$

which implies that $v \in L^\infty(\mathbb{R}^2_+)$. It's easy to see that

$$\begin{aligned} v_x(x,y) &= \int_0^y [\overline{u}_x(x,t)\overline{u}_{xx}(x,t) - \overline{u}_y(x,t)\overline{u}_{xy}(x,t)]t^{1-2s}\ dt, \\ v_y(x,y) &= \frac{1}{2}[\overline{u}_x^2(x,y) - \overline{u}_y^2(x,y)]y^{1-2s}. \end{aligned}$$

Since div $[y^{1-2s}\nabla\overline{u}(x,y)] = 0$, we get

$$\overline{u}_{xx}(x,y) = -y^{2s-1}D_y[y^{1-2s}\overline{u}_y(x,y)], \quad \forall (x,y) \in \mathbb{R}^2_+.$$

Use integration by parts, we obtain

$$\begin{aligned} v_x(x,y) &= \int_0^y \left[-\overline{u}_x(x,t)t^{2s-1}D_y[t^{1-2s}\overline{u}_y(x,y)] - \overline{u}_y(x,t)\overline{u}_{xy}(x,t)\right] t^{1-2s}\ dt \\ &= -\int_0^y \overline{u}_x(x,t)D_y[t^{1-2s}\overline{u}_y(x,t)]\ dt - \int_0^y \overline{u}_y(x,t)\overline{u}_{xy}(x,t)t^{1-2s}\ dt \\ &= \overline{u}_x(x,0)\lim_{t\searrow 0} t^{1-2s}\overline{u}_y(x,t) - y^{1-2s}\overline{u}_y(x,y)\overline{u}_x(x,y) \\ &= -\overline{u}_x(x,0)d_s\left[\mu\overline{u}_x(x,0) + f(\overline{u}(x,0))\right] - y^{1-2s}\overline{u}_y(x,y)\overline{u}_x(x,y) \\ &= d_s\left[-\mu[\overline{u}_x(x,0)]^2 + \frac{d}{dx}G(\overline{u}(x,0))\right] - y^{1-2s}\overline{u}_y(x,y)\overline{u}_x(x,y). \end{aligned} \tag{2.3.4}$$

Consider the function

$$w(x,y) = d_s\left[-\mu\int_{-\infty}^x [\overline{u}_x(r,0)]^2\ dr + G(\overline{u}(x,0))\right] - v(x,y), \quad \forall (x,y)\in\overline{\mathbb{R}^2_+}.$$

By Proposition 2.3.2, we have $\lim\limits_{y\to\infty} w(x,y) = 0$ for all $x\in\mathbb{R}$, and

$$\lim_{x\to\infty}\left[-\mu\int_{-\infty}^x [\overline{u}_x(r,0)]^2\ dr + G(\overline{u}(x,0))\right] = 0.$$

By (2.3.3), we know that $w(x,y)\to 0$ uniformly in $y\geq 0$, as $|x|\to\infty$. Since $v\in L^\infty(\mathbb{R}^2_+)$, $|\overline{u}(x,0)| = |u(x)|\leq 1$ in $\mathbb{R}$, and $\int_{\mathbb{R}}|\overline{u}_x(r,0)|^2\ dr = \int_{\mathbb{R}}|u'(r)|^2\ dr < \infty$, we get $w\in L^\infty(\mathbb{R}^2_+)$.

Claim I: For all $(x,y)\in\mathbb{R}^2_+$, we have

$$\begin{cases} \text{div}\ [y^{1-2s}\nabla w(x,y)] = (2s-1)y^{1-4s}\overline{u}_x^2(x,y), \\ \text{div}\ [y^{2s-1}\nabla w(x,y)] = (2s-1)y^{-1}\overline{u}_y^2(x,y). \end{cases} \tag{2.3.5}$$

In fact, for $\nabla w(x,y)$, we have

$$\begin{aligned} w_x(x,y) &= d_s\left[-\mu[\overline{u}_x(x,0)]^2 + \frac{d}{dx}G(\overline{u}(x,0))\right] - v_x(x,y) \\ &= y^{1-2s}\cdot\overline{u}_x(x,y)\cdot\overline{u}_y(x,y) \quad \text{By (2.3.4)} \\ w_y(x,y) &= -v_y(x,y) \\ &= \frac{y^{1-2s}}{2}\cdot[\overline{u}_y^2(x,y)-\overline{u}_x^2(x,y)]. \end{aligned}$$

For $D^2w(x,y)$, we have

$$\begin{aligned} w_{xx}(x,y) &= y^{1-2s}[\overline{u}_{xx}(x,y)\overline{u}_y(x,y)+\overline{u}_x(x,y)\overline{u}_{xy}(x,y)] \\ w_{yy}(x,y) &= \frac{(1-2s)y^{-2s}}{2}\cdot[\overline{u}_y^2(x,y)-\overline{u}_x^2(x,y)] \\ &\quad +\frac{y^{1-2s}}{2}[2\overline{u}_y(x,y)\overline{u}_{yy}(x,y)-2\overline{u}_x(x,y)\overline{u}_{xy}(x,y)] \\ &= \frac{(1-2s)y^{-2s}}{2}\cdot[\overline{u}_y^2(x,y)-\overline{u}_x^2(x,y)] \\ &\quad +y^{1-2s}[\overline{u}_y(x,y)\overline{u}_{yy}(x,y)-\overline{u}_x(x,y)\overline{u}_{xy}(x,y)] \\ \Delta w(x,y) &= y^{1-2s}[\overline{u}_{xx}(x,y)\overline{u}_y(x,y)+\overline{u}_x(x,y)\overline{u}_{xy}(x,y)] \\ &\quad +\frac{(1-2s)y^{-2s}}{2}\cdot[\overline{u}_y^2(x,y)-\overline{u}_x^2(x,y)] \\ &\quad +y^{1-2s}[\overline{u}_y(x,y)\overline{u}_{yy}(x,y)-\overline{u}_x(x,y)\overline{u}_{xy}(x,y)] \\ &= y^{1-2s}\overline{u}_y(x,y)\Delta\overline{u}(x,y)+\frac{(1-2s)y^{-2s}}{2}\cdot[\overline{u}_y^2(x,y)-\overline{u}_x^2(x,y)]. \end{aligned}$$

Since div $[y^{1-2s}\nabla\overline{u}(x,y)]=0$, we know that $\Delta\overline{u}(x,y)=(2s-1)y^{-1}\overline{u}_y(x,y)$. For div $[y^{1-2s}\nabla w(x,y)]$, we have

$$\begin{aligned} &\text{div } [y^{1-2s}\nabla w(x,y)] \\ =\ & y^{1-2s}\Delta w(x,y)+(1-2s)y^{-2s}w_y(x,y) \end{aligned}$$

$$
\begin{aligned}
&= y^{1-2s}\left[y^{1-2s}\overline{u}_y(x,y)\Delta\overline{u}(x,y) + \frac{(1-2s)y^{-2s}}{2}\cdot[\overline{u}_y^2(x,y)-\overline{u}_x^2(x,y)]\right] \\
&\quad +(1-2s)y^{-2s}\cdot\frac{y^{1-2s}}{2}\cdot[\overline{u}_y^2(x,y)-\overline{u}_x^2(x,y)] \\
&= y^{2-4s}\overline{u}_y(x,y)\Delta\overline{u}(x,y) + (1-2s)y^{1-4s}\overline{u}_y^2(x,y) + (2s-1)y^{1-4s}\overline{u}_x^2(x,y) \\
&= (2s-1)y^{1-4s}\overline{u}_y^2(x,y) + (1-2s)y^{1-4s}\overline{u}_y^2(x,y) + (2s-1)y^{1-4s}\overline{u}_x^2(x,y) \\
&= (2s-1)y^{1-4s}\overline{u}_x^2(x,y).
\end{aligned}
$$

For div $[y^{2s-1}\nabla w(x,y)]$, we have

$$
\begin{aligned}
&\text{div } [y^{2s-1}\nabla w(x,y)] \\
&= y^{2s-1}\Delta w(x,y) + (2s-1)y^{2s-2}w_y(x,y) \\
&= y^{2s-1}\left[y^{1-2s}\overline{u}_y(x,y)\Delta\overline{u}(x,y) + \frac{(1-2s)y^{-2s}}{2}\cdot[\overline{u}_y^2(x,y)-\overline{u}_x^2(x,y)]\right] \\
&\quad +(2s-1)y^{2s-2}\cdot\frac{y^{1-2s}}{2}\cdot[\overline{u}_y^2(x,y)-\overline{u}_x^2(x,y)] \\
&= \overline{u}_y(x,y)\Delta\overline{u}(x,y) \\
&= (2s-1)y^{-1}\overline{u}_y(x,y).
\end{aligned}
$$

Claim II: w is not a constant function in $\mathbb{R}^2_+$.

If Claim II is not true, that is, $w(x,y) \equiv w(0,0)$ in $\overline{\mathbb{R}^2_+}$. Since $v(x,0) \equiv 0$ in $\mathbb{R}$, we have

$$
w(0,0) \equiv d_s\left[-\mu\int_{-\infty}^{x}\overline{u}_x^2(r,0)\ dr + G(\overline{u}(x,0))\right], \quad \forall x\in\mathbb{R}.
$$

This implies that

$$
\mu|\overline{u}_x(x,0)|^2 = \frac{d}{dx}G(\overline{u}(x,0)) = -f(\overline{u}(x,0))\overline{u}_x(x,0), \quad \forall x\in\mathbb{R}.
$$

Since $\overline{u}_x(x,0) = u'(x) > 0$ and $\overline{u}(x,0) = u(x)$ in $\mathbb{R}$, then we get $\mu u'(x) = -f(u(x))$ in $\mathbb{R}$, which implies that $(-\Delta)^s u(x) = 0$ in $\mathbb{R}$. Since $u \in L^\infty(\mathbb{R})$, by Liouville's theorem for the fractional harmonic functions, then u is a constant function in $\mathbb{R}$, which contradicts with $\lim_{x\to\pm\infty} u(x) = \pm 1$.

Claim III: $w(x,y) > A := \inf_{z\in\mathbb{R}^2_+} w(z)$ for all $(x,y) \in \overline{\mathbb{R}^2_+}$.

If Claim III is not true, then there exists some $(x^0, y^0) \in \overline{\mathbb{R}^2_+}$ such that

$$w(x^0, y^0) = A = \inf_{(x,y)\in\mathbb{R}^2_+} w(x,y).$$

Since $u'(x) > 0$ in $\mathbb{R}$, then $\overline{u}_x(x,y) > 0$ for all $(x,y) \in \overline{\mathbb{R}^2_+}$. It's easy to see that $w_x(x,y) = y^{1-2s}\overline{u}_y(x,y)\overline{u}_x(x,y)$ in $\mathbb{R}^2_+$, then we have $\overline{u}_y(x,y) = y^{2s-1}\cdot\dfrac{w_x(x,y)}{\overline{u}_x(x,y)}$. By (2.3.5), we get

$$\text{div}\,[y^{2s-1}\nabla w(x,y)] = (2s-1)y^{2s-2}\cdot\frac{\overline{u}_y(x,y)}{\overline{u}_x(x,y)}\cdot w_x(x,y), \quad \text{in } \mathbb{R}^2_+.$$

Hence w satisfies an locally uniformly elliptic equation in $\mathbb{R}^2_+$. By the strong maximum principle and Claim II, we know that $(x^0, y^0) \in \partial\mathbb{R}^2_+$, that is, $(x^0, 0)$ is a boundary minimum point of w in $\overline{\mathbb{R}^2_+}$, in particular, $w_x(x^0, 0) = 0$, that is,

$$0 = -f(\overline{u}(x^0,0))\overline{u}_x(x^0,0) - \mu\overline{u}^2_x(x^0,0).$$

Since $\overline{u}_x(x,y) > 0$ in $\overline{\mathbb{R}^2_+}$, then $\mu\overline{u}_x(x^0,0) + f(\overline{u}(x^0,0)) = 0$, which implies that

$$\lim_{y\searrow 0} -y^{1-2s}\overline{u}_y(x^0, y) = 0. \tag{2.3.6}$$

We consider the following two cases:

Case I: $0 < s \le \frac{1}{2}$. By (2.3.5), then div $[y^{1-2s}\nabla w(x,y)] \le 0$ in $\mathbb{R}^2_+$. Since $w(x,y) > w(x^0,0)$ for all $(x,y) \in \mathbb{R}^2_+$, by Hopf Lemma, Proposition 4.11 in [18], we know $\lim_{y\searrow 0} -y^{1-2s}\overline{u}_y(x^0,y) < 0$, which contradicts with (2.3.6).

Case II: $1 > s > \frac{1}{2}$. By the extension, we know that $\lim_{y\searrow 0} \overline{u}_y(x,y) = 0$ in $\mathbb{R}$. Since $w(x,y) > w(x^0,0)$ in $\mathbb{R}^2_+$, then

$$0 \ge \liminf_{y\searrow 0} \; -y^{2s-1}w_y(x^0,y).$$

On the other hand, it's easy to see that $w_y(x,y) = \dfrac{1}{2}[\overline{u}_y^2(x,y) - \overline{u}_x^2(x,y)]y^{1-2s}$ in $\mathbb{R}^2_+$, which implies that

$$\begin{aligned} 0 &\ge \liminf_{y\searrow 0} \; [\overline{u}_x^2(x^0,0) - \overline{u}_y^2(x^0,y)] \\ &= \overline{u}_x^2(x^0,0) - \limsup_{y\searrow 0} \; \overline{u}_y^2(x^0,0) \\ &= \overline{u}_x^2(x^0,0) > 0. \end{aligned}$$

Therefore, we get a contradiction. In summary, we know that for all $0 < s < 1$, $w(x,y) > A$ in $\overline{\mathbb{R}^2_+}$.

Since $w(x,y) \to 0$ uniformly in $y \ge 0$, as $|x| \to \infty$, and $w(x,y) \to 0$, as $y \to \infty$, then $\lim_{|(x,y)|\to\infty} w(x,y) = 0$. By Claim III, we must have $A \ge 0$ and $w(x,y) > 0$ in $\overline{\mathbb{R}^2_+}$, that is, for all $(x,y) \in \overline{\mathbb{R}^2_+}$, we have

$$\frac{1}{2}\int_0^y [\overline{u}_x^2(x,t) - [\overline{u}_y^2(x,t)]t^{1-2s}\; dt < d_s\left[-\mu\int_{-\infty}^x \overline{u}_x^2(r,0)\; dr + G(\overline{u}(x,0))\right].$$

□

Remark 2.3.5. *Since* $\lim_{x\to\pm\infty} u(x) = \pm 1$, *by (2.3.2), we know that nonnegative speed*

$\mu \geq 0$ implies that $G(t) > 0$ in $(-1,1)$, and

$$\int_{-\infty}^{x} (-\Delta)^s u(r)u'(r)\ dr = -G(u(x)) + \mu \int_{-\infty}^{x} |u'(r)|^2\ dr < 0, \quad \forall x \in \mathbb{R}. \qquad (2.3.7)$$

2.4 Nondegeneracy

The following proposition the nondegeneracy of traveling wave solution, which allows to use the implicit function theorem near the traveling wave solution.

Proposition 2.4.1. *Let $\mu \in \mathbb{R}$ and $u \in C^2(\mathbb{R})$ be a solution to (2.0.1) with μ. Assume $h \in \mathbb{R}$ and $\phi \in C^2(\mathbb{R})$ satisfy*

$$\begin{cases} (-\Delta)^s \phi(x) - \mu\phi'(x) - f'(u(x))\phi(y) + hu'(x) = 0, \quad \forall x \in \mathbb{R}, \\ \phi(0) = 0, \\ \lim\limits_{|x|\to\infty} \phi(x) = 0. \end{cases}$$

Then $h = 0$ and $\phi(x) \equiv 0$ in $\mathbb{R}$.

Proof. By Proposition 2.1.2, we know that $v(x) := u'(x) > 0$ in $\mathbb{R}$. We consider the following two cases:

Case I: $h \geq 0$. For any $\delta > 0$, let $w_\delta(x) = v(x) - \delta\phi(x)$ in $\mathbb{R}$, then w_δ satisfies

$$\begin{cases} (-\Delta)^s w_\delta(x) - \mu w_\delta'(x) - f'(u(x))w_\delta(y) = \delta h u'(x) \geq 0, \quad \forall x \in \mathbb{R}, \\ w_\delta(0) > 0, \\ \lim\limits_{|x|\to\infty} w_\delta(x) = 0. \end{cases}$$

Since $f'(\pm 1) < 0$ and $\lim\limits_{|x|\to\infty} u(x) = \pm 1$, we can find some fixed large $R_0 > 0$ such that $-f'(u(x)) > 0$ for all $|x| \geq R_0$. Since $v(x) > 0$ in $\mathbb{R}$, there exists some fixed small $\epsilon_0 > 0$ such that $0 < |\delta| \leq \epsilon_0$ and all $|x| \leq R_0 + 1$, we have $w_\delta(x) > 0$. By Lemma 2.1.1, we know that $w_\delta(x) > 0$ in $\mathbb{R}$. Hence, we can define

$$\Lambda = \sup \ \{\epsilon > 0 : w_\delta(x) > 0 \text{ in } \mathbb{R}, \quad \forall \delta \in (0, \epsilon)\} \geq \epsilon_0.$$

If $\Lambda = \infty$, that is, for all $\delta > 0$, we have $w_\delta(x) > 0$ in $\mathbb{R}$, which implies that $\phi(x) \leq 0$ in $\mathbb{R}$. Since $\phi(0) = 0$. By Proposition 2.1.1, we know that $\phi(y) \equiv 0$ in $\mathbb{R}$. Hence $h \equiv 0$.

If $\Lambda < \infty$, then $w_\Lambda(x) \geq 0$ in $\mathbb{R}$. Since $\phi(0) = 0$, $v(x) > 0$ in $\mathbb{R}$, by Lemma 2.1.1, we know $w_\Lambda(x) > 0$ in $\mathbb{R}$. Replace v by w_Λ in the previous argument, we can find a larger $\Lambda' > \Lambda$ such that for all $\delta \in (0, \Lambda')$, we have $w_\delta(y) > 0$ in $\mathbb{R}$, which contradicts with the definition of Λ.

Hence, in this case, we have $h = 0$ and $\phi(x) \equiv 0$ in $\mathbb{R}$.

Case II: $h \leq 0$. In this case, let $k = -h \geq 0$ and $\psi(x) = -\phi(x)$ in $\mathbb{R}$, then it's easy to see that k and ψ satisfies $\lim_{|x| \to \infty} \psi(x) = 0$ and

$$(-\Delta)^s \psi(x) - \mu \psi'(x) - f'(u(x))\psi(x) + ku'(x) = 0, \quad \forall x \in \mathbb{R}.$$

Applying the result of Case I, we know that $k = 0$ and $\psi(x) \equiv 0$ in $\mathbb{R}$, which implies that, $h = 0$ and $\phi(x) \equiv 0$ in $\mathbb{R}$.

In summary, we can conclude that $h = 0$ and $\phi(x) \equiv 0$ in $\mathbb{R}$.

$\square$

Chapter 3

Estimates of Traveling Speeds

In this chapter, we will estimate the traveling speed in terms of potential function f, which is crucial in the proof of the existence of traveling waves. In the chapter, we always assume that $0 < s < 1$, f is a bistable nonlinearity, that is, f satisfies (1.1.3), and $G(t) = -\int_{-1}^{t} f(u)\, du$.

3.1 Laplacian Case

In this section, let's get some motivations from the case of usual Laplacian. As I mentioned in Section 1.1 (see [61, 66] for more details), there exists a unique pair (μ, ϕ) up to translation such that there exists some constant $\nu > 0$, we have

$$\begin{cases} -\phi''(x) - \mu\phi'(x) = f(\phi(x)), \quad \forall x \in \mathbb{R}, \\ \phi'(x) \sim e^{-\nu|x|}, \quad \text{as } |x| \to \infty, \\ \lim\limits_{x\to\pm\infty} \phi(x) = \pm 1. \end{cases}$$

If $\mu < 0$, consider $\psi(x) = -\phi(-x)$ in $\mathbb{R}$, then $\psi'(x) = \phi'(-x)$ in $\mathbb{R}$ and $\lim_{x\to\pm\infty} \psi(x) = \pm 1$. Let $\overline{f}(t) = -f(-t)$ in $\mathbb{R}$, it's easy to see that ψ satisfies

$$-\psi''(x) - (-\mu)\psi'(x) = \overline{f}(\psi(x)), \quad \forall x \in \mathbb{R}$$

In summary, without loss of generality, we can assume the speed $\mu \geq 0$, and consider the following problem:

$$\begin{cases} -\phi''(x) - \mu\phi'(x) = f(\phi(x)), \quad \forall x \in \mathbb{R} \\ \phi'(x) \sim e^{-\nu|x|}, \quad \text{as } |x| \to \infty, \\ \phi'(x) > 0, \quad \forall x \in \mathbb{R} \\ \lim_{x\to\pm\infty} \phi(x) = \pm 1, \qquad \mu \geq 0 \end{cases} \tag{3.1.1}$$

Multiply $\phi'(x)$ on both sides of the first equation in (3.1.1) and integrate from $-\infty$ to y, we get

$$-\int_{-\infty}^{y} \phi''(x)\phi'(x)\ dx - \mu \int_{-\infty}^{y} [\phi'(x)]^2\ dx = \int_{-\infty}^{y} f(\phi(x))\phi'(x)\ dx.$$

Since $\lim_{x\to-\infty} \phi'(x) = 0$ and $\dfrac{dG(\phi(x))}{dx} = -f(\phi(x))\phi'(x)$, we get

$$\frac{1}{2}[\phi'(y)]^2 + \mu \int_{-\infty}^{y} [\phi'(x)]^2\ dx = G(\phi(y)), \quad \forall y \in \mathbb{R}. \tag{3.1.2}$$

Since $\lim_{x\to\infty} \phi'(x) = 0$ and $\lim_{x\to\infty} \phi(x) = 1$, by taking $y \to \infty$ in (3.1.2), we get the following Hamiltonian identity:

$$\mu \int_{-\infty}^{\infty} [\phi'(x)]^2\ dx = G(1) \tag{3.1.3}$$

Since $\mu \geq 0$, we have the Modica type estimate:

$$\frac{1}{2}[\phi'(y)]^2 \leq G(\phi(y)), \quad \forall y \in \mathbb{R}, \tag{3.1.4}$$

which implies that $G(t) > 0$ in $(-1, 1)$.

Now we are ready to estimate the speed μ. If $G(1) = 0$, by (3.1.3), $\mu = 0$, there is nothing to do. In the following, we assume $G(1) > 0$, that is, $\mu > 0$, so we only need to find the upper bound of μ. In the following, we introduce three different approaches.

Since f satisfies (1.1.3), we know that G is strictly increasing on $[-1, t_0]$, and strictly decreasing on $[t_0, 1]$, in particular, there exists some $t_1 \in (-1, t_0)$ such that

$$0 < G(1) < G(t), \quad \forall t \in (t_1, 1), \quad \text{and} \quad G(t_0) = \max_{t \in [-1,1]} G(t).$$

By translation, without loss of generality, we can assume that $\phi(0) = t$. Take $y = 0$ in (3.1.2), we have

$$\begin{aligned}
\frac{1}{2}[\phi'(0)]^2 &= G(t) - \mu \int_{-\infty}^{0} [\phi'(x)]^2 \, dx \\
&\geq G(t) - \mu \int_{-\infty}^{\infty} [\phi'(x)]^2 \, dx \quad \text{Since } \mu > 0 \\
&= G(t) - G(1), \quad \text{By (3.1.3).}
\end{aligned}$$

By (3.1.3), we have

$$\frac{1}{2}[\phi'(0)]^2 \geq G(t) - G(1). \tag{3.1.5}$$

Proposition 3.1.1. *For any $t \in (t_1, 1)$, we have*

$$0 < \mu < \frac{\|f\|_{L^\infty([-1,1])}}{\sqrt{2[G(t) - G(1)]}}.$$

Proof. Since $\lim_{|x|\to\infty} \phi'(x) = 0$, by (3.1.5), we know that there exists some $x_0 \in \mathbb{R}$ such that

$$\phi'(x_0) = \sup_{x\in\mathbb{R}} \phi'(x) \geq \phi'(0) \geq \sqrt{2[G(t) - G(1)]},$$

which implies that $\phi''(x_0) = 0$. By (3.1.1), we get

$$\mu = -\frac{f(\phi(x_0))}{\phi'(x_0)} \leq \frac{\|f\|_{C([-1,1])}}{\phi'(0)} \leq \frac{\|f\|_{C([-1,1])}}{\sqrt{2[G(t) - G(1)]}}.$$

□

Remark 3.1.2. *The approach of Proposition 3.1.1 involves the Hamiltonian identity (3.1.5) and the relation between $\phi'(x)$ and $\phi''(x)$ which is very special for the Laplacian.*

Proposition 3.1.3. *For any $t \in (t_1, 1)$, we have*

$$0 < \mu \leq \frac{B + \sqrt{B^2 + 4AC}}{2A},$$

Where

$$A = G(t) - G(1), \ \ B = |f(t)|\sqrt{2G(t)}, \ \text{and } C = \|f'\|_{L^\infty([-1,t])} \cdot G(1).$$

In particular, we have

$$0 < \mu \leq \sqrt{\|f'\|_{L^\infty([-1,t_0])} \cdot \frac{G(1)}{G(t_0) - G(1)}}.$$

Proof. Multiply $\phi''(x)$ on (3.1.1) and integrate over $(-\infty, 0)$, use the integration by parts, then

$$\begin{aligned}
\int_{-\infty}^{0} [\phi''(x)]^2 \, dx &= -\int_{-\infty}^{0} \phi''(x)\phi'(x) \, dx - \int_{-\infty}^{0} f(\phi(x))\phi''(x) \, dx \\
&= -\frac{\mu}{2}[\phi'(0)]^2 - \int_{-\infty}^{0} f(\phi(x))\phi''(x) \, dx \\
&\geq 0.
\end{aligned}$$

Since $\mu > 0$, by (3.1.5), we obtain

$$\mu[G(t) - G(1)] \leq \frac{\mu}{2}[\phi'(0)]^2 \leq -\int_{-\infty}^{0} f(\phi(x))\phi''(x) \, dx$$

Let's estimate $-\int_{-\infty}^{0} f(\phi(x))\phi''(x) \, dx$. Use the integration by parts, since $\lim_{x\to-\infty} \phi(x) = -1$ and $f(-1) = 0$, we obtain

$$\begin{aligned}
-\int_{-\infty}^{0} f(\phi(x))\phi''(x) \, dx &= -f(\phi(x))\phi'(x)|_{-\infty}^{0} + \int_{-\infty}^{0} \phi'(x)f'(\phi(x))\phi'(x) \, dx \\
&= -f(t)\phi'(0) + \int_{-\infty}^{0} f'(\phi(x))[\phi'(x)]^2 \, dx
\end{aligned}$$

For $-f(t)\phi'(0)$, by (3.1.4), we get

$$|-f(t)\phi'(0)| \leq |f(t)|\sqrt{2G(t)}$$

For $\int_{-\infty}^{0} f'(\phi(x))[\phi'(x)]^2 \, dx$, then

$$\begin{aligned}\int_{-\infty}^{0} |f'(\phi(x))|[\phi'(x)]^2 \, dx &\leq \|f'\|_{L^\infty([-1,t])} \int_{-\infty}^{0} [\phi'(x)]^2 \, dx \\ &\leq \|f'\|_{L^\infty([-1,t])} \int_{\mathbb{R}} [\phi'(x)]^2 \, dx \\ &= \|f'\|_{L^\infty([-1,t])} \cdot \frac{G(1)}{\mu}, \quad \text{By (3.1.4).}\end{aligned}$$

In summary, we have

$$\mu[G(t) - G(1)] \leq |f(t)|\sqrt{2G(t)} + \|f'\|_{L^\infty([-1,t])} \cdot \frac{G(1)}{\mu}.$$

Since $\mu > 0$, we get

$$[G(t) - G(1)]\mu^2 - |f(t)|\sqrt{2G(t)}\mu - \|f'\|_{L^\infty([-1,t])} \cdot G(1) \leq 0$$

Let $A = G(t) - G(1)$, $B = |f(t)|\sqrt{2G(t)}$ and $C = \|f'\|_{L^\infty([-1,t])} \cdot G(1)$, then we have $A\mu^2 - B\mu - C \leq 0$, which implies that

$$\mu \leq \frac{B + \sqrt{B^2 + 4AC}}{2A}$$

□

The third approach is quite similar to Proposition 3.1.3, but it gives different estimate.

Proposition 3.1.4. *For any $t \in (t_1, 1)$, we have*

$$0 < \mu \leq \frac{B + \sqrt{B^2 + 4AC}}{2A},$$

Where

$$A = [G(t) - G(1)]^2, \;\; B = |f(t)|G(1)\sqrt{2G(t)}, \;\text{ and } C = \|f'\|_{L^\infty([-1,t])} \cdot [G(1)]^2.$$

In particular, we have

$$0 < \mu \leq \sqrt{\|f'\|_{L^\infty([-1,t_0])}} \cdot \frac{G(1)}{G(t_0) - G(1)}.$$

Proof. Multiply $\phi''(x)$ on the both sides of (3.1.1) and integrate over $(-\infty, 0)$, use the integration by parts, then

$$\begin{aligned}
\int_{-\infty}^{0} [\phi''(x)]^2 \; dx &= -\mu \int_{-\infty}^{0} \phi''(x)\phi'(x) \; dx - \int_{-\infty}^{0} f(\phi(x))\phi''(x) \; dx \\
&= -\frac{\mu}{2}|\phi'(x)|^2\Big|_{-\infty}^{0} - \int_{-\infty}^{0} f(\phi(x))\phi''(x) \; dx \\
&= -\frac{\mu}{2}[\phi'(0)]^2 - \int_{-\infty}^{0} f(\phi(x))\phi''(x) \; dx \\
&< -\int_{-\infty}^{0} f(\phi(x))\phi''(x) \; dx, \quad \text{Since } \mu, \phi'(0) > 0.
\end{aligned}$$

By the proof of Proposition 3.1.3, we have

$$-\int_{-\infty}^{0} f(\phi(x))\phi''(x) \; dx \leq |f(t)|\sqrt{2G(t)} + \|f'\|_{L^\infty([-1,t])} \cdot \frac{G(1)}{\mu}.$$

So we get

$$\int_{-\infty}^{0} [\phi''(x)]^2 \, dx \leq |f(t)|\sqrt{2G(t)} + \|f'\|_{L^\infty([-1,t])} \cdot \frac{G(1)}{\mu}.$$

By (3.1.5), use the Cauchy-Schwarz's inequality, we get

$$\begin{aligned}
G(t) - G(1) &\leq \frac{1}{2}[\phi'(0)]^2 \\
&= -\int_{-\infty}^{0} \phi''(x)\phi'(x) \, dx \quad \text{Since } \lim_{x \to -\infty} \phi'(x) = 0 \\
&\leq \left[\int_{-\infty}^{0} [\phi''(x)]^2 \, dx\right]^{\frac{1}{2}} \left[\int_{-\infty}^{0} [\phi'(x)]^2 \, dx\right]^{\frac{1}{2}} \\
&\leq \left[\int_{-\infty}^{0} [\phi''(x)]^2 \, dx\right]^{\frac{1}{2}} \left[\int_{\mathbb{R}} [\phi'(x)]^2 \, dx\right]^{\frac{1}{2}} \\
&\leq \left[|f(t)|\sqrt{2G(t)} + \|f'\|_{L^\infty([-1,t])} \cdot \frac{G(1)}{\mu}\right]^{\frac{1}{2}} \left[\frac{G(1)}{\mu}\right]^{\frac{1}{2}}, \quad \text{By (3.1.3).}
\end{aligned}$$

Since $\mu > 0$, we get

$$[G(t) - G(1)]\mu^2 - |f(t)|G(1)\sqrt{2G(t)}]\mu - \|f'\|_{L^\infty([-1,t])} \cdot [G(1)]^2 \leq 0.$$

Let $A = [G(t) - G(1)]^2$, $B = |f(t)|G(1)\sqrt{2G(t)}$ and $C = \|f'\|_{L^\infty([-1,t])} \cdot [G(1)]^2$, then we have $A\mu^2 - B\mu - C \leq 0$, which implies that

$$\mu \leq \frac{B + \sqrt{B^2 + 4AC}}{2A}.$$

□

Remark 3.1.5. *The approaches of Proposition 3.1.3 and Proposition 3.1.4 involve* *the Hamiltonian identity (3.1.5) Modica type estimate (3.1.4),. These two approaches will be used in Section 3.2 to estimate the speeds with fractional Laplacians.*

Remark 3.1.6. *For the lower bound of μ, by the Hamiltonian identity (3.1.3) and Modica type estimate (3.1.4), since $\phi'(x) > 0$ in $\mathbb{R}$ and $\lim_{x\to\pm\infty} \phi(x) = \pm 1$, we have*

$$\mu = \frac{G(1)}{\int_{\mathbb{R}} |\phi'(x)|^2 dx} \geq \frac{G(1)}{\int_{\mathbb{R}} 2\sqrt{G(\phi(x))} \cdot \phi'(x)\ dx} = \frac{G(1)}{2\int_{-1}^{1} \sqrt{G(t)}\ dt}.$$

3.2 Fractional Laplacian Case

By the discussion for the Laplacian case in Section 3.1, in this section, we can assume that f is a unbalanced bistable nonlinearity such that $-\int_{-1}^{1} f(t)\ dt > 0$. So we only need to consider the following problem:

$$\begin{cases} (-\Delta)^s u(x) - \mu u'(x) = f(u(x)), \quad \forall x \in \mathbb{R}, \\ u'(x) > 0, \quad \forall x \in \mathbb{R}, \\ \lim_{x\to\pm\infty} u(x) = \pm 1, \quad \mu > 0. \end{cases} \tag{3.2.1}$$

Since $\mu > 0$, by (2.3.2), we have $G(1) > 0$. Since f satisfies (1.1.3), we know that G is strictly increasing on $[-1, t_0]$, and strictly decreasing on $[t_0, 1]$, in particular, there exists some $t_1 \in (-1, t_0)$ such that

$$0 < G(1) < G(t), \quad \forall t \in (t_1, 1), \quad \text{and} \quad G(t_0) = \max_{t\in[-1,1]} G(t).$$

For any $t \in (t_1, 1)$, by translation, without of generality, we can assume $u(0) = t$. Multiply $u'(x)$ on the first equation of (3.2.1) and integrate over $(-\infty, 0)$, use the integration by parts, we have

$$\begin{aligned}
-\int_{-\infty}^{0}(-\Delta)^s u(x)u'(x)\ dx &= -\int_{-\infty}^{0} f(u(x))u'(x)\ dx - \mu\int_{-\infty}^{0}|u'(x)|^2\ dx \\
&= G(u(x))|_{-\infty}^{0} - \mu\int_{-\infty}^{0}|u'(x)|^2\ dx \\
&= G(t) - \mu\int_{-\infty}^{0}|u'(x)|^2\ dx \\
&\geq G(t) - \mu\int_{\mathbb{R}}|u'(x)|^2\ dx.
\end{aligned}$$

By Proposition 2.3.2, we have

$$G(t) - G(1) \leq -\int_{-\infty}^{0}(-\Delta)^s u(x)u'(x)\ dx. \tag{3.2.2}$$

Remark 3.2.1. *The formula (3.2.2) is the counterpart of (3.1.5) in the Laplacian case. Notice that in the Laplacian case, we have*

$$-\int_{-\infty}^{0}[-\phi''(x)]\phi'(x)\ dx = \frac{1}{2}[\phi'(0)]^2.$$

Now we are ready to estimate the speed, but we need to separate to three parts: $0 < s < 1/2$ (supercritcal), $s = 1/2$ (critical) and $1/2 < s < 1$ (subcritical).

Proposition 3.2.2. *Let $0 < s < \frac{1}{2}$, then for all $t \in (t_1, 1)$, we have*

$$G(t) - G(1) \leq \frac{2C_{1,s}}{s}R^{-2s}[t+1] + \frac{2C_{1,s}}{1-2s}\cdot R^{1-2s}\cdot\frac{G(1)}{\mu}, \quad \forall R > 0.$$

In particular, we know that

$$0 < \mu \leq \left[\frac{4C_{1,s}}{1-2s} \cdot \left(\frac{t}{s} \right)^{1-2s} \cdot \frac{1}{G(t) - G(1)} \right]^{\frac{1}{2s}} \cdot G(1).$$

Proof. For any $R > 0$, then

$$\begin{aligned}
-(-\Delta)^s u(x) &= -C_{1,s} \text{P.V.} \int_{\mathbb{R}} \frac{u(x) - u(y)}{|x-y|^{1+2s}} \, dy \\
&= -C_{1,s} \int_{\mathbb{R}} \frac{u(x) - u(y)}{|x-y|^{1+2s}} \, dy \quad \text{Since } 0 < s < \frac{1}{2} \\
&= C_{1,s} \int_{\mathbb{R}} \frac{u(x+z) - u(x)}{|z|^{1+2s}} \, dz \qquad \text{Let } y = x + z \\
&= C_{1,s} \int_{|z|<R} \frac{u(x+z) - u(x)}{|z|^{1+2s}} \, dz + C_{1,s} \int_{|z| \geq R} \frac{u(x+z) - u(x)}{|z|^{1+2s}} \, dz
\end{aligned}$$

For $C_{1,s} \int_{|z| \geq R} \frac{u(x+z) - u(x)}{|z|^{1+2s}} \, dz$, since $|u(x)| \leq 1$ in $\mathbb{R}$, then

$$\begin{aligned}
C_{1,s} \int_{|z| \geq R} \frac{u(x+z) - u(x)}{|z|^{1+2s}} \, dz &\leq 2C_{1,s} \int_{|z| \geq R} \frac{1}{|z|^{1+2s}} \, dz \\
&= 4C_{1,s} \int_R^{\infty} \frac{1}{z^{1+2s}} \, dz \\
&= \frac{2C_{1,s}}{s} R^{-2s}.
\end{aligned}$$

For $C_{1,s} \int_{|z|<R} \frac{u(x+z) - u(x)}{|z|^{1+2s}} \, dz$, since $u'(x) > 0$ in $\mathbb{R}$, then

$$\begin{aligned}
C_{1,s} \int_{|z|<R} \frac{u(x+z) - u(x)}{|z|^{1+2s}} \, dz &= C_{1,s} \int_{|z|<R} \frac{\int_0^1 u'(x+tz)z \, dt}{|z|^{1+2s}} \, dz \\
&\leq C_{1,s} \int_0^1 \int_{|z|<R} u'(x+tz) \cdot \frac{1}{|z|^{2s}} \, dz dt
\end{aligned}$$

Hence we know that

$$-(-\Delta)^s u(x) \leq \frac{2C_{1,s}}{s} R^{-2s} + C_{1,s} \int_0^1 \int_{|z|<R} u'(x+tz) \cdot \frac{1}{|z|^{2s}} \, dzdt, \quad \forall x \in \mathbb{R}$$

Since $u'(x) > 0$ in $\mathbb{R}$, multiply $u'(x)$ on the both sides of the above inequality, and integrate over $(-\infty, 0)$, we get

$$\begin{aligned} & -\int_{-\infty}^0 (-\Delta)^s u(x) u'(x) \, dx \\ \leq & \int_{-\infty}^0 \left[\frac{2C_{1,s}}{s} R^{-2s} + C_{1,s} \int_0^1 \int_{|z|<R} u'(x+tz) \cdot \frac{1}{|z|^{2s}} \, dzdt \right] \cdot u'(x) \, dx \\ = & \frac{2C_{1,s}}{s} R^{-2s} [t+1] + C_{1,s} \int_{|z|<R} \frac{1}{|z|^{2s}} \left[\int_0^1 \int_{-\infty}^0 u'(x+tz) \cdot u'(x) \, dxdt \right] \, dz \end{aligned}$$

For $\int_{-\infty}^0 u'(x+tz) \cdot u'(x) \, dx$, by Cauchy-Schwarz's inequality, then

$$\begin{aligned} \int_{-\infty}^0 u'(x+tz) \cdot u'(x) \, dx & \leq \left[\int_{-\infty}^0 [u'(x+tz)]^2 \, dx \right]^{\frac{1}{2}} \cdot \left[\int_{-\infty}^0 [u'(x)]^2 \, dx \right]^{\frac{1}{2}} \\ & \leq \left[\int_{\mathbb{R}} [u'(x+tz)]^2 \, dx \right]^{\frac{1}{2}} \cdot \left[\int_{\mathbb{R}} [u'(x)]^2 \, dx \right]^{\frac{1}{2}} \\ & = \int_{\mathbb{R}} [u'(x)]^2 \, dx \\ & = \frac{G(1)}{\mu}, \quad \text{By Proposition 2.3.2.} \end{aligned}$$

So we get

$$\begin{aligned} & C_{1,s} \int_{|z|<R} \frac{1}{|z|^{2s}} \left[\int_0^1 \int_{-\infty}^0 u'(x+tz) \cdot u'(x) \, dxdt \right] \, dz \\ \leq & C_{1,s} \int_{|z|<R} \frac{1}{|z|^{2s}} \left[\int_0^1 \frac{G(1)}{\mu} \, dt \right] \, dz \end{aligned}$$

$$\begin{aligned} &= 2C_{1,s} \cdot \frac{G(1)}{\mu} \int_0^R \frac{1}{z^{2s}} \, dz \\ &= \frac{2C_{1,s}}{1-2s} \cdot R^{1-2s} \cdot \frac{G(1)}{\mu}, \quad \text{Since } 2s < 1. \end{aligned}$$

In summary, we get

$$-\int_{-\infty}^{0} (-\Delta)^s u(x) u'(x) \, dx \leq \frac{2C_{1,s}}{s} R^{-2s}[t+1] + \frac{2C_{1,s}}{1-2s} \cdot R^{1-2s} \cdot \frac{G(1)}{\mu}, \quad \forall R > 0$$

By (3.2.2), we get

$$\begin{aligned} G(t) - G(1) &\leq -\int_{-\infty}^{0} (-\Delta)^s u(x) u'(x) \, dx \\ &\leq \frac{2C_{1,s}}{s} R^{-2s}[t+1] + \frac{2C_{1,s}}{1-2s} \cdot R^{1-2s} \cdot \frac{G(1)}{\mu}, \quad \forall R > 0. \end{aligned}$$

Let's take $\frac{2C_{1,s}}{s} R^{-2s}[t+1] = \frac{2C_{1,s}}{1-2s} \cdot R^{1-2s} \cdot \frac{G(1)}{\mu}$, that is, $R = \frac{\mu[t+1]}{sG(1)}$, then we obtain

$$\begin{aligned} G(t) - G(1) &\leq \frac{4C_{1,s}}{1-2s} \cdot R^{1-2s} \cdot \frac{G(1)}{\mu} \\ &= \frac{4C_{1,s}}{1-2s} \cdot \left(\frac{\mu[t+1]}{sG(1)} \right)^{1-2s} \cdot \frac{G(1)}{\mu} \\ &= \frac{4C_{1,s}}{1-2s} \cdot \left(\frac{t+1}{s} \right)^{1-2s} \cdot \left(\frac{\mu}{G(1)} \right)^{-2s}, \end{aligned}$$

which implies that

$$\mu \leq \left[\frac{4C_{1,s}}{1-2s} \cdot \left(\frac{t+1}{s} \right)^{1-2s} \cdot \frac{1}{G(t) - G(1)} \right]^{\frac{1}{2s}} \cdot G(1).$$

□

Remark 3.2.3. *When $0 < s < 1/2$, the point-wise estimate of $(-\Delta)^s u(x)$ is the most important part in the proof of Proposition 3.2.2*

For the case $s = 1/2$, the following lemma tells us the energies of $\overline{u}_x$ and $\overline{u}_y$ in x direction are the same, which is crucial in the estimate of speed μ.

Lemma 3.2.4. *Let $s = \frac{1}{2}$, and $\overline{u}$ be the 1/2-harmonic extension in $\mathbb{R}^2_+$ of u. Then we have*

$$\int_{\mathbb{R}} [\overline{u}_x(x,y)]^2 \, dx = \int_{\mathbb{R}} [\overline{u}_y(x,y)]^2 \, dx, \quad \forall y \geq 0.$$

Proof. Since $-\overline{u}_y(x,0) = (-\Delta)^{\frac{1}{2}} u(x) = \mu u'(x) + f(u(x))$ in $\mathbb{R}$, $u' \in L^2(\mathbb{R})$ and Remark 2.2.3, we know that $\overline{u}_y(\cdot,0) \in L^2(\mathbb{R})$. Since $\overline{u}$ is harmonic in $\mathbb{R}^2_+$, then $\overline{u}_y(\cdot,y) \in L^2(\mathbb{R})$ for all $y \geq 0$. In particular, we can consider the function

$$\psi(y) = \int_{\mathbb{R}} [\overline{u}_y^2(x,y) - \overline{u}_x^2(x,y)] dx, \quad \forall y \geq 0.$$

Differentiating ψ and using integration by parts, we get

$$\begin{aligned} \psi'(y) &= 2\int_{\mathbb{R}} \overline{u}_y(x,y)\overline{u}_{yy}(x,y) \, dx - 2\int_{\mathbb{R}} \overline{u}_x(x,y)\overline{u}_{xy}(x,y) \, dx \\ &= 2\int_{\mathbb{R}} \overline{u}_y(x,y)\overline{u}_{yy}(x,y) \, dx + 2\int_{\mathbb{R}} \overline{u}_y(x,y)\overline{u}_{xx}(x,y) \, dx \\ &= 2\int_{\mathbb{R}} \overline{u}_y(x,y)\Delta\overline{u}(x,y) \, dx \\ &= 0, \quad \forall y \geq 0. \end{aligned}$$

<u>Claim I</u>: $\lim_{y\to\infty} \int_{\mathbb{R}} |\nabla \overline{u}(x,y)|^2 \, dx = 0.$

Let $P(x,y) = P_{1,\frac{1}{2}}(x,y) = \frac{1}{\pi} \cdot \frac{y}{x^2+y^2}$ in $\mathbb{R}^2_+$, we get

$$\begin{aligned} \|P(\cdot,y)\|^2_{L^2(\mathbb{R})} &= \frac{1}{\pi^2}\int_{\mathbb{R}} \frac{y^2}{[x^2+y^2]^2}\,dx \\ &= \frac{1}{2\pi y} \to 0, \quad \text{as } x \to \infty. \end{aligned}$$

By Young's inequality, we have

$$\begin{aligned} \|\overline{u}_x(\cdot,y)\|_{L^2(\mathbb{R})} &= \|P(\cdot,y) * u'\|_{L^2(\mathbb{R})} \\ &\le \|P(\cdot,y)\|_{L^2(\mathbb{R})} \cdot \|u'\|_{L^1(\mathbb{R})} \\ &\to 0, \quad \text{as } y \to \infty. \end{aligned}$$

Look at $\overline{u}_y(x,y)$, by Proposition 2.2.2, we know that there exists some constant $C_1 > 0$ such that

$$|\overline{u}_y(x,0)| = |(-\Delta)^{\frac{1}{2}}u(x)| = |\mu u'(x) + f(u(x))| \le \frac{C_1}{1+|x|}, \quad \forall x \in \mathbb{R}.$$

Since $\overline{u}_y$ is harmonic in $\mathbb{R}^2_+$, so there exists some constant $C > C_1 > 0$ such that

$$|\overline{u}_y(x,y)| \le \frac{C}{1+y}, \quad \forall (x,y) \in \mathbb{R}^2_+.$$

Since $\frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$, we know that $\frac{x}{x^2+y^2}$ and $\frac{y}{x^2+y^2}$ are harmonic in $\mathbb{R}^2_+$. Let $v(x,y) = \frac{C(x+y+1)}{x^2+(y+1)^2}$, we know that v is harmonic in $\mathbb{R}^2$ and $\pm\overline{u}_y(x,0) \le \frac{C_1}{1+x} \le \frac{C(1+x)}{1+x^2} = v(x,0)$ for all $x \ge 0$ and $\pm\overline{u}_y(0,y) \le \frac{C}{1+y} = v(0,y)$ for all

$x \geq 0$. By the weak maximum principle, we know that

$$\pm\overline{u}_y(x,y) \leq v(x,y) = \frac{C(x+y+1)}{x^2+(y+1)^2}, \quad \forall x, y \geq 0.$$

Let $w(x,y) = \dfrac{C(y+1-x)}{x^2+(y+1)^2}$, we get w is harmonic in $\mathbb{R}^2$ and $\pm\overline{u}_y(x,0) \leq \dfrac{C_1}{1-x} \leq \dfrac{C(1-x)}{1+x^2} = w(x,0)$ for all $x \leq 0$ and $\pm\overline{u}_y(0,y) \leq \dfrac{C}{1+y} = w(0,y)$ for all $y \geq 0$. By the weak maximum principle, we know that

$$\pm\overline{u}_y(x,y) \leq w(x,y) = \frac{C(y+1-x)}{x^2+(y+1)^2}, \quad \forall y \geq 0,\ \forall x \leq 0.$$

In summary, we have

$$|\overline{u}_y(x,y)| \leq \frac{C(y+1+|x|)}{x^2+(y+1)^2} \leq \frac{C}{|(x,y+1)|}, \quad \forall (x,y) \in \mathbb{R}^2_+.$$

Notice that

$$\int_{\mathbb{R}} \frac{1}{|(x,y+1)|^2}\, dx \leq \int_{\mathbb{R}} \frac{1}{x^2+y^2}\, dx = \frac{\pi}{y} \to 0, \quad \text{as } y \to \infty.$$

So we know that $\|\overline{u}_y(\cdot,y)\|_{L^2(\mathbb{R})} \to 0$, as $y \to \infty$. In summary, we have

$$\lim_{y\to\infty} \int_{\mathbb{R}} |\nabla \overline{u}(x,y)|^2\, dx = 0.$$

By Claim I, we get $\psi(y) \equiv 0$ in $\mathbb{R}^+$, which completes the proof.

□

Remark 3.2.5. *Lemma 3.2.4 tells us that* $\int_{\mathbb{R}}[(-\Delta)^{\frac{1}{2}}u(x)\, dx$ *is controlled by* $\int_{\mathbb{R}} |u'(x)|^2\, dx$*,* *in fact, they are equal.*

Proposition 3.2.6. *Let $s = \frac{1}{2}$, then for any $t \in (t_1, 1)$, we have*

$$0 < \mu \le \frac{G(1)}{G(t) - G(1)} \le \frac{G(1)}{G(t_0) - G(1)}.$$

Proof. Let $\overline{u}$ be the 1/2-harmonic extension of u, by Cauchy-Schwarz's inequality, then we know that

$$\begin{aligned}
-\int_{-\infty}^{0} (-\Delta)^{\frac{1}{2}} u(x)u'(x)\ dx &= \int_{-\infty}^{0} \overline{u}_y(x,0)\cdot u'(x)\ dx \\
&\le \left[\int_{-\infty}^{0} [\overline{u}_y(x,0)]^2\ dx\right]^{\frac{1}{2}} \cdot \left[\int_{-\infty}^{0} [u'(x)]^2\ dx\right]^{\frac{1}{2}} \\
&\le \left[\int_{\mathbb{R}} [\overline{u}_y(x,0)]^2\ dx\right]^{\frac{1}{2}} \cdot \left[\int_{\mathbb{R}} [u'(x)]^2\ dx\right]^{\frac{1}{2}} \\
&= \left[\int_{\mathbb{R}} [\overline{u}_x(x,0)]^2\ dx\right]^{\frac{1}{2}} \cdot \left[\int_{\mathbb{R}} [u'(x)]^2\ dx\right]^{\frac{1}{2}} \quad \text{By Lemma 3.2.4} \\
&= \int_{\mathbb{R}} [u'(x)]^2\ dx \\
&= \frac{G(1)}{\mu}, \quad \text{By Proposition 2.3.2.}
\end{aligned}$$

So we get

$$\begin{aligned}
\mu &\le \frac{G(1)}{-\int_{-\infty}^{0} (-\Delta)^{\frac{1}{2}} u(x)u'(x)\ dx} \\
&\le \frac{G(1)}{G(t) - G(1)}, \quad \text{By (3.2.2).}
\end{aligned}$$

□

Proposition 3.2.7. *Let $\frac{1}{2} < s < 1$, then for any $R > 0$, we have*

$$\mu \leq \frac{B + \sqrt{B^2 + 4AC}}{2A}$$

where

$$A = G(t_0) - G(1), \;\; B = \frac{4C_{1,s}\|f\|_{C([-1,1])}}{2s-1} \cdot R^{1-2s}, \text{ and } C = \frac{G(1)C_{1,s}\|f'\|_{C([-1,1])}}{1-s} \cdot R^{2-2s}.$$

Moreover, we have

$$0 < \mu \leq \left[\frac{\left[\|f\|_{C^1([-1,1])} \cdot \frac{8C_{1,s}}{2s-1}\right]^{\frac{1}{2}} \cdot \left(\frac{1-s}{2s-1}\right)^{\frac{1-2s}{2}}}{G(t_0) - F(1)} \right]^{\frac{1}{s}} \cdot G(1).$$

Proof. Multiply $(-\Delta)^s u(x)$ on the both sides of (3.2.1) and integrate over $(-\infty, 0)$, we have

$$\int_{-\infty}^0 [(-\Delta)^s u(x)]^2 \, dx = \mu \int_{-\infty}^0 (-\Delta)^s u(x) u'(x) \, dx + \int_{-\infty}^0 f(u(x))(-\Delta)^s u(x) \, dx.$$

Since $u(0) = t_0$ and $\mu > 0$, by (3.2.2), we have

$$\begin{aligned} 0 < \mu[G(t_0) - G(1)] &\leq -\mu \int_{-\infty}^0 (-\Delta)^s u(x) u'(x) \, dx \\ &= \int_{-\infty}^0 [-(-\Delta)^s u(x) + f(u(x))](-\Delta)^s u(x) \, dx \\ &= \int_{-\infty}^0 f(u(x))(-\Delta)^s u(x) \, dx - \int_{-\infty}^0 [(-\Delta)^s u(x)]^2 \, dx \\ &\leq \int_{-\infty}^0 f(u(x))(-\Delta)^s u(x) \, dx \end{aligned}$$

In particular, we know that

$$\int_{-\infty}^{0} [(-\Delta)^s u(x)]^2 \; dx < \int_{-\infty}^{0} f(u(x))(-\Delta)^s u(x) \; dx.$$

Now let us estimate $\int_{-\infty}^{0} f(u(x))(-\Delta)^s u(x) \; dx$, for any $R > 0$, we have

$$\begin{aligned} \int_{-\infty}^{0} f(u(x))(-\Delta)^s u(x) \; dx &= C_{1,s} \Bigg[\int_{-\infty}^{0} f(u(x)) \int_{|x-y|\geq R} \frac{u(x)-u(y)}{|x-y|^{1+2s}} \; dydx \\ &\quad + \int_{-\infty}^{0} f(u(x)) \text{ P.V. } \int_{|x-y|\leq R} \frac{u(x)-u(y)}{|x-y|^{1+2s}} \; dydx \Bigg]. \end{aligned}$$

For $\int_{-\infty}^{0} f(\phi(x)) \int_{|x-y|\geq R} \frac{u(x)-u(y)}{|x-y|^{1+2s}} \; dydx$, we have

$$\int_{-\infty}^{0} f(u(x)) \int_{|x-y|\geq R} \frac{u(x)-u(y)}{|x-y|^{1+2s}} \; dydx$$

$$\begin{aligned} &= \int_{-\infty}^{0} f(u(x)) \int_{|z|\geq R} \frac{u(x)-u(x+z)}{|z|^{1+2s}} \; dzdx \quad \text{Let } z = y - x \\ &= \int_{|z|\geq R} \int_{-\infty}^{0} \frac{f(u(x))[u(x)-u(x+z)]}{|z|^{1+2s}} \; dxdz \\ &= -\int_{|z|\geq R} \int_{-\infty}^{0} \int_{0}^{1} \frac{f(u(x))u'(x+tz)z}{|z|^{1+2s}} \; dtdxdz \\ &\leq \|f\|_{C([-1,1])} \int_{|z|\geq R} |z|^{-2s} \left[\int_{0}^{1} \int_{-\infty}^{0} u'(x+tz) \; dxdt \right] \; dz \\ &\leq \|f\|_{C([-1,1])} \int_{|z|\geq R} |z|^{-2s} \left[\int_{0}^{1} \int_{\mathbb{R}} u'(x+tz) \; dxdt \right] \; dz \\ &= 2\|f\|_{C([-1,1])} \int_{|z|\geq R} |z|^{-2s} \; dz \quad \text{Since } \lim_{x\to\pm\infty} u(x) = \pm 1 \\ &= \frac{4\|f\|_{C([-1,1])}}{2s-1} \cdot R^{1-2s}, \quad \text{Since } 2s > 1. \end{aligned}$$

For $\int_{-\infty}^{0} f(u(x))$ P.V. $\int_{|x-y|\le R} \frac{u(x)-u(y)}{|x-y|^{1+2s}}\, dydx$, then

$$\begin{aligned}
&\int_{-\infty}^{0} f(u(x)) \text{ P.V. } \int_{|x-y|\le R} \frac{u(x)-u(y)}{|x-y|^{1+2s}}\, dydx \\
= &\int_{-\infty}^{0} f(u(x)) \text{ P.V. } \int_{|z|\le R} \frac{u(x)-u(x+z)}{|z|^{1+2s}}\, dzdx \quad \text{Let } z=y-x \\
= &-\int_{-\infty}^{0} f(u(x)) \int_{|z|<R} \frac{u(x+z)-u(x)-u'(z)z}{|z|^{1+2s}}\, dzdx \\
= &-\int_{-\infty}^{0} f(u(x)) \int_{|z|<R} \int_{0}^{1} \frac{u'(x+tz)-u'(x)}{|z|^{1+2s}} \cdot z\ dtdzdx \\
= &-\int_{-\infty}^{0} f(u(x)) \int_{|z|<R} \int_{0}^{1} \int_{0}^{1} \frac{u'(x+rtz)}{|z|^{1+2s}} \cdot tz \cdot z\ drdtdzdx \\
= &-\int_{-\infty}^{0} f(u(x)) \int_{|z|<R} \int_{0}^{1} \int_{0}^{1} \frac{u''(x+rtz)}{|z|^{1+2s}} \cdot tz^2\ drdtdzdx \\
= &-\int_{|z|<R} \int_{0}^{1} \int_{0}^{1} t|z|^{1-2s} \int_{-\infty}^{0} f(u(x))u''(x+rtz)dxdrdtdz.
\end{aligned}$$

For $\int_{-\infty}^{0} f(u(x))u''(x+rtz)dx$,since $\lim_{x\to-\infty} f(u(x)) = f(-1) = 0$, use integration by parts, then

$$\begin{aligned}
&-\int_{-\infty}^{0} f(u(x))u''(x+rtz)dx \\
= &-f(u(x))u'(x+rtz)|_{-\infty}^{0} + \int_{-\infty}^{0} f'(u(x))u'(x)u'(x+rtz)\ dx \\
= &-f(u(0))u'(rtz) + \int_{-\infty}^{0} f'(u(x))u'(x)u'(x+rtz)\ dx \\
= &-f(t_0)u'(rtz) + \int_{-\infty}^{0} f'(u(x))u'(x)u'(x+rtz)\ dx \\
= &\int_{-\infty}^{0} f'(u(x))u'(x)u'(x+rtz)\ dx, \quad \text{Since } f(t_0)=0.
\end{aligned}$$

So we get

$$
\begin{aligned}
&\int_{-\infty}^{0} f(u(x)) \text{ P.V. } \int_{|x-y|\leq R} \frac{u(x)-u(y)}{|x-y|^{1+2s}} \, dydx \\
\leq\ & -\int_{|z|<R}\int_0^1\int_0^1 t|z|^{1-2s}\int_{-\infty}^{0} f'(u(x))u'(x)u'(x+rtz) \, dxdrdtdz.
\end{aligned}
$$

By Cauchy-Schwarz's inequality, we get

$$
\begin{aligned}
&\int_{-\infty}^{0} f(u(x)) \text{ P.V. } \int_{|x-y|\leq R} \frac{u(x)-u(y)}{|x-y|^{1+2s}} \, dydx \\
=\ & \int_{|z|<R}\int_0^1\int_0^1 t|z|^{1-2s}\int_{-\infty}^{0} f'(u(x))\phi'(x)u'(x+rtz) \, dxdrdtdz \\
\leq\ & \|f'\|_{C([-1,1])}\int_{|z|<R}\int_0^1\int_0^1 |z|^{1-2s}\int_{\mathbb{R}} u'(x)u'(x+rtz)dxdrdtdz \\
\leq\ & \|f'\|_{C([-1,1])}\int_{|z|<R}\int_0^1\int_0^1 |z|^{1-2s}\left[\int_{\mathbb{R}}[u'(x)]^2 \, dx\right]^{\frac{1}{2}}\cdot\left[\int_{\mathbb{R}}[u'(x+rtz)]^2 \, dx\right]^{\frac{1}{2}} \, drdtdz
\end{aligned}
$$

$$
\begin{aligned}
=\ & \|f'\|_{C([-1,1])}\int_{|z|<R} |z|^{1-2s} \, dz\cdot\|u'\|^2_{L^2(\mathbb{R})} \\
\leq\ & \frac{\|f'\|_{C([-1,1])}}{1-s}\cdot R^{2-2s}\cdot\frac{G(1)}{\mu}, \quad \text{By Proposition 2.3.2.}
\end{aligned}
$$

In summary, we get

$$
\mu[G(t_0)-G(1)] \leq C_{1,s}\cdot\left[\frac{4\|f\|_{C([-1,1])}}{2s-1}\cdot R^{1-2s}+\frac{\|f'\|_{C([-1,1])}}{1-s}\cdot R^{2-2s}\cdot\frac{G(1)}{\mu}\right].
$$

Let

$$
A = G(t_0)-G(1), \ B = \frac{4C_{1,s}\|f\|_{C([-1,1])}}{2s-1}R^{1-2s}, \ \text{and } C = \frac{C_{1,s}G(1)\|f'\|_{C([-1,1])}}{1-s}R^{2-2s}.
$$

Then we have $\mu A \leq B + \dfrac{C}{\mu}$, which implies that

$$\mu \leq \frac{B + \sqrt{B^2 + 4AC}}{2A}.$$

By the previous proof, we know that

$$\begin{aligned}
& \int_{-\infty}^{0} [(-\Delta)^s u(x)]^2 \; dx \\
\leq \; & \int_{-\infty}^{0} f(u(x))(-\Delta)^s u(x) \; dx \\
\leq \; & C_{1,s} \cdot \left[\frac{4\|f\|_{C([-1,1])}}{2s-1} R^{1-2s} + \frac{\|f'\|_{C([-1,1])}}{1-s} R^{2-2s} \cdot \frac{G(1)}{\mu} \right] \\
= \; & \|f\|_{C^1([-1,1])} \cdot \frac{4C_{1,s}}{2s-1} \cdot \left[R^{1-2s} + \frac{2s-1}{1-s} R^{2-2s} \cdot \frac{G(1)}{\mu} \right].
\end{aligned}$$

Let $R^{1-2s} = \dfrac{2s-1}{1-s} R^{2-2s} \cdot \dfrac{G(1)}{\mu}$, that is, $R = \dfrac{1-s}{2s-1} \cdot \dfrac{\mu}{G(1)}$, then

$$\begin{aligned}
\int_{-\infty}^{0} [(-\Delta)^s \phi(x)]^2 \; dx \;\; & \leq \;\; 2\|f\|_{C^1([-1,1])} \cdot \frac{4C_{1,s}}{2s-1} \cdot R^{1-2s} \\
& = \;\; \|f\|_{C^1([-1,1])} \cdot \frac{8C_{1,s}}{2s-1} \cdot \left(\frac{1-s}{2s-1} \cdot \frac{\mu}{G(1)} \right)^{1-2s}.
\end{aligned}$$

By (3.2.2), use the Cauchy-Schwarz's inequality, then

$$\begin{aligned}
G(t_0) - G(1) \;\; & \leq \;\; \left[\int_{-\infty}^{0} [(-\Delta)^s u(x)]^2 \; dx \right]^{\frac{1}{2}} \cdot \left[\int_{-\infty}^{0} [u'(x)]^2 \; dx \right]^{\frac{1}{2}} \\
& \leq \;\; \left[\|f\|_{C^1([-1,1])} \cdot \frac{8C_{1,s}}{2s-1} \cdot \left(\frac{1-s}{2s-1} \cdot \frac{\mu}{G(1)} \right)^{1-2s} \right]^{\frac{1}{2}} \cdot \left[\int_{\mathbb{R}} [u'(x)]^2 \; dx \right]^{\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
&= \left[\|f\|_{C^1([-1,1])}\cdot\frac{8C_{1,s}}{2s-1}\right]^{\frac{1}{2}}\cdot\left(\frac{1-s}{2s-1}\cdot\frac{\mu}{G(1)}\right)^{\frac{1-2s}{2}}\cdot\sqrt{\frac{G(1)}{\mu}} \\
&\qquad\qquad \text{By Proposition 2.3.2} \\
&= \left[\|f\|_{C^1([-1,1])}\cdot\frac{8C_{1,s}}{2s-1}\right]^{\frac{1}{2}}\cdot\left(\frac{1-s}{2s-1}\right)^{1-2s}\cdot\left(\frac{G(1)}{\mu}\right)^{\frac{2s-1}{2}+\frac{1}{2}} \\
&= \left[\|f\|_{C^1([-1,1])}\cdot\frac{8C_{1,s}}{2s-1}\right]^{\frac{1}{2}}\cdot\left(\frac{1-s}{2s-1}\right)^{\frac{1-2s}{2}}\cdot\left(\frac{G(1)}{\mu}\right)^{s},
\end{aligned}
$$

which implies that

$$
0<\mu\leq\left[\frac{\left[\|f\|_{C^1([-1,1])}\cdot\frac{8C_{1,s}}{2s-1}\right]^{\frac{1}{2}}\cdot\left(\frac{1-s}{2s-1}\right)^{\frac{1-2s}{2}}}{G(t_0)-G(1)}\right]^{\frac{1}{s}}\cdot G(1).
$$

□

Remark 3.2.8. *The key part in the proof of Proposition 3.2.7 is the estimate of* $\int_{-\infty}^{0}[(-\Delta)^s u(x)]^2\,dx$*, this approach is similar to Proposition 3.1.4. It's interesting to know whether we can get a similar estimate in terms of* $t\in(t_1,1)$ *not only* t_0*, because we use* $f(t_0)=0$*, but* $f(t)\neq 0$ *other* $t\in(t_1,1)$*.*

Remark 3.2.9. *All proofs in Proposition 3.2.2, Proposition 3.2.6 and Proposition 3.2.7 only use the Hamiltonian identity in Proposition 2.3.2.*

By Proposition 3.2.2, Proposition 3.2.6 and Proposition 3.2.7, we have the following summary.

Theorem 3.2.10. *For any* $0<s<1$*, there exists a constant* $C>0$ *which only depends on* s*, the upper bound of* $\|f\|_{C^1([-1,1])}$ *and the positive lower bound of* $G(t_0)-G(1)$ *such that*

$$0\leq\mu\leq CG(1).$$

Remark 3.2.11. *For the lower bound of speed μ, by Theorem 3.2.10 and the regularity theory for the fractional Laplacians, we know that there exists some constant $C >$ 0 such that $\|u\|_{C^1(\mathbb{R})} \leq C$. By the Hamiltonian identity, Proposition 2.3.2, since $\lim_{x \to \pm\infty} u(x) = \pm 1$, we know that*

$$\mu = \frac{G(1)}{\int_{\mathbb{R}} |u'(x)|^2 \ dx} \geq \frac{G(1)}{C \int_{\mathbb{R}} u'(x) \ dx} = \frac{G(1)}{2C}.$$

Chapter 4

Existence of Traveling Wave Solution

In this chapter, we use a continuation argument to prove our main theorem, Theorem 4.2.1, which shows the existence of traveling wave solution to all bistable nonlinearity (balanced or unbalanced). By using the estimate in Chapter 3, we will show that the traveling waves uniformly converge at infinity if we perturb the bistable nonlinearity linearly, see Section 4.1 for more details. This result will allow us to control the decays of a family of traveling waves.

4.1 Uniform Decays at Infinity, and Linear Dependence of Speed

In this section, we assume that $f, g \in C^3(\mathbb{R})$, f has only three zeros $m_- < m_0 < m_+$ in $\mathbb{R}$, $f'(m_\pm) < 0$ and $f'(m_0) > 0$. Let $F(t) = -\int_{-m_-}^{t} f(u)\, du$ and $G(t) = -\int_{-m_-}^{t} g(u)\, du$.

For any $h \in [0, H]$, let $F_h(u) = F(u) + hG(u)$ and $f_h(u) = -F_h'(u) = f(u) - hg(u)$ for all $u \in \mathbb{R}$. By the Implicit Function Theorem, there exist some small constants $\beta_0 > 0$ and $1 \gg \delta_0, H_0 > 0$, and differentiable functions m_-, m_0 and m_+ on $[-H_0, H_0]$ such that for all $h \in [-H_0, H_0]$, we have

- The function $f_h(u) := f(u) - hg(u)$ has only three zeros $m_-(h) < m_0(h) < m_+(h)$ in $[m_- - 1, m_+ + 1]$.
- $m_- - \delta_0 < m_-(h) < m_- + \delta_0 < m_+ - \delta_0 < m_+(h) < m_+ + \delta_0$.
- $f_h'(u) \leq -\beta_0, \quad \forall u \in [m_- - \delta_0, m_- + \delta_0] \bigcup [m_+ - \delta_0, m_+ + \delta_0]$.
- $\max\limits_{u \in [m_- - \delta_0, m_+ + \delta_0]} |F_h(u) - F_h(m_+(h))| \geq \dfrac{1}{2} \max\limits_{u \in [m_- - \delta_0, m_+ + \delta_0]} |F(u) - F(m_+)| > 0.$

For any $s \in (0, 1)$, $h \in [-H_0, H_0]$, let (μ_h, ϕ_h) be the solution of the following problem:

$$\begin{cases} (-\Delta)^s \phi_h(x) - \mu_h \phi_h'(x) = f_h(\phi_h(x)), \quad \forall x \in \mathbb{R}, \\ \phi_h'(x) > 0, \quad \forall x \in \mathbb{R}, \\ \lim\limits_{x \to \pm\infty} \phi_h(x) = m_\pm(h), \quad \phi_h(0) = m_- + \delta_0. \end{cases} \tag{4.1.1}$$

By Theorem 3, there exists some constant $C_0 > 0$ such that

$$|\mu_h| \leq C_0, \quad \forall h \in [-H_0, H_0]. \tag{4.1.2}$$

Lemma 4.1.1. *There exists some $R_0 > 0$ and $H_1 \in (0, H)$ such that for all $h \in [-H_1, H_1]$, we have*

$$\phi_h(x) \geq m_+ - \delta_0, \quad \forall x \geq R_0.$$

Proof. Assume the statement is not true, that is, there exists some sequences $\{h_k\}$ and $\{x_k\}$ such that $h_k \to 0$ and $x_k \nearrow \infty$, as $k \to \infty$, and $\phi_{h_k}(x_k) < m_+ - \delta_0$ for all $k \geq 1$. Since $|\mu_k| \leq C_0$ for all $k \geq 1$, by the regularity theories for the fractional Laplacian (see [11, 45, 46, 57, 58]) and the Ascoli-Arzela theorem, for any $\alpha \in (0;1)$, up to subsequence, we know that there exists some constant μ and function $\phi \in C^2(\mathbb{R})$ such that $\mu_{h_k} \to \mu$ and $\phi_k \to \phi$ in $C^{2,\alpha}_{loc}(\mathbb{R})$, as $k \to \infty$, which implies that ψ satisfies

$$\begin{cases} (-\Delta)^s \phi(x) - \mu\phi'(x) = f(\phi(x)), \quad \forall x \in \mathbb{R}, \\ \phi'(x) \geq 0, \quad \forall x \in \mathbb{R}, \\ m_- - \delta_0 \leq \phi(x) \leq m_+ + \delta_0, \quad \forall x \in \mathbb{R}, \\ \phi(0) = m_- + \delta_0. \end{cases}$$

Since $\phi'(x) \geq 0$ and $m_- - \delta_0 \leq \phi(x) \leq m_+ + \delta_0$ in $\mathbb{R}$, then there exist some $L^+, L^- \in [m_- - \delta_0, m_+ + \delta_0]$ such that

$$\lim_{x \to \pm\infty} \phi(x) = L^{\pm}.$$

Since f has only three zeros m_-, m_0 and m_+, and $\phi(0) = m_- + \delta_0$, by a compactness argument, we know that

$$L^- = m_-, \quad \text{and} \quad L^+ \in \{m_0, m_+\}.$$

If $L^+ = m_0$, since $f'(m_-) < 0$ and $f'(m_0) > 0$, that is, f is a Fisher-KPP type nonlinearity in $[m_-, m_0]$, so we get a traveling wave solution for a Fisher-KPP type nonlinearity, which is a contradiction with the result in [16, 38]. So we must have $L^+ = m_+$.

Since $\psi_k \to \phi$ in $C^{2,\alpha}_{loc}(\mathbb{R})$, as $k \to \infty$, we know that there exists some large $R \gg 1$ and $K \gg 1$ such that $\phi_k(R) \geq m_+ - \delta_0$ for all $k \geq K$. On the other hand, since

$x_k \to \infty$ as $k \to \infty$, then there exists some large $K' \geq K \gg 1$ such that $x_k \geq R$ for all $k \geq K'$. Since $\phi_k'(x) > 0$ in $\mathbb{R}$, we have $\phi_k(x_k) \geq \phi_k(R) \geq m_+ - \delta_0$ for all $k \geq K'$, which contradicts with our assumption.

□

Theorem 4.1.2. *There exists some constant $C > 0$ and $R > 0$ such that for all $h \in [-H_1, H_1]$, we have*

$$0 < \phi_h'(x) \leq \frac{C}{|x|^{1+2s}}, \quad \forall |x| \geq 1.$$

In particular, we have

$$\lim_{h\to 0} \int_{\mathbb{R}} |\phi_h'(x)|^2 \, dx = \int_{\mathbb{R}} |\phi_0'(x)|^2 \, dx.$$

Proof. Let $\psi_h(x) = \phi_h'(x) > 0$ in $\mathbb{R}$, by Claim I and the assumption of β_0, we have

$$(-\Delta)^s \psi_h(x) - \mu_h \psi_h'(x) = f'(\phi_h(x))\psi_h(x) \leq -\beta_0 \psi_h(x), \quad \forall |x| \geq R_0.$$

Let v_t, φ_t and f_t be the functions in Section 2.2, since $|\mu_h| \leq C_0$ for all $h \in [-H_1, H_1]$, then

$$\begin{aligned} & (-\Delta)^s \varphi_t(x) - \mu_h \varphi_t'(x) + \beta_0 \varphi_t(x) \\ = \; & f_t(v_t(x))\varphi_t(x) - \mu_h \varphi_t'(x) + \beta_0 \varphi_t(x) \\ \geq \; & [\beta_0 + f_t(v_t(x))]\varphi_t(x) - C_0 \varphi_t'(x), \quad \forall x \in \mathbb{R}. \end{aligned}$$

Since $\beta_0 > 0$, $\lim_{x\to\pm\infty} v_t(x) = \pm 1$ and $f_t'(\pm 1) = -\frac{1}{t}$, then there exist some large

$R_1 > R_0$ and large $T_0 \gg 1$ such that

$$\beta_0 + f_{T_0}(v_{T_0}(x)) \geq \frac{\beta_0}{2}, \quad \forall |x| \geq R_1.$$

Since $\varphi_{T_0}(x) > 0$ in $\mathbb{R}$, by (2.2.1) and Lemma 2.2.1, then there exists some large $R > R_1$ such that

$$\frac{\beta_0}{2}\varphi_{T_0}(x) - C_0\varphi'_{T_0}(x) \geq 0, \quad \forall |x| \geq R.$$

Hence we have

$$(-\Delta)^s\varphi_{T_0}(x) - \mu_h\varphi'_{T_0}(x) + \beta_0\varphi_{T_0}(x) \geq 0, \quad \forall |x| \geq R.$$

For any $\delta > 0$, let $w_\delta(x) = \delta\varphi_{T_0}(x) - \phi'_h(x)$ in $\mathbb{R}$, since $|\mu| \leq C_0$ in $[-H_1, H_1]$, by the regularity theories for the fractional Laplacian, there exists some constant $C_1 > 0$ such that $\|\phi_h\|_{C^1(\mathbb{R})} \leq C_1$ for all $h \in [-H_1, H_1]$, which implies that we can take $\delta \gg 1$ large enough such that

$$\inf_{|x| \leq R+1} w_\delta(x) > 0, \quad \forall h \in [-H_1, H_1].$$

So w_δ satisfies the following problem:

$$\begin{cases} (-\Delta)^s w_\delta(x) - \mu_h w'_\delta(x) + \beta_0 w_\delta(x) > 0, \quad \forall |x| \geq R, \\ w_\delta(x) > 0, \quad \forall |x| \leq R+1, \\ \lim\limits_{|x|\to\infty} w_\delta(x) = 0. \end{cases}$$

By the same proof as Claim I in the proof of Proposition 2.2.2, we know that $w_\delta(x) \geq 0$ in $\mathbb{R}$, that is, for any $h \in [-H_1, H_1]$, we have $\phi'_h(x) \leq \delta\varphi_{T_0}(x)$ in $\mathbb{R}$. So by

(2.2.1), we know that there exists some constant $C > 0$ such that

$$0 < \phi_h'(x) \le \frac{C}{|x|^{1+2s}}, \quad \forall |h| \le H_1, \text{ and } \forall |x| \ge 1.$$

□

Theorem 4.1.3. *If f is a balanced bistable nonlinearity, that is, $F(m_+) = F(m_-) = 0$, then*

$$\lim_{h\to 0} \frac{\mu_h}{h} = \frac{G(m_+)}{\int_{\mathbb{R}} |\phi_0(x)|^2 \, dx}.$$

Proof. Notice that since $F(m_-) = F(m_+)$ and $f(m_\pm) = 0$ by L'Hospital's rule, we have

$$\begin{aligned}
& \lim_{h\to 0} \frac{F_h(m_+(h)) - F_h(m_-(h))}{h} \\
= & \lim_{h\to 0} \left(\frac{F(m_+(h)) - F(m_-(h))}{h} + [G(m_+(h)) - G(m_-(h))] \right) \\
= & -f(m_+)m_+'(0) + f(m_-)m_-'(0) + \lim_{h\to 0} [G(m_+(h)) - G(m_-(h))] \\
= & \ G(m_+) - G(m_-).
\end{aligned}$$

By Theorem 4.1.2, we have

$$\lim_{h\to 0} \frac{\mu_h}{h} = \frac{G(m_+)}{\int_{\mathbb{R}} |\phi_0(x)|^2 \, dx}.$$

□

Remark 4.1.4. *The proofs of Lemma 4.1.1, Theorem 4.1.2, Theorem 4.1.3 also works for the Laplacian. For Theorem 4.1.2, we can consider the function $v(x) = \tanh \frac{x}{\sqrt{2}}$*

and $\varphi(x) = v'(x)$ *in* $\mathbb{R}$*, where* v *is a layer solution to the Allen-Cahn equation:* $-v''(x) = f(v(x))$ *in* $\mathbb{R}$ *with* $f(t) = t - t^3$*, then we get a uniform exponential decay at infinity.*

Remark 4.1.5. *For the case of phase field, that is, for any balanced bistable potential* F *and* $G(u) = u + 1$ *in* $\mathbb{R}$*, we have*

$$\lim_{h \to 0} \frac{\mu_h}{h} = \frac{m_+ - m_-}{\int_{\mathbb{R}} |\phi_0'(x)|^2 \, dx}.$$

Remark 4.1.6. *For the case of the convex linear combination, that is, for any balanced bistable potential* F *and unbalanced bistable potential* $\tilde{F}$*, let* $G(u) = \tilde{F}(u) - F(u)$ *in* $\mathbb{R}$*, then we have*

$$\lim_{h \to 0} \frac{\mu_h}{h} = \frac{\tilde{F}(m_+)}{\int_{\mathbb{R}} |\phi_0'(x)|^2 \, dx}.$$

Example 4.1.7. *Let* $f(u) = 2u(1 - u^2)$ *for all* $u \in \mathbb{R}$*, then* f *is a balanced bistable nonlinearity. By the Implicit Function Theorem and L'Hospital's rule, for sufficiently small* $h > 0$*, there exists* $m_-(h) < m_0(h) < m_+(h)$ *such that*

$$\begin{aligned} f(m_-(h)) &= f(m_0(h)) = f(m_+(h)) = h \\ \lim_{h \searrow 0} \frac{m_0(h)}{h} &= m_0'(0) = \frac{1}{f'(0)} = \frac{1}{2} \\ \lim_{h \searrow 0} \frac{m_-(h) + m_+(h)}{h} &= m_-'(-1) + m_+'(1) = \frac{1}{f'(-1)} + \frac{1}{f'(1)} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2} \end{aligned}$$

Define

$$\begin{aligned} \mu_h &= 2m_0(h) - m_+(h) - m_-(h), \\ \phi_h(x) &= m_-(h) + \frac{m_+(h) - m_-(h)}{1 + \exp\left(-[m_+(h) - m_-(h)]x\right)}, \ \forall x \in \mathbb{R}. \end{aligned}$$

Direct computations show that

$$\begin{cases} -\phi_h''(x) - \mu_h \phi_h'(x) = f(\phi_h(x)) - h, \quad \forall x \in \mathbb{R} \\ \phi_h'(x) = \dfrac{[m_+(h) - m_-(h)]^2}{[1 + \exp\,(-[m_+(h) - m_-(h)]x)]^2} \cdot \exp\,(-[m_+(h) - m_-(h)]x) > 0, \quad \forall x \in \mathbb{R} \\ \lim\limits_{x \to \pm\infty} \phi_h(x) = m_\pm(h). \end{cases}$$

And we have

$$\lim_{h \searrow 0} \frac{\mu_h}{h} = \lim_{h \searrow 0} \frac{2m_0(h) - m_+(h) - m_-(h)}{h} = \frac{3}{2}$$

Since $\lim\limits_{h \searrow 0} [m_+(h) - m_-(h)] = 2$*, by dominated convergence theorem, we know that*

$$\begin{aligned} \lim_{h \searrow 0} \int_{\mathbb{R}} |\phi_h'(x)|^2 \, dx &= \int_{\mathbb{R}} |\phi_0'(x)|^2 \, dx \\ &= \int_{\mathbb{R}} \left(\frac{4e^{-2x}}{[1 + e^{-2x}]^2} \right)^2 \, dx \\ &= 16 \int_{\mathbb{R}} \frac{e^{4x}}{[e^{2x} + 1]^4} \, dx \\ &= 8 \int_0^\infty \frac{y}{[y+1]^4} \, dy \\ &= 8 \left[\int_0^\infty \frac{1}{[1+y]^3} \, dy - \int_0^\infty \frac{1}{[1+y]^4} \, dy \right] \\ &= 8 \cdot \left[\frac{1}{2} - \frac{1}{3} \right] \\ &= \frac{4}{3}. \end{aligned}$$

Hence we have

$$\frac{m_+ - m_-}{\int_{\mathbb{R}} |\phi_0'(x)|^2 \, dx} = \frac{2}{\frac{4}{3}} = \frac{3}{2} = \lim_{h \searrow 0} \frac{\mu_h}{h}.$$

4.2 Existence of Traveling Wave Solution

Theorem 4.2.1. *For any* $0 < s < 1$ *and bistable nonlinearity* $f \in C^2(\mathbb{R})$*, i.e.,* f *satisfies (1.1.3), then there exists a unique pair* (μ, u) *as the solution to (1.2.3). Moreover,* $u'(x) > 0$ *for all* $x \in \mathbb{R}$ *and there exists some constant* $C > 0$ *which only depends on* s *and* f *such that*

$$\frac{C^{-1}}{|x|^{1+2s}} \leq u'(x) \leq \frac{C}{|x|^{1+2s}}, \quad \forall\ |x| > 1.$$

Let $G(t) = -\int_{-1}^{t} f(u)\, du$, by the result in [19, 17], Theorem 1.2.1, and the discussion of the sign of μ in Section 3.1, in the following, we can assume f is unbalanced and $G(1) > 0$.

Now take any fixed $f_0 = -G_0' \in C^2(\mathbb{R})$ which is a balanced double well potential such that $G_0(t) \equiv G(t)$ for all $t \in [-1, t_0]$. Let $G_1 = G$, for any $0 \leq \theta \leq 1$, we let $G_\theta = (1-\theta)G_0 + \theta G_1$ and $f_\theta = -G_\theta'$. It is easy to see that for all $\theta \in (0, 1)$, G_θ is a unbalanced bistable potential, i.e., $f_\theta = -G_\theta'$ satisfies (1.1.3). Moreover, $G_\theta(1) = \theta G(1) > 0$ for all $\theta \in (0, 1]$.

Since G_0 is a balanced bistable potential, by the result in [19, 17], Theorem 1.2.1, we know that there exists a a unique solution $g = u_0 \in C^2(\mathbb{R})$ of the following problem:

$$\begin{cases} (-\Delta)^s g(x) = f_0(g(x)), \quad \forall x \in \mathbb{R}, \\ g'(x) > 0, \quad \forall x \in \mathbb{R}, \\ \lim\limits_{x \to \pm\infty} g(x) = \pm 1, \quad g(t_0) = 0. \end{cases}$$

Hence we will consider the following problem:

$$\begin{cases} (-\Delta)^s u_\theta(x) - \mu_\theta u_\theta'(x) = f_\theta(u_\theta(x)), \quad \forall x \in \mathbb{R}, \\ u_\theta'(x) > 0, \quad \forall x \in \mathbb{R}, \\ \lim\limits_{y \to \pm\infty} u_\theta(y) = \pm 1, \quad u_\theta(0) = t_0. \end{cases} \tag{4.2.1}$$

Let $v_\theta = u_\theta - u_0$ in $\mathbb{R}$, then (4.2.1) is equivalent to the following problem:

$$\begin{cases} (-\Delta)^s v_\theta - \mu_\theta v_\theta' - \mu_\theta g' + (-\Delta)^s g - f_\theta\big(v_\theta + g\big)\big) = 0, \quad x \in \mathbb{R}, \\ \lim\limits_{|x| \to \infty} v_\theta(x) = 0, \\ v_\theta(0) = 0. \end{cases}$$

For any $0 < \alpha < 1$, we consider the function space:

$$C_g^{1,\alpha}(\mathbb{R}) = \left\{ v \in C^{1,\alpha}(\mathbb{R}) : \ v(0) = 0, \ \lim_{|x| \to \infty} v(x) = 0, \ -1 \le v(x) + g(x) \le 1, \text{ in } \mathbb{R} \right\}.$$

We define the following operator $S : [0,1] \times \mathbb{R} \times C_g^{1,\alpha}(\mathbb{R}) \to C^\alpha(\mathbb{R})$ given by: for any $0 \le \theta \le 1$, $\mu \in \mathbb{R}$ and $v \in C_g^{1,\alpha}(\mathbb{R})$, we have

$$S(\theta, \mu, v) = v - \mu(-\Delta)^{-s} v' - \mu(-\Delta)^{-s} g' + g - (-\Delta)^{-s}[f_\theta(v+g)].$$

It is easy to see that S is C^1, and for all $h \in \mathbb{R}$ and $\phi \in C_g^{1,\alpha}(\mathbb{R})$, we have

$$D_{\mu,v} S(\theta,\mu,v)[h,\phi] = \phi - h(-\Delta)^{-s} v' - \mu(-\Delta)^{-s}\phi' - h(-\Delta)^{-s} g' - (-\Delta)^{-s}[f_\theta'(v+g)\phi].$$

Proof of Theorem 4.2.1. Let's define the solution set $\Sigma \subset [0,1]$: $\theta \in \Sigma$ if there exists $\mu_\theta \in \mathbb{R}$ and $v_\theta \in C_g^{1,\alpha}(\mathbb{R})$ such that $S(\theta, \mu_\theta, v_\theta) = 0$. By taking $\theta = \mu = 0$ and $v(x) \equiv 0$

in $\mathbb{R}$, by the assumption of g, we see that $S(0,0,0) = 0$, that is, $(0,0,0) \in \sum$, in particular, we know that $\sum$ is a nonempty subset of $[0,1]$.

Claim I: $\sum$ is open in $[0,1]$.

If $\theta \in \sum$, let $u_\theta(x) = v_\theta(x) + g(x)$ in $\mathbb{R}$, that is, u_θ is a solution of (4.2.1). Assume that there exist $h \in \mathbb{R}$ and $\phi \in C_g^{1,\alpha}(\mathbb{R})$ such that $D_{\mu,v_\theta} S(\theta, \mu_\theta, v_\theta)[h, \phi] = 0$, that is,

$$(-\Delta)^s \phi(x) - \mu_\theta \phi'(x) - h u_\theta'(x) - f_\theta'(u_\theta(x))\phi(x) = 0, \quad x \in \mathbb{R}.$$

By Proposition 2.4.1, we have $h = 0$ and $\phi(y) \equiv 0$ in $\mathbb{R}$, which implies that $D_{\mu,v} S(\theta, \mu_\theta, u_\theta)$ is injective. By the implicit function theorem, $\sum$ is open in $[0,1]$.

Claim II: $\sum$ is closed in $[0,1]$.

Assume that there exists a sequence $\{\theta_k\}_{k=1}^\infty \subset \sum$ such that $\theta_k \to \theta$, as $k \to \infty$. If $\theta = 0$, are done. If $\theta > 0$. Let $u_k(x) = v_{\theta_k}(x) + g(x)$ in $\mathbb{R}$. By Theorem 2.1.3, we know that for each θ_k, the speed μ_{θ_k} and the solution v_{θ_k} are unique. By Proposition 2.3.2, we know that $\mu_k \geq 0$ for all $k \geq 1$.

Since $G_0(t) \equiv G(t)$ for all $t \in [-1, t_0]$, then $G_{\theta_k}(t_0) - G_{\theta_k}(1) = G(t_0) - \theta_k G(1) \geq G(t_0) - G(1) > 0$. By Theorem 3.2.10, we know that there exists some constant $C_1 > 0$ which just depends on s, G_0 and G_1 such that $0 < \mu_{\theta_k} \leq C_1 < \infty$ for all $k \geq 1$. By the regularity theory for fractional Laplacians (see [11, 45, 46, 57, 58]) and the bootstrap method, we know that there exists some constant $C > 0$ which just depends on s, G_0 and G_1 such that

$$\|u_k\|_{C^{2,\alpha}(\mathbb{R}^n)} \leq C, \quad \forall k \geq 1.$$

Taking any fixed $t_* \in (t_0, 1)$, since $\lim\limits_{x \to \pm\infty} u_k(x) = \pm 1$, there exists some $y_k > 0$

such that $u_k(x_k) = t_*$ for all $k \geq 1$. Let $w_k(x) = u_k(x + x_k)$ in $\mathbb{R}$, w_k solves the following problem:

$$\begin{cases} (-\Delta)^s w_k(x) - \mu_{\theta_k} w_k'(x) = f_{\theta_k}(w_k(x)), \quad x \in \mathbb{R}, \\ w_k'(x) > 0, \quad x \in \mathbb{R}, \\ \lim_{x \to \pm\infty} w_k(x) = \pm 1, \quad w_k(0) = t_*. \end{cases}$$

Moreover, we know that $\|v_k\|_{C^{2,\alpha}(\mathbb{R}^n)} \leq C$ for all $k \geq 1$. By Ascoli-Arzela Theorem, there exists a subsequence of $\mu_\theta \geq 0$ and $\{w_k\}_{k=1}^\infty$, which is still denoted the same, such that $\mu_{\theta_k} \to \mu_\theta$ and $w_k \to w$ in $C^2_{loc}(\mathbb{R})$, as $k \to \infty$. In particular, w solves the problem:

$$\begin{cases} (-\Delta)^s w(x) - \mu_\theta w'(x) = f_\theta(w(x)), \quad x \in \mathbb{R}, \\ |w(x)| \leq 1, \quad x \in \mathbb{R}, \\ w'(x) \geq 0, \quad x \in \mathbb{R}, \quad w(0) = t_*. \end{cases}$$

Since $w'(x) \geq 0$ and $|w(x)| \leq 1$ in $\mathbb{R}$, there exist some constants $L^\pm$ such that $w(x) \to L^\pm$, as $x \to \pm\infty$, and $-1 \leq L^- \leq t_1 \leq L^+ \leq 1$. By a compactness argument, we also see that $f(L^\pm) = 0$. Hence $L^+ = 1$. Since $\mu_\theta \geq 0$, by Proposition 2.3.2 we get

$$\mu_\theta \int_{\mathbb{R}} |w'(x)|^2 \, dy = G_\theta(L^+) - G_\theta(L^-).$$

By a compactness argument, we know that both L^+ and L^- should be zeros of G_θ, which implies that $L^\pm \in \{-1, t_0, 1\}$. Since $\mu_\theta \geq 0$, we get $L^- = -1$ and $L^+ = 1$. Hence w is a solution to, that is, $\theta \in \sum$. Hence $\sum$ is closed in $[0, 1]$.

By Claim I and Claim II, we know that $\sum = [0, 1]$. That is, for any $\theta \in [0, 1]$,

$S(\theta, \cdot, \cdot)$ has solution, which implies that (4.2.1) has solutions. The asymptotic behaviors at infinity of solution to (4.2.1) follow directly from Proposition 2.2.2.

In summary, we know that for any bistable nonlinearity $f \in C^2(\mathbb{R})$ (balanced or unbalacned), i.e., f satisfies (1.1.3), then there exists a unique pair (μ, u) as the solution to (1.2.3). Moreover, $u'(x) > 0$ for all $x \in \mathbb{R}$ and there exists some constant $C > 0$ which only depends on s and f such that

$$\frac{C^{-1}}{|x|^{1+2s}} \leq u'(x) \leq \frac{C}{|x|^{1+2s}}, \quad \forall\ |x| > 1.$$

□

Remark 4.2.2. *The continuation arguments have also been used in the study of nonlocal problems in [5, 30], where a family of operators is used to connect nonlocal operators to the classical elliptic operators.*

Remark 4.2.3. *Our main results, Theorem 4.1.2 and Theorem 4.2.1, are exactly the Assumption 2 in [44] which studied the phase field theory for the fractional Laplacians.*

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