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Dimension Theory of Conformal Iterated Function Systems

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Abstract

This thesis is an expository investigation of the conformal iterated function system (CIFS) approach to fractals and their dimension theory. Conformal maps distort regions, subject to certain constraints, in a controlled way. Let $\mathcal{S} = (X, E, \{\phi_e\}_{e \in E})$ be an iterated function system where X is a compact metric space, E is a countable index set, and $\{\phi_e\}_{e \in E}$ is a family of injective and uniformly contracting maps. If the family of maps $\{\phi_e\}_{e \in E}$ is also conformal and satisfies the open set condition, then the distortion properties of conformal maps can be extended to the system $\mathcal{S}$. The behavior of the system can be modeled via thermodynamic formalism, which introduces notions such as the topological pressure and the Perron-Frobenius operator. Both are critical to developing numerical approximations for the dimension of the limit set of the system. Finally, we provide examples of fractals which are well-described by the CIFS framework.

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Contents

Chapter 1

An Overview of Conformal Maps in Euclidean Space

I could tell you how many steps make up the streets... and the degree of the arcades' curves ... but I already know this would be the same as telling you nothing. The city does not consist of this, but of relationships between the measurements of its space and the events of its past ...
∼ Italo Calvino, "Invisible Cities"

Within math, conformal maps appear frequently in analysis and geometry, and even underlie important structures in other areas such as number theory. Outside of math, conformal maps appear as stereographic projections in cartography and photography. They are abundant in physics – in studies of general relativity, electrostatics, aerodynamics, and fluid flow.

We are interested in conformal maps in the context of fractals and fratcal-like systems that can be generated by countable iterations of conformal maps. In this section, we introduce conformal maps, and see exactly what the conformal maps in Euclidean space are. We also provide an overview of distortion properties of conformal maps, which are later critical to characterizing the behavior of conformal iterated function systems. As this section is meant as a prelude to our discussion of these systems, we provide a general overview of results which we state without proof. For more details, see Chapter 7 of [2].

1.1 Definitions and Properties

Although there are many situations where conformal maps may be considered in general and non-Euclidean spaces, for our purposes we consider conformal maps in $\mathbb{R}^n$. We denote the usual inner product for $x, y \in \mathbb{R}^n$ as $\langle x, y \rangle = \sum_{i=1}^n x_i y_i$, and the Euclidean norm as $|x| := \langle x, x \rangle^{1/2}$. We denote the angle between $x, y \in \mathbb{R}^n \setminus \{0\}$ by $\angle(x, y) \in [0, \pi]$. For a function $f : U \to f(U) \in \mathbb{R}^n$ that is differentiable at $x \in U \subset \mathbb{R}^n$, we denote $Df(x)$ as its derivative at x.

For $x, y \in \mathbb{R}^n$, we record the following relation between $\angle(x, y)$, $|x|$, and $\langle x, y \rangle$:

$$\angle(x, y) = \arccos\left(\frac{\langle x, y \rangle}{|x||y|}\right). \tag{1.1}$$

Definition 1.1.1. A nonsingular linear map is *conformal* if it preserves angles between vectors. In other words, $T : \mathbb{R}^n \to \mathbb{R}^n$ conformal if

$$\angle(x, y) = \angle(T(x), T(y)) \text{ for all } x, y \in \mathbb{R}^n \setminus \{0\}. \tag{1.2}$$

The following proposition highlights some important properties of linear conformal maps.

Proposition 1.1.1. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. Then the following are equivalent,*

1. *T is conformal,*
2. *There exists some $\lambda > 0$ such that $\langle T(x), T(y) \rangle = \lambda \langle x, y \rangle$,*
3. *There exists some $\lambda > 0$ such that $|T(x)| = \lambda |x|$,*

Recall that a linear map is *orthogonal* if it preserves inner products. Then, the equivalence (1) $\iff$ (2) can be reformulated as: A linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ is conformal if and only if there exists an orthogoanl transformation $O : \mathbb{R}^n \to \mathbb{R}^n$ and a $\lambda > 0$ such that $T(x) = \lambda O(x)$.

Before defining conformal maps in $\mathbb{R}^n$, we recall the following definition.

Definition 1.1.2. Let $U \subset \mathbb{R}^n$ be open, and $k \in \mathbb{N}$. Then a bijection $f : U \to f(U) \subset \mathbb{R}^m$ is a C^k-*diffeomorphism* if f and f^{-1} are of class C^k.

For a conformal map, we consider a C^1-diffeomorphism $f : U \to \mathbb{R}^n$ for an open set U. We say f is conformal at a point if its derivative map at that point is a linear conformal map. Formally, we have the following definition.

Definition 1.1.3. Let $U \subset \mathbb{R}^n$ be an open set. A C^1-diffeomorphism $f : U \to \mathbb{R}^n$ is called *conformal* at $x \in U$ if $Df(x) : \mathbb{R}^n \to \mathbb{R}^n$ is a linear conformal map. A map is conformal at U if it is conformal at all $x \in U$.

From Proposition 1.1.1, it follows that a C^1-diffeomorphism $f : U \to \mathbb{R}^n$ is conformal if and only if

1. There exists a function $h : U \to \mathbb{R}$ such that for all $x \in U$,
$$\langle Df(x)(u), Df(x)(v)\rangle = e^{h(x)}\langle u, v\rangle \text{ for all } x, y \in \mathbb{R}^n,$$
2. For all $x \in U$ there exist orthogonal transformations O_x and parameters $\lambda_x > 0$ such that
$$Df(x) = \lambda_x O_x.$$

Definition 1.1.4. Let U be an open set, and suppose that $f : U \to \mathbb{R}^m$ is differentiable at $x \in U$. The *operator norm* of the linear map $Df(x)$ is

$$\begin{aligned} \|Df(x)\| &:= \sup\{|Df(x)(p)| : p \in \mathbb{R}^n, |p| \le 1\} \\ &= \sup\{|Df(x)(p)| : p \in \mathbb{R}^n, |p| = 1\} \\ &= \sup\left\{\frac{|Df(x)(p)|}{|p|} : p \in \mathbb{R}^n \setminus \{0\}\right\}. \end{aligned} \tag{1.3}$$

Note that the operator norm here is defined exactly as the standard operator norm for a general linear map.

Using this definition, we may formulate the following proposition relating the operator norm of $Df(x)$ to the inner product. The proposition also concerns two conformal maps, f and g, for which the target space of f is contained in the domain of g.

Proposition 1.1.2. *Let $U, V \subset \mathbb{R}^n$ be two open sets and let $f : U \to \mathbb{R}^n$ and $g : V \to \mathbb{R}^n$ be two conformal maps such that $f(U) \subset V$. Then $g \circ f : U \to \mathbb{R}^n$ is conformal and for all $x \in U$ and $p, q \in \mathbb{R}^n$,*

1. $|Df(x)(p)| = \|Df(x)\| \, |p|$,
2. $\langle Df(x)(p), Df(x)(q)\rangle = \|Df(x)\|^2 \langle p, q\rangle$,
3. $|\det J_{f(x)}| = \|Df(x)\|^n$,
4. $\|D(g \circ f)(x)\| = \|Dg(f(x))\| \, \|Df(x)\|$.

Proposition 1.1.3. *Let $U \subset \mathbb{R}^n$ be an open set, and $f : U \to f(U) \subset \mathbb{R}^n$ be a conformal map. Then $f^{-1} : f(U) \to U$ is also conformal.*

1.2 The Conformal Maps in $\mathbb{R}^n$

A natural question might be what the conformal maps in $\mathbb{R}^n$ actually are. It turns out that in $\mathbb{R}$, every C^1- diffeomorphism is conformal: this follows from Proposition (1.1.1). In $\mathbb{R}^2$, we have the following result.

Theorem 1.2.1. *Let $U \subset \mathbb{R}^2$ be an open set. A C^1-diffeomorphism $f : U \to \mathbb{R}^2$ is conformal if and only if f is holomorphic or antiholomorphic.*

For $n \geq 3$, Liouville's Theorem shows that every conformal map is of a particular form.

Theorem 1.2.2. *Let $n \geq 3$, and suppose that $U \subset \mathbb{R}^n$ is open and connected. Then every conformal map is a Möbius transformation; i.e. a complex map of the form*

$$f(z) = \frac{az + b}{cz + d}$$

where $ad - bc = 0$.

1.3 Distortion Theorems

We now state a number of theorems and propositions that reveal how conformal maps "distort" regions in Euclidean space. These results are critical in proofs of results in the following chapter, where we consider families of conformal maps, and countable iterations of these maps over sets.

Theorem 1.3.1 (Koebe Distortion Theorem)**.** *There exists a monotone increasing continuous function $K : [0,1) \to [1,\infty)$ such that $K(0) = 1$ and with the following property. If $w \in \mathbb{C}, R > 0$ and $f : B(w,R) \to \mathbb{C}$ is an arbitrary univalent analytic function, then*

$$\left| \frac{|f'(z)|}{|f'(w)|} - 1 \right| \leq K(r/R)|z - w| \tag{1.4}$$

for every $r \in [0,R]$ and for all $z \in \bar{B}(w,r)$.

The Koebe Distortion Theorem, together with the results stated in the previous section that characterize all the conformal maps in $\mathbb{R}^n$, prove the following distortion result for conformal maps.

Proposition 1.3.1. *Let $U \subset \mathbb{R}^n, n \geq 2$, be open and connected and let $S \subset U$ be a compact and connected subset of U. If $f : U \to \mathbb{C}$ is conformal then*

$$\left| \frac{\|Df(x)\|}{\|Df(y)\|} - 1 \right| \leq K|x - y| \text{ for all } x, y \in S, \tag{1.5}$$

where K depends only on S and dist(S, U^c).

The following inequality is an immediate corollary of the previous proposition.

Proposition 1.3.2 (Harnack Inequality for conformal maps)**.** *Let $U \subset \mathbb{R}^n, n \geq 2$, be open and connected and let $S \subset U$ be a compact and connected subset of U. If $f : U \to \mathbb{C}$ is conformal then*

$$\|Df(x)\| \leq K\|Df(y)\| \text{ for all } x, y \in S, \tag{1.6}$$

where K depends only on S and dist(S, U^c).

The next distortion theorem establishes that, so long as a ball is separated from the boundary of a domain, a conformal map distorts the ball in a controlled way.

Theorem 1.3.2 (Egg Yolk Principle for conformal maps)**.** *Let $U \subset \mathbb{R}^n, n \geq 2$, be an open and connected set and let $f : U \to \mathbb{R}^n$ be a conformal map. Let also $S \subset U$ be a compact and connected set. If $B(p,r) \subset S$ then*

$$B(f(p), K^{-1}\|Df\|_S r) \subset f(B(p,r)) \subset B(f(p), \|Df\|_S r), \tag{1.7}$$

where K depends only on S and dist(S, U^c).

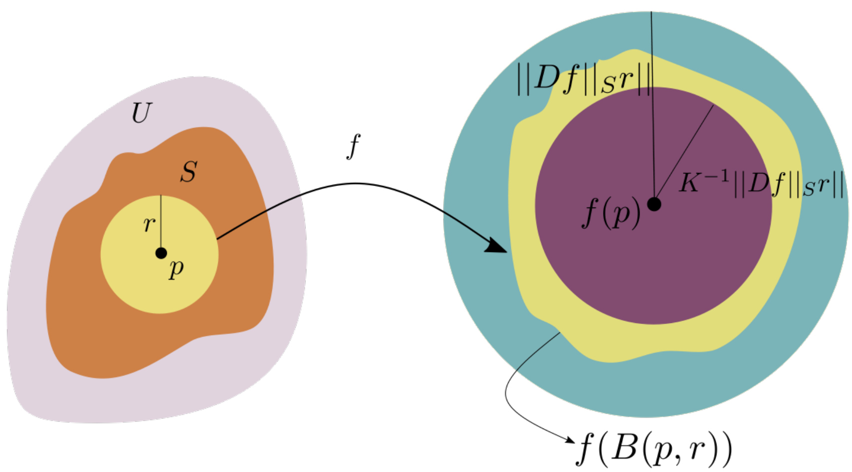

Figure 1.1: The Egg Yolk Principle.

Corollary 1.3.1. *Let $U \subset \mathbb{R}^n, n \geq 2$, be an open and connected set and let $f : U \to \mathbb{R}^n$ be a conformal map. Suppose that there exist sets*

$$X \subset Int(S) \subset S \subset U$$

such that X is compact and S is compact and connected. Then, if $\eta = dist(X, S^c)$, for all $p \in X$ and for all $r < \eta$ it holds that

$$B(f(p), K^{-1}\|Df\|_S r) \subset f(B(p,r)) \subset B(f(p), \|Df\|_S r), \tag{1.8}$$

where K depends only on S and dist(S, U^c).

Remark 1.3.1. In Theorem 1.3.2 (and Corollary 1.3.1) for $n = 1$, it can be seen that if we also include the assumption there exists some constant $K \geq 1$ such that $|f'(x)| \leq K|f'(y)|$ for all $x, y \in S$, then (1.7) continues to hold. This can be seen from the proof of Theorem 1.3.2, which uses the fact that conformal maps in $\mathbb{R}^n, n \geq 2$, automatically satisfy Harnack's inequality away from their singularities.

Remark 1.3.2. Under the assumptions of Theorem 1.3.2, (1.6) and (1.7) imply that for all $p \in X$ and for all $r < \eta$ it holds that

$$B(f(p), K^{-1}\|Df\|_X r) \subset f(B(p,r)) \subset B(f(p), K\|Df\|_X r). \tag{1.9}$$

Chapter 2

Conformal Iterated Function Systems

This is a landscape so dynamic that its very changeability leads to innumerable moments of recognition.
∼Amitav Ghosh, "The Great Derangement"

2.1 Iterated Function Systems

Definition 2.1.1. A (countable) *iterated function system* (IFS) $\mathcal{S} = (X, E, \{\phi_e\}_{e \in E})$ consists of

1. A compact metric space (X, d),
2. A countable index set E,
3. A family $\{\phi_e : X \to X\}_{e \in E}$ of injective and *uniformly contracting maps*; that is there exists some $s < 1$ such that

$$d(\phi_e(x), \phi_e(y)) \leq s d(x, y) \text{ for all } e \in E, x, y \in X.$$

Let ω be a word in E^n. Consider ϕ_ω, the map coded by ω :

$$\phi_\omega := \phi_{\omega_1} \circ \cdots \circ \phi_{\omega_n} : X \to X. \tag{2.1}$$

Observe that

$$d(\phi_\omega(x), \phi_\omega(y)) \leq s^n d(x, y), \tag{2.2}$$

because we have a composition of n uniformly contracting maps. Suppose that we have an infinite word $\omega \in E^{\mathbb{N}}$. Then the family

$$\{\phi_{\omega|_n}(X)\}_{n=1}^{\infty}$$

is a descending sequence of non-empty compact sets, in the sense of inclusion. Thus, it follows from Cantor's Intersection Theorem that

$$\bigcap_{n\in\mathbb{N}} \phi_{\omega|_n}(X) \neq \emptyset.$$

Furthermore, since $\mathrm{diam}(\phi_{\omega|_n}(X)) \overset{(2.2)}{\leq} s^n \mathrm{diam}(X), \quad$ for all $n \in \mathbb{N}$, we may conclude that for all infinite words $\omega \in E^{\mathbb{N}}$ the intersection

$$\bigcap_{n\in\mathbb{N}} \phi_{\omega|_n}(X)$$

is a singleton, which we denote by $\pi(\omega)$. From this, we define the coding map as follows.

Definition 2.1.2. The *coding map*, or *natural projection*, is the map $\pi : E^{\mathbb{N}} \to X$, where $\pi(\omega) = \bigcap_{n\in\mathbb{N}} \phi_{\omega|_n}(X)$ for $\omega \in E^{\mathbb{N}}$.

We may also consider the union of all natural projections associated to infinite words on E, which we define to be the limit set, or attractor, of the IFS.

Definition 2.1.3. The *limit set* of the IFS $\mathcal{S}$ is the set

$$J = J_{\mathcal{S}} := \pi(E^{\mathbb{N}}) = \bigcup_{\omega\in E^{\mathbb{N}}} \pi(\omega) = \bigcup_{\omega\in E^n} \bigcap_{n\in\mathbb{N}} \phi_{\omega|_n}(X).$$

Proposition 2.1.1. *The limit set is invariant with respect to $\mathcal{S}$.*

Proof. We will show that

$$J = \bigcup_{e \in E} \phi_e(J).$$

If $x \in J$, then there exists a word $\omega \in E^{\mathbb{N}}$ such that $x = \pi(\omega) = \cap_{n \in \mathbb{N}} \phi_{\omega|_n}(X)$. Then

$$x = \phi_{\omega_1}(\cap_{n \geq 1} \phi_{\sigma(\omega)|_n}(X)) = \phi_{\omega_1}(\pi(\sigma(\omega))),$$

where $\sigma : E^{\mathbb{N}} \to E^{\mathbb{N}}$ is the shift map given by

$$\sigma\left((\omega_n)_{n=1}^{\infty}\right) = \left((\omega_{n+1})_{n=1}^{\infty}\right).$$

Thus $x \in \phi_{\omega_1}(J)$. For the other direction, suppose $x \in \cup_{e \in E} \phi_e(J)$. Then there exists some $e \in E$ and $\omega \in E^n$ such that $x = \phi_e(\omega) = \pi(e\omega) \in J$. □

Remark 2.1.1. Since the limit set J is contained in the metric space X, it is always bounded. Furthermore, if the index set E is finite, J is compact (see Chapter 6, [2]). Note, however, that this is not the case when E is countable infinite.

2.2 Conformal Iterated Function Systems

We will now restrict our attention to *conformal* iterated function systems, for which the maps $\{\phi_e : X \to X\}_{e \in E}$ are conformal.

Definition 2.2.1. An iterated function system $\{X, E, \{\phi_e : X \to X\}_{e \in E}\}$ in $\mathbb{R}^n$ is called *conformal (CIFS)* if the following conditions are satisfied:

1. $\overline{\text{Int}(X)} = X$.

2. (*Open set condition* or OSC). For all distinct $a, b \in E$,

$$\phi_a(\text{Int}(X)) \cap \phi_b(\text{Int}(X)) = \emptyset. \tag{2.3}$$

3. There exists an open and connected set $W \supset X$ such that the maps $\{\phi_e\}_{e \in E}$, extend to uniformly contracting conformal maps of W into W.

4. (*Bounded Distortion Property* or BDP) There exists a compact and connected set S such that

$$X \subset \mathrm{Int}S \subset S \subset W$$

and two constants $L \geq 1$ and $\alpha > 0$ such that

$$\left| \frac{\|D\phi_e(x)\|}{\|D\phi_e(y)\|} - 1 \right| \leq L|x-y|^\alpha \tag{2.4}$$

for every $e \in E$ and every pair of points $x, y \in S$.

When $n \geq 2$, the definition for a CIFS may be simplified: in Definition 2.2.1 (3) we do not have to assume that the maps $\{\phi_e\}_{e\in E}$ extend to uniformly contracting maps; and we do not have to assume the Bounded Distortion Property since it is implied from Definition 2.2.1 (3). To show this, we begin with the following proposition.

Proposition 2.2.1. *Let $\mathcal{S} = \{\phi_e\}_{e\in E}$ be a CIFS. There exists a constant L_0 depending only on $\mathcal{S}$ (actually, only depending on L and s), such that for all $\omega \in E^* := \bigcup_{n=0}^{\infty} E^n$. and $p, q \in S$,*

$$|\log\|D\phi_\omega(p)\| - \log\|D\phi_\omega(q)\|| \leq L_0|p-q|^\alpha. \tag{2.5}$$

Proof. Observe from the Bounded Distortion Property (4) in Definition 2.2.1 that for all $x, y \in S$ and for all $e, j \in E$,

$$\frac{\|D\phi_e(x)\|}{\|D\phi_e(y)\|} \leq 1 + L|x-y|^\alpha \leq 1 + L\mathrm{diam}(X)^\alpha := K. \tag{2.6}$$

Now let

$$p_k = \phi_{\omega_{k+1}} \circ \phi_{\omega_{k+2}} \circ \cdots \circ \phi_{\omega_n}(p), \text{ for } 1 \leq k < n.$$

for every $\omega \in E^*$, say $\omega \in E^n$, and every $p \in W$, and let $p_n = p$. The maps $\phi_e : W \to W$ are uniformly contracting by Definition 2.2.1(3), so there exists an $s \in (0,1)$ such that

$$|\phi_e(x) - \phi_e(y)| \leq s|x-y| \text{ for all } e \in E, x, y \in W.$$

Therefore, for $p, q \in W$, $\omega \in E^n$ and $k \in [1, n]$,

$$|p_k - q_k| \le s^{n-k}|p - q|. \tag{2.7}$$

Now observe from Proposition 1.1.2 that (4)

$$\begin{aligned}\log \|D\phi_\omega(p)\| &= \log(\|D\phi_{\omega_1}(p_1)\| \cdot \|D\phi_{\omega_2}(p_2)\| \dots \|D\phi_{\omega_n}(p_n)\|) \\ &= \sum_{j=1}^{n} \log \|D\phi_{\omega_j}(p_{n-j})\|.\end{aligned}$$

Then for any $p, q \in S$, we have

$$\begin{aligned}|| \log \|D\phi_\omega(p)\| - \log \|D\phi_\omega(q)\|| &\le \sum_{j=1}^{n} \big| \log \|D\phi_{\omega_j}(p_j)\| - \log \|D\phi_{\omega_j}(q_j)\| \big| \\ &\overset{MVT}{\le} \sum_{j=1}^{n} \frac{\big|\|D\phi_{\omega_j}(p_j)\| - \|D\phi_{\omega_j}(q_j)\|\big|}{\min\{\|D\phi_{\omega_j}(p_{n-j})\|, \|D\phi_{\omega_j}(q_j)\|\}} \\ &\overset{(2.6)}{\le} \sum_{j=1}^{n} K \frac{\big|\|D\phi_{\omega_j}(p_j)\| - \|D\phi_{\omega_j}(q_j)\|\big|}{||D\phi_{\omega_j}||_\infty} \\ &\overset{(2.4)}{\le} \sum_{j=1}^{n} LK|p_j - q_j|^\alpha \\ &\overset{(2.7)}{\le} LK \sum_{j=1}^{n} (s^\alpha)^{n-j}|p - q| \\ &\le \frac{LK}{1 - s^\alpha}|p - q|.\end{aligned}$$

Note that the last inequality follows because in the previous line we have a finite geometric series. Thus, setting $L_0 = \frac{LK}{1-s^\alpha}$, we have proven (2.5). □

Corollary 2.2.1. *Let $\mathcal{S} = \{\phi_e\}_{e \in E}$ be a CIFS. Then there exists a constant $K \ge 1$ such that for all $\omega \in E^*$*

$$\|D\phi_\omega(x)\| \le K\|D\phi_\omega(y)\| \text{ for all } x, y \in S. \tag{2.8}$$

Proof. This follows directly from Proposition (2.5). Suppose we have any $\omega \in E^*$ and $x, y \in S$. Then applying (2.5) and raising both sides to a power of e, we obtain

$$\frac{\|D\phi_\omega(x)\|}{\|D\phi_\omega(y)\|} \overset{(2.5)}{\le} e^{L_0 \operatorname{diam}(S)^\alpha},$$

so the corollary is proven with $K = e^{L_0 \operatorname{diam}(S)^\alpha}$. □

Lemma 2.2.1. *Let $X \subset W \subset \mathbb{R}^n$ such that X is compact and W is open and connected. Then there exists a compact and connected set S, such that*

$$X \subset Int(S) \subset S \subset W.$$

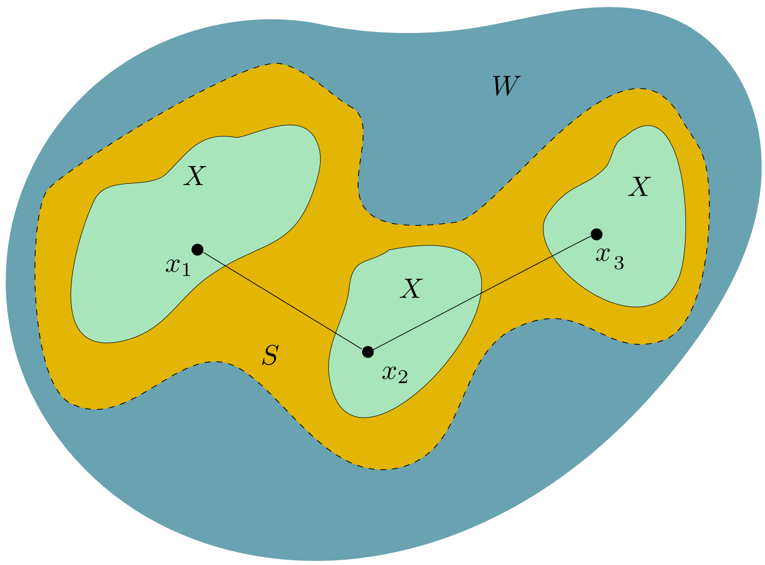

Figure 2.1: The sets in 2.2.1 and the points x_i in the proof below.

Proof. Let $\delta = \text{dist}(X, W^c)/2$. Note that every connected component of the ball $B(X, \delta) := \cup_{x \in X} B(x, \delta)$ contains a ball of the form $B(x, \delta)$ for some $x \in X$. To see this, let C be a connected component of $B(X, \delta)$ and let $y \in C$. Then there exists an $x \in X$ such that $|y - x| < \delta$, so $B(x, \delta) \cap C \neq \emptyset$.

Since C is a connected component, $B(x, \delta) \subset C$. Moreover, since $B(X, \delta)$ is bounded, by definition its connected components are disjoint, and since each connected component contains a ball of radius δ,$B(X, \delta)$ has finitely many connected components.

Label these components $C_1, \ldots, C_m$, and note that that $C_i \cap X \neq \emptyset$ for all $i = 1, \ldots, m$.. Therefore, we may choose points $x_i \in C_i \cap X$. Note that the set W is path connected, because it is open and connected. Therefore, there exist continuous maps $\gamma_i : [0, 1] \to \mathbb{R}^n$, $i = 1, \ldots, m - 1$, such that

1. $\Gamma_i := \gamma_i([0,1]) \subset W$,

2. $\gamma_i(0) = x_i$ and $\gamma_i(1) = x_{i+1}$.

Each Γ_i is compact. Thus, there exists $\delta_i > 0, i = 1, \ldots, m-1$, such that $B(\Gamma_i, \delta_i) \subset W$, and the set

$$T = B(X, \delta) \cup \cup_{i=1}^{m-1} B(\Gamma_i, \delta_i)$$

is open, path connected (and thus connected), and bounded. We can then take $S = \overline{T}$. □

Thus, for $n \geq 2$, we can replace Definition 2.2.1 (3) with the condition

(3∗) There exists an open and connected set $W \supset X$ such that for every $e \in E$, the maps ϕ_e extend to conformal maps of W into W.

In other words, we do not have to assume that the maps ϕ_e are uniformly contracting. Furthermore, we have the following proposition.

Proposition 2.2.2. *For $n \geq 2$:*

$$\textit{Definition } 2.2.1(3) \implies \textit{Definition } 2.2.1\,(4) \wedge \textit{ Corollary } 2.2.1.$$

Proof. By Lemma 2.2.1 there exists a compact and connected set S such that

$$X \subset \mathrm{Int}(S) \subset S \subset W.$$

By Definition 2.2.1 (3) $\phi_\omega : W \to W$ is conformal. Therefore, (2.4) with $\alpha = 1$ follows immediately by Theorem 1.5, and Corollary 2.2.1 follows by Proposition 1.3.2. □

Remark 2.2.1. It is also true for the case when $n = 1$, that Condition (3∗) can replace condition (4) from Definition 3 when X is connected (see Chapter 8, [2]).

2.3 Distortion Properties of CIFSs

We will now discuss distortion properties of CIFS's, many of which are naturally inherited from the distortion properties of conformal maps which were introduced in the previous chapter. In the following, given $\omega \in E^*$ we denote

$$\|D\phi_\omega\|_\infty := \|D\phi_\omega\|_X.$$

Theorem 2.3.1 (Distortion Properties of CIFS). *Let $\mathcal{S} = \{\phi_e\}_{e \in E}$ be a CIFS.*

1. *(Egg Yolk Principle) For all $\omega \in E^*, x \in X$ and $0 < r < \eta := dist(X, S^c)$,*

$$B(f(x), K^{-1}\|D\phi_\omega\|_\infty r) \subset f(B(x,r)) \subset B(f(x), K\|D\phi_\omega\|_\infty r). \tag{2.9}$$

2. *(Chain Rule) For any $\omega, \tau \in E^*$,*

$$K^{-1}\|D\phi_\omega\|_\infty \|D\phi_\tau\|_\infty \leq \|D\phi_{\omega\tau}\|_\infty \leq \|D\phi_\omega\|_\infty \|D\phi_\tau\|_\infty. \tag{2.10}$$

3. *There exists a constant L_1 depending only on $\mathcal{S}$ (actually, it only depends on X, S and K) such that for all $\omega \in E^*$ and $x, y \in X$,*

$$|\phi_\omega(x) - \phi_\omega(y)| \leq L_1 \|D\phi_\omega\|_\infty |x - y|. \tag{2.11}$$

4. *There exists a constant L_2 depending only on $\mathcal{S}$ (actually, it only depends on X and S) such that for all $\omega \in E^*$ such that*

$$L_2^{-1}\|D\phi_\omega\|_\infty \leq diam(\phi_\omega(X)) \leq L_2\|D\phi_\omega\|_\infty. \tag{2.12}$$

Proof. (1) follows by Corollaries 1.3.1 and 2.2.1, and Remarks 1.3.1 and 1.3.2. (2) follows immediately by Proposition 1.1.2 (4) and Remark 2.2.1.

To prove (3), we begin by noting that there exists a compact and connected set S' such that

$$X \subset \text{Int}S' \subset S' \subset \text{Int}S,$$

as in Lemma 2.2.1.

Let $\eta_1 = \text{dist}(S', S^c)$. If $p, q \in S'$ and $|p - q| < \eta_1/2$, by the Mean Value Theorem,

$$\begin{aligned} |\phi_\omega(p) - \phi_\omega(q)| &\leq \|D\phi_\omega\|_{B(p,\eta_1)}|p - q| \\ &\leq \|D\phi_\omega\|_S|p - q| \overset{(2.8)}{\leq} K\|D\phi_\omega\|_\infty|p - q|. \end{aligned} \tag{2.13}$$

Now suppose that $|p - q| \geq \eta_1/2$. Since S' is compact and connected, as we saw in in the proof of Proposition 1.3.1 there exists a finite "chain" of balls $B_i = B(x_i, \eta_1), i = 1, \ldots, m$, (with m depending only on S') such that

1. $S' \subset \cup_{i=1}^m B_i \subset S$,
2. $B_i \cap B_{i+1} \neq \emptyset$ for $i = 1, \ldots, m - 1$.

Therefore, after a possible relabeling, there exist $\{x_i\}_{i=1}^{k+1} \in S, k \leq m$, such that

1. $p := x_1 \in B_1$,
2. $q = x_{k+1} \in B_k$,
3. $x_i \in B_i \cap B_{i+1} \neq \emptyset$ for $i = 2, \ldots, k - 1$.

Observe that $x_i, x_{i+1} \in B_i$ for $i = 1, \ldots, k - 1$. Thus,

$$\begin{aligned} |\phi_\omega(p) - \phi_\omega(q)| &= |\phi_\omega(x_1) - \phi_\omega(x_{n+1})| \\ &\leq \sum_{i=1}^n |\phi_\omega(x_{i+1}) - \phi_\omega(x_i)| \\ &\leq \sum_{i=1}^n \|D\phi_\omega\|_{B_i}|x_{i+1} - x_i| \\ &\leq \sum_{i=1}^n \|D\phi_\omega\|_S 2\eta_1 \\ &\overset{(2.8)}{\leq} 2nK\|D\phi_\omega\|_\infty|p - q|. \end{aligned} \tag{2.14}$$

Then combining (2.13) and (2.14), we obtain (3) with $L_1 = 2Kn$.

Finally, we will prove (4). The inequality on the right follows by (3):

$$\operatorname{diam}(\phi_\omega(X)) = \sup\{|\phi_\omega(x) - \phi_\omega(y)| : x, y \in X\} \overset{(2.11)}{\leq} L_1 \|D\phi_\omega\|_\infty \operatorname{diam}(X). \tag{2.15}$$

To prove the inequality on the left, let R_0 and $x_0 \in X$ be the radius and the center of the largest inscribed ball in $\operatorname{Int}(X)$. Then, by Proposition 1.3.2, Remarks 1.3.1 and 1.3.2, and Corollary 2.2.1 we have

$$\phi_\omega(\operatorname{Int}(X)) \supset \phi_\omega(B(x_0, R_0)) \supset B(\phi_\omega(x_0), K^{-1}\|D\phi_\omega\|_\infty R_0). \tag{2.16}$$

Since $\overline{\operatorname{Int}(X)} = X$,

$$\begin{aligned} \operatorname{diam}(\phi_\omega(X)) &= \operatorname{diam}(\phi_\omega(\overline{\operatorname{Int}(X)})) = \operatorname{diam}(\overline{\phi_\omega(\operatorname{Int}(X))}) \\ &= \operatorname{diam}(\phi_\omega(\operatorname{Int}(X))) \overset{(2.16)}{\geq} K^{-1}R_0\|D\phi_\omega\|_\infty. \end{aligned} \tag{2.17}$$

Combining (2.15) and (2.17) we obtain (4) with $L_2 := \max\{L_1\operatorname{diam}(X), KR_0^{-1}\}$.

□

Chapter 3

Thermodynamic Formalism

And in the poet's plangent dream I saw...
... an algebra of lyricism
which I am still deciphering.
Lawrence Ferlinghetti, "A Coney Island of the Mind"

In this section, we introduce a number of concepts that will be crucial to developing the dimension theory of CIFS's. We introduce the topological pressure of a conformal iterated function system, which is a decreasing, convex, and continuous function of t beyond a certain value of t. Then, we introduce the Perron-Frobenius operator, which we use to obtain a measure that will later be used to develop an approximation of the size of the limit set.

The phrase "thermodynamic formalism" originates from statistical mechanics, for which many of these concepts were first developed. The comparison can be helpful with developing intuition for CIFS thermodynamic formalism, as discussed in Chapter 5 of [5]. In statistical mechanics one might consider systems of particles at different positions. We might label the possible positions of the individual particles, from which we may consider "coding" possible energy states of the whole system by writing down possible combinations of positions for the particles. Similarly, with CIFS's we consider a system over a certain set (such as the interval $[0, 1]$). Infinite words on an index set E code compositions of maps from the family $\{\phi_e\}_{e\in E}$, which when applied to the set corresponds to a single point in the limit set of the system.

In statistical mechanics, the partition function is related to different possible energy states of the system under different configurations of the particles. Similarly, we define a partition function for CIFS's. In a physical system of particles, there is also a natural notion of the pressure of a system, which is related to the temperature. Similarly, for CIFS's we define topological pressure, which is related to the dimension of the limit set of the system.

3.1 Topological Pressure

Let $\mathcal{S} = \{\phi_e\}_{e\in E}$ be a CIFS. As before, for $\omega \in E^*$ define $||D\phi_\omega||_\infty := ||D\phi_\omega||_X$.

For $t \geq 0$ define the partition functions

$$Z_n(t) = \sum_{w\in E^n} ||D\phi_\omega||_\infty^t.$$

Lemma 3.1.1. *$Z_{m+n}(t) \leq Z_m(t)Z_n(t)$*

Proof. Note that $\phi_\omega = \phi_1 \circ \cdots \circ \phi_n = \phi_{\omega|_m} \circ \phi_{\omega|_{m+1}^n}$. Then, we have

$$\begin{aligned}
Z_{m+n}(t) &= \sum_{\omega\in E^{m+n}} ||D\phi_\omega||_\infty^t \\
&= \sum_{\omega\in E^{m+n}} ||D\phi_{\omega|_m \omega|_{m+1}^n}||_\infty^t \\
&\leq \sum_{\omega\in E^{m+n}} ||D\phi_{\omega|_m}||_\infty^t ||D\phi_{\omega_{|_{m+1}^n}}||_\infty^t \text{ by the Chain Rule} \\
&= \sum_{\tau\in E^m} \sum_{\rho\in E^n} ||D\phi_\tau||_\infty^t ||D\phi_\rho||_\infty^t \\
&= Z_m(t)Z_n(t).
\end{aligned}$$

□

Corollary 3.1.1. *The sequence $(\log Z_n(t))_{n=1}^\infty$ is subadditive.*

Then $\lim_{n\to\infty} \frac{1}{n} \log Z_n(t)$ exists, and it is equal to $\inf \left\{\frac{1}{n} \log Z_n(t)\right\}$. See Chapter 6, [2] for more details.

Definition 3.1.1. Define the *topological pressure* to be

$$\mathcal{P}(t) := \lim_{n\to\infty} \frac{1}{n} \log Z_n(t) = \inf \left\{\frac{1}{n} \log Z_n(t)\right\}.$$

Proposition 3.1.1. *Let $n \in \mathbb{N}$. The following are equivalent:*

1. $Z_1(t) < \infty$.
2. $Z_n(t) < \infty$.
3. $\mathcal{P}(t) < \infty$.

Proof. From Lemma 3.1.1,

$$Z_n(t) \leq (Z_1(t))^n. \tag{3.1}$$

Therefore, if $Z_1(t) < \infty$, then $Z_n(t) < \infty$. Next, we will show the converse. Assume that $Z_n(t) < \infty$, and choose $e \in E$. Then we have

$$\begin{aligned} Z_{n+1}(t) &= \sum_{\omega\in E^{n+1}} ||D\phi_\omega||_\infty^t \\ &\geq \sum_{\omega\in E^n} ||D\phi_{\omega e}||_\infty^t \\ &\overset{2.10}{\geq} K^{-t} \sum_{\omega\in E^n} ||D\phi_\omega||_\infty^t ||D\phi_e||_\infty^t \\ &= K^{-t} ||D\phi_e||_\infty^t Z_n(t). \end{aligned}$$

Therefore,

$$Z_n(t) \geq K^{-t(n-1)} ||D\phi_e||_\infty^{t(n-1)} Z_1(t), \tag{3.2}$$

so $Z_1(t) < \infty$ if $Z_n(t) < \infty$.

Next, we will show $Z_n(t) < \infty$ if and only if $\mathcal{P}(t) < \infty$. By 3.1 and 3.2

$$K^{-t(n-1)} ||D\phi_e||_\infty^{t(n-1)} Z_1(t) \leq Z_n(t) \leq (Z_1(t))^n.$$

Taking the log over the inequality and dividing by n, we obtain

$$\frac{-t(n-1)}{n}\log K + \frac{t(n-1)}{n}\cdot\log||D\phi_e||_\infty + \frac{\log Z_1(t)}{n} \le \frac{\log Z_n(t)}{n} \le \log Z_1(t)$$

Taking the limit as $n \to \infty$, we see that

$$-t\log K + t\log||D\phi_e||_\infty \le \mathcal{P}(t) \le \log Z_1(t).$$

It follows, then that $\mathcal{P}(t) < \infty$ if and only if $Z_1(t) < \infty$. □

We wish to approximate $\mathcal{P}(t)$ by finite subsets of E. First, we recall the following definition.

Definition 3.1.2. A uniformly continuous function $f : E^{\mathbb{N}} \to \mathbb{R}$ is called *acceptable* if

$$\operatorname{osc}(f) := \sup_{i\in E}\{\sup(f|_{[i]}) - \inf(f|_{[i]})\} < \infty. \tag{3.3}$$

Theorem 6.8 from [2] says the following:

Theorem 3.1.1. *If $f : E^{\mathbb{N}} \to \mathbb{R}$ is acceptable, then*

$$\mathcal{P}^\sigma(f) = \sup\left\{\mathcal{P}^\sigma_F(f) : F \subset E, F \text{ finite}\right\}.$$

Therefore, if we can find a suitable function f_t that is uniformly continuous, if we can show that this function is acceptable, and if we can also show that $\mathcal{P}^\sigma(f_t) = \mathcal{P}(t)$, then Theorem 3.1.1 will hold for $\mathcal{P}(t)$. This is exactly what we will do, with the function f_t below.

Theorem 3.1.2. *Let $t \ge 0$ and $\omega \in \mathbb{N}$. Let*

$$f_t(\omega) = t\log||D\phi_{\omega_1}(\pi(\sigma(\omega))||.$$

Then:

1. *f_t is Hölder continuous and acceptable*

2. $\mathcal{P}^\sigma(f_t) = \mathcal{P}(t)$

Proof. Let $t \geq 0$, and let $\omega, \tau \in E^{\mathbb{N}}$. Then

$$\begin{aligned} |f_t(\omega) - f_t(\tau)| &= t\,|\log ||D\phi_{\omega_1}(\pi(\sigma(\omega))|| - \log ||D\phi_{\tau_2}(\pi(\sigma(\tau))|||| \\ &\overset{(2.5)}{\leq} t|\pi(\sigma(\omega)) - \pi(\sigma(\tau))|^\alpha \\ &\leq d(\omega, \tau)^\beta \\ &\leq 1. \end{aligned}$$

For the penultimate inequality, see Proposition 8.3 in [2]. Therefore, f_t is $\alpha-$Hölder. Note than $\pi(\sigma(\omega)), \pi(\sigma(\tau)) \in X$. We will now show that f_t is acceptable. Take $i \in E$. Then for all $\tau, \rho \in E^{\mathbb{N}}$

$$\begin{aligned} |f_t(i\tau) - f_t(i\rho)| &\leq e^{-|(i\tau)\wedge(i\rho)|^\alpha} \\ &\leq e^{-\alpha} \text{ because } |(i\tau) \wedge (i\rho)| \geq 1. \end{aligned}$$

Therefore, $f_t(i\tau) \leq f_t(i\rho) + e^{-\alpha}$, so for all ρ in $E^{\mathbb{N}}$ we have

$$\begin{aligned} \sup(f_t|_{[i]}) &\leq f_t(i\rho) + e^{-\alpha} \\ \sup(f_t|_{[i]}) - e^{-\alpha} &\leq f_t(i\rho) \\ \sup(f_t|_{[i]}) - e^{-\alpha} &\leq \inf(f_t|_{[i]}) \\ \sup(f_t|_{[i]}) - \inf(f_t|_{[i]}) &\leq e^{-\alpha} < \infty. \end{aligned}$$

Therefore, f_t is acceptable.

Next, we will show that $\mathcal{P}^\sigma(f_t) = \mathcal{P}(t)$.

From the partition function as defined above, we have the following.

$$\begin{aligned}
Z_n(t) &= \sum_{\omega\in E^n} \exp(\overline{S_n}(f_t)([w])) \\
&= \sum_{\omega\in E^n} \exp\left(\sup_{\tau\in[\omega]} \sum_{j=0}^{n-1} f_t\circ\sigma^j(t)\right) \\
&= \sum_{\omega\in E^n} \exp\left(\sup_{\tau\in[\omega]} \sum_{j=0}^{n-1} \log ||D\phi_{\sigma^j(\tau)_1}(\pi(\sigma^{j+1}(\tau)))||^t\right) \\
&= \sum_{\omega\in E^n} \exp\left(\sup_{\tau\in[\omega]} \log\left[\prod_{j=0}^{n-1} ||D\phi_{\tau_{j+1}}(\pi(\sigma^{j+1}(\tau)))||^t\right]\right) \\
&= \sum_{\omega\in E^n} \exp\left(\log \sup_{\tau\in[\omega]} \left[\prod_{j=1}^{n} ||D\phi_{\tau_j}(\pi(\sigma^{j}(\tau)))||^t\right]\right) \\
&= \sum_{\omega\in E^n} \sup_{\tau\in[\omega]} \prod_{j=1}^{n} ||D\phi_{\tau_j}(\pi(\sigma^{j}(\tau)))||^t. \qquad (3.4)
\end{aligned}$$

Note that

$$\prod_{j=1}^{n} ||D\phi_{\tau_j}(\pi(\sigma^{j}(\tau)))||^t = ||D\phi_{\tau|_n}(\pi(\sigma^n(\tau)))||^t. \qquad (3.5)$$

This follows from the Chain Rule. For example, if $n = 2$ we have

$$\begin{aligned}
||D\phi_{\tau|_2}(\pi(\sigma^2(\tau))|| &= ||D(\phi_{\tau_1}\circ\phi_{\tau_2})(\pi(\sigma^2(\tau))|| \\
&= ||D\phi_{\tau_1}(\phi_{\tau_2}(\pi(\sigma^2(\tau)))||||D\phi_{\tau_2}(\pi(\sigma^2(\tau))||
\end{aligned}$$

Then, by (3.4) and (3.5), we have

$$\begin{aligned}
Z_n(t) &= \sum_{\omega\in E^n} \sup_{\tau\in[\omega]} ||D\phi_{\tau|_n}(\pi(\sigma^n(\tau)))||^t \\
&= \sum_{\omega\in E^n} \sup_{\tau\in[\omega]} ||D\phi_{\omega}(\pi(\sigma^n(\tau)))||^t. \qquad (3.6)
\end{aligned}$$

Since $\pi(\sigma^n(t)) \in X$ for all $\tau \in E^{\mathbb{N}}$, by Corollary 2.2.1 there exists a constant $K \geq 1$ such that for all $\tau \in [\omega]$

$$K^{-1}||D\phi_\omega||_\infty \leq ||D\phi_\omega(\pi(\sigma^n(\tau))|| \leq ||D\phi_\omega||_\infty. \qquad (3.7)$$

Then by (3.6) and (3.7) we obtain

$$K^{-1} \sum_{\omega \in E^n} ||D\phi_\omega||_\infty^t \leq Z_n(f_t) \leq \sum_{\omega \in E^n} ||D\phi_\omega||_\infty^t = Z_n(t).$$

Then

$$\frac{\log(K^{-1})}{n} + \frac{\log Z_n(t)}{n} \leq \frac{\log Z_n(f_t)}{n} \leq \frac{\log Z_n(t)}{n}.$$

Note that the $\log(K^{-1})$ is a constant, so letting $n \to \infty$ we have $\frac{\log(K^{-1})}{n} \to 0$ and $\mathcal{P}(t) = \mathcal{P}^\sigma(f_t)$. □

Now that we have established the previous theorem, we can formally state the finite approximation result for $\mathcal{P}(t)$:

Theorem 3.1.3.

$$\mathcal{P}(t) = \sup\{\mathcal{P}_F(t) : F \subset E, F \textit{ finite}\}. \tag{3.8}$$

Definition 3.1.3. Denote by $\mathrm{Fin}(\mathcal{S})$ the set of all nonnegative values for which the topological pressure is finite. More explicitly,

$$\mathrm{Fin}(\mathcal{S}) = \{t \geq 0 : \mathcal{P}(t) < +\infty\}.$$

We define $\theta = \inf \mathrm{Fin}(\mathcal{S})$.

We note the following from the definitions above:

1. $\theta = \inf\{t \geq 0 : Z_n(t) < +\infty\}$
2. $\mathrm{Fin}(\mathcal{S}) \neq \emptyset$.

The first follows from Proposition 3.1.1.The second follows as in Lemma 4.18

[8]. We have:

$$
\begin{aligned}
|\mathrm{Int}(X)| &\geq \sum_{e\in E} |\phi_e(\mathrm{Int}(X)| \\
&= \sum_{e\in E} \int_{\mathrm{Int}(X)} |\mathrm{det} J_{\phi_e(y)}| dy \\
&= \sum_{e\in E} \int_{\mathrm{Int}(X)} ||D\phi_e(y)||^n dy \\
&\geq \sum_{e\in E} \int_{\mathrm{Int}(X)} K^{-n} ||D\phi_e||_\infty^n \\
&\geq K^{-n}|x| \sum_{e\in E} ||D\phi_e||_\infty^n.
\end{aligned}
$$

Therefore, $\sum_{e\in E} ||D\phi_e||_\infty^n \leq K^n$, so $n \in \mathrm{Fin}(\mathcal{S})$.

The following proposition establishes some useful properties of the topological pressure $\mathcal{P}(t)$.

Proposition 3.1.2. *The function $t \mapsto \mathcal{P}(t)$, $t \geq 0$ is*

1. *Non-increasing on $[0, +\infty)$ and strictly decreasing on $[\theta, +\infty)$,*
2. $\lim_{t\to+\infty} \mathcal{P}(t) = -\infty$,
3. *Convex on $(\theta, +\infty)$,*
4. *Continuous on $(\theta, +\infty)$, and right continuous at θ.*

Proof. 1. Since $\mathcal{P}(t) = +\infty$ for all $t \in [0, \theta)$, we take t_1 and t_2 such that $t_2 > t_1 \geq \theta$. In the case that $\mathcal{P}(\theta) = +\infty$ and $t_2 = \theta$, then trivially $\mathcal{P}(t_1) > \mathcal{P}(t_2)$, so we can assume that $\mathcal{P}(t_1) < +\infty$. Then by 3.1.1, $Z_n(t_1) < +\infty$ for all

$n \in \mathbb{N}$. In particular, we have

$$\begin{aligned}
Z_n(t_2) &= \sum_{\omega \in E^n} ||D\phi_\omega||_\infty^{t_2} \\
&= \sum_{\omega \in E^n} ||D\phi_\omega||_\infty^{t_2 - t_2} ||D\phi_\omega||_\infty^{t_1} \\
&\leq \sum_{\omega \in E^n} s^{n(t_2 - t_1)} ||D\phi_\omega||_\infty^{t_1} \\
&= s^{n(t_2 - t_1)} Z_n(t_1).
\end{aligned}$$

Note that the second the equality on the second-to-last line is because $||D\phi_e|| \leq s < 1$ for all $e \in E$ and 2.10. Therefore,

$$\begin{gathered}
\frac{\log Z_n(t_2}{n} \leq \frac{\log s^{n(t_2-t_1)}}{n} + \frac{\log Z_n(t_1)}{n} \\
\inf \left\{ \frac{\log Z_n(t_2)}{n} \right\} \leq (t_2 - t_1) \log s + \inf \left\{ \frac{\log Z_n(t_1)}{n} \right\} \\
\mathcal{P}(t_2) \leq (t_2 - t_1) \log s + \mathcal{P}(t_1).
\end{gathered}$$

Note that since $t_2 - t_1 > 0$ and $\log s < 0$, the first term on the right side of the inequality is negative. Therefore, we obtain

$$\mathcal{P}(t_2) < \mathcal{P}(t_1).$$

2. Since we saw that $\mathrm{Fin}(\mathcal{S}) \neq \emptyset$, it is clear that $0 \leq \theta < +\infty$. Fix $t_1 > \theta$. Then from the proof of the previous part of the proposition,

$$\mathcal{P}(t) \leq (t - t_1) \log s + \mathcal{P}(t_1).$$

Taking the limit as $t \to +\infty$, we see that

$$\lim_{t \to +\infty} (t - t_1) \log s + \mathcal{P}(t_1) = -\infty,$$

so it follows that

$$\lim_{t \to +\infty} \mathcal{P}(t) = -\infty.$$

3. The functions $t \mapsto ||D\phi_\omega||_\infty^t$ for $\omega \in E*$ are convex in $\mathbb{R}$. Therefore, the partition function $Z_n(t) = \sum_{\omega \in E^n} ||D\phi_\omega||_\infty^t$ is convex in $(\theta, +\infty)$. However,

we can go further and show that more is true: the partition functions $Z_n(t)$ are log-convex in $(\theta, +\infty)$. Let $\lambda \in [0,1]$ and $x, y \in (\theta, +\infty)$. Then

$$\begin{aligned} Z_n(\lambda x + (1-\lambda)y) &= \sum_{\omega \in E^n} ||D\phi_\omega||_\infty^{\lambda x + (1-\lambda)y} \\ &= \sum_{\omega \in E^n} ||D\phi_\omega||_\infty^{\lambda x} ||D\phi_\omega||_\infty^{(1-\lambda)y} \\ &\leq \left(\sum_{\omega \in E^n} ||D\phi_\omega||_\infty^{\lambda x \cdot \frac{1}{\lambda}} \right)^\lambda \left(\sum_{\omega \in E^n} ||D\phi_\omega||_\infty^{(1-\lambda)y \cdot \frac{1}{1-\lambda}} \right)^{1-\lambda} \\ &= \left(\sum_{\omega \in E^n} ||D\phi_\omega||_\infty^{x} \right)^\lambda \left(\sum_{\omega \in E^n} ||D\phi_\omega||_\infty^{y} \right)^{1-\lambda} \\ &= Z_n(x)^\lambda Z_n(y)^{1-\lambda}. \end{aligned}$$

The first inequality in the above follows from observing that for $p = \frac{1}{\lambda}$ and $q = \frac{1}{1-\lambda}$

$$\frac{1}{p} + \frac{1}{q} = \lambda + (1-\lambda) = 1,$$

and applying Hölder's Inequality.

Continuing from above, we have

$$\begin{aligned} \log(Z_n(\lambda x + (1-\lambda)y)) &\leq \log(Z_n(x)^\lambda Z_n(y)^{1-\lambda}) \\ &= \lambda \log Z_n(x) + (1-\lambda) Z_n(y), \end{aligned}$$

and $t \mapsto Z_n(t)$ is log-convex in $(\theta, +\infty)$. Therefore, $\mathcal{P}(t) = \lim_{n \to +\infty} \frac{1}{n} \log Z_n(t)$ is convex in $(\theta, +\infty)$.

4. Continuity on $(\theta, +\infty)$ follows from convexity on $(\theta, +\infty)$. To show that $\mathcal{P}$ is right continuous at θ, we will show

$$\lim_{t \to \theta^+} \mathcal{P}(t) = \mathcal{P}(\theta).$$

Assume that $P(\theta) < +\infty$. If $\mathcal{P}(\theta) = +\infty$, then the proof is similar and we omit it. We know from (3.8) that

$$\mathcal{P}(\theta) = \sup \left\{ \mathcal{P}(F, \theta) : F \subset E, F \text{ finite} \right\},$$

so letting $\epsilon > 0$, we can choose a finite set $F \subset E$ such that

$$\mathcal{P}(\theta) - \mathcal{P}(F, \theta) < \frac{\epsilon}{2}. \tag{3.9}$$

Observe that $t \mapsto Z_n(F, t) = \sum_{\omega \in E^n} ||D\phi_\omega||_\infty^t$ is log-convex in $(\theta, +\infty) = (-\infty, \infty)$ because $\sum_{\omega \in E^n} ||D\phi_\omega||_\infty^t < +\infty$ for all $t \in \mathbb{R}$.

Therefore, $\mathcal{P}(F, t) = \lim_{n \to +\infty} \frac{1}{n} \log Z_n(F, t)$ is continuous in $\mathbb{R}$, and there exists a $t_0 > \theta$ such that if $t \in [\theta, t_0)$

$$\mathcal{P}(F, \theta) - \mathcal{P}(F, t) < \frac{\epsilon}{2}. \tag{3.10}$$

Hence, if $t \in [\theta, t_0)$ we have

$$\begin{aligned} \mathcal{P}(\theta) - \mathcal{P}(t) &\leq \mathcal{P}(\theta) - \mathcal{P}(F, t) \text{ since } \mathcal{P}(t) \geq \mathcal{P}(F, t) \\ &= \mathcal{P}(\theta) - \mathcal{P}(F, \theta) + \mathcal{P}(F, \theta) - \mathcal{P}(F, t) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \text{ by (3.9) and (3.10)} \\ &= \epsilon, \end{aligned}$$

and $\mathcal{P}$ is right continuous at θ.

$\square$

We showed in the proposition above that $\mathcal{P}$ is convex on $(\theta, +\infty)$; it can also be shown that $\mathcal{P}$ is convex on $[\theta, +\infty)$. Let $x_n \to \theta, x_n > \theta, y > \theta$, and $t \in [0, 1]$. Then

$$\begin{aligned} \mathcal{P}(t\theta + (1-t)y) &= \lim_{n \to +\infty} \mathcal{P}(tx_n + (1-t)y) \\ &\leq \lim_{n \to +\infty} (t\mathcal{P}x_n + (1-t)\mathcal{P}(y)) \\ &= t \lim_{n \to +\infty} \mathcal{P}x_n + (1-t)\mathcal{P}(y) \\ &= t\mathcal{P}(\theta) + (1-t)\mathcal{P}(y), \end{aligned}$$

so $\mathcal{P}$ is convex in $[\theta, +\infty)$.

3.2 The Perron-Frobenius Operator

The Perron-Frobenius operator is a technical construction used to obtain a measure that will become useful later on in developing the dimension theory of CIFS's.

Definition 3.2.1. Let $t \in \text{Fin}(\mathcal{S})$. Define the Perron-Frobenius operator as $\mathcal{L}_t g(\omega) = \sum_{e \in E} g(e\omega)||D\phi_e(\pi(\omega))||^t$ for $\omega \in E^{\mathbb{N}}$ and $g \in C_b(E^{\mathbb{N}})$, where

$C_b(E^{\mathbb{N}}) = \{g : E^{\mathbb{N}} \to \mathbb{R} : g$ is bounded and continuous with respect to $(E^{\mathbb{N}}, d_1)\}$.

Note that $d_1 : E^{\mathbb{N}} \times E^{\mathbb{N}} \to [0, \infty)$ is the metric defined by $d_1(\omega, \tau) = e^{-|\omega \wedge \tau|}$, as in Chapter 6 of [2]. The Perron-Frobenius operator has the following properties.

Proposition 3.2.1. *Let $t \in Fin(\mathcal{S})$. Then*

1. $\mathcal{L}_t : C_b(E^{\mathbb{N}}) \to C_b(E^{\mathbb{N}})$,
2. $||\mathcal{L}_t||_{op} \leq Z_1(t)$, *and*
3. *For* $g \in C_b(E^{\mathbb{N}})$, $\mathcal{L}_t^n g(\omega) = \sum_{\tau \in E^n} g(\tau\omega)||D\phi_\tau(\pi(\omega))||_\infty^t$.

Proof. 1. Let $f \in C_b(E^{\mathbb{N}})$ where $|f| \leq B$. Let $\epsilon > 0$, and let $\omega_n \xrightarrow{d} \omega$ for $\omega_n \in E^{\mathbb{N}}$ and $\omega \in E$. Choose $F \subset E$ such that

$$\sum_{e \in E \setminus F} ||D\phi_e||_\infty^t < \epsilon. \tag{3.11}$$

Since f is continous, for all $e \in E$, there exists $n(e)$ such that

$$|f(e\omega^n) - f(e\omega)| < \epsilon \tag{3.12}$$

whenever $n > n(e)$. Let $N = \max_{e \in F} n(e)$.

We know that $\pi : E^{\mathbb{N}} \to E^{\mathbb{N}}$ is Hölder continuous and that the maps $x \mapsto ||D\phi_e(x)||$ are continuous for all $e \in E$ and $x \in X$. Therefore, for all $e \in E$ there exists an $m(e)$ such that

$$\left| ||D\phi_e(\pi(\omega^m))||^t - ||D\phi_e(\pi(\omega))||^t \right| < ||D\phi_e||_\infty^t \epsilon \tag{3.13}$$

whenever $m > m(e)$. Let $M = \max_{e \in F} m(e)$, and let $K = \max\{M, N\}$. Take $n > K$. Then

$$\begin{aligned}
|\mathcal{L}_t f(\omega^n) - \mathcal{L}_t f(\omega)| &= \left| \sum_{e \in E} f(e\omega^n) ||d\phi_e(\pi(\omega^n))||^t - \sum_{e \in E} f(e\omega) ||d\phi_e(\pi(\omega))||^t \right| \\
&\leq \sum_{e \in E} |f(e\omega^n) - f(e\omega)| ||D\phi_e(\pi(\omega^n))||^t \\
&+ \sum_{e \in E} |f(e\omega)| \left| ||D\phi_e(\pi(\omega^n))||^t - ||D\phi_e(\pi(\omega))||^t \right| \\
&\overset{(3.11)}{\leq} \sum_{e \in F} |f(e\omega^n) - f(e\omega)| ||D\phi_e||_\infty^t + 2B\epsilon \\
&+ \sum_{e \in F} |f(e\omega)| \left| ||D\phi_e(\pi(\omega^n))||^t - ||D\phi_e(\pi(\omega))||^t \right| + 2B\epsilon \\
&\overset{(3.12) \wedge (3.13)}{<} \epsilon \sum_{e \in F} ||D\phi_e||_\infty^t + \sum_{e \in F} B ||D\phi_e||_\infty^t \epsilon + 4B\epsilon \\
&= \epsilon(Z_1(t) + BZ_1(t) + 4B).
\end{aligned}$$

Thus, we have shown that

$$\mathcal{L}_t f(\omega^n) \to \mathcal{L}_t f(\omega),$$

so the function $\mathcal{L}_t f$ is continuous. Next, we will show that $\mathcal{L}_t f$ is bounded:

$$\begin{aligned}
|\mathcal{L}_t f(\omega)| &\leq \sum_{e \in E} |f(e\omega)| ||D\phi_e(\pi(\omega))||^t \\
&\leq B \sum_{e \in E} ||D\phi_e||_\infty^t \\
&= B \cdot Z_1(t) < +\infty.
\end{aligned}$$

Therefore, $\mathcal{L}_t f \in C_b(E^{\mathbb{N}})$.

2. Let $f \in C_b(E^{\mathbb{N}})$ with $||f||_\infty \leq 1$ (Recall that $||\mathcal{L}_t||_{op} = \sup\{||\mathcal{L}_t(f)|| : ||f|| \leq 1\}$).

Then

$$\begin{aligned}||\mathcal{L}_t(f)||_\infty &= \sup_{\omega\in E^{\mathbb{N}}} |\mathcal{L}_t(f)(\omega)| \\ &\le \sup_{\omega\in E^{\mathbb{N}}} \sum_{e\in E} |f(e\omega)|||D\phi_e(\pi(\omega))||^t \\ &\le \sum_{e\in E} ||f||_\infty ||D\phi_e||_\infty^t \\ &= ||f||_\infty Z_1(t) \\ &\le Z_1(t).\end{aligned}$$

3. Consider the case when $n = 2$ (the proof of general case is similar). We have the following:

$$\begin{aligned}\mathcal{L}_t^2 g(\omega) &= \mathcal{L}_t(\mathcal{L}_t g)(\omega) \\ &= \sum_{e\in E} \mathcal{L}_t g(e\omega)||D\phi_e(\pi(\omega))|^t \\ &= \sum_{e\in E}\sum_{i\in E} g(ie\omega)||D\phi_i(\pi(\omega))|^t||D\phi_e(\pi(\omega))|^t \\ &= \sum_{\tau\in E^2} g(\tau\omega)||D\phi_\tau(\pi(\omega))||^t.\end{aligned}$$

The last equality follows from the Chain Rule, since

$$\begin{aligned}||D(\phi_i\circ\phi_e)\pi(\omega)|| &= ||D\phi_i(\phi_e(\pi(\omega))||||D\phi_e(\pi(\omega)|| \\ &= ||D\phi_i(\pi(e\omega))||||D\phi_e(\pi(\omega))||.\end{aligned}$$

□

3.3 The Schauder-Tychonoff Theorem

We will now use the Perron-Frobenius operator to obtain a particular Borel probability measure on $E^{\mathbb{N}}$. Let $\mathcal{P}(E^{\mathbb{N}})$ denote the set of all Borel probability measures on $E^{\mathbb{N}}$. The Riesz Representation theorem provides a functional $I_\mu \in C(E^{\mathbb{N}})$ which we use to construct a map $T : I(\mathcal{P}(E^{\mathbb{N}})) \to C(E^{\mathbb{N}})^*$. We will show that T

satisfies the conditions of the Schauder-Tychonoff theorem, which will yields the desired measure, and results in a simple expression for the topological pressure.

From now on, we assume that E is finite. Then $(E^{\mathbb{N}}, d_1)$ is compact.

Let $t \in \mathrm{Fin}(\mathcal{S})$. Since $\mathcal{L}_t : C(E^{\mathbb{N}}) \to C(E^{\mathbb{N}})$ is a bounded linear operator, the adjoint

$$\mathcal{L}_t^* : C(E^{\mathbb{N}})^* \to C(E^{\mathbb{N}})^*$$

given by

$$\mathcal{L}_t^*(F)(f) = F(\mathcal{L}_t(f)) \text{ for all } F \in C(E^{\mathbb{N}})^*, f \in C(E^{\mathbb{N}})$$

is also a bounded linear operator. This follows from Theorem 3.20 in [3].

By the Riesz Representation Theorem(Theorem 1.33 [3]), we have the isometry

$$\left(C(E^{\mathbb{N}})^*, ||\ ||_*\right) \cong \left(\mathcal{M}(E^{\mathbb{N}}), ||\ ||_{Tv}\right),$$

where $||\ ||_{Tv}$ is the total variation and

$$\mathcal{M}(E^{\mathbb{N}}) = \{\text{signed Radon measures on } E^{\mathbb{N}}\}.$$

For $\mu \in \mathcal{M}(E^{\mathbb{N}})$, the total variation $||\mu||$ is given by $||\mu|| = |\mu|(E^{\mathbb{N}})$, where $|\mu| : \mathcal{B}(E^{\mathbb{N}}) \to [0, +\infty]$ is the variation defined as

$$|\mu|(A) = \sup\left\{\sum_{k=1}^{m} |\mu(A_k)| : \{A_k\}_{k=1}^{m} \text{ is a finite Borel partition of } A\right\}.$$

For more details, see Definition 1.28 and Proposition 1.29 in [3]. More explicitly, the isometry above is given by the map

$$\mathcal{M}(E^{\mathbb{N}}) \ni \mu \mapsto I_\mu \in C(E^{\mathbb{N}}),$$

where

$$I_\mu(f) = \int f d\mu = \int f \frac{d\mu}{d|\mu|} d|\mu|.$$

Note that $C(E^{\mathbb{N}}) \subset L^1(\mu)$ so $f \in L^1(\mu)$, and for $|\mu| - a.e.$ we have $\frac{d\mu}{d|\mu|} = 1$. Therefore, we may identify any measure $\mu \in \mathcal{M}(E^{\mathbb{N}})$ with the functional I_μ.

Definition 3.3.1. Denote by $\mathcal{P}(E^{\mathbb{N}})$ the set of all Borel probability measures on $E^{\mathbb{N}}$.

Theorem 3.3.1. *All Borel probability measures are Radon; i.e.*

$$\mathcal{P}(E^{\mathbb{N}}) \subset \mathcal{M}(E^{\mathbb{N}}).$$

For proof, see Theorem 1.2 in [6].

Note that $\mathcal{P}(E^{\mathbb{N}})$ is convex. Indeed, taking $\mu, \nu \in \mathcal{P}(E^{\mathbb{N}})$ and $t \in [0,1]$ it is easy to verify that $t\mu + (1-t)\nu \in \mathcal{P}(E^{\mathbb{N}})$.

Proposition 3.3.1. *A topological vector space X is locally convex if and only if* 0 *has a neighborhood basis consisting of open and convex sets.*

For more details see Theorem 6.19 in [3]. We shall abbreviate locally convex topological vector space with LCTVS. By Example 6.12 in [3], $(C(E^{\mathbb{N}})^*, \sigma(C(E^{\mathbb{N}})^*, C(E^{\mathbb{N}}))$ is a LCTVS. By Theorem 3.17 [2], since $E^{\mathbb{N}}$ is compact $\mathcal{P}(E^{\mathbb{N}})$ is also compact, with respect to the weak topology of measures. Therefore, $I(\mathcal{P}(E^{\mathbb{N}}))$ is compact in $\sigma(C(E^{\mathbb{N}})^*, C(E^{\mathbb{N}}))$ and

$$\mu_\lambda \rightharpoonup \mu \iff \int f d\mu_\lambda \to \int f d\mu \text{ for all } f \in C(E^{\mathbb{N}}) \iff I(\mu_\lambda) \overset{*}{\rightharpoonup} I(\mu).$$

where $(\mu_\lambda)_{\lambda \in \Lambda}$ is a sequence of measures in $\mathcal{P}(E^{\mathbb{N}})$. Therefore, the map

$$I : (\mathcal{P}(E^{\mathbb{N}}), \sigma(C(E^{\mathbb{N}}), C(E^{\mathbb{N}})^*) \to (I(\mathcal{P}(E^{\mathbb{N}}), \sigma(C(E^{\mathbb{N}})^*, C(E^{\mathbb{N}})) \subset C(E^{\mathbb{N}})^*$$

is a homoemorphism. Thus,

(a) $\mathcal{P}(E^{\mathbb{N}})$ is compact in $\sigma(C(E^{\mathbb{N}})^*, C(E^{\mathbb{N}}))$, and

(b) $I(\mathcal{P}(E^{\mathbb{N}}))$ is convex:

Proof. Take $\mu, \nu \in \mathcal{P}(E^{\mathbb{N}})$ and $t \in [0,1]$. Then

$$\begin{aligned}
(tI(\mu) + (1-t)I(\nu))(f) &= tI(\mu)(f) + (1-t)I(\nu)(f) \\
&= t\int f d\mu + (1-t)\int f d\nu \\
&= \int f(td\mu + (1-t)d\nu) \\
&= I(t\mu + (1-t)\nu)(f) \text{ for all } f \in C(E^{\mathbb{N}}).
\end{aligned}$$

By a previous observation, $(t\mu + (1-t)\nu) \in \mathcal{P}(E^{\mathbb{N}})$, so

$$I(t\mu + (1-t)\nu) \in I(\mathcal{P}(E^{\mathbb{N}}).$$

□

Definition 3.3.2. Define $T : I(\mathcal{P}(E^{\mathbb{N}})) \to C(E^{\mathbb{N}})^*$ by

$$T(I_\mu)(f) = \frac{\mathcal{L}_t^*(I_\mu)(f)}{\mathcal{L}_t^*(I_\mu)(1)} = \frac{\int \mathcal{L}_t(f)d\mu}{\int \mathcal{L}_t(1)d\mu}$$

for $f \in C(E^{\mathbb{N}})$, and where $1 := \chi_{E^{\mathbb{N}}}$. (Note that the last equality follows since $\mathcal{L}_t^*(I_\mu)(f) = I_\mu(\mathcal{L}_t(f)) = \int \mathcal{L}_t(f)d\mu$).

Fix $\mu \in \mathcal{P}(E^{\mathbb{N}})$. Then note that

- $T(I_\mu)$ is linear,
- $T(I_\mu)(f) \geq 0$ if $f \geq 0$, and
- $T(I_\mu)(1) = 1$.

Then Theorem 5.8 in [6] implies that

$$\int \mathcal{L}_t(1)d\mu = \int \sum_{e\in E} ||D\phi_e(\pi(\omega))||^t d\mu(\omega) > 0.$$

Moreover, there exists $\mu' \in \mathcal{P}(E^{\mathbb{N}})$ such that $T(I_\mu) = I_{\mu'}$. Therefore,

(c) $T : I(\mathcal{P}(E^{\mathbb{N}})) \to I(\mathcal{P}(E^{\mathbb{N}}))$, and

(d) $T : (I(\mathcal{P}(E^{\mathbb{N}})), \sigma(C(E^{\mathbb{N}})^*, C(E^{\mathbb{N}})))$ is continuous.

Proof. Take $(\mu_\lambda)_{\lambda\in\Lambda}, \mu_\lambda \in \mathcal{P}(E^{\mathbb{N}})$, and $\mu \in \mathcal{P}(E^{\mathbb{N}})$ such that $I(\mu_\lambda) \overset{*}{\rightharpoonup} I(\mu)$. But

$$\begin{aligned} I(\mu)_\lambda \overset{*}{\rightharpoonup} I(\mu) &\iff I(\mu_\lambda)(f) \to I(\mu)(f) \text{ for all} f \in C(E^{\mathbb{N}}) \\ &\iff \int f d\mu \to \int f d\mu \text{ for all} f \in C(E^{\mathbb{N}}). \end{aligned}$$

Let $f \in C(E^{\mathbb{N}})$. Then

$$T(I_{\mu_\lambda})(f) = \frac{\int \mathcal{L}_t(f)d\mu_\lambda}{\int \mathcal{L}_t(f)(1)d\mu_\lambda}.$$

Since $\mathcal{L}_t : C(E^{\mathbb{N}}) \to C(E^{\mathbb{N}})$, the sequence of biconditionals above applied to the continuous functions $\mathcal{L}_t(f)$ and $\mathcal{L}_t(1)$ implies that

$$\int \mathcal{L}_t(f) d\mu_\lambda \to \mathcal{L}_t(f) d\mu \text{ and } \int \mathcal{L}_t(1) d\mu)\lambda \to \int \mathcal{L}_t(1) d\mu,$$

which together imply

$$T(I_{\mu\lambda})(f) \to T(I_\mu)(f) \text{ for all } f \in C(E^{\mathbb{N}}).$$

Therefore,

$$T(I_{\mu_\lambda})(f) \to T(I_\mu)(f) \iff T(I_{\mu_\lambda}) \xrightharpoonup{*} T(I_\mu),$$

so $T : (I(\mathcal{P}(E^{\mathbb{N}})), \sigma(C(E^{\mathbb{N}})^*, C(E^{\mathbb{N}})))$ is continuous. □

Theorem 3.3.2. *(Schauder-Tychonoff) Let X be a LCTVS. If $K \subset X$ is compact and convex and $T : K \to K$ is continuous, then there exists an $x \in K$ such that $T(x) = x$.*

Since $(C(E^{\mathbb{N}})^*, \sigma(C(E^{\mathbb{N}})^*, C(E^{\mathbb{N}}))$ is continuous, (a), (b), (c), and (d) allow us to apply Schauder-Tychonoff with $K = I(\mathcal{P}(E^{\mathbb{N}}))$ to obtain some $\tilde{m}_t \in \mathcal{P}(E^{\mathbb{N}})$, $t \in \mathrm{Fin}(\mathcal{S})$ such that

$$T(I_{\tilde{m}_t}) = \frac{\mathcal{L}_t^*(I_{\tilde{m}_t})}{\mathcal{L}_t^*(I_{\tilde{m}_t})(1)} = I_{\tilde{m}_t}.$$

Definition 3.3.3. Define

$$\begin{aligned} \lambda_t &:= \mathcal{L}_t^*(I_{\tilde{m}_t})(1) \\ &= \int \mathcal{L}_t(1) d\tilde{m}_t > 0. \end{aligned}$$

Then if $f \in C(E^{\mathbb{N}})$,

$$\begin{aligned} &\mathcal{L}_t^*(I_{\tilde{m}_t})(f) = \lambda_t I_{\tilde{m}_t}(f) \\ \iff &I_{\tilde{m}_t}(\mathcal{L}_t f) = \lambda_t \int f d\tilde{m}_t \\ \iff &\int \mathcal{L}_t(f) d\tilde{m}_t = \lambda_t \int f d\tilde{m}_t. \end{aligned}$$

Therefore,

$$\int \mathcal{L}_t(f) d\tilde{m}_t = \lambda_t \int f d\tilde{m}_t \text{ for all } f \in C(E^{\mathbb{N}}). \tag{3.14}$$

Since $\mathcal{L}_t : C(E^{\mathbb{N}}) \to C(E^N)$, we deduce that for all $n \in \mathbb{N}$

$$\int \mathcal{L}_t^n(f) d\tilde{m}_t = \lambda_t^n \int f d\tilde{m}_t \text{ for all } f \in C(E^{\mathbb{N}}). \tag{3.15}$$

Lemma 3.3.1. *Let $\omega \in E^*$. Then $f := \chi_{[\omega]} \in C(E^{\mathbb{N}})$.*

Proof. Let $U \subseteq \mathbb{R}$ be an open set. If $0 \notin U$ and $1 \notin U$, then $f^{-1}(U) = \emptyset$. If $0 \in U$ but $1 \notin U$, $f^{-1}(U) = E^{\mathbb{N}} \setminus [\omega]$, which is open. If $0 \notin U$ but $1 \in U$, $f^{-1}(U) = [\omega]$, which is also open. Finally, if $0 \in U$ and $1 \in U$ then $f^{-1}(U) = E^{\mathbb{N}}$, which is open. Since in all these cases the pre-image under f of U was open, we deduce that $f := \chi_{[\omega]}$ is continuous. □

Proposition 3.3.2. *For all words $\omega \in E^n$,*

$$\lambda_t^{-n} K^{-t} ||D\phi_\omega||_\infty^t \leq \tilde{m}_t([\omega]) \leq \lambda_t^{-n} ||D\phi_\omega||_\infty^t.$$

Proof. Let $\omega \in E^{\mathbb{N}}$. By 3.14 and the previous lemma,

$$\begin{aligned} \int \chi_{[\omega]} d\tilde{m}_t &= \lambda_t^{-n} \int \mathcal{L}_t^n \left(\chi_{[\omega]}\right)(\rho) d\tilde{m}_t(\rho) \\ &\overset{3.2.1(3)}{=} \lambda_t^{-n} \int \sum_{\tau \in E^n} \chi_{[\omega]}(\tau\rho) ||D\phi_\tau(\pi(\rho))||^t d\tilde{m}_t(\rho) \\ &= \lambda_t^{-n} \int ||D\phi_\omega(\pi(\rho))||^t d\tilde{m}_t(\rho). \end{aligned}$$

Therefore, $\tilde{m}_t([\omega]) = \lambda_t^{-n} \int ||D\phi_\omega(\pi(\rho))||^t d\tilde{m}_t(\rho)$, so

$$\tilde{m}_t([\omega]) \leq \lambda_t^{-n} ||D\phi_\omega||_\infty^t.$$

By the Bounded Distortion Property, the left-hand inequality follows and

$$\tilde{m}_t([\omega]) \geq \lambda_t^{-n} K^{-t} ||D\phi_\omega||_\infty^t.$$

□

Theorem 3.3.3. $\mathcal{P}(t) = \log \lambda_t$.

Proof.

$$
\begin{aligned}
\lambda_t^n &= \lambda_t^n \chi_{E^{\mathbb{N}}} \\
&= \lambda_t^n \int 1 d\tilde{m}_t \\
&\overset{3.15}{=} \int \mathcal{L}_t^n(1) d\tilde{m}_t \\
&= \int_{E^{\mathbb{N}}} \sum_{\tau \in E^n} ||D\phi_\tau(\pi(\omega))||^t d\tilde{m}_t(\omega) \\
&\leq \int_{E^{\mathbb{N}}} \sum_{\tau \in E^n} ||D\phi_\tau||_\infty d\tilde{m}_t \\
&\leq Z_n(t).
\end{aligned}
$$

Therefore,

$$\mathcal{P}(t) = \lim_{n \to +\infty} \frac{1}{n} \log Z_n(t) \geq \lim_{n \to +\infty} \frac{1}{n} \log \lambda_t^n = \log \lambda_t.$$

For the other direction,

$$
\begin{aligned}
Z_n(t) &= \sum_{\omega \in E^n} ||D\phi_\omega||^t \\
&\overset{3.3.2}{\leq} \lambda_t^{-n} K^{-t} \sum_{\omega \in E^n} m_t([\omega]) \\
&= \lambda_t^{-n} K^{-t}.
\end{aligned}
$$

The last equality follows because the words $\omega \in E^n$ are mutually incomparable, so the cylinder $[\omega]$ for $\omega \in E^{\mathbb{N}}$ partitions $E^{\mathbb{N}}$. Therefore,

$$
\begin{aligned}
\mathcal{P}(t) &= \lim_{n \to +\infty} \frac{1}{n} \log Z_n(t) \\
&\leq \lim_{n \to +\infty} \left(\frac{1}{n} \log \lambda_t^n + \frac{1}{n} \log K^{-t} \right) \\
&= \log \lambda_t.
\end{aligned}
$$

Since we have proved both inequalities, the theorem follows. □

Chapter 4

Bowen's Formula

What story could he tell? Besides, the guests demanded marvels, while the marvelous was perhaps incommnuicable...
∼Jorge Luis Borges, "Averroës' Search"

The Hausdorff dimension provides a way to reasonably quantify the "size" of the limit set of a CIFS. Bowen's formula relates the topological pressure to the Hausdorff dimension, which yields a concrete way to estimate the dimension of self-similar systems.

4.1 Hausdorff Measure and Dimension

Definition 4.1.1. Let (X, d) be a metric space, $A \subset X, s \geq 0$ and $\delta > 0$. We define

$$\mathcal{H}^s_\delta(A) = \inf \left\{ \sum_{i=1}^{\infty} \operatorname{diam}(E_i)^s : A \subset \bigcup_{i=1}^{\infty} E_i, \operatorname{diam}(E_i) \leq \delta \right\},$$

and

$$\mathcal{H}^s(A) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(A) = \sup_{\delta > 0} \mathcal{H}^s_\delta(A).$$

It can be shown that $\mathcal{H}^s_\delta$ is a metric outer measure. We call $\mathcal{H}^s_\delta$ the *Hausdorff measure.*

The following are important properties of the Hausdorff measure. For more details and proofs, see Chapter 4 [4].

1. The Hausdorff measure $\mathcal{H}^s$ is Borel regular.
2. Let $A \subset X$ and $0 \leq s < t < \infty$. Then
   (a) If $\mathcal{H}^s(A) < \infty$, then $\mathcal{H}^t(A) = 0$.
   (b) If $\mathcal{H}^t(A) > 0$, then $\mathcal{H}^s(A) = \infty$.
3. The Hausdorff measure $\mathcal{H}^s(A)$ equals 0 if and only if for all $\epsilon > 0$ there exists a cover $\{E_i\}_{i\in\mathbb{N}}$ of A such that $\sum_{i=1}^{\infty} \operatorname{diam}(E_i) < \epsilon$.
4. If there exists a Borel finite measure μ on A and $c, r_0 > 0$ such that
$$c^{-1}r^h \leq \mu(B(x,r)) \leq cr^h \text{ for all } x \in A, r < r_0$$
   then there exists a $c_0 > 0$ such that
$$c_0^{-1}\mu(A) \leq \mathcal{H}^h(A) \leq c_0\mu(A) \text{ for all Borel sets } A. \tag{4.1}$$
5. In $\mathbb{R}^n$, the $n-$dimensional Hausdorff measure is a rescaling of the $n-$dimensional Lebesgue measure. More explicitly, we have that $\mathcal{H}^n = c\mathcal{L}^n$ in $\mathbb{R}^n$.

We may now introduce the notion of Hausdorff dimension, shown in the figure below.

Definition 4.1.2. The *Hausdorff dimension* $\dim_{\mathcal{H}}(A)$ of $A \subset X$ is

$$\begin{aligned}\dim_{\mathcal{H}}(A) &= \inf\{t \geq 0 : \mathcal{H}^t(A) = 0\}\\ &= \inf\{t \geq 0 : \mathcal{H}^t(A) < \infty\}\\ &= \sup\{s \geq 0 : \mathcal{H}^s(A) > 0\}\\ &= \sup\{s \geq 0 : \mathcal{H}^s(A) = \infty\}.\end{aligned}$$

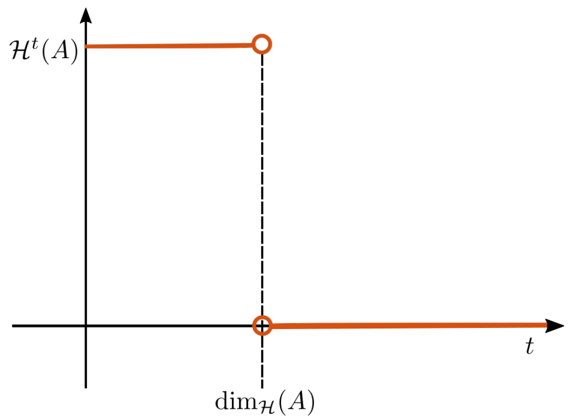

Figure 4.1: The Hausdorff dimension.

4.2 Bowen's Parameter

Definition 4.2.1. Bowen's parameter is defined to be the value

$$h = \inf\{t \geq 0 : \mathcal{P}(t) \leq 0\}.$$

Recall from the previous chapter that we may obtain a Borel probability measure $\tilde{m_h}$ on $E^{\mathbb{N}}$ for which

$$T(I_{\tilde{m_h}}) = I_{\tilde{m_h}}.$$

By (3.3.2),

$$M_h||D\phi_\omega||_\infty^h \leq \tilde{m}_h([\omega]) \leq ||D\phi_\omega||_\infty^h. \tag{4.2}$$

Let $m_h = \tilde{m}_h \circ \pi^{-1}$.

Theorem 4.2.1. *Let $\mathcal{S} = \{\phi_e\}_{e\in E}$ be a CIFS and E a finite set. Then there exists a $c > 0$ such that*

$$c^{-1}r^h \leq m_h(B(x,r)) \leq cr^h \textit{ for all } 0 < r \leq 1 \textit{ and } x \in J,$$

and thus $m_h(J) \approx \mathcal{H}^h(J)$.

Before proving the theorem, we will prove the following lemma, where we consider a set of words corresponding to a fixed point x in our metric space. We define the set to be all words for which the diameter of the corresponding ϕ_ω is within fixed constants. It turns out that the number of words in this set is bounded above by a constant that depends only on the chosen point x and the fixed constants.

Lemma 4.2.1. *Let $\mathcal{S}$ be a CIFS. Then for all $0 < k_1 < k_2 < +\infty, r > 0, x \in X$ there exists a $C_1 = const(x, k_1, k_2)$ such that if F is the set of all words such that*

1. $k_1 r \leq diam(\phi_\omega(x)) \leq k_2 r$ *and*
2. $B(x, r) \cap \phi_\omega \neq \emptyset$,

then $\#F \leq C_1$.

Proof. Let R be the maximal radius such that $B(x_0, R) \subset \text{Int}(X)$ for some $x_0 \in X$. Then

$$\phi(X) \subset B(x, r + \text{diam}(\phi_\omega(x)) \subset B(x, (1 + k_2)r), \text{ and we have}$$

$$\begin{aligned}
|B(0,1)|(1+k_2)^n r^n &= |B(x,(1+k_2)r)| \\
&\geq \left| \bigcup_{\omega \in F} \phi_\omega(X) \right| \\
&\geq \left| \bigcup_{\omega \in F} \phi_\omega(\text{Int}(X) \right| \text{ because } X \supset \text{Int}(X) \\
&\overset{(2.3)}{=} |\phi_\omega(\text{Int}(X)| \\
&\geq \sum_{\omega \in F} |\phi_\omega(B(x_0, R)| \\
&\geq \sum_{\omega \in F} |B(\phi_\omega(x_0), K^{-1}||D\phi_\omega||_\infty R)| \\
&\geq \sum_{\omega \in F} |B(\phi_\omega(x_0), (KL)^{-1}\text{diam}\phi_\omega(X)R)| \\
&\geq \sum_{\omega \in F} |B(\phi_\omega(x_0), (KL)^{-1}k_1 rR)| \\
&= \#F(KL)^{-n}(k_1 rR)^n.
\end{aligned}$$

Therefore,

$$\#F \leq \frac{(KL)^n(1+k_2)^n|B(0,1)|^n}{(k_1R)^n} = \left(\frac{KL(1+k_2)|B(0,1)|}{k_1R}\right)^n,$$

and the proof is complete. □

We are now ready to prove Theorem 4.2.1.

Proof. Let $\xi = \inf\{||D_{\phi e}||_\infty : e \in E\}$, $x \in J_{\mathcal{S}}$ such that $x = \pi(\rho)$ for some $\rho \in E^{\mathbb{N}}$, and $r > 0$. Let $n = n(\rho)$ be the largest $n \in N$ such that

$$\phi_{\rho|_n}(X) \subset B(x,r).$$

Then

$$\begin{aligned} m_h(B(x,r)) &\geq m_h(\phi_{\rho|_n}(X)) \\ &= \tilde{m}_h(\pi^{-1}(\phi_{\rho|_n}(X)) \\ &\geq \tilde{m}_h([\rho|_n]) \\ &\overset{(4.2)}{\geq} M_h||D\phi_{\rho|_n}||_\infty^h, \end{aligned}$$

so

$$m_h(B(x,r)) \geq M_h||D\phi_{\rho|_n}||_\infty^h. \tag{4.3}$$

Therefore, $[\rho|_n] \subset \pi^{-1}(\phi_{\rho|_n}(X))$ if and only if $\pi([\rho|_n]) \subset \phi_{\rho|_n}(X)$. Observe that (1) $\phi_{\tau|_{n-1}}(X) \not\subset B(x,r)$, and (2) $x \in \phi_{\tau|_{n-1}}(X)$, which together imply that

$$\operatorname{diam}(\phi_{\rho|_{n-1}}(X)) \geq r, \tag{4.4}$$

and with the Bounded Distortion Property imply that

$$\operatorname{diam}(\phi_{\rho|_{n-1}}(X)) \leq L||D\phi_{\rho|_{n-1}}||_\infty. \tag{4.5}$$

Therefore, by (4.3), (4.4), and (4.5),

$$m_h(B(x,r)) \geq M_hL^{-h}r^h. \tag{4.6}$$

Next, let Z be the family of all minimal length words $\omega \in E^*$ such that

$$B(x,r) \cap \phi_\omega(x) \neq \emptyset, \text{ and} \tag{4.7}$$

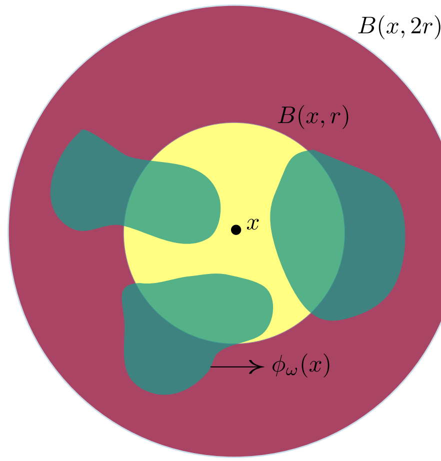

Figure 4.2: ω satisfies (4.7) and (4.8).

$$\phi_\omega(x) \subset B(x, 2r). \tag{4.8}$$

Observe that Z contains mutually incomparable words. Indeed, suppose ω and τ belong to Z and are mutually comparable. Then $\phi_\omega \cap \phi_\tau \neq \emptyset$, which implies that there exists a word ρ that is smaller than ω and τ satisfying (4.7) and (4.8). However, this contradicts the minimality of ω and τ.

Let $\omega \in Z$ with $|\omega| = n$. Then

$$\text{diam}(\phi_\omega(x)) \leq 4r \text{ and} \tag{4.9}$$

$$\text{diam}(\phi_\omega|_{n-1}(x)) \geq r. \tag{4.10}$$

(4.10) follows by noting that since $\omega \in Z$,

$$\phi_\omega(x) \cap B(x, r) \neq \emptyset.$$

Then, because $\phi_\omega|_{n-1}(x) \supset \phi_\omega(x)$, we have

$$\phi_\omega|_{n-1}(x) \cap B(x, r) \neq \emptyset. \tag{4.11}$$

Since Z is a set of minimal words for the defined properties, we also have that

$$\phi_\omega|_{n-1}(x) \not\subset B(x, 2r). \tag{4.12}$$

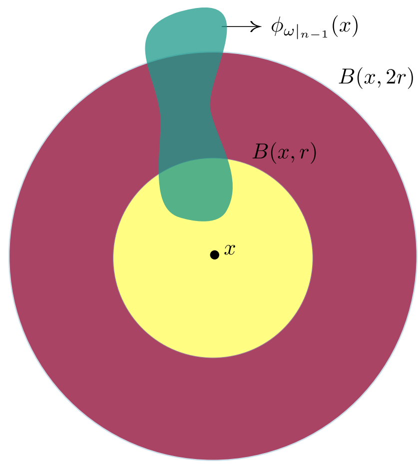

Figure 4.3: ω satisfies (4.12).

Therefore, we obtain (4.10) by (4.11) and (4.12).

Now, let R be the maximal radius such that $B(x_0, R) \subset \text{Int}(X)$ for some $x_0 \in X$. Then for all $\omega \in E^*$,

$$\phi_\omega(\text{Int}(X)) \supset \phi_\omega(B(x_0, R)) \overset{BDP}{\supset} B(\phi_\omega(x_0), RK^{-1}||D\phi_\omega||_\infty.$$

Since $X = \overline{\text{Int}(X)}$, we have that

$$\text{diam}(\phi_\omega(X)) \geq 2RK^{-1}||D\phi_\omega||_\infty. \tag{4.13}$$

Note that since E is finite $\xi = \min\{||D\phi_e||_\infty : e \in E\} > 0$. Therefore,

$$\begin{aligned}
\text{diam}(\phi_\omega(X)) &\geq 2RK^{-1}||D\phi_\omega||_\infty \\
&\geq 2RK^{-2}||D\phi_{\omega|_{n-1}}||_\infty||D\phi_{\omega_n}||_\infty \\
&\geq 2RK^{-2}\xi||D\phi_{\omega|_{n-1}}||_\infty \\
&\geq 2RK^{-2}\xi\text{diam}(\phi_{\omega_{n-1}}(X)) \\
&\overset{4.10}{\geq} 2RK^{-2-1}r.
\end{aligned}$$

Therefore, for all $\omega \in Z$

1. $r \leq \text{diam}(\phi_\omega(x)) \leq 4r$ and
2. $\phi_\omega(x) \cap B(x, r) \neq \emptyset$.

Then from Lemma 4.2.1, there exists a c_1 not depending on x or r such that

$$\#Z \le c_1. \tag{4.14}$$

Before completing the proof, we will show that

$$\pi^{-1}(B(x,r)) \subset \bigcup_{\omega \in Z} [\omega]. \tag{4.15}$$

Let $\tau \in \pi^{-1}(B(x,r))$. Then $y = \pi(\tau) = \bigcap_{n=1}^{\infty} \phi_{\tau|_n}(X) \in B(x,r)$, so the set

$$\{n \in \mathbb{N} : \phi_{\tau|_n}(x) \subset B(x,2r), \phi_{\tau|_n}(x) \cap B(x,r) \neq \emptyset\}$$

is nonempty. Thus we may choose a minimal n such that

$$\phi_{\tau|_n}(x) \subset B(x,2r) \text{ and } \phi_{\tau|_n}(x) \cap B(x,r) \neq \emptyset. \tag{4.16}$$

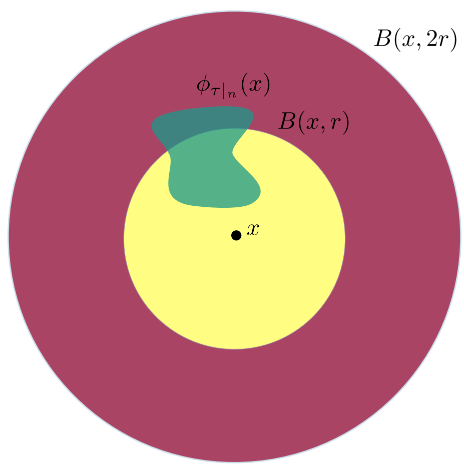

Figure 4.4: τ satisfies (4.12).

Therefore $\tau|_n \in Z$, and $\tau \in [\tau|_n]$ so we have shown (4.16) is true.

Now, we may complete the proof. We have

$$\begin{aligned}
m_h(B(x,r)) &= \tilde{m}_h \circ \pi^{-1}(B(x,r)) \\
&\overset{(4.15)}{\leq} \tilde{m}_h \left(\bigcup_{\omega \in Z} [\omega] \right) \\
&= \sum_{\omega \in Z} \tilde{m}_h([\omega]) \\
&\leq \sum_{\omega \in Z} ||D\phi_\omega||_\infty^h \\
&\overset{(4.13)}{\leq} \sum_{\omega \in Z} K^h (2R)^{-h} \text{diam}(\phi_\omega(x))^h \\
&\overset{(4.9)}{\leq} K^h (2R)^{-h} 4^h \sum_{\omega \in Z} r^h \\
&= K^h 2^h R^{-h} |Z| r^h \\
&\leq K^h 2^h R^{-h} c_1 r^h.
\end{aligned}$$

Letting $c = K^h 2^h R^{-h} c_1$, the proof of the inequality is complete. The final statement of the theorem that $m_h(J) \approx \mathcal{H}^h(J)$ follows from (4.1). □

4.3 Bowen's Formula

The following theorem, proved by Rufus Bowen in 1979, relates the topological pressure to the Hausdorff dimension $\dim_{\mathcal{H}}(A)$.

Theorem 4.3.1. *Let $\mathcal{S} = \{\phi_e\}_{e \in E}$ be a CIFS. Then*

$$h = \dim_{\mathcal{H}}(J) = \sup\{\dim_{\mathcal{H}}(J_F) : F \subset E \textit{ finite } \},$$

where h is Bowen's parameter.

Proof. Let $h_\infty = \sup\{\dim_{\mathcal{H}}(J_F) : F \subset E \text{ finite } \}$. Fix $t > h$. Then $\mathcal{P}(t) < 0$. Since $\mathcal{P}(t) < 0$ and $\mathcal{P}(t) = \lim_{n \to +\infty} \frac{1}{n} \log Z_n(t)$, there exists an $n_0 \in \mathbb{N}$ such that

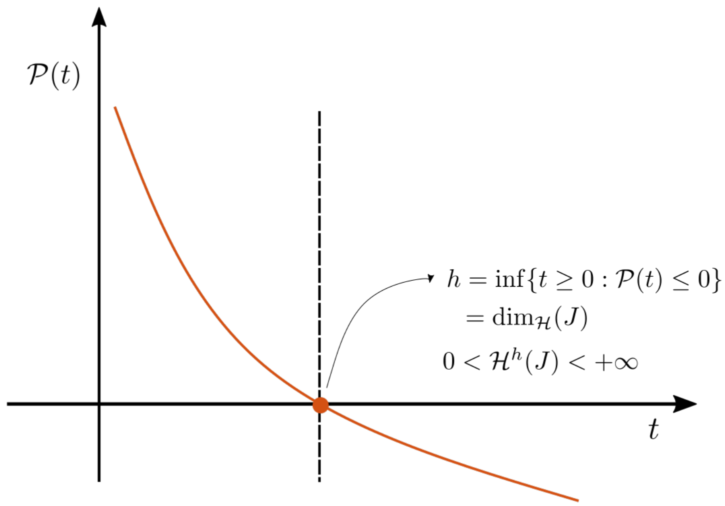

Figure 4.5: Bowen's Formula.

for all $n \geq n_0$

$$\frac{1}{n} \log Z_n(t) < \frac{\mathcal{P}(t)}{2}, \implies \sum_{\omega \in E^n} ||D\phi_\omega||_\infty^t = Z_n(t) < e^{\frac{n\mathcal{P}(t)}{2}}.$$

Therefore, for all $n \geq n_0$ we have

$$\sum_{\omega \in E^n} (\text{diam}\phi_\omega(X))^t \overset{BDP}{\leq} L^t \sum_{\omega \in E^n} ||D\phi_\omega||_\infty^t < L^t e^{\frac{n\mathcal{P}(t)}{2}}.$$

Note that the family $\{\phi_\omega(X)\}_{\omega \in E^n}$ covers the limit set J for all $n \in N$, and that for all $\epsilon > 0$ there exists an $n > 0$ such that

$$\sum_{\omega \in E} \text{diam}(\phi_\omega(X))^t \leq L^t e^{\frac{n\mathcal{P}(t)}{2}} < \epsilon.$$

Then it follows from the 3rd property listed of the Hausdorff measure that $\mathcal{H}^t(J) = 0$. Since we have shown this for $t > h$, we deduce that $h \geq \inf\{t \geq 0 : \mathcal{H}^t(J) = 0\} = \dim_{\mathcal{H}}(J)$.

For the other direction, let $F \subset E$ be a finite set. By the previous theorem, for all such sets $h_F = \dim_{\mathcal{H}}(J_F) \leq h_\infty$, so $\mathcal{P}_F(h_\infty) \leq 0$. Then by $\mathcal{P}(h_\infty) = \sup\{\mathcal{P}_F(h_) : F \subset E \text{ finite}\} \leq 0$, which implies that $h_\infty \geq h$ since $J_F \subset J$. Since $h_\infty \leq \dim_{\mathcal{H}}(J)$, it follows that $\dim_{\mathcal{H}}(J) \geq h$. □

Corollary 4.3.1. *If $\mathcal{S}$ consists of similarities, then*

$$h = \dim_{\mathcal{H}}(J_{\mathcal{S}}) = \inf\left\{t \geq 0 : \sum_{e \in E} ||D\phi_e||_\infty^t \leq 1\right\}.$$

Proof. Since $\mathcal{S}$ consists of similarities, $||D\phi_\omega|| = r_\omega$ for $\omega \in E^n$, where r_ω is a constant less than 1. Recall also that

$$||D\phi_\omega||_\infty = ||D\phi_{\omega_1}||_\infty \ldots ||D\phi_{\omega_n}||_\infty.$$

Therefore,

$$Z_n(t) = \sum_{\omega \in E^n} ||D\phi_\omega||_\infty^t = \left(\sum_{e \in E} ||D\phi_e||_\infty^t \right)^n = (Z_1(t))^n,$$

so $Z_n(t) = (Z_1(t))^n$. Taking the log of both sides, dividing both sides by n, and letting $n \to \infty$, we obtain that $\mathcal{P}(t) = \log Z_1(t)$. Thus,

$$\begin{aligned} \dim_{\mathcal{H}}(J) &= \inf\{t \geq 0 : \mathcal{P}(t) \leq 0\} \\ &= \inf\{t \geq 0 : \log Z_1(t) \leq 0\} \\ &= \inf\{t \geq 0 : Z_1(t) \leq 1\} \\ &= \inf \left\{ t \geq 0 : \sum_{e \in E} ||D\phi_e||_\infty^t \leq 1 \right\}. \end{aligned}$$

Furthermore, if E is finite, $h = \inf \left\{ t \geq 0 : \sum_{e \in E} ||D\phi_e||_\infty^t = 1 \right\}$. □

Chapter 5

Examples

When one dives into endlessness...[She] must divide [her] universe in distances of a specific length, in compartments that repeat themselves in endless series...Let me give a more tangible example...Long before there were people on earth, crystals grew in the earth's crust. One day a human being saw for the first time such a glimmering piece of regularity lying about, or he hit it with his stone pickax, and it broke off and fell in front of his feet and he picked it up and looked at it in his open hand and he was amazed.

M.C. Escher, "Approaching Infinity"

5.1 Self-Similar Fractals

The framework developed in the previous chapters for CIFS's and their dimension covers all linearly contracting self-similar sets that satisfy the open set condition. We illustrate this with some of the "classic" examples of fractals.

As a brief note, these types of fractals can also be described without the IFS framework, and their dimension can be estimated with the fractal dimension as in Chapter 4 of [1], which gives a numerical approximation equivalent to the Hausdorff dimension via a more simple mechanism. However, we include these examples to illustrate the robustness of the IFS framework, and as a precursor to further examples that are very well described by it.

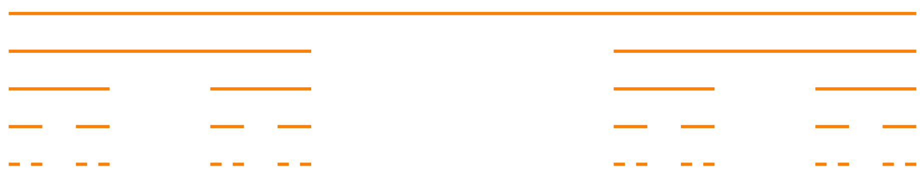

Figure 5.1: The first four iterations of the Cantor set.

5.1.1 Ternary Cantor Set

The ternary Cantor set is constructed on the unit interval $[0, 1]$ by removing the middle-third of every line segment at each iteration. As a CIFS with $E = \{1, 2\}$ it can be written as:

$$\mathcal{S} = \left\{\phi_n : [0,1] \to [0,1] : \phi_1(x) = \frac{1}{3}x, \phi_2(x) = \frac{1}{3}x + \frac{2}{3}\right\}$$

The maps ϕ_1 and ϕ_2 satisfy the open set condition, and are uniformly contracting by a constant value of $1/3$. Therefore, by Corollary 4.3.1, the Hausdorff dimension h of the Cantor set is the solution to

$$2\left(\frac{1}{3}\right)^h = 1,$$

from which we obtain

$$\dim_{\mathcal{H}}(J_{\mathcal{S}}) = \frac{\log 2}{\log 3} \approx 0.6309.$$

Similarly, we can consider higher dimensional versions of the ternary Cantor set. In 2-dimensions, with $E = \{1, 2, 3, 4\}$ we have the following maps for the corresponding CIFS, where $\phi_n : [0, 1] \times [0, 1] \to [0, 1] \times [0, 1]$ for all $n \in E$:

$$\begin{aligned}
\phi_1(x, y) &= \left(\frac{1}{3}x, \frac{1}{3}y\right) \\
\phi_2(x, y) &= \left(\frac{1}{3}x + \frac{2}{3}, \frac{1}{3}y\right) \\
\phi_3(x, y) &= \left(\frac{1}{3}x, \frac{1}{3}y + \frac{2}{3}\right) \\
\phi_4(x, y) &= \left(\frac{1}{3}x + \frac{2}{3}, \frac{1}{3}y + \frac{2}{3}\right).
\end{aligned}$$

The Hausdorff dimension h of the attractor of this set is the solution to

$$4\left(\frac{1}{3}\right)^h = 1,$$

from which we obtain

$$\dim_{\mathcal{H}}(J_{\mathcal{S}}) = \frac{\log 4}{\log 3} \approx 1.2619.$$

Likewise, in 3−dimensions we could construct 8 maps ϕ_i, each with a contracting factor of 1/3, and each a map from the unit cube to itself. Then the same framework applies, and the Hausdorff dimension is

$$\dim_{\mathcal{H}}(J_{\mathcal{S}}) = \frac{\log 8}{\log 3} \approx 1.8928.$$

We could apply the same techniques to determining the Hausdorff dimension of other Cantor sets as well which can be described by contracting, self-similar mappings.

5.1.2 Sierpinski Triangle

The Sierpinski triangle is constructed by beginning with an equilateral triangle, and then removing the middle third of each triangle at each iteration (see Figure 5.2). Let $E = \{1, 2, 3\}$. We have the corresponding contraction maps $\phi_n : T \to T$ for triangle $T \subset \mathbb{R}^2$ with vertices $(0,0), (1,0)$, and $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$:

$$\begin{aligned} \phi_1(x,y) &= \left(\frac{1}{2}x, \frac{1}{2}y\right) \\ \phi_2(x,y) &= \left(\frac{1}{2}x + \frac{1}{2}, \frac{1}{2}y\right) \\ \phi_3(x,y) &= \left(\frac{1}{2}x + \frac{1}{4}, \frac{1}{2}y + \frac{\sqrt{3}}{4}\right). \end{aligned}$$

These maps are uniformly contracting, and satisfy the open set condition. Since the contractions are similarities, by Corollary 4.3.1, the Hausdorff dimension of

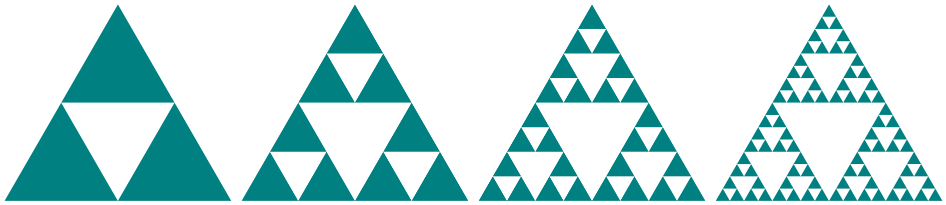

Figure 5.2: The first four iterations of the Sierpinski triangle

the Sierpinski triangle is the solution to

$$3\left(\frac{1}{2}\right)^h = 1,$$

and thus we obtain

$$\dim_{\mathcal{H}}\left(J_{\mathcal{S}}\right) = \frac{\log 3}{\log 2} \approx 1.5850.$$

5.1.3 Von Koch Curves

Von Koch curves can also be described by families of uniformly contracting, self-similar maps which satisfy the open set condition. The "classic" von Koch curve is formed by removing the middle third from each line segment, and then drawing an equilateral triangle pointing outwards from the line segment with the removed section as its base, at each iteration. The first four iterations are shown in Figure 5.3.

As a CIFS, the von Koch curve can be described by the following four contracting, self-similar maps with $E = \{1, 2, 3, 4\}$:

$$\begin{aligned}
\phi_1(x,y) &= \frac{1}{3}\begin{pmatrix} x \\ y \end{pmatrix} \\
\phi_2(x,y) &= \frac{1}{3}\begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1/3 \\ 0 \end{pmatrix} \\
\phi_3(x,y) &= \frac{1}{3}\begin{pmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1/2 \\ \sqrt{3}/6 \end{pmatrix} \\
\phi_4(x,y) &= \frac{1}{3}\begin{pmatrix} x \\ y \end{pmatrix}.
\end{aligned}$$

Figure 5.3: The first four iterations of the von Koch curve.

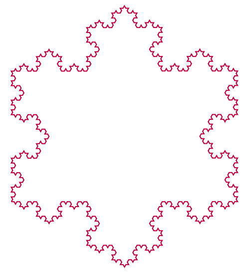

Figure 5.4: Three von Koch curves form a Koch snowflake.

Then it is clear that

$$\dim_{\mathcal{H}}(J_{\mathcal{S}}) = \frac{\log 4}{\log 3} \approx 1.2619.$$

There are variations on the von Koch curve, such as the Quadratic von Koch curves of Type 1 and 2 (Type 2 is also known as the Minkowski curve).

The Quadratic von Koch curve of Type 1 is formed by removing the middle third of the initial segment, and replacing it with three sides of a square pointing upwards (see Figure 5.5).

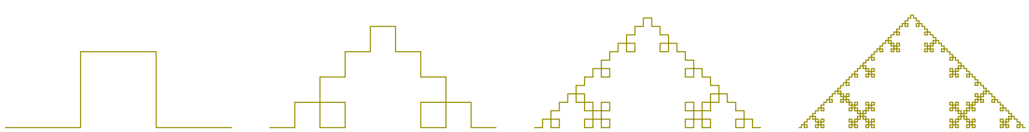

Figure 5.5: The Quadratic Type 1 von Koch Curve.

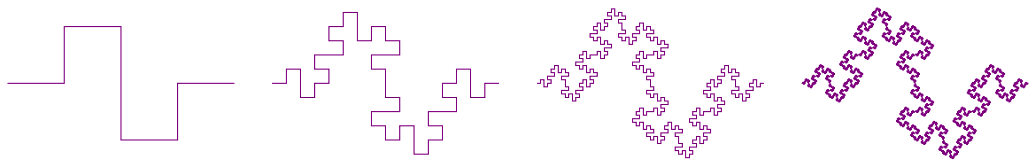

Figure 5.6: The Quadratic Type 2/Minkowski Curve.

As a CIFS, it is described by the following maps:

$$\begin{aligned}
\phi_1(x,y) &= \frac{1}{3}\begin{pmatrix} x \\ y \end{pmatrix} \\
\phi_2(x,y) &= \frac{1}{3}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1/3 \\ 0 \end{pmatrix} \\
\phi_3(x,y) &= \frac{1}{3}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1/3 \\ 1/3 \end{pmatrix} \\
\phi_4(x,y) &= \frac{1}{3}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix} \\
\phi_5(x,y) &= \frac{1}{3}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2/3 \\ 0 \end{pmatrix}.
\end{aligned}$$

As before, since all the maps contract by a factor of 1/3, and since there are 5 maps, the Hausdorff dimension is

$$\dim_{\mathcal{H}}(J_{\mathcal{S}}) = \frac{\log 5}{\log 3} \approx 1.4649.$$

The Minkowski curve is constructed similarly. The middle half of the segment is removed, and one quarter is replaced by a square pointing upwards, and the other quarter is replaced by a square pointing downwards (see Figure 5.6). Similarly to the other von Koch curves, one could construct maps to describe each of the resulting segments at each iteration. Since at each iteration each segment is replaced by 8 segments that are 1/4 of the original segment's length, we obtain

$$\dim_{\mathcal{H}}(J_{\mathcal{S}}) = \frac{\log 8}{\log 4} = 1.50.$$

5.2 Continued Fractions

Continued fractions can also be studied as conformal iterated function systems. Let $p \in [0,1]$ be an irrational number. It is known that p has a unique representation as a continued fraction,

$$p = \frac{1}{e_1 + \frac{1}{e_2 + \frac{1}{e_3 + \dots}}}$$

where $e_i \in \mathbb{N}$ for all $i \in \mathbb{N}$. We can represent the set of all irrationals $(0,1) \setminus \mathbb{Q}$ as the attractor of a CIFS. Note that every irrational number p corresponds to an infinite word

$$\omega = (e_1, e_2, \dots) \in \mathbb{N}^{\mathbb{N}},$$

and that for the maps $\phi_n(x) = \frac{1}{n+x}$

$$p = \bigcap_{n \in \mathbb{N}} \phi_{e_1} \circ \phi_{e_2} \circ \dots \phi_{e_n}([0,1]) = \bigcap_{n \in \mathbb{N}} \phi_{\omega|_n}([0,1])$$

The maps are uniformly contracting, and satisfy the open set condition. Therefore, we may consider the following CIFS,

$$\mathcal{CF}_{\mathbb{N}} = \left\{ \phi_n : [0,1] \to [0,1] : \phi_n = \frac{1}{n+x} \text{ for } n \in \mathbb{N} \right\},$$

where the attractor is

$$\mathcal{J}_{\mathcal{CF}_{\mathbb{N}}} = \bigcup_{\omega \in \mathbb{N}^{\mathbb{N}}} \bigcap_{n \in \mathbb{N}} \phi_{\omega|_n}([0,1]) = (0,1) \setminus \mathbb{Q}.$$

We could similarly define $\mathcal{CF}_E$ for any $E \subset \mathbb{N}$. [7] gives estimates for $\dim_{\mathcal{H}}(\mathcal{CF}_E)$, which relies on mechanisms beyond the scope of this thesis.

5.2.1 Complex Continued Fractions

As discussed in [9], one may also consider complex continued fractions, represented by the CIFS

$$\mathcal{CF}_E = \left\{ \phi_e : \overline{B}(1/2) \to \overline{B}(1/2) \right\}_{e \in E}$$

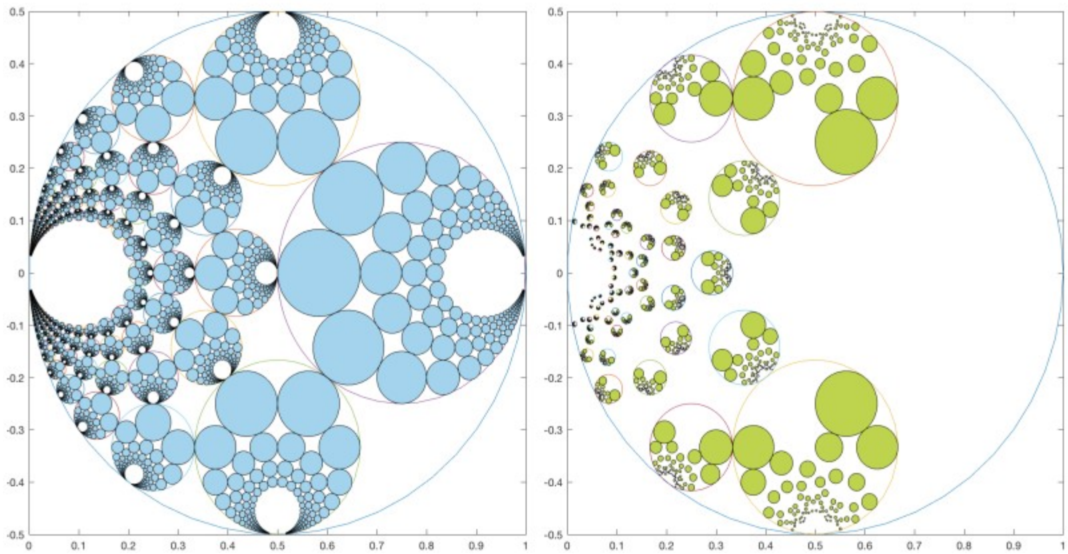

Figure 5.7: Approximation of the limit sets for the complex continued fraction CIFS's described.

where

$$E = \{m + ni : (m, n) \in \mathbb{N} \times \mathbb{Z}\} \text{ and } \phi_e(z) = \frac{1}{e + z}.$$

It is also a subject of interest to consider CIFS's where E is the set of Gaussian primes. The attractors of these sets are shown below, in Figures from [9] (used with permission from the author).

The Hausdorff dimension of the attractors of these systems is more difficult to approximate, and finding rigorous estimates is a subject of ongoing research.

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