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Meyers Inequality for Stable-like Operators and Pathwise Uniqueness of Stochastic Differential Equations with Jumps

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Meyers Inequality for Stable-like Operators and Pathwise Uniqueness of Stochastic Differential Equations with Jumps

Hua Ren, Ph.D.

University of Connecticut, 2013

ABSTRACT

We consider a class of symmetric stable-like operators of order $\alpha \in (0, 2)$. Let

$$\mathcal{E}(u, u) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(y) - u(x))^2 \frac{A(x, y)}{|x - y|^{d+\alpha}} dy dx$$

be the Dirichlet form for a stable-like operator, let

$$\Gamma u(x) = \left(\int_{\mathbb{R}^d} (u(y) - u(x))^2 \frac{A(x, y)}{|x - y|^{d+\alpha}} dy \right)^{1/2},$$

let \mathcal{L} be the associated infinitesimal generator, and suppose $A(x, y)$ is jointly measurable, symmetric, bounded, and bounded below by a positive constant. We prove that if u is the weak solution to $\mathcal{L}u = h$, then $\Gamma u \in L^p$ for some $p > 2$. As an application, we prove strong stability results for stable-like operators. If A is perturbed slightly, we give explicit bounds on how much the semigroup and fundamental solution are perturbed.

For $\alpha \in (0, 1)$, we consider the one-dimensional jump stochastic differential equation driven by one-sided stable processes of order α :

$$dX_t = \phi(X_{t-}) dZ_t.$$

We prove that pathwise uniqueness holds for this equation under the assumptions that ϕ is continuous, non-decreasing and positive on \mathbb{R} . A counter-example is given to show that the positivity of ϕ is crucial for pathwise uniqueness to hold.

Meyers Inequality for Stable-like Operators and Pathwise Uniqueness of Stochastic Differential Equations with Jumps

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B.S. Mathematics, Nankai University, Tianjin, China, 2009

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Submitted in Partial Fulfillment of the

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at the

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2013

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Hua Ren

2013

APPROVAL PAGE

Doctor of Philosophy Dissertation

**Meyers Inequality for Stable-like
Operators and Pathwise Uniqueness of
Stochastic Differential Equations with
Jumps**

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Chapter 1

Introduction

1.1 Stochastic processes with jumps

As it is well-known, stochastic processes have been widely applied in modeling in many areas; for example, the Ornstein-Uhlenbeck process and geometric Brownian motion are popular models in physics and financial mathematics. However, these types of continuous models suffer from some serious defects. For example, stock prices will at times decrease or increase too fast to be followed by a geometric Brownian motion. A model that better fits the data is a geometric Brownian motion with jumps at random times. This, as well as for other reasons, led to an intense interest in recent years in studying stochastic processes with jumps.

The infinitesimal generators associated with continuous stochastic processes are given by differential operators, which are also called as local operators, while the ones associated with jump stochastic processes are integro-differential operators. Integro-

differential operators are not nearly as well understood as differential operators, and to study them it makes sense to first look at the extreme case, that of purely integral operators.

Recall that divergence form elliptic operators are given by the following:

$$\mathcal{L}_d f(x) = \sum_{i,j=1}^d \frac{\partial}{\partial x_i} \left(a_{ij}(\cdot) \frac{\partial f}{\partial x_j}(\cdot) \right) (x). \quad (1.1.1)$$

These have been studied even when the a_{ij} 's are only bounded and measurable, and to make sense of the operator in this case, one looks at the corresponding Dirichlet form:

$$\mathcal{E}_d(f, f) = \int_{\mathbb{R}^d} \sum_{i,j=1}^d a_{ij}(x) \frac{\partial f}{\partial x_i}(x) \frac{\partial f}{\partial x_j}(x) dx.$$

The most basic jump process is the Poisson process, which is the building block for Lévy processes. A larger class of integral operators, stable operators, have the following as infinitesimal generators:

$$\mathcal{L}f(x) = \int_{\mathbb{R}^d} (f(y) - f(x)) \frac{c}{|x - y|^{d+\alpha}} dy,$$

where $\alpha \in (0, 2)$ and c is a constant. As in the case for divergence form elliptic operators, it is useful to look at the associated Dirichlet form

$$\mathcal{E}(f, f) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (f(y) - f(x))^2 \frac{c}{|x - y|^{d+\alpha}} dy dx.$$

We also consider operators of the following form:

$$\mathcal{L}f(x) = \int_{\mathbb{R}^d} (f(y) - f(x)) \frac{A(x, y)}{|x - y|^{d+\alpha}} dy, \quad (1.1.2)$$

where $\alpha \in (0, 2)$ and $A(x, y)$ satisfies some suitable conditions. The associated Dirichlet form is given by

$$\mathcal{E}(f, f) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (f(y) - f(x))^2 \frac{A(x, y)}{|x - y|^{d+\alpha}} dy dx. \quad (1.1.3)$$

These operators are usually called as stable-like operators, and they bear the same relationship to the fractional Laplacian as divergence form operators do to the Laplacian.

1.2 Stochastic differential equations (SDEs)

The theory of stochastic differential equations (SDEs), which was developed by Itô, provides a very important tool for constructing stochastic processes. A one-dimensional SDE driven by Brownian motion with a drift is of the following form

$$dX_t = \sigma(X_t) dB_t + b(X_t) dt. \quad (1.2.1)$$

For example, the Ornstein-Uhlenbeck processes and geometric Brownian motions mentioned in Section 1.1 can be generated through (1.2.1) by some specific choices of σ and b . Equations of the form (1.2.1) are usually called as continuous stochastic differential equations.

For the reasons stated in Section 1.1, it is useful to study SDEs driven by jump stochastic processes. One can get the jump type analogue of (1.2.1) by replacing the Brownian motion by a Lévy process with jumps, for example, an α -stable process Z_t :

$$dX_t = \sigma(X_{t-}) dZ_t + b(X_t) dt.$$

For more background information on jump SDEs, see the books [2], [45], [46], [47] and a survey paper [10] by Bass .

In order to generate a new stochastic process (either a diffusion or a jump process) through a solution to some stochastic differential equation, it is necessary to verify the existence and uniqueness for solutions of the given SDEs.

There is a long history of studying the existence and uniqueness for solutions of SDEs, not only in an analytic way through the theory of differential equations, but also in a probabilistic way through transformations such as time changes or successive approximations. In this thesis, we study two types of uniqueness for solutions to a given SDE: one is pathwise uniqueness and the other one is weak uniqueness (see the definitions in Section 2.2). For continuous SDEs, if the coefficients are assumed to be Lipschitz continuous, then the existence of strong solutions and pathwise uniqueness can be easily obtained by Picard iteration and Gronwall's inequality. This condition has been greatly improved by Yamada and Watanabe [51], who showed that pathwise uniqueness holds when σ is Hölder continuous of order $\frac{1}{2}$. In contrast to pathwise uniqueness, weak uniqueness, also known as uniqueness in law, requires fewer smooth-

ness conditions on the coefficients. A result by Engelbert and Schmidt [26] says that if $\sigma^2(x) > 0$ on \mathbb{R} , then weak uniqueness holds.

In the recent years, with the intense interest on studying the SDEs driven by jump processes, many more results on the existence and uniqueness for solutions of jump SDEs have been obtained. Some of them are parallel to the results in the continuous case. A few of these results on this topic will be discussed in detail in Section 1.4.

1.3 Meyers inequality

As mentioned in Section 1.1, divergence form operators are the ones of the following form:

$$\mathcal{L}_d f(x) = \sum_{i,j=1}^d \frac{\partial}{\partial x_i} \left(a_{ij}(\cdot) \frac{\partial f}{\partial x_j}(\cdot) \right)(x). \quad (1.3.1)$$

When the a_{ij} are only bounded and measurable, one looks at the corresponding Dirichlet form:

$$\mathcal{E}_d(f, f) = \int_{\mathbb{R}^d} \sum_{i,j=1}^d a_{ij}(x) \frac{\partial f}{\partial x_i}(x) \frac{\partial f}{\partial x_j}(x) dx.$$

One says that u is a weak solution of $\mathcal{L}_d u = h$ if $\mathcal{E}_d(u, v) = -(h, v)$ for all v in a suitably large class, where $(h, v) = \int_{\mathbb{R}^d} h(x)v(x) dx$.

An inequality of Meyers ([41]) says that if the a_{ij} are uniformly elliptic and u is a weak solution to $\mathcal{L}_d u = h$, then not only is ∇u locally in L^2 but it is locally in L^p for some $p > 2$.

The Meyers inequality has many applications. One is to the stability of solutions to $\mathcal{L}_d u = h$. Suppose one perturbs the coefficients a_{ij} slightly. How does this affect the associated semigroup? What about the fundamental solution associated with the operator \mathcal{L}_d ? These are natural questions since the coefficients a_{ij} might themselves be only estimated or approximated. In [24] these issues were resolved, with an explicit bound on how large the difference between the semigroups and solutions associated with two operators \mathcal{L}_d and $\tilde{\mathcal{L}}_d$ can be in terms of the difference of the coefficients a_{ij} and \tilde{a}_{ij} .

Our purpose in Chapter 3 is to examine the analogues of these results for stable-like processes. The operator is the one given by (1.1.2) and the associated Dirichlet form is given by (1.1.3).

The bulk of Chapter 3 is devoted to proving a Meyers inequality for weak solutions to $\mathcal{L}u = h$ when h is in L^2 . Define

$$\Gamma u(x) = \left(\int_{\mathbb{R}^d} \frac{(u(y) - u(x))^2}{|x - y|^{d+\alpha}} dy \right)^{\frac{1}{2}}. \quad (1.3.2)$$

Our main result in Chapter 3 is that there exists $p > 2$ such that the L^p norm of Γu is bounded in terms of the L^2 norms of u and h ; see Theorem 3.2.5.

Once one has the Meyers inequality for \mathcal{E} , strong stability results can be proved along the lines of [24]. Suppose $\tilde{\mathcal{E}}$ is defined in terms of $\tilde{A}(x, y)$ analogously to (1.1.3). We obtain explicit bounds on the L^p norm of $P_t f - \tilde{P}_t f$ and on the L^∞ norm of

$p(t, x, y) - \tilde{p}(t, x, y)$ in terms of

$$G(x) = \sup_{y \in \mathbb{R}^d} |A(x, y) - \tilde{A}(x, y)|,$$

where P_t and $p(t, \cdot, \cdot)$ are the semigroup and fundamental solution associated with \mathcal{L} and \tilde{P}_t and $\tilde{p}(t, \cdot, \cdot)$ are defined similarly. See Theorems 3.3.1, 3.3.3, and 3.3.4.

For other papers on stable-like operators and on closely related operators, see [4] – [16], [19] – [23], [28], [29], [36] and [48].

1.4 Previous results for SDEs with jumps

In Chapter 4, we study pathwise uniqueness of a type of jump SDE, namely, the ones driven by one-sided stable processes of order $\alpha \in (0, 1)$.

To provide the setting for our result, in this section we recall some results on some closely related equations. We consider the following one-dimensional differential equation:

$$dX_t = \sigma(X_{t-}) dZ_t, \tag{1.4.1}$$

where Z_t is a (symmetric) stable process of order $\alpha \in (0, 2)$ taking values on \mathbb{R} .

Bass proved a pathwise uniqueness result for (1.4.1) when Z_t is a symmetric stable process of order $\alpha \in (1, 2)$.

Theorem 1.4.1 (Bass [9]). *Suppose $\alpha \in (1, 2)$, σ is bounded and continuous. Suppose ρ is a nondecreasing continuous function on $[0, \infty)$ with $\rho(0) = 0$ and $|\sigma(x) - \sigma(y)| \leq \rho(|x - y|)$ for all $x, y \in \mathbb{R}$. If $\int_{0+} \rho(x)^{-\alpha} dx = \infty$, then the solution to (1.4.1) is pathwise unique.*

This is the exact analogue of the Yamada-Watanabe condition for SDEs driven by Brownian motions; see [51]. In particular, if σ is Hölder continuous of order $\frac{1}{\alpha}$, then pathwise uniqueness holds for (1.4.1). See also Komatsu [37].

Bass, Burdzy and Chen proved the condition in the above theorem is sharp, based on the idea from Barlow [3].

Theorem 1.4.2 (Bass, Burdzy and Chen [11]). *If $\beta < \frac{1}{\alpha} \wedge 1$, there exists σ that is bounded above and below by strictly positive finite constants and such that σ is Hölder continuous of order β , but where pathwise uniqueness fails for (1.4.1).*

Based on the above result, we conclude that when Z_t is a symmetric α -stable process, and $\alpha \in (1, 2)$, then σ has to be at least Hölder continuous of order $\frac{1}{\alpha}$ for pathwise uniqueness to hold for (1.4.1), while when $\alpha \in (0, 1)$, σ has to be almost Lipschitz continuous.

For the existence of weak solutions and weak uniqueness of equation (1.4.1), we mention the following results by Zanzotto in [52] and [53].

Theorem 1.4.3 (Zanzotto [52]). *Let x be a real number and f denote the transition density function of Z_t and suppose σ satisfies the following condition*

$$\int_0^t \int_{|y|<L} \sigma(x+y)^{-\alpha} f(s,y) dy ds < \infty \text{ for all } t > 0, L > 0.$$

1. *Consider (1.4.1) with respect to a stable process of order $\alpha \in (1,2)$. Under the above assumption there exists a nontrivial weak solution.*
2. *Consider (1.4.1) with respect to a stable process of order $\alpha \in (0,1)$. Assume in addition that there exists a real number $U > 0$ such that $m(B_U) < \infty$ where $B_U = \{y \in \mathbb{R} : |\sigma(x+y)| > U\}$ and m denotes Lebesgue measure. Then there exists a nontrivial weak solution.*

Zanzotto also provided a sufficient and necessary condition for the existence of a weak solution when α is between 1 and 2.

Theorem 1.4.4 (Zanzotto [52]). *For $\alpha \in (1,2)$, there exists a nontrivial solution of (1.4.1) if and only if $|\sigma|^{-\alpha}$ is locally integrable.*

The above results has been improved by Zanzotto in 2002 to the following:

Theorem 1.4.5 (Zanzotto [53]). *Define*

$$I = \{x \in \mathbb{R} : \int_{-\epsilon}^{\epsilon} |\sigma(x+y)|^{-\alpha} dy = \infty \text{ for any } \epsilon > 0\}$$

and

$$N = \{x \in \mathbb{R} : \sigma(x) = 0\}.$$

Then, for $\alpha \in (1,2)$, equation (1.4.1) has a weak solution if and only if $I \subseteq N$. Furthermore, weak uniqueness holds if $N = I$.

Usually, by adding some monotonicity conditions on σ , one may lower the requirements on the smoothness of σ to obtain pathwise uniqueness. The following result comes from the paper by Li and Mytnik ([38]).

Theorem 1.4.6 (Li and Mytnik [38]). *Let Z_t denote a spectrally positive stable process of order $\alpha \in (1, 2)$. Suppose σ is non-decreasing, bounded, and with modulus of continuity $\rho(z)$ satisfying $\int_{0+} \rho(z)^{-\frac{\alpha}{\alpha-1}} dz = \infty$. Then pathwise uniqueness holds for (1.4.1).*

In particular, the above results says if σ is Hölder continuous of order $1 - \frac{1}{\alpha}$, then pathwise uniqueness holds for (1.4.1). See [32], [38] and [39] for more results on the general case. The result from the above theorem has been improved in [17] and [39].

Fournier proved pathwise uniqueness when Z_t is a symmetric stable process of order $\alpha \in (0, 1)$ under non-Lipschitz conditions. However, instead of equation (1.4.1), he studied another equation

$$dX_t = \int_{\mathbb{R}^d \setminus \{0\}} \int_{\mathbb{R}^d \setminus \{0\}} z [1_{\{0 < u < \gamma(X_{s-})\}} - 1_{\{\gamma(X_{s-}) < u < 0\}}] M(ds dz du), \quad (1.4.2)$$

where $\gamma(x) = \text{sign}(\sigma(x)) \cdot |\sigma(x)|^\alpha$. Refer to [30] for more details.

Theorem 1.4.7 (Fournier [30]). *Suppose Z_t is a symmetric stable process of order $\alpha \in (0, 1)$. Suppose σ is bounded below from zero and Hölder continuous of order α . Then pathwise uniqueness holds for (1.4.2).*

We mention another result from [30] here:

Theorem 1.4.8 (Fournier [30]). *Suppose $\alpha \in (0, 1)$ and Z_t is a one-sided α -stable process. Suppose σ is bounded below from zero. Suppose σ is Hölder continuous of order α if $\alpha \in (0, \frac{1}{2}]$ and non-increasing and Hölder continuous of order $1 - \alpha$ if $\alpha \in (\frac{1}{2}, 1)$. Then pathwise uniqueness holds for (1.4.2).*

Our main result in Chapter 4 improves the above result. Instead of considering equation (1.4.2), we proved pathwise uniqueness for equation (1.4.1) where σ is continuous. See Theorem 4.1.2.

See [9], [11], [17], [26], [30], [32]–[34], [36]–[39], [43], [44], [50], [52] and [53] for more discussions on the existence and uniqueness for the solutions of SDEs with jumps.

Chapter 2

Preliminaries

2.1 Dirichlet forms and Spectral theory for Stable-like operators

Suppose \mathcal{S} is a locally compact separable metric space together with a σ -finite measure m defined on the Borel subsets of \mathcal{S} . In this section and also in Chapter 3, we will take $\mathcal{S} = \mathbb{R}^d$ and let m be Lebesgue measure on \mathbb{R}^d . Suppose there exists a dense subset $\mathcal{D} = \mathcal{D}(\mathcal{E})$ of $L^2(\mathcal{S}, m)$ and a non-negative bilinear symmetric form \mathcal{E} defined on $\mathcal{D} \times \mathcal{D}$, which means

1. $\mathcal{E}(u, v) = \mathcal{E}(v, u)$,
2. $\mathcal{E}(u + v, h) = \mathcal{E}(u, h) + \mathcal{E}(v, h)$,
3. $\mathcal{E}(au, v) = a\mathcal{E}(u, v)$,
4. $\mathcal{E}(u, u) \geq 0$,

for $u, v, h \in \mathcal{D}$, $a \in \mathbb{R}$.

We write $(u, v) = \int u(x)v(x) m(dx)$. Define

$$\mathcal{E}_1(u, v) = \mathcal{E}(u, v) + (u, v), \quad \text{for all } u, v \in \mathcal{D}(\mathcal{E}).$$

We say \mathcal{E} is closed if \mathcal{D} is complete with respect to the norm induced by \mathcal{E}_1 .

We say \mathcal{E} is Markovian if whenever $u \in \mathcal{D}$, then $v = 0 \vee (u \wedge 1) \in \mathcal{D}$ and $\mathcal{E}(v, v) \leq \mathcal{E}(u, u)$.

A Dirichlet form \mathcal{E} is a non-negative bilinear symmetric form that is closed and Markovian. We call $\mathcal{D}(\mathcal{E})$ as the domain of \mathcal{E} .

Let $\mathcal{C}_0(\mathcal{S})$ be the continuous functions on \mathcal{S} with compact support. \mathcal{C} is a core of \mathcal{E} if $\mathcal{C} \subset \mathcal{D}(\mathcal{E}) \cap \mathcal{C}_0(\mathcal{S})$, \mathcal{C} is dense in $\mathcal{D}(\mathcal{E})$ with respect to the \mathcal{E}_1 norm, and dense in \mathcal{C}_0 with respect to the sup norm. We say \mathcal{E} is regular if \mathcal{E} possesses a core.

According to [31], for every Dirichlet form, there is an associated semigroup and infinitesimal generator. Furthermore, if the Dirichlet form is regular, then there exists an associated Hunt process which is a strong Markov process. See [31] for more details.

We use the letter c with or without subscripts to denote a finite positive constant whose exact value is unimportant and which can vary from place to place. We use $B(x, r)$ for the open ball in \mathbb{R}^d with center x and radius r . When the center is clear

from the context, we will also write B_r . The Lebesgue measure of $B(x, r)$ will be denoted $|B(x, r)|$.

In this section and Chapter 3, let $\alpha \in (0, 2)$ and suppose the dimension d is greater than α . We let $A(x, y)$ be a jointly measurable symmetric function on $\mathbb{R}^d \times \mathbb{R}^d$ and suppose there exists $\Lambda > 0$ such that

$$\Lambda^{-1} \leq A(x, y) \leq \Lambda, \quad x, y \in \mathbb{R}^d.$$

We define the Dirichlet form \mathcal{E} with domain $\mathcal{D}(\mathcal{E}) = \mathcal{F}$ by

$$\mathcal{E}(u, v) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(y) - u(x))(v(y) - v(x)) \frac{A(x, y)}{|x - y|^{d+\alpha}} dy dx, \quad (2.1.1)$$

$$\mathcal{F} = \{u \in L^2(\mathbb{R}^d) : \mathcal{E}(u, u) < \infty\}.$$

Observe that $\mathcal{F} = W^{\alpha/2, 2}(\mathbb{R}^d)$, the fractional Sobolev space of order $\alpha/2$ defined by

$$W^{\alpha/2, 2}(\mathbb{R}^d) = \left\{ u \in L^2(\mathbb{R}^d) : \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(u(x) - u(y))^2}{|x - y|^{d+\alpha}} dy dx < \infty \right\}.$$

For $u \in W^{\alpha/2, 2}(\mathbb{R}^d)$, the norm of u is defined by the following:

$$\|u\|_{W^{\alpha/2, 2}(\mathbb{R}^d)} = \|u\|_{L^2(\mathbb{R}^d)} + \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(u(x) - u(y))^2}{|x - y|^{d+\alpha}} dy dx \right)^{\frac{1}{2}}.$$

See [1] for more details. It is well-known that $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^2(\mathbb{R}^d)$. The strong Markov symmetric process X associated with $(\mathcal{E}, \mathcal{F})$ is called as

a stable-like process. Let $\{P_t\}_{t \geq 0}$ be the semigroup corresponding to $(\mathcal{E}, \mathcal{F})$.

For $u \in \mathcal{F}$ define

$$\Gamma u(x) = \left(\int_{\mathbb{R}^d} \frac{(u(y) - u(x))^2}{|x - y|^{d+\alpha}} dy \right)^{\frac{1}{2}}. \quad (2.1.2)$$

Since $\int |\Gamma u(x)|^2 dx = \mathcal{E}(u, u) < \infty$, then $\Gamma u \in L^2$, and in particular $\Gamma u(x)$ exists for almost every x .

Let \mathcal{L} be the infinitesimal generator corresponding to \mathcal{E} . There are a number of known results that follow from the spectral theorem. We collect these in the following lemma. Let $\{E_\lambda\}$, $\lambda \geq 0$, be the spectral representation of $-\mathcal{L}$. For $f \in \mathcal{F}$, we have

$$\mathcal{E}(f, f) = \int_0^\infty \lambda d(E_\lambda f, E_\lambda f);$$

see [31].

Lemma 2.1.1. (1) For $t > 0$, $f \in L^2(D)$, we have

$$\mathcal{E}(P_t f, P_t f) \leq ct^{-1} \|f\|_2^2.$$

(2) If $g \in L^2$, then $P_t g$ is in $\mathcal{D}(\mathcal{L})$, the domain of \mathcal{L} .

(3) If $f, g \in \mathcal{F}$, then

$$\frac{d}{dt}(P_t f, g) = -\mathcal{E}(P_t f, g).$$

(4) If $f \in \mathcal{F}$, then

$$\mathcal{E}(P_t f, P_t f) \leq \mathcal{E}(f, f). \quad (2.1.3)$$

Proof. (1) This follows from

$$\begin{aligned}\mathcal{E}(P_t f, P_t f) &= \int_0^\infty \lambda e^{-2\lambda t} d(E_\lambda f, E_\lambda f) \\ &\leq ct^{-1} \int_0^\infty d(E_\lambda f, E_\lambda f) = ct^{-1} \|f\|_2^2,\end{aligned}$$

since $\lambda e^{-2\lambda t} \leq ct^{-1}$ for all $\lambda \geq 0$.

(2) By the spectral representation of $-\mathcal{L}$, we have

$$\frac{P_h(P_t g) - P_t g}{h} = \frac{P_{t+h} g - P_t g}{h} = \int_0^\infty \frac{e^{-\lambda(t+h)} - e^{-\lambda t}}{h} dE_\lambda g.$$

Let $H = -\int_0^\infty \lambda e^{-\lambda t} dE_\lambda g$. Note $\|H\|_{L^2}$ is finite because $\lambda^2 e^{-2\lambda t}$ is bounded. Then

$$\left\| \frac{P_h(P_t g) - P_t g}{h} - H \right\|_{L^2}^2 = \int_0^\infty \left[\frac{e^{-\lambda(t+h)} - e^{-\lambda t}}{h} + \lambda e^{-\lambda t} \right]^2 d(E_\lambda g, E_\lambda g),$$

which tends to 0 as $h \rightarrow 0$ by dominated convergence. Therefore $P_t g \in \mathcal{D}(\mathcal{L})$ and $\mathcal{L}(P_t g) = H$.

(3) For any $g \in \mathcal{F}$, we have

$$(P_t f, g) = \int_0^\infty e^{-\lambda t} d(E_\lambda f, g),$$

and so

$$\frac{d}{dt}(P_t f, g) = -\int_0^\infty \lambda e^{-\lambda t} d(E_\lambda f, g).$$

On the other hand,

$$\mathcal{E}(P_t f, g) = \int_0^\infty \lambda d(E_\lambda P_t f, g) = \int_0^\infty \lambda e^{-\lambda t} d(E_\lambda f, g),$$

which proves the assertion.

(4) We prove this by writing

$$\int_0^\infty \lambda e^{-2\lambda t} d(E_\lambda f, E_\lambda f) \leq \int_0^\infty \lambda d(E_\lambda f, E_\lambda f),$$

which translates to (2.1.3). □

2.2 Existence and Uniqueness of SDEs with jumps

Suppose we are given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $\{\mathcal{F}_t\}$ which satisfies the “usual conditions”, i.e., \mathcal{F}_t is right continuous and \mathcal{F}_0 contains all the null sets. We say a process X is \mathcal{F}_t -adapted if X_t is \mathcal{F}_t -measurable. A process X_t is right continuous with left limits if there exists a null set \mathcal{N} such that if $w \notin \mathcal{N}$, then $\lim_{u \downarrow t} X_u(w) = X_t(w)$ and $\lim_{s \uparrow t} X_s(w)$ exists for all $t \geq 0$. Such a process X_t is called a càdlàg process. For a càdlàg process X_t , let $X_{t-} = \lim_{s \uparrow t} X_s$ and $\Delta X_t = X_t - X_{t-}$.

Let \mathcal{B} be the Borel σ -field on \mathbb{R} and λ an infinite measure on $(\mathcal{B}, \mathbb{R})$. A Poisson point process μ is a measurable mapping from $\Omega \times [0, \infty) \times \mathcal{B} \rightarrow \mathbb{R}$ which satisfies the following two conditions:

1. for each $A \in \mathcal{B}$, with $\lambda(A) < \infty$, $\mu([0, t] \times A)$ is a Poisson process with parameter $\lambda(A)$;
2. if A_1, \dots, A_n are disjoint with $\lambda(A_i) < \infty$ for each i , then $\mu([0, t] \times A_i)$ are independent processes.

Let $\nu([0, t] \times A) = t\lambda(A)$. Then for each A with $\lambda(A) < \infty$, we have that the process $(\mu - \nu)([0, t] \times A)$ is a martingale. We say μ is a Poisson point process with compensator ν . Then an SDE of pure jump type is one driven by a compensated Poisson point process:

$$dX_t = F(X_{t-}, z) (\mu(dz, dt) - \nu(dz)dt). \quad (2.2.1)$$

One may think of this equation as follows: if μ assigns mass one to z at time t , then X_t jumps at this time and the size of the jump is $F(X_{t-}, z)$. Refer to [47] for more details.

For a_-, a_+ in $[0, \infty)$ and $\alpha \in (0, 2)$, let

$$\nu_{a_-, a_+}^\alpha(dz) = |z|^{-\alpha-1} \left[a_- 1_{\{z < 0\}} + a_+ 1_{\{z > 0\}} \right] dz, \quad (2.2.2)$$

and $\mu(dz, ds)$ be the Poisson point process with intensity $\nu_{a_-, a_+}^\alpha(dz)ds$.

Let

$$Z_t = \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} z \mu(dz, ds) \quad \text{for } \alpha \in (0, 1);$$

$$Z_t = \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} z \left[\mu(dz, ds) - \nu_{a_-, a_+}^\alpha(dz)ds \right] \quad \text{for } \alpha \in [1, 2).$$

Then Z_t is a stable process of order α . Z_t is said to be symmetric if $a_- = a_+$. See [45] for more discussions on stable processes.

In this thesis, instead of studying the general SDEs given by (2.2.1), we focus only

on the ones driven by one-dimensional stable processes. For $\alpha \in (0, 2)$, we consider the following equation:

$$dX_t = F(X_{t-}) dZ_t. \quad (2.2.3)$$

where Z_t is a α -stable process on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ and taking values on \mathbb{R} .

Definition 2.2.1 (Strong solution). *A càdlàg process X_t is said to be a strong solution of (2.2.3) if X_t satisfies equation (2.2.3) and is \mathcal{F}_t -adapted.*

Definition 2.2.2 (Weak solution). *We say a weak solution exists for (2.2.3) if there exists a probability space $(\Omega', \mathcal{F}', \{\mathcal{F}'_t\}_{t \geq 0}, \mathbb{P}')$ carrying a pair of $\{\mathcal{F}'_t\}$ -adapted processes X'_t, Z'_t such that Z'_t is an α -stable process and (X', Z') satisfies (2.2.3).*

Definition 2.2.3 (Pathwise uniqueness). *If*

$$\mathbb{P}(X_t^1 = X_t^2, \text{ for all } t \geq 0) = 1,$$

where X_t^1 and X_t^2 are strong solutions of (2.2.3), then we say that pathwise uniqueness holds for (2.2.3).

Definition 2.2.4 (Weak uniqueness). *We say weak uniqueness holds for (2.2.3) if whenever we have two weak solutions (X_t, Z_t) and (X'_t, Z'_t) such that $X_0 = X'_0$, then the law of X_t is the same as the law of X'_t .*

Based on the above definitions, it is clear that the existence of a strong solution always implies the existence of a weak solution. It can be shown that whenever pathwise uniqueness holds for (2.2.3), so does weak uniqueness, see, e.g., Corollary 140 in [46]. Roughly speaking, the existence of a strong solution and pathwise uniqueness is

known to hold for (2.2.3) only when F is reasonably smooth. However weak existence and weak uniqueness can be obtained under relatively loose regularity conditions on F .

Chapter 3

Meyers Inequality and Strong Stability for Stable-like Operators

In this chapter, we consider a class of symmetric stable-like operators of order $\alpha \in (0, 2)$. Let

$$\mathcal{E}(u, u) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(y) - u(x))^2 \frac{A(x, y)}{|x - y|^{d+\alpha}} dy dx$$

be the associated Dirichlet form, let

$$\Gamma u(x) = \left(\int_{\mathbb{R}^d} (u(y) - u(x))^2 \frac{A(x, y)}{|x - y|^{d+\alpha}} dy \right)^{1/2},$$

let \mathcal{L} be the associated infinitesimal generator, and suppose $A(x, y)$ is jointly measurable, symmetric, bounded, and bounded below by a positive constant. We prove the Meyers inequality for this type of operators, that is, if u is the weak solution to $\mathcal{L}u = h$, then $\Gamma u \in L^p$ for some $p > 2$. As an application, we prove strong stability results for these operators. If A is perturbed slightly, we give explicit bounds on how much the semigroup and fundamental solution are perturbed.

Our proof of the Meyers inequality begins by first proving a Caccioppoli inequality. However there are considerable differences between the stable-like case and the divergence form case. For example, as one would expect, our Caccioppoli inequality is not a local one; the integral of $|\Gamma u|^2$ on a ball depends on values of u outside the ball. This makes proving the Meyers inequality considerably more difficult and requires the introduction of some new ideas, such as localization, use of the Hardy-Littlewood maximal function, and use of the Sobolev-Besov embedding theorem.

3.1 Caccioppoli inequality

In this section, we will derive a Caccioppoli inequality for the weak solution of the equation

$$\mathcal{L}u(x) = h(x), \quad x \in \mathbb{R}^d, \quad (3.1.1)$$

where $h \in L^2(\mathbb{R}^d)$. A function $u \in W^{\frac{\alpha}{2}, 2}(\mathbb{R}^d)$ is called a weak solution of (3.1.1) if

$$\mathcal{E}(u, v) = -(h, v) \quad \text{for all } v \in W^{\frac{\alpha}{2}, 2}(\mathbb{R}^d), \quad (3.1.2)$$

where $(h, v) = \int_{\mathbb{R}^d} h(x)v(x) dx$.

Let's first recall the Caccioppoli inequality for divergence form operators.

Let $u \in W^{1,2}(\mathbb{R}^d)$ be the weak solution of

$$\mathcal{L}_d u(x) = h(x), \quad x \in \mathbb{R}^d. \quad (3.1.3)$$

where \mathcal{L}_d is the divergence operator given by (1.3.1) and $h \in L^2(\mathbb{R}^d)$.

Theorem 3.1.1 (Caccioppoli inequality for divergence form operators). *For all $x_0 \in \mathbb{R}^d$, and all r, R with $0 < r < R < \infty$, we have*

$$\int_{B_r(x_0)} |\nabla u|^2 dx \leq \frac{c}{(R-r)^2} \left[\int_{B_R(x_0)} |u - u_R|^2 dx + \int_{B_R(x_0)} h^2(x) dx \right],$$

where $u_R = \frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} u(x) dx$.

Roughly speaking, the Caccioppoli inequality is the reverse of the Sobolev inequality.

For our stable-like operators, we prove the Caccioppoli inequality in the following theorem:

Theorem 3.1.2. *Let $x_0 \in \mathbb{R}^d$. Suppose $u(x)$ satisfies (3.1.2). There exists a constant c_1 depending only on Λ, α , and d such that*

$$\begin{aligned} \int_{B_{R/2}} \int_{\mathbb{R}^d} (u(y) - u(x))^2 \frac{A(x, y)}{|x - y|^{d+\alpha}} dy dx \\ \leq c_1 \int_{\mathbb{R}^d} u^2(y) \psi(y) dy + \int_{B_R} |h(y)u(y)| dy, \end{aligned} \quad (3.1.4)$$

where

$$\psi(x) = R^{-\alpha} \wedge \frac{R^d}{|x - x_0|^{d+\alpha}}.$$

Proof. We define a cutoff function $\varphi(x) : \mathbb{R}^d \rightarrow [0, 1]$ such that $\varphi = 1$ on $B_{R/2}$, $\varphi = 0$ on B_R^c , and

$$|\varphi(x) - \varphi(y)| \leq c \frac{|x - y|}{R}.$$

For example, we can take

$$\varphi(x) = 1 - \left(\frac{\text{dist}(x, B(x_0, R/2))}{R/2} \wedge 1 \right).$$

In what follows the constants may depend on R .

Let $v(x) = \varphi^2(x)u(x)$. Since $|v| \leq |u|$ and $u \in L^2$, then $v \in L^2$. Since

$$v(y) - v(x) = (u(y) - u(x))\varphi^2(y) + u(x)(\varphi^2(y) - \varphi^2(x)),$$

then

$$\begin{aligned} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(v(y) - v(x))^2}{|x - y|^{d+\alpha}} dy dx &\leq 2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(u(y) - u(x))^2 \varphi^4(y)}{|x - y|^{d+\alpha}} dy dx \\ &\quad + 2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{u^2(x)(\varphi^2(y) - \varphi^2(x))^2}{|x - y|^{d+\alpha}} dy dx. \end{aligned}$$

The first term on the right hand side is finite because $\varphi \leq 1$ and $u \in \mathcal{F}$. The second term is bounded by

$$c \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{u^2(x)(1 \wedge |y - x|^2/R^2)}{|x - y|^{d+\alpha}} dy dx \leq c \int_{\mathbb{R}^d} u^2(x) dx,$$

which is finite since $u \in L^2$. Therefore $v \in \mathcal{F}$.

We write

$$\begin{aligned}
-(h, v) &= \mathcal{E}(u, v) \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(y) - u(x))(\varphi^2(y)u(y) - \varphi^2(x)u(x)) \frac{A(x, y)}{|x - y|^{d+\alpha}} dy dx \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(y) - u(x))^2 \varphi^2(x) \frac{A(x, y)}{|x - y|^{d+\alpha}} dy dx \\
&\quad + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} [(u(y) - u(x))(\varphi(y) - \varphi(x))(\varphi(y) + \varphi(x))u(y)] \\
&\quad \quad \times \frac{A(x, y)}{|x - y|^{d+\alpha}} dy dx \\
&= I_1 - I_2.
\end{aligned}$$

Then

$$\begin{aligned}
I_1 &= I_2 - \int_{\mathbb{R}^d} h(y) \varphi^2(y) u(y) dy \\
&\leq I_2 + \int_{B_R} |h(y) u(y)| dy.
\end{aligned} \tag{3.1.5}$$

Using the inequality $ab \leq \frac{1}{8}a^2 + 2b^2$, symmetry, and the fact that $0 \leq \varphi(x) \leq 1$,

we have

$$\begin{aligned}
I_2 &\leq \frac{1}{8} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(y) - u(x))^2 (\varphi(y) + \varphi(x))^2 \frac{A(x, y)}{|x - y|^{d+\alpha}} dy dx \\
&\quad + 2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (\varphi(y) - \varphi(x))^2 u^2(y) \frac{A(x, y)}{|x - y|^{d+\alpha}} dy dx \\
&\leq \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(y) - u(x))^2 \varphi^2(x) \frac{A(x, y)}{|x - y|^{d+\alpha}} dy dx \\
&\quad + 2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (\varphi(y) - \varphi(x))^2 u^2(y) \frac{A(x, y)}{|x - y|^{d+\alpha}} dy dx \\
&= \frac{1}{2} I_1 + 2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (\varphi(y) - \varphi(x))^2 u^2(y) \frac{A(x, y)}{|x - y|^{d+\alpha}} dy dx.
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{1}{2} I_1 &\leq 2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (\varphi(y) - \varphi(x))^2 u^2(y) \frac{A(x, y)}{|x - y|^{d+\alpha}} dy dx \\
&\quad + \int_{B_R} |h(y)u(y)| dy. \tag{3.1.6}
\end{aligned}$$

Next, using $|\varphi(y) - \varphi(x)| \leq c(1 \wedge |x - y|/R)$, some calculus shows that

$$\int_{\mathbb{R}^d} (\varphi(y) - \varphi(x))^2 \frac{A(x, y)}{|x - y|^{d+\alpha}} dx \leq cR^{-\alpha}, \quad y \in \mathbb{R}^d. \tag{3.1.7}$$

If $y \notin B_{2R}$, then

$$\int_{\mathbb{R}^d} (\varphi(y) - \varphi(x))^2 \frac{A(x, y)}{|x - y|^{d+\alpha}} dx \leq c \int_{B_R} \frac{dx}{|y - x_0|^{d+\alpha}} = c \frac{R^d}{|y - x_0|^{d+\alpha}}.$$

Hence the first term on the right hand side of (3.1.6) is bounded by

$$c \int u(y)^2 \psi(y) dy. \quad (3.1.8)$$

Combining (3.1.6) and (3.1.8) with the fact that

$$I_1 \geq \int_{B_{R/2}} \int_{\mathbb{R}^d} (u(y) - u(x))^2 \frac{A(x, y)}{|x - y|^{d+\alpha}} dy dx$$

completes the proof. \square

For another approach to the Caccioppoli inequality for non-local operators, see [35].

3.2 Meyers inequality

Recall that Meyers inequality for divergence form operators is given by the following theorem (see [41]):

Theorem 3.2.1 (Meyers inequality for divergence form operators). *Let Ω be a bounded C^1 -smooth domain in \mathbb{R}^d . We can find a constant $\beta = \beta(\Omega, \lambda) > 2$ such that if $1 < p < \min\{\frac{\beta}{2}, \frac{d}{d-2}\}$, there exists a constant $c = c(\Omega, \lambda, \beta) > 0$ such that*

$$\|\nabla u\|_{L^{2p}} \leq c \|h(x)\|_2.$$

Let $h \in L^2$. We consider the weak solution $u(x)$ of (3.1.2) and we will show that Γu is in L^p for some $p > 2$. We suppose throughout this section that $d > \alpha$. This

will always be the case if $d \geq 2$.

Let

$$u_R = \frac{1}{|B_R|} \int_{B_R} u(y) dy.$$

Using Theorem 3.1.2 with u replaced by $u - u_R$, we have

$$\begin{aligned} \|\Gamma u\|_{L^2(B_{R/2})}^2 &\leq c \int_{\mathbb{R}^d} (u(x) - u_R)^2 \psi(x) dx \\ &\quad + \int_{B_R} |h(x)(u(x) - u_R)| dx. \end{aligned} \quad (3.2.1)$$

Lemma 3.2.2. *Suppose $u \in W^{\frac{\alpha}{2}, q}(B_R)$, $1 < q \leq 2$. Suppose $x_0 \in \mathbb{R}^d$ and $R > 0$. Let $p = 2dq/(2d - q\alpha)$. Then $u \in L^p(B_R)$ and there exists a constant c_1 depending only on d, α , and q such that*

$$\|u - u_R\|_{L^p(B_R)} \leq c_1 \left[\int_{B_R} \int_{B_R} \frac{(u(y) - u(x))^q}{|x - y|^{d + \frac{\alpha}{2}q}} dy dx \right]^{\frac{1}{q}}. \quad (3.2.2)$$

Proof. We first do the case $R = 1$. By the Sobolev-Besov embedding theorem (see Theorem 7.57 in [1] or Section 2.3.3 in [25]), we know

$$\begin{aligned} \|u - u_R\|_{L^p(B_1)} &\leq c \|u - u_R\|_{W^{\frac{\alpha}{2}, q}(B_1)} \\ &= c \left\{ \|u - u_R\|_{L^q(B_1)} + \left[\int_{B_1} \int_{B_1} \frac{(u(y) - u(x))^q}{|x - y|^{d + \frac{\alpha}{2}q}} dy dx \right]^{\frac{1}{q}} \right\} \end{aligned} \quad (3.2.3)$$

On the other hand, the fractional Poincaré inequality for $u \in W^{\frac{\alpha}{2}, q}(B_1)$ (see equation (4.2) in [42]) tells us

$$\|u - u_R\|_{L^q(B_1)} \leq c \left[\int_{B_1} \int_{B_1} \frac{(u(y) - u(x))^q}{|x - y|^{d + \frac{\alpha}{2}q}} dy dx \right]^{\frac{1}{q}}. \quad (3.2.4)$$

Combining (3.2.3) and (3.2.4) proves the lemma in the case $R = 1$.

The case for general R follows by a scaling argument, that is, by a change of variables. The $dy dx$ expression in the right hand side of (3.2.2) contributes a factor R^{2d} and the denominator contributes a factor $R^{-(d+\alpha q/2)}$, so the right hand side of (3.2.2) is equal to

$$c(R^{d-\alpha q/2})^{1/q} \left[\int_{B_1} \int_{B_1} \frac{(v(y) - v(x))^q}{|x - y|^{d+\frac{\alpha}{2}q}} dy dx \right]^{\frac{1}{q}},$$

where $v(z) = u(Rz)$. Similarly the left hand side of (3.2.2) is equal to

$$R^{d/p} \|v - v_1\|_{L^p(B_1)}.$$

Inequality (3.2.2) then follows by the preceding paragraph and our choice of p . \square

Proposition 3.2.3. *There exists $q_1 \in (1, 2)$ and a constant c_1 depending on d, α , and q_1 such that if $x_0 \in \mathbb{R}^d$ and $R > 0$, then*

$$\|u - u_R\|_{L^2(B_R)} \leq c_1 R^{(\alpha-\alpha_1)/2} \|\Gamma u\|_{L^{q_1}(B_R)}, \quad (3.2.5)$$

where $\alpha_1 = (2 - q_1)d/q_1$.

Proof. Again we may suppose $R = 1$ and obtain the general case by a scaling argument as in the last paragraph of the proof of Lemma 3.2.2. Take $\alpha_1 < \alpha$ and let $q_1 = 2d/(d + \alpha_1)$. Note that $q_1 \in (1, 2)$. By Lemma 3.2.2

$$\|u - u_R\|_{L^2(B_R)} \leq c \left[\int_{B_R} \int_{B_R} \frac{(u(y) - u(x))^{q_1}}{|x - y|^{d+\alpha_1 q_1/2}} dy dx \right]^{\frac{1}{q_1}}. \quad (3.2.6)$$

Fix x for the moment. Using Hölder's inequality with respect to the measure $|x - y|^{-d} dy$,

$$\begin{aligned}
& \int_{B_R} \frac{(u(y) - u(x))^{q_1}}{|x - y|^{d + \alpha_1 q_1/2}} dy \\
&= \int_{B_R} \frac{(u(y) - u(x))^{q_1}}{|x - y|^{\alpha q_1/2}} \frac{1}{|x - y|^{(\alpha_1 - \alpha) q_1/2}} \frac{1}{|x - y|^d} dy \\
&\leq \left[\int_{B_R} \left(\frac{(u(y) - u(x))^{q_1}}{|x - y|^{\alpha q_1/2}} \right)^{\frac{2}{q_1}} \frac{1}{|x - y|^d} dy \right]^{\frac{q_1}{2}} \\
&\quad \times \left[\int_{B_R} \left(\frac{1}{|x - y|^{(\alpha_1 - \alpha) q_1/2}} \right)^{\frac{2}{2 - q_1}} \frac{1}{|x - y|^d} dy \right]^{\frac{2 - q_1}{2}} \\
&= \left[\int_{B_R} \frac{(u(y) - u(x))^2}{|x - y|^{d + \alpha}} dy \right]^{\frac{q_1}{2}} \left[\int_{B_R} \frac{1}{|x - y|^{(\alpha_1 - \alpha) \frac{q_1}{2 - q_1} + d}} dy \right]^{\frac{2 - q_1}{2}} \\
&\leq c \left[\int_{B_R} \frac{(u(y) - u(x))^2}{|x - y|^{d + \alpha}} dy \right]^{\frac{q_1}{2}} \\
&\leq c |\Gamma u(x)|^{q_1}.
\end{aligned}$$

Integrating over $x \in B_R$, taking the q_1^{th} root, and combining with (3.2.6) yields (3.2.5).

□

Proposition 3.2.4. *There exists $p \in (2, 4d/(2d - \alpha))$ and a constant c_1 depending on Λ , d , α , and p such that if u satisfies (3.1.2), then*

$$\|\Gamma u\|_{L^p(\mathbb{R}^d)} \leq c_1 \left(\mathcal{E}(u, u)^{\frac{1}{2}} + \|h\|_{L^2(\mathbb{R}^d)} + \|u\|_{L^p(\mathbb{R}^d)} + \|u\|_{L^{2p/(4-p)}(\mathbb{R}^d)} \right).$$

Proof. Set $x_0 = 0$ and $R = 1$ for now. From (3.2.1) we know that

$$\begin{aligned}
\|\Gamma u\|_{L^2(B_{R/2})}^2 &\leq c \int_{\mathbb{R}^d} (u(x) - u_R)^2 \psi(x) dx + \int_{B_R} |h(x)(u(x) - u_R)| dx \\
&\leq c \int_{B_R} (u(x) - u_R)^2 dx + c \int_{B_R^c} u(x)^2 \psi(x) dx \\
&\quad + c \int_{B_R^c} u_R^2 \psi(x) dx + \int_{B_R} |h(x)(u(x) - u_R)| dx \\
&= J_1 + J_2 + J_3 + J_4.
\end{aligned} \tag{3.2.7}$$

We proceed to bound $J_1, J_2, J_3,$ and J_4 .

Using Proposition 3.2.3, we have

$$J_1 = \int_{B_R} (u(x) - u_R)^2 dx \leq c \left(\int_{B_R} \Gamma u(x)^{q_1} dx \right)^{\frac{2}{q_1}} \tag{3.2.8}$$

for $q_1 \in (1, 2)$.

Note that $\psi(x) = 1 \wedge \frac{1}{|x-x_0|^{d+\alpha}}$ when $R = 1$. For any $y \in B_R$ and $x \in B_R^c$, we have $|x - y| < 2|x - x_0|$. Letting $\rho(x) = 1 \wedge \frac{1}{|x|^{d+\alpha}}$, we observe that

$$\begin{aligned}
J_2 &= \int_{B_R^c} u(x)^2 \psi(x) dx \leq c \int_{B_R^c} u(x)^2 \left(1 \wedge \frac{1}{|x - y|^{d+\alpha}} \right) dx \\
&\leq c((u^2) * \rho)(y).
\end{aligned}$$

Using Theorem 2 in Section 2.2 of Chapter 3 in [49], it follows that

$$\begin{aligned}
J_2 &\leq c((u^2) * \rho)(y) \leq c \left(\int_{\mathbb{R}^d} \rho(x) dx \right) M(u^2)(y) \\
&\leq c M(u^2)(y),
\end{aligned}$$

where M is the Hardy-Littlewood maximal operator:

$$Mf(x) = \sup_{r>0} \frac{1}{|B_r|} \int_{B(x,r)} |f(y)| dy.$$

For any $y \in B_R$, by Jensen's inequality

$$\begin{aligned} u_R^2 &= \left(\frac{1}{|B_R|} \int_{B_R} u(x) dx \right)^2 \leq \frac{1}{|B_R|} \int_{B_R} u(x)^2 dx \\ &\leq \frac{|B_{2R}|}{|B_R|} \cdot \frac{1}{|B_{2R}|} \int_{B(y,2R)} u(x)^2 dx \\ &\leq 2^d M(u^2)(y). \end{aligned}$$

Hence

$$J_3 = \int_{B_R^c} u_R^2 \psi(x) dx \leq cM(u^2)(y) \int_{B_R^c} \psi(x) dx \leq cM(u^2)(y).$$

Similarly, $|u_R| \leq cMu(x)$ for all $x \in B_R$. Since $|B(x,s)|^{-1} \int_{B(x,s)} u(y) dy$ converges to $u(x)$ as $s \rightarrow 0$ for almost every x and is bounded by $Mu(x)$, we have $|u(x)| \leq Mu(x)$ a.e. Thus

$$\begin{aligned} J_4 &= \int_{B_R} |h(x)(u(x) - u_R)| dx \leq \int_{B_R} |h(x)u(x)| dx + \int_{B_R} |h(x)Mu(x)| dx \\ &\leq c \int_{B_R} |h(x)|Mu(x) dx. \end{aligned}$$

Combining our bounds for J_1, J_2, J_3 , and J_4 ,

$$\begin{aligned} \|\Gamma u\|_{L^2(B_{R/2})}^2 &\leq c\|\Gamma u\|_{L^{q_1}(B_R)}^2 + cM(u^2)(y) \\ &\quad + c \int_{B_R} |h(x)|Mu(x) dx. \end{aligned} \tag{3.2.9}$$

Integrating both sides of (3.2.9) over $y \in B_R$, we conclude that

$$\begin{aligned} \int_{B_{R/2}} \Gamma u(x)^2 dx &\leq c \left(\int_{B_R} \Gamma u(x)^{q_1} dx \right)^{\frac{2}{q_1}} \\ &\quad + c \int_{B_R} M(u^2)(x) dx + c \int_{B_R} |h(x)| M u(x) dx. \end{aligned} \quad (3.2.10)$$

Let

$$g(x) = \Gamma u(x)^{q_1}$$

and

$$f(x) = \left(M(u^2)(x) + |h(x)| M u(x) \right)^{\frac{q_1}{2}}.$$

We can rewrite (3.2.10) as

$$\begin{aligned} \frac{1}{|B(x_0, R)|} \int_{B(x_0, R/2)} g^{\frac{2}{q_1}}(x) dx \\ \leq c \left(\frac{1}{|B(x_0, R)|} \int_{B(x_0, R)} g(x) dx \right)^{\frac{2}{q_1}} + c \frac{1}{|B(x_0, R)|} \int_{B(x_0, R)} f^{\frac{2}{q_1}}(x) dx. \end{aligned} \quad (3.2.11)$$

By a scaling and translation argument, (3.2.11) holds for all $R > 0$ and all $x_0 \in \mathbb{R}^d$.

We now apply the reverse Hölder inequality (see Theorem 4.1 in [18]). Thus there exists $\varepsilon > 0$ and $c_1 > 0$ such that $g(x) \in L^t(B(x_0, R/2))$ for all $t \in [\frac{2}{q_1}, \frac{2}{q_1} + \varepsilon)$ and

$$\begin{aligned} \left(\frac{1}{|B(x_0, R/2)|} \int_{B(x_0, R/2)} g^t(x) dx \right)^{\frac{1}{t}} &\leq c \left(\frac{1}{|B(x_0, R)|} \int_{B(x_0, R)} g^{\frac{2}{q_1}}(x) dx \right)^{\frac{q_1}{2}} \\ &\quad + c \left(\frac{1}{|B(x_0, R)|} \int_{B(x_0, R)} f^t(x) dx \right)^{\frac{1}{t}}. \end{aligned}$$

This leads to

$$\begin{aligned}
& \left(\frac{1}{|B(x_0, R/2)|} \int_{B(x_0, R/2)} \Gamma u(x)^{q_1 t} dx \right)^{\frac{1}{t}} \\
& \leq c \left(\frac{1}{|B(x_0, R)|} \int_{B(x_0, R)} \Gamma u(x)^2 dx \right)^{\frac{q_1}{2}} \\
& \quad + c \left(\frac{1}{|B(x_0, R)|} \int_{B(x_0, R)} (M(u^2))^{t q_1 / 2}(x) dx \right)^{\frac{1}{t}} \\
& \quad + c \left(\frac{1}{|B(x_0, R)|} \int_{B(x_0, R)} (|h|Mu)^{t q_1 / 2} dx \right)^{\frac{1}{t}}.
\end{aligned}$$

Choose $t \in (2/q_1, 2/q_1 + \varepsilon)$ so that $q_1 t < 4d/(d - \alpha)$ and set $p = q_1 t$.

Now set $R = 2\sqrt{d}$ for the remainder of the proof. Taking q_1^{th} roots and using the inequality $(a + b)^{1/q_1} \leq a^{1/q_1} + b^{1/q_1}$,

$$\begin{aligned}
\|\Gamma u\|_{L^p(B(x_0, R/2))} & \leq c \|\Gamma u\|_{L^2(B(x_0, R))} + c \|M(u^2)\|_{L^{p/2}(B(x_0, R))}^{1/2} \\
& \quad + c \|h(Mu)\|_{L^{p/2}(B(x_0, R))}^{1/2}.
\end{aligned}$$

For $k \in \mathbb{Z}^d$, let $C_k = B(k, \sqrt{d})$ and $D_k = B(k, 2\sqrt{d})$. Note that $\mathbb{R}^d \subset \cup_{k \in \mathbb{Z}^d} C_k$ and that there exists an integer N depending only on the dimension d such that no point of \mathbb{R}^d is in more than N of the D_k . This can be expressed as $\sum_{k \in \mathbb{Z}^d} \chi_{D_k} \leq N$.

Using $\sum a_k^{p/2} \leq (\sum a_k)^{p/2}$ when each $a_k \geq 0$ and $p/2 \geq 1$, we write

$$\begin{aligned}
\int_{\mathbb{R}^d} |\Gamma u(x)|^p dx &\leq \sum_{k \in \mathbb{Z}^d} \int_{C_k} |\Gamma u(x)|^p dx \\
&\leq c \sum_k \left(\int_{D_k} |\Gamma u(x)|^2 dx \right)^{p/2} + c \sum_k \int_{D_k} (M(u^2)(x))^{p/2} dx \\
&\quad + c \sum_k \int_{D_k} (|h(x)|Mu(x))^{p/2} dx \\
&\leq c \left(\sum_k \int_{D_k} |\Gamma u(x)|^2 dx \right)^{p/2} + c \sum_k \int_{D_k} (M(u^2)(x))^{p/2} dx \\
&\quad + c \sum_k \int_{D_k} (|h(x)|Mu(x))^{p/2} dx \\
&= c \left(\int_{\mathbb{R}^d} |\Gamma u(x)|^2 \sum_k \chi_{D_k}(x) dx \right)^{p/2} \\
&\quad + c \int_{\mathbb{R}^d} (M(u^2)(x))^{p/2} \sum_k \chi_{D_k}(x) dx \\
&\quad + c \int_{\mathbb{R}^d} (|h(x)|Mu(x))^{p/2} \sum_k \chi_{D_k}(x) dx.
\end{aligned}$$

We thus obtain

$$\begin{aligned}
\int_{\mathbb{R}^d} |\Gamma u|^p &\leq c \left(\int_{\mathbb{R}^d} |\Gamma u|^2 dx \right)^{p/2} + c \int_{\mathbb{R}^d} (M(u^2))^{p/2} dx \\
&\quad + c \int_{\mathbb{R}^d} (|h|Mu)^{p/2} dx.
\end{aligned} \tag{3.2.12}$$

Letting $r = 4/p$ and $s = 4/(4-p)$, Hölder's inequality and the inequality $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$ shows

$$\begin{aligned}
\int (|h|Mu)^{p/2} &\leq \left(\int |h|^{pr/2} \right)^{1/r} \left(\int (Mu)^{ps/2} \right)^{1/s} \\
&\leq \frac{1}{2} \left(\int |h|^2 \right)^{p/2} + \frac{1}{2} \left(\int (Mu)^{2p/(4-p)} \right)^{(4-p)/2}.
\end{aligned} \tag{3.2.13}$$

Since M is a bounded operator on $L^{p'}$ for each $p' > 1$ and we know that $2p/(4-p) > 1$, the second term on the last line of (3.2.13) is bounded by

$$c \left(\int |u|^{2p/(4-p)} \right)^{(4-p)/2}.$$

Similarly, since $p > 2$, the second term on the right hand side of the first line of (3.2.12) is bounded by

$$c \int (|u|^2)^{p/2} = c \int |u|^p.$$

Therefore

$$\begin{aligned} \int_{\mathbb{R}^d} |\Gamma u|^p &\leq c \left(\int |\Gamma u|^2 \right)^{p/2} + c \int |u|^p + c \left(\int |h|^2 \right)^{p/2} \\ &\quad + c \left(\int |u|^{2p/(4-p)} \right)^{(4-p)/2}. \end{aligned}$$

Taking p^{th} roots and using $(a+b)^{1/p} \leq a^{1/p} + b^{1/p}$, we obtain

$$\begin{aligned} \|\Gamma u\|_{L^p(\mathbb{R}^d)} &\leq c \|\Gamma u\|_{L^2(\mathbb{R}^d)} + c \|u\|_{L^p(\mathbb{R}^d)} + c \|h\|_{L^2(\mathbb{R}^d)} \\ &\quad + c \|u\|_{L^{2p/(4-p)}(\mathbb{R}^d)}. \end{aligned}$$

This completes the proof of the proposition. □

We now bound the L^p and $L^{2p/(4-p)}$ norms of u .

Theorem 3.2.5. (1) *Suppose $d > \alpha$ and (3.1.2) holds. There exists $p > 2$ and a constant c_1 depending on Λ, p, d , and α such that*

$$\|\Gamma u\|_{L^p(\mathbb{R}^d)} \leq c_1 \left(\mathcal{E}(u, u)^{\frac{1}{2}} + \|h\|_{L^2(\mathbb{R}^d)} + \|u\|_{L^2(\mathbb{R}^d)} \right).$$

(2) If in addition $u \in \mathcal{D}(\mathcal{L})$, there exists a constant c_2 such that

$$\|\Gamma u\|_{L^p(\mathbb{R}^d)} \leq c_2 \left(\|h\|_{L^2(\mathbb{R}^d)} + \|u\|_{L^2(\mathbb{R}^d)} \right).$$

Proof. Let $p_1 = 2d/(d - \alpha)$. Let C_k be defined as in the previous proof.

By Lemma 3.2.2 with $q = 2$

$$\int_{C_k} |u - u_{C_k}|^{p_1} \leq c \left(\int_{C_k} |\Gamma u(x)|^2 dx \right)^{p_1/2}.$$

Here $u_{C_k} = (1/|C_k|) \int_{C_k} u$. Then

$$\begin{aligned} \sum_{k \in \mathbb{Z}^d} \int_{C_k} |u - u_{C_k}|^{p_1} &\leq c \sum_k \left(\int_{C_k} |\Gamma u(x)|^2 dx \right)^{p_1/2} \\ &\leq c \left(\sum_k \int_{C_k} |\Gamma u(x)|^2 dx \right)^{p_1/2} \\ &\leq c \left(\int_{\mathbb{R}^d} |\Gamma u(x)|^2 \sum_k \chi_{C_k}(x) dx \right)^{p_1/2} \\ &\leq c \left(\int_{\mathbb{R}^d} |\Gamma u(x)|^2 dx \right)^{p_1/2}. \end{aligned}$$

Also,

$$\int_{C_k} |u_{C_k}|^{p_1} = c |u_{C_k}|^{p_1} \leq c \left(\int_{C_k} |u|^2 \right)^{p_1/2}$$

by Jensen's inequality. Similarly to the above,

$$\sum_k \int_{C_k} |u_{C_k}|^{p_1} \leq c \left(\int_{\mathbb{R}^d} u^2 \right)^{p_1/2}.$$

Hence

$$\begin{aligned} \int |u|^{p_1} &\leq \sum_k \int_{C_k} |u|^{p_1} \leq c \sum_k \int_{C_k} |u - u_{C_k}|^{p_1} + \sum_k \int_{C_k} |u_{C_k}|^{p_1} \\ &\leq c \left(\int |\Gamma u|^2 \right)^{p_1/2} + c \left(\int u^2 \right)^{p_1/2}. \end{aligned}$$

Taking p_1 th roots, we have

$$\|u\|_{L^{p_1}(\mathbb{R}^d)} \leq c \|\Gamma u\|_{L^2(\mathbb{R}^d)} + c \|u\|_{L^2(\mathbb{R}^d)}.$$

If $2 \leq r \leq p_1$, there exists $\theta \in [0, 1]$ depending only on r and p_1 such that $\|u\|_{L^r} \leq \|u\|_{L^2}^\theta \|u\|_{L^{p_1}}^{1-\theta}$; see, e.g., Proposition 6.10 of [27]. Combining with the inequality $a^\theta b^{1-\theta} \leq a + b$ yields

$$\|u\|_{L^r} \leq \|u\|_{L^2} + \|u\|_{L^{p_1}}.$$

We thus obtain

$$\|u\|_{L^r(\mathbb{R}^d)} \leq c \|\Gamma u\|_{L^2(\mathbb{R}^d)} + c \|u\|_{L^2(\mathbb{R}^d)}.$$

Applying this with r first equal to p and then with r equal to $2p/(4-p)$ and using Proposition 3.2.4, we obtain (1).

Suppose now that $u \in \mathcal{D}(\mathcal{L})$ and that $h = \mathcal{L}u$. Let $\{E_\lambda\}$ be the spectral resolution of the operator $-\mathcal{L}$. Then for $u \in L^2$,

$$u = \int_0^\infty dE_\lambda u, \quad \|u\|_{L^2(\mathbb{R}^d)} = \int_0^\infty d(E_\lambda u, E_\lambda u).$$

If $u \in \mathcal{D}(\mathcal{L})$ and $h = \mathcal{L}u$, then

$$h = \int_0^\infty \lambda dE_\lambda u, \quad \|h\|_{L^2(\mathbb{R}^d)} = \int_0^\infty \lambda^2 d(E_\lambda u, E_\lambda u).$$

It then follows that

$$\begin{aligned} \|\Gamma u\|_{L^2(\mathbb{R}^d)}^2 &= \mathcal{E}(u, u) \\ &= \int_0^\infty \lambda d(E_\lambda u, E_\lambda u) \\ &= \int_0^1 \lambda d(E_\lambda u, E_\lambda u) + \int_1^\infty \lambda d(E_\lambda u, E_\lambda u) \\ &\leq \int_0^1 d(E_\lambda u, E_\lambda u) + \int_1^\infty \lambda^2 d(E_\lambda u, E_\lambda u) \\ &\leq \|u\|_{L^2(\mathbb{R}^d)}^2 + \|h\|_{L^2(\mathbb{R}^d)}^2. \end{aligned}$$

This and (1) prove (2). □

3.3 Strong stability

Let

$$G(x) = \sup_{y \in \mathbb{R}^d} |\tilde{A}(x, y) - A(x, y)|.$$

Theorem 3.3.1. *Suppose $d > \alpha$. There exist $q \geq 2d/\alpha$ and a constant c_1 depending on Λ, d, α , and q such that if $f \in L^2(\mathbb{R}^d)$, then*

$$\|P_t f - \tilde{P}_t f\|_{L^2}^2 \leq c_1 (t^{-\frac{1}{2}} + t^{\frac{1}{2}}) \|G\|_{L^{2q}} \|f\|_{L^2}^2. \quad (3.3.1)$$

Proof. For $t > 0$, let $u = P_t f - \tilde{P}_t f$. By Lemma 2.1.1(1), we know that $P_t f$ and $\tilde{P}_t f$

are both in $\mathcal{F} = W^{\frac{\alpha}{2}, 2}(\mathbb{R}^d)$, so $u \in W^{\frac{\alpha}{2}, 2}(\mathbb{R}^d)$.

We write

$$\begin{aligned} \|P_t f - \tilde{P}_t f\|_{L^2}^2 &= (P_t f - \tilde{P}_t f, u) \\ &= \int_0^t \frac{d}{ds} (P_s \tilde{P}_{t-s} f, u) ds. \end{aligned}$$

This, Lemma 2.1.1(3), and routine calculations show that

$$\|P_t f - \tilde{P}_t f\|_{L^2}^2 = \int_0^t \left(-\mathcal{E}(\tilde{P}_{t-s} f, P_s u) + \tilde{\mathcal{E}}(\tilde{P}_{t-s} f, P_s u) \right) ds. \quad (3.3.2)$$

Using (3.3.2), Lemma 2.1.1(1) and Hölder's inequality, we obtain

$$\begin{aligned} &\|P_t f - \tilde{P}_t f\|_{L^2}^2 \\ &= \int_0^t \left(-\mathcal{E}(\tilde{P}_{t-s} f, P_s u) + \tilde{\mathcal{E}}(\tilde{P}_{t-s} f, P_s u) \right) ds \\ &= \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left(\tilde{P}_{t-s} f(y) - \tilde{P}_{t-s} f(x) \right) \left(P_s u(y) - P_s u(x) \right) \\ &\quad \times \frac{\tilde{A}(x, y) - A(x, y)}{|x - y|^{d+\alpha}} dy dx ds \\ &\leq c \int_0^t \left[\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left(\tilde{P}_{t-s} f(y) - \tilde{P}_{t-s} f(x) \right)^2 \frac{1}{|x - y|^{d+\alpha}} dy dx \right]^{\frac{1}{2}} \\ &\quad \times \left[\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left(P_s u(y) - P_s u(x) \right)^2 \frac{|\tilde{A}(x, y) - A(x, y)|^2}{|x - y|^{d+\alpha}} dy dx \right]^{\frac{1}{2}} ds \\ &\leq c \int_0^t \left[\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left(\tilde{P}_{t-s} f(y) - \tilde{P}_{t-s} f(x) \right)^2 \frac{\tilde{A}(x, y)}{|x - y|^{d+\alpha}} dy dx \right]^{\frac{1}{2}} \\ &\quad \times \left[\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left(P_s u(y) - P_s u(x) \right)^2 \frac{|\tilde{A}(x, y) - A(x, y)|^2}{|x - y|^{d+\alpha}} dy dx \right]^{\frac{1}{2}} ds \\ &\leq c \int_0^t \left[\tilde{\mathcal{E}}(\tilde{P}_{t-s} f, \tilde{P}_{t-s} f) \right]^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned}
& \times \left[\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\left(P_s u(y) - P_s u(x)\right)^2}{|x-y|^{d+\alpha}} dy G^2(x) dx \right]^{\frac{1}{2}} ds \\
& \leq c \int_0^t (t-s)^{-\frac{1}{2}} \|f\|_{L^2} \\
& \quad \times \left\{ \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} \frac{\left(P_s u(y) - P_s u(x)\right)^2}{|x-y|^{d+\alpha}} dy \right]^{p'} dx \right\}^{\frac{1}{2p'}} \\
& \quad \times \left\{ \int_{\mathbb{R}^d} G^{2q'}(x) dx \right\}^{\frac{1}{2q'}} ds
\end{aligned} \tag{3.3.3}$$

$$= c \|f\|_{L^2} \|G\|_{L^{2q'}} \int_0^t (t-s)^{-\frac{1}{2}} \|\Gamma(P_s u)(x)\|_{L^{2p'}} ds, \tag{3.3.4}$$

where p' and q' are conjugate exponents.

We choose p' so that $2p'$ is equal to the p in Theorem 3.2.5(2). By that theorem,

$$\|\Gamma(P_s u)\|_{L^{2p'}} \leq c \|P_s u\|_{L^2} + c \|\mathcal{L}(P_s u)\|_{L^2}. \tag{3.3.5}$$

Since P_s, P_t , and \tilde{P}_t are contractions,

$$\|P_s u\|_{L^2} \leq \|u\|_{L^2} = \|P_t f - \tilde{P}_t f\|_{L^2} \leq 2\|f\|_{L^2}. \tag{3.3.6}$$

To estimate $\mathcal{L}(P_s u)$, we note $P_{s/2} u \in \mathcal{D}(\mathcal{L})$ by Lemma 2.1.1(2) and then use Lemma

2.1.1(4). Then

$$\begin{aligned}
\|\mathcal{L}(P_s u)\|_{L^2} &= \|(-\mathcal{L})^{1/2} P_{s/2} (-\mathcal{L})^{1/2} (P_{s/2} u)\|_{L^2} & (3.3.7) \\
&\leq c s^{-1/2} \|(-\mathcal{L})^{1/2} (P_{s/2} u)\|_{L^2} \\
&= c s^{-1/2} \mathcal{E}(P_{s/2} u, P_{s/2} u)^{1/2} \\
&\leq c s^{-1/2} \mathcal{E}(u, u)^{1/2} \\
&\leq c s^{-1/2} [\mathcal{E}(P_t f, P_t f)^{1/2} + \mathcal{E}(\tilde{P}_t f, \tilde{P}_t f)^{1/2}] \\
&\leq c (st)^{-1/2} \|f\|_{L^2},
\end{aligned}$$

where Lemma 2.1.1(1) is used in the first and last inequalities. Combining (3.3.4), (3.3.5), (3.3.6), and (3.3.7) yields our result. \square

Remark 3.3.2. A scaling argument allows one to improve (3.3.1) to

$$\|P_t f - \tilde{P}_t f\|_{L^2}^2 \leq c_1 t^{-d/2q\alpha} \|G\|_{L^{2q}} \|f\|_{L^2}^2. \quad (3.3.8)$$

Proof of Remark 3.3.2. For $a > 0$. Define $Y_t = aX_{a^{-\alpha}t}$ and $\mathbb{Q}^x = \mathbb{P}_a^{\frac{x}{a}}$. Since (\mathbb{P}^x, X_t) is a strong Markov process and Y_t is a constant multiple of a time change of X_t , then (\mathbb{Q}^x, Y_t) is a strong Markov process.

Let \mathcal{L}_Y be the generator for Y in the sense of the martingale problem. Then, by Proposition 2.2 in [12],

$$\mathcal{L}_Y f(x) = \int_{\mathbb{R}^d} (f(y) - f(x)) \frac{B(x, y)}{|x - y|^{d+\alpha}} dy dx,$$

where $B(x, y) = A(x/a, y/a)$ and $A(x, y)$ is the function in (2.1.1).

And then, the Dirichlet form for Y is given by

$$\mathcal{E}(f, f) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (f(y) - f(x))^2 \frac{B(x, y)}{|x - y|^{d+\alpha}} dy dx.$$

Let Q_t be the semigroup for Y . We have

$$\begin{aligned} P_t f(x) &= \mathbb{E}^x f(X_t) = \mathbb{E}^x \left(f\left(\frac{1}{a} Y_{a^\alpha t}\right) \right) \\ &= \mathbb{E}^{x^*} g(Y_{a^\alpha t}) = Q_{a^\alpha t} g(x^*), \end{aligned}$$

where $g(z) = f(\frac{1}{a}z)$. If X_t starts at x , then Y_t starts at ax . So $x^* = ax$.

Therefore, $P_t f(x) = Q_{a^\alpha t} g(ax)$.

Suppose we define \tilde{Q}_t and \tilde{B} in terms of \tilde{P}_t similarly and let

$$H(x) = \sup_{y \in \mathbb{R}^d} |B(x, y) - \tilde{B}(x, y)|.$$

Fix $t > 0$ and set $a = t^{-1/\alpha}$ so that $a^\alpha = t^{-1}$. An application of Theorem 3.3.1

yields

$$\begin{aligned}
\|P_t f - \tilde{P}_t f\|_{L^2}^2 &= \|Q_{a^{\alpha t}} g(ax) - \tilde{Q}_{a^{\alpha t}} g(ax)\|_{L^2}^2 \\
&= \|Q_1 g(ax) - \tilde{Q}_1 g(ax)\|_{L^2}^2 \\
&= a^{-d} \|Q_1 g(z) - \tilde{Q}_1 g(z)\|_{L^2}^2 \\
&\leq ca^{-d} \|H\|_{2q} \|g\|_{L^2}^2.
\end{aligned}$$

Straightforward calculations show that

$$\|H\|_{L^{2q}} = a^{d/2q} \|G\|_{L^{2q}}$$

and

$$\|g\|_{L^2}^2 = a^d \|f\|_{L^2}^2.$$

Combining the above gives (3.3.8). □

Let $p(t, x, y)$ and $\tilde{p}(t, x, y)$ be the heat kernels corresponding to P_t and \tilde{P}_t . We have the following two theorems. The proofs are similar to the ones in [24].

Theorem 3.3.3. *Let $t > 0$. There exist $q > 1$ and a constant c_1 depending on $t, \Lambda, \gamma, d, \alpha$, and q such that for any $x, y \in \mathbb{R}^d$*

$$|p(t, x, y) - \tilde{p}(t, x, y)| \leq c_1 \|G\|_{2q}^{\frac{\gamma}{2(d+\gamma)}}.$$

Proof. Notice that

$$\begin{aligned}
& \frac{1}{|B_r|^2} |(P_t 1_{B_r(x)}, 1_{B_r(y)}) - (\tilde{P}_t 1_{B_r(x)}, 1_{B_r(y)})| \\
&= \frac{1}{|B_r|^2} |((P_t - \tilde{P}_t) 1_{B_r(x)}, 1_{B_r(y)})| \\
&\leq \frac{1}{|B_r|^2} \|P_t 1_{B_r(x)} - \tilde{P}_t 1_{B_r(x)}\|_2 \|1_{B_r(y)}\|_2.
\end{aligned}$$

By Theorem 3.3.1, we know that

$$\|P_t 1_{B_r(x)} - \tilde{P}_t 1_{B_r(x)}\|_2^2 \leq c \|1_{B_r(x)}\|_2^2 \|G(x)\|_{2q}.$$

Thus, combining the above two inequalities gives

$$\begin{aligned}
& \frac{1}{|B_r|^2} |(P_t 1_{B_r(x)}, 1_{B_r(y)}) - (\tilde{P}_t 1_{B_r(x)}, 1_{B_r(y)})| \\
&\leq \frac{c}{|B_r|^2} \|1_{B_r(x)}\|_2 \|G(x)\|_{2q}^{\frac{1}{2}} \|1_{B_r(y)}\|_2 \\
&\leq \frac{c}{r^d} \|G(x)\|_{2q}^{\frac{1}{2}}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& \left| p(t, x, y) - \frac{1}{|B_r|^2} (P_t 1_{B_r(x)}, 1_{B_r(y)}) \right| \\
&\leq \frac{1}{|B_r|^2} \int_{B_r(x) \times B_r(y)} |p(t, x, y) - p(t, z, v)| \, dv \, dz.
\end{aligned}$$

By Theorem 4.14 in [4], we know there exist $\gamma > 0$ and a constant c such that

$$|p(t, x, y) - p(t, z, v)| \leq c t^{-\frac{d+\gamma}{\alpha}} (|x - z| + |y - v|)^\gamma.$$

Hence,

$$\frac{1}{|B_r|^2} \int_{B_r(x) \times B_r(y)} |p(t, x, y) - p(t, z, v)| \, dv \, dz \leq cr^\gamma.$$

Therefore,

$$\begin{aligned} & |p(t, x, y) - \tilde{p}(t, x, y)| \\ & \leq \left| p(t, x, y) - \frac{1}{|B_r|^2} (P_t 1_{B_r(x)}, 1_{B_r(y)}) \right| + \left| \tilde{p}(t, x, y) - \frac{1}{|B_r|^2} (\tilde{P}_t 1_{B_r(x)}, 1_{B_r(y)}) \right| \\ & \quad + \frac{1}{|B_r|^2} |(P_t 1_{B_r(x)}, 1_{B_r(y)}) - (\tilde{P}_t 1_{B_r(x)}, 1_{B_r(y)})| \\ & \leq cr^\gamma + \frac{c}{r^d} \|G(x)\|_{2q}^{\frac{1}{2}}. \end{aligned}$$

Letting $r^\gamma = \frac{1}{r^d} \|G(x)\|_{2q}^{\frac{1}{4}}$, then $r = \|G(x)\|_{2q}^{\frac{1}{2(d+\gamma)}}$, and therefore

$$|p(t, x, y) - \tilde{p}(t, x, y)| \leq c \|G(x)\|_{2q}^{\frac{\gamma}{2(d+\gamma)}}. \quad (3.3.9)$$

□

Theorem 3.3.4. *Let $t > 0$. There exist $q > 1$ and a constant c_2 depending on $t, \Lambda, \gamma, d, \alpha$, and q such that for any $p \in [1, \infty]$, we have*

$$\|P_t f - \tilde{P}_t f\|_{L^p} \leq c_2 \|G\|_{2q}^{\frac{\gamma\alpha}{2(d+\gamma)(d+\alpha)}} \|f\|_{L^p}.$$

Proof. Let $\beta_t(x) = \int_{\mathbb{R}^d} |p(t, x, y) - \tilde{p}(t, x, y)| \, dy$ and $\beta_t = \sup_{x \in \mathbb{R}^d} \beta_t(x)$.

By Theorem 1.1 in [4], there exist constants c and \tilde{c} such that

$$\begin{aligned} c \min \left\{ t^{-\frac{d}{\alpha}}, \frac{t}{|x-y|^{d+\alpha}} \right\} &\leq p(t, x, y) \\ &\leq \tilde{c} \min \left\{ t^{-\frac{d}{\alpha}}, \frac{t}{|x-y|^{d+\alpha}} \right\}. \end{aligned} \quad (3.3.10)$$

For $x \in R^d$, $r > 0$, by applying (3.3.9) and (3.3.10) we have

$$\begin{aligned} \beta_t(x) &= \int_{B(x,r)} |p(t, x, y) - \tilde{p}(t, x, y)| dy + \int_{B(x,r)^c} |p(t, x, y) - \tilde{p}(t, x, y)| dy \\ &\leq c \|G(x)\|_{2q}^{\frac{\gamma}{2(d+\gamma)}} r^d + c \int_{B(x,r)^c} \frac{t}{|x-y|^{d+\alpha}} dy \\ &= c \|G(x)\|_{2q}^{\frac{\gamma}{2(d+\gamma)}} r^d + c \int_r^\infty \frac{1}{\xi^{d+\alpha}} \cdot \xi^{d-1} d\xi \\ &= c \|G(x)\|_{2q}^{\frac{\gamma}{2(d+\gamma)}} r^d + cr^{-\alpha}. \end{aligned}$$

Choose r such that $r^{-\alpha} = \|G(x)\|_{2q}^{\frac{\gamma}{2(d+\gamma)}} r^d$, that is $r^d = \|G(x)\|_{2q}^{-\frac{d\gamma}{2(d+\gamma)(d+\alpha)}}$.

Therefore $\beta_t(x) \leq c \|G(x)\|_{2q}^{\frac{\gamma\alpha}{2(d+\gamma)(d+\alpha)}}$.

Now for $p \in [1, \infty)$ and $f \in L^p(\mathbb{R}^d)$,

$$\begin{aligned}
& \|P_t f - \tilde{P}_t f\|_p \\
& \leq \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |p(t, x, y) - \tilde{p}(t, x, y)| |f(y)| dy \right)^p dx \right)^{\frac{1}{p}} \\
& = \left(\int_{\mathbb{R}^d} \beta_t(x)^p \left(\int_{\mathbb{R}^d} \beta_t(x)^{-1} |p(t, x, y) - \tilde{p}(t, x, y)| |f(y)| dy \right)^p dx \right)^{\frac{1}{p}} \\
& \leq \left(\int_{\mathbb{R}^d} \beta_t(x)^p \left(\int_{\mathbb{R}^d} \beta_t(x)^{-1} |p(t, x, y) - \tilde{p}(t, x, y)| |f(y)|^p dy \right) dx \right)^{\frac{1}{p}} \\
& = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \beta_t(x)^{p-1} |p(t, x, y) - \tilde{p}(t, x, y)| dx \right) |f(y)|^p dy \right)^{\frac{1}{p}} \\
& \leq \beta_t \|f\|_p .
\end{aligned}$$

For $p = \infty$, since

$$\|(P_t - \tilde{P}_t)f\|_\infty \leq \int_{\mathbb{R}^d} |p(t, x, y) - \tilde{p}(t, x, y)| dy \|f\|_{L^\infty} \leq \beta_t \|f\|_{L^\infty}.$$

Hence, $\|P_t f - \tilde{P}_t f\|_{L^p} \leq c \|G\|_{2q}^{\frac{\gamma\alpha}{2(d+\gamma)(d+\alpha)}} \|f\|_{L^p}$ for $p \in [1, \infty]$.

□

As in Remark 3.3.2, one could use scaling to obtain an explicit bound on how the constants depend on t .

Chapter 4

Pathwise Uniqueness of SDEs with Jumps

In this chapter, we consider the following one-dimensional jump stochastic differential equation driven by one-sided stable processes of order $\alpha \in (0, 1)$:

$$dX_t = \phi(X_{t-}) dZ_t.$$

In Section 4.1, we prove that pathwise uniqueness holds for this equation if ϕ is continuous, non-decreasing and positive on \mathbb{R} . A counter-example is given in Section 4.2 to show that the positivity of ϕ is crucial for pathwise uniqueness to hold.

4.1 SDEs driven by one-sided stable processes

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a filtered probability space. For $\alpha \in (0, 1)$, let Z_t be a one-sided α -stable process adapted to $\{\mathcal{F}_t\}_{t \geq 0}$ which only has positive jumps, i.e., $a_- = 0$ and $a_+ \in (0, \infty)$ in (2.2.2).

In this section, we will study the following stochastic differential equation

$$dX_t = \phi(X_{t-}) dZ_t, \quad (4.1.1)$$

where $\phi(x)$ is time-independent and satisfies the following conditions:

Assumption 4.1.1. 1. $\phi(\cdot)$ is Borel measurable and continuous on \mathbb{R} ;

2. $\phi(\cdot)$ is non-decreasing on \mathbb{R} ;

3. $\phi(\cdot)$ is positive on \mathbb{R} ;

We prove that under Assumption 4.1.1, pathwise uniqueness holds for equation (4.1.1).

Theorem 4.1.2. *Suppose ϕ satisfies Assumption 4.1.1. Then the solution to equation (4.1.1) is pathwise unique.*

To prove Theorem 4.1.2, our strategy is to first construct a strong solution X_t to equation (4.1.1) and then show that weak uniqueness holds for (4.1.1). Once we finish these two steps, pathwise uniqueness will follow.

Proposition 4.1.3. *Suppose ϕ satisfies Assumption 4.1.1. Then there exists a strong solution to equation (4.1.1).*

Proof. For each $n \in \mathbb{N}$, we define

$$Z_t^n = \sum_{s \leq t} \Delta Z_s 1_{\{\Delta Z_s \geq \frac{1}{n}\}}.$$

Then Z_t^n is adapted to \mathcal{F}_t . Recall that Z_t only has positive jumps and no continuous part, so for each $t \geq 0$, $\{Z_t^n\}_{n \geq 1}$ is a non-decreasing process and $Z_t^n \rightarrow Z_t$ \mathbb{P} -a.s. as $n \rightarrow \infty$.

For any $x_0 \in \mathbb{R}$, let X^n be the solution to

$$dX_t^n = \phi(X_{t-}^n) dZ_t^n, \quad X_0^n = x_0.$$

Recall that there are only finitely many jumps on a finite time interval, so it is easy to see the existence of the solution X^n and that the solution is uniquely determined by the initial condition: X_t^n will stay constant until the first jump of Z_t^n , at which point X_t^n will jump $\phi(X_{t-}^n) \Delta Z_t$.

It is clear that X_t^n is adapted to \mathcal{F}_t for each $n \in \mathbb{N}$.

For each $t \geq 0$, we show that $\{X_t^n\}_{n \geq 1}$ is also a non-decreasing sequence.

Take $n, m \in \mathbb{N}$ such that $n > m$. We claim that $X_t^n \geq X_t^m$ \mathbb{P} -a.s. for $t \geq 0$. If not, let

$$S = \inf\{t \geq 0 : X_t^n < X_t^m\}.$$

Then $X_{S-}^n \geq X_{S-}^m$.

Since $X_S^n = X_{S-}^n + \phi(X_{S-}^n)\Delta Z_S$ and $X_S^m = X_{S-}^m + \phi(X_{S-}^m)\Delta Z_S$, remembering that ϕ is non-decreasing, then $\phi(X_{S-}^n) \geq \phi(X_{S-}^m)$. Therefore $X_S^n \geq X_S^m$. Clearly S must be a jump time of Z if $S < \infty$. But if $S < \infty$, then Z , and therefore X , is constant for a positive length of time after time S , and we conclude that $S = \infty$ \mathbb{P} -a.s.

This implies $\{X_t^n\}_{n \geq 1}$ is a non-decreasing sequence for $t \geq 0$.

Let $X_t = \lim_{n \rightarrow \infty} X_t^n$ and note that X_t is adapted to \mathcal{F}_t . We have

$$X_t^n = x_0 + \sum_{s \leq t} \phi(X_{s-}^n) 1_{\{\Delta Z_s \geq \frac{1}{n}\}} \Delta Z_s.$$

As $n \rightarrow \infty$, the right hand side converges to

$$x_0 + \sum_{s \leq t} \phi(X_{s-}) \Delta Z_s$$

by monotone convergence. Since Z is non-decreasing and has no continuous part, we conclude that

$$X_t = x_0 + \int_0^t \phi(X_{s-}) dZ_s.$$

□

Next, we will show that weak uniqueness holds for equation (4.1.1).

Proposition 4.1.4. *Suppose ϕ satisfies Assumption 4.1.1. Then the solution to equation (4.1.1) is unique in law.*

Proof. Let X_t denote the strong solution to equation (4.1.1). Then by Theorem 4.1 in [33], for any real number x , there exists a process \tilde{Z}_t such that \tilde{Z} has the same law as Z and $X_t = x + \tilde{Z}_{\tau_t}$ where $\tau_t = \int_0^t \phi(X_s)^\alpha ds$.

Let $B_t = \inf\{s \geq 0 : \tau_s > t\}$. We will show that

$$B_t = \int_0^t \phi(\tilde{Z}_s)^{-\alpha} ds \quad (4.1.2)$$

and that $\tau_t = \inf\{s \geq 0 : \int_0^s \phi(\tilde{Z}_u)^{-\alpha} du > t\}$ \mathbb{P} -a.s. Therefore the distribution of X_t will be determined by the one of \tilde{Z} for any given initial value x .

By Lemma 1.6 in [44], for any $t \geq 0$, we have

$$\begin{aligned} B_t &\geq \int_0^{B_t} 1_{\{\phi(X_s) \neq 0\}} ds = \int_0^{t \wedge \tau_\infty} \phi(X_{B_s})^{-\alpha} ds \\ &= \int_0^{t \wedge \tau_\infty} \phi(x + \tilde{Z}_s)^{-\alpha} ds. \end{aligned} \quad (4.1.3)$$

Since ϕ is positive, then (4.1.3) gives us

$$B_t = \int_0^{t \wedge \tau_\infty} \phi(x + \tilde{Z}_s)^{-\alpha} ds. \quad (4.1.4)$$

Also, because of the fact that ϕ is positive, then τ_t is strictly increasing on $[0, \infty]$.

If $t \geq \tau_\infty$, then $B_t = \inf\{s \geq 0 : \tau_s > t\} = \infty$.

One the other hand, by (4.1.4),

$$\int_0^t \phi(x + \tilde{Z}_s)^{-\alpha} ds \geq \int_0^{t \wedge \tau_\infty} \phi(x + \tilde{Z}_s)^{-\alpha} ds = B_t = \infty,$$

and then $\int_0^t \phi(x + \tilde{Z}_s)^{-\alpha} ds = \infty$. Therefore (4.1.2) holds with both sides equal to ∞ .

If $t < \tau_\infty$, then

$$B_t = \int_0^{t \wedge \tau_\infty} \phi(x + \tilde{Z}_s)^{-\alpha} ds = \int_0^t \phi(x + \tilde{Z}_s)^{-\alpha} ds,$$

which is just (4.1.2).

Hence (4.1.2) has been proved. Therefore we can conclude that for any given initial value x , the law of X_t is unique.

□

Now we are ready to give the proof of Theorem 4.1.2.

Proof of Theorem 4.1.2. According to Proposition 3.2.2, there exists a strong solution X_t to equation (4.1.1). Therefore there exists a measurable map $H : Z \mapsto X$. Suppose X'_t is another strong solution. Then by Proposition 4.1.4, the laws of X and X' are the same. Since $Z_t = \int_0^t \phi(X_{s-})^{-1} dX_s$ and $Z_t = \int_0^t \phi(X'_{s-})^{-1} dX'_s$, then the joint laws of (Z, X) and (Z, X') are the same. Since $X = H(Z)$, then $X' = H(Z)$. Therefore, $X' = H(Z) = X$.

□

4.2 A counter-example

In this section, we give a counter-example to show that the positivity condition on ϕ is crucial to obtaining pathwise uniqueness.

For $\alpha \in (0, 1)$ and $\beta \in (0, 1)$, let Z_t be a one-sided α -stable process which starts from zero and only has positive jumps. Let $\phi(x) = x^\beta$. Then $\phi(x)$ is continuous and increasing on $\mathbb{R}^+ \cup \{0\}$.

Define $B_t = \int_0^t \phi(Z_{s-})^{-\alpha} ds$. Then,

$$\mathbb{E}B_t = \int_0^t \int_{\mathbb{R}^+} \phi(y)^{-\alpha} f(s, y) dy ds,$$

where $f(s, y)$ denotes the transition density function of Z_t . It's well-known that there exist positive constants c_1 and c_2 such that

$$f(s, y) \leq c_1 s^{-\frac{1}{\alpha}} \wedge c_2 \frac{s}{y^{1+\alpha}}, \quad \text{for } s \geq 0, y \in \mathbb{R}^+.$$

Therefore, since $0 < \beta < 1$,

$$\begin{aligned} \mathbb{E}B_t &\leq c_1 \int_0^t \int_{0 < y < s^{\frac{1}{\alpha}}} \phi(y)^{-\alpha} s^{-\frac{1}{\alpha}} dy ds + c_2 \int_0^t \int_{y \geq s^{\frac{1}{\alpha}}} \phi(y)^{-\alpha} \frac{s}{y^{1+\alpha}} dy ds \\ &= c_1 \int_0^t \int_{0 < y < s^{\frac{1}{\alpha}}} y^{-\alpha\beta} s^{-\frac{1}{\alpha}} dy ds + c_2 \int_0^t \int_{y \geq s^{\frac{1}{\alpha}}} y^{-\alpha\beta-\alpha-1} s dy ds \\ &< \infty. \end{aligned}$$

Hence B_t is finite \mathbb{P} -a.s. for $t \geq 0$.

It is clear that $B_1 > 0$, \mathbb{P} -a.s., and by the scaling property of Z_t , we can show that B_t has the same law as $t^{1-\beta}B_1$. Therefore for any $M > 0$,

$$\mathbb{P}(B_t \leq M) = \mathbb{P}(B_1 \leq t^{\beta-1}M) \rightarrow 0 \text{ as } t \rightarrow \infty,$$

recalling that $\beta \in (0, 1)$. We then conclude $B_t \rightarrow \infty$, \mathbb{P} -a.s., as $t \rightarrow \infty$. Let γ_t represent the inverse of B_t .

Define

$$Y_t = \int_0^t \phi(Z_{s-}) dZ_s, \quad V_t = Y_{\gamma_t} \quad \text{and} \quad X_t = Z_{\gamma_t}.$$

Then by Theorem 3 in [34], V_t has the same law as Z_t , i.e., V_t is also a one-sided α -stable process with $V_0 = 0$. Some calculus shows that

$$dX_t = \phi(X_{t-}) dV_t.$$

Notice that X_t is non-zero, while the identically zero process is another solution. Therefore pathwise uniqueness does not hold.

This example also shows that it is not true that there is necessarily uniqueness to the ordinary differential equation

$$dy(x) = \phi(y(x-)) z(dx),$$

even when z is a positive purely atomic measure and ϕ is non-negative and non-

decreasing.

Bibliography

- [1] R.A. Adams, *Sobolev Spaces*. New York, Academic Press, 1975.
- [2] D. Applebaum, *Lèvy Processes and Stochastic Calculus*. Cambridge Univ. Press, Cambridge, 2004.
- [3] M.T. Barlow, One-dimensional stochastic differential equations with no strong solution. *J. London Math. Soc.* **26** (1982), 335-347.
- [4] M.T. Barlow, A. Grigor'yan and T. Kumagai, Heat kernel upper bounds for jump processes and the first exit time. *J. Reine Angew. Math.* **626** (2006), 135-157.
- [5] M.T. Barlow, R.F. Bass, Z.-Q. Chen, and M. Kassmann, Non-local Dirichlet forms and symmetric jump processes. *Trans. Amer. Math. Soc.* **361** (2009), 1963–1999.
- [6] M.T. Barlow, A. Grigor'yan and T. Kumagai, On the equivalence of parabolic Harnack inequalities and heat kernel estimates. *J. Math. Soc. Japan* **64** (2012), 1091–1146.
- [7] R.F. Bass, Uniqueness in law for pure jump Markov processes. *Probab. Theory Relat. Fields* **79** (1988), 271–287.

- [8] R.F. Bass and D.A. Levin, Harnack inequalities for jump processes. *Potential Anal.* **17** (2002), 375–388.
- [9] R.F. Bass, Stochastic differential equations driven by symmetric stable processes. *Séminaire de Probabilités, XXXVI*, 302-313, Lecture Notes in Math., 1801, Springer, Berlin, 2003.
- [10] R.F. Bass, Stochastic differential equations with jumps. *Probab. Surv.* **1** (2004), 1-19.
- [11] R.F. Bass, K. Burdzy and Z-Q. Chen, Stochastic differential equations driven by stable process for which pathwise uniqueness fails. *Stochastic Process. Appl.* **111** (2003), 1-15.
- [12] R.F. Bass and D.A. Levin, Transition probabilities for symmetric jump processes. *Trans. Amer. Math. Soc.* **354** (2002), 2933-2953.
- [13] R.F. Bass and M. Kassmann, Harnack inequalities for non-local operators of variable order. *Trans. Amer. Math. Soc.* **357** (2005), 837–850.
- [14] R.F. Bass and M. Kassmann, Hölder continuity of harmonic functions with respect to operators of variable order. *Comm. PDE* **30** (2005), 1249–1259.
- [15] R.F. Bass and H. Tang, The martingale problem for a class of stable-like processes. *Stoch. Proc. and their Applic.* **119** (2009), 1144–1167.
- [16] R.F. Bass, M. Kassmann, and T. Kumagai, Symmetric jump processes: localization, heat kernels, and convergence. *Ann. de l'Institut H. Poincaré* **46** (2010), 59–71.

- [17] M. Benabdallah, S. Bouhadou, and Y. Ouknine, On the pathwise uniqueness of solutions of one-dimensional stochastic differential equations with jumps. Preprint.
- [18] Y.-Z. Chen and L.-C. Wu, *Second Order Elliptic Systems and Equations*. AMS , Providence, 1992.
- [19] Z.-Q. Chen, P. Kim, and T. Kumagai, Global heat kernel estimates for symmetric jump processes. *Trans. Amer. Math. Soc.* **363** (2011), 5021–5055.
- [20] Z.-Q. Chen, P. Kim, and R. Song, Two-sided heat kernel estimates for censored stable-like processes. *Probab. Th. Relat. Fields* **146** (2010), 361-399.
- [21] Z.-Q. Chen and T. Kumagai, Heat kernel estimate for stable-like processes on d -sets. *Stoch. processes and their Applic.* **108** (2003), 27–62.
- [22] Z.-Q. Chen and T. Kumagai, Heat kernel estimates for jump processes of mixed types on metric measure spaces. *Probab. Theory Relat. Fields* **140** (2008), 277–317.
- [23] Z.-Q. Chen and T. Kumagai, A priori Hölder estimate, parabolic Harnack principle and heat kernel estimates for diffusions with jumps. *Rev. Mat. Iberoamericana*, **26** (2010), 551–589.
- [24] Z.-Q. Chen, Z. Qian, Y. Hu and W. Zheng, Stability and approximations of symmetric diffusion semigroups and kernels. *J. Functional Analysis* **152** (1998), 255–280.
- [25] D.E. Edmunds and H. Triebel, *Functions Spaces, Entropy Numbers, Differential Operators*. Cambridge University Press, 1996.

- [26] H.J. Engelbert and W. Schmidt, On solutions of one-dimensional stochastic differential equations without drift. *Z. Wahrsch. Verw. Gebiete* **68** (1985), 287-314.
- [27] G.B. Folland, *Real Analysis: Modern Techniques and their Applications*, 2nd ed., Wiley, New York, 1999.
- [28] M. Foondun, Harnack inequalities for a class of integro-differential operators. *Potential Anal.* **31** (2009), 21–44.
- [29] M. Foondun, Heat kernel estimates and Harnack inequalities for some Dirichlet forms with non-local parts. *Electron. J. Probab.* **14** (2009), Paper No. 11, 314–340.
- [30] N. Fournier, On the pathwise uniqueness for stochastic differential equations driven by stable Lévy processes. *Ann. Inst. H. Poincaré Probab. Statist.* **49** (2013), 138-159
- [31] M. Fukushima, Y. Oshima, and M. Takeda, *Dirichlet Forms and Symmetric Markov Processes*, 2nded. de Gruyter, Berlin, 2001.
- [32] Z.F. Fu and Z.H. Li, Stochastic equations of non-negative processes with jumps. *Stochastic Process. Appl.* **120** (2010), 306-330.
- [33] O. Kallenberg, Some time change representations of stable integrals, via predictable transformations of local martingales. *Stochastic Process. Appl.* **40** (1992), 199-223.
- [34] J. Kallsen and A.V. Shiryaev, Time change representation of stochastic integrals. *Theory Probab. Appl.* **46** (2001), no.3, 522-528.

- [35] M. Kassmann, *Analysis of symmetric Markov processes. A localization technique for non-local operators*, Habilitation Thesis, Universität Bonn, 2007.
- [36] V. Kolokoltsov, Symmetric stable laws and stable-like jump-diffusions. *Proc. London Math. Soc.* **80** (2000), 725–768.
- [37] T. Komatsu, On the pathwise uniqueness of solutions of one-dimensional stochastic differential equations of jump type. *Proc. Japan Acad.* **58 Ser. A** (1982), 383-386.
- [38] Z.H. Li and L. Mytnik, Strong solutions for stochastic differential equations with jumps. *Ann. Inst. H. Poincaré Probab. Statist.* **47** (2011), 1055-1067.
- [39] Z.H. Li and F. Pu, Strong solutions of jump-type stochastic equations. Preprint.
- [40] D. Luo, Pathwise uniqueness of multi-dimensional stochastic differential equations with Hölder diffusion coefficients. *Front. Math. China* **6**(1) (2011), 129-136.
- [41] N.G. Meyers, An L^p -estimate for the gradient of solutions of second order elliptic divergence equations. *Ann. Scuola Norm. Sup. Pisa* **17** (1963), 189–206.
- [42] G. Mingione, The singular set of solutions to non-differentiable elliptic systems. *Arch. Rational Mech. Anal.* **166** (2003), 287–301.
- [43] H. Pragarauskas and P.A. Zanzotto, On one-dimensional stochastic differential equations driven by stable processes. *Lithuanian Math. J.* **40** (2000), no. 3, 277-295.
- [44] J. Rosiński and W.A. Wogczyński, On Itô stochastic integration with respect to p-stable motion: inner clock, integrability of sample paths, double and multiple integrals. *Ann. Probab.* **14** (1986), 271-286.

- [45] K. Sato, *Lévy Processes and Infinitely Divisible Distributions*. Cambridge Univ. Press, Cambridge, 1999.
- [46] R. Situ, *Theory of Stochastic Differential Equations with Jumps and Applications*. Springer, Berlin, 2005.
- [47] A.V. Skorokhod, *Studies in the Theory of Random Process*. Addison-Wesley, Reading, MA, 1965.
- [48] R. Song and Z. Vondraček, Harnack inequality for some classes of Markov processes. *Math. Z.* **246** (2004), 177–202.
- [49] E.M. Stein, *Singular Integrals and Differentiability Properties of Functions*. Princeton University Press, 1970.
- [50] T. Tsuchiya, On the pathwise uniqueness of solutions of stochastic differential equations driven by multi-dimensional symmetric α -stable class. *J. Math. Kyoto Univ.* **46** (2006), 107-121.
- [51] T. Yamada and S. Watanabe, On the uniqueness of solutions of stochastic differential equations. *J. Math. Kyoto Univ.* **11** (1971), 155-167.
- [52] P.A. Zanzotto, On solutions of one-dimensional stochastic differential equations driven by stable Lévy motion. *Stochastic Process. Appl.* **68** (1997) 209-228.
- [53] P.A. Zanzotto, On stochastic differential equations driven by a Cauchy process and other stable Lévy motions. *Ann. Probab.* **30** (2002), no.2, 802-825.