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Topology Optimization of Space Frames via Geometry Projection

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Topology Optimization of Space Frames via Geometry Projection

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Topology Optimization of Space Frames via Geometry Projection

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Abstract

There is growing interest in space agencies to design lunar habitats for different stages of habitation. In this work, we present a topology optimization method to design a class of these structures consisting of rigid space frames with elements transported from earth and assembled in situ. Since the space frame must be made of stock material for ease of transportation and assembly, this work employs the geometry projection method, which is a topology optimization technique that render designs made exclusively of geometric primitives. This work makes three contributions to the geometry projection method that are necessary for the design of rigid space frames. The first is accommodating designs where the frame struts are connected at their ends throughout the optimization, as opposed to using ‘floating’ struts that can result in overlaps that are difficult to manufacture. The second is to consider a weighted sum of the structural compliances corresponding to multiple load cases as the objective for the optimization. The third contribution consists of the imposition of a constraint in the optimization to enforce a maximum strut length, which facilitates transportation. Several numerical examples demonstrate the effectiveness of the method in incorporating these three capabilities.
Chapter 1

Introduction

1.1 Motivation

Global space agencies have created a plan for space exploration into the future [1], with a vision to explore and live on Mars. The moon is a stepping stone to achieve this goal, allowing humans to use it as a place to learn more about extraterrestrial exploration and habitation. Designing habitation structures for extraterrestrial surfaces requires substantial consideration for the origin and location of building materials. The National Aeronautics and Space Agency (NASA) defines three classes of structures based on these considerations.

Class I structures are completely manufactured and assembled on Earth. A limitation of these class of structures is their transportation. Given their weight and size, their transportation cost makes them impractical for long-term habitation. Class II structures are fabricated on Earth, but can be assembled in the final location, with the possibility of disassembly and transportation to another location. This reduces the space required to deliver the structure to its intended location. The challenge with this class of structures is their assembly at the site, which requires tools and possibly machines. Class III structures are constructed from a few parts fabricated on Earth and primarily from \textit{in situ} materials, for example, regolith (lunar soil) on the moon. This class of structures drastically minimizes the weight and space required for transportation. However, substantial research on fabricating and assembling these materials on site is needed to make this a feasible option [2, 3, 4, 5, 6]. Class II structures are currently within the grasp of existing technology and will be pursued and studied here.

Class II structures are divided into four types: inflatable, cable, rigid and underground structures [7]. Inflatable structures will easily collapse if punctured and are thus not suitable for long-term habitation; cable structures are difficult to construct due to the necessary prestressing of the cables in a different gravitational field; and underground construction using existing natural caverns is challenging due to lack of knowledge in the geology of these natural structures and their stability. Rigid structures have been deemed the most practical of these types of structures [7].

Building structures somewhere other than on Earth presents many challenges. The lunar surface has a temperature fluctuation of 250 K, which will cause frequent temperature cycling and can lead to structural fatigue. Meteoroid impact is another factor that must be
considered. Places like the moon have a much thinner atmosphere than earth, and provide no shielding against harmful radiation. To protect the structure from meteoroid collisions and provide radiation shielding, a layer of regolith can be placed on top of the structure to minimize their impact. Consequently, the structure design must account for the weight of this protective layer [8, 9, 6, 10].

The function of a lunar habitat can be primarily expressed in terms of maximizing its livable volume. Its fitness is determined by the structure’s ability to withstand the loads without failure during its design life. These loads include the internal pressurization for the livable areas, the weight of the regolith layer, potential impacts, and the cyclic thermal loading. The cost of this structure is dominated by the transportation costs and the on-site assembly costs. Given the complex combination of loads and the geometry of the available design region (for example, ‘pillow-shaped’), it is not straightforward to design these structures based solely on intuition, therefore we must resort to computational design tools. Topology optimization, and in particular the geometry projection method, are computational design techniques that can greatly aid in exploring design concepts for these structures.

1.2 Objective

The objective of this thesis is to advance a computational method to design rigid space frames in the presence of multiple loads and with consideration for the length of the struts to facilitate packing and thus transportation. This capability is a stepping stone to build a more comprehensive computational framework for the design of lightweight Class II rigid structures that are easy to manufacture and assemble. Specifically, this work advances a topology optimization scheme that designs a 3-dimensional frame structure made of cylindrical bars for stiffness. The simplicity of these structural elements allows for habitats that can be easily manufactured. This work advances necessary capabilities to design these structures, however several important considerations are out of scope and deferred to future work. These include other structural criteria to prevent important failure modes such as plastic deformation, resonance, impact, buckling, thermomechanical fatigue and fracture; and the design-dependent nature of the loading arising from the internal pressure and the weight of the protective layer.

1.3 Contribution of this Thesis

The work presented in this thesis consists of three major contributions. The first is the extension of the geometry projection method developed in [11, 12] for the design of 2-dimensional structures made of flat bars to 3-dimensional structures made of cylindrical struts. In addition, we modify the foregoing formulation to accommodate struts that remain connected at all times during the optimization. The second contribution is the incorporation of an objective function for the design optimization that consists of a weighted sum of the structural compliance for different loading conditions that the habitat may be subjected to. Finally, the third contribution consists of imposing a constraint on the maximum length of any strut in the frame, to ensure that struts can be more easily packed and transported, thus decreasing transportation cost. Each of these contributions consists of two parts: a mathematical
formulation, including the derivation of analytical sensitivities that allow us to use efficient gradient-based optimization algorithms, and the implementation of these formulations into our group’s C++ code, which is built on the deal.II library [13]. We demonstrate these formulations with several numerical experiments.

1.4 Outline

The rest of this thesis is organized as follows. Chapter 2 introduces topology optimization and provides an overview of the approaches available in the literature. In Chapter 3, the geometry projection method is explained in further detail, and the formulation and implementation of the weighted compliance and maximum length constraint are discussed. Chapter 4 presents various examples to demonstrate the proposed formulation. Chapter 5 summarizes this work and draws conclusions, and it discusses future work.
Chapter 2

Review of Existing Approaches

2.1 Topology Optimization

Topology optimization is a method used to generate conceptual designs of structures. It determines the optimal material distribution \( \rho \) in a design region \( \Omega \) to minimize some function of interest \( f(u(\rho), \rho) \) subject to \( M \) resource and/or structural constraints \( g_j(u(\rho), \rho) \). These functions depend on the design \( \rho \) explicitly and/or implicitly through the structural response \( u(\rho) \). In this work, the topology optimization problem is formulated as the following nonlinear program (NLP):

\[
\min_{\rho} f(u(\rho), \rho) \\
\text{subject to} \\
g_j(u(\rho), \rho) \leq 0, \quad j = 1, ..., M \\
0 \leq \rho_i \leq 1, \quad i = 1, ..., N
\]  

(2.1)

In this problem, the material distribution is parameterized via the \( N \)-vector \( \rho \). The type of parameterization employed leads to two prominent classes of methods: the density method and the level set method. Both types of methods employ a fixed finite element mesh for the analysis, which circumvents the need to re-mesh every time the design changes.

2.2 Density Method

In the density method [14], the design parameters are typically element-wise densities, hence there are as many design parameters as elements in the finite element analysis mesh. Each elemental density can take a value of 1 or 0, indicating whether material should be placed in that element or removed from the optimal design, respectively. The ensuing optimization problem is discrete, which makes it impractical to solve given the large number of design parameters. Therefore, in the density method, the elemental densities are relaxed so that they can continuously take any value in between 0 and 1. Nevertheless, the optimal design must have elemental densities that are near zero or near unity because regions with intermediate densities cannot be manufactured, i.e., we want a 0-1 design.

To achieve 0-1 designs, density methods penalize intermediate density values. A widely
used penalization scheme is the solid isotropic material with penalization (SIMP). In this scheme, a power law is used to modify the material’s modulus for element $i$ as

$$E(\rho_i) = \rho_i^s E_0,$$  \hspace{1cm} (2.2)

where $s > 1$ is a penalization power and $E_0$ is the Young’s modulus of the material in the structure. The volume of the structure is computed as

$$V = \sum_i \rho_i v_i,$$  \hspace{1cm} (2.3)

where $v_i$ is the volume of element $i$. Since the elemental density is raised to the power $s$ in Eq. 2.2 but not in Eq. 2.3, the overall effect is that intermediate density values render a poor stiffness-to-volume ratio, as shown in Fig. 2-1, which causes the optimizer to favor 0 or 1 density values. A value of $s = 3$ is typically used in the literature [15, 14].

![Figure 2-1: Penalized density in SIMP method.](image)

To use efficient gradient-based optimization methods in order to solve problem 2.1, design sensitivities of the objective and constraint functions are required. Since the topology optimization problem has a large number of design variables, the adjoint differentiation method is typically used [15]. In the density method, the calculation of these sensitivities is done over the entire design region $\Omega$.

### 2.3 Level Set Method

The level set method defines the boundary of the structure using the zero-level set of a function $\phi$. This function is defined via nodal values in the finite element mesh, hence there are as many design parameters as nodes in the mesh. To perform the analysis, two methods
are primarily used. In ersatz methods, the nodal level set values $\Phi^{(j)}$ of an element $j$ are used to determine a linear approximation of the zero-level set inside that element. Then, an elemental density $\rho_j$ is computed as the volume fraction of the solid portion, and the modulus is computed as $E_j = \rho_j E_0$. Alternatively, the nodal values of $\rho_j$ are simply averaged to obtain $\rho_j$. The second analysis method uses the extended finite element method (XFEM), which introduces additional degrees of freedom inside the elements that intersect the zero-level set, so that the interface is captured exactly (i.e., there are no ‘gray’ regions) [14].

Unlike the density method, design sensitivities of the optimization functions are computed over the boundaries of the structure (i.e., the zero-level set) via the divergence theorem. Then, the design is updated either using nonlinear programming (NLP) methods, or by advancing the location of the zero-level set using the Hamilton-Jacobi equation:

$$\frac{\partial \phi}{\partial t} = -V \mathbf{n} \cdot \nabla \phi$$

$$\mathbf{n} = \frac{\nabla \phi}{|\nabla \phi|}$$

In the expression above, $V(x)$ is the sensitivity of the objective function with respect to an infinitesimal change in the location of the zero-level set at $x$ along the normal vector $\mathbf{n}(x)$. A time step in the solution of this equation corresponds to a design update. In solving this equation, the Courant-Friedrichs-Levy (CFL) condition needs to be satisfied to ensure stability of the numerical solution, which restricts the time step size and therefore how much the design changes at each iteration. This can lead to slower convergence in comparison to using NLP methods.

Unlike density methods, level set methods have a ‘sharp’ representation of the design at all times given by the zero-level set, hence they nearly attain a 0-1 design at every optimization iteration without any penalization. As the boundaries evolve, holes can naturally merge. However, without an external mechanism, they cannot introduce holes in the design.

2.4 Ground Structure Methods

The ground structure method is an early form of topology optimization that works with structural elements such as bars that can be modeled with one-dimensional (two-force or beam) elements. As such, it ensures manufacturability and ease of assembly because it only considers stock material. To construct the initial design, the domain is scattered with uniformly or randomly spaced nodes, and each node is connected to a set of neighboring nodes to form the bar elements. Each element has a uniform cross-sectional area that is a design variable. An element whose area becomes zero in the optimization is essentially removed from the structure [15]. A limitation of this method pertains to the evaluation of stresses, as the one-dimensional elements cannot capture the three-dimensional stress state at joints among structural elements.
2.5 Geometry Projection

In the foregoing methods, the designs can take any form without regard to the manufacturability of the optimal design. In this thesis, we are interested in structures made of stock bars, such as space frames. Therefore, we want the topology optimization to render designs that can be readily made from bars. To this end, we use the geometry projection method \[16, 11, 12, 17\]. In this method, the geometry is parameterized using high-level features, which are then projected onto a fixed finite element mesh for analysis. Instead of the previously described elemental densities or nodal level set values, the design variables are high-level geometric parameters that describe the length, position and orientation of a bar. The projection consists of a smooth map between the design parameters of the bar and a continuous density field. This density field simplifies the primal and sensitivity analyses, as in density methods. Fig. 2-2 shows an example of the geometry projection of a bar onto a fixed grid. A unique feature of the geometry projection method is that it ascribes a size variable to each bar in the frame. This variable is penalized in the same spirit of SIMP, so that if it attains a value of unity in the optimal design it indicates the bar is needed, however if it attains a value of zero it signifies the bar must be removed from the design. This is explained in more detail in the following chapter.

A similar family of methods consists of the Moving Morphable Components (MMC) method \[18, 19\]. The design variables are again high-level geometric parameters that describe the starting point, length, thickness and orientation of each bar. The bars are described by level set functions, and the analysis is performed on a fixed finite element grid using XFEM, which captures the location of the boundary. The use of level sets precludes the possibility of using a penalized size variable for each bar as in geometry projection techniques. Therefore, for a bar to be removed in the MMC method, it has to be engulfed in another bar or collapse to a size small enough that it is not reflected in the analysis.

The geometry projection of a space frame does not suffer from the aforementioned limitation of ground structure methods, because it can capture three-dimensional stress states. Also, it does not require bars to be connected in a predefined pattern, and, in fact, bars can ‘float’ within the design envelope without being connected to other bars at their endpoints. It is also more advantageous than the MMC method in that it can readily remove bars by making their size variables equal zero, which greatly improves convergence.
Chapter 3

Methodology

3.1 Geometry Projection

The geometry projection method \cite{12, 17} takes a parametric representation of the geometry of a structural component (for instance, a primitive) and renders a pseudo density field on a fixed finite element grid. Roughly speaking, the pseudo density at a point is an indication of how much material is present at that point, with zero indicating that no geometric component intersects the point, and unity indicating at least one geometric component intersects the point. To compute the geometry projection at a point \( p \) for a single geometric component, we consider a sampling window of radius of \( r \) and centered at \( p \), which we define as \( B_p^r := \{ x \mid \| p - x \| \leq r \} \). Thus, the sampling window is circular or spherical for 2-d and 3-d problems respectively. The area (or volume) ratio of the portion of \( B_p^r \) that intersects the component corresponds to the projected density at \( p \), i.e.:

\[
\rho(p, r) := \frac{|B_p^r \cap \omega|}{|B_p^r|} \quad (3.1)
\]

where \( \omega \) denotes the domain occupied by the geometric component. We can approximate this ratio as a function of the signed distance from \( p \) to the boundary of the component \( \partial \omega \) if we assume that \( B_p^r \) is small enough that \( \partial \omega \cap B_p^r \) is close to a straight line (in 2-d) or a circle (in 3-d):

\[
\rho^{2D}_q(\phi_q(p, z_q), r) := \begin{cases} 
0 & \text{if } \phi_q > r \\
\frac{1}{\pi r^2} \left[ r^2 \arccos \left( \frac{\phi_q}{r} \right) - \phi_q \sqrt{r^2 - \phi_q^2} \right] & \text{if } -r \leq \phi_q \leq r \\
1 & \text{if } \phi_q < -r 
\end{cases} \quad (3.2)
\]

\[
\rho^{3D}_q(\phi_q(p, z_q), r) := \begin{cases} 
0 & \text{if } \phi_q > r \\
\frac{1}{2} + \frac{\phi_q^2}{4r^2} - \frac{3\phi_q}{4r} & \text{if } -r \leq \phi_q \leq r \\
1 & \text{if } \phi_q < -r 
\end{cases} \quad (3.3)
\]

The signed distance \( \phi_q \) is in turn a function of the parameters \( z_q \) that describe each geometric component \( q \), which are described in the next section. In this work, the radius \( r \) is considered
fixed, corresponding to, for example, the radius of the circle or the sphere that circumscribes the element in 2-d and 3-d, respectively. Consequently, we henceforth omit \( r \) as an argument of the projected density for brevity.

Another important aspect of this formulation is the assignment of a size variable \( \alpha_q \) to each geometric component that determines whether component \( q \) is present in the design. If \( \alpha_q = 1 \), the component is fully present, whereas if \( \alpha_q = 0 \) the component is effectively removed from the design. This size variable is penalized as in the SIMP method described in the previous section to preclude intermediate values. Using this size variable, we define an effective density \( \hat{\rho}_q \) as:

\[
\hat{\rho}_q(p, z_q, s) := \alpha_q \rho_q(\phi_q(p, z_q)) \tag{3.4}
\]

where \( s \) is the penalization power. When \( p \) intersects more than one component, we define a composite density as the maximum of the effective densities at \( p \) of all the components:

\[
\tilde{\rho}(p, Z, s) := \max_q \hat{\rho}_q(p, Z, s) \tag{3.5}
\]

This is equivalent to a Boolean union of the components [20]. In this expression, \( Z \) is the vector of design variables for the entire structure, which is described in detail in the next section. The maximum function is not differentiable, which precludes the use of efficient gradient-based optimization algorithms. Therefore, we replace it with a smooth approximation to ensure differentiability. Here, we use a modified \( p \)-norm [17], so that the composite density is now given by

\[
\tilde{\rho}(p, Z, s) := \left[ \rho_{\text{min}}^p + (1 - \rho_{\text{min}}^p) \sum_{q=1}^{N_q} (\hat{\rho}_q(p, Z, s))^p \right]^{\frac{1}{p}} \tag{3.6}
\]

where \( N_q \) is the total number of geometric components. When all the bar effective densities are \( \hat{\rho}_q = 0 \), the composite density equals a minimum density \( \rho_{\text{min}} \) that prevents an ill-posed analysis. Also, the above expression ensures that the derivative is well defined when all \( \hat{\rho}_q = 0 \), as otherwise there is a division by zero. As in density-based topology optimization, we employ an ersatz material for the analysis, whereby the material properties at a given point are a function of its density. The modified material elasticity tensor is given by:

\[
\mathbb{C}(p, Z, s) := \tilde{\rho}(p, Z, s)\mathbb{C}_0 \tag{3.7}
\]

where \( \mathbb{C}_0 \) denotes the elasticity tensor of the solid material. In the finite element analysis, we assign a uniform \( \mathbb{C}(x_e, z_q, s) \) to element \( e \), where \( x_e \) is the element centroid. Consequently, the composite, effective and projected densities are discretized in an element-wise manner as \( \tilde{\rho}_e \), \( \hat{\rho}_q^e \) and \( \rho_q^e \) respectively. Here and henceforth, the superscript \( e \) denotes an element-wise quantity.
3.2 Connected Bars

In this work, the geometric components are cylindrical bars with semi-spherical ends. Each bar is described by the coordinates of the endpoints of its medial axis, which we denote by \( x_{q_0} \) and \( x_{q_f} \), the size variable \( \alpha_q \), and the bar’s width \( w \). Although \( w \) can be readily considered as a design variable, here we assume it is fixed as we wish to design structures made of stock material with the same width, as this will facilitate packing and transporting of the structural elements. The vector of design variables for bar \( q \) is thus given by

\[
\begin{align*}
z_q := [x_{q_0}, x_{q_f}, \alpha_q]^T
\end{align*}
\]  

Contrary to the previous work by our group [12], in this work bars may be connected at their medial axis endpoints. We denote by \( \chi_i \) the position vector of point \( i \), and thus \( \chi_i \) may correspond to \( x_{q_0} \) or \( x_{q_f} \) for more than one bar. Consequently, a change in \( \chi_i \) will in general affect the effective density \( \hat{\rho}_q \) of more than one bar. The vector of design variables \( Z \) for the entire structure is therefore given by

\[
Z = \left[ \bigcup_{q=1}^{N_q} z_q \right] = [\chi_1, \cdots, \chi_{N_p}, \alpha_1, \cdots, \alpha_{N_q}]^T
\]  

where \( N_p \) is the number of points. The signed distance from point \( p \) to bar \( q \) is given by

\[
\phi_q(p, z_q) = d_q(p, z_q) - \frac{w}{2}
\]  

where \( d_q \) is the unsigned distance from \( p \) to the medial axis \( x_{q_0} \overrightarrow{x_{q_f}} \), and is calculated as:

\[
d_q(p, z_q) = \begin{cases} 
\|b\| & \text{if } a \cdot b \leq 0 \\
\|g\| & \text{if } 0 < a \cdot b < a \cdot a \\
\|e\| & \text{if } a \cdot b > a \cdot a 
\end{cases}
\]

where

\[
\begin{align*}
a & := x_{q_f} - x_{q_0} \\
b & := p - x_{q_0} \\
e & := p - x_{q_f} \\
g & := \left[ I - \frac{1}{\|a\|^2} a \otimes a \right] b
\end{align*}
\]  

with \( I \) the identity matrix. The arguments \( p \) and \( z_q \) have been omitted from these expressions for brevity. These foregoing vectors are shown in Fig. 3-1.

In this work, we refer to bars whose medial axis endpoints are independent from other bars as floating bars, and bars whose medial axis endpoints are shared with other bars as connected bars. As shown Section 3.5, the sensitivity analysis must take into account the shared design variables for connected bars. An example of how endpoints and bars for the
3.3 Weighted Compliance

Structures are often subjected to multiple load cases. A frequently used objective function to account for these multiple loadings is the weighted compliance, which corresponds to a linear combination of the compliance corresponding to each load case and is given by

\[ \theta(Z) := \sum_{i=1}^{N_l} w_i \theta_i = \sum_{i=1}^{N_l} w_i \int_{\Gamma_i} u_i(Z) \cdot t_i \, dv \]  

(3.13)

where for load case \( i \), \( w_i \) is the weight, \( u_i \) is the displacement, \( t_i \) is the surface traction applied on the structural boundary \( \Gamma_i \) (which here we assume to be design-independent), and \( \theta_i = \int_{\Gamma_i} u_i \cdot t_i \, dv \) is the structural compliance. \( N_l \) denotes the number of load cases. The weights are prescribed by the designer according to the importance of the load cases relative to each other.

3.4 Length Constraint

When designing with stock material, it is important to prevent very long members from developing in the optimal design since it may simply not be possible to acquire or transport stock bars longer than a certain value. We ensure this by constraining the maximum length of each and every bar. To avoid the square root and simplify the derivations, we work with the squared length of bar \( q \), given by:

\[ l^2_q := (x_{q_f} - x_{q_o})^T (x_{q_f} - x_{q_o}) \]  

(3.14)

However, a large number of bars will incur in an equally large number of constraints, which will impede convergence and cause the optimization to slow down. To circumvent this problem, we can replace all the length constraints for all bars with a single constraint on the
longest (i.e., maximum length) bar. Since the maximum function does not have a continuous derivative, a \( p \)-norm approximation is used. The maximum of the squared length of all bars is thus approximated as

\[
\ell_{\text{max}}^p := \left[ \frac{1}{N_q} \sum_{q=1}^{N_q} (\alpha_q l_q^2)^p \right]^{\frac{1}{p}}
\]  

(3.15)

In the above expression above, the squared distance for bar \( q \) is weighted with the size variable \( \alpha_q \), because if a bar has \( \alpha_q = 0 \), we need not impose a length constraint on that bar.

The \( p \)-norm approximates the true maximum value from above; the larger the value of \( p \), the closer the approximation to the true maximum. However, if we use too large a value of \( p \), the approximation becomes highly nonlinear. This means that the approximations of this constraint made by gradient-based optimizers that use sequences of approximated problems are only good within a small neighborhood of the current design. Therefore, the optimization would take a large number of iterations to converge as it can only take small design steps at each iteration. As a result of this, in our method we use a value of \( p \) that is not too high (we use \( p = 8 \) in all of our examples).

Since the value of \( p \) is not large, the \( p \)-norm approximation will not be close to the true maximum, which prevents the method from exactly enforcing the length constraint. To address this, we use the adaptive constraint scaling formulation presented by [22]. The idea of this continuation technique is to successively scale the limit of the constraint on the \( p \)-norm during the optimization so that an active \( p \)-norm constraint renders the desired limit on the true maximum. If we denote by \( \tilde{g} \) the approximated constraint and \( b \) the desired limit on the true maximum, the \( p \)-norm constraint is scaled as:

\[
C^{(I)} \tilde{g} \leq b
\]  

(3.16)

where \( C^{(I)} \) is the scaling factor at iteration \( I \) given by:

\[
C^{(I)} := \gamma^{(I)} \frac{g^{(I-1)}}{\tilde{g}^{(I-1)}} + (1 - \gamma^{(I)}) C^{(I-1)}
\]  

(3.17)

The parameter \( \gamma \) is used to control oscillations. A value of \( \gamma^{(I)} = 0.5 \) is chosen if \( C^{(I)} \) oscillates between iterations, otherwise \( \gamma^{(I)} = 1 \).

The length constraint presented in this section can only be readily accommodated by techniques like the geometry projection method, where a high-level parametric representation of the structure is available. This representation makes it possible to directly compute the length of the bars from the design variables. Although this is a simple concept, imposing a length constraint is notably impossible in density-based and level-set methods, where the geometric representation is such that the concept of length of a structural element is simply not available.

### 3.5 Sensitivity Analysis

To employ efficient gradient-based optimizers, we must compute sensitivities of the objective and constraint functions. Since the number of design variables in the optimization (namely,
the geometric parameters and size variables of the bars) is larger than the number of functions in the optimization, we employ the adjoint differentiation method to more efficiently compute the sensitivities of functions that depend on the structural response.

### 3.5.1 Weighted Compliance

We first consider the weighted compliance of Eq. 3.13. A direct application of the chain rule renders

$$\frac{d\theta}{dZ_j} = \sum_{i=1}^{N_1} w_i \frac{d\theta_i}{dZ_j}$$

where $Z_j$ denotes a single design variable. Applying again the chain rule, the sensitivity of each individual compliance $\theta_i$ is given by

$$\frac{d\theta_i}{dZ_j} = \sum_{e=1}^{N_{elem}} \frac{\partial \theta_i}{\partial \rho^e} \frac{\partial \rho^e}{\partial Z_j}$$

(3.19)

The term $\partial \theta_i/\partial \rho^e$ is essentially the same sensitivity computed in density-based topology optimization methods. Using Eq. 3.7 and adjoint differentiation, we can readily obtain the well-known result [15]:

$$\frac{\partial \theta}{\partial \rho^e} = -\frac{1}{\rho^e} \mathbf{u}^e \cdot \mathbf{K}^e \mathbf{u}^e$$

(3.20)

where $\mathbf{K}^e$ and $\mathbf{u}^e$ denote the stiffness matrix and vector of nodal displacements for element $e$, respectively. To obtain the design sensitivity $\partial \rho^e/\partial Z_j$ of the composite density of element $e$ in Eq. 3.19, we differentiate Eq. 3.6 to obtain:

$$\frac{\partial \rho^e}{\partial Z_j} = \begin{cases} s \alpha_q \left( 1 - \rho_{min}^e \right)^{p-1} \frac{\partial \rho_q^e}{\partial Z_j} & \text{if } Z_j = \alpha_q \\ \alpha_q \frac{\partial \rho_q^e}{\partial Z_j} & \text{if } Z_j \in x_{q0} \text{ or } Z_j \in x_{qf} \end{cases}$$

(3.21)

The difference in the sensitivity analysis between floating and connected bars lies in the foregoing expression. For floating bars, $\partial \rho_q^e/\partial Z_j$ is zero when $Z_j \notin z_q$, i.e. when $Z_j$ does not correspond to one of the coordinates of the endpoints of the medial axis of bar $q$, or to the size variable $\alpha_q$. For connected bars, on the other hand, $Z_j$ may correspond to a coordinate of a point $\chi$, that is shared by two or more bars, and therefore we must carry out the sum over all bars as shown above. We find the sensitivities for the effective density by differentiating Eq. 3.4:

$$\frac{\partial \rho_q^e}{\partial Z_j} = \begin{cases} s \alpha_q^{s-1} \rho_q^e & \text{if } Z_j = \alpha_q \\ \alpha_q s \frac{\partial \phi_q^e}{\partial Z_j} & \text{if } Z_j \in x_{q0} \text{ or } Z_j \in x_{qf} \end{cases}$$

(3.22)

Using the chain rule, the design sensitivity of the projected density is subsequently given by

$$\frac{\partial \rho_q^e}{\partial Z_j} = \frac{\partial \rho_q^e}{\partial \phi_q^e} \frac{\partial \phi_q^e}{\partial Z_j}$$

(3.23)
with the derivative of the projected density with respect to the signed distance given by differentiating Eqs. 3.2 and 3.3 for 2-d and 3-d, respectively:

\[
\frac{\partial \rho^e_{q}}{\partial \phi^e_{q}} = \begin{cases} 
0 & \text{if } |\phi^e_q| > r \\
-\frac{2}{\pi r} \sqrt{r^2 - (d^e_q)^2} & \text{otherwise}
\end{cases}
\] (3.24)

\[
\frac{\partial \rho^e_{q}}{\partial \phi^e_{q}} = \begin{cases} 
0 & \text{if } |\phi^e_q| > r \\
\frac{3}{4r} \left[ \left( \frac{d^e_q}{r} \right)^2 - 1 \right] & \text{otherwise}
\end{cases}
\] (3.25)

Since they only differ by subtraction of a constant, the design sensitivities of the unsigned and signed distance are equivalent (cf., Eq. 3.10):

\[
\frac{\partial \phi^e_q}{\partial Z_j} = \frac{\partial d^e_q}{\partial Z_j}
\] (3.26)

For bar \( q \), \( Z_j \in z_q \), that is, \( Z_j \) may correspond to one of the coordinates of the bar’s medial axis endpoints \( x_{q_0} \) and \( x_{q_f} \), or to the bar size variable \( \alpha_q \). The corresponding design sensitivities of the unsigned distance of Eq. 3.11 are obtained as:

\[
\frac{\partial d^e_q}{\partial x_{q_0}} = \begin{cases} 
-\frac{b}{\|b\|} & \text{if } a \cdot b \leq 0 \\
\frac{1}{\|g\|} \left[ \frac{a}{\|a\|} - I \right] g & \text{if } 0 < a \cdot b < a \cdot a \\
0 & \text{if } a \cdot b > a \cdot a
\end{cases}
\] (3.27)

\[
\frac{\partial d^e_q}{\partial x_{q_f}} = \begin{cases} 
0 & \text{if } a \cdot b \leq 0 \\
\frac{a}{\|g\|} g & \text{if } 0 < a \cdot b < a \cdot a \\
-\frac{e}{\|e\|} & \text{if } a \cdot b > a \cdot a
\end{cases}
\] (3.28)

\[
\frac{\partial d^e_q}{\partial \alpha_q} = 0
\] (3.29)

We provide the latter equation for completeness, however this quantity is never computed in the code.

### 3.5.2 Volume Fraction

The volume fraction constraint is defined as the fraction of the sum of the volume of the structure over the volume of the entire design region \( \Omega \):

\[
v_f := \frac{1}{|\Omega|} \int_{\Omega} \tilde{\rho}(p, Z, s) \, dv = \frac{1}{|\Omega|} \sum_{e=1}^{N_{\text{elem}}} \tilde{\rho}^e v^e
\] (3.30)

where \( v^e \) is the element volume. Similarly to the sensitivity of the weighted compliance, the design sensitivity of the volume fraction is obtained using the chain rule:
\[
\frac{\partial v_f}{\partial Z_j} = \sum_{e=1}^{N_{\text{elem}}} \frac{\partial v_f}{\partial \tilde{\rho}^e} \frac{\partial \tilde{\rho}^e}{\partial Z_j} = \frac{1}{|\Omega|} \sum_{e=1}^{N_{\text{elem}}} v_e \frac{\partial \tilde{\rho}^e}{\partial Z_j} \tag{3.31}
\]

The term \( \partial \tilde{\rho}^e / \partial Z_j \) is computed as before (cf., Eq. 3.21).

### 3.5.3 Length Constraint

The length constraints depend explicitly on the geometric design parameters and thus require no adjoint finite element solution. The length constraint sensitivities are directly derived from Eq. 3.15 using the chain rule:

\[
\frac{\partial l_{\text{max}}^2}{\partial Z_j} = \frac{1}{N_q(l_{\text{max}}^2)^{p-1}} \sum_{q=1}^{N_q} (\alpha_q l_q^2)^{p-1} \frac{\partial}{\partial Z_j} (\alpha_q l_q^2) \tag{3.32}
\]

where

\[
\frac{\partial (\alpha_q l_q^2)}{\partial x_{q_0}} = -2\alpha_q (x_{q_0} - x_{q_0}) \\
\frac{\partial (\alpha_q l_q^2)}{\partial x_{q_f}} = 2\alpha_q (x_{q_0} - x_{q_0}) \\
\frac{\partial (\alpha_q l_q^2)}{\partial \alpha_q} = l_q^2 \tag{3.33}
\]

### 3.6 Optimization Problem

The optimization problem we consider in this work consists of minimizing the weighted compliance of the structure subject to a constraint on the volume fraction of the structure and to bounds on the design variables, and is given by:

\[
\min_{\theta(Z)} \theta(Z) \tag{3.34}
\]

subject to

\[
v_f(Z) - v_f^* \leq 0 \\
C(l)^2 - l^2 \leq 0 \\
\chi_k \in \Omega, \quad k = 0, \ldots, N_p \\
0 \leq \alpha_q \leq 1
\]

where \( v_f^* \) is the limit on the volume fraction. As in the SIMP method, we note that for the penalization of the bar size variables to be effective, it is necessary that the penalization powers of Eq. 3.4 used in the ersatz material model of Eq. 3.7 is higher than the penalization power used to compute the volume of Eq. 3.30. Here, as customary in the SIMP method, we use \( s = 3 \) and \( s = 1 \), respectively. The length constraint limit is denoted as \( l^2 \). For
simplicity, in this work we only consider cuboid design domains, and so to ensure that bars
entirely remain within the design domain, we simply enforce bounds on the points $\chi_k$. We
note, however, that it is possible to consider general-shape, non-convex design domains in
geometry projection methods [23].

To improve convergence of the optimization, and as in previous works [12, 17], we scale
the design variables so that the scaled variables $\hat{Z}_j$ lie within the range $[0, 1]$, and we impose a
move limit $M$ on each scaled variable. For the points $\chi_k$, we scale the coordinates by dividing
by the respective dimensions of the design regions; the variables $\alpha_q$ need not scaling. The
move limits are thus imposed by modifying the lower and upper bounds of the scaled variables
at iteration $I$ as

$$
\max(\hat{Z}_j - M, \hat{Z}_{j_{\text{lower}}}) \leq \hat{Z}_j \leq \max(\hat{Z}_j + M, \hat{Z}_{j_{\text{upper}}})
$$

(3.35)

where $\hat{Z}_{j_{\text{lower}}}$ and $\hat{Z}_{j_{\text{upper}}}$ correspond to the lower and upper bounds on the design variables
in problem 3.34.
Chapter 4

Examples

In this chapter we present several examples to demonstrate our formulation. In all examples, the bars are made of an isotropic, homogenous and linearly elastic material with Young’s modulus of 1E5 and Poisson’s ratio of 0.3. The design domain for all examples is a rectangular prism that is discretized for analysis using a uniform mesh of cube (hexahedral) elements created in the free software Gmsh [24]. A minimum density $\rho_{\text{min}} = 1E^{-3}$ is used in Eq. 3.7. A size variable value of $\alpha_q = 0.5$ is assigned to all bars in the initial design.

To perform the finite element analysis, we use the deal.II library [13]. The library’s capabilities allow us to parallelize our code using the message passing interface (MPI) and the numerical linear algebra routines in PETSc [25, 26, 27]. The solution of the system of linear equations arising from the finite element formulation is performed using an iterative solver (pre-conditioned conjugate gradient). The pre-conditioning of the global stiffness matrix is achieved using the algebraic multigrid (AMG) preconditioner available in deal.II. The iterative solution is stopped when the magnitude of the residual is $\|r\| \leq 1E^{-7}$ or when the number of iterations exceeds $1E^6$ (the latter did not occur in any of the examples). It is worth noting that we found in our numerical experiments that the choice of preconditioner is very important for this problem, because the sparse 3-d structures produced by our method tend to render highly ill-conditioned stiffness matrices. We initially employed a Jacobi preconditioner, but later found that the AMG preconditioner consistently performs better, and therefore we use it for all the examples in this work.

To perform the optimization, we use the method of moving asymptotes (MMA) [28, 29, 30]. We use a move limit of $M = 0.05$ in Eq. 3.35 for all problems. The optimization is stopped when the magnitude of the relative change between objective function values in consecutive iterations is less than $1E^{-3}$. We note that, as with any optimizer, it is important (and ideal) to have a set of parameters for the optimization method that works consistently well for all problems. We found that the default MMA parameters reported in [30] led to good results and therefore we used those in our examples.

4.1 Vertical Beam with No Length Constraints

Our first example consists of a vertical beam subject to multiple loads. The dimensions of the beam are $8 \times 4 \times 4$. We consider three different loading scenarios with two, three and four
concentrated loads applied at various locations of the beam, as shown in Fig. 4-1. Force 3 is distributed over the top face. Loading scenario 1 has forces 1 and 2; scenario 2 has forces 1-3; and scenario 3 has all forces. All the loading scenarios share the same displacement boundary conditions, corresponding to a fixed bottom face. Forces 1 and 2 have a magnitude of 10, force 3 has a magnitude of 5 and forces 4, which simulate a torsion load, have a magnitude of 10. The mesh is made of 320,769 elements of size 0.075. We employ equal weights for the weighted compliance of Eq. 3.13 in each of three loading scenarios. The bars for this example are connected, and we do not impose constraints on their lengths.

![Figure 4-1: Forces and displacement boundary conditions for the vertical beam design example.](image)

4.1.1 20 Bars

We choose an arbitrary initial design with 20 bars, shown in Fig. 4-2. All bars have a fixed width of \( w = 0.25 \). The volume fraction constraint limit is \( v_f^* = 0.05 \). The results of the optimization are shown in Figs. 4-3 and 4-4 using ParaVIEW and FreeCAD, respectively. The former corresponds to the composite density of Eq. 3.6, and the latter to the CAD representation of the bars. The three loading scenarios result in different designs, as expected. In general, the optimization tends to create triangles throughout the structure, which is expected for stiff frames. As seen in the different loading scenarios, the added forces cause the optimizer to redistribute material to more efficiently transfer the applied loads. The history plots in Fig. 4-5 for all the load combinations show the optimization converges and the volume fraction constraint is inactive for all the designs presented, meaning these designs are feasible. The weighted compliance and volume fraction values for the optimal designs for the different loadings are: a) \( \theta = 0.6921, \ v_f = 0.0273 \) for scenario 1; b) \( \theta = 58.85, \ v_f = 0.0328 \) for scenario 2; and \( \theta = 7.4256, \ v_f = 0.0357 \) for scenario 3. The weighted compliance for the second loading scenario is much larger than for third scenario; we posit this is due to the optimization prematurely locking into a poor local minimum. A recent
strategy to effectively escape poor local minima in geometry projection methods has been introduced in [31]; this is, however, outside of the scope of this work. It is also worth noticing that, unlike density-based topology optimization, due to the more restrictive representation of the design in our method, it is possible that the volume fraction constraint is inactive, as discussed in [12].

Figure 4-2: Initial design of 20-bar vertical beam example in a CAD representation.

Figure 4-3: Composite density iso-surface ($\tilde{\rho} = 0.5$) of optimal design of the 20-bar vertical beam structure under loading scenarios 1 to 3 (from left to right).
4.1.2 60 bars

We perform the same design, but this time with 60 bars in the design. The initial design is shown in Fig. 4-6. The composite density iso-surface ($\tilde{\rho} = 0.5$) and the CAD geometry of the optimal designs for all three loading scenarios are shown in Figs. 4-7 and 4-8, respectively. The weighted compliance and volume fraction values for the optimal designs for the different loadings are: a) $\theta = 0.2858$, $v_f = 0.05$ for scenario 1; b) $\theta = 0.2899$, $v_f = 0.05$ for scenario 2; and $\theta = 1.0443$, $v_f = 0.05$ for scenario 3. As expected, having more bars renders stiffer designs. The iteration histories for these optimization solutions are shown in Fig. 4-9. The volume fraction constraint in these cases is active, due to the design representation being less restrictive than in the previous example as we have more available bars.

4.2 Horizontal Beam with Length Constraints

This example is a horizontal cantilever with two loading scenarios applied as shown in Fig. 4-10, each corresponding to a single force. Forces 1 and 2, with a magnitude of 10, correspond to loading scenarios 1 and 2, respectively. The displacement boundary conditions, which are the same for both loadings, correspond to a fixed face as shown in the figure. The dimensions of the cantilever are $4 \times 4 \times 16$. Three different length constraints are applied to this problem, and we also solve it with no length constraint for comparison. The mesh is made of 132,104 elements of size 0.125. We employ equal weights for the weighted compliance as before. The applied length constraints are 4, 8 and 16 (corresponding to squared values of $l^2$ of 16, 64 and 256 in Eq. 2.1, respectively). We choose an initial design made of 40 floating bars with near-zero length (i.e., near-spheres), shown in Fig. 4-11. All bars have a fixed width of $w = 0.5$. The volume fraction constraint limit is $v_f^* = 0.06$.

The corresponding composite density iso-surfaces are presented in Fig. 4-12 and their CAD representation counterparts are shown in Fig. 4-13. The weighted compliance values
Figure 4-5: History plots of the 20-bar structure under the different loading scenarios.
Figure 4-6: CAD representation of initial design of 60-bar vertical beam example.

Figure 4-7: Composite density iso-surface ($\bar{\rho} = 0.5$) of optimal design of the 60-bar vertical beam structure under loading scenarios 1 to 3 (from left to right).
for the optimal designs for the different length constraints are: a) $\theta = 12.5867$ for $\sqrt{l^2} = 4$; b) $\theta = 11.7866$ for $\sqrt{l^2} = 16$; c) $\theta = 11.8063$ for $\sqrt{l^2} = 64$; and d) $\theta = 11.7916$ without a length constraint. All designs attain a tight volume fraction constraint of $v_f = 0.06$. The design without a length constraint is clearly stiffer than the design with the shortest length constraint ($\sqrt{l^2} = 4$). However, the design with $\sqrt{l^2} = 16$ is actually slightly stiffer than both the design with $\sqrt{l^2} = 64$ and the unconstrained design. We posit this is due to the latter two converging prematurely to suboptimal local minima.

As can be seen in the history plots of Fig. 4-14, the length constraint is active for the length constraints of 4 and 8, but it is inactive for the constraint with a maximum bar length of 16. This means the proposed formulation to incorporate the length constraints, including the adaptive constraint scaling, are successful in enforcing the length limits on the bars.

One interesting note, is that due to the more restrictive design representation in geometry projection techniques, the optimal designs may not be symmetric even if the design region and the loading and boundary conditions are [12]. This is consistent with similar findings in the design of truss structures using 1-dimensional elements [32]. To ensure symmetry in the design, it would be necessary to impose symmetry constraints, as demonstrated in [33, 21]; this is outside of the scope of the current work. Nevertheless, it is interesting to note that the obtained designs are somewhat symmetric with respect to the parallelepiped diagonal plane on which the resultant of the applied forces for both loadings lie.
Figure 4-9: History plots of the 60-bar structure under the different loading scenarios.
Figure 4-10: Forces and displacement boundary conditions for the design example with length constraints.

Figure 4-11: Initial design for the horizontal beam with length constraints.
Figure 4-12: Composite density iso-surface ($\tilde{\rho} = 0.5$) of the optimal horizontal beam for different length constraints.

Figure 4-13: CAD representation of the optimal horizontal beam for different length constraints.
Figure 4-14: History plots of horizontal beam design for different length constraints.

(a) $\sqrt{l^2} = 4$

(b) $\sqrt{l^2} = 8$

(c) $\sqrt{l^2} = 16$

(d) No length constraint
Chapter 5

Conclusions and Future Work

This thesis makes contributions to the geometry projection method presented in [12]. These contributions include 1) the ability to design frame structures whose bars are connected for any value of the design variables, 2) consideration for a weighted compliance response function, and 3) the formulation of a constraint on the maximum bar length. These formulations were implemented and tested in our group’s research code. The examples presented in this work demonstrate the effectiveness of the proposed method. Although lunar habitats were not designed in this thesis, the contributions of this work make the geometry projection method more suitable for designing those structures. Having connected bars renders frame structures that are more easily manufactured, as floating bars can lead to overlaps that cannot be readily fabricated. Constraints on the bar lengths are also very important, as the transport of these elements to the moon imposes limits on the dimensions of stock material. Finally, considering weighted compliance is important as these habitats are subjected to multiple loadings.

One important consideration to be made in the future is to account for design-dependent distributed loads. This is important in the design of lunar habitats, as distributed loads can simulate the addition of a regolith layer on top of the habitat for radiation shielding. Another important design-dependent loading corresponds to the internal pressurization of the habitat necessary for human occupation. In addition to stiffness, the design of these habitats must also account for other potential failure modes such as buckling, plastic deformation, crack initiation, etc. Therefore, the optimization needs to incorporate constraints that account for relevant structural responses that can ensure the structure is fit with regards to these failure modes. Another important consideration in the design of lunar habitats is the loads induced by the large temperature changes. Finally, there must be a consideration for how to join these bars and how to anchor the habitat to the lunar surface.
Appendix

In this Appendix we provide an example of the text input files required by our code to define the initial design in the optimization. To this end, our code reads an ASCII file defining the bars and another one defining the points. To illustrate the syntax for these input files, we provide the files corresponding to the design shown in Fig 5-1.

The file Points.txt contains the information about the points that make up the nodes of the frame structure, and is shown in Fig. 5-2. The first line contains the number of points, and every line afterwards contains a space-separated list of the node ID (integer) and its corresponding coordinates in the global coordinate system. Comments are allowed at the end of any line preceded by the # symbol.

The file Bars.txt contains the information about the bars, and is shown in Fig. 5-3. The first line contains the number of bars, and every subsequent line is a space-separated list with the bar ID, the IDs of the points (in the Points.txt file) that correspond to the endpoints of the medial axis of the bar, the initial value of the bar’s size variable, and the value of its width.
Figure 5-1: CAD representation of the example geometry input files.

```plaintext
14  # number of points
0 0 0 0  # point id number, x, y, z
1 10 0 0
2 10 10 0
3 0 10 0
4 0 0 10
5 10 0 10
6 10 10 10
7 0 10 10
8 0 5 10
9 0 0 14
10 0 10 14
11 0 0 18
12 0 5 18
13 0 10 18
```

Figure 5-2: Input file example `Points.txt` with point coordinates for design shown in Fig. 5-1
```
19  #number of bars
0 0 4 0.5 0.5  #bar number, point id x0, point id xf, alpha, width
1 1 5 0.5 0.5
2 2 6 0.5 0.5
3 3 7 0.5 0.5
4 4 5 0.5 0.5
5 5 6 0.5 0.5
6 6 7 0.5 0.5
7 7 8 0.5 0.5
8 8 4 0.5 0.5
9 4 9 0.5 0.5
10 8 9 0.5 0.25
11 8 10 0.5 0.25
12 7 10 0.5 0.5
13 10 13 0.5 0.5
14 10 12 0.5 0.25
15 9 12 0.5 0.25
16 9 11 0.5 0.5
17 11 12 0.5 0.5
18 12 13 0.5 0.5
```

Figure 5-3: Input file example `Bars.txt` with bar connectivity for design shown in Fig. 5-1
Bibliography


