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Homogeneous Representations of Khovanov-Lauda-Rouquier Algebras

Gabriel Feinberg
University of Connecticut - Storrs, gabriel.feinberg@gmail.com

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ABSTRACT

The Khovonov-Lauda-Rouquier (KLR) algebra arose out of attempts to categorify quantum groups. Kleshchev and Ram proved a result reducing the representation theory of these algebras to the study of irreducible cuspidal representations. In finite types, these cuspidal representations are part of a larger class of homogeneous representations, which are related to fully commutative elements of Coxeter groups.

For KLR algebras of types A_n and D_n, we classify and enumerate these homogeneous representations.

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Gabriel Feinberg

M.S. Mathematics, Drexel University, Philadelphia, Pennsylvania, 2008

B.S. Mathematics, Muhlenberg College, Allentown, Pennsylvania, 2004

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Homogeneous Representations of Khovanov-Lauda-Rouquier Algebras

Presented by

Gabriel Feinberg, B.S. Math., M.S. Math.

Major Advisor ______________________________
Kyu-Hwan Lee

Associate Advisor ______________________________
Ralf Schiffler

Associate Advisor ______________________________
Álvaro Lozano-Robledo

University of Connecticut

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Chapter 1

Background & Definitions

1.1 Khovanov-Lauda-Rouquier algebras

1.1.1 History

Discovered by Khovanov and Lauda [KL09] and independently by Rouquier [Rou08], the Khovanov-Lauda-Rouquier (KLR) algebras (also known as quiver Hecke algebras) have been the focus of many recent studies. In both of the works mentioned, the application was the categorification of the lower half of a quantum group. That is, the Cartan datum associated with a Lie algebra, $\mathfrak{g}$, gives rise to a KLR algebra, R. The category of finite dimensional graded projective modules of this algebra can be given a Hopf algebra structure by taking the Grothendieck group, and taking the induction and restriction functors as multiplication and co-multiplication. To say that this algebra categorifies the quantum group $U_q^-(\mathfrak{g})$, is to say that this Hopf algebra is isomorphic to Lusztig's integral form on it.

Another major result which followed soon after was given by Brundan and Kleshchev [BK09], who found that blocks of cyclotomic Hecke algebras can be naturally identified with KLR algebras. These KLR algebras contain more structure than a Hecke algebra so, in particular, this provided a natural **Z**-grading on these Hecke algebra blocks.

In the wonderful work [KR10b], Kleshchev and Ram significantly reduce the problem of describing the irreducible representations of the KLR algebras. They identify a class of "cuspidal" representations, and show that, for most finite types, *every* irreducible representation appears as the head of some induction of these cuspidals. Further, these cuspidal representations can be identified by their highest weights, found combinatorially in the dual canonical basis. Hill, Melvin, and Mondragon, in [HMM12] extended these results to all classical finite types, and re-frame them in a slightly more unified manner.

When first introducing these algebras, Khovanov and Lauda made a "cyclotomic" conjecture: that the quotient algebra obtained from a special ideal will categorify a corresponding highest weight representation of the quantum group associated with the same Cartan datum. This conjecture was first proved by Kang and Kashiwara [KK12], who showed it to be true for all symmetrizable Kac-Moody algebras.

Lauda and Vazirani later imposed a crystal structure on the isomorphism classes of irreducible representations of both ordinary and cyclotomic KLR algebras. They showed in [LV11] that these crystals are, respectively, $B(\infty)$ and $B(\Lambda)$ for a corresponding highest weight Λ. Crystals are also used by Benkart, Kang, Oh, and Park in [BKOP12] to give a new approach towards the work of Kleshchev and Ram on cuspidal representations.

In the process of constructing the cuspidal modules, Kleschev and Ram define a class of graded representations known as homogeneous representations [KR10a], those that are fixed in a single degree. These will be the focus of the present work. Homogeneous representations are more easily described and, for finite types, include all of the cuspidal representations. Because of this, they have been used in more complex types to approach the representation theory of KLR algebras (see [BKM12], [Kle12], [HMM12]). We study here the combinatorics of these representations. In type A_n we give a natural way to generate them, and in type D_n we find that they can be counted by a classic sequence. After the following definitions, we will present these ideas more thoroughly.

1.1.2 Definitions

To generate a KLR algebra, we begin with a quiver, Γ. In this work, we will focus mainly on quivers of Dynkin type A_n and D_n, but for the definition, any finite quiver with no double bonds will suffice. Let I be the set indexing the vertices of Γ, and for indices $i \neq j$, we will say that i and j are neighbors if $i \to j$ or $i \leftarrow j$. Define $Q_+ = \bigoplus_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$ as the non-negative lattice with basis $\{\alpha_i | i \in I\}$.

Note that a quiver encodes the information of a generalized Cartan matrix, and vice versa. We could have alternatively begun with such a matrix and proceeded similarly.

For a fixed $\alpha = \sum_{i \in I} c_i \alpha_i \in Q_+$, let I^α be the set of words $\boldsymbol{w}$ on the alphabet I such that each $i \in I$ occurs exactly c_i times in $\boldsymbol{w}$. Denote by d the height of α, $\sum_{i \in I} c_i$. We will write $\boldsymbol{w} = [w_{i_1}, w_{i_2}, \ldots, w_{i_d}]$.

Now, fix an arbitrary ground field $\mathbb{F}$ and choose an element $\alpha \in Q_+$. Then the *Khovanov-Lauda-Rouquier Algebra*, R_α, is the associative $\mathbb{F}$-algebra generated by:

- idempotents $\{e(\boldsymbol{w}) \mid \boldsymbol{w} \in I^\alpha\}$
- symmetric generators $\{\psi_1, \ldots, \psi_{d-1}\}$, where d is the height of the root α
- polynomial generators $\{y_1, \ldots, y_d\}$ where, again, d is the height of the root α

subject to relations (taken from [KR10b])

$$e(\boldsymbol{w})e(\boldsymbol{v}) = \delta_{\boldsymbol{w}\boldsymbol{v}}e(\boldsymbol{w}), \qquad \sum_{\boldsymbol{w}\in I^\alpha} e(\boldsymbol{w}) = 1 \tag{1.1.1}$$

$$y_k e(\boldsymbol{w}) = e(\boldsymbol{w}) y_k, \tag{1.1.2}$$

$$\psi_k e(\boldsymbol{w}) = e(s_k\boldsymbol{w})\psi_k, \tag{1.1.3}$$

$$y_k y_\ell = y_\ell y_k, \tag{1.1.4}$$

$$y_k \psi_\ell = \psi_\ell y_k, \text{(for } k \neq \ell, \ell+1) \tag{1.1.5}$$

$$(y_{k+1}\psi_k - \psi_k y_k)e(\boldsymbol{w}) = \begin{cases} e(\boldsymbol{w}) & \text{if } w_k = w_{k+1}, \\ 0 & \text{otherwise,} \end{cases} \tag{1.1.6}$$

$$(\psi_k y_{k+1} - y_k\psi_k)e(\boldsymbol{w}) = \begin{cases} e(\boldsymbol{w}) & \text{if } w_k = w_{k+1}, \\ 0 & \text{otherwise,} \end{cases} \tag{1.1.7}$$

$$\psi_k^2 e(\boldsymbol{w}) = \begin{cases} 0 & \text{if } w_k = w_{k+1}, \\ (y_k - y_{k+1})e(\boldsymbol{w}) & \text{if } w_k \to w_{k+1}, \\ (y_{k+1} - y_k)e(\boldsymbol{w}) & \text{if } w_k \leftarrow w_{k+1}, \\ e(\boldsymbol{w}) & \text{otherwise,} \end{cases} \tag{1.1.8}$$

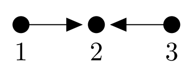
$$\psi_k\psi_\ell = \psi_\ell\psi_k, \text{(for } |k-\ell| > 1) \tag{1.1.9}$$

$$(\psi_{k+1}\psi_k\psi_{k+1} - \psi_k\psi_{k+1}\psi_k)e(\boldsymbol{w}) = \begin{cases} e(\boldsymbol{w}) & \text{if } w_{k+2} = w_k \to w_{k+1}, \\ -e(\boldsymbol{w}) & \text{if } w_{k+2} = w_k \leftarrow w_{k+1}, \\ 0 & \text{otherwise,} \end{cases} \tag{1.1.10}$$

Where $\delta_{\boldsymbol{w}\boldsymbol{v}}$ in 1.1.1 is the Kronecker delta and, in 1.1.3, s_k is the k^{th} simple transposition in the symmetric group S_d, acting on the word $\boldsymbol{w}$ by swapping the letters in the k^{th} and $(k+1)^{\text{st}}$ positions.

If Γ is a Dynkin-type quiver, we will say that R_α is a KLR algebra of that type.

1.1.1 Example. Suppose we begin with the A_3 quiver

$\Gamma =$ 

We then have $I = \{1, 2, 3\}$, and we can choose the element $\alpha = \alpha_1 + 2\alpha_2 + \alpha_3 \in Q_+$. The generators of the type A_3 KLR algebra R_α are

- The twelve idempotents $\{e([1,2,2,3]), e([1,2,3,2]), e([1,3,2,2]),$
  $e([2,1,2,3]), e([2,1,3,2]), e([2,2,1,3]), e([2,2,3,1]),$
  $e([2,3,1,2]), e([2,3,2,1]), e([3,1,2,2]), e([3,2,1,2]),$
  $e([3,2,2,1])\}$
- The four polynomial generators $\{y_1, y_2, y_3, y_4\}$
- The three symmetric generators $\{\psi_1, \psi_2, \psi_3\}$

We can check that the relations imply, for example, that

$$e([2,2,1,3])y_3\psi_3 \tag{1.1.11}$$

$$= y_3 e([2,2,1,3])\psi_3 \quad \text{(by relation 1.1.2)} \tag{1.1.12}$$

$$= y_3\psi_3 e([2,2,3,1]) \quad \text{(by relation 1.1.3)} \tag{1.1.13}$$

We impose a $\mathbb{Z}$-grading on R_α by declaring

$$\deg(e(\boldsymbol{w})) = 0 \tag{1.1.14}$$

$$\deg(y_i) = 2 \tag{1.1.15}$$

$$\deg(\psi_i e(\boldsymbol{w})) = \begin{cases} -2 & \text{if } w_i = w_{i+1} \\ 1 & \text{if } w_i, w_{i+1} \text{ are neighbors in } \Gamma \\ 0 & \text{if } w_i, w_{i+1} \text{ are not neighbors in } \Gamma \end{cases} \tag{1.1.16}$$

1.1.2 Example. Note that, in equation 1.1.11, the degree of the element shown is 2.

Elements in this algebra can be expressed nicely, thanks to a theorem of Khovanov and Lauda. For any reduced $\sigma = s_1 s_2 \cdots s_r \in S_d$, define the product of symmetric generators $\psi_\sigma = \psi_1\psi_2\cdots\psi_r$.

1.1.3 Theorem ([KL09], Theorem 2.5)**.** *The set* $\{\psi_\sigma y_1^{k_1}\cdots y_d^{k_d} e(\boldsymbol{w}) : \sigma \in S_d; k_i \in \mathbb{Z}_{\geq 0}; \boldsymbol{w} \in I^\alpha\}$ *gives an* $\mathbb{F}$*-basis of* R_α*.*

We often have the opportunity to consider a larger KLR-algebra associated to the quiver Γ, but without fixing a specific root α. We denote this $R = \bigoplus_{\alpha\in Q_+} R_\alpha$.

To the Cartan datum encoded in the quiver Γ, one can construct a quantum group given by a deformation of the universal enveloping algebra of the corresponding Kac-Moody algebra. This quantum group has a dual canonical basis equipped with a shuffle product (See [Gre97]). Words $w \in I^\alpha$ whose images are maximal under this shuffle product are defined to be *good words*. Further, a word that is lexicographically smaller than all of its right factors is known as a *Lyndon word*. Using the techniques developed by Leclerc in [Lec04], Kleshchev and Ram, and then Melvin, Mondragon, and Hill show

1.1.4 Theorem ([KR10b], 3.4; [HMM12], 4.1.1)**.** *The good Lyndon words $\mathbf{W}_+^\alpha$ parameterize the cuspidal representations of the KLR algebra R_α. In turn, these cuspidal representations induce all irreducible graded R_α-modules up to isomorphism and degree shift.*

1.2 Homogeneous representations

We define a *homogeneous representation* of a KLR algebra as an irreducible, graded representation fixed in a single degree (with respect to the $\mathbb{Z}$-grading described in equations 1.1.14 - 1.1.16). Immediately it is clear from equation 1.1.15 that, under such a representation, the polynomial generators must act as 0. To completely describe these, we will need some more terminology, described here as in [KR10a]. In this section, we will assume that Γ is a simply-laced quiver (no multiple edges, but loops are permitted).

Fix an $\alpha \in Q_+$ and let d be the height of α. For any word $\boldsymbol{w} \in I^\alpha$, we say that the simple transposition $s_r \in S_d$ is an *admissible transposition for* $\boldsymbol{w}$ if the letters w_r and w_{r+1} are neither equal nor neighbors in the quiver Γ. We follow Kleshchev and Ram and define the *weight graph* G_α with vertices given by I^α. Two words, $\boldsymbol{w}$ and $\boldsymbol{v}$, are connected by an edge if there is some admissible transposition s_r such that $s_r\boldsymbol{w} = \boldsymbol{v}$.

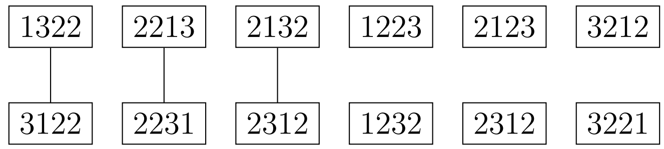
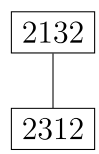


FIGURE 1.2.1: The weight graph $G_{\alpha_1+2\alpha_2+\alpha_3}$ for Γ, a type A_3 quiver.

We say that a connected component, C, of the weight graph G_α is *homogeneous* if the following property holds for every $\boldsymbol{w} \in C$:

$$\begin{aligned} \text{If } w_r &= w_s \text{ for some } 1 \le r < s \le d, \text{ then there exist } t, u \\ &\text{with } r < t < u < s \text{ such that } w_r \text{ is neighbors with both } w_t \text{ and } w_u. \end{aligned} \tag{1.2.1}$$

1.2.1 Example. For the weight graph shown in figure 1.2.1, the only homogeneous component is



In this case, the two instances of the letter [2] have the neighbors [1] and [3] occurring between them. Note that, for some $\alpha \in Q_+$ we may have that every component is homogeneous ($\alpha = \alpha_1 + \alpha_2 + \alpha_3$ for example), while for others we may see that no components are homogeneous (e.g. $\alpha = 2\alpha_1 + \alpha_2$).

A main theorem of [KR10a] shows that the homogeneous components of G_α exactly parameterize the homogeneous representations of the KLR algebra, R_α:

1.2.2 Theorem ([KR10a], Theorem 3.4)**.** *Let C be a homogeneous component of the weight graph G_α. Define an $\mathbb{F}$-vector space $S(C)$ with basis $\{v_{\boldsymbol{w}} \mid \boldsymbol{w} \in C\}$ labeled by* *the words in C. Then we have an R_α-action on $S(C)$ given by*

$$\begin{aligned} e(\boldsymbol{w}')v_{\boldsymbol{w}} &= \delta_{\boldsymbol{w},\boldsymbol{w}'}v_{\boldsymbol{w}} \quad (\boldsymbol{w}' \in I^\alpha, \boldsymbol{w} \in C) \\ y_r v_{\boldsymbol{w}} &= 0 \quad (1 \leq r \leq d, \boldsymbol{w} \in C) \\ \psi_r v_{\boldsymbol{w}} &= \begin{cases} v_{s_r\boldsymbol{w}} & \text{if } s_r\boldsymbol{w} \in C \\ 0 & \text{else} \end{cases} \qquad (1 \leq r \leq d-1, \boldsymbol{w} \in C) \end{aligned}$$

which gives $S(C)$ the structure of a homogeneous, irreducible R_α-module. Further $S(C) \not\cong S(C')$ if $C \neq C'$ and this gives all of the irreducible homogeneous modules, up to isomorphism.

So, the task of identifying homogeneous modules of a KLR algebra is reduced to identifying homogeneous components in a weight graph. This is simplified further by the following lemma:

1.2.3 Lemma ([KR10a], lemma 3.3)**.** *A connected component, C, of the weight graph G_α is homogeneous if and only if some element $\boldsymbol{w} \in C$ satisfies the condition 1.2.1.*

With this in mind, let us call any word satisfying condition 1.2.1 a *homogeneous word*. These homogeneous words have other combinatorial properties, which we explore in the next section.

1.3 Fully commutative elements of Coxeter groups

Note that, while the letters represented in a word $\boldsymbol{w} \in I^\alpha$ depend on the choice of α, the homogeneity of $\boldsymbol{w}$ depends only on the underlying quiver. Even the orientation of the arrows is irrelevant, so it is natural to consider Dynkin diagrams and the corresponding Coxeter groups.

1.3.1 Coxeter groups

We recall the definitions of a (simply-laced) Coxeter groups (see [BB05] or [Hum90] for more details). Given a graph with no double edges (such as a type A_n or type D_n Dynkin diagram), we can construct the corresponding Coxeter group. We include a generator s_i for each $i \in I$, and impose the following relations:

(Reflection Relation)	$s_i^2 = 1$	
(Commuting relation)	$s_i s_j = s_j s_i$	if nodes i and j are not neighbors
(Braid relation)	$s_i s_j s_i = s_j s_i s_j$	if nodes i and j are neighbors

When an element in a Coxeter group is expressed as a product of the fewest generators possible, we say that this expression is *reduced*. A given element may have many reduced expressions.

1.3.1 Example. In the type A_4 Coxeter group, we have

$$s_3 s_1 s_2 s_3 s_4 = s_1 s_3 s_2 s_3 s_4 = s_1 s_2 s_3 s_2 s_4 = s_1 s_2 s_3 s_4 s_2$$

are all reduced expressions for the same element. Notice that in order to generate this list, we employ both the commuting relations and braid relations.

As a short-hand notation, we will express these as words on the alphabet I:

$$[3, 1, 2, 3, 4] = [1, 3, 2, 3, 4] = [1, 2, 3, 2, 4] = [1, 2, 3, 4, 2]$$

and denote the identity element by the empty word [ ].

It is well known ([BB05],[Hum90]) that a Coxeter group of type A_n has size $(n+1)!$, while a type D_n Coxeter group contains $2^{n-1}n!$ elements.

We recognize that, for simply laced types, words appearing in a connected component of the weight graph G_α are reduced expressions for the same element of a Coxeter group whose graph is the associated Dynkin diagram. The notion of a homogeneous component agrees completely with the concept of a fully commutative element in a Coxeter group.

1.3.2 Background

Stembridge [Ste96] defined a fully commutative element in a Coxeter group as an element whose reduced words could be obtained using only commutativity relations. He recognized that these elements had not only combinatorial applications, but also representation theoretic implications. He classified all of the groups that have finitely many fully commutative elements. This completed the work of Fan, [Fan96], who had done this for the simply-laced types.

1.3.2 Theorem ([Ste96], Theorem 4.1)**.** *The Coxeter groups with finitely many fully commutative elements are exactly* A_n $(n \geq 1)$, B_n $(n \geq 2)$, D_n $(n \geq 4)$, E_n $(n \geq 6)$, F_n $(n \geq 4)$, H_n $(n \geq 3)$, *and* $I_2(m)$ $(m \geq 5)$

We recognize now that a homogeneous component in the weight graph of a KLR algebra will correspond to a fully commutative element in the Coxeter group based on the same graph (ignoring the orientation of the arrows). By the definition of fully commutative elements, the nodes in a homogeneous component of G_α will be all of the reduced expressions for a single fully commutative element.

1.3.3 Corollary. *Any KLR algebra of type A_n $(n \geq 1)$, D_n $(n \geq 4)$, or E_n $(n \geq 6)$ has finitely many irreducible homogeneous representations.*

Fan [Fan96] also showed that these elements parameterized natural bases for corresponding quotients of Hecke algebras. In particular, in type A_n, these give rise to the Temperley-Lieb algebras (see [Jon87]).

In another work, [Ste98], Stembridge sets out to enumerate the set of fully commutative elements in each of the groups he previously identified. We shall see that his enumeration has some applications to the representation theory of KLR algebras, but is too coarse to be practical. He found, for example

1.3.4 Theorem ([Fan96], 3; [Ste98], 2.6)**.** *Let C_n be the n^{th} Catalan number, $C_n = \frac{1}{n+1}\binom{2n}{n}$.*

A Coxeter group of type A_n has C_{n+1} fully commutative elements.

A Coxeter group of type D_n has $\frac{n+3}{2}C_n - 1$ fully commutative elements.

We immediately have then,

1.3.5 Corollary. *A KLR algebra $R = \bigoplus_{\alpha \in I^\alpha} R_\alpha$ of type A_n has C_{n+1} irreducible, homogeneous representations, while a KLR algebra of type D_n has $\frac{n+3}{2}C_n - 1$ irreducible homogeneous representations.*

We will extend these results to give a finer enumeration of homogeneous representations in types A_n and D_n.

In chapter 2, we focus on homogeneous representations of KLR algebras of type A_n, and our main result is a fine bijection between such irreducible representations and the set of Dyck paths, a well known combinatorial object. This bijection can be used to quickly enumerate the fully commutative words of a given length, and hence the attached homogeneous representations.

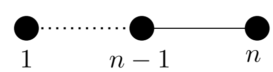
In the last chapter, we shift our focus to homogeneous representations of KLR algebras of type D_n. Our main theorem 3.3.4 in this chapter proves that these homogeneous representations can be organized naturally, and counted by Catalan's triangle, a sequence similar to Pascal's triangle.

Note that the results in chapter 2 and chapter 3 contribute to the combinatorics of the representation theory of KLR algebras, but could also be framed independently in the study of fully commutative elements of Coxeter groups. In this work, we chose to keep the language of homogeneous representations.

Chapter 2

Homogeneous Representations of a Type-A_n KLR Algebra

In this chapter, we describe all of the homogeneous representations of a KLR algebra of type A_n. That is, we suppose that we have a quiver of the form



We begin with some lemmas about the form of a type A_n homogeneous word. With those, we prove a bijection from the set of homogeneous components, which provides our enumerative results. We end with some examples of the algorithm in practice.

2.1 Canonical reduced words

In order to show our bijection, we begin by establishing a "canonical representative" for each homogeneous component. We define the decreasing segments

$$T_i^j = \begin{cases} [j, j-1, \ldots, i+1, i] & \text{for } i < j \\ [i] & \text{for } i = j \\ [\,] & \text{for } i > j \end{cases}$$

These segments will be fundamental, so we record several facts here that we will use freely.

2.1.1 Lemma. *Let T_i^j be a segment, as defined above. Then,*

i. T_i^j *is a homogeneous word.*

ii. If $i-1 = j' \geq i'$ *then* $T_i^j T_{i'}^{j'} = T_{i'}^j$*. In particular,* $T_i^j[i-1] = T_{i-1}^j$ *and* $[j+1]T_i^j = T_i^{j+1}$.

iii. If $j' < i-1$ *then* $T_i^j T_{i'}^{j'} = T_{i'}^{j'} T_i^j$*. In particular, for* $k > j+1$ *or* $k < i-1$, $[k]T_i^j = T_i^j[k]$.

Proof. These statements follow directly from the definition. ■

Bokut and Shiao use these segments to give a canonical form for all words in the Coxeter group of type A_n.

2.1.2 Lemma ([BS01], lemma 3.2)**.** *Every reduced word in* A_n *can be presented in the form*

$$T_{i_1}^1 T_{i_2}^2 \cdots T_{i_n}^n$$

where $1 \leq i_j \leq j+1$ *for all* $1 \leq j \leq n$.

2.1.3 Remark. Notice that there are $(n+1)!$ choices for the i_j's in this form, and $(n+1)!$ elements in a type A_n Coxeter group, so expressions of this form are necessarily unique.

Since this holds for all elements of the Coxeter group, it of course is true for the homogeneous words. Regarding those elements, though, we can make sharper statements about this canonical form.

2.1.4 Lemma. *Suppose that $\boldsymbol{w}$ is a word of the form $\boldsymbol{w} = T_{i_1}^{m_1} \cdots T_{i_\ell}^{m_\ell}$ where $1 \leq i_1 < i_2 \cdots < i_\ell \leq n$ and $i_j \leq m_j$. Then $\boldsymbol{w}$ is a homogeneous word if and only if $m_1 < m_2 < \cdots < m_\ell$.*

Proof. ($\Rightarrow$) Suppose that $\boldsymbol{w}$ is a homogeneous word. For the sake of contradiction assume that $m_i \geq m_j$ for some $i < j$. Without loss of generality, suppose that $m_1 \geq m_2$. Then $\boldsymbol{w}$ has as a subword $T_{i_1}^{m_1} T_{i_2}^{m_2} = [m_1, \ldots m_2, m_2 - 1, \ldots, i_1; m_2, \ldots]$. But this subword has two occurrences of the letter $[m_2]$ separated by only one neighbor, $m_2 - 1$, therefore violating the homogeneity assumption.

($\Leftarrow$) Now, suppose that $\boldsymbol{w}$ is of this form with $m_1 < m_2 < \cdots < m_\ell$, and that the letter $[k]$ appears at least twice. Let $w_r = w_s = [k]$ with $r < s$ and no other occurrences of k between them. Then there exist j_1, j_2 with $i_{j_1} \leq k \leq m_{j_1}$ and $i_{j_2} \leq k \leq m_{j_2}$, that is, with $i_{j_1} < i_{j_2} \leq k \leq m_{j_1} < m_{j_2}$. So we necessarily have $i_{j_1} \leq k - 1$ and $m_{j_2} \geq k + 1$.

Then we have a subword of $\boldsymbol{w}$, $[m_{j_1}, \ldots, k, k-1, \ldots, i_{j_1}, m_{j_2}, \ldots, k+1, k, \ldots, i_{j_2}]$. In particular, there are two neighbors, $k - 1$ and $k + 1$, occurring between the two instances of k. Since this is true for any pair of instances of a letter that appears at least twice, $\boldsymbol{w}$ is homogeneous. ■

2.1.5 Corollary. *A homogeneous word of type A_n in the form of lemma 2.1.4 contains at most one instance of each of the letters $[n]$ and $[1]$, at most two instances of each of the letters $[n-1]$ and $[2]$, and so on. In general, there are at most k occurrences of each of the letters $[n+1-k]$ and $[k]$.*

Proof. By the definition of homogeneity, all reduced expressions for a homogeneous word contain the same number of instances of any given letter. This is a consequence of the fact that a homogeneous word may present opportunities to apply a commutation relation, but never a braid relation.

The letter $[k]$ can appear in a segment T_i^j only if $i \leq k$, and since $1 \leq i_1 < i_2 < \cdots$, there can be at most k many segments where this is the case. In the other direction, the letter $[n+1-k]$ appears in a segment T_i^j only if $n+1-k \leq j$. Because we have $\cdots < m_{\ell-1} < m_\ell \leq n$, there are at most k many segments in which this can be the case. ∎

The previous lemma suggests a powerful method for recognizing and generating homogeneous words, but we can give an even more complete formulation. We prove now that *e*very homogeneous word has a representative of this form.

2.1.6 Lemma. *Every homogeneous component contains a unique word of the form*

$$T_{i_1}^{m_1} T_{i_2}^{m_2} \cdots T_{i_\ell}^{m_\ell}$$

where $1 \leq i_1 < i_2 < \cdots < i_\ell \leq n$, $i_j \leq m_j$ and $m_1 < m_2 < \cdots < m_n \leq n$. Consider the empty word as the case where $\ell = 0$.

Proof. We will prove this by induction on n. Clearly, this result holds for the group A_1: There are only two reduced words, both homogeneous: $[\,] = T_1^0$ and $[1] = T_1^1$.

Now suppose that this result holds for the group A_{n-1}, and let $\boldsymbol{w}$ be a reduced homogeneous word in the group A_n.

Note that, by homogeneity and corollary 2.1.5, $\boldsymbol{w}$ can contain *at most* one occurrence of the letter $[n]$. If $\boldsymbol{w}$ does not contain any instances of the letter n, then $\boldsymbol{w}$ has the form of a word in A_{n-1} and, by induction, we are done. Suppose instead that there is one occurrence of this letter. That is, we can write $\boldsymbol{w} = \boldsymbol{v}_1[n]\boldsymbol{v}_2$, where neither $\boldsymbol{v}_1$ nor $\boldsymbol{v}_2$ contain the letter n. Both of these words can be written in the form of lemma 2.1.6 . We examine three cases:

Case 1: $\boldsymbol{v}_2$ does not contain the letter $[n-1]$. (Then $\boldsymbol{v}_1$ might, but there can be at most one instance in that factor.) Because the letters in $\boldsymbol{v}_2$ are all less than $n-1$, we can write

$$\begin{aligned} \boldsymbol{w} &= \boldsymbol{v}_1[n]\boldsymbol{v}_2 \\ &= \boldsymbol{v}_1\boldsymbol{v}_2[n] \\ &= \boldsymbol{v}_1\boldsymbol{v}_2T_n^n \end{aligned}$$

Because the left factor $\boldsymbol{v}_1\boldsymbol{v}_2$ is a homogenous word that doesn't contain the letter $[n]$, by induction, it has an expression in A_{n-1} in the desired form, and the result is shown.

Case 2: $\boldsymbol{v}_1$ does not contain the letter $[n-1]$, but $\boldsymbol{v}_2$ does. That is, we can write

$$\begin{aligned}\boldsymbol{w} &= \boldsymbol{v}_1[n]\boldsymbol{v}_2 \\ &= [n]\boldsymbol{v}_1\boldsymbol{v}_2 \\ &= [n]\left(T_{i_1}^{m_1}\cdots T_{i_{\ell-1}}^{m_{\ell-1}}T_{i_\ell}^{n-1}\right) \quad \text{(By the induction hypothesis)}\end{aligned}$$

where, by the lemma 2.1.4, we have $m_1 < \cdots < m_{\ell-1} < n-1$. We recognize that the one instance of the letter $[n-1]$ must be in the final segment, $T_{i_\ell}^{n-1}$. Thus the letter $[n]$ will commute with all of the segments before the last, and we can write

$$\begin{aligned}\boldsymbol{w} &= T_{i_1}^{m_1}\cdots T_{i_{\ell-1}}^{m_{\ell-1}}[n]T_{i_\ell}^{n-1} \\ &= T_{i_1}^{m_1}\cdots T_{i_{\ell-1}}^{m_{\ell-1}}T_{i_\ell}^{n}\end{aligned}$$

as desired.

Case 3: Both $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ each contain one occurrence of the letter $[n-1]$. We can express each of them in the canonical form, noting that they may have different numbers of segments.

$$\begin{aligned} \boldsymbol{w} &= \boldsymbol{v}_1[n]T_{i_1}^{m_1}\cdots T_{i_{k-1}}^{m_{k-1}}T_{i_k}^{n-1} \\ &= \boldsymbol{v}_1 T_{i_1}^{m_1}\cdots T_{i_{k-1}}^{m_{k-1}}[n]T_{i_k}^{n-1} \\ &= \left(\boldsymbol{v}_1 T_{i_1}^{m_1}\cdots T_{i_{k-1}}^{m_{k-1}}\right)T_{i_k}^{n} \end{aligned}$$

Since the left factor does not contain any instances of the letter $[n]$, it can be expressed as a word in A_{n-1} and, by induction, can be placed in the desired canonical form:

$$\boldsymbol{w} = \left(T_{i'_1}^{m'_1}\cdots T_{i'_{\ell-1}}^{m'_{k-1}}T_{i'_\ell}^{n-1}\right)T_{i_k}^{n}$$

At this point, it is clear that $m'_1 < \cdots < m'_{k-1} < n-1 < n$ and that $i'_1 < \cdots < i'_\ell$. Thus, if we show that $i'_\ell < i_k$, then we will be done. Assume for the sake of a contradiction that $i'_\ell \geq i_k$. Then $\boldsymbol{w}$ contains the factor $T_{i'_\ell}^{n-1}T_{i_k}^{n} = [n-1, \ldots, i'_\ell, n, i'_\ell + 1, i_\ell, \ldots i_k]$. But in this factor, the letter $[i'_\ell]$ appears twice with only one neighbor, $[i'_\ell - 1]$, occurring between the two instances of it, which violates our assumption of the homogeneity of $\boldsymbol{w}$.

Notice that by corollary 2.1.5, there can be at most two instances of the letter $[n-1]$, so we have exhausted all of the possibilities, and the existence of this form is shown.

Elements in this form agree with their expansion in terms of lemma 2.1.2, so uniqueness is immediate. ■

Given a homogeneous word in the canonical form $\boldsymbol{w} = T_{i_1}^{m_1} T{i_2}^{m_2} \cdots T_{i_\ell}^{m_\ell}$, we say that a factor $T_{i_j}^{m_j} T{i_{j+1}}^{m_{j+1}} \cdots T_{i_k}^{m_k}$ is *connected* if, for every pair of adjacent segments $T_{i_r}^{m_r}$ and $T{i_{r+1}}^{m_{r+1}}$, we have $m_{i_r} + 1 \geq i_{r+1}$. For example, in A_5, $T_1^3 T_4^5 = [3, 2, 1, 5, 4]$ is connected while $T_1^2 T_4^5 = [2, 1, 5, 4]$ is not.

2.2 Dyck paths

As in [Deu99], we define a Dyck path as a lattice path in the first quadrant consisting of steps $\langle 1, 1\rangle$ (north-east) and $\langle 1, -1\rangle$ (south-east), beginning at the origin and ending at the point $(2n, 0)$. We refer to n as the semi-length of the path. By a *peak* we shall mean a rise $\langle 1, 1\rangle$ followed by a fall $\langle 1, -1\rangle$, while a *valley* is a fall, followed by a rise.

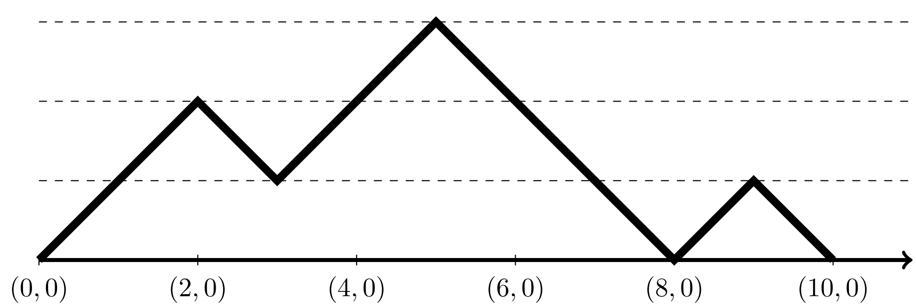


Figure 2.2.1: An example of a Dyck path of semilength 5.

Denote by $\mathcal{D}_{n,k}$ the set of all Dyck paths of semi-length n with the property that (*sum of peak heights*) - (*number of peaks*) $= k$. For the example path shown in figure 2.2.1 we have $k = (2 + 3 + 1) - 3 = 3$.

Let $T(n, k)$ be the cardinality of the set $\mathcal{D}_{n,k}$, as defined in [Deu08]. It is known that $T(n, k) = 0$ when $k > 1 + \lfloor \frac{n^2}{4} \rfloor$, so often it is convenient to display the sequence of non-zero values as a triangle with the entry $T(n, k)$ in the n^{th} row from the top (starting with $n = 0$) and the k^{th} column (also beginning with $k = 0$). The top of

1;									
1;									
1,	1;								
1,	2,	2;							
1,	3,	5,	4,	1;					
1,	4,	9,	12,	10,	4,	2;			
1,	5,	14,	25,	31,	26,	16,	9,	4,	1;

TABLE 2.2.1: The beginning of the sequence $T(n,k)$

the triangle is shown here.

It is well known that the number of Dyck paths of semi-length n is equal to the n^{th} Catalan number, $C_n = \dfrac{1}{n+1}\dbinom{2n}{n}$, so we have the rows adding to the entries of that sequence.

2.3 A bijection

Let $\mathcal{C}_{n,k}$ be the set of homogeneous components of length k for a type-A_n quiver. Suppose that we begin with *any* homogeneous component in $\mathcal{C}_{n,k}$. By the lemma 2.1.6, we can choose a representative word, $\boldsymbol{w} = T_{i_1}^{m_1} T_{i_2}^{m_2} \cdots T_{i_\ell}^{m_\ell}$. We will give an algorithm to construct the corresponding Dyck path.

To define the map, Φ: $\mathcal{C}_{n,k} \to \mathcal{D}_{n+1,k}$

1. Consider the left-most connected factor of $\boldsymbol{w}$, say $\boldsymbol{v} = T_{i_1}^{m_{i_1}} T_{i_2}^{m_{i_2}} \cdots T_{i_k}^{m_{i_k}}$.

2. Construct a lattice as shown in figure 2.3.1, ranging (horizontally) from $(0,0)$ to $(2n+2,0)$. Our Dyck path will travel along the edges of this lattice.

3. Follow the null path (*up, down, up, down,...*) from the origin to the point $(2i_1 - 2, 0)$, to the left of the i_1 block. From there, go north-east $\langle 1,1\rangle$.

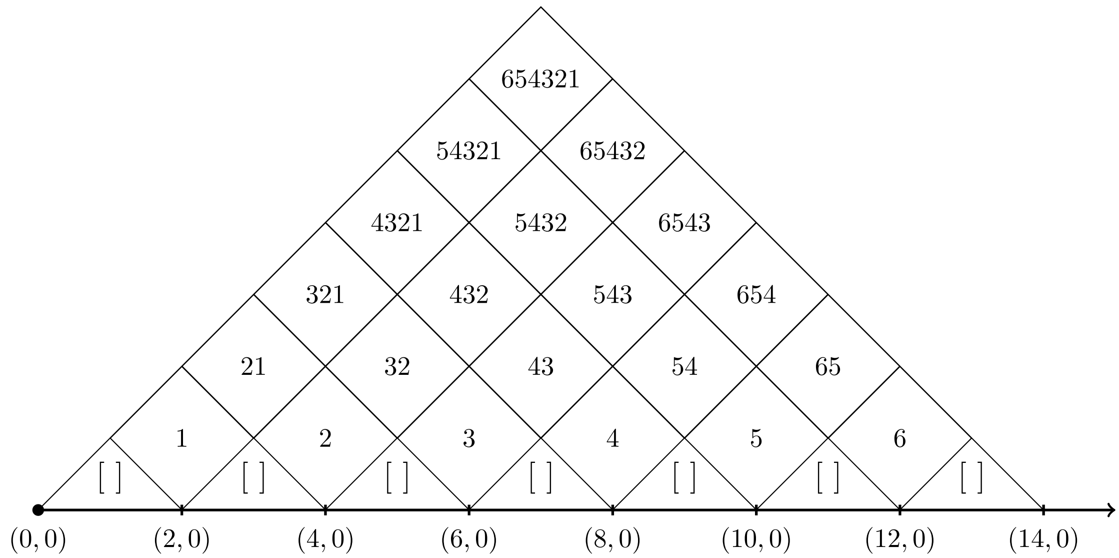


FIGURE 2.3.1: The triangular lattice for tracing Dyck paths

4. To get to the first peak: Go north-east $\langle (m_{i_1} - i_1 + 1), (m_{i_1} - i_1 + 1) \rangle$ to the peak of the $T_{i_1}^{m_{i_1}}$ block.

5. To form the first valley: Go south-east $\langle (i_2 - i_1), (i_1 - i_2) \rangle$, then north-east $\langle (m_{i_2} - m_{i_1}), (m_{i_2} - m_{i_1}) \rangle$ to the peak of $T_{i_2}^{m_{i_2}}$.

6. For $j = 2, \cdots, k-1$:

(a) Go south-east $i_{j+1} - i_j$ blocks (to the peak of $T_{i_{j+1}}^{m_{i_j}}$).

(b) Go north-east $m_{i_{j+1}} - m_{i_j}$ blocks (to the peak of $T_{i_{j+1}}^{m_{i_{j+1}}}$).

(c) Repeat the previous 2 steps for the next j.

7. Go down $m_{i_k} - i_k + 2$ blocks to the x-axis.

8. Repeat steps 5 & 6 for the other connected factors, tracing the "null path" between paths for connected factors.

9. After finishing with the last connected factor, follow the null path to the point $(2n+2,0)$.

To define the inverse, $\Psi : \mathcal{D}_{n+1,k} \to \mathcal{C}_{n,k}$

1. Superimpose the Dyck path of semi-length n onto a lattice such as the one shown above.

2. From left to right, read the words contained in the block at each peak of the Dyck path. These will be the T_i^m's.

2.3.1 Proposition. *The map Φ is a bijection with inverse Ψ.*

Proof. We begin by showing that the map Φ is well defined. Let $C \in \mathcal{C}_{n,k}$ be a homogeneous component with canonical representative $\boldsymbol{w} = T_{i_1}^{m_{i_1}} T_{i_2}^{m_{i_2}} \cdots T_{i_j}^{m_{i_j}}$, and without loss of generality, assume that $\boldsymbol{w}$ is a connected factor. Following the steps of the algorithm above, we see that, in the resulting path, the first peak has a height of $1 + m_{i_1} - i_1 + 1 = 2 + (m_{i_1} - i_1)$. The first "valley," following this peak, occurs at a height of $2 + (m_{i_1} - i_1) - (i_2 - i_1) = 2 + m_{i_1} - i_2$. Observe that because $\boldsymbol{w}$ is a connected factor, $m_{i_1} + 1 \geq i_2$, so this valley is above the x-axis.

Inductively, we see that the p^{th} peak in this path (for $1 \leq p \leq j$) has a height of $2 + m_{i_p} - i_p$, while the p^{th} valley (for $1 \leq p < j$) is at height $2 + m_{i_p} - i_{p+1}$. Again, because $\boldsymbol{w}$ is connected, this valley has a positive height.

The final peak of $\Phi(C)$ (corresponding to the factor $T_{i_j}^{m_{i_j}}$) occurs at height $2 + m_{i_j} - i_j$, so step 6 in the algorithm ends the path on the x-axis. Thus, we have $\Phi(C)$ begins on the x-axis, never dips into the 4th quadrant, and ends on the x-axis and, as such, is in fact a Dyck path.

It is clear from the algorithm that the blocks in the lattice that form the peaks of $\Phi(C)$ contain the words, respectively, $T_{i_1}^{m_{i_1}}, T_{i_2}^{m_{i_2}}, \ldots, T_{i_j}^{m_{i_j}}$. Thus, we see that $\Psi(\Phi(C)) = C$, and that Ψ is a left inverse of Φ.

Now, suppose that $D \in \mathcal{D}$ is a Dyck path that has been superimposed on a triangular root lattice, as per the algorithm, and say D has j non-degenerate peaks, for $j > 0$ (that is, we're considering peaks that have height greater than 1). Call the blocks occurring at each of these peaks $T_{i_1}^{m_{i_1}}, T_{i_2}^{m_{i_2}}, \ldots, T_{i_j}^{m_{i_j}}$. We notice that, by construction, $i_1 < i_2 < \cdots < i_j$ and likewise $m_{i_1} < m_{i_2} < \cdots < m_{i_j}$.

Consider the word $\Psi(D)$. It is clear that this word is in the canonical form described above, so we will show that this word represents a homogeneous component. That is, we show that if a letter, $r \in \{1, 2, \ldots, n\}$ appears in two segments, $T_{i_\ell}^{m_{i_\ell}}$ and $T_{i_{\ell'}}^{m_{i_{\ell'}}}$ for $\ell < \ell'$, then the letters $r-1, r+1$ appear between the 2 occurrences of r. First, assume that $\ell + 1 < \ell'$, and that r occurs in both segments $T_{i_\ell}^{m_{i_\ell}}$ and $T_{i_{\ell'}}^{m_{i_{\ell'}}}$. This implies that $i_\ell < i_{\ell+1} < i_{\ell'} \leq r \leq m_{i_\ell} < m_{i_{\ell+1}} < m_{i_{\ell'}}$, so r also occurs in the segment $T_{i_{\ell+1}}^{m_{i_{\ell+1}}}$.

So we can limit ourselves, without loss of generality, to the case in which r occurs in adjacent segments, $T_{i_\ell}^{m_{i_\ell}}$ and $T_{i_{\ell+1}}^{m_{i_{\ell+1}}}$, of the word $\Psi(D)$. It follows that $i_\ell < i_{\ell+1} \leq r \leq m_{i_\ell} < m_{i_{\ell+1}}$ By definition, $T_{i_\ell}^{m_{i_\ell}} T_{i_{\ell+1}}^{m_{i_{\ell+1}}} = [m_{i_\ell}, \ldots, r, r-1, \ldots, i_\ell, m_{i_{\ell+1}}, \ldots, r+1, r, \ldots, i_{\ell+1}]$. This factor or $\Psi(D)$ satisfies the homogeneity condition, since the letters $r-1, r+1$ occur between the two instances of the letter r. Since this holds for every pair of adjacent segments and any repeated letter, we see that the word $\Psi(D)$, and thus the component it represents, is homogeneous.

Consider now the Dyck path $\Phi(\Psi(D))$. Its non-degenerate peaks are precisely those defined by the blocks in the lattice $T_{i_1}^{m_{i_1}}, T_{i_2}^{m_{i_2}}, \ldots, T_{i_j}^{m_{i_j}}$, and the peaks of height 1 are recovered by following the null path between the connected factors of the word

$\Phi(D)$. We see, thus, that $\Phi(\Psi(D)) = D$, and so Ψ is in fact a two-sided inverse of Φ, proving the bijection. ■

We now give our main result for this chapter.

2.3.2 Theorem. *There exists a bijection between the irreducible homogeneous representations of a KLR-Algebra of type A_n and the Dyck paths of semi-length $n+1$. Further, if such a representation is given by a homogeneous component of words with length k, the corresponding Dyck path will have (Sum of Peak Heights) - (Number of Peaks) $= k$.*

Proof. The bijection has been shown by proposition 2.3.1. It remains only to show that, as claimed, a homogenous component $C \in \mathcal{C}_{n,k}$ is mapped to a Dyck path $D \in \mathcal{D}_{n+1,k}$. It's clear from the algorithm that, if $C \in C_{n,k}$, the Dyck path $\Phi(C)$ has semi-length $n+1$.

Now, suppose that C is represented by a word in canonical form, $\boldsymbol{w} = T_{i_1}^{m_{i_1}} T_{i_2}^{m_{i_2}} \cdots T_{i_j}^{m_{i_j}}$. Note that each segment $T_{i_\ell}^{m_{i_\ell}}$ contains $m_{i_\ell} - i_\ell + 1$ letters. Since $\boldsymbol{w}$ has k letters, we have

$$\begin{aligned} k &= \sum_{\ell=1}^{j} [(m_{i_\ell} - i_\ell) + 1] \\ &= j + \sum_{\ell=1}^{j} (m_{i_\ell} - i_\ell). \end{aligned}$$

But in the path $\Phi(C)$, each of the j segments, $T_{i_\ell}^{m_{i_\ell}}$, corresponds to a (non-degenerate) peak with height $(m_{i_\ell} - i_\ell) + 2$. We then have

$$
\begin{aligned}
\text{(peak heights) - (\# of peaks)} &= \sum_{\ell=1}^{j}[(m_{i_\ell} - i_\ell) + 2] - j \\
&= \sum_{\ell=1}^{j}(m_{i_\ell} - i_\ell) + 2j - j \\
&= j + \sum_{\ell=1}^{j}(m_{i_\ell} - i_\ell). \\
&= k
\end{aligned}
$$

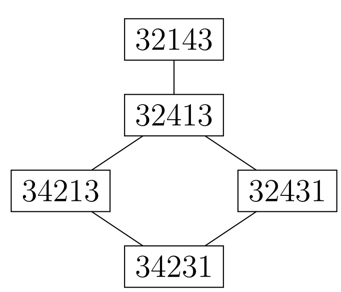
as desired. ■

2.4 Examples

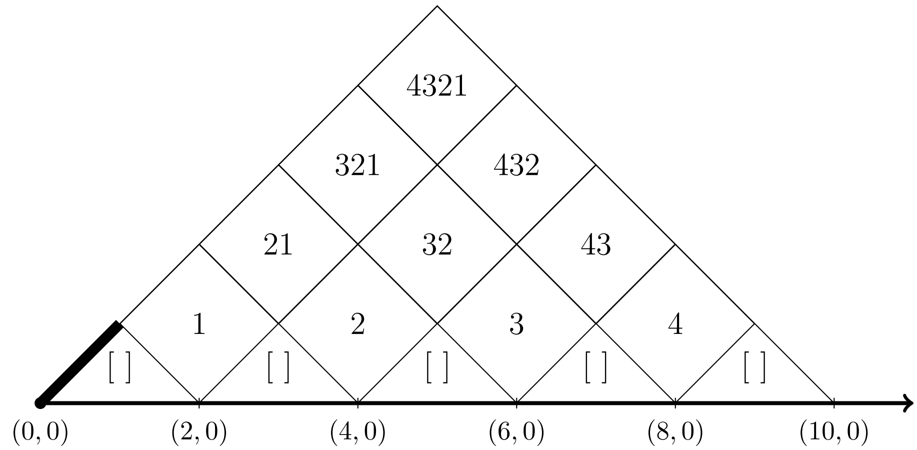
We show here an application of the bijection in either direction.

2.4.1 The map Φ

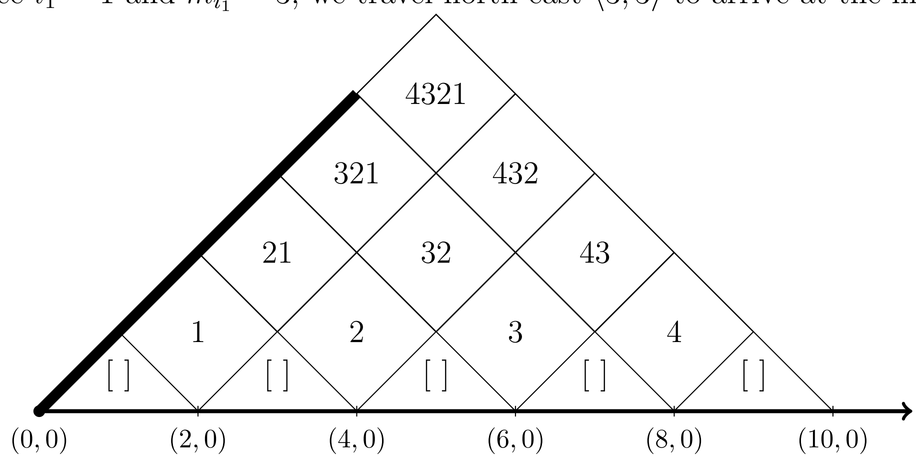
Suppose the quiver Γ is of type A_4, and the homogeneous component C is



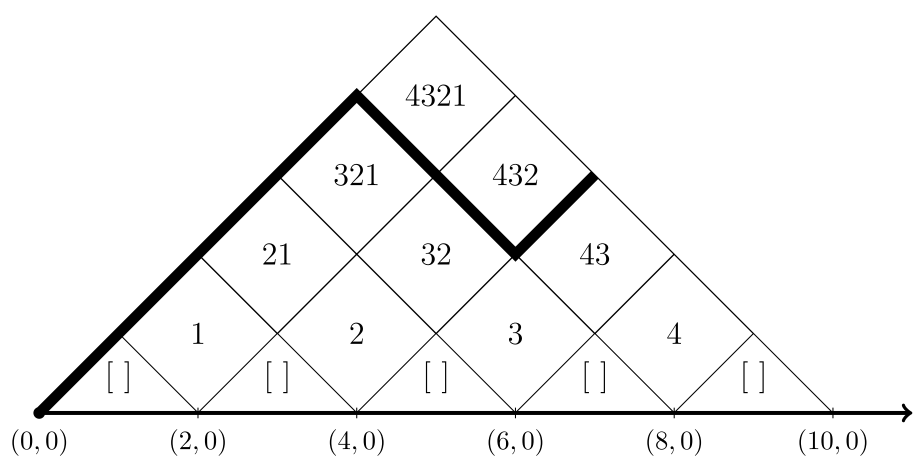
Then the canonical representative of this component is $\boldsymbol{w} = [3, 2, 1, 4, 3] = T_1^3 T_3^4$. We begin with the lattice as shown in figure 2.3.1.



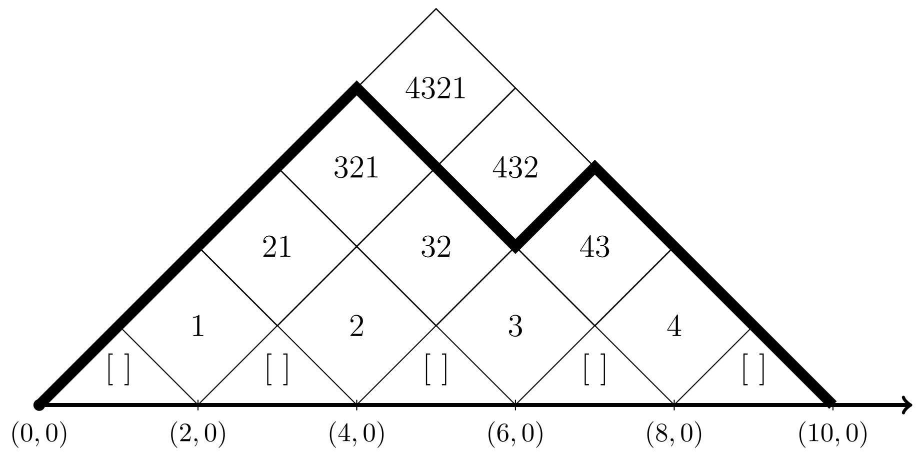
Since $i_1 = 1$ and $m_{i_1} = 3$, we travel north east $\langle 3, 3\rangle$ to arrive at the first peak:



We then go south-east $3 - 1 = 2$ blocks, and northeast $4 - 3 = 1$ block:

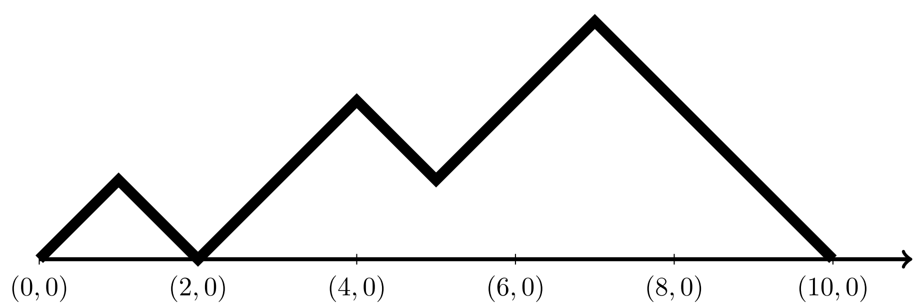


Since there are no more T_i^m's, we skip to step 7 and complete the Dyck path $\Phi(C)$:

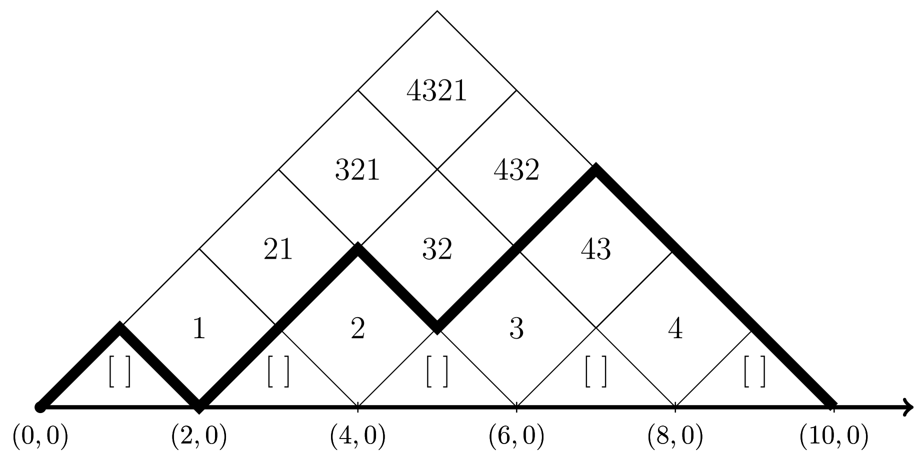


2.4.2 The map Ψ

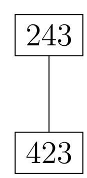
To see an example of the application of Ψ, suppose that we have the Dyck path D:



We superimpose this on the lattice:



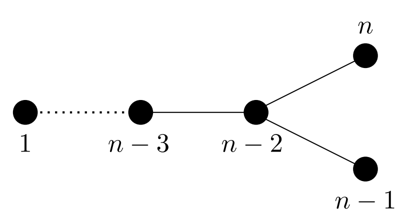
Reading, from left to right, the segments contained in the peaks, we see that the component $\Psi(D)$ is represented by the word $[2, 4, 3] = T_2^2 T_3^4$. This is the homogeneous component



Chapter 3

Homogeneous Words in a Group of Type D_n

In this chapter, we will focus on representations of KLR-algebras of type D_n. That is, we shall assume that Γ is a quiver of the form:



While it is clear that any homogeneous word in a group of type A_n is also a homogeneous word in D_{n+1}, in this setting we see some new phenomena. We begin, though, as we did in the type A_n case.

3.1 Canonical reduced words

Let us define shorthand for sequences of letters that appear in reduced words. For $1 \leq i \leq n-1$, we define s_{ij} similarly as we do for type A:

$$s_{ij} = \begin{cases} [i, i-1, \ldots, j] & \text{if } i > j \\ [i] & \text{if } i = j \\ [\,] & \text{if } i < j \end{cases} \qquad s_{nj} = \begin{cases} [n, n-2, \ldots, j] & \text{if } j \leq n-2 \\ [n] & \text{if } j = n, n-1 \\ [\,] & \text{if } j > n \end{cases}$$

3.1.1 Lemma ([BS01], Lemma 5.2)**.** *Any element of the Coxeter group D_n can be presented in the reduced form*

$$s_{1i_1} s_{2i_2} \cdots s_{nj_1} s_{n-1j_2} s_{nj_3} s_{n-1j_4} \cdots$$

where, $i_1 \leq 2, \cdots, i_{n-1} \leq n$; $1 \leq j_1 < j_2 < \cdots < j_\ell \leq n-1$, and $\ell \geq 0$.

We will refer to the *prefix*, the left factor $s_{1i_1} s_{2i_2} \cdots s_{n-1i_{n-1}}$ and *suffix* of a word, the right factor $s_{nj_1} s_{n-1j_2} s_{nj_3} s_{n-1j_4} \cdots$. Note that we will require the suffix of a word to be of maximal length, e.g. in a D_5 group, the word $[2, 1, 3, 4, 5, 3, 2, 4, 3, 5]$ has suffix $[5, 3, 2, 4, 3, 5]$ even though $[5]$ is a right factor and admissible suffix by the above lemma. Given a reduced word, $\boldsymbol{w}$, in canonical form, we will denote by $\boldsymbol{w}_0$ the prefix of $\boldsymbol{w}$ and by $\boldsymbol{w}'$ the suffix. We clearly then have $\boldsymbol{w} = \boldsymbol{w}_0, \boldsymbol{w}'$.

3.1.2 Remark. Notice that choosing an admissible suffix is equivalent to choosing a (possibly empty) subset of $1, 2, \ldots, n-1$. There are 2^{n-1} ways to do this. There are $n!$ prefixes, so we have $n!2^{n-1}$ elements of this form. There are the same number of elements in a type D_n Coxeter group, so expressions of this form are unique.

We will then declare this form to be the canonical reduced word for any element in this group.

3.1.3 Remark. It is easy to check from the definition that, on its own, every admissible suffix is a homogeneous word.

For a fixed n, the homogeneous elements of a Coxeter group of type D_n can be collected based on the suffix of their canonical reduced word expressions.

3.1.4 Definition. A **collection** labeled by an admissible suffix, $\mathbf{c}^n_{\boldsymbol{w}'}$ described above will refer to the set of homogeneous elements whose canonical reduced expressions end with this suffix. By remark 3.1.3, there is a non-empty collection for each admissible suffix.

3.1.5 Lemma. *Any collection whose suffix begins with* $[n, n-2, \ldots, k+2, k+1, n-1, \ldots]$*, that is, for which* $\ell > 1$ *and* $j_1 = k+1$*, with* $0 \leq k \leq n-3$*, has the same number of elements. Further, the prefixes in these collections are exactly the same.*

Proof. Suppose that $\boldsymbol{w}'$ and $\boldsymbol{w}''$ are the suffixes of the form $[n, n-2, \cdots, k+1, n-1, \cdots]$ and that $\boldsymbol{w}''$ has $\boldsymbol{w}'$ as a proper left factor. Since removing letters from the end of a word will not affect its homogeneity, it is clear that any prefix appearing in the collection $\mathbf{c}^n_{\boldsymbol{w}''}$ also appears in $\mathbf{c}^n_{\boldsymbol{w}'}$. We need to show, then, the opposite inclusion.

Suppose now that $\boldsymbol{w}_0$ is a prefix appearing in the collection labeled by $\boldsymbol{w}'$, and that there is some letter, r which appears in both $\boldsymbol{w}_0$ and $\boldsymbol{w}'$. The prefix and suffix are individually homogeneous words, but the fact that $\boldsymbol{w}_0, \boldsymbol{w}'$ is homogeneous requires that two neighbors of r appear between the last instance of r in $\boldsymbol{w}_0$ and the first instance of r in $\boldsymbol{w}'$.

We claim that adding a letter to the end of $\boldsymbol{w}'$ in order to obtain a new admissible suffix preserves the homogeneity. Recognize that if the letter added does not occur in

$\boldsymbol{w}_0$, or if it already occurs in $\boldsymbol{w}'$, there is nothing to check. That is, we need to examine the case in which the added letter appears in $\boldsymbol{w}_0$ but not previously in $\boldsymbol{w}'$. The case in which this can occur is very limited: lemma 3.1.1 implies that this arises when $\boldsymbol{w}' = s_{nj_1}s_{n-1j_2}$ for $j_2 > j_1+1$, and the letter $[j_2-1]$ occurs in $\boldsymbol{w}_0$. If we add this letter to the end of the suffix, we get the word $\boldsymbol{w}_0[n, n-2, \ldots, j_2, \cdots, j_1, n-1, \ldots, j_2, j_2-1]$. The letter j_2 is a neighbor of j_2-1, and appears twice between the instances of $[j_2-1]$ in the prefix and suffix. Thus, homogeneity is preserved.

Each admissible suffix of the form specified in this lemma can be obtained from a smaller one by adding a letter, so the result is shown. ■

3.1.6 Definition. A suffix that is exactly $s_{nk} = [n, n-2, \ldots, k+1, k]$ is said to be of **type 1**. A collection labeled by such a suffix will be called a **type 1 collection**.

A suffix of the type described in lemma 3.1.5 is said to be of **type 2**. Likewise, a collection labeled by such a suffix is said to be a **type 2 collection**.

3.1.7 Lemma. *For* $1 \leq k < n-2$, *any type 1 collection whose suffix is exactly* $[n, n-2, \ldots, k+1, k]$, *that is, for which* $j_1 = k$ *and* $\ell = 1$, *has the same number of words as any type 2 collection whose suffix begins with* $[n, n-2, \ldots, k+2, k+1, n-1, \ldots]$.

Proof. (of Lemma 3.1.7) We prove this by establishing a bijection between the words in $\mathbf{c}^n_{[n,\ldots,k]}$ and the words in $\mathbf{c}^n_{[n,\ldots,k+1,n-1]}$, which is sufficient by lemma 3.1.5. Before defining the map between these collections, we record some observations.

Suppose that $\boldsymbol{w} = \boldsymbol{w}_0, \boldsymbol{w}' = s_{1i_1} \cdots s_{n-1i_{n-1}}\boldsymbol{w}'$ is a word in the homogeneous collection $\mathbf{c}^n_{[n,\ldots,k]}$. We claim that there are three cases for the value of i_{n-1}:

(i) $i_{n-1} < k$, in which case $\boldsymbol{w}_0$ ends with a descending segment,

(ii) $i_{n-1} = n-1$, which implies that $\boldsymbol{w}_0$ ends with the letter $[n-1]$, or

(iii) $i_{n-1} = n$, in which case there is no instance of $[n-1]$ in the prefix $\boldsymbol{w}_0$.

To see why this is the case, suppose that $k \leq i_{n-1} < n-1$. Then $\boldsymbol{w}$ contains the factor $s_{n-1 i_{n-1}} s_{nk} = [n-1, \ldots, i_{n-1}, n, n-2, \ldots, i_{n-1}+1, i_{n-1}, \ldots]$. If $i_{n-1} = n-2$, then its neighbors are $n-3, n-1$, and n. If $i_{n-1} < n-2$, its neighbors are $i_{n-1}+1$ and $i_{n-1}-1$. In either case, only one neighbor appears between the two occurrences of i_{n-1} in this factor, thus violating homogeneity.

Now we can define a map $\sigma : \mathbf{c}^n_{[n,\ldots,k]} \to \mathbf{c}^n_{[n,\ldots,k+1,n-1]}$. If we have either case (i) or case (iii) from the last paragraph, the map σ will simply replace the type 1 suffix $[n, n-2, \ldots, k]$ with the type 2 suffix $[n, n-2, \ldots, k+1, n-1]$. To see that this image is actually in $\mathbf{c}^n_{[n,\ldots,k+1,n-1]}$, we need to check that changing the last letter of the suffix to $[n-1]$ does not violate homogeneity. In the first case, we have $s_{n-1 i_{n-1}}[n, \ldots, k+1, n-1] = [n-1, n-2, \ldots, i_{n-1}, n, n-2, \ldots, k+1, n-1]$ and clearly the neighbor $[n-2]$ appears twice between the two occurrences of $[n-1]$. In the third case, there is no occurrence of the letter $[n-1]$ in the prefix, thus the homogeneity condition is trivially satisfied.

If, instead, we have $i_{n-1} = n-1$, then this map is a bit more involved to describe. In this case, $\boldsymbol{w}_0$ ends with some ascending string. We can choose $m \geq k$ to be the smallest letter such that this string $[m, m+1, \ldots, n-1]$ is as long as possible. Since, by assumption, $i_{n-1} = n-1$, there is some such m with $k \leq m \leq n-1$. In this case the map σ will, in addition to changing the suffix, act by replacing the factor $[m, m+1, \ldots, n-1]$ with the segment $s_{mk} = [m, m-1, \ldots, k]$. Some examples of the action of σ are shown in table 3.1.1. We show now that, under this action of σ, the image

$$\sigma(\boldsymbol{w}) = s_{1i_1} \cdots s_{m-1 i_{m-1}} s_{mk}[n, n-2, \ldots, k+1, n-1]$$

is still a homogeneous word.

$\boldsymbol{w} \in \mathbf{c}^5_{[5,3,2]}$	$\sigma(\boldsymbol{w}) \in \mathbf{c}^5_{[5,3,4]}$
$[3,2,1,5,3,2]$	$[3,2,1,5,3,4]$
$[4,3,2,1,5,3,2]$	$[4,3,2,1,5,3,4]$
$[1,2,3,4,5,3,2]$	$[1,2,5,3,4]$
$[2,1,4,5,3,2]$	$[2,1,4,3,2,5,3,4]$

TABLE 3.1.1: Some examples of the map $\sigma : \mathbf{c}^5_{[5,3,2]} \to \mathbf{c}^5_{[5,3,4]}$, where $k = 2$

It is clear to check that the right factor $s_{mk}[n, \ldots, k+1, n-1]$ is homogeneous, but suppose that some letter $r \leq n-2$ appears in both the segment s_{mk}, (so $k \leq r < m$) and in some segment s_{ji_j}, (so $i_j \leq r \leq j$). Since $r < m$, the letter $[r]$ did not appear in the ascending string $[m, \ldots, n-1]$ in the word $\boldsymbol{w}$, but since $k \leq r < n-2$, it did appear in the suffix, $[n, \ldots, r+1, r, \ldots, k]$, along with one neighbor. Since $\boldsymbol{w}$ is homogeneous, there must have been another neighbor in the prefix $\boldsymbol{w}_0$ which was untouched by the action of σ. Thus $\sigma(\boldsymbol{w})$ is also a homogeneous word, and we have shown that $\sigma(\mathbf{c}^n_{[n,\ldots,k]}) \subset \mathbf{c}^n_{[n,\ldots,k+1,n-1]}$. To summarize, we have

$$\sigma(\boldsymbol{w}) = \begin{cases} s_{1i_1} \cdots s_{mk}[n \ldots, k+1, n-1] & \text{if } \boldsymbol{w}_0 \text{ ends with } [m, m+1, \ldots, n-1] \\ \boldsymbol{w}_0[n, \ldots, k+1, n-1] & \text{otherwise} \end{cases}$$

where $m \geq k$.

Next, we define a map that goes in the other direction, $\tau : \mathbf{c}^n_{[n,\ldots,k+1,n-1]} \to \sigma : \mathbf{c}^n_{[n,\ldots,k]}$. Suppose that $\boldsymbol{w} = \boldsymbol{w}_0[n, n-2, \ldots, k+1, n-1] \in \mathbf{c}^n_{[n,\ldots,k+1,n-1]}$. By the condition of homogeneity, $\boldsymbol{w}_0$ must end with one of the letters $1, 2, \ldots, k$. If $\boldsymbol{w}_0$ does not end with the letter $[k]$, then τ acts by simply replacing the suffix $[n, \ldots, k+1, n-1]$ with the suffix $[n, \ldots, k+1, k]$. To see that this results in a homogeneous word, we need to check only that inclusion of the letter $[k]$ at the end of the suffix does not violate the homogeneity condition. Consider the last non-empty segment, s_{jr} for

$r \le j$, of $\boldsymbol{w}_0$. Since $\boldsymbol{w}_0$ ends with a letter smaller than k, we have either $r < k \le j$ with $s_{jr} = [j, \ldots, k, k-1, \ldots, r]$, or $r \le j < k$, in which case the prefix does not contain the letter k. In the first case, there are two neighbors ($k+1$ and $k-1$) between the last two occurrences of $[k]$, and in the first case, there are no new pairs of $[k]$'s to check. In either case, homogeneity is preserved.

If, on the other hand, $\boldsymbol{w}_0$ ends with the letter k, then the final non-empty segment of the prefix is s_{jk} for some $k \le j \le n-1$. We declare that τ acts by replacing the suffix $[n, \ldots, k+1, n-1]$ with $[n, \ldots, k+1, k]$, and replacing s_{jk} with the ascending string $[j, j+1, \ldots, n-1]$. Table 3.1.2 gives some examples of the action of τ. It remains to see that $\tau(\boldsymbol{w})$ is in fact a homogeneous word. Notice that the left factor of $\boldsymbol{w}_0$, $s_1 i_1 \cdots s_{j-1 i_{j-1}}$ and the ascending string $[j, \ldots, n-1]$ have no letters in common, so there is nothing here to check. Also the right factor $[j, \ldots, n-1, n, n-2, \ldots, k]$ is easily checked to be homogeneous.

$\boldsymbol{w} \in \mathbf{c}^5_{[5,3,4]}$	$\tau(\boldsymbol{w}) \in \mathbf{c}^5_{[5,3,2]}$
$[2,1,5,3,4]$	$[2,1,5,3,2]$
$[2,5,3,4]$	$[2,3,4,5,3,2]$
$[1,4,3,2,5,3,4]$	$[1,4,5,3,2]$

Table 3.1.2: Some examples of the map $\tau : \mathbf{c}^5_{[5,3,4]} \to \mathbf{c}^5_{[5,3,2]}$, where $k = 2$

Suppose that some letter $[r]$ occurs in the left factor $s_1 i_1 \cdots s_{j-1 i_{j-1}}$, and also in the suffix $[n, \ldots, r+1, r, \ldots, k]$. Since $[r]$ doesn't appear in the ascending string $[j, \ldots, n-1]$ but does appear in the suffix $[n, n-2, \ldots, k]$, it follows that $k \le r < j$, so the letter $[r]$ appears in the segment $s_{j,k}$ in $\boldsymbol{w}_0$. Since the word $\boldsymbol{w} = \boldsymbol{w}_0, \boldsymbol{w}'$ was assumed to be homogeneous, it must be the case that two neighbors of $[r]$ appear between these two instances in $\boldsymbol{w}_0$. One of them may be in the segment s_{jk} which is replaced by the map τ, but at least one of them must be in the left factor which remains fixed under τ. Clearly a second neighbor occurs in the suffix, so the condition for homogeneity is satisfied. To summarize, we have, for $\boldsymbol{w} = \boldsymbol{w}_0, \boldsymbol{w}' \in \mathbf{c}^n_{[n,\ldots,k+1,n-1]}$

$$\tau(\boldsymbol{w}) = \begin{cases} s_{1i_1} \cdots s_{j-1i_{j-1}}[j, j+1, \ldots, n-1]s_{nk} & \text{if } \boldsymbol{w}_0 \text{ ends with } s_{jk} \\ \boldsymbol{w}_0 s_{nk} & \text{otherwise} \end{cases}$$

where $k \leq j \leq n-1$.

At this point it is clear that τ is both a left and a right inverse of σ, so the bijection is shown.

■

3.2 Packets

We have so far established that the type 1 collection labeled by suffix s_{nk} and any of the type 2 collections with suffix of the form $[n, n-2, \ldots, k+2, k+1, n-1, \ldots]$ have the same cardinality. It is natural, then, to identify them together.

3.2.1 Definition. For $1 \leq k \leq n-3$, the set containing the type 1 collection labeled by suffix s_{nk} and all of the the type 2 collections with suffix of the form $[n, n-2, \ldots, k+2, k+1, n-1, \ldots]$ will be known as the $(\mathbf{n}, \mathbf{k})$**-packet**.

We extend this definition to $0 \leq k \leq n$ by declaring:

- The $(n, 0)$-packet contains all of the type 2 collections labeled by $[n, n-2, \ldots, 1, n-1, \ldots]$.
- The $(n, n-2)$-packet contains the type 1 collection labeled by $[n, n-2]$.
- The $(n, n-1)$-packet contains the type 1 collection labeled by $[n]$.

- The (n,n)-packet contains the collection labeled by an empty suffix [ ]. We will consider this collection to be of type 1.

For the sake of brevity, we will often denote by $\mathcal{P}(n,k)$ the (n,k)-packet.

As an example, table 3.2.1 shows all of the packets in the case of D_4.

3.2.2 Proposition. *The size of the packet $\mathcal{P}(n,k)$ is*

$$|\mathcal{P}(n,k)| = \begin{cases} 2^{n-2}-1 & \text{if } k=0 \\ 2^{n-k-2} & \text{if } 0<k<n-1 \\ 1 & \text{if } k=n-1,n \end{cases}$$

It may be convenient to visualize these values in the array in table 3.2.2, where the row is given by n (starting at 0), and the column is given by k (also beginning at 0).

Proof. It is clear that, for $1 \le k \le n$, the packet $\mathcal{P}(n,k)$ contains exactly one collection of type 1. It remains to show, then, that when $0 \le k \le n-2$, the packet $\mathcal{P}(n,k)$ contains $2^{n-k-2}-1$ type 2 collections, i.e. that there are $2^{n-k-2}-1$ suffixes that begin with $[n, n-2, \ldots, k+1, n-1, \ldots]$.

Notice that, by lemma 3.1.1, a suffix is completely determined by the increasing list $(j_1, j_2, \ldots, j_\ell)$, where $1 \le j_1,\ j_\ell \le n-1$, and $0 \le \ell \le n-1$. In particular, a type 2 suffix in $\mathcal{P}(n,k)$ is determined by a list $(k+1, j_2, \ldots, j_\ell)$. Choosing such a list, though, is equivalent to choosing a non-empty subset of $\{k+2, k+3, \cdots, n-1\}$. There are $2^{n-k-2}-1$ such subsets, and our result follows. ∎

3.2.3 Corollary. *For a Coxeter group of type D_n, with $n \ge 3$, there are 2^{n-1} suffixes, and these are split into n packets.*

$\mathcal{P}(4,0)$

$\underline{\mathbf{c}^4_{[4,2,1,3]}}$	$\underline{\mathbf{c}^4_{[4,2,1,3,2]}}$	$\underline{\mathbf{c}^4_{[4,2,1,3,2,1]}}$
$[4,2,1,3]$	$[4,2,1,3,2]$	$[4,2,1,3,2,1]$

$\mathcal{P}(4,1)$

$\underline{\mathbf{c}^4_{[4,2,1]}}$		$\underline{\mathbf{c}^4_{[4,2,3]}}$	
$[4,2,1]$	$[2,3,4,2,1]$	$[3,2,1,4,2,3]$	$[1,4,2,3]$
$[3,4,2,1]$	$[1,2,3,4,2,1]$	$[2,1,4,2,3]$	$[4,2,3]$

$\mathcal{P}(4,2)$

	$\underline{\mathbf{c}^4_{[4,2]}}$	
$[4,2]$	$[1,4,2]$	$[3,4,2]$
$[3,2,1,4,2]$	$[2,1,3,4,2]$	$[2,3,4,2]$
$[2,1,4,2]$	$[1,3,4,2]$	$[1,2,3,4,2]$

$\mathcal{P}(4,3)$

	$\underline{\mathbf{c}^4_{[4]}}$		
$[1,2,3,4]$	$[1,4]$	$[2,3,4]$	$[3,4]$
$[1,2,4]$	$[2,1,3,2,4]$	$[2,4]$	$[4]$
$[1,3,2,4]$	$[2,1,3,4]$	$[3,2,1,4]$	
$[1,3,4]$	$[2,1,4]$	$[3,2,4]$	

$\mathcal{P}(4,4)$

	$\underline{\mathbf{c}^4_{[\,]}}$		
$[1,2,3]$	$[1]$	$[2,3]$	$[3]$
$[1,2]$	$[2,1,3,2]$	$[2]$	$[\,]$
$[1,3,2]$	$[2,1,3]$	$[3,2,1]$	
$[1,3]$	$[2,1]$	$[3,2]$	

TABLE 3.2.1: The packets of D_4

1								
1	1							
1	1	1						
1	1	1	1					
3	2	1	1	1				
7	4	2	1	1	1			
15	8	4	2	1	1	1		
31	16	8	4	2	1	1	1	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

TABLE 3.2.2: Triangle of Packet Sizes

Proof. This is found easily by expanding the expressions from the previous lemma as a geometric series. ■

3.3 Catalan's triangle

In this section, we show the size of any collection in a given packet, allowing us to classify and enumerate all homogeneous representations. We begin by presenting a seemingly unrelated sequence:

3.3.1 Definition. The array shown below is known as **Catalan's Triangle**. The entry in the n^{th} row and k^{th} column is denoted by $C(n, k)$, for $0 \leq k \leq n$. It can be built recursively by declaring that the first entry $C(0, 0) = 1$, and that each subsequent entry is given by the sum of the entry above it and the entry to the left. We can imagine that all entries outside of the range $0 \leq k \leq n$ for $n \geq 0$ are filled with 0's.

1								
1	1							
1	2	2						
1	3	5	5					
1	4	9	14	14				
1	5	14	28	42	42			
1	6	20	48	90	132	132		
1	7	27	75	165	297	429	429	
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

TABLE 3.3.1: Catalan's Triangle

More precisely, for $n \geq 0$ and $0 \leq k \leq n$,

$$C(n,k) = \begin{cases} 1 & \text{if } n = 0 \\ C(n,k-1) + C(n-1,k) & \text{if } 0 < k < n \\ C(n-1,0) & \text{if } k = 0 \\ C(n,n-1) & \text{if } k = n \end{cases} \tag{3.3.1}$$

The closed form for entries in this triangle is well known [OEI]. For $n \geq 0$ and $0 \leq k \leq n$, we have

$$C(n,k) = \frac{(n+k)!(n-k+1)}{k!(n+1)!} \tag{3.3.2}$$

3.3.2 Remark. We note that the elements on the diagonal of this array give the sequence of Catalan numbers:

$$\begin{aligned} C(n,n-1) &= \frac{(2n-1)!(2)}{(n-1)!(n+1)!} = \frac{1}{n+1}\frac{(2n-1)!(2n)}{(n-1)!n(n)!} = \frac{1}{n+1}\binom{2n}{n} = C_n \\ C(n,n) &= \frac{(2n)!(1)}{n!(n+1)!} = \frac{1}{n+1}\binom{2n}{n} = C_n \end{aligned}$$

3.3.3 Lemma. *For any $n \geq 4$, the collections in the packets $\mathcal{P}(n,n-1)$ and $\mathcal{P}(n,n)$*

each contain $C(n, n-1) = C(n, n) = C_n$ *homogeneous words.*

Proof. We recognize that each of these packets contains only one collection: $\mathcal{P}(n, n-1) = \{\mathbf{c}^n_{[n]}\}$ and $\mathcal{P}(n, n) = \{\mathbf{c}^n_{[\ ]}\}$. Since neither of the suffixes labeling these packets contain any of the letters $1, 2, \ldots, n-1$, there are no restrictions on the homogeneous prefixes that can be included. Therefore, any type-A_{n-1} homogeneous word can act as a prefix. But we know from the work in the previous chapter and from theorem 1.3.4 that there are exactly C_n of these. It is easy to verify that, in equation 3.3.2, $C(n, n-1) = C(n, n) = C_n$. ■

The previous lemma showed that the diagonal and subdiagonal of Catalan's triangle do in fact count the size of collections in the corresponding packets. We now give the main theorem of this chapter, extending this result to the rest of the triangle, and proving the tool allowing us to classify the homogeneous words.

3.3.4 Theorem. *Any collection in the packet* $\mathcal{P}(n, k)$ *contains exactly* $C(n, k)$ *distinct reduced words.*

Proof. Note that lemma 3.3.3 treats the case when $k = n-1$ or n. We next consider the packets $\mathcal{P}(n, 0)$: the sets of collections labeled by the prefixes $[n, n-2, \ldots, 1, n-1, \ldots]$. We claim that the only prefix appearing in any of these collections is $[\ ]$, the empty word. Suppose that one of these collections contains a non-empty prefix $\boldsymbol{w}_0$ ending with the letter $[r]$ for $1 \leq r \leq n-1$. That is, we have a homogeneous word containing the factor $[\ldots, r, n, n-2, \ldots, 1, n-1, \ldots]$. We immediately reach a contradiction: If $1 \leq r \leq n-2$, there is only one neighbor between the two occurrences of r in this factor. If $r = n$, then $\boldsymbol{w}$ is not reduced. Finally if, $r = n-1$, there is only one occurrence of the neighbor $[n-2]$ between the two instances of $[r]$. Thus, every collection in this packet contains $C(n, 0)$ reduced words, but $C(n, 0) = 1$.

Just as the entries of Catalan's triangle are determined uniquely by fixing a particular row of the triangle and a recursion rule, we prove this result for the remaining packets by establishing a single row of the triangle and showing that the collection size of the packets satisfy the relations given in equation 3.3.1. By the definition of a packet, it will be enough to check that these relations hold for a single collection in each of them.

We see in table 3.2.1 that, in the case of D_4, any collection in the packet $\mathcal{P}(4,k)$ contains exactly $C(4,k)$ elements. It remains to show that the recursion given in equation 3.3.1 holds. To that end, let $n > 4$ and $1 \leq k \leq n-2$. We give an explicit bijection from the words in $\mathbf{c}^{n}_{[n,\ldots,k,n-11]} \cup \mathbf{c}^{n-1}_{[n-1,\ldots,k]}$ to the type 1 collection $\mathbf{c}^{n}_{[n,\ldots,k]}$. That is, we will have a map from a type 2 collection in $\mathcal{P}(n-1,k)$ and another from the type 1 collection in $\mathcal{P}(n,k-1)$ into the type 1 collection in $\mathcal{P}(n,k)$. By our choice of n and k, we are guaranteed that these collections are non-empty.

Define the map $\varphi_1 : \mathbf{c}^{n}_{[n,\ldots,k,n-1]} \to \mathbf{c}^{n}_{[n,\ldots,k]}$ by

$$\varphi_1(\boldsymbol{w}_0[n,\cdots,k,n-1]) = \boldsymbol{w}_0[n,\cdots,k].$$

Clearly, removing the last letter of the suffix will not affect the homogeneity of a word, so φ_1 does in fact map $\mathbf{c}^{n}_{[n,\ldots,k,n-1]}$ into $\mathbf{c}^{n}_{[n,\ldots,k]}$. We can see that this map is injective, as words with different prefixes will have different images.

Similarly, we define the map $\varphi_2 : \mathbf{c}^{n-1}_{[n-1,\ldots,k]} \to \mathbf{c}^{n}_{[n,\ldots,k]}$ via

$$\varphi_2(\boldsymbol{w}_0[n-1,n-3,\ldots,k]) = \boldsymbol{w}_0[n-1,n,n-2,n-3\ldots,k].$$

Figure 3.3.1 shows an example of the maps φ_1 and φ_2 in the case of $n=5$ and $k=2$.

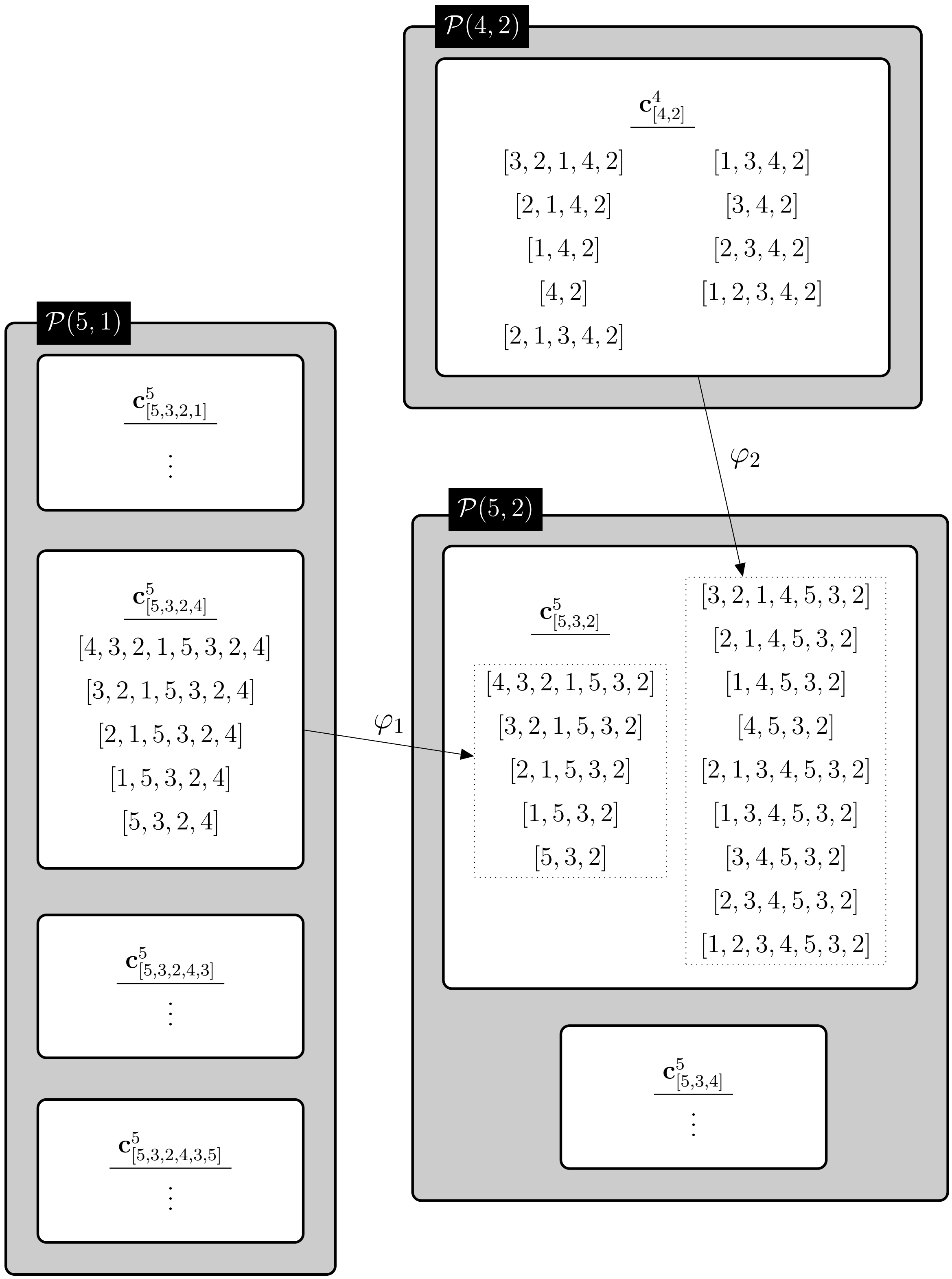


FIGURE 3.3.1: The maps φ_1 and φ_2 into the packet $\mathcal{P}(5,2)$.

There is a bit of subtlety in checking that the image of this map is homogeneous in D_n since the original word was homogeneous in D_{n-1}. It is clear that $\boldsymbol{w}_0$ doesn't contain the letters $[n-1]$ or $[n]$, by lemma 3.1.1. Notice that, in D_{n-1}, $[n-3]$ and $[n-1]$ are neighbors. Still, if $[n-3]$ occurs in the prefix $\boldsymbol{w}_0$, it must be followed by another neighbor (on the quiver D_{n-1}), $[n-2]$ or $[n-4]$, in $\boldsymbol{w}_0$. In either case, this is sufficient for homogeneity in $\boldsymbol{w}_0[n-1, n, n-2, n-3\ldots, k]$. On the other hand, if $\boldsymbol{w}_0$ contains the letter $[n-2]$, it is easy to see that the image of this map is also homogeneous. Thus, the map φ_2 is well defined and, for the same reasons as above, injective. It is important to note that the images of φ_1 and φ_2 are disjoint: the words in the image of φ_2 all have prefixes that end with the letter $[n-1]$, while none of the words in the image of φ_1 do.

To show that these homogeneous words are in bijection, we define inverse maps. Let $\rho : \mathbf{c}^n_{[n,\ldots,k]} \to \mathbf{c}^n_{[n,\ldots,k,n-1]} \cup \mathbf{c}^{n-1}_{[n-1,\ldots,k]}$ be given by the rule

$$\rho(\boldsymbol{w}_0[n, n-2, \ldots, k]) = \begin{cases} [\boldsymbol{w}_0, n-3, \ldots, k] \in \mathbf{c}^{n-1}_{[n-1,\ldots,k]} & \text{if } \boldsymbol{w}_0 \text{ ends with } [n-1] \\ [\boldsymbol{w}_0 s_{nk}, n-1] \in \mathbf{c}^n_{[n,\ldots,k,n-1]} & \text{otherwise.} \end{cases}$$

Note that in the case where $\boldsymbol{w}_0$ ends with $[n-1]$, there are no issues with homogeneity. Even if $\boldsymbol{w}_0$ contains the letter $[n-3]$, remember that by passing to D_{n-1}, $[n-1]$ and $[n-3]$ are neighbors. In the case where $\boldsymbol{w}_0$ does not end with $[n-1]$, it is possible that $\boldsymbol{w}_0$ contains this letter, but since it would be part of a descending sequence at the end of the prefix, there will always be two instances of the neighbor $[n-2]$ between the two occurrences of $[n-1]$. Thus the map ρ is well defined. If we restrict ρ to the words whose prefixes end with $[n-1]$, then we obtain a right inverse for φ_2, while if we restrict to the prefixes *not* ending in $[n-1]$, we have a right inverse for φ_1.

Our result is thus proven: the size of the type 1 collection in the packet $\mathcal{P}(n,k)$ is given by the sum of the collection sizes in $\mathcal{P}(n-1,k)$ and $\mathcal{P}(n,k-1)$, and by lemma 3.1.7, all of the collections in this packet have that size. This shows that the relationship in equation 3.3.1 holds, and so the Catalan Triangle indeed counts the size of the collections in the corresponding packet. ■

3.3.5 Corollary. *For a type D_n KLR algebra $R = \bigoplus_{\alpha \in Q_+} R_\alpha$, the sequence in Catalan's triangle counts the number of homogeneous representations given by words with a given suffix. That is, the number of homogeneous words ending with s_{nk} or with $s_{n(k+1)}[n-1,\ldots]$ is given by the entry in the n^{th} row and k^{th} column.*

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