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Neutron Interference in the Gravitational Field of a Ring Laser

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Neutron Interferometry in the Gravitational Field of a Ring Laser

Robert D. Fischetti, PhD

University of Connecticut 2014

Measuring the gravitational inertial frame dragging effect has been attempted in the recent Gravity Probe B experiment. With the accuracy of the results under scrutiny from the community, and the expense of repeating the experiment in the near future too high, new ways of demonstrating gravitational frame dragging could result in more accurate results sooner[1]. A number of analyses of neutron interference effects due to various metric perturbations have been found in the literature [12, 11]. However, the approach of each author depends on a specific metric. I will present a new general technique giving the Foldy-Wouthuysen transformed Hamiltonian for a Dirac particle in the most general linearized space-time metric. I will then apply this new technique to calculate the phase shift on a neutron beam interferometer due to the gravitational field of a ring laser which contains the gravitational frame dragging effect[3].

Neutron Interferometry in the Gravitational Field of a Ring Laser

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B.A., Physics, SUNY New Paltz, 2006

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Neutron Interferometry in the Gravitational Field of a Ring
Laser

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Constants and Conventions

$$c = 3 \times 10^8 \frac{m}{s}$$

$$G = 6.67 \times 10^{-11} \frac{m^3}{kg \cdot s^2}$$

$$k = \frac{8\pi G}{c^4}$$

$$m_n = 1.675 \times 10^{-27} kg$$

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

$$\hbar = 6.63 \times 10^{-34} \frac{m^2 kg}{s}$$

Introduction

Neutron interferometry has opened up a whole new domain of research utilizing the wavelike nature of neutrons to explore quantum mechanical effects. The general technique uses at its core some of the more basic fundamental principles of quantum mechanics involving the de Broglie wavelength and the interference of neutrons. The effect of the Newtonian gravitational potential on the interference of two neutron beams was tested in the Colella-Overhauser-Werner (COW) experiment and found to agree with predictions. [2]

The development of the ring laser has led to numerous applications in many areas of physics. Mallett [3] solved the linear Einstein field equations to obtain the gravitational field produced by the electromagnetic radiation of a unidirectional ring laser. It was shown that a massive neutral spinning particle at the center of the ring laser exhibited gravitational inertial frame dragging. The post Newtonian phenomenon of inertial frame dragging is usually associated with the gravitational field generated by rotating matter. An example of this is the prediction that a satellite in a polar orbit around the earth should be dragged around by the gravitational field generated by the rotation of the earth. The recent results of the Gravity Probe B experiment seems to indicate the existence of this effect. However, to our knowledge, no experimental demonstration has yet been carried out quantitatively verifying the gravitational influence of circulating light on matter.

For my thesis, I propose to develop a new general technique giving the Foldy-Wouthuysen transformed Hamiltonian for a Dirac particle in the most general linearized space-time metric. I will then apply this new technique to calculate the phase shift due to the interference of two neutron beams in the gravitational field of a ring laser.

Chapter 1

Background

1.1 Neutron Interferometry

The neutron split-beam interferometer has proven to be useful in measuring small Newtonian gravitational effects by using quantum principles. Greenberger and Overhauser [5] had shown that for a spinless Shrödinger particle, a small energy perturbation would cause a phase shift of the wave function along the path of a free particle which would yield an interference pattern shift for the split beam interferometer. This technique ignores spin-gravity coupling but with some forethought it was decided any spin effects would be of much smaller magnitude on the phase than the inertial effects.

To find the phase shift for a Shrödinger particle due to a small, slowly varying, and position dependent energy perturbation $U(\vec{x})$ we write down the Shrödinger equation

$$i\hbar \frac{\partial \Psi}{\partial t} = H\Psi = \{H_0 + U\} \Psi \quad (1.1)$$

and use, as a trial wave function, the perturbed particle solution

$$\Psi = \Psi_0 e^{i\delta\phi} = e^{i(\vec{k}_0 \cdot \vec{x} + \delta\phi - \omega_0 t)} \quad (1.2)$$

Substituting this solution back into 1.1 we find

$$\begin{aligned} E_0 \Psi &= \left\{ -\frac{\hbar^2}{2m} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) + U \right\} e^{i(\vec{k}_0 \cdot \vec{x} + \delta\phi - \omega_0 t)} \\ &= \left\{ \frac{\hbar^2}{2m} \left(|k_0|^2 + k_0^i \frac{\partial \delta\phi}{\partial x^i} + i\nabla^2 \delta\phi \right) + U \right\} \Psi \end{aligned} \quad (1.3)$$

Multiplying equation 1.3 on left by $\Psi^\dagger$ while using the relation $E_0 = \frac{\hbar^2 |k_0|^2}{2m}$ yields

$$0 = \frac{\hbar^2}{m} \left(k_0^i \frac{\partial \delta\phi}{\partial x^i} + i\nabla^2 \delta\phi \right) + U \quad (1.4)$$

The $i\nabla^2 \delta\phi$ term can be neglected for large k_0 in a slowly varying potential. In interpreting the free particle wave number as the classical particles momentum trajectory, we integrate equation 1.4 over time to get

$$\begin{aligned} \int_{t=0}^{t=T} dt \frac{\hbar}{m} k_0^i \frac{\partial \delta\phi}{\partial x^i} &= \int_{t=0}^{t=T} dt \frac{dx_0^i}{dt} \frac{\partial \delta\phi}{\partial x^i} = \int_{t=0}^{t=T} dt \frac{\partial \delta\phi}{\partial t} \\ &= \delta\phi(T) = -\frac{1}{\hbar} \int_{t=0}^{t=T} dt U \end{aligned} \quad (1.5)$$

Equation 1.5 represents a time integral which is integrated along the path of the free classical particle, due to the dot product of the free classical particle velocity $\frac{dx_0^i}{dt}$ with the gradient of $\delta\phi$, and not the perturbed path. In this way, the phase shift is defined on the trajectory of the free particle only. Therefore, for a split beam interferometer, each path will experience a different effect due to it's unique trajectory and may have a different phase shift upon entering the recombination point. It should be noted that the prescription in equation 1.5 has been shown to be incorrect by Mannheim[6], for the reason that a particle moving in a perturbing potential will not travel along the free particle path, but a close by alternate trajectory. It will be discussed in the appendix section 3.1 that the phase shift along the perturbed trajectory is equivalent to the phase shift along the free particle trajectory at the same parameterized coordinate distance traveled.

The interference pattern is measured as a function of the total difference in phase $= \phi_A - \phi_B$ which, normalizing to the free particle phase difference for the same paths, yields

Figure 1.1: Diagram of the Greenberger and Overhauser neutron interferometer representing the two paths, labeled I & II, splitting at point a and recombining at point d . The acceleration due to gravity is in the $\vec{Z}$ direction.

$$\text{phase difference} = \delta\phi_A - \delta\phi_B \quad (1.6)$$

Greenberger and Overhauser[5] used equation 1.5 to find the phase shift along each beam path in the interferometer shown in figure 1.1. Their calculation indicated an induced phase difference caused by the Newtonian gravitational potential in the z direction expressed as

$$\delta\phi_I - \delta\phi_{II} = - \left(\frac{2mgl^2}{\hbar v} \right) \left(1 + \frac{a}{L} \right) \tan \theta \quad (1.7)$$

where a is the width of the center crystal, L is the length of free space between the surfaces of different crystals, and m and v are the neutrons rest mass and velocity.

The experimental confirmation of this result by Colella, Overhauser, and Werner[2], has shown that neutron interferometry is sensitive to small gravitational effects.

1.2 Post-Newtonian dynamics

With the introduction of Einsteins theory of general relativity there has been the need to confirm the implications it has over the standard of Newtonian dynamics. Just as in the theory of electrodynamics where there are three domains which separately have their own implications on charged particles and field dynamics, general relativity (GR) has three separately distinct domains.

The electrostatic domain, where a charged stationary particle imprints a field in it's vicinity which describes a force on an imposing test particle, is analogous to a stationary massive object. In GR, it is in this domain where test particles experience the relativistic effects of the precession of perihelion of orbits, the red-shift of light particles, and the bending of light. The metric field of a stationary mass in the weak field limit has the form [4]

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

$$h_{\mu\nu} = \begin{pmatrix} -\frac{2GM}{\kappa r} & 0 & 0 & 0 \\ 0 & -\frac{2GM}{\kappa r} & 0 & 0 \\ 0 & 0 & -\frac{2GM}{\kappa r} & 0 \\ 0 & 0 & 0 & -\frac{2GM}{\kappa r} \end{pmatrix} \quad (1.8)$$

The magneto-static domain, where a rotating charged particle generates a magnetic field in it's vicinity which describes the force on a moving test particle, is analogous to a rotating massive object which creates a metric field in it's vicinity with nonzero off diagonal time-space cross terms. An example of this metric field is one a rotating spherical mass known as the Kerr or Lens-Thirring solution

$$h_{\mu\nu} = \begin{pmatrix} -\frac{2GM}{\kappa r} & -\frac{2GS_z y}{\kappa r^3} & -\frac{2GS_z x}{\kappa r^3} & 0 \\ -\frac{2GS_z y}{\kappa r^3} & -\frac{2GM}{\kappa r} & 0 & 0 \\ -\frac{2GS_z x}{\kappa r^3} & 0 & -\frac{2GM}{\kappa r} & 0 \\ 0 & 0 & 0 & -\frac{2GM}{\kappa r} \end{pmatrix} \quad (1.9)$$

where we can see that the angular momentum energy of the system S_z generates non-zero time-space cross terms in the metric. Since in Newtonian dynamics, the rotating earth does not cause any additional effects on a distant body beyond those present for a similar non-rotating body of the same rest mass, any effect due to these off diagonal metric components are strictly post-Newtonian and are referred to as “gravitational frame-dragging” effects.

The fully dynamic case of electromagnetic radiation has an analogy in GR as well. The homogeneous linearized Einstein equations are wave equations with plane wave solutions to yield a time dependent metric of the typical form

$$h_{\mu\nu} = \epsilon_{\mu\nu} \cos k_\alpha x^\alpha \quad (1.10)$$

where the $\epsilon_{\mu\nu}$ are constants.

1.3 The ring laser

The ring laser [3] has many illuminating features as a solution of Einsteins linearized field equations. It’s primary function for this theoretical paper is that it is an analytic solution that generates the frame-dragging terms in it’s resultant metric that have been historically hard to validate through experimental tests. The second insightful feature of it’s design is that the generation of this metric is unlike any frame-dragging generating system considered for experiment before in that it is composed of only light energy. In the larger picture of the history of GR, all experimentally verified, and nearly all predicted, post-Newtonian effects of GR have been based on a metric field generated by matter. An overview of the gravitational field generated by the ring laser is given here.

The linearized Einstein gravitational field equations in the Hilbert gauge $\partial_\mu (h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}h) = 0$ are

$$\partial_\lambda \partial^\lambda \left(h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}h \right) = -\kappa T^{\mu\nu} \quad (1.11)$$

where for electromagnetic radiation

Figure 1.2: Ring Laser

$$\tau^{\mu\nu} = -\frac{1}{4\pi} \left(f^{\mu\alpha} f_{\alpha}^{\nu} - \frac{1}{4} \eta^{\mu\nu} f^{\alpha\beta} f_{\alpha\beta} \right) \quad (1.12)$$

where $f^{\alpha\beta}$ is the Maxwell field tensor. Since the trace of 1.12 τ_{μ}^{μ} is zero, 1.11 can be rewritten as

$$\partial_{\lambda} \partial^{\lambda} h^{\mu\nu} = -\kappa \tau^{\mu\nu} \quad (1.13)$$

For a thin laser beam in the configuration of Figure 1.2 with a polarization in the z direction, the Maxwell tensor $f^{\mu\nu}$ components are

$$f_{(1)}^{30} = f_{(2)}^{30} = f_{(3)}^{30} = f_{(4)}^{30} = E_z \quad (1.14)$$

$$f_{(1)}^{13} = -f_{(3)}^{13} = -B_y \quad (1.15)$$

$$f_{(2)}^{32} = -f_{(4)}^{32} = B_x \quad (1.16)$$

while all other components are zero.

After assuming an infinitely thin laser beam of linear energy density ρ , the nonzero metric components for the ring laser were shown to be

$$h^{00} = -\frac{\kappa\rho}{4\pi} [\phi_{(1)} + \phi_{(2)} + \phi_{(3)} + \phi_{(4)}] \quad (1.17)$$

$$h^{01} = -\frac{\kappa\rho}{4\pi} [\phi_{(1)} - \phi_{(3)}] \quad (1.18)$$

$$h^{02} = -\frac{\kappa\rho}{4\pi} [\phi_{(2)} - \phi_{(4)}] \quad (1.19)$$

$$h^{11} = -\frac{\kappa\rho}{4\pi} [\phi_{(1)} + \phi_{(3)}] \quad (1.20)$$

$$h^{22} = -\frac{\kappa\rho}{4\pi} [\phi_{(2)} + \phi_{(4)}] \quad (1.21)$$

with the definitions

$$\phi_{(1)} = \ln \left\{ \frac{-x + a + [(x-a)^2 + y^2 + z^2]^{\frac{1}{2}}}{-x + [x^2 + y^2 + z^2]^{\frac{1}{2}}} \right\} \quad (1.22)$$

$$\phi_{(2)} = \ln \left\{ \frac{-y + a + [(x-a)^2 + (y-a)^2 + z^2]^{\frac{1}{2}}}{-y + [(x-a)^2 + y^2 + z^2]^{\frac{1}{2}}} \right\} \quad (1.23)$$

$$\phi_{(3)} = \ln \left\{ \frac{-x + a + [(x-a)^2 + (y-a)^2 + z^2]^{\frac{1}{2}}}{-x + [x^2 + (y-a)^2 + z^2]^{\frac{1}{2}}} \right\} \quad (1.24)$$

$$\phi_{(4)} = \ln \left\{ \frac{-y + a + [x^2 + (y-a)^2 + z^2]^{\frac{1}{2}}}{-y + [x^2 + y^2 + z^2]^{\frac{1}{2}}} \right\} \quad (1.25)$$

These functions, as they appear in the metric above, are independently solutions of Einsteins linearized field equations with the subscript denoting the side of the ring laser that prompted their generation. This can be shown graphically in figures 1.3 through 1.6 where the source of each function is clearly depicted by it's gradient.

Figure 1.3: The function ϕ_1 for a ring laser of side length $a = 100$ units. The bold contour outlining the plotted area is at a distance of 1 unit from each side of the ring laser.

Figure 1.4: The function ϕ_2 for a ring laser of side length $a = 100$ units. The bold contour outlining the plotted area is at a distance of 1 unit from each side of the ring laser.

Figure 1.5: The function ϕ_3 for a ring laser of side length $a = 100$ units. The bold contour outlining the plotted area is at a distance of 1 unit from each side of the ring laser.

Figure 1.6: The function ϕ_4 for a ring laser of side length $a = 100$ units. The bold contour outlining the plotted area is at a distance of 1 unit from each side of the ring laser.

Chapter 2

New approach to the Dirac equation in a linearized gravitational field

2.1 The Dirac equation in curved space

To begin to solve the problem of how a spinning neutron will propagate in curved space, we need a covariant set of quantum equations. By starting from the covariant Dirac equation and casting it into a Hamiltonian wave equation form in the rest frame of the interferometer apparatus, we can solve the wave equation, as Greenberger and Overhauser did, using a similar perturbed solution that is space but not time dependent. By using the Dirac equation rather than the Klein-Gordon wave equation, we are not assuming that spin effects are delegable.

The Dirac equation in flat space is a Lorentz covariant equation with a Dirac spinor type wave function as a solution which can be written as [9]

$$i\hbar\gamma^{(\alpha)}\frac{\partial}{\partial x^{(\alpha)}}\psi - mc\psi = 0 \tag{2.1}$$

or in a more convenient form

$$\left(\gamma^{(\alpha)} \frac{\partial}{\partial x^{(\alpha)}} + k \right) \psi = 0 \quad (2.2)$$

defining k .

To satisfy Lorentz covariance, in a Lorentz transformed coordinate system equation 2.2 must take the form

$$\left(\gamma^{(\alpha)'} \frac{\partial}{\partial x^{(\alpha)'}} + k \right) \psi' = 0 \quad (2.3)$$

and since the Dirac spinor is a one column matrix it is assumed to have the linear transformation property's

$$\psi'(\vec{x}') = \psi'(a\vec{x}) = S\psi(\vec{x}) = S\psi(a^{-1}\vec{x}') \quad (2.4)$$

where $S(a)$ is some transformation matrix, for which we need to find the solution of, which depends on the Lorentz transformation a .

Using the property $S(a^{-1}) = S^{-1}(a)$ we can write $\psi = S^{-1}(a)\psi'$.

$$\left(\gamma^{(\alpha)} \frac{\partial}{\partial x^{(\alpha)}} + k \right) S^{-1}(a)\psi' = 0 \quad (2.5)$$

Under a Lorentz transformation $x_{(\beta)'} = a_{(\beta)'}^{(\alpha)} x_{(\alpha)}$ the derivative $\frac{\partial}{\partial x^{(\alpha)}}$ transforms as a covariant Lorentz vector and upon multiplying on the left by $S(a)$ and rewriting $\frac{\partial}{\partial x^{(\alpha)}} = a_{(\alpha)}^{(\beta)'} \frac{\partial}{\partial x^{(\beta)'}}$ equation 2.5 becomes

$$\left(S(a)\gamma^{(\alpha)}S^{-1}(a)a_{(\alpha)}^{(\beta)'} \frac{\partial}{\partial x^{(\beta)'}} + k \right) \psi' = 0 \quad (2.6)$$

If we make no distinction between $\gamma^{(\alpha)'}$ and $\gamma^{(\alpha)}$, since they will be functionally the same in any inertial coordinate system we find that the characteristic equation for $S(a)$ is

$$S(a)\gamma^{(\alpha)}S^{-1}(a)a_{(\alpha)}^{(\beta)'} = \gamma^{(\beta')} \quad (2.7)$$

or

$$S(a)\gamma^{(\alpha)}S^{-1}(a) = a_{(\beta)'}^{(\alpha)}\gamma^{(\beta')} \quad (2.8)$$

If we restrict our Lorentz coordinate systems to Cartesian ones, $\gamma^{(\alpha)}$ and $\gamma^{(\alpha')}$ will be functionally equivalent and we can drop the distinction and 2.8 becomes

$$S(a)\gamma^{(\alpha)}S^{-1}(a) = a_{(\beta)'}^{(\alpha)}\gamma^{(\beta)} \quad (2.9)$$

If a solution for S can be found using equation 2.9, then the spinor theory of 2.2 is Lorentz covariant with ψ transforming as in 2.4 and γ transforming as in 2.7 .

The usual way of extending a Tensor theory into curved space is by way of replacing all Lorenz covariant tensors with their generally covariant counterparts and replacing all partial derivatives with covariant derivatives. For instance, the Klein-Gordan equation with the wave function being an ordinary scalar has the Lorentz covariant form

$$\{\nabla^{(\alpha)}\nabla_{(\alpha)} + K\}\psi = 0 \quad (2.10)$$

where the D'lambertian, K , and ψ are considered Lorentz scalars. In curved space the derivatives are replaced with covariant derivatives to give

$$g'^{\gamma\beta}\psi_{;\gamma\beta} + K\psi = 0 \quad (2.11)$$

where $\psi_{;\alpha\beta}$ is a covariant rank 2 tensor, K and ψ are both scalars which makes 2.11 generally covariant.

For 2.2 we do not simply have scalars and partial derivatives but also a spinor which behaves unlike a tensor under Lorentz transformations. We can however use the wave functions Lorentz transformation properties to develop a generally covariant Dirac theory if we treat it as a scalar under a general coordinate transformation but as a spinor under a Lorentz transformation. We can define a coordinate transformation which, at each point in space, transforms a vector to a locally inertial coordinate system given by

$$\begin{aligned}
\epsilon_{(\beta)}^\nu \epsilon_{(\alpha)}^\mu g_{\nu\mu} &= \eta_{(\alpha)(\beta)} \\
\epsilon_{(\beta)}^\nu v_\nu &= v_{(\beta)} \\
\epsilon_\nu^{(\beta)} v^\nu &= v^{(\beta)}
\end{aligned} \tag{2.12}$$

where the flat space indices α are raised and lowered by η and the general coordinate indices by g .

Rewriting the four-gradient in 2.2 we get

$$\left(\gamma^{(\alpha)} \epsilon_{(\alpha)}^\nu \frac{\partial}{\partial x^\nu} + k \right) \psi = 0 \tag{2.13}$$

Equation 2.13 is form invariant with respect to a general coordinate transformation of the ν indice with $K\psi$ and $\gamma^{(\alpha)}$ coordinate scalars and $\epsilon_{(\alpha)}^\nu$ and $\frac{\partial}{\partial x^\nu}$ coordinate vectors, but it should also be form invariant with respect to a position dependent Lorentz transformation in the indice (α) which as shown here it is not.

$$\begin{aligned}
\left(\gamma^{(\alpha)} \epsilon_{(\alpha)}^\nu \frac{\partial}{\partial x^\nu} + k \right) \psi &= \left(a_{(\beta')}^{(\alpha)} \gamma^{(\beta')} a_{(\alpha)}^{(\gamma')} \epsilon_{(\gamma')}^\nu \frac{\partial}{\partial x^\nu} + k \right) S^{-1}(a(x)) \psi' \\
&= \left(\gamma^{(\beta')} \epsilon_{(\beta')}^\nu \frac{\partial}{\partial x^\nu} + k \right) S^{-1} \psi' = \left(\gamma^{(\beta')} \epsilon_{(\beta')}^\nu \frac{\partial}{\partial x^\nu} + k \right) S^{-1} \psi' \\
&= 0
\end{aligned} \tag{2.14}$$

Multiplying by $S(a(x))$ gives

$$\left(S \gamma^{(\beta')} S^{-1} \epsilon_{(\beta')}^\nu \frac{\partial}{\partial x^\nu} + k + S \gamma^{(\beta')} \epsilon_{(\beta')}^\nu \frac{\partial S^{-1}}{\partial x^\nu} \right) \psi' = 0 \tag{2.15}$$

It appears from the last term in 2.15 that the derivative $\frac{\partial \psi}{\partial x^{(\alpha)}}$ is not form invariant with respect to position dependent Lorentz transformations. What we need is a derivative $D_{(\alpha)}$ which has the property

$$D_{(\beta')} \psi' = a_{(\beta')}^{(\alpha)} S(a(x)) D_{(\alpha)} \psi \tag{2.16}$$

than our field equation would transform as

$$\begin{aligned}
S \left(\gamma^{(\alpha)} D_{(\alpha)} + k \right) \psi &= S \left(\gamma^{(\alpha)} a_{(\alpha)}^{(\beta')} D_{(\beta')} + k \right) S^{-1} \psi' \\
&= \left(S \gamma^{(\alpha)} a_{(\alpha)}^{(\beta')} S^{-1} D_{(\beta')} + k \right) \psi' \\
&= \left(\gamma^{(\beta')} D_{(\beta')} + k \right) \psi'
\end{aligned} \tag{2.17}$$

which is Lorentz invariant. It has been shown that a derivative that has the properties in equation 2.16 is [7]

$$D_{(\alpha)} = \epsilon_{(\alpha)}^\nu \left(\frac{\partial}{\partial x^\nu} - \frac{i}{4} \sigma^{(\beta)(\gamma)} \epsilon_{(\beta)}^\mu \epsilon_{(\gamma)\mu;\nu} \right) \tag{2.18}$$

where the σ 's are objects with the commutation property

$$\left[\sigma^{(\alpha)(\beta)}, \sigma^{(\gamma)(\delta)} \right] = \eta^{(\gamma)(\beta)} \sigma^{(\alpha)(\delta)} - \eta^{(\gamma)(\alpha)} \sigma^{(\beta)(\delta)} + \eta^{(\delta)(\beta)} \sigma^{(\gamma)(\alpha)} - \eta^{(\delta)(\alpha)} \sigma^{(\gamma)(\beta)} \tag{2.19}$$

It can be shown that a suitable form is

$$\sigma^{(\alpha)(\beta)} = \frac{i}{2} \left[\gamma^{(\alpha)}, \gamma^{(\beta)} \right] \tag{2.20}$$

With this new derivative inserted into 2.13 our generally covariant Dirac field equation now takes the form

$$\left\{ \gamma^{(\alpha)} \epsilon_{(\alpha)}^\nu \left(\frac{\partial}{\partial x^\nu} - \frac{i}{4} \sigma^{(\beta)(\gamma)} \epsilon_{(\beta)}^\mu \epsilon_{(\gamma)\mu;\nu} \right) + k \right\} \psi = 0 \tag{2.21}$$

2.21 is a field equation which is form invariant under general coordinate transformations of the ν indices, with all other quantities being treated as scalars, and form invariant under a Lorenz transformation of the locally flat (α) coordinates as in 2.17.

2.2 Foldy Wouthuysen Transformation

It has been discussed that the original Dirac theory in flat space contains results that are difficult to relate to classical physics. For instance, for a free particle 2.1 can be written

$$i\hbar \frac{\partial}{\partial t} \psi = H\psi = (mc^2 + c\vec{\alpha} \cdot \vec{p}) \psi \quad (2.22)$$

In this representation the expectation value of the velocity of the particle $\frac{\partial \vec{x}}{\partial t}$ should, in the non-relativistic limit, go to $\frac{\vec{p}}{m}$. However what we get is that the velocity of a wave packet is actually

$$\frac{\partial \vec{x}}{\partial t} = \frac{1}{i\hbar} [\vec{x}, H] = c\vec{\alpha} \quad (2.23)$$

There are other issues with this representation, such as mixing positive and negative energy states, due to the inability of having 2-component spinor solutions to 2.23 instead of 4-component ones. To illustrate this, we look at the free particle solutions to 2.22

$$\begin{aligned} \psi_1 &= \begin{pmatrix} 1 \\ 0 \\ \frac{cp_z}{\sqrt{c^2p^2+m^2c^4+mc^2}} \\ \frac{c(p_x+ip_y)}{\sqrt{c^2p^2+m^2c^4+mc^2}} \end{pmatrix} e^{i(\vec{k} \cdot \vec{x} - \omega t)}, \psi_2 = \begin{pmatrix} 0 \\ 1 \\ \frac{c(p_x-ip_y)}{\sqrt{c^2p^2+m^2c^4+mc^2}} \\ -\frac{cp_z}{\sqrt{c^2p^2+m^2c^4+mc^2}} \end{pmatrix} e^{i(\vec{k} \cdot \vec{x} - \omega t)} \\ \psi_3 &= \begin{pmatrix} \frac{cp_z}{-\sqrt{c^2p^2+m^2c^4-mc^2}} \\ \frac{c(p_x+ip_y)}{-\sqrt{c^2p^2+m^2c^4-mc^2}} \\ 1 \\ 0 \end{pmatrix} e^{i(\vec{k} \cdot \vec{x} - \omega t)}, \psi_4 = \begin{pmatrix} \frac{c(p_x-ip_y)}{-\sqrt{c^2p^2+m^2c^4-mc^2}} \\ -\frac{cp_z}{-\sqrt{c^2p^2+m^2c^4-mc^2}} \\ 0 \\ 1 \end{pmatrix} e^{i(\vec{k} \cdot \vec{x} - \omega t)} \end{aligned} \quad (2.24)$$

The first two of these solutions have energy $E_{1,2} = \sqrt{m^2c^4 + c^2p^2}$ while the second two have energy $E_{3,4} = -\sqrt{m^2c^4 + c^2p^2}$. Solutions 1 and 3 have z-direction spin angular momentum eigenvalues $\frac{\hbar}{2}\sigma^3\psi_{1,3} = +\frac{\hbar}{2}\psi_{1,3}$ (in the low energy limit) while solutions 2 and 4 have eigenvalues $\frac{\hbar}{2}\sigma^3\psi_{2,4} = -\frac{\hbar}{2}\psi_{2,4}$.

In order to find a velocity eigenfunction solution to 2.23 in say the x-direction, we

necessarily must superpose a positive energy solution with a negative one. There are 4 ways in which we can do this.

$$\begin{aligned}
\psi_{v1} &= \psi_1 + \psi_3 \\
\psi_{v2} &= \psi_1 - \psi_3 \\
\psi_{v3} &= \psi_2 + \psi_4 \\
\psi_{v4} &= \psi_2 - \psi_4
\end{aligned} \tag{2.25}$$

These solutions represent four possibilities of a spin-up or spin-down particle with velocity eigenvalue $+c$ or $-c$. It is unsettling that this type of result where a physical situation such as a free particle with a constant velocity can only be realized by including both particles and antiparticles. In the low energy limit it is possible to have separate positive and negative energy particle solutions for the velocity eigenvalue equation, since the energy eigenfunctions 2.24 become 2 component spinors, but the problem of Ehrenfest's theorem is not resolved still.

The issue lies in the form of the Dirac Hamiltonian. In fact, Foldy and Wouthuysen had found that, by applying a unitary transformation on the system, these problems can be alleviated and the theory is now capable of independent 2-component solutions having definite energy[8]. The form of this transformation for a free particle is

$$\begin{aligned}
\psi' &= e^{iS}\psi \\
H' &= e^{iS}He^{-iS} \\
O' &= e^{iS}Oe^{-iS} \\
S &= -i\frac{\beta}{2mc}\alpha \cdot \vec{p}\theta
\end{aligned} \tag{2.26}$$

Expanding the transformation matrix gives

$$\begin{aligned} e^{iS} &= \cos\left(\frac{|p|\theta}{2mc}\right) + \beta \frac{\vec{\alpha} \cdot \vec{p}}{|p|} \sin\left(\frac{|p|\theta}{2mc}\right) \\ e^{-iS} &= \cos\left(\frac{|p|\theta}{2mc}\right) - \beta \frac{\vec{\alpha} \cdot \vec{p}}{|p|} \sin\left(\frac{|p|\theta}{2mc}\right) \end{aligned} \quad (2.27)$$

And the transformed Hamiltonian becomes

$$\begin{aligned} H' &= \left(\cos\left(\frac{|p|\theta}{2mc}\right) + \beta \frac{\vec{\alpha} \cdot \vec{p}}{|p|} \sin\left(\frac{|p|\theta}{2mc}\right) \right) (mc^2 + c\vec{\alpha} \cdot \vec{p}) \left(\cos\left(\frac{|p|\theta}{2mc}\right) - \beta \frac{\vec{\alpha} \cdot \vec{p}}{|p|} \sin\left(\frac{|p|\theta}{2mc}\right) \right) \\ &= (mc^2 + c\vec{\alpha} \cdot \vec{p}) \left(\cos\left(\frac{|p|\theta}{2mc}\right) - \beta \frac{\vec{\alpha} \cdot \vec{p}}{|p|} \sin\left(\frac{|p|\theta}{2mc}\right) \right)^2 \\ &= e^{-2iS} = (mc^2 + c\vec{\alpha} \cdot \vec{p}) \left(\cos\left(\frac{|p|\theta}{mc}\right) - \beta \frac{\vec{\alpha} \cdot \vec{p}}{|p|} \sin\left(\frac{|p|\theta}{mc}\right) \right) \\ &= \beta \left(mc^2 \cos\left(\frac{|p|\theta}{mc}\right) + c|p| \sin\left(\frac{|p|\theta}{mc}\right) \right) + c\vec{\alpha} \cdot \vec{p} \left(\cos\left(\frac{|p|\theta}{mc}\right) - \frac{mc^2}{|p|c} \sin\left(\frac{|p|\theta}{mc}\right) \right) \end{aligned} \quad (2.28)$$

Since the mixing of positive and negative energy issues stem from the “odd” $\vec{\alpha}$ operator, we choose θ such that the second term above is zero. This choice is

$$\begin{aligned} \theta &= \frac{mc}{|p|} \tan^{-1}\left(\frac{|p|}{mc}\right) \\ \cos\left(\frac{|p|\theta}{mc}\right) &= \frac{mc}{\sqrt{p^2 + m^2c^2}} \\ \sin\left(\frac{|p|\theta}{mc}\right) &= \frac{|p|}{\sqrt{p^2 + m^2c^2}} \end{aligned} \quad (2.29)$$

and the Hamiltonian is now an “even” function of the β matrix.

$$H' = \beta \left(mc^2 \frac{mc}{\sqrt{p^2 + m^2c^2}} + |p|c \frac{|p|}{\sqrt{p^2 + m^2c^2}} \right) = \beta \sqrt{c^2p^2 + m^2c^4} \quad (2.30)$$

giving us the special relativistic relation between energy and momentum which, in the low energy limit, goes to

$$H' = \beta \left(mc^2 + \frac{p^2}{2m} \right) \quad (2.31)$$

Also, since $\vec{p}$ and $\frac{\partial}{\partial t}$ both commute with e^{iS} , their operator forms in the new representation are the same.

2.31 has 4 independent spinor wave function solutions of the simple form

$$\begin{aligned} \psi_1 &= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{i(\vec{k} \cdot \vec{x} - \omega t)}, \psi_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} e^{i(\vec{k} \cdot \vec{x} - \omega t)} \\ \psi_3 &= \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{i(\vec{k} \cdot \vec{x} - \omega t)}, \psi_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} e^{i(\vec{k} \cdot \vec{x} - \omega t)} \end{aligned} \quad (2.32)$$

The the first two of these solutions have energy $E_{1,2} = \left(mc^2 + \frac{\hbar^2 k^2}{2m} \right)$ while the second two have energy $E_{3,4} = - \left(mc^2 + \frac{\hbar^2 k^2}{2m} \right)$. Solutions 1 and 3 have z-direction spin angular momentum eigenvalues $\frac{\hbar}{2} \sigma^3 \psi_{1,3} = +\frac{\hbar}{2} \psi_{1,3}$ while solutions 2 and 4 have eigenvalues $\frac{\hbar}{2} \sigma^3 \psi_{2,4} = -\frac{\hbar}{2} \psi_{2,4}$. Also, since the new Hamiltonian 2.31 no longer contains any odd terms and only the even β , Ehrenfest's theorem for velocity now gives us the more realistic

$$\frac{\partial \vec{x}}{\partial t} = \frac{1}{i\hbar} [\vec{x}, H'] = \frac{\vec{p}}{m} \quad (2.33)$$

Now that we know the odd terms in the free particle Hamiltonian were at fault for the apparent discrepancy between classical and quantum physics for the Dirac particle, we will cast the procedure for the free particle into a form including a perturbing potential. We can see that for the free particle the even term in 2.22 was βmc^2 and the odd term was

$c\vec{\alpha} \cdot \vec{p}$ and so the transformation 2.26 can be written

$$\begin{aligned}
H &= \beta mc^2 + O \\
\psi' &= e^{iS} \psi \\
H' &= e^{iS} H e^{-iS} \\
A' &= e^{iS} A e^{-iS} \\
S &= -i \frac{\beta}{2mc^2} O
\end{aligned} \tag{2.34}$$

To extend this formalism to a small non-explicitly time-dependent perturbing potential which has the possibility of having even and odd parts, we can try the transformation

$$\begin{aligned}
H &= \beta mc^2 + h(E) + (c\vec{\alpha} \cdot \vec{p} + h(O)) = \beta mc^2 + E + O \\
\psi' &= e^{iS} \psi \\
H' &= e^{iS} H e^{-iS} \\
A' &= e^{iS} A e^{-iS} \\
S &= -i \frac{\beta}{2mc^2} O
\end{aligned} \tag{2.35}$$

If the odd term is smaller than mc^2 we can expand the exponential to low order in which case it can be shown that after the transformation the new Hamiltonian, keeping terms of order $\frac{1}{m^3 c^6}$, has the form

$$\begin{aligned}
H' &= \beta \left(mc^2 + \frac{O^2}{2mc^2} - \frac{O^4}{8m^3 c^6} \right) + E - \frac{1}{8m^2 c^4} [O, [O, E]] \\
&\quad + \frac{\beta}{2mc^2} [O, E] - \frac{O^3}{3m^2 c^4} \\
&= \beta mc^2 + E' + O'
\end{aligned} \tag{2.36}$$

The effect of the transformation is such that odd terms appear in the new Hamiltonian of order $\frac{1}{mc^2}$ or higher. By applying successive transformations, one can achieve higher

order relativistic corrections with the ability to drop the lowest order odd terms leaving behind a purely even Hamiltonian capable of 2-component spinor solutions with definite energies. After two more transformations the Hamiltonian achieves a purely even form of order $\frac{1}{m^2c^4}$.

$$H''' = \beta \left(mc^2 + \frac{O^2}{2mc^2} \right) + E - \frac{1}{8m^2c^4} [O, [O, E]] \quad (2.37)$$

It is this form in which we wish to cast a non-relativistic or semi-relativistic particle in order to theorize an experiment such a neutron interferometer in which we have solely positive energy particles.

2.3 New technique

We have a procedure for expressing the Dirac equation in a generally covariant form 2.21 which is influenced by the metric in which you are working $h_{\mu\nu}$. Further, in the event that all $h_{\mu\nu}$ go to zero we then have the flat space free particle Hamiltonian. This leads us to interpret the additional terms arising from the metric perturbations as a perturbing potential. We can rearrange 2.21 to the form

$$i\hbar \frac{\partial}{\partial t} \psi = H\psi = (mc^2 + c\vec{\alpha} \cdot \vec{p} + \phi(h)) \psi \quad (2.38)$$

If we now apply 3 FW transformations to the Hamiltonian in 2.38, we will have a physical theory in which we can find easy to interpret observables such as average position, momentum, velocity and spin for solutions that have positive definite energy.

Previous author's have been interested in how GR can be incorporated in to the dynamics of a Dirac particle. For comparison I will briefly discuss the goals and techniques of 2 such authors which together illustrate the benefit of a more general approach.

In [11], the authors posed the question if the the equivalence principal will hold for a quantum particle. This is a natural question to ask and a good candidate for the techniques of applying gravity to the Dirac equation. The authors analyzed the Schwarzschild solution in Cartesian coordinates.

$$ds^2 = (1-2\phi)c^2dt^2 - \left[\left(1 + 2\frac{g_1x^1}{c^2}\right)(dx^1)^2 + \left(1 + 2\frac{g_2x^2}{c^2}\right)(dx^2)^2 + \left(1 + 2\frac{g_3x^3}{c^2}\right)(dx^3)^2 \right] \\ - \frac{2}{c^2} [(g_1x^2 + g_2x^1)dx^1dx^2 + (g_2x^3 + g_3x^2)dx^2dx^3 + (g_3x^1 + g_1x^3)dx^3dx^1] \quad (2.39)$$

where $g_i = -\frac{\partial\phi}{\partial x^i}c^2$ and $\phi = \frac{GM_s}{rc^2}$.

In this example, the authors were able to ignore all time-space components in the metric $h_{0i} = 0$ while applying the generalization procedure.

In [12], the authors aimed to use an accelerating and rotating frame of reference in a flat metric to simulate, via the equivalence principle, the effects of the earths Schwarzschild-like gravity (this technique leaves out effects due to the momentum density of the rotating earth) on a particle in a laboratory frame on the earths surface. The accelerating and rotating frame metric used was of the form

$$ds^2 = (dx^0)^2 \left[1 + \frac{2\vec{a} \cdot \vec{x}}{c^2} + \left(\frac{\vec{a} \cdot \vec{x}}{c^2} \right)^2 + \left(\frac{\vec{\omega} \cdot \vec{x}}{c} \right)^2 - \left(\frac{\vec{\omega} \cdot \vec{\omega}}{c^2} \right) (\vec{x} \cdot \vec{x}) \right] \\ - \frac{2}{c} dx^0 d\vec{x} \cdot \vec{\omega} \times \vec{x} - d\vec{x} \cdot d\vec{x} \quad (2.40)$$

where $\vec{a}$ is the proper 3-acceleration and $\vec{\omega}$ the proper rotation experienced by the moving frame.

In this example the author was able avoid use of all spacial perturbation components h_{ij} in the generalization procedure for the Dirac equation.

It is clear that the general techniques share similarities between authors in finding a physical Dirac Hamiltonian in that gravity is incorporated into the theory through the metric field then the resulting Hamiltonian is subjected to a FW transformation. Since, in many cases, having the Hamiltonian in this form is the goal, it would prove useful for future investigations of various metrics of unknown form at this time, to carry this generalization procedure through for an unknown form, meaning to keep all 16 components (10 independent) of the metric perturbation field $h_{\mu\nu}$ and express the FW transformed

Hamiltonian in curved space in terms of $h_{\mu\nu}$ and it's derivatives.

Since we are not eliminating any parts of the metric $h_{\mu\nu}$ we proceed to separate the Dirac equation 2.21 according to whether an object contains the energy term h_{00} , the rotation time-space terms h_{0i} , or the space-space cross terms h_{ij} including the diagonal $i = j$.

To start we will make use of the properties of the tetrad fields 2.12 with a linearized metric

$$\begin{aligned}
\epsilon_{(\beta)}^\nu \epsilon_{(\alpha)}^\mu g_{\nu\mu} &= \epsilon_{(\beta)}^\nu \epsilon_{(\alpha)}^\mu (\eta_{\nu\mu} + h_{\mu\nu}) \\
&= \eta_{(\alpha)(\beta)} \\
e_{(\alpha)}^\mu &= \delta_\alpha^\mu - \frac{1}{2} h_\alpha^\mu \\
e_\mu^{(\alpha)} &= \delta_\mu^{(\alpha)} + \frac{1}{2} h_\mu^{(\alpha)} \\
e_{(\alpha)\mu} &= \eta_{(\alpha)\mu} + \frac{1}{2} h_{(\alpha)\mu} \\
e^{(\alpha)\mu} &= \eta^{(\alpha)\mu} - \frac{1}{2} h^{(\alpha)\mu}
\end{aligned} \tag{2.41}$$

and find the covariant derivative of the tetrad field that appears in 2.21

$$\begin{aligned}
e_{(\beta)\mu;\alpha} &= \frac{\partial e_{(\beta)\mu}}{\partial x^\alpha} - \Gamma_{\mu\alpha}^\lambda e_{(\beta)\lambda} \\
&= \frac{\partial}{\partial x^\alpha} \left(\eta_{(\beta)\mu} + \frac{1}{2} h_{(\beta)\mu} \right) - \frac{1}{2} g^{\lambda\delta} \left(\frac{\partial g_{\delta\mu}}{\partial x^\alpha} + \frac{\partial g_{\delta\alpha}}{\partial x^\mu} - \frac{\partial g_{\alpha\mu}}{\partial x^\delta} \right) \left(\eta_{(\beta)\lambda} + \frac{1}{2} h_{(\beta)\lambda} \right) \\
&= \frac{1}{2} \frac{\partial h_{(\beta)\mu}}{\partial x^\alpha} - \frac{1}{2} \eta^{\lambda\delta} \eta_{(\beta)\lambda} \left(\frac{\partial h_{\delta\mu}}{\partial x^\alpha} + \frac{\partial h_{\delta\alpha}}{\partial x^\mu} - \frac{\partial h_{\alpha\mu}}{\partial x^\delta} \right) \\
&= \frac{1}{2} \frac{\partial h_{(\beta)\mu}}{\partial x^\alpha} - \frac{1}{2} \delta_\beta^\delta \left(\frac{\partial h_{\delta\mu}}{\partial x^\alpha} + \frac{\partial h_{\delta\alpha}}{\partial x^\mu} - \frac{\partial h_{\alpha\mu}}{\partial x^\delta} \right) \\
&= \frac{1}{2} \frac{\partial h_{(\beta)\mu}}{\partial x^\alpha} - \frac{1}{2} \left(\frac{\partial h_{\beta\mu}}{\partial x^\alpha} + \frac{\partial h_{\beta\alpha}}{\partial x^\mu} - \frac{\partial h_{\alpha\mu}}{\partial x^\beta} \right) \\
&= \frac{1}{2} \left(\frac{\partial h_{\alpha\mu}}{\partial x^\beta} - \frac{\partial h_{\beta\alpha}}{\partial x^\mu} \right)
\end{aligned} \tag{2.42}$$

As a result

$$\begin{aligned}
-\frac{i}{4}\sigma^{(\sigma)(\beta)}e_{(\sigma)}^\mu e_{(\beta)\mu;\alpha} &= -\frac{i}{4}\sigma^{(\sigma)(\beta)}\left(\delta_\sigma^\mu - \frac{1}{2}h_\sigma^\mu\right)\frac{1}{2}\left(\frac{\partial h_{\alpha\mu}}{\partial x^\beta} - \frac{\partial h_{\beta\alpha}}{\partial x^\mu}\right) \\
&= -\frac{i}{8}\sigma^{(\sigma)(\beta)}\delta_\sigma^\mu\left(\frac{\partial h_{\alpha\mu}}{\partial x^\beta} - \frac{\partial h_{\beta\alpha}}{\partial x^\mu}\right) \\
&= -\frac{i}{8}\sigma^{(\sigma)(\beta)}\left(\frac{\partial h_{\alpha\sigma}}{\partial x^\beta} - \frac{\partial h_{\beta\alpha}}{\partial x^\sigma}\right) \\
&= -\frac{i}{4}\sigma^{(\sigma)(\beta)}\frac{\partial h_{\sigma\alpha}}{\partial x^\beta}
\end{aligned} \tag{2.43}$$

The last step owing to the anti-symmetric nature of $\sigma^{(\sigma)(\beta)}$.

Inserting 2.43 into 2.21 we get

$$\begin{aligned}
0 &= \left\{ \gamma^{(\gamma)}\epsilon_{(\gamma)}^\alpha \left(\frac{\partial}{\partial x^\alpha} - \frac{i}{4}\sigma^{(\sigma)(\beta)}\epsilon_{(\sigma)}^\mu \epsilon_{(\beta)\mu;\alpha} \right) + k \right\} \psi \\
&= \left\{ \gamma^{(\gamma)} \left(\delta_\gamma^\alpha - \frac{1}{2}h_\gamma^\alpha \right) \left(\frac{\partial}{\partial x^\alpha} - \frac{i}{4}\sigma^{(\sigma)(\beta)}\frac{\partial h_{\sigma\alpha}}{\partial x^\beta} \right) + k \right\} \psi \\
&= \left\{ \left(\gamma^{(\alpha)} - \frac{1}{2}\gamma^{(\gamma)}h_\gamma^\alpha \right) \left(\frac{\partial}{\partial x^\alpha} - \frac{i}{4}\sigma^{(\sigma)(\beta)}\frac{\partial h_{\sigma\alpha}}{\partial x^\beta} \right) + k \right\} \psi \\
&= \left\{ \left(\gamma^{(\alpha)} - \frac{1}{2}\gamma^{(\gamma)}h_\gamma^\alpha \right) \frac{\partial}{\partial x^\alpha} - \frac{i}{4}\left(\gamma^{(\alpha)} - \frac{1}{2}\gamma^{(\gamma)}h_\gamma^\alpha \right) \sigma^{(\sigma)(\beta)}\frac{\partial h_{\sigma\alpha}}{\partial x^\beta} + k \right\} \psi \\
&= \left\{ \left(\gamma^{(\alpha)} - \frac{1}{2}\gamma^{(\gamma)}h_\gamma^\alpha \right) \frac{\partial}{\partial x^\alpha} - \frac{i}{4}\gamma^{(\alpha)}\sigma^{(\sigma)(\beta)}\frac{\partial h_{\sigma\alpha}}{\partial x^\beta} + k \right\} \psi \\
&= \left\{ \left(\gamma^{(0)} - \frac{1}{2}\gamma^{(\gamma)}h_\gamma^0 \right) \frac{\partial}{\partial x^0} + \left(\gamma^{(i)} - \frac{1}{2}\gamma^{(\gamma)}h_\gamma^i \right) \frac{\partial}{\partial x^i} - \frac{i}{4}\gamma^{(\alpha)}\sigma^{(\sigma)(\beta)}\frac{\partial h_{\sigma\alpha}}{\partial x^\beta} + k \right\} \psi \\
&= \left\{ \left(\gamma^{(0)} \left(1 - \frac{1}{2}h_0^0 \right) - \frac{1}{2}\gamma^{(j)}h_j^0 \right) \frac{\partial}{\partial x^0} + \left(\gamma^{(i)} - \frac{1}{2}\gamma^{(\gamma)}h_\gamma^i \right) \frac{\partial}{\partial x^i} - \frac{i}{4}\gamma^{(\alpha)}\sigma^{(\sigma)(\beta)}\frac{\partial h_{\sigma\alpha}}{\partial x^\beta} + k \right\} \psi
\end{aligned} \tag{2.44}$$

With the end goal of obtaining a field equation of the form $i\hbar\frac{\partial\psi}{\partial t} = H\psi$ we rewrite equation 2.44 as

$$\left(\gamma^{(0)} \left(1 - \frac{1}{2}h_0^0 \right) - \frac{1}{2}\gamma^{(j)}h_j^0 \right) \frac{\partial\psi}{\partial x^0} = \left\{ - \left(\gamma^{(i)} - \frac{1}{2}\gamma^{(\gamma)}h_\gamma^i \right) \frac{\partial}{\partial x^i} + \frac{i}{4}\gamma^{(\alpha)}\sigma^{(\sigma)(\beta)}\frac{\partial h_{\sigma\alpha}}{\partial x^\beta} - k \right\} \psi$$

and then multiply by the inverse matrix

$$i\hbar c \left(\gamma^{(0)} \left(1 - \frac{1}{2} h_{00} \right) - \frac{1}{2} \gamma^{(j)} h_j^0 \right)^{-1} = i\hbar c (1 + h_{00}) \left(\gamma^{(0)} \left(1 - \frac{1}{2} h_{00} \right) - \gamma^{(i)} \frac{h_i^0}{2} \right)$$

giving for the Dirac equation

$$\begin{aligned} \frac{\partial \psi}{c \partial t} &= i\hbar c (1 + h_{00}) \left(\gamma^{(0)} \left(1 - \frac{1}{2} h_{00} \right) - \gamma^{(i)} \frac{h_i^0}{2} \right) \left\{ - \left(\gamma^{(i)} - \frac{1}{2} \gamma^{(\gamma)} h_{\gamma}^i \right) \frac{\partial}{\partial x^i} \right. \\ &\quad \left. + \frac{i}{4} \gamma^{(\alpha)} \sigma^{(\sigma)(\beta)} \frac{\partial h_{\sigma\alpha}}{\partial x^{\beta}} - k \right\} \psi \\ H &= i\hbar c (1 + h_{00}) \left(\gamma^{(0)} \left(1 - \frac{1}{2} h_{00} \right) - \gamma^{(i)} \frac{h_i^0}{2} \right) \left\{ - \left(\gamma^{(i)} - \frac{1}{2} \gamma^{(\gamma)} h_{\gamma}^i \right) \frac{\partial}{\partial x^i} \right. \\ &\quad \left. + \frac{i}{4} \gamma^{(\alpha)} \sigma^{(\sigma)(\beta)} \frac{\partial h_{\sigma\alpha}}{\partial x^{\beta}} - k \right\} \end{aligned} \tag{2.45}$$

Working now with the first term on the right of 2.45 we find

$$\begin{aligned}
& -i\hbar c (1 + h_{00}) \left(\gamma^{(0)} \left(1 - \frac{1}{2} h_{00} \right) - \gamma^{(i)} \frac{h_i^0}{2} \right) \left(\gamma^{(i)} - \frac{1}{2} \gamma^{(\gamma)} h_\gamma^i \right) \frac{\partial}{\partial x^i} \\
&= -i\hbar c \left(\gamma^{(0)} \left(1 + \frac{1}{2} h_{00} \right) - \gamma^{(i)} \frac{h_i^0}{2} \right) \left(\gamma^{(i)} - \frac{1}{2} \gamma^{(\gamma)} h_\gamma^i \right) \frac{\partial}{\partial x^i} \\
&= -i\hbar c \gamma^{(0)} \left(1 + \frac{1}{2} h_{00} \right) \left(\gamma^{(i)} - \frac{1}{2} \gamma^{(\gamma)} h_\gamma^i \right) \frac{\partial}{\partial x^i} + i\hbar c \gamma^{(i)} \frac{h_i^0}{2} \gamma^{(i)} \frac{\partial}{\partial x^i} \\
&= -i\hbar c \gamma^{(0)} \left(\gamma^{(i)} \left(1 + \frac{1}{2} h_{00} \right) - \frac{1}{2} \gamma^{(\gamma)} h_\gamma^i \right) \frac{\partial}{\partial x^i} + i\hbar c \gamma^{(i)} \frac{h_i^0}{2} \gamma^{(i)} \frac{\partial}{\partial x^i} \\
&= -i\hbar c \left(\gamma^{(0)} \gamma^{(i)} \left(1 + \frac{1}{2} h_{00} \right) - \frac{1}{2} \gamma^{(0)} \gamma^{(\gamma)} h_\gamma^i \right) \frac{\partial}{\partial x^i} + i\hbar c \gamma^{(i)} \frac{h_i^0}{2} \gamma^{(i)} \frac{\partial}{\partial x^i} \\
&= -i\hbar c \left(\beta \beta \alpha^{(i)} \left(1 + \frac{1}{2} h_{00} \right) - \frac{1}{2} \beta \left(\gamma^{(0)} h_0^i + \gamma^{(j)} h_j^i \right) \right) \frac{\partial}{\partial x^i} + i\hbar c \gamma^{(i)} \frac{h_i^0}{2} \gamma^{(i)} \frac{\partial}{\partial x^i} \\
&= -i\hbar c \left(\beta \beta \alpha^{(i)} \left(1 + \frac{1}{2} h_{00} \right) - \frac{1}{2} \left(\beta \beta h_0^i + \beta \beta \alpha^{(j)} h_j^i \right) \right) \frac{\partial}{\partial x^i} + i\hbar c \beta \alpha^{(i)} \frac{h_i^0}{2} \beta \alpha^{(i)} \frac{\partial}{\partial x^i} \\
&= -i\hbar c \left(\alpha^{(i)} \left(1 + \frac{1}{2} h_{00} \right) - \frac{1}{2} (h_0^i + \alpha h_j^i) \right) \frac{\partial}{\partial x^i} - i\hbar c \alpha^{(i)} \frac{h_i^0}{2} \alpha^{(i)} \frac{\partial}{\partial x^i} \\
&= -i\hbar c \left(\left(1 + \frac{1}{2} h_{00} \right) \alpha^{(i)} \frac{\partial}{\partial x^i} - \frac{1}{2} (h_0^i + \alpha h_j^i) \frac{\partial}{\partial x^i} \right) - i\hbar c \alpha^{(i)} \frac{h_i^0}{2} \alpha^{(i)} \frac{\partial}{\partial x^i} \\
&= -i\hbar c \left(1 + \frac{1}{2} h_{00} \right) \alpha^{(i)} \frac{\partial}{\partial x^i} + \frac{1}{2} i\hbar c (h_0^i + \alpha h_j^i) \frac{\partial}{\partial x^i} - i\hbar c \alpha^{(i)} \frac{h_i^0}{2} \alpha^{(i)} \frac{\partial}{\partial x^i} \\
&= i\hbar c \left(1 + \frac{1}{2} h_{00} \right) \vec{\alpha} \cdot \vec{\nabla} - \frac{1}{2} i\hbar c \vec{h} \cdot \vec{\nabla} + \frac{1}{2} i\hbar c \vec{\alpha} \cdot \overleftrightarrow{h} \cdot \vec{\nabla} - \frac{1}{2} i\hbar c \vec{\alpha} \cdot \vec{h} \vec{\alpha} \cdot \vec{\nabla} \\
&= c \left(1 + \frac{1}{2} h_{00} \right) \vec{\alpha} \cdot \vec{p} - \frac{1}{2} c \vec{h} \cdot \vec{p} + \frac{1}{2} c \vec{\alpha} \cdot \overleftrightarrow{h} \cdot \vec{p} - \frac{1}{2} c \vec{\alpha} \cdot \vec{h} \vec{\alpha} \cdot \vec{p}
\end{aligned} \tag{2.46}$$

where we have introduced a common convention that a dot product of two 3-vectors $\vec{A} \cdot \vec{B}$ is taken to be the negative of the sum of a contravariant and a covariant index

$$\vec{A} \cdot \vec{B} = -A^i B_i = \Sigma A^i B^i \tag{2.47}$$

and by raising and lowering all indices with $\eta_{\mu\nu}$, the momentum vector is defined as

$$\vec{p} = p^i = i\hbar \frac{\partial}{\partial x_i} = -i\hbar \frac{\partial}{\partial x^i} \tag{2.48}$$

We can simplify the last term in 2.46 additionally by expanding

$$\begin{aligned}
& -\frac{1}{2}c\vec{\alpha} \cdot \vec{h}\vec{\alpha} \cdot \vec{p} \\
= & -\frac{1}{2}c (\alpha^1 h^1 + \alpha^2 h^2 + \alpha^3 h^3) (\alpha^1 p^1 + \alpha^2 p^2 + \alpha^3 p^3) \\
= & -\frac{c}{2} (\alpha^1 h^1 \alpha^1 p^1 + \alpha^2 h^2 \alpha^2 p^2 + \alpha^3 h^3 \alpha^3 p^3) \\
& -\frac{c}{2} (\alpha^1 h^1 \alpha^2 p^2 + \alpha^2 h^2 \alpha^1 p^1 + \alpha^3 h^3 \alpha^1 p^1 + \alpha^1 h^1 \alpha^3 p^3 + \alpha^2 h^2 \alpha^3 p^3 + \alpha^3 h^3 \alpha^2 p^2) \\
= & -\frac{1}{2}c (h^1 p^1 + h^2 p^2 + h^3 p^3) \\
& -\frac{c}{2} (\alpha^1 \alpha^2 (h^1 p^2 - h^2 p^1) + \alpha^3 \alpha^1 (h^3 p^1 - h^1 p^3) + \alpha^2 \alpha^3 (h^2 p^3 - h^3 p^2)) \\
= & -\frac{1}{2}c\vec{h} \cdot \vec{p} - \frac{c}{2} (i\sigma^3 (h^1 p^2 - h^2 p^1) + i\sigma^2 (h^3 p^1 - h^1 p^3) + i\sigma^1 (h^2 p^3 - h^3 p^2)) \\
= & -\frac{1}{2}c\vec{h} \cdot \vec{p} - \frac{ic}{2}\vec{\sigma} \cdot \vec{h} \times \vec{p} \tag{2.49}
\end{aligned}$$

which makes 2.46

$$\begin{aligned}
& c \left(1 + \frac{1}{2}h_{00}\right) \vec{\alpha} \cdot \vec{p} - \frac{1}{2}c\vec{h} \cdot \vec{p} + \frac{1}{2}c\vec{\alpha} \cdot \overleftrightarrow{\vec{h}} \cdot \vec{p} - \frac{1}{2}c\vec{\alpha} \cdot \vec{h}\vec{\alpha} \cdot \vec{p} \\
= & c \left(1 + \frac{1}{2}h_{00}\right) \vec{\alpha} \cdot \vec{p} - \frac{1}{2}c\vec{h} \cdot \vec{p} + \frac{1}{2}c\vec{\alpha} \cdot \overleftrightarrow{\vec{h}} \cdot \vec{p} - \frac{1}{2}c\vec{h} \cdot \vec{p} - \frac{ic}{2}\vec{\sigma} \cdot \vec{h} \times \vec{p} \\
= & c \left(1 + \frac{1}{2}h_{00}\right) \vec{\alpha} \cdot \vec{p} - c\vec{h} \cdot \vec{p} + \frac{1}{2}c\vec{\alpha} \cdot \overleftrightarrow{\vec{h}} \cdot \vec{p} - \frac{ic}{2}\vec{\sigma} \cdot \vec{h} \times \vec{p} \tag{2.50}
\end{aligned}$$

Continuing on with the second term in 2.45 we find

$$\begin{aligned}
& i\hbar c (1 + h_{00}) \left(\gamma^{(0)} \left(1 - \frac{1}{2} h_{00} \right) - \gamma^{(i)} \frac{h_i^0}{2} \right) \left\{ \frac{i}{4} \gamma^{(\alpha)} \sigma^{(\sigma)(\beta)} \frac{\partial h_{\sigma\alpha}}{\partial x^\beta} \right\} \\
= & i\hbar c \gamma^{(0)} \left\{ \frac{i}{4} \gamma^{(\alpha)} \sigma^{(\sigma)(\beta)} \frac{\partial h_{\sigma\alpha}}{\partial x^\beta} \right\} \\
= & -\hbar c \gamma^{(0)} \frac{1}{4} \gamma^{(\alpha)} \sigma^{(\sigma)(\beta)} \frac{\partial h_{\sigma\alpha}}{\partial x^\beta} \tag{2.51} \\
= & -\frac{\hbar c}{4} \beta \gamma^{(\alpha)} \sigma^{(\sigma)(i)} \frac{\partial h_{\sigma\alpha}}{\partial x^i} \\
= & -\frac{\hbar c}{4} \beta \gamma^{(0)} \sigma^{(\sigma)(i)} \frac{\partial h_{\sigma 0}}{\partial x^i} - \frac{\hbar c}{4} \beta \gamma^{(j)} \sigma^{(\sigma)(i)} \frac{\partial h_{\sigma j}}{\partial x^i} \\
= & -\frac{\hbar c}{4} \beta \gamma^{(0)} \sigma^{(0)(i)} \frac{\partial h_{00}}{\partial x^i} - \frac{\hbar c}{4} \beta \gamma^{(j)} \sigma^{(0)(i)} \frac{\partial h_{0j}}{\partial x^i} - \frac{\hbar c}{4} \beta \gamma^{(0)} \sigma^{(k)(i)} \frac{\partial h_{k0}}{\partial x^i} - \frac{\hbar c}{4} \beta \gamma^{(j)} \sigma^{(k)(i)} \frac{\partial h_{kj}}{\partial x^i} \\
= & -\frac{\hbar c}{4} \beta \beta \sigma^{(0)(i)} \frac{\partial h_{00}}{\partial x^i} - \frac{\hbar c}{4} \beta \beta \alpha^{(j)} \sigma^{(0)(i)} \frac{\partial h_{0j}}{\partial x^i} - \frac{\hbar c}{4} \beta \beta \sigma^{(k)(i)} \frac{\partial h_{k0}}{\partial x^i} - \frac{\hbar c}{4} \beta \beta \alpha^{(j)} \sigma^{(k)(i)} \frac{\partial h_{kj}}{\partial x^i} \\
= & -\frac{i\hbar c}{8} \left(\left[\gamma^{(0)}, \gamma^{(i)} \right] \frac{\partial h_{00}}{\partial x^i} + \alpha^{(j)} \left[\gamma^{(0)}, \gamma^{(i)} \right] \frac{\partial h_{0j}}{\partial x^i} + \left[\gamma^{(k)}, \gamma^{(i)} \right] \frac{\partial h_{k0}}{\partial x^i} + \alpha^{(j)} \left[\gamma^{(k)}, \gamma^{(i)} \right] \frac{\partial h_{kj}}{\partial x^i} \right)
\end{aligned}$$

We can further simplify 2.51 using the property $[\gamma^{(\alpha)}, \gamma^{(\beta)}] = 2\gamma^{(\alpha)}\gamma^{(\beta)} - 2\eta^{(\alpha)(\beta)}$ which gives

$$\begin{aligned}
& i\hbar c (1 + h_{00}) \left(\gamma^{(0)} \left(1 - \frac{1}{2} h_{00} \right) - \gamma^{(i)} \frac{h_i^0}{2} \right) \left\{ \frac{i}{4} \gamma^{(\alpha)} \sigma^{(\sigma)(\beta)} \frac{\partial h_{\sigma\alpha}}{\partial x^\beta} \right\} \\
= & -\frac{i\hbar c}{8} \left(\left(2\gamma^{(0)} \gamma^{(i)} - 2\eta^{(0)(i)} \right) \frac{\partial h_{00}}{\partial x^i} + \alpha^{(j)} \left(2\gamma^{(0)} \gamma^{(i)} - 2\eta^{(0)(i)} \right) \frac{\partial h_{0j}}{\partial x^i} \right) \\
& -\frac{i\hbar c}{8} \left(\left[\gamma^{(k)}, \gamma^{(i)} \right] \frac{\partial h_{k0}}{\partial x^i} + \alpha^{(j)} \left[\gamma^{(k)}, \gamma^{(i)} \right] \frac{\partial h_{kj}}{\partial x^i} \right) \\
= & -\frac{i\hbar c}{8} \left(2\gamma^{(0)} \gamma^{(i)} \frac{\partial h_{00}}{\partial x^i} + \alpha^{(j)} 2\gamma^{(0)} \gamma^{(i)} \frac{\partial h_{0j}}{\partial x^i} + \left(2\gamma^{(k)} \gamma^{(i)} - 2\eta^{(k)(i)} \right) \frac{\partial h_{k0}}{\partial x^i} \right) \\
& -\frac{i\hbar c}{8} \left(\alpha^{(j)} \left(2\gamma^{(k)} \gamma^{(i)} - 2\eta^{(k)(i)} \right) \frac{\partial h_{kj}}{\partial x^i} \right) \\
= & -\frac{i\hbar c}{8} \left(2\beta\beta\alpha^{(i)} \frac{\partial h_{00}}{\partial x^i} + \alpha^{(j)} 2\beta\beta\alpha^{(i)} \frac{\partial h_{0j}}{\partial x^i} + \left(2\beta\alpha^{(k)} \beta\alpha^{(i)} - 2\eta^{(k)(i)} \right) \frac{\partial h_{k0}}{\partial x^i} \right) \\
& -\frac{i\hbar c}{8} \left(\alpha^{(j)} \left(2\beta\alpha^{(k)} \beta\alpha^{(i)} - 2\eta^{(k)(i)} \right) \frac{\partial h_{kj}}{\partial x^i} \right) \\
= & -\frac{i\hbar c}{4} \left\{ \alpha^{(i)} \frac{\partial h_{00}}{\partial x^i} + \alpha^{(j)} \alpha^{(i)} \frac{\partial h_{0j}}{\partial x^i} + \left(-\beta\beta\alpha^{(k)} \alpha^{(i)} - \eta^{(k)(i)} \right) \frac{\partial h_{k0}}{\partial x^i} \right. \\
& \left. + \alpha^{(j)} \left(\beta\alpha^{(k)} \beta\alpha^{(i)} - \eta^{(k)(i)} \right) \frac{\partial h_{kj}}{\partial x^i} \right\} \\
= & -\frac{i\hbar c}{4} \left(\alpha^{(i)} \frac{\partial h_{00}}{\partial x^i} - \frac{\partial h_0^i}{\partial x^i} + \alpha^{(j)} \beta\alpha^{(k)} \beta\alpha^{(i)} \frac{\partial h_{kj}}{\partial x^i} - \alpha^{(j)} \eta^{(k)(i)} \frac{\partial h_{kj}}{\partial x^i} \right) \\
= & -\frac{i\hbar c}{4} \left(\alpha^{(i)} \frac{\partial h_{00}}{\partial x^i} - \frac{\partial h_0^i}{\partial x^i} - \alpha^{(j)} \alpha^{(k)} \alpha^{(i)} \frac{\partial h_{kj}}{\partial x^i} - \alpha^{(j)} \eta^{(k)(i)} \frac{\partial h_{kj}}{\partial x^i} \right) \\
= & -\frac{i\hbar c}{4} \left(-\vec{\alpha} \cdot \vec{\nabla} h_{00} + \vec{\nabla} \cdot \vec{h} - \alpha^{(j)} \left(-\alpha^{(i)} \alpha^{(k)} - \eta^{(i)(k)} \right) \frac{\partial h_{kj}}{\partial x^i} - \alpha^{(j)} \frac{\partial h_j^i}{\partial x^i} \right) \\
= & -\frac{i\hbar c}{4} \left(-\vec{\alpha} \cdot \vec{\nabla} h_{00} + \vec{\nabla} \cdot \vec{h} + \alpha^{(j)} \alpha^{(i)} \alpha^{(k)} \frac{\partial h_{kj}}{\partial x^i} + \alpha^{(j)} \eta^{(i)(k)} \frac{\partial h_{kj}}{\partial x^i} - \alpha^{(j)} \frac{\partial h_j^i}{\partial x^i} \right) \\
= & -\frac{i\hbar c}{4} \left(-\vec{\alpha} \cdot \vec{\nabla} h_{00} + \vec{\nabla} \cdot \vec{h} + \left(-\alpha^{(i)} \alpha^{(j)} - \eta^{(i)(j)} \right) \alpha^{(k)} \frac{\partial h_{kj}}{\partial x^i} + \alpha^{(j)} \eta^{(i)(k)} \frac{\partial h_{kj}}{\partial x^i} - \vec{\nabla} \cdot \overleftrightarrow{h} \cdot \vec{\alpha} \right) \\
= & -\frac{i\hbar c}{4} \left(-\vec{\alpha} \cdot \vec{\nabla} h_{00} + \vec{\nabla} \cdot \vec{h} - \alpha^{(i)} \alpha^{(j)} \alpha^{(k)} \frac{\partial h_{kj}}{\partial x^i} - \vec{\nabla} \cdot \overleftrightarrow{h} \cdot \vec{\alpha} \right) \\
= & -\frac{i\hbar c}{4} \left(-\vec{\alpha} \cdot \vec{\nabla} h_{00} + \vec{\nabla} \cdot \vec{h} + \vec{\alpha} \cdot \vec{\nabla} \vec{\alpha} \cdot \overleftrightarrow{h} \cdot \vec{\alpha} - \vec{\nabla} \cdot \overleftrightarrow{h} \cdot \vec{\alpha} \right) \\
= & \frac{i\hbar c}{4} \left(\vec{\alpha} \cdot \vec{\nabla} h_{00} - \vec{\nabla} \cdot \vec{h} - \left(\vec{\alpha} \cdot \vec{\nabla} \right) \left(\vec{\alpha} \cdot \overleftrightarrow{h} \cdot \vec{\alpha} \right) + \vec{\nabla} \cdot \overleftrightarrow{h} \cdot \vec{\alpha} \right) \tag{2.52}
\end{aligned}$$

Continuing on with the last piece of equation 2.45

$$\begin{aligned}
& -i\hbar c (1 + h_{00}) \left(\gamma^{(0)} \left(1 - \frac{1}{2} h_{00} \right) - \gamma^{(i)} \frac{h_i^0}{2} \right) k \\
&= -i\hbar c \left(\gamma^{(0)} \left(1 + \frac{1}{2} h_{00} \right) - \gamma^{(i)} \frac{h_i^0}{2} \right) k \\
&= -i\hbar c \left(\beta \left(1 + \frac{1}{2} h_{00} \right) - \beta \alpha^{(i)} \frac{h_i^0}{2} \right) k \\
&= -i\hbar c \beta \left(\left(1 + \frac{1}{2} h_{00} \right) + \frac{\vec{\alpha} \cdot \vec{h}^{(i)}}{2} \right) \frac{imc}{\hbar} \\
&= \beta mc^2 \left(\left(1 + \frac{1}{2} h_{00} \right) + \frac{\vec{\alpha} \cdot \vec{h}}{2} \right)
\end{aligned} \tag{2.53}$$

Using this notation and equations 2.46 through 2.53 the Dirac Hamiltonian with a general metric perturbation 2.45 can now be written as

$$\begin{aligned}
H &= \left(1 + \frac{1}{2} h_{00} \right) c \vec{\alpha} \cdot \vec{p} - c \vec{h} \cdot \vec{p} + \frac{c}{2} \vec{\alpha} \cdot \overleftrightarrow{h} \cdot \vec{p} - \frac{ic}{2} \vec{\sigma} \cdot \vec{h} \times \vec{p} \\
&+ \frac{i\hbar c}{4} \left(\vec{\alpha} \cdot \vec{\nabla} h_{00} - \vec{\nabla} \cdot \vec{h} - \left(\vec{\alpha} \cdot \vec{\nabla} \right) \left(\vec{\alpha} \cdot \overleftrightarrow{h} \cdot \vec{\alpha} \right) + \vec{\nabla} \cdot \overleftrightarrow{h} \cdot \vec{\alpha} \right) \\
&+ \beta mc^2 \left(\left(1 + \frac{1}{2} h_{00} \right) + \frac{\vec{\alpha} \cdot \vec{h}}{2} \right)
\end{aligned} \tag{2.54}$$

Algebraically, equation 2.54 is exactly the same as a flat space free particle piece plus a small perturbing energy. It therefore inherits all of the issues of the mixing of energy states and the inconsistent operator definitions due to the persistent odd terms with odd powers of the $\vec{\alpha}$ matrices. In order to interpret any real particle mechanics from this Hamiltonian, it is necessary to perform a FW transformation to attain an acceptable working theory.

Our last step before we begin to look at particle dynamics for a general metric is to apply three FW transformations 2.37 to the Dirac Hamiltonian 2.54. First we separate the Hamiltonian into its odd and even components as

$$\begin{aligned}
H &= \beta mc^2 + E + O \\
E &= \frac{\beta mc^2}{2} h_{00} - c \vec{h} \cdot \vec{p} - \frac{ic}{2} \vec{\sigma} \cdot \vec{h} \times \vec{p} - \frac{i\hbar c}{4} \vec{\nabla} \cdot \vec{h} \\
O &= \beta mc^2 \frac{\vec{\alpha} \cdot \vec{h}}{2} + \left(1 + \frac{1}{2} h_{00}\right) c \vec{\alpha} \cdot \vec{p} + \frac{c}{2} \vec{\alpha} \cdot \overleftrightarrow{h} \cdot \vec{p} \\
&\quad + \frac{i\hbar c}{4} \left(\vec{\alpha} \cdot \vec{\nabla} h_{00} - \left(\vec{\alpha} \cdot \vec{\nabla} \right) \left(\vec{\alpha} \cdot \overleftrightarrow{h} \cdot \vec{\alpha} \right) + \vec{\nabla} \cdot \overleftrightarrow{h} \cdot \vec{\alpha} \right)
\end{aligned} \tag{2.55}$$

We now proceed, while keeping terms linear in h , in finding each component of 2.37.

$$\begin{aligned}
\beta \frac{O^2}{2mc^2} &= \frac{\beta}{2mc^2} \left\{ \beta mc^2 \frac{\vec{\alpha} \cdot \vec{h}}{2} + \left(1 + \frac{1}{2} h_{00}\right) c \vec{\alpha} \cdot \vec{p} \right. \\
&\quad \left. + \frac{c}{2} \vec{\alpha} \cdot \overleftrightarrow{h} \cdot \vec{p} + \frac{i\hbar c}{4} \left(\vec{\alpha} \cdot \vec{\nabla} h_{00} - \left(\vec{\alpha} \cdot \vec{\nabla} \right) \left(\vec{\alpha} \cdot \overleftrightarrow{h} \cdot \vec{\alpha} \right) + \vec{\nabla} \cdot \overleftrightarrow{h} \cdot \vec{\alpha} \right) \right\}^2 \\
&= \frac{\beta}{2mc^2} c \vec{\alpha} \cdot \vec{p} \left\{ \beta mc^2 \frac{\vec{\alpha} \cdot \vec{h}}{2} + \left(1 + \frac{1}{2} h_{00}\right) c \vec{\alpha} \cdot \vec{p} + \frac{c}{2} \vec{\alpha} \cdot \overleftrightarrow{h} \cdot \vec{p} \right. \\
&\quad \left. + \frac{i\hbar c}{4} \left(\vec{\alpha} \cdot \vec{\nabla} h_{00} - \left(\vec{\alpha} \cdot \vec{\nabla} \right) \left(\vec{\alpha} \cdot \overleftrightarrow{h} \cdot \vec{\alpha} \right) + \vec{\nabla} \cdot \overleftrightarrow{h} \cdot \vec{\alpha} \right) \right\} \\
&\quad + \frac{\beta}{2mc^2} \left\{ \beta mc^2 \frac{\vec{\alpha} \cdot \vec{h}}{2} + \frac{1}{2} h_{00} c \vec{\alpha} \cdot \vec{p} + \frac{c}{2} \vec{\alpha} \cdot \overleftrightarrow{h} \cdot \vec{p} \right. \\
&\quad \left. + \frac{i\hbar c}{4} \left(\vec{\alpha} \cdot \vec{\nabla} h_{00} - \left(\vec{\alpha} \cdot \vec{\nabla} \right) \left(\vec{\alpha} \cdot \overleftrightarrow{h} \cdot \vec{\alpha} \right) + \vec{\nabla} \cdot \overleftrightarrow{h} \cdot \vec{\alpha} \right) \right\} c \vec{\alpha} \cdot \vec{p} \\
&= \frac{\beta}{2mc^2} \left\{ \beta mc^2 \left(c \vec{\alpha} \cdot \vec{p} \frac{\vec{\alpha} \cdot \vec{h}}{2} + \frac{\vec{\alpha} \cdot \vec{h}}{2} c \vec{\alpha} \cdot \vec{p} \right) \right. \\
&\quad + \left(c \vec{\alpha} \cdot \vec{p} \left(1 + \frac{1}{2} h_{00}\right) c \vec{\alpha} \cdot \vec{p} + \frac{1}{2} h_{00} c \vec{\alpha} \cdot \vec{p} c \vec{\alpha} \cdot \vec{p} \right) \\
&\quad + \left(c \vec{\alpha} \cdot \vec{p} \frac{c}{2} \vec{\alpha} \cdot \overleftrightarrow{h} \cdot \vec{p} + \frac{c}{2} \vec{\alpha} \cdot \overleftrightarrow{h} \cdot \vec{p} c \vec{\alpha} \cdot \vec{p} \right) + \frac{i\hbar c}{4} \left(c \vec{\alpha} \cdot \vec{p} \vec{\alpha} \cdot \vec{\nabla} h_{00} + \vec{\alpha} \cdot \vec{\nabla} h_{00} c \vec{\alpha} \cdot \vec{p} \right) \\
&\quad - \frac{i\hbar c}{4} \left(c \vec{\alpha} \cdot \vec{p} \left(\vec{\alpha} \cdot \vec{\nabla} \right) \left(\vec{\alpha} \cdot \overleftrightarrow{h} \cdot \vec{\alpha} \right) + \left(\vec{\alpha} \cdot \vec{\nabla} \right) \left(\vec{\alpha} \cdot \overleftrightarrow{h} \cdot \vec{\alpha} \right) c \vec{\alpha} \cdot \vec{p} \right) \\
&\quad \left. + \frac{i\hbar c}{4} \left(c \vec{\alpha} \cdot \vec{p} \vec{\nabla} \cdot \overleftrightarrow{h} \cdot \vec{\alpha} + \vec{\nabla} \cdot \overleftrightarrow{h} \cdot \vec{\alpha} c \vec{\alpha} \cdot \vec{p} \right) \right\}
\end{aligned} \tag{2.56}$$

In simplifying equation 2.56 further, we shall need frequently the commutation property of

$$\begin{aligned}
[\alpha^i, \alpha^j] &= \alpha^i \alpha^j - \alpha^j \alpha^i = 2\alpha^i \alpha^j + 2\eta^{ij} \\
\alpha^i \alpha^j &= -\alpha^j \alpha^i - 2\eta^{ij}
\end{aligned} \tag{2.57}$$

We can now rewrite the following objects.

$$\begin{aligned}
\left(c\vec{\alpha} \cdot \vec{p} \frac{\vec{\alpha} \cdot \vec{h}}{2} + \frac{\vec{\alpha} \cdot \vec{h}}{2} c\vec{\alpha} \cdot \vec{p} \right) &= \frac{c}{2} \left(\alpha^i p^i \alpha^j h^j + \vec{\alpha} \cdot \vec{h} \vec{\alpha} \cdot \vec{p} \right) \\
&= \frac{c}{2} \left(\alpha^i i\hbar (\alpha^j \nabla^i h^j + \alpha^j h^j \nabla^i) + \vec{\alpha} \cdot \vec{h} \vec{\alpha} \cdot \vec{p} \right) \\
&= \frac{c}{2} \left(i\hbar \alpha^i \alpha^j \nabla^i h^j + \alpha^i \alpha^j h^j i\hbar \nabla^i + \vec{\alpha} \cdot \vec{h} \vec{\alpha} \cdot \vec{p} \right) \\
&= \frac{c}{2} \left(i\hbar \vec{\alpha} \cdot \vec{\nabla} \vec{\alpha} \cdot \vec{h} + (-\alpha^j \alpha^i - 2\eta^{ij}) h^j p^i + \vec{\alpha} \cdot \vec{h} \vec{\alpha} \cdot \vec{p} \right) \\
&= \frac{c}{2} \left(i\hbar \vec{\alpha} \cdot \vec{\nabla} \vec{\alpha} \cdot \vec{h} - \vec{\alpha} \cdot \vec{h} \vec{\alpha} \cdot \vec{p} - 2\eta^{ij} h^j p^i + \vec{\alpha} \cdot \vec{h} \vec{\alpha} \cdot \vec{p} \right) \\
&= \frac{c}{2} \left(i\hbar \vec{\alpha} \cdot \vec{\nabla} \vec{\alpha} \cdot \vec{h} + 2\vec{h} \cdot \vec{p} \right)
\end{aligned} \tag{2.58}$$

$$\begin{aligned}
&\left(c\vec{\alpha} \cdot \vec{p} \left(1 + \frac{1}{2} h_{00} \right) c\vec{\alpha} \cdot \vec{p} + \frac{1}{2} h_{00} c\vec{\alpha} \cdot \vec{p} c\vec{\alpha} \cdot \vec{p} \right) \\
&= \left(c\vec{\alpha} \cdot \vec{p} c\vec{\alpha} \cdot \vec{p} + \frac{c}{2} \vec{\alpha} \cdot \vec{p} h_{00} c\vec{\alpha} \cdot \vec{p} + \frac{1}{2} h_{00} c\vec{\alpha} \cdot \vec{p} c\vec{\alpha} \cdot \vec{p} \right) \\
&= \left(c^2 (\vec{\alpha} \cdot \vec{p})^2 + \frac{c^2}{2} i\hbar \vec{\alpha} \cdot \vec{\nabla} h_{00} \vec{\alpha} \cdot \vec{p} + \frac{c^2}{2} h_{00} \vec{\alpha} \cdot \vec{p} \vec{\alpha} \cdot \vec{p} + \frac{c^2}{2} h_{00} \vec{\alpha} \cdot \vec{p} \vec{\alpha} \cdot \vec{p} \right) \\
&= \left(c^2 (\vec{\alpha} \cdot \vec{p})^2 + \frac{c^2}{2} i\hbar \vec{\alpha} \cdot \vec{\nabla} h_{00} \vec{\alpha} \cdot \vec{p} + c^2 h_{00} (\vec{\alpha} \cdot \vec{p})^2 \right)
\end{aligned} \tag{2.59}$$

$$\begin{aligned}
& \left(c\vec{\alpha} \cdot \vec{p} \frac{c}{2} \vec{\alpha} \cdot \overleftrightarrow{h} \cdot \vec{p} + \frac{c}{2} \vec{\alpha} \cdot \overleftrightarrow{h} \cdot \vec{p} c\vec{\alpha} \cdot \vec{p} \right) \\
= & \frac{c^2}{2} \left(\alpha^i p^i \alpha^j h^{jk} p^k + \vec{\alpha} \cdot \overleftrightarrow{h} \cdot \vec{p} \vec{\alpha} \cdot \vec{p} \right) \\
= & \frac{c^2}{2} \left(\alpha^i \alpha^j i\hbar \nabla^i h^{jk} p^k + \alpha^i \alpha^j h^{jk} p^i p^k + \vec{\alpha} \cdot \overleftrightarrow{h} \cdot \vec{p} \vec{\alpha} \cdot \vec{p} \right) \\
= & \frac{c^2}{2} \left(i\hbar \vec{\alpha} \cdot \vec{\nabla} \vec{\alpha} \cdot \overleftrightarrow{h} \cdot \vec{p} + (-\alpha^j \alpha^i - 2\eta^{ij}) h^{jk} p^i p^k + \vec{\alpha} \cdot \overleftrightarrow{h} \cdot \vec{p} \vec{\alpha} \cdot \vec{p} \right) \\
= & \frac{c^2}{2} \left(i\hbar \vec{\alpha} \cdot \vec{\nabla} \vec{\alpha} \cdot \overleftrightarrow{h} \cdot \vec{p} - \vec{\alpha} \cdot \overleftrightarrow{h} \cdot \vec{p} \vec{\alpha} \cdot \vec{p} + 2 \overleftrightarrow{h} \cdot \vec{p} \cdot \vec{p} + \vec{\alpha} \cdot \overleftrightarrow{h} \cdot \vec{p} \vec{\alpha} \cdot \vec{p} \right) \\
= & \frac{c^2}{2} \left(i\hbar \vec{\alpha} \cdot \vec{\nabla} \vec{\alpha} \cdot \overleftrightarrow{h} \cdot \vec{p} + 2 \overleftrightarrow{h} \cdot \vec{p} \cdot \vec{p} \right) \tag{2.60}
\end{aligned}$$

$$\begin{aligned}
& \left(c\vec{\alpha} \cdot \vec{p} \vec{\alpha} \cdot \vec{\nabla} h_{00} + \vec{\alpha} \cdot \vec{\nabla} h_{00} c\vec{\alpha} \cdot \vec{p} \right) \\
= & \left(ci\hbar \vec{\alpha} \cdot \vec{\nabla} \vec{\alpha} \cdot \vec{\nabla} h_{00} + c\alpha^i \alpha^j \nabla^j h_{00} p^i + \vec{\alpha} \cdot \vec{\nabla} h_{00} c\vec{\alpha} \cdot \vec{p} \right) \\
= & \left(ci\hbar \vec{\alpha} \cdot \vec{\nabla} \vec{\alpha} \cdot \vec{\nabla} h_{00} + c(-\alpha^j \alpha^i - 2\eta^{ij}) \nabla^j h_{00} p^i + \vec{\alpha} \cdot \vec{\nabla} h_{00} c\vec{\alpha} \cdot \vec{p} \right) \\
= & \left(ci\hbar \vec{\alpha} \cdot \vec{\nabla} \vec{\alpha} \cdot \vec{\nabla} h_{00} - c\alpha^j \alpha^i \nabla^j h_{00} p^i - 2c\eta^{ij} \nabla^j h_{00} p^i + \vec{\alpha} \cdot \vec{\nabla} h_{00} c\vec{\alpha} \cdot \vec{p} \right) \\
= & \left(ci\hbar \vec{\alpha} \cdot \vec{\nabla} \vec{\alpha} \cdot \vec{\nabla} h_{00} - \vec{\alpha} \cdot \vec{\nabla} h_{00} c\vec{\alpha} \cdot \vec{p} + 2c\vec{\nabla} h_{00} \cdot \vec{p} + \vec{\alpha} \cdot \vec{\nabla} h_{00} c\vec{\alpha} \cdot \vec{p} \right) \\
= & \left(ci\hbar (\vec{\alpha} \cdot \vec{\nabla})^2 h_{00} + 2c\vec{\nabla} h_{00} \cdot \vec{p} \right) \tag{2.61}
\end{aligned}$$

[illegible]

$$\begin{aligned}
& \left(c\vec{\alpha} \cdot \vec{p} \vec{\nabla} \cdot \overleftrightarrow{h} \cdot \vec{\alpha} + \vec{\nabla} \cdot \overleftrightarrow{h} \cdot \vec{\alpha} c\vec{\alpha} \cdot \vec{p} \right) \\
= & \left(ci\hbar \vec{\alpha} \cdot \vec{\nabla} \vec{\nabla} \cdot \overleftrightarrow{h} \cdot \vec{\alpha} + c\alpha^i \nabla^k h^{kj} \alpha^j p^i + \vec{\nabla} \cdot \overleftrightarrow{h} \cdot \vec{\alpha} c\vec{\alpha} \cdot \vec{p} \right) \\
= & \left(ci\hbar \vec{\alpha} \cdot \vec{\nabla} \vec{\nabla} \cdot \overleftrightarrow{h} \cdot \vec{\alpha} + c\nabla^k h^{kj} (-\alpha^j \alpha^i - 2\eta^{ij}) p^i + \vec{\nabla} \cdot \overleftrightarrow{h} \cdot \vec{\alpha} c\vec{\alpha} \cdot \vec{p} \right) \\
= & \left(ci\hbar \vec{\alpha} \cdot \vec{\nabla} \vec{\nabla} \cdot \overleftrightarrow{h} \cdot \vec{\alpha} - c\nabla^k h^{kj} \alpha^j \alpha^i p^i - c\nabla^k h^{kj} 2\eta^{ij} p^i + \vec{\nabla} \cdot \overleftrightarrow{h} \cdot \vec{\alpha} c\vec{\alpha} \cdot \vec{p} \right) \\
= & \left(ci\hbar \vec{\alpha} \cdot \vec{\nabla} \vec{\nabla} \cdot \overleftrightarrow{h} \cdot \vec{\alpha} - \vec{\nabla} \cdot \overleftrightarrow{h} \cdot \vec{\alpha} c\vec{\alpha} \cdot \vec{p} + 2c\vec{\nabla} \cdot \overleftrightarrow{h} \cdot \vec{p} + \vec{\nabla} \cdot \overleftrightarrow{h} \cdot \vec{\alpha} c\vec{\alpha} \cdot \vec{p} \right) \\
= & \left(ci\hbar \vec{\alpha} \cdot \vec{\nabla} \vec{\nabla} \cdot \overleftrightarrow{h} \cdot \vec{\alpha} + 2c\vec{\nabla} \cdot \overleftrightarrow{h} \cdot \vec{p} \right) \tag{2.63}
\end{aligned}$$

Using the additional properties

$$\begin{aligned}
(\vec{\alpha} \cdot \vec{\nabla})^2 &= \alpha^i \nabla^i \alpha^j \nabla^j \\
&= (\alpha^1 \nabla^1)^2 + (\alpha^2 \nabla^2)^2 + (\alpha^3 \nabla^3)^2 + \nabla^1 \nabla^2 (\alpha^1 \alpha^2 + \alpha^2 \alpha^1) \\
&\quad + \nabla^1 \nabla^3 (\alpha^1 \alpha^3 + \alpha^3 \alpha^1) + \nabla^3 \nabla^2 (\alpha^3 \alpha^2 + \alpha^2 \alpha^3) \\
&= \vec{\nabla} \cdot \vec{\nabla} = (\nabla)^2
\end{aligned} \tag{2.64}$$

$$(\vec{\alpha} \cdot \vec{p})^2 = \vec{p} \cdot \vec{p} = (p)^2 \tag{2.65}$$

and equations 2.57 through 2.63, equation 2.56 can be written in the simplified form

$$\begin{aligned}
\beta \frac{O^2}{2mc^2} &= \frac{\beta}{2mc^2} \left\{ \beta mc^2 \frac{c}{2} \left(i\hbar \vec{\alpha} \cdot \vec{\nabla} \vec{\alpha} \cdot \vec{h} + 2\vec{h} \cdot \vec{p} \right) \right. \\
&\quad + \left(c^2 (\vec{\alpha} \cdot \vec{p})^2 + \frac{c^2}{2} i\hbar \vec{\alpha} \cdot \vec{\nabla} h_{00} \vec{\alpha} \cdot \vec{p} + c^2 h_{00} (\vec{\alpha} \cdot \vec{p})^2 \right) \\
&\quad + \frac{c^2}{2} \left(i\hbar \vec{\alpha} \cdot \vec{\nabla} \vec{\alpha} \cdot \overleftrightarrow{h} \cdot \vec{p} + 2\overleftrightarrow{h} \cdot \vec{p} \cdot \vec{p} \right) + \frac{i\hbar c}{4} \left(ci\hbar (\vec{\alpha} \cdot \vec{\nabla})^2 h_{00} + 2c\vec{\nabla} h_{00} \cdot \vec{p} \right) \\
&\quad - \frac{i\hbar c}{4} \left(ci\hbar (\vec{\alpha} \cdot \vec{\nabla})^2 (\vec{\alpha} \cdot \overleftrightarrow{h} \cdot \vec{\alpha}) + 2c\vec{\nabla} (\vec{\alpha} \cdot \overleftrightarrow{h} \cdot \vec{\alpha}) \cdot \vec{p} \right) \\
&\quad \left. + \frac{i\hbar c}{4} \left(ci\hbar \vec{\alpha} \cdot \vec{\nabla} \vec{\nabla} \cdot \overleftrightarrow{h} \cdot \vec{\alpha} + 2c\vec{\nabla} \cdot \overleftrightarrow{h} \cdot \vec{p} \right) \right\} \\
&= \left\{ \frac{c}{4} \left(i\hbar \vec{\alpha} \cdot \vec{\nabla} \vec{\alpha} \cdot \vec{h} + 2\vec{h} \cdot \vec{p} \right) \right. \\
&\quad + \frac{\beta c^2}{2mc^2} \left((p)^2 + \frac{1}{2} i\hbar \vec{\alpha} \cdot \vec{\nabla} h_{00} \vec{\alpha} \cdot \vec{p} + h_{00} (p)^2 \right) \\
&\quad + \frac{\beta}{4m} \left(i\hbar \vec{\alpha} \cdot \vec{\nabla} \vec{\alpha} \cdot \overleftrightarrow{h} \cdot \vec{p} + 2\overleftrightarrow{h} \cdot \vec{p} \cdot \vec{p} \right) + \frac{i\hbar c^2 \beta}{8mc^2} \left(i\hbar (\nabla)^2 h_{00} + 2\vec{\nabla} h_{00} \cdot \vec{p} \right) \\
&\quad - \frac{i\hbar \beta c^2}{8mc^2} \left(i\hbar (\nabla)^2 (\vec{\alpha} \cdot \overleftrightarrow{h} \cdot \vec{\alpha}) + 2\vec{\nabla} (\vec{\alpha} \cdot \overleftrightarrow{h} \cdot \vec{\alpha}) \cdot \vec{p} \right) \\
&\quad \left. + \frac{i\hbar c^2 \beta}{8mc^2} \left(i\hbar \vec{\alpha} \cdot \vec{\nabla} \vec{\nabla} \cdot \overleftrightarrow{h} \cdot \vec{\alpha} + 2\vec{\nabla} \cdot \overleftrightarrow{h} \cdot \vec{p} \right) \right\}
\end{aligned} \tag{2.66}$$

Next we need to compute to first order in $\hbar$ and power $\frac{1}{mc^2}$ the commutator in equation 2.37. Since the even E terms in equation 2.55 are all of order $\hbar$, for the sake of calculating this commutator we can only take the odd terms which are of zeroth order in $\hbar$ which means

$O_c = c\vec{\alpha} \cdot \vec{p}$. Also, since the commutator has a factor of $\frac{1}{m^2c^4}$, we need terms in E to be of order mc^2 or larger which means for this commutator $E_c = \frac{\beta mc^2}{2}h_{00}$. This yields to the correct order for our calculation

$$\begin{aligned}
-\frac{1}{8m^2c^4} [O_c, [O_c, E_c]] &= -\frac{1}{8m^2c^4} [O_c, O_c E_c - E_c O_c] \\
&= -\frac{1}{8m^2c^4} \{O_c O_c E_c - O_c E_c O_c - (O_c E_c O_c - E_c O_c O_c)\} \\
&= -\frac{1}{8m^2c^4} \{O_c^2 E_c - 2O_c E_c O_c + E_c O_c^2\} \\
&= -\frac{1}{8m^2c^4} \left\{ c^2 (\vec{\alpha} \cdot \vec{p})^2 \frac{\beta mc^2}{2} h_{00} - 2c\vec{\alpha} \cdot \vec{p} \frac{\beta mc^2}{2} h_{00} c\vec{\alpha} \cdot \vec{p} + \frac{\beta mc^2}{2} h_{00} c^2 (p)^2 \right\} \\
&= -\frac{\beta}{16m} \{ (\vec{\alpha} \cdot \vec{p})^2 h_{00} - 2\vec{\alpha} \cdot \vec{p} h_{00} \vec{\alpha} \cdot \vec{p} + h_{00} (p)^2 \} \\
&= -\frac{\beta}{16m} \left\{ i\hbar (\vec{\alpha} \cdot \vec{p}) \vec{\alpha} \cdot \vec{\nabla} h_{00} + \vec{\alpha} \cdot \vec{p} h_{00} \vec{\alpha} \cdot \vec{p} - 2\vec{\alpha} \cdot \vec{p} h_{00} \vec{\alpha} \cdot \vec{p} + h_{00} (p)^2 \right\} \\
&= -\frac{\beta}{16m} \{ i\hbar \alpha^i (i\hbar \nabla^i \alpha^j \nabla^j h_{00} + \alpha^j \nabla^j h_{00} p^i) \\
&\quad - (i\hbar \vec{\alpha} \cdot \vec{\nabla} h_{00} \vec{\alpha} \cdot \vec{p} + h_{00} (\vec{\alpha} \cdot \vec{p})^2) + h_{00} (p)^2 \} \\
&= -\frac{\beta}{16m} \left\{ (i\hbar)^2 (\vec{\alpha} \cdot \vec{\nabla})^2 h_{00} + i\hbar (-\alpha^j \alpha^i - 2\eta^{ij}) \nabla^j h_{00} p^i \right. \\
&\quad \left. - i\hbar \vec{\alpha} \cdot \vec{\nabla} h_{00} \vec{\alpha} \cdot \vec{p} - h_{00} (\vec{\alpha} \cdot \vec{p})^2 + h_{00} (p)^2 \right\} \\
&= -\frac{\beta}{16m} \left\{ (i\hbar)^2 (\nabla)^2 h_{00} + i\hbar (-\alpha^j \alpha^i - 2\eta^{ij}) \nabla^j h_{00} p^i - i\hbar \vec{\alpha} \cdot \vec{\nabla} h_{00} \vec{\alpha} \cdot \vec{p} \right\} \\
&= -\frac{\beta}{16m} \left\{ (i\hbar)^2 (\nabla)^2 h_{00} - i\hbar \vec{\alpha} \cdot \vec{\nabla} h_{00} \vec{\alpha} \cdot \vec{p} + i\hbar 2\vec{\nabla} h_{00} \cdot \vec{p} - i\hbar \vec{\alpha} \cdot \vec{\nabla} h_{00} \vec{\alpha} \cdot \vec{p} \right\} \\
&= \frac{\beta}{16m} \left\{ \hbar^2 (\nabla)^2 h_{00} + 2i\hbar \vec{\alpha} \cdot \vec{\nabla} h_{00} \vec{\alpha} \cdot \vec{p} - i\hbar 2\vec{\nabla} h_{00} \cdot \vec{p} \right\} \tag{2.67}
\end{aligned}$$

Finally, the three consecutive FW transformed Hamiltonian of equation 2.55 can be

rewritten using equations 2.66 and 2.67 as

$$\begin{aligned}
H''' &= \beta \left(mc^2 + \frac{O^2}{2mc^2} \right) + E - \frac{1}{8m^2c^4} [O, [O, E]] \\
&= \beta mc^2 + \frac{c}{4} \left(i\hbar \vec{\alpha} \cdot \vec{\nabla} \vec{\alpha} \cdot \vec{h} + 2\vec{h} \cdot \vec{p} \right) \\
&\quad + \frac{\beta c^2}{2mc^2} \left((p)^2 + \frac{1}{2} i\hbar \vec{\alpha} \cdot \vec{\nabla} h_{00} \vec{\alpha} \cdot \vec{p} + h_{00} (p)^2 \right) \\
&\quad + \frac{\beta}{4m} \left(i\hbar \vec{\alpha} \cdot \vec{\nabla} \vec{\alpha} \cdot \overleftrightarrow{h} \cdot \vec{p} + 2\overleftrightarrow{h} \cdot \vec{p} \cdot \vec{p} \right) + \frac{i\hbar c^2 \beta}{8mc^2} \left(i\hbar (\nabla)^2 h_{00} + 2\vec{\nabla} h_{00} \cdot \vec{p} \right) \\
&\quad - \frac{i\hbar \beta c^2}{8mc^2} \left(i\hbar (\nabla)^2 \left(\vec{\alpha} \cdot \overleftrightarrow{h} \cdot \vec{\alpha} \right) + 2\vec{\nabla} \left(\vec{\alpha} \cdot \overleftrightarrow{h} \cdot \vec{\alpha} \right) \cdot \vec{p} \right) \\
&\quad + \frac{i\hbar c^2 \beta}{8mc^2} \left(i\hbar \vec{\alpha} \cdot \vec{\nabla} \vec{\nabla} \cdot \overleftrightarrow{h} \cdot \vec{\alpha} + 2\vec{\nabla} \cdot \overleftrightarrow{h} \cdot \vec{p} \right) \\
&\quad + \frac{\beta mc^2}{2} h_{00} - c\vec{h} \cdot \vec{p} - \frac{ic}{2} \vec{\sigma} \cdot \vec{h} \times \vec{p} - \frac{i\hbar c}{4} \vec{\nabla} \cdot \vec{h} \\
&\quad + \frac{\beta}{16m} \left\{ \hbar^2 (\nabla)^2 h_{00} + 2i\hbar \vec{\alpha} \cdot \vec{\nabla} h_{00} \vec{\alpha} \cdot \vec{p} - i\hbar 2\vec{\nabla} h_{00} \cdot \vec{p} \right\} \\
&= \beta mc^2 \left(1 + \frac{1}{2} h_{00} \right) + \frac{\beta}{2m} (1 + h_{00}) (p)^2 + \frac{c}{4} i\hbar \vec{\alpha} \cdot \vec{\nabla} \vec{\alpha} \cdot \vec{h} - \frac{c}{2} \vec{h} \cdot \vec{p} \\
&\quad + \frac{\beta}{4m} i\hbar \vec{\alpha} \cdot \vec{\nabla} \vec{\alpha} \cdot \overleftrightarrow{h} \cdot \vec{p} + \frac{\beta}{2m} \overleftrightarrow{h} \cdot \vec{p} \cdot \vec{p} + \frac{\hbar^2 \beta}{8m} (\nabla)^2 \left(\vec{\alpha} \cdot \overleftrightarrow{h} \cdot \vec{\alpha} \right) \\
&\quad - \frac{i\hbar \beta}{4m} \vec{\nabla} \left(\vec{\alpha} \cdot \overleftrightarrow{h} \cdot \vec{\alpha} \right) \cdot \vec{p} - \frac{\hbar^2 \beta}{8m} \vec{\alpha} \cdot \vec{\nabla} \vec{\nabla} \cdot \overleftrightarrow{h} \cdot \vec{\alpha} + \frac{i\hbar \beta}{4m} \vec{\nabla} \cdot \overleftrightarrow{h} \cdot \vec{p} \\
&\quad - \frac{ic}{2} \vec{\sigma} \cdot \vec{h} \times \vec{p} - \frac{i\hbar c}{4} \vec{\nabla} \cdot \vec{h} + \frac{3\beta}{8m} i\hbar \vec{\alpha} \cdot \vec{\nabla} h_{00} \vec{\alpha} \cdot \vec{p} + \frac{\beta}{8m} i\hbar \vec{\nabla} h_{00} \cdot \vec{p} - \frac{\hbar^2 \beta}{16m} (\nabla)^2 h_{00} \\
&= \beta mc^2 \left(1 + \frac{1}{2} h_{00} \right) + \frac{\beta}{2m} (1 + h_{00}) (p)^2 + \frac{c}{4} i\hbar \left(\vec{\nabla} \cdot \vec{h} + i\vec{\sigma} \cdot \vec{\nabla} \times \vec{h} \right) - \frac{c}{2} \vec{h} \cdot \vec{p} \\
&\quad + \frac{\beta}{4m} i\hbar \vec{\alpha} \cdot \vec{\nabla} \vec{\alpha} \cdot \overleftrightarrow{h} \cdot \vec{p} + \frac{\beta}{2m} \overleftrightarrow{h} \cdot \vec{p} \cdot \vec{p} + \frac{\hbar^2 \beta}{8m} (\nabla)^2 \left(\vec{\alpha} \cdot \overleftrightarrow{h} \cdot \vec{\alpha} \right) \\
&\quad - \frac{i\hbar \beta}{4m} \vec{\nabla} \left(\vec{\alpha} \cdot \overleftrightarrow{h} \cdot \vec{\alpha} \right) \cdot \vec{p} - \frac{\hbar^2 \beta}{8m} \vec{\alpha} \cdot \vec{\nabla} \vec{\nabla} \cdot \overleftrightarrow{h} \cdot \vec{\alpha} + \frac{i\hbar \beta}{4m} \vec{\nabla} \cdot \overleftrightarrow{h} \cdot \vec{p} \\
&\quad - \frac{ic}{2} \vec{\sigma} \cdot \vec{h} \times \vec{p} - \frac{i\hbar c}{4} \vec{\nabla} \cdot \vec{h} + \frac{3\beta}{8m} i\hbar \vec{\alpha} \cdot \vec{\nabla} h_{00} \vec{\alpha} \cdot \vec{p} + \frac{\beta}{8m} i\hbar \vec{\nabla} h_{00} \cdot \vec{p} - \frac{\hbar^2 \beta}{16m} (\nabla)^2 h_{00} \\
&= \beta mc^2 \left(1 + \frac{1}{2} h_{00} \right) + (1 + h_{00}) \frac{\beta}{2m} (p)^2 - \frac{c}{2} \vec{h} \cdot \vec{p} - \frac{ic}{2} \vec{\sigma} \cdot \vec{h} \times \vec{p} - \frac{c}{4} \hbar \vec{\sigma} \cdot \vec{\nabla} \times \vec{h} \\
&\quad + \frac{3\beta}{8m} i\hbar \vec{\alpha} \cdot \vec{\nabla} h_{00} \vec{\alpha} \cdot \vec{p} + \frac{\beta}{8m} i\hbar \vec{\nabla} h_{00} \cdot \vec{p} - \frac{\hbar^2 \beta}{16m} (\nabla)^2 h_{00} \\
&\quad + \frac{\beta}{4m} i\hbar \vec{\alpha} \cdot \vec{\nabla} \vec{\alpha} \cdot \overleftrightarrow{h} \cdot \vec{p} + \frac{\beta}{2m} \overleftrightarrow{h} \cdot \vec{p} \cdot \vec{p} + \frac{\hbar^2 \beta}{8m} (\nabla)^2 \left(\vec{\alpha} \cdot \overleftrightarrow{h} \cdot \vec{\alpha} \right) \\
&\quad - \frac{i\hbar \beta}{4m} \vec{\nabla} \left(\vec{\alpha} \cdot \overleftrightarrow{h} \cdot \vec{\alpha} \right) \cdot \vec{p} - \frac{\hbar^2 \beta}{8m} \vec{\alpha} \cdot \vec{\nabla} \vec{\nabla} \cdot \overleftrightarrow{h} \cdot \vec{\alpha} + \frac{i\hbar \beta}{4m} \vec{\nabla} \cdot \overleftrightarrow{h} \cdot \vec{p} \tag{2.68}
\end{aligned}$$

2.4 Application to neutron interference in the gravitational field of a ring laser

Now that we have an “even” Hamiltonian for a Dirac particle perturbed by a general gravitational metric perturbation 2.68, we can calculate the phase shift of a free particle due to this perturbation. We do this in the same manner as [5] with the purpose of finding the phase shift of two particle paths of a split beam interferometer for comparison at a recombination point. The phase shift is of the form

$$\Psi = \Psi_0 e^{i\delta\phi} = M_0 e^{i(\vec{k}_0 \cdot \vec{x} + \delta\phi - \omega_0 t)} \quad (2.69)$$

where M_0 is a 2-component spinor column matrix with norm $M_0^\dagger M_0 = 1$.

Using 2.69, the Dirac equation for a small potential can now be written

$$i\hbar \frac{\partial}{\partial t} \Psi = E_0 \Psi = \{H_0 + H'\} \Psi \quad (2.70)$$

The $H_0 \Psi$ term can be expanded as

$$H_0 \Psi = \left\{ \beta mc^2 + \frac{\beta}{2m} p^2 \right\} M_0 e^{i(\vec{k}_0 \cdot \vec{x} + \delta\phi - \omega_0 t)} \quad (2.71)$$

and

$$\begin{aligned}
& \frac{\beta}{2m} p^2 M_0 e^{i(\vec{k}_0 \cdot \vec{x} + \delta\phi - \omega_0 t)} \\
= & \frac{\beta}{2m} \vec{p} \left\{ i\hbar M_0 \vec{\nabla}(i\delta\phi) e^{i(\vec{k}_0 \cdot \vec{x} + \delta\phi - \omega_0 t)} \right. \\
& \left. + i\hbar M_0 \vec{\nabla}(i\vec{k}_0 \cdot \vec{x}) e^{i(\vec{k}_0 \cdot \vec{x} + \delta\phi - \omega_0 t)} \right\} \\
= & -\frac{\beta\hbar^2}{2m} \left\{ M_0 \vec{\nabla} \cdot \vec{\nabla}(i\delta\phi) e^{i(\vec{k}_0 \cdot \vec{x} + \delta\phi - \omega_0 t)} + M_0 \vec{\nabla}(i\delta\phi) \vec{\nabla}(i\delta\phi) e^{i(\vec{k}_0 \cdot \vec{x} + \delta\phi - \omega_0 t)} \right. \\
& + M_0 \vec{\nabla}(i\delta\phi) \vec{\nabla}(i\vec{k}_0 \cdot \vec{x}) e^{i(\vec{k}_0 \cdot \vec{x} + \delta\phi - \omega_0 t)} + M_0 \vec{\nabla} \cdot \vec{\nabla}(i\vec{k}_0 \cdot \vec{x}) e^{i(\vec{k}_0 \cdot \vec{x} + \delta\phi - \omega_0 t)} \\
& \left. + M_0 \vec{\nabla}(i\vec{k}_0 \cdot \vec{x}) \cdot \vec{\nabla}(i\delta\phi) e^{i(\vec{k}_0 \cdot \vec{x} + \delta\phi - \omega_0 t)} + M_0 \vec{\nabla}(i\vec{k}_0 \cdot \vec{x}) \cdot \vec{\nabla}(i\vec{k}_0 \cdot \vec{x}) e^{i(\vec{k}_0 \cdot \vec{x} + \delta\phi - \omega_0 t)} \right\} \\
= & -\frac{\beta\hbar^2}{2m} \left\{ iM_0 \vec{\nabla} \cdot \vec{\nabla}(\delta\phi) e^{i(\vec{k}_0 \cdot \vec{x} + \delta\phi - \omega_0 t)} + M_0 \vec{\nabla}(\delta\phi) \cdot \vec{k}_0 e^{i(\vec{k}_0 \cdot \vec{x} + \delta\phi - \omega_0 t)} \right. \\
& \left. - M_0 \vec{k}_0 \cdot \vec{k}_0 e^{i(\vec{k}_0 \cdot \vec{x} + \delta\phi - \omega_0 t)} + M_0 \vec{\nabla}(\delta\phi) \cdot \vec{k}_0 e^{i(\vec{k}_0 \cdot \vec{x} + \delta\phi - \omega_0 t)} \right\} \\
= & -\frac{\beta\hbar^2}{2m} \left\{ iM_0 \vec{\nabla} \cdot \vec{\nabla}(\delta\phi) e^{i(\vec{k}_0 \cdot \vec{x} + \delta\phi - \omega_0 t)} - M_0 |k_0|^2 e^{i(\vec{k}_0 \cdot \vec{x} + \delta\phi - \omega_0 t)} \right. \\
& \left. + 2M_0 \vec{\nabla}(\delta\phi) \cdot \vec{k}_0 e^{i(\vec{k}_0 \cdot \vec{x} + \delta\phi - \omega_0 t)} \right\} \tag{2.72}
\end{aligned}$$

Multiplying equation 2.70 on the left by $\Psi^\dagger$, using equations 2.71, 2.72 and the relation $E_0 = mc^2 + \frac{\hbar^2 |k_0|^2}{2m}$ for 2-component spinors with positive energy yields

$$\begin{aligned}
E_0 &= \Psi^\dagger \{H_0 + H'\} \Psi \\
&= mc^2 + H' - \frac{\hbar^2}{2m} \left\{ i\vec{\nabla} \cdot \vec{\nabla}(\delta\phi) - |k_0|^2 + 2\vec{\nabla}(\delta\phi) \cdot \vec{k}_0 \right\} \tag{2.73}
\end{aligned}$$

and

$$H' = \frac{\hbar^2}{2m} \left\{ i\vec{\nabla} \cdot \vec{\nabla}(\delta\phi) + 2\vec{\nabla}(\delta\phi) \cdot \vec{k}_0 \right\} \tag{2.74}$$

Dropping the second derivative term for large $\vec{k}_0$, as was done in the Shrödinger case, and using the semi classical approximation for $\vec{k}_0$ gives

$$\begin{aligned}
H' &= \frac{\hbar^2}{m} \vec{\nabla}(\delta\phi) \cdot \vec{k}_0 = -\frac{\hbar^2}{m} \nabla_i(\delta\phi) k_0^i \\
&= -\frac{\hbar}{m} \frac{\partial(\delta\phi)}{\partial x^i} \frac{m \partial x_0^i}{\partial t} \\
&= -\hbar \frac{\partial(\delta\phi)}{\partial t} \tag{2.75}
\end{aligned}$$

This gives a solution for $\delta\phi$ in the same manner as the Shrödinger case, in the form of a time integral over the path of a free particle trajectory.

$$\delta\phi(\vec{x}) = -\frac{1}{\hbar} \int_0^{T(x_0)} dt' H'(t') \quad (2.76)$$

The path of each particle beam can, in a semi-classical approach, be approximated by the path of a classical free particle. We can then integrate equation 2.76 along this trajectory to find the phase shift of each particle beam at the recombination point.

This means inserting H' from equation 2.68 using the ring laser metric from equations 1.17 through 1.25. Considering this metric does not have off diagonal space-space components h^{ij} , all $\overleftrightarrow{h}$ terms in 2.68 can be simplified. Using the first order effect $h^{\mu\nu} p^i = h^{\mu\nu} i\hbar \nabla^i (k_0^1 x^1 + k_0^2 x^2 + k_0^3 x^3) = h^{\mu\nu} \hbar k_0^i$, and for the same reason we dropped $\vec{\nabla} \cdot \vec{\nabla}(\delta\phi)$ in equation 2.73, we also drop first and second derivatives of $h^{\mu\nu}$ for large k_0 , meaning, $\frac{\partial^2 h^{\mu\nu}}{\partial(|k_0|x^i)\partial(|k_0|x^j)} \ll \frac{\partial h^{\mu\nu}}{\partial(|k_0|x^k)} \ll h^{\mu\nu}$.

With these approximations, the perturbed energy function can be written in the simplified form

$$\begin{aligned} H' &= mc^2 \frac{1}{2} h_{00} + h_{00} \frac{\beta}{2m} (p)^2 - \frac{c}{2} \vec{h} \cdot \vec{p} - \frac{ic}{2} \vec{\sigma} \cdot \vec{h} \times \vec{p} + \frac{\beta}{2m} \overleftrightarrow{h} \cdot \vec{p} \cdot \vec{p} \\ &= \frac{mc^2}{2} h_{00} + \frac{\beta}{2m} h_{00} (p)^2 - \frac{c\hbar}{2} (h^{01} k_0^1 + h^{02} k_0^2) - \frac{ic\hbar}{2} \vec{\sigma} \cdot \vec{h} \times \vec{k}_0 + \frac{\beta\hbar^2}{2m} \{h^{11} (k_0^1)^2 + h^{22} (k_0^2)^2\} \end{aligned} \quad (2.77)$$

For a particle initially moving in the $+x$ direction or the $+y$ direction, this simplifies further to

$$H'_x = \frac{mc^2}{2} h_{00} + \frac{\hbar^2}{2m} h_{00} (k_0)^2 - \frac{c\hbar}{2} h^{01} k_0 + \frac{ic\hbar}{2} (\sigma^3 h^{02} k_0) + \frac{\hbar^2}{2m} h^{11} (k_0)^2 \quad (2.78)$$

$$H'_y = \frac{mc^2}{2} h_{00} + \frac{\hbar^2}{2m} h_{00} (k_0)^2 - \frac{c\hbar}{2} h^{02} k_0 - \frac{ic\hbar}{2} \sigma^3 h^{01} k_0 + \frac{\hbar^2}{2m} h^{22} (k_0)^2 \quad (2.79)$$

Since to first order, the integral in equation 2.76 will be taken over the path of the free

particle, it can be rewritten as

$$\delta\phi(\vec{x}) = -\frac{1}{\hbar} \int_0^{T(x_0)} dt' H'(t') \frac{k_0}{k_0} = -\frac{1}{\hbar k_0} \int_0^{T(x_0)} dt' H'(t') \frac{m}{\hbar} \frac{dx'}{dt'} = -\frac{m}{\hbar^2 k_0} \int_0^{x_0} dx' H'(x) \quad (2.80)$$

We now aim to find the phase difference of two paths of a split beam interferometer for the scenario depicted in figure 2.1. Here we have a neutron beam split at point A , at which point we will assume the free particle will enter the perturbing region, and is the point where we consider the two beams to be coherent. The two beams will then travel their separate paths, accumulating unique phase shifts found using the line integral of equation 2.80, where at the recombination point D , we are interested in the expression which represents our interference result which is the measurable difference

$$\begin{aligned} \text{Measurable_Phase_Difference} &= \delta\phi_{ABD} - \delta\phi_{ACD} \\ &= -\frac{m}{\hbar^2 k_0} \left\{ \int_{ABD} dx' H'(x) - \int_{ACD} dx' H'(x) \right\} \quad (2.81) \end{aligned}$$

It should be noted that a possible additional effect, shown by Mannheim [6], due to the acceleration of the reflecting surfaces at points B and C has been considered in appendix 3.2 and shown to not to contribute in this specific setup.

To understand the results of these integrals, it will be useful to look at the position dependence of the functions in the integrands. According to equations 2.78 and 2.79, whether the particle is traveling in the x direction or the y direction it will always experience a phase shift due to the $\left\{ \frac{mc^2}{2} + \frac{\hbar^2}{2m} (k_0)^2 \right\} h_{00}$ term in equation 2.80. That this term will go to zero can be shown visually by expressing the line integral in equation 2.80 as an average of the integrand. This can be written

$$\int_0^{x_0} dx' H'(x) = \bar{H}' X_{tot} \quad (2.82)$$

where X_{tot} is the total length of the line integral. This means that we can express the measurable phase difference between the paths due to the h_{00} term, since paths ABD and

Figure 2.1: Ring Laser

Figure 2.2: Function $\frac{4\pi}{\kappa\rho}h^{00} = -[\phi_{(1)} + \phi_{(2)} + \phi_{(3)} + \phi_{(4)}]$ for a ring laser of side length $a = 100$ units. The bold contour outlining the plotted area is at a distance of 1 unit from each side of the ring laser.

ACD travel the same coordinate distance, as

$$\begin{aligned}
 & -\frac{m}{\hbar^2 k_0} \left\{ \frac{mc^2}{2} + \frac{\hbar^2}{2m} (k_0)^2 \right\} \left\{ \int_{ABD} dx' h'_{00}(x) - \int_{ACD} dx' h'_{00}(x) \right\} \\
 & = -\frac{m}{\hbar^2 k_0} \left\{ \frac{mc^2}{2} + \frac{\hbar^2}{2m} (k_0)^2 \right\} X_{ACD} \{ \bar{h}_{00-ABD} - \bar{h}_{00-ACD} \} \quad (2.83)
 \end{aligned}$$

Looking now at the function h_{00} in figure 2.2 we can see that from the symmetry of the function along the two paths (the bold contour), the average value is the same and the difference thus is zero if we consider a symmetric interferometer where it travels an equal distance along each path. The h_{00} part of the metric, in this experiment, will cause no measurable phase difference.

$$-\frac{m}{\hbar^2 k_0} \left\{ \frac{mc^2}{2} + \frac{\hbar^2}{2m} (k_0)^2 \right\} \left\{ \int_{ABD} dx' h'_{00}(x) - \int_{ACD} dx' h'_{00}(x) \right\} = 0 \quad (2.84)$$

It is a simple extension now to analyze the contribution to the quantity in equation 2.81 of the h^{11} and h^{22} parts of the metric. For the same arguments as above, we can write their

Figure 2.3: The function $\frac{4\pi}{\kappa\rho}h^{11} = -[\phi_{(1)} + \phi_{(3)}]$ for a ring laser of side length $a = 100$ units. The bold contour outlining the plotted area is at a distance of 1 unit from each side of the ring laser.

contribution to the measurable phase difference as

$$\begin{aligned}
 & -\frac{m}{\hbar^2 k_0} \left\{ \frac{\hbar^2}{2m} (k_0)^2 \right\} \left\{ \int_{ABD} dx' (h'^{11} + h'^{22}) - \int_{ACD} dx' (h'^{11} + h'^{22}) \right\} \\
 & -\frac{m}{\hbar^2 k_0} \left\{ \frac{\hbar^2}{2m} (k_0)^2 \right\} \left\{ X_{AB} (\bar{h}_{AB}^{11} - \bar{h}_{CD}^{11}) - Y_{BD} (\bar{h}_{BD}^{22} - \bar{h}_{AC}^{22}) \right\} = 0 \quad (2.85)
 \end{aligned}$$

from the symmetry of the functions shown in figures 2.3 and 2.4.

In a similar way, h^{01} and h^{02} , will contribute differently to the phase difference whether the particle is traveling along the x direction or the y direction, as seen from equations 2.78 and 2.79, however, their contributions do not cancel in this case. Let us follow the integrals more carefully by explicitly showing the paths to be followed. The only nonzero

Figure 2.4: The function $\frac{4\pi}{\kappa\rho}h^{22} = -[\phi_{(2)} + \phi_{(4)}]$ for a ring laser of side length $a = 100$ units. The bold contour outlining the plotted area is at a distance of 1 unit from each side of the ring laser.

contributions to the phase shift are now

$$\begin{aligned}
& \delta\phi_{ABD} - \delta\phi_{ACD} \\
&= -\frac{m}{\hbar^2 k_0} \left\{ \int_{ABD} dx' H'(x) - \int_{ACD} dx' H'(x) \right\} \\
&= -\frac{m}{\hbar^2 k_0} \left\{ \int_{AB} dx' \left\{ -\frac{c\hbar}{2} h'^{01} k_0 + \frac{ic\hbar}{2} (\sigma^3 h'^{02} k_0) \right\} + \int_{BD} dy' \left\{ -\frac{c\hbar}{2} h'^{02} k_0 - \frac{ic\hbar}{2} \sigma^3 h'^{01} k_0 \right\} \right. \\
&\quad \left. + \frac{m}{\hbar^2 k_0} \int_{AC} dy' \left\{ -\frac{c\hbar}{2} h'^{02} k_0 - \frac{ic\hbar}{2} (\sigma^3 h'^{01} k_0) \right\} + \int_{CD} dx' \left\{ -\frac{c\hbar}{2} h'^{01} k_0 + \frac{ic\hbar}{2} \sigma^3 h'^{02} k_0 \right\} \right\} \\
&= \frac{m}{\hbar^2 k_0} \frac{c\hbar}{2} k_0 \{ X_{AB} \bar{h}_{AB}^{01} + Y_{BD} \bar{h}_{BD}^{02} - X_{CD} \bar{h}_{CD}^{01} - Y_{AC} \bar{h}_{AC}^{02} \} \\
&\quad + \frac{m}{\hbar^2 k_0} \frac{ic\hbar}{2} (\sigma^3 k_0) \{ -X_{AB} \bar{h}_{AB}^{02} + Y_{BD} \bar{h}_{BD}^{01} + X_{CD} \bar{h}_{CD}^{02} - Y_{AC} \bar{h}_{AC}^{01} \} \\
&= \frac{mc}{2\hbar} \{ X_{AB} (\bar{h}_{AB}^{01} - \bar{h}_{CD}^{01}) + Y_{BD} (\bar{h}_{BD}^{02} - \bar{h}_{AC}^{02}) \} \\
&\quad + \frac{imc\sigma^3}{2\hbar} \{ -X_{AB} (\bar{h}_{AB}^{02} - \bar{h}_{CD}^{02}) + Y_{BD} (\bar{h}_{BD}^{01} - \bar{h}_{AC}^{01}) \} \tag{2.86}
\end{aligned}$$

On comparing our final result equation 2.86 with figures 2.5 and 2.6, we can see that the momentum term increases in magnitude along each leg, due to opposite path sides of

Figure 2.5: The function $\frac{4\pi}{\kappa\rho}h^{01} = -[\phi_{(1)} - \phi_{(3)}]$ for a ring laser of side length $a = 100$ units. The bold contour outlining the plotted area is at a distance of 1 unit from each side of the ring laser.

the neutron beam experiencing the opposite sign strength of the function, however, the spin terms go to zero not from the additive nature of the expression in equation 2.86, but from the figures it can be seen that the values $\bar{h}_{AB}^{02}, \bar{h}_{CD}^{02}, \bar{h}_{BD}^{01}, \bar{h}_{AC}^{01}$ go to zero individually. From equations 2.78 and 2.79, this evidently comes from the result of the spin being precessed by the frame dragging momentum energy of the gravity generating source traveling perpendicularly with respect to the momentum of the spinning particle. Therefore, during a trip along any one leg of the neutron beams path in figure 2.1, the particle feels the momentum of the ring laser section behind it for the first half of the leg, and in front of it for the last half, canceling the result overall since opposite sides of the ring laser have energy momentum in opposite directions.

Turning now back to equation 2.86 we can express the only non-zero contribution to the total phase difference along the two paths of the interferometer as, from the symmetry of

Figure 2.6: The function $\frac{4\pi}{\kappa\rho}h^{02} = -[\phi_{(2)} - \phi_{(4)}]$ for a ring laser of side length $a = 100$ units. The bold contour outlining the plotted area is at a distance of 1 unit from each side of the ring laser.

the functions, four times the magnitude of the effect along any one leg to thus give

$$\begin{aligned}
\delta\phi_{ABD} - \delta\phi_{ACD} &= \frac{mc}{2\hbar} \{X_{AB} (\bar{h}_{AB}^{01} - \bar{h}_{CD}^{01}) + Y_{BD} (\bar{h}_{BD}^{02} - \bar{h}_{AC}^{02})\} \\
&= \frac{2mcL_n}{\hbar} \bar{h}_{AB}^{01} \\
&= -\frac{2mcL_n}{\hbar} \frac{\kappa\rho}{4\pi} \overline{[\phi_{(1)} - \phi_{(3)}]}_{AB} \\
&= \frac{mc8\pi G\rho}{\hbar 2\pi c^4} \left\{ -L_n \overline{[\phi_{(1)} - \phi_{(3)}]}_{AB} \right\} \\
&= \frac{4mGP(W)}{\hbar c^4} \left\{ -L_n \overline{[\phi_{(1)} - \phi_{(3)}]}_{AB} \right\} \tag{2.87}
\end{aligned}$$

Where L_n is the length of one leg of the neutron beam and $P(W)$ is the power of the ring laser beams in Watts.

The average function $-L_n \overline{[\phi_{(1)} - \phi_{(3)}]}$, which is the average value along the outside contour near the x axis in figure 2.5, is a function of the relative dimensions of the ring laser and the interferometer. This is a tunable parameter in our result, equation 2.87. Also tunable is the power of the ring laser which is generating the gravitational field. This is a satisfying result for our theoretical work which makes an experiment to confirm these effects more likely.

2.5 Conclusions

Motivated by the predictions and results of the COW experiment for the phase shift of a Shrödinger particle due to a newtonian potential, we had set out to find the analogous procedure for a Dirac particle in the gravitational field of a unidirectional ring laser. In pursuit of this goal, it was necessary to evaluate the methods previously published in calculating gravitational perturbations to the Dirac wave function.

The procedure for expressing the Dirac Hamiltonian in curved space used in our work has been utilized by a number of authors already. In our opinion, our presentation of the procedure is much more general. It has often been the topic in these papers to compare the Hamiltonian resulting from their chosen metric to the work of others using different metrics. The general form of equation 2.81 represents the final result of any of those papers, but without the specific metric components included or excluded. An alternative approach

in the future if one would like to compare the Dirac Hamiltonian in a curved space to a previously published one, they may use this new equation with considerably less work. One may also endeavor to compare at this level, what types of metrics make each term in equation 2.81 either vanish or stay relevant for the purpose of gaining more insight on the importance of weak gravity in quantum mechanics. It is our plan to look at this relationship in future research.

In Einsteins metric theory all energy, whether electromagnetic or material, will generate a gravitational field. In this thesis, new techniques were developed and applied to the calculation of the interference of two neutron beams in the gravitational field of a ring laser. The result of the interference phase shift of a neutron interferometer due to the gravitational inertial frame dragging of the ring laser in equation 2.87 is the first of it's kind and was expected to exist as a theoretical result. This gravitational frame dragging effect is a parallel to the effect that a rotating mass has on an orbiting satellite and provides an alternate method for possible experimental verification for gravitational frame dragging.

Chapter 3

Appendix

3.1 Phase shift along a perturbed path

To evaluate the deflection and time delay issues pointed out in Mannheim[6], we will take a closer look at the validness of the WKB approximation in the interferometer. To find the change in the particle's phase $\delta\phi$ in the perturbed wave function in equation 1.2, you can perform a line integral parameterized by the time variable t of the classical particle. The meaning of this parameter is the semi-classical relation $\hbar\vec{k} = m\frac{\partial\vec{x}(t)}{\partial t}$. Therefore, the defining equation for the phase shift along a particles actual path, and not the original unperturbed one, is

$$\delta\phi(\vec{x}(t)) = \int_0^t k'^i \frac{\partial\delta\phi'}{\partial x'^i} dt' = \frac{\hbar}{m} \int_0^t \frac{\partial x'^i(t')}{\partial t'} \frac{\partial\delta\phi'}{\partial x'^i} dt' \quad (3.1)$$

Expanding the trajectory $\vec{x}(t)$ about the unperturbed path with the function $\delta\vec{x} = \vec{x}(t) - \vec{x}_0(t)$ we can rewrite the integral as

$$\delta\phi(\vec{x}(t)) = \frac{\hbar}{m} \int_0^t \frac{\partial(x_0'^i(t') + \delta x'^i(t'))}{\partial t'} \frac{\partial\delta\phi'}{\partial x'^i} dt' \approx \frac{\hbar}{m} \int_0^t \frac{\partial x_0'^i(t')}{\partial t'} \frac{\partial\delta\phi'}{\partial x'^i} dt' = \frac{\hbar}{m} \int_0^t dx_0'^i(t') \frac{\partial\delta\phi'}{\partial x'^i} \quad (3.2)$$

This expression can be interpreted as a line integral over the path of the free particle. In full, we can say that the phase shift at some time along the path of the particle $\delta\phi(\vec{x}(t))$ is the same as the phase shift as calculated by the line integral above from the point $\vec{x}(0) = \vec{x}_0(0)$

to the point the unperturbed particle travels in the same time $\vec{x}_0(t)$. As a note, any first order difference in the coordinate length traveled by the two paths in the same coordinate time interval would have a second order effect on $\delta\phi$ and can be neglected.

The above exercise is useful in assessing the validness of the WKB approximation in an interferometer experiment where small first order deflections may change the reflection locations at the endpoints of each leg of the paths. However, the accumulated phase change is not sensitive to these first order path differences and they may be neglected completely. Changing the incident point A along one particle path to make the classical particle beams intersect at point D is also of no additional consequence for this reason.

The argument above hinges upon the importance of the statement in Mannheim[6] that the beam width must be large enough to accommodate any actual deflections of the beam that changes the end point D on the detecting crystal or else there will be no intersection of the beams. Also, it may be somewhat misleading when comparing the calculations in this paper with Mannheim[6] specifically with respect to the importance in their calculation of the exact time elapsed during the trip around each path. In our anzatz quantum wave function in equation 2.69, we compare at point D two wave functions that were coherent in their quantum phase at point A but are now incoherent with the calculated phase difference $\delta\phi_{ABD} - \delta\phi_{ACD}$. This is essentially a snapshot of the whole system using the coordinate time parameter. This should not be confused with saying the the same particle left point A and arrived at point D at the same time. On the contrary, the quantum wave gives no such reference to the exact location of a single particle but in effect relies on the coherent nature to ignore it. In this way, our result does not specify, nor does it depend on, the actual perturbed path of the particles nor the time difference it takes a classical particle to traverse the perturbed path.

3.2 Reflection rules in curved space

To show whether or not there will be an additional affect of the effective acceleration of the interferometer instrument on the phase, it is useful to investigate how the neutron beam is reflected at the crystal surfaces. To do this, we will be transforming to a local

inertial coordinate system in which the reflecting point is instantaneously stationary at the reflecting surface in the manner demonstrated by Mannheim[6]. First, transforming the proper velocity of the point $B^\mu = (ct, \frac{l+a}{2}, 0, 0)$ in the curved space-time of the fixed apparatus to the local flat space time defined by the tetrads from equation 2.41 we have

$$\begin{aligned}\dot{B}^{(\alpha)} &= e_\mu^{(\alpha)} \dot{B}^\mu = e_\mu^{(\alpha)} \frac{\partial B^\mu}{\partial \tau} = e_\mu^{(\alpha)} \frac{\partial B^\mu}{\sqrt{g_{00}} \partial t} = e_\mu^{(\alpha)} (\frac{c}{\sqrt{g_{00}}}, 0, 0, 0) \\ &= (\delta_\mu^{(\alpha)} + \frac{1}{2} h_\mu^{(\alpha)}) \dot{B}^\mu = (\frac{c(1 + \frac{1}{2} h_0^0)}{\sqrt{g_{00}}}, \frac{ch_0^1}{2\sqrt{g_{00}}}, \frac{ch_0^2}{2\sqrt{g_{00}}}, 0) \approx (c, \frac{ch_0^1}{2}, \frac{ch_0^2}{2}, 0) \quad (3.3)\end{aligned}$$

where $\gamma = (1 - \frac{v^2}{c^2})^{-\frac{1}{2}} \approx 1$ to first order in h . Also, the covariant wavenumber k_{in}^μ of the incoming particle will transform to the same inertial coordinate system as

$$\begin{aligned}k_{in}^{(\alpha)} &= e_\mu^{(\alpha)} k_{in}^\mu = e_\mu^{(\alpha)} (\frac{\omega}{c}, k, 0, 0) \\ &= (\delta_\mu^{(\alpha)} + \frac{1}{2} h_\mu^{(\alpha)}) k_{in}^\mu = (\frac{\omega}{c} (1 + \frac{1}{2} h_0^0) + k \frac{h_1^0}{2}, k(1 + \frac{1}{2} h_1^1) + \frac{\omega}{c} \frac{h_0^1}{2}, \frac{\omega}{c} \frac{h_0^2}{2}, 0) \quad (3.4)\end{aligned}$$

From these equations, it is clear that the apparatus is not fixed with respect this inertial coordinate system and that the apparatus will have a different proper velocity at different points in the tetrad frame. This is necessarily so from the very fact that it is a rigid structure in a complex gravitational field where tidal forces will be present. To further our analysis, we will apply a Lorentz transformation to counter the velocity of point B . To do this, we note that the proper time of the apparatus point B can be written in the new primed coordinate system as

$$c^2 d\tau_B^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 = c^2 dt^2 \gamma^{-2} \approx c^2 dt^2 = c^2 dt'^2 \quad (3.5)$$

To clarify, this says that to first order in the velocity $v^{(i)} = \frac{\partial x^i}{\partial t} = \frac{ch_0^i}{2}$, the proper time is the same as the coordinate time in the inertial coordinate system $x^{(\alpha)}$ and the coordinate system in which the velocity of point B of the apparatus is zero $x^{(\beta)'}$. The complete Lorentz

transformation to the $x^{(\beta) \prime}$ coordinate system so that $\dot{B}^{(i) \prime} = 0$ can be written as

$$\begin{aligned}
a_{(0)}^{(0) \prime} &= \gamma \approx 1 \\
a_{(0)}^{(i) \prime} &= -\gamma \frac{v^{(i)}}{c} \approx -\frac{v^{(i)}}{c} \\
a_{(i)}^{(0) \prime} &= -\gamma \frac{v^{(i)}}{c} \approx -\frac{v^{(i)}}{c} \\
a_{(j)}^{(i) \prime} &= \delta_j^i + v^{(i)} v^{(j)} \frac{(\gamma - 1)}{v^2} \approx \delta_j^i
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
\dot{B}^{(\beta) \prime} &= a_{(\alpha)}^{(\beta) \prime} \dot{B}^{(\alpha)} = \left(a_{(\alpha)}^{(0) \prime} \dot{B}^{(\alpha)}, a_{(\alpha)}^{(1) \prime} \dot{B}^{(\alpha)}, a_{(\alpha)}^{(2) \prime} \dot{B}^{(\alpha)}, a_{(\alpha)}^{(3) \prime} \dot{B}^{(\alpha)} \right) \\
\dot{B}^{(0) \prime} &= a_{(\alpha)}^{(0) \prime} \dot{B}^{(\alpha)} = a_{(0)}^{(0) \prime} \dot{B}^{(0)} + a_{(i)}^{(0) \prime} \dot{B}^{(i)} = c - \frac{cv^{(2)}}{c^2} \approx c \\
\dot{B}^{(i) \prime} &= a_{(\alpha)}^{(i) \prime} \dot{B}^{(\alpha)} = a_{(0)}^{(i) \prime} \dot{B}^{(0)} + a_{(j)}^{(i) \prime} \dot{B}^{(j)} = -\frac{v^{(i)}}{c} c + \delta_j^i v^{(j)} = -v^{(i)} + v^{(i)} = 0
\end{aligned} \tag{3.7}$$

The wavenumber in equation 3.4 in the primed coordinate systems is then

$$\begin{aligned}
k_{in}^{(\beta) \prime} &= a_{(\alpha)}^{(\beta) \prime} k_{in}^{(\alpha)} = \left(a_{(\alpha)}^{(0) \prime} k_{in}^{(\alpha)}, a_{(\alpha)}^{(1) \prime} k_{in}^{(\alpha)}, a_{(\alpha)}^{(2) \prime} k_{in}^{(\alpha)}, a_{(\alpha)}^{(3) \prime} k_{in}^{(\alpha)} \right) \\
k_{in}^{(0) \prime} &= a_{(0)}^{(0) \prime} k_{in}^{(0)} + a_{(i)}^{(0) \prime} k_{in}^{(i)} = \frac{\omega}{c} \left(1 + \frac{1}{2} h_0^0 \right) + k \frac{h_1^0}{2} - \frac{v^{(1)}}{c} \left(k \left(1 + \frac{1}{2} h_1^1 \right) + \frac{\omega}{c} \frac{h_0^1}{2} \right) - \frac{v^{(2)}}{c} \left(\frac{\omega}{c} \frac{h_0^2}{2} \right) \\
&\approx \frac{\omega}{c} \left(1 + \frac{1}{2} h_0^0 \right) + k \frac{h_1^0}{2} - \frac{ch_0^1}{2c} k = \frac{\omega}{c} \left(1 + \frac{1}{2} h_0^0 \right) - k \frac{h_0^1}{2} - \frac{h_0^1}{2} k = \frac{\omega}{c} \left(1 + \frac{1}{2} h_0^0 \right) - kh_0^1 \\
k_{in}^{(1) \prime} &= a_{(0)}^{(1) \prime} k_{in}^{(0)} + a_{(i)}^{(1) \prime} k_{in}^{(i)} = -\frac{v^{(1)}}{c} \left(\frac{\omega}{c} \left(1 + \frac{1}{2} h_0^0 \right) + k \frac{h_1^0}{2} \right) + \delta_j^1 k_{in}^{(j)} \\
&\approx -\frac{ch_0^1}{2c} \frac{\omega}{c} + k \left(1 + \frac{1}{2} h_1^1 \right) + \frac{\omega}{c} \frac{h_0^1}{2} = k \left(1 + \frac{1}{2} h_1^1 \right) \\
k_{in}^{(2) \prime} &= a_{(0)}^{(2) \prime} k_{in}^{(0)} + a_{(i)}^{(2) \prime} k_{in}^{(i)} = -\frac{v^{(2)}}{c} \left(\frac{\omega}{c} \left(1 + \frac{1}{2} h_0^0 \right) + k \frac{h_1^0}{2} \right) + \delta_j^2 k_{in}^{(j)} \\
&\approx -\frac{v^{(2)}}{c} \frac{\omega}{c} + \frac{\omega}{c} \frac{h_0^2}{2} \\
k_{in}^{(3) \prime} &= a_{(0)}^{(3) \prime} k_{in}^{(0)} + a_{(i)}^{(3) \prime} k_{in}^{(i)} = -\frac{v^{(3)}}{c} \left(\frac{\omega}{c} \left(1 + \frac{1}{2} h_0^0 \right) + k \frac{h_1^0}{2} \right) + \delta_j^3 k_{in}^{(j)} = 0
\end{aligned} \tag{3.8}$$

Now then, an incident particle striking a reflecting crystal undergoing Bragg reflection at an angle of 45° will be emitted from the surface with no change in the energy but with

a change in direction as

$$k_{out}^{(\beta)'} = \left(k_{in}^{(0)'}, k_{in}^{(2)'}, k_{in}^{(1)'}, k_{in}^{(3)'} \right) = \left(\frac{\omega}{c} \left(1 + \frac{1}{2} h_0^0 \right) - k h_0^1, -\frac{v^{(2)}}{c} \frac{\omega}{c} + \frac{\omega}{c} \frac{h_0^2}{2}, k \left(1 + \frac{1}{2} h_1^1 \right), 0 \right) \quad (3.9)$$

To now transform back to the unprimed inertial coordinate system, we have to apply the reverse boost

$$\begin{aligned} a_{(0)'}^{(0)} &= \gamma \approx 1 \\ a_{(0)'}^{(i)} &= \gamma \frac{v^{(i)}}{c} \approx \frac{v^{(i)}}{c} \\ a_{(i)'}^{(0)} &= \gamma \frac{v^{(i)}}{c} \approx \frac{v^{(i)}}{c} \\ a_{(j)'}^{(i)} &= \delta_j^i + v^{(i)} v^{(j)} \frac{(\gamma - 1)}{v^2} \approx \delta_j^i \end{aligned} \quad (3.10)$$

$$\begin{aligned} k_{out}^{(\alpha)} &= a_{(\beta)'}^{(\alpha)} k_{out}^{(\beta)'} = \left(a_{(\beta)'}^{(0)} k_{out}^{(\beta)'}, a_{(\beta)'}^{(1)} k_{out}^{(\beta)'}, a_{(\beta)'}^{(2)} k_{out}^{(\beta)'}, a_{(\beta)'}^{(3)} k_{out}^{(\beta)'} \right) \\ k_{out}^{(0)} &= a_{(\beta)'}^{(0)} k_{out}^{(\beta)'} = \frac{\omega}{c} \left(1 + \frac{1}{2} h_0^0 \right) - k h_0^1 + \frac{v^{(1)}}{c} \left(-\frac{v^{(2)}}{c} \frac{\omega}{c} + \frac{\omega}{c} \frac{h_0^2}{2} \right) + \frac{v^{(2)}}{c} \left(k \left(1 + \frac{1}{2} h_1^1 \right) \right) \\ &\approx \frac{\omega}{c} \left(1 + \frac{1}{2} h_0^0 \right) - k h_0^1 + \frac{v^{(2)}}{c} k = \frac{\omega}{c} \left(1 + \frac{1}{2} h_0^0 \right) - k \left(h_0^1 - \frac{h_0^2}{2} \right) \\ k_{out}^{(1)} &= a_{(\beta)'}^{(1)} k_{out}^{(\beta)'} = \frac{v^{(1)}}{c} \left(\frac{\omega}{c} \left(1 + \frac{1}{2} h_0^0 \right) - k h_0^1 \right) - \frac{v^{(2)}}{c} \frac{\omega}{c} + \frac{\omega}{c} \frac{h_0^2}{2} \\ &\approx \frac{v^{(1)}}{c} \frac{\omega}{c} - \frac{h_0^2}{2} \frac{\omega}{c} + \frac{\omega}{c} \frac{h_0^2}{2} = \frac{h_0^1}{2} \frac{\omega}{c} \\ k_{out}^{(2)} &= a_{(\beta)'}^{(2)} k_{out}^{(\beta)'} = \frac{v^{(2)}}{c} \left(\frac{\omega}{c} \left(1 + \frac{1}{2} h_0^0 \right) - k h_0^1 \right) + k \left(1 + \frac{1}{2} h_1^1 \right) \\ &\approx \frac{h_0^2}{2} \frac{\omega}{c} - k h_0^1 + k \left(1 + \frac{1}{2} h_1^1 \right) = \frac{h_0^2}{2} \frac{\omega}{c} + k \left(1 - h_0^1 + \frac{1}{2} h_1^1 \right) \\ k_{out}^{(3)} &= a_{(\beta)'}^{(3)} k_{out}^{(\beta)'} = 0 \end{aligned} \quad (3.11)$$

Finally, to transform back to the curved space coordinate system where the entire ap-

paratus is stationary, we apply the reverse tetrad transformation so that

$$\begin{aligned}
k_{out}^\mu &= e_{(\alpha)}^\mu k_{out}^{(\alpha)} = \left(\delta_\alpha^\mu - \frac{1}{2} h_\alpha^\mu \right) k_{out}^{(\alpha)} \\
k_{out}^0 &= \left(\delta_\alpha^0 - \frac{1}{2} h_\alpha^0 \right) k_{out}^{(\alpha)} = \left(1 - \frac{1}{2} h_0^0 \right) \left(\frac{\omega}{c} \left(1 + \frac{1}{2} h_0^0 \right) - k \left(h_0^1 - \frac{h_0^2}{2} \right) \right) \\
&\quad - \frac{1}{2} h_1^0 \frac{h_0^1 \omega}{2c} - \frac{1}{2} h_2^0 \left(\frac{h_0^2 \omega}{2c} + k \left(1 - h_0^1 + \frac{1}{2} h_1^1 \right) \right) \\
&\approx \frac{\omega}{c} - k \left(h_0^1 - \frac{h_0^2}{2} \right) - \frac{1}{2} h_2^0 k = \frac{\omega}{c} - k \left(h_0^1 - \frac{h_0^2}{2} \right) + \frac{1}{2} h_0^2 k = \frac{\omega}{c} + k (h_0^2 - h_0^1) \\
k_{out}^1 &= \left(\delta_\alpha^1 - \frac{1}{2} h_\alpha^1 \right) k_{out}^{(\alpha)} = -\frac{1}{2} h_0^1 \left(\frac{\omega}{c} \left(1 + \frac{1}{2} h_0^0 \right) - k \left(h_0^1 - \frac{h_0^2}{2} \right) \right) + \left(1 - \frac{1}{2} h_1^1 \right) \frac{h_0^1 \omega}{2c} \\
&= -\frac{1}{2} h_0^1 \frac{\omega}{c} + \frac{h_0^1 \omega}{2c} = 0 \\
k_{out}^2 &= \left(\delta_\alpha^2 - \frac{1}{2} h_\alpha^2 \right) k_{out}^{(\alpha)} = -\frac{1}{2} h_0^2 \left(\frac{\omega}{c} \left(1 + \frac{1}{2} h_0^0 \right) - k \left(h_0^1 - \frac{h_0^2}{2} \right) \right) \\
&\quad + \left(1 - \frac{1}{2} h_2^2 \right) \left(\frac{h_0^2 \omega}{2c} + k \left(1 - h_0^1 + \frac{1}{2} h_1^1 \right) \right) \\
&= -\frac{1}{2} h_0^2 \frac{\omega}{c} + \frac{h_0^2 \omega}{2c} + k \left(1 - h_0^1 + \frac{1}{2} h_1^1 \right) - k \frac{1}{2} h_2^2 = k \left(1 - h_0^1 + \frac{1}{2} (h_1^1 - h_2^2) \right) \\
k_{out}^3 &= \left(\delta_\alpha^3 - \frac{1}{2} h_\alpha^3 \right) k_{out}^{(\alpha)} = 0
\end{aligned} \tag{3.12}$$

that, when evaluated at the corner of point B in the ring laser metric of equation 1.17, due to the symmetry of the space becomes simply

$$k_{out}^\mu = \left(\frac{\omega}{c}, 0, k, 0 \right) \tag{3.13}$$

We can see now that for particles reflecting near the corners of the ring laser apparatus will have the same reflection rules as in flat space with the angle of incidence equal to the angle of reflection and with no change in the wavenumber energy. This differs from the result that Mannheim[6] arrived at for his particular metric in that he found a shift in the frequency of the particle on reflection when the particle started or ended the reflection process in the direction of acceleration. We find no such shift here only due to the fact that the acceleration of the reflecting mirror contains some symmetry in both the incident direction and the outgoing particle direction, in this case the x and y directions, at the point of reflection. That this is the reason is directly seen in the expression for k_{out}^0 in equation

3.12 where the velocity terms resulting from acceleration $k(h_0^2 - h_0^1)$ go to zero at the point in space.

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