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A consistent classical relativistic model of a finite size particle

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Abstract

An exactly solvable classical field theoretical model of a stable particle of finite size is studied. The model consists of a "swarm" of matter particles bound by an interplay of three static fields: one electromagnetic field, a massive scalar field, and a massive vector field. The internal forces due to the three fields balance each other to form a stable particle. The model parameters can be chosen such that the described particle has the mass and radius of a proton. The model is used to study the energy-momentum tensor of the proton. Numerical results are obtained for the energy density, pressure and shear forces. A comparison is made to other models and experimental studies. The model qualitatively reproduces the pressure distribution inside a proton in agreement with recent experimental findings.

Contents

1 Introduction

The understanding of the internal structure of the proton, neutron and atomic nuclei is at the forefront of current research in modern nuclear physics. The goal of this thesis is to explore a consistent classical model for the description of the proton structure to study quantities related to the energy-momentum tensor. The energy-momentum tensor contains information about the most fundamental properties of a particle, such as the well-known mass, but also the property known as the D-term. The D-term is experimentally nearly completely unknown, but has received considerable interest in recent literature [1].

The model studied in this thesis consists of a "swarm" or "dust" of constituent particles which interact with three classical fields in order to describe an extended particle. The constituent particles have an electric charge, e, and interact with a photon field A_μ, a charge of g_s which interacts with a scalar field ϕ, and a charge g_w which interacts with a vector field W_μ. This "swarm" of particles inhabits a finite "bag" with radius R and is described by a phase distribution function given by: $f(\vec{r}, \vec{p}, t)$. The scalar field, ϕ, creates attractive forces. If there was only a scalar field, the system would collapse, as nothing would be present to prevent the collapse of the system. Attraction is needed at larger distances so that the particles do not escape to infinity. However, repulsive forces located at the center and inner regions are also necessary to prevent the system from collapsing. A massive vector field can supply this "short range repulsion". Additionally, we need the interaction with the photon field. This is because we want to describe a charged particle (the goal will be to apply the model to the description of a proton). All three of these fields interact with the dust particles. This system is not stabilized by "ad hoc" forces, but instead by the self-consistent internal dynamics of the system. This model was introduced in Ref. [2].

Electric charges and currents generate electric and magnetic fields. Special relativity is important, as electric and magnetic fields depend on the observer. In this model, the particles (the "dust") with mass m swirl around (in the later solution remain static, however), carrying an electric charge, e. This generates electric and magnetic fields. In the relation: $\partial_\mu F^{\mu\nu} = ej^\nu$, ej^ν represents the electric current in 4-vector notation, which depends on the distribution of the particles and their motion. The electromagnetic potential, A^μ, describes the "photon field." Photons are massless, $m_\gamma = 0$, as required by gauge invariance. (Note: Light moves at the speed of light, but particles with a non-zero rest mass cannot move at the speed of light.) The vector field, W^μ, resembles a photon but differs in two key ways. First, it has a mass $m_w \neq 0$. Additionally, it does not interact with the particles through their electric charges, but instead through the charge g_w.

The energy-momentum tensor is of interest, because it contains information on the internal forces inside the proton including the pressure distribution [1]. Just recently, Burkert, Elouadrhiri, and Girod in their article, "The pressure distribution inside the proton," measured the pressure inside the proton for the first time [3]. The measurement was not direct: it was conducted using form-factors of the energy-momentum tensor at Jefferson Lab. Although the measurement was not direct, it provides the first experimentally inferred radial pressure distribution inside the proton. This result can be used as a benchmark to check the validity of several models of the proton, including the one described in this thesis. This topic is very current; the results of this theoretical thesis will help to improve the understanding of the proton structure and will play a helpful role for the interpretation of modern experiments.

The outline of this thesis is as follows: In Secs. 2 and 3 the notations of Relativistic Mechanics and Electrodynamics are introduced, and in Sec. 4 the gauge invariance of Electrodynamics is reviewed. In Secs. 5 and 6 the concepts of the forces due to massive vector and scalar fields are motivated. In Sec. 7 the classical model is introduced, and in Sec. 8 its solution in terms of static fields is presented. In Sec. 9 the parameters used in the numerical calculations are defined. Sec. 10 contains a general discussion of the energy-momentum tensor, and Sec. 11 contains the results for the energy-momentum tensor in the classical model. In Sec. 12 some numerical results from Ref. [2] are reproduced. Sec. 13 contains new results on the energy density, pressure and shear forces in this model. Sec. 14 contains some comments on the D-term and Sec. 15 contains the conclusions.

2 Relativistic Mechanics

In order to introduce the relativistic notation, we follow Ref. [4], Sec. 12. As we know from special relativity, the proper time, aka, the time an observer observes is given by:

$$d\tau = \sqrt{1-\tfrac{u^2}{c^2}}\; dt\ . \tag{1}$$

In this case, τ is the time the observer observes, t is the proper time, u is the velocity of the object relative to the ground observer ($u = \frac{dl}{dt}$), and c is the speed of light. If we want to know the distance per unit of proper time, we define a new quantity, which is the proper velocity:

$$\eta \equiv \tfrac{dl}{dt}. \tag{2}$$

From special relativity, we know that the proper velocity is also equal to:

$$\eta = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} u. \tag{3}$$

Here, η is the spatial part of a 4-vector, which is given by:

$$\eta^\mu \equiv \tfrac{dx^\mu}{d\tau}\ . \tag{4}$$

In this relation, dx^μ is the displacement 4-vector and $d\tau$ is the invariant. The zeroth component of this vector is:

$$\eta^0 = \tfrac{dx^0}{d\tau} = c\tfrac{dt}{d\tau} = \frac{c}{\sqrt{1-\frac{u^2}{c^2}}}\ . \tag{5}$$

As it has always been defined, momentum is equal to the mass times velocity. If we apply this principle to relativity, we get the following expression for proper momentum,

$$p \equiv m\eta = \frac{mu}{\sqrt{1-\frac{u^2}{c^2}}} \tag{6}$$

which is the relativistic momentum of an object (with mass m) traveling at a velocity of u. The relativistic momentum, p^μ, can also be written in the 4-vector form,

$$p^\mu \equiv m\eta^\mu \tag{7}$$

which is called the energy-momentum four vector. The zeroth component of the momentum is given by:

$$p^0 = m\eta^0 = \frac{mc}{\sqrt{1-\frac{u^2}{c^2}}}. \tag{8}$$

The relativistic energy, p^0c is:

$$E \equiv \frac{mc^2}{\sqrt{1-\frac{u^2}{c^2}}}. \tag{9}$$

The rest energy is given by:

$$E_{rest} \equiv mc^2. \tag{10}$$

If we calculate the scalar product of p^μ and $p_{\mu,}$ we get,

$$\left(p^0\right)^2 - (\vec{p} \cdot \vec{p}) = m^2c^2 \tag{11}$$

which is equal to:

$$E^2 - \vec{p}^2c^2 = m^2c^4 \tag{12}$$

in terms of relativistic energy and momentum.

3 Electrodynamics in Relativistic Notation

In order to introduce our relativistic model of a particle, we must first introduce the four-derivative,

$$\partial^\mu = \left(\frac{\partial}{\partial t}, -\nabla\right) \tag{13}$$

as well as:

$$\partial_\mu = \left(\frac{\partial}{\partial t}\ ,\ \nabla\right) \tag{14}$$

The four current, j^μ, is defined as,

$$j^\mu = \left(c\rho, \vec{j}\right) \tag{15}$$

where $\vec{j}$ is a vector and ρ is the zeroth component of a 4-vector. In this work we will use the natural units where $c = 1$ (and $\hbar = 1$). Because, in this case $c = 1$, we are left with:

$$j^\mu = (\rho, j)\,. \tag{16}$$

Since the current is conserved,

$$\partial_\mu j^\mu = \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0. \tag{17}$$

We also have the electromagnetic field tensor which is defined as,

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{18}$$

where A^μ is the electromagnetic 4-potential. The electromagnetic field tensor expressed in terms of the components of the electric and magnetic fields is given by:

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{bmatrix}. \tag{19}$$

A^μ, the electromagnetic 4-potential, is defined as:[1]

$$A^\mu = \left(A^0, A^i\right). \tag{20}$$

We define this 4-potential in a way such that we automatically satisfy the homogeneous Maxwell's equations:

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \tag{21}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{22}$$

Let us briefly review how $\vec{E}$ and $\vec{B}$ are expressed in terms of the electromagnetic potential. Since equation (22) holds true, we automatically know:

$$\vec{B} = \vec{\nabla} \times \vec{A} \tag{23}$$

If we take equation (21) and substitute in equation (23), we get:

$$\vec{\nabla} \times \vec{E} + \frac{\partial\left(\vec{\nabla} \times \vec{A}\right)}{\partial t} = 0, \tag{24}$$

$$\vec{\nabla} \times \vec{E} + \vec{\nabla} \times \frac{\partial \vec{A}}{\partial t} = 0, \tag{25}$$

$$\vec{\nabla} \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0. \tag{26}$$

The solution to this equation will be,

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla} A^0 \tag{27}$$

[1] In Electricity and Magnetism, it is customary to call $A^0 = \phi$ but in this work the symbol ϕ will be used later for the scalar field. To avoid confusion, the electrostatic potential will always be denoted by A^0.

where A^0 is the electrostatic potential. The field tensor, then, will be given by equation (19).

If we take the inhomogeneous Maxwell's equations,

$$\vec{\nabla} \cdot \vec{E} = \rho \tag{28}$$

$$\vec{\nabla} \times \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{1}{c}\vec{j} \tag{29}$$

and combine j^i and ρ into a 4-vector, we get:

$$j^\mu = (j^0 = c\rho, j^i)\,. \tag{30}$$

Now, the inhomogeneous Maxwell's equations can be represented as,

$$\partial_\nu F^{\mu\nu} = \frac{1}{c} j^\mu, \tag{31}$$

which relates $\partial_\nu F^{\mu\nu}$ to j^μ . If we set $\mu = 0$, we see,

$$\partial_\nu F^{0\nu} = \frac{1}{c} j^0 = \rho \tag{32}$$

which can also be written as $\partial_i F^{0i} = \rho$ or as $\vec{\nabla} \cdot \vec{E} = \rho$. Hence, $\vec{\nabla} \cdot \vec{E} = \rho$. If we set: $\mu = i$, we see that,

$$\partial_0 F^{i0} + \partial_j F^{ij} = \frac{1}{c} j^i \tag{33}$$

which is equivalent to equation (29), the second Maxwell equation.

Additionally, the homogeneous Maxwell's equations can be written as $\partial_\nu \widetilde{F}^{\mu\nu} = 0$, using the dual field tensor $\widetilde{F}^{\mu\nu}$:

$$\widetilde{F}^{\mu\nu} = \begin{bmatrix} 0 & B^1 & B^2 & B^3 \\ -B^1 & 0 & -E^3 & E^2 \\ -B^2 & E^3 & 0 & -E^1 \\ -B^3 & -E^2 & E^1 & 0 \end{bmatrix} \tag{34}$$

(The dual field tensor can be found by substituting $E_m \to -B_m$ and $B_m \to E_m$ into the field tensor equation.) If we substitute $v = 0$ into the dual field tensor, we get, $\partial_i \widetilde{F}^{\mu\nu} = \partial_i \widetilde{F}^{10} + \partial_i \widetilde{F}^{20} + \partial_i \widetilde{F}^{30} = \vec{\nabla} \cdot \vec{B} = 0$, which is the first homogeneous Maxwell equation.

4 Gauge Invariance in Electricity and Magnetism

If we look at the following Maxwell equation, $\vec{\nabla} \cdot \vec{B} = 0$, we can see that there automatically exists a solution: $\vec{B} = \vec{\nabla} \times \vec{A}$. If we now introduce an arbitrary Lorentz- scalar function,

$\Lambda(x,t)$, and replace $\vec{A}$ with $\vec{A}+\vec{\nabla}\Lambda$, the magnetic field will remain unchanged. The field tensor under this "gauge transformation" can be written as,

$$A^{\mu} \longrightarrow A'^{\mu} = A^{\mu} - \partial^{\mu}\Lambda \tag{35}$$

in 4-vector notation.

Additionally, if we consider the following Maxwell equation: $\vec{\nabla}\times\ \vec{E} = -\frac{\partial\vec{B}}{\partial t}$, there exists a solution:

$$\vec{E} = -\vec{\nabla}A^{0} - \tfrac{\partial\vec{A}}{\partial t}. \tag{36}$$

We can substitute A^0 for $A^0 - \frac{\partial\Lambda}{\partial t}$ without changing the electric field. Then, the Maxwell equation, $\partial_{\mu}F^{\mu\nu} = ej^{\nu}$ becomes equivalent to:

$$\partial_{\mu}F^{\mu\nu} = \partial_{\mu}\left(\partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}\right) = \Box A^{\nu} - \partial^{\nu}\left(\partial_{\mu}A^{\mu}\right) = ej^{\nu}. \tag{37}$$

Because A^{μ} is not unique, we can choose the Lorentz condition, $\partial_{\mu}A^{\mu} = 0$, so that the 4-vector potential is

$$\Box A^{\nu} = ej^{\nu}. \tag{38}$$

The Lorentz condition is an example of a "gauge condition." If we consider the solution to the previous equation in the absence of external currents and charge densities, $j^{\mu} = 0$, we call this a "free photon." One solution to this equation is the plane-wave, which is given by:

$$A^{\nu} = \varepsilon^{\nu}(k)\, e^{ik_{\alpha}x^{\alpha}}, \tag{39}$$

where $\varepsilon^{\nu}(k)$ is the polarization vector of the electromagnetic wave. If we take $\Box A^{\mu} = ik_{\mu}ik^{\mu}\varepsilon^{ik_{\mu}x^{\mu}} = 0$, we get a solution of $k_{\mu}k^{\mu} = 0$, which corresponds to a massless particle of $k_0^2 - \vec{k}^2 = 0$.

The Lorentz condition given earlier implies, $\partial_{\mu}A^{\mu} = ik_{\mu}\varepsilon^{\mu}(k)\, e^{ik_{\alpha}x^{\alpha}} = 0$, where $k_{\mu}\varepsilon^{\mu}(k) = 0$. Only one component of A^{μ} is removed by this solution, so we still have a "residual gauge freedom" due to $\partial_{\mu}A^{\mu} = \Box\Lambda = 0$ because Λ is arbitrary as long as it satisfies $\Box\Lambda = 0$. If we remove this "residual gauge freedom," we completely fix A^{μ} .

5 Theory of Massive Vector Fields

Now we generalize the concept of the photon field A^{μ} to a massive vector field W^{μ}. Since this field is allowed to be massive, there is no gauge invariance in this theory. This massive vector field satisfies the following equation of motion for a free particle:

$$\left(\Box + m_w^2\right) W^{\mu} = 0 \tag{40}$$

The plane wave solution describing a free particle with no force acting on it is given by:

$$W^{\mu} = \varepsilon^{\mu}(k)\, e^{ik_{\alpha}x^{\alpha}}. \tag{41}$$

Inserting this plane wave solution into equation (40) yields:

$$(\Box + m_w^2)\, W^\mu = (-k_\nu k^\nu + m_w^2)\, W^\mu = 0. \tag{42}$$

If we solve this equation, we get a solution:

$$-k_0^2 + \vec{k}^2 + m_w^2 = 0 \tag{43}$$

which is equal to $k_0^2 = \vec{k}^2 + m_w^2$. This solution corresponds to,

$$E^2 = (c\vec{p})^2 + (m_w c^2)^2 \tag{44}$$

where $E = c\hbar k_0$ and $\vec{p} = \hbar k$ (where we use the natural units: $\hbar = c = 1$).

The relativistic equation of motion for a free W^μ field is given by,

$$\partial_\mu G^{\mu\nu} + m_w^2 W^\nu = 0 \tag{45}$$

where

$$G^{\mu\nu} = \partial^\mu W^\nu - \partial^\nu W^\mu \tag{46}$$

From this, we obtain: $\partial_\nu \partial_\mu G^{\mu\nu} + m_w^2 \partial_\nu W^\nu = 0$. Because $\partial_\nu \partial_\mu G^{\mu\nu} = 0$ vanishes identically, the solution to this equation is: $m_w^2 \partial_\mu W^\mu = 0$. Since we are using a massive field, $m_w \neq 0$, so $\partial_\mu W^\mu = 0$. In this case, there is no "gauge invariance," and this condition removes one degree of freedom of the vector field W^μ which therefore has 3 degrees of freedom. The 3 degrees of freedom correspond to the 3 polarization states of a spin-1 particle: $S_z = -1, 0, 1$.

In order to couple this field to an external current j^μ, we introduce the coupling constant g_w and the equation of motion becomes:

$$\partial_\alpha G^{\alpha\beta} + m_w^2 W^\beta = g_w j^\beta \tag{47}$$

6 Theory of a Massive Scalar Field

We now introduce a free scalar field:

$$(\Box + m_s^2)\, \phi = 0 \tag{48}$$

A plane wave solution can be defined by,

$$\phi(x) = \phi_0 e^{ik_\alpha x^\alpha} \tag{49}$$

which can also be written as:

$$(\Box + m_s^2)\, \phi = (-k_\nu k^\nu + m_s^2)\, \phi = 0 \tag{50}$$

If we solve this equation, we get a solution of,

$$-k_0^2+\vec{k}^2+m_s^2=0 \tag{51}$$

which is equal to $k_0^2=\vec{k}^2+m_s^2$. The field can be coupled to a scalar density ρ with the coupling constant g_s. The equation of motion is then:

$$\left(\Box+m_s^2\right)\phi=g_s\rho. \tag{52}$$

7 Formulation of the Classical Model

In this section, the model of Ref. [2] is introduced. To describe the motion of a relativistic dust of particles interacting in a self-consistent manner with three fields $(c=\hbar=1)$, we start off with the following expressions:

$$\left[\left(m-g_s\phi\right)\left(\partial_t+v\cdot\nabla\right)+mF\cdot\partial_p\right]f\left(r,p,t\right)=0 \tag{53}$$

$$\partial_\mu F^{\mu\nu}=ej^\nu \tag{54}$$

$$\left(\Box+m_s^2\right)\phi=g_s\rho \tag{55}$$

$$\partial_\mu G^{\mu\nu}+m_w^2W^\nu=g_wj^\nu \tag{56}$$

where ∇ and ∂_p are the derivatives with respect to p and r. If we assign m_s and m_w a dimension of inverse length and use them only as measures of forces due to scalar and vector fields, equations (54)-(56) can be viewed as classical.

The spatial parts of the 4-velocity, u^μ and the 4-force f^μ are defined as:

$$u^\mu=\frac{p^\mu}{m} \tag{57}$$

$$f^\mu=eF^{\mu\nu}u_\nu-g_s\left(\partial^\mu-u^\mu u^\nu\partial_\nu\right)\phi+g_wG^{\mu\nu}u_\nu. \tag{58}$$

As per usual, $v^i=\frac{u^i}{u^0}=\frac{p^i}{E_p}$ and $F^i=\frac{f^i}{u^0}$.

The anti-symmetrical field tensors, $F^{\mu\nu}$ and $G^{\mu\nu}$ are constructed from their corresponding four vectors, A_μ and G_μ, as defined by equations (18) and (46).

The 4-current j^μ and scalar density ρ are defined as:

$$j^\mu\left(r,t\right)=\int\frac{\partial^3p}{E_p}p^\mu f\left(r,\vec{p},t\right), \tag{59}$$

$$\rho\left(r,t\right)=\int\frac{\partial^3p}{E_p}mf\left(r,\vec{p},t\right). \tag{60}$$

In general, this scalar density should not be confused with the zeroth component of $j^\mu = \left(\rho, \vec{j}\right)$. Notice that in the static case below, this distinction becomes unnecessary.

When all of the particles are at rest, the static case is given by:

$$f\left(r, \vec{p}, t\right) = \delta\left(\vec{p}\right)\rho\left(r\right) \tag{61}$$

and equations (54)-(56) reduce to:

$$-\Delta A_0 = e\rho \tag{62}$$

$$\left(-\Delta + m_s^2\right)\phi = g_s\rho \tag{63}$$

$$\left(-\Delta + m_w^2\right)W_0 = g_w\rho \tag{64}$$

In order to satisfy equation (53), we must assume the following, which says that the net force on each particle in the bag is zero:

$$\rho\vec{F} \equiv -\rho\vec{\nabla}\left(eA_0 - g_s\phi + g_w W_0\right) = 0. \tag{65}$$

This equilibrium condition is only imposed where matter is present. We also impose the following normalization condition:

$$\int d^3r\rho = 1 \tag{66}$$

8 The Solution of the Model

Inside the bag, when $r \leq R$, we get the following solutions for equations (54)-(56):

$$\rho = f_+ - f_- \tag{67}$$

$$eA_0 = e^2\left(\frac{f_+}{k_+^2} - \frac{f_-}{k_-^2}\right) - V_0 \tag{68}$$

$$g_s\phi = g_s^2\left(\frac{f_+}{k_+^2+m_s^2} - \frac{f_-}{k_-^2+m_s^2}\right) \tag{69}$$

$$g_w W_0 = g_w^2\left(\frac{f_+}{k_+^2+m_w^2} - \frac{f_-}{k_-^2+m_w^2}\right). \tag{70}$$

where

$$f_\pm = \frac{d_\pm}{4\pi}\frac{\sin(k_\pm r)}{r}, \tag{71}$$

and $k_\pm$ is:

$$k_\pm^2 = \frac{B\pm\sqrt{D}}{2}. \tag{72}$$

D and B are defined as:

$$D = B^2 - 4eQ^2m_s^2m_w^2, \tag{73}$$

B is:

$$B = \left(g_s^2 - e^2\right)m_w^2 - \left(g_w^2 + e^2\right)m_s^2, \tag{74}$$

and Q is:

$$Q^2 = e^2 - g_s^2 + g_w^2. \tag{75}$$

We can check equations (62), (67) and (68) by substituting equations (67) and (68) into equation (62). Following equation (62), we know:

$$A_0 = \frac{e^2}{e}\left(\frac{f_+}{k_+^2} - \frac{f_-}{k_-^2}\right) - \frac{V_0}{e} = e\left(\frac{f_+}{k_+^2} - \frac{f_-}{k_-^2}\right) - \frac{V_0}{e}. \tag{76}$$

Plugging equation (67) into (68), we obtain:

$$-\Delta A_0 = e\left(f_+ - f_-\right). \tag{77}$$

We now substitute Equation (76) into Equation (77), use the definitions of $f_\pm$ and $k_\pm^2$ and simplify:

$$-\Delta\left[e\left(\frac{f_+}{k_+^2} - \frac{f_-}{k_-^2}\right) - \frac{V_0}{e}\right] = e\left(f_+ - f_-\right) \tag{78}$$

$$-e\Delta\left[\left(\frac{f_+}{k_+^2} - \frac{f_-}{k_-^2}\right) - \frac{V_0}{e^2}\right] = e\left(f_+ - f_-\right) \tag{79}$$

$$-e\Delta\left[\frac{\frac{d_+}{4\pi}\frac{\sin(k_+ r)}{r}}{\frac{B+\sqrt{D}}{2}} - \frac{\frac{d_-}{4\pi}\frac{\sin(k_- r)}{r}}{\frac{B-\sqrt{D}}{2}}\right] = e\left(f_+ - f_-\right) \tag{80}$$

$$e\left[\frac{d_+}{4\pi}\frac{\sin(k_+ r)}{r} - \frac{d_-}{4\pi}\frac{\sin(k_- r)}{r}\right] = e\left(f_+ - f_-\right) \tag{81}$$

$$e\left(f_+ - f_-\right) = e\left(f_+ - f_-\right) \tag{82}$$

Therefore, Equation (62) holds true.

In order to show that equation (63) is true, we use equation (69) to solve for ϕ and then simplify:

$$\phi = g_s\left(\frac{f_+}{k_+^2 + m_s^2} - \frac{f_-}{k_-^2 + m_s^2}\right), \tag{83}$$

So,

$$\left(-\Delta + m_s^2\right) g_s\left(\frac{f_+}{k_+^2 + m_s^2} - \frac{f_-}{k_-^2 + m_s^2}\right) = g_s\rho, \tag{84}$$

$$\left(-\Delta + m_s^2\right)\left(\frac{\frac{d_+}{4\pi}\frac{\sin(k_+ r)}{r}}{k_+^2 + m_s^2} - \frac{\frac{d_-}{4\pi}\frac{\sin(k_- r)}{r}}{k_-^2 + m_s^2}\right) = \rho. \tag{85}$$

Taking the Laplacian, and factoring out the constant $\frac{1}{4\pi r}$, we are left with:

$$\frac{1}{4\pi r}\left[\left(k_+^2 + m_s^2\right)\frac{d_+\sin(k_+ r)}{k_+^2 + m_s^2}\right] - \frac{1}{4\pi r}\left[\left(k_-^2 + m_s^2\right)\frac{d_-\sin(k_- r)}{k_-^2 + m_s^2}\right] = \rho. \tag{86}$$

Cancelling out terms reveals:

$$\frac{1}{4\pi}\left[\frac{d_+\sin(k_+ r)}{r} - \frac{d_-\sin(k_- r)}{r}\right] = \rho \tag{87}$$

And finally, $\rho = \rho$.

In order to show that equation (64) is true, we use equation (70) and repeat the similar steps as above:

$$W_0 = g_w\left(\frac{f_+}{k_+^2+m_w^2} - \frac{f_-}{k_-^2+m_w^2}\right) \tag{88}$$

$$\left(-\Delta + m_w^2\right) g_w\left(\frac{f_+}{k_+^2+m_w^2} - \frac{f_-}{k_-^2+m_w^2}\right) = g_w\rho \tag{89}$$

$$\left(-\Delta + m_w^2\right)\left(\frac{\frac{d_+}{4\pi}\frac{\sin(k_+r)}{r}}{k_+^2+m_w^2} - \frac{\frac{d_-}{4\pi}\frac{\sin(k_-r)}{r}}{k_-^2+m_w^2}\right) = \rho \tag{90}$$

$$\frac{1}{4\pi r}\left[\left(k_+^2 + m_w^2\right)\frac{d_+\sin(k_+r)}{k_+^2+m_w^2}\right] - \frac{1}{4\pi r}\left[\left(k_-^2 + m_w^2\right)\frac{d_-\sin(k_-r)}{k_-^2+m_w^2}\right] = \rho \tag{91}$$

$$\frac{1}{4\pi}\left[\frac{d_+\sin(k_+r)}{r} - \frac{d_-\sin(k_-r)}{r}\right] = \rho \tag{92}$$

$$\rho = \rho$$

Therefore, the equation holds true.

Outside of the bag, (when r>R), we obtain the following solutions:

$$\rho = 0 \tag{93}$$

$$eA_0 = \frac{e^2}{4\pi r} \tag{94}$$

$$g_s\phi = \frac{b_s}{4\pi r}\ e^{-m_s(r-R)} \tag{95}$$

$$g_w W_0 = \frac{b_w}{4\pi r}e^{-m_w(r-R)} \tag{96}$$

Equation (93) is true because there are no dust particles outside by construction. We also know $eA_0 = \frac{e^2}{4\pi r}$ due to Coulomb's Law, which confirms equation (94). The solutions for the scalar and vector fields (95 and 96) are obtained by solving the equations (54) - (56) analytically (and if $m_s,\ m_w \to 0$, it is also $\frac{1}{r}$ potential which is a feature of massless fields). The six parameters $b_w,\ b_s,\ d_+,\ d_-,\ V_0$ and R are determined from requiring that the fields $A_0(r), W_0(r)$, and $\phi(r)$ be continuous and differentiable at $r = R$. This can be implemented in Mathematica.

9 Parameter Fixing

In order to describe a proton, the following parameters were proposed in [2] and will be used later in this thesis to produce numerical results:

$$m = 938\,\mathrm{MeV} \tag{97}$$

$$e = \sqrt{\frac{4\pi\hbar c}{137}} \tag{98}$$

$$m_s = 550\frac{\mathrm{MeV}}{\hbar c} \tag{99}$$

$$m_w = 783 \tfrac{\mathrm{MeV}}{\hbar c} \tag{100}$$

$$g_s^2 = 91.64\, \hbar c \tag{101}$$

$$g_w^2 = 136.2\, \hbar c \tag{102}$$

For these parameters, the requirements of continuity and differentiability of $\phi(r), A_0(r), W_0(r)$ fixes the constants $b_w, b_s, d_+, d_-, V_0,$ and R to have the following values:

$$b_w = 1354.13\,\mathrm{MeV\,fm} \tag{103}$$

$$b_s = 1786.38\,\mathrm{MeV\,fm} \tag{104}$$

$$d_+ = 2.02477/\mathrm{fm}^2 \tag{105}$$

$$d_- = -3.93639/\mathrm{fm}^2 \tag{106}$$

$$V_0 = 31.42\,\mathrm{MeV} \tag{107}$$

$$R = 1.05\,\mathrm{fm} \tag{108}$$

10 The Energy Momentum Tensor

The energy momentum tensor, $T^{\mu\nu}$, can be derived from the Lagrangian of a theory, and describes how the fields couple to gravity in general relativity. The energy momentum tensor contains fundamental information describing a particle: $T^{00}(r)$ is the energy density and $\int d^3r\ T^{00}(r)$ tells us the mass of the particle. The T^{0k} components, if non-zero, are related to the spin of a particle. If $T^{0k} = 0$, the described particle has spin zero. In a classical system, $T^{0k} = 0$ indicates that there is no rotation in the system. The T^{ij} components of the tensor are related to the internal forces as described by the stress tensor in continuum mechanics. T^{ij} is given by,

$$T^{ij} = \left(e_r^i e_r^j - \tfrac{1}{3}\delta^{ij}\right) s(r) + \delta^{ij} p(r) \tag{109}$$

where $s(r)$ is the shear force, $p(r)$ is the pressure and ${e_r}^i$ is the unit vector in the radial direction. The energy momentum tensor is conserved, $\partial_\mu T^{\mu\nu} = 0$, therefore the following relations hold [1]:

$$\tfrac{2}{3}s'(r) + \tfrac{2}{r}s(r) + p(r) = 0 \tag{110}$$

and,

$$\textstyle\int_0^\infty dr\ r^2 p(r) = 0 \tag{111}$$

Equation (111) tells us that the internal forces inside a system must exactly balance each other. If the integral in this equation gives a positive result, then the system explodes. If the result is negative, the system implodes. It is a necessary (though not sufficient) condition for stability that the integral in Equation (111) is exactly zero.

11 The Energy Momentum Tensor in our Model

In our model, the total energy momentum tensor is defined as [2]:

$$\begin{aligned} T^{\mu\nu} = & \left(m - g_s\phi\right)\rho u^\mu u^\nu + F^{\mu\lambda}F^\nu_\lambda + \frac{1}{4}g^{\mu\nu}F_{\lambda p}F^{\lambda p} + \partial^\mu\phi\partial^\nu\phi - g^{\mu\nu}\frac{1}{2}\left(\partial_\lambda\phi\partial^\lambda\phi - m_s^2\phi^2\right) \\ & + G^{\mu\lambda}G^\nu_\lambda + m_\nu^2 W^\mu W^\nu + g^{\mu\nu}\left(\frac{1}{4}G_{\lambda p}G^{\lambda p} - \frac{1}{2}m_\nu^2 W_\lambda W^\lambda\right) \end{aligned} \tag{112}$$

We can find the T^{00} component of this tensor by substituting $\mu = v = 0$ into the previous equation and simplifying terms. If we do this process component by component, we get:

[1] : $\left(m - g_s\phi\right)\rho u^0 u^0 = \left(m - g_s\phi\right)\rho$ because $u^0 = 1$.

[2] : $F^{0\lambda}F^0_\lambda = \left(\partial^0 A^\lambda - \partial^\lambda A^0\right)\left(\partial_\lambda A^0 - \partial^0 A_\lambda\right) = \left(-\partial^\lambda A^0\right)\left(\partial_\lambda A^0\right) = \left(\nabla A^0\right)^2$, which occurs because $\partial^0 A^\lambda = 0$.

[3] : $\frac{1}{4}g^{00}F_{\lambda p}F^{\lambda p} = \frac{1}{4}g^{00}\left(-2\vec{E}^2\right) = \frac{1}{4}\left(-2\left(\nabla A_0\right)^2\right) = -\frac{1}{2}\left(\nabla A^0\right)^2$. (Note: This result comes from $F^{\mu\nu} = -F^{\nu\mu}$ and $F^{00} = F^{11} = F^{22} = F^{33} = 0$.)

[4]: $\partial^0\phi\partial^0\phi = 0$, since the solutions are all static in this model.

[5]: $-g^{00}\frac{1}{2}\left(-\left(\nabla\phi\right)^2 - m_s^2\phi^2\right) = \frac{1}{2}\left(\nabla\phi\right)^2 + \frac{1}{2}\left(m_s^2\phi^2\right)$

[6]: $G^{0\lambda}G^0_\lambda = \left(\partial^0 W^\lambda - \partial^\lambda W^0\right)\left(\partial_\lambda W^0 - \partial^0 W_\lambda\right) = \left(-\partial^\lambda W^0\right)\left(\partial_\lambda W^0\right) = \left(\nabla W_0^2\right)$

[7]: $m_w^2 W^0 W^0 = m_w^2 W_0^2$

[8]: $g^{00}\left(\frac{1}{4}G_{\lambda p}G^{\lambda p} - \frac{1}{2}m_w^2 W_\lambda W^\lambda\right) = -\frac{1}{2}\left(\nabla W_0\right)^2 - \frac{1}{2}m_w^2 W_0^2$. (Note: We got this result by taking the zero component of the G tensor and simplifying, paralleling the method used in step 3).

By adding the individual components found, we obtain the following:

$$T^{00} = \left(m - g_s\phi\right)\rho + \tfrac{1}{2}\left(\nabla A_0\right)^2 + \tfrac{1}{2}\left(\nabla\phi\right)^2 + \tfrac{1}{2}m_s^2\phi^2 + \tfrac{1}{2}\left(\nabla W_0\right)^2 + \tfrac{1}{2}m_w^2 W_0^2 \tag{113}$$

If we integrate this quantity over space, we obtain the combined energy of the three fields:

$$\begin{aligned} E_{tot} &= \int d^3r\left(m - g_s\phi\right)\rho + \tfrac{1}{2}\int d^3r\left[\left(\nabla A_0\right)^2 + \left(\nabla\phi\right)^2 + m_s^2\phi^2 + \left(\nabla W_0\right)^2 + m_w^2 W_0^2\right] \\ &= \int d^3r\left(m - g_s\phi\right)\rho + \tfrac{1}{2}\int d^3r\left[-A_0\Delta A_0 - \phi\Delta\phi + m_s^2\phi^2 - W_0\Delta W_0 + m_w^2 W_0^2\right] \end{aligned} \tag{114}$$

If we substitute equations (62)-(64) into this integral, we get:

$$E_{tot} = m + \tfrac{1}{2}\int d^3r\left[eA_0\rho - g_s\phi\rho + g_w W_0\rho\right] \tag{115}$$

If we set the solution of the integral equal to V_0, we get:

$$E_{tot} = m - \tfrac{V_0}{2} \ . \tag{116}$$

We can similarly find the T^{ij} component, undergoing a similar process:

$[1] : (m - g_s\phi)\, \rho u^i u^j = 0$ (Note: This term drops out because u^i and u^j only have a zero component).

$[2] \ : \ F^{i\lambda}F^j_\lambda \ = \ F^{i0}F^j_0 + F^{\mu j}F^i_j \ = \ F^{i0}F^{ij} \ = \ (-\nabla^i A^0 - \partial_0 A^i)\,(\partial_0 A^j + \nabla^j A_0) \ = -\,(\nabla^i A^0)\,(\nabla^j A^0) = -A^{0'}\,(r)^2\, e^i_r e^j_r$

$[3] : \frac{1}{4} g^{ij} F_{\lambda p}F^{\lambda p} = \frac{1}{4}\left(-\delta^{ij}\right)\left(-2\vec{E}^2\right) = \frac{1}{2}\left(\delta^{ij}\right)\left(\nabla A^0\right)^2$ (Note: $g^{ij} = -\delta^{ij}$).

[4]: $\partial^i\phi\partial^j\phi = \left(-\nabla^i\phi\,(r)\right)\left(-\nabla^i\phi\,(r)\right) = \phi'\,(r)^2\, e^i_r e^j_r$

[5]: $-g^{ij}\frac{1}{2}\left(-\left(\nabla\phi\right)^2 - m_s^2\phi^2\right) = -\frac{1}{2}\delta^{ij}\left[\left(\nabla\phi\right)^2 + m_s^2\phi^2\right]$

[6]: $G^{0\lambda}G^0_\lambda = G^{i\lambda}G^j_\lambda = G^{i0}F^j_0 + G^{\mu j}F^i_j = F^{i0}F^{ij} = (-\nabla^i W^0 - \partial_0 W^i)\,(\partial_0 W^j + \nabla^j W_0) = -\,(\nabla^i W^0)\,(\nabla^j W^0) = -W_0'\,(r)^2\, e^i_r e^j_r$

[7]: $m_w^2 W^i W^j = 0$ (Note: This term drops out because the W field only has a zero component).

[8]: $g^{ij}\left(\frac{1}{4}G_{\lambda p}G^{\lambda p} - \frac{1}{2}m_w^2 W_\lambda W^\lambda\ \right)$
$= -\delta^{ij}\left(\frac{1}{4}\left(\partial^0 W^\lambda - \partial^\lambda W^0\right)\left(\partial_\lambda W^0 - \partial^0 W_0\right) - \frac{1}{2}m_w^2 W_\lambda W^\lambda\ \right)$
$= -\delta^{ij}\left(\frac{1}{4}\left(-\partial^\lambda W^0\right)\left(\partial_\lambda W^0\right) - \frac{1}{2}m_w^2 W_\lambda W^\lambda\ \right)$
$= \frac{1}{2}\delta^{ij}\left[\left(\nabla W_0\right)^2 + m_w^2 W_0^2\right]$

When we add up all of these individual components, we obtain the following:

$$\begin{aligned} T^{\mathrm{ij}}(r) &= e^i_r e^j_r\left(\phi'(r)^2 - A_0'(r)^2 - W_0'(r)^2\right) \\ &+ \frac{\delta^{ij}}{2}\left((\nabla A^0)^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m_s^2\phi^2 + \frac{1}{2}(\nabla W_0)^2 + \frac{1}{2}m_w^2 W_0^2\right) \end{aligned} \tag{117}$$

Since T^{ij} is given by:

$$T^{ij} = \left(e^i_r e^j_r - \frac{1}{3}\delta^{ij}\right) s\,(r) + p\,(r)\,\delta^{ij}$$

or

$$T^{ij} = e^i_r e^j_r s\,(r) + \left(-\frac{1}{3}s\,(r) + p\,(r)\right)\delta^{ij}\,.$$

We can compare the structures of our calculated energy-momentum tensor value to the general structure of T^{ij} and solve for the shear force, $s\,(r)$ and the pressure, $p\,(r)$. When we do this, we obtain the following results:[2]

$$s\left(r\right)=\phi'\left(r\right)^{2}-W_{0}'\left(r\right)^{2}-A_{0}'\left(r\right)^{2} \tag{118}$$

and

$$p\left(r\right)=\tfrac{1}{6}A_{0}'\left(r\right)^{2}-\tfrac{1}{6}\phi'\left(r\right)^{2}-\tfrac{1}{2}m_{s}^{2}\phi^{2}+\tfrac{1}{6}W_{0}'\left(r\right)^{2}+\tfrac{1}{2}m_{w}^{2}W_{0}^{2}. \tag{119}$$

12 Numerical Results from the Model

In this section we reproduce some of the results presented in Ref. [2].

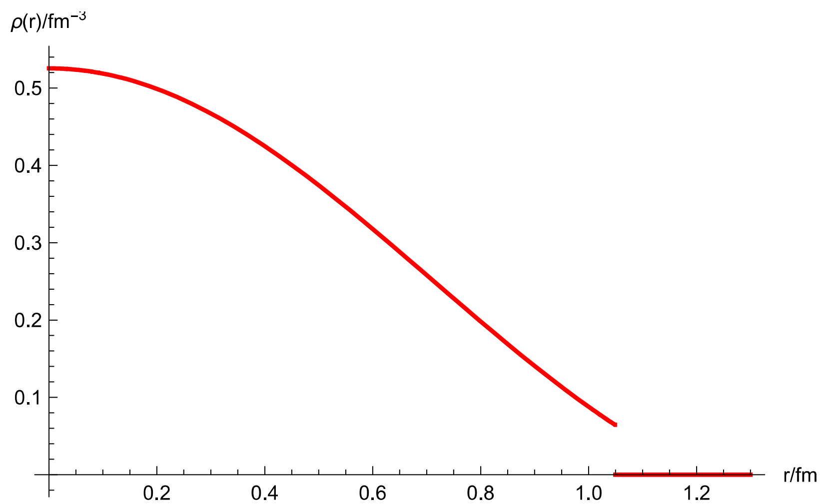

Figure 1: The density of matter, $\rho\left(r\right)$, vs. the radius r.

In Figure 1, we graph the matter distribution $\rho\left(r\right)$ with respect to r. This plot shows us how the density of matter changes at different points in the bag, which is spherically symmetric. It is important to note that there is a jump at 1.05 fm, which is the radius of the bag, because the dust particles are only inside the bag and there is no matter outside.

[2] In order to arrive at this result, we needed to use the following relations: $\nabla^{i}W^{0}=W_{0}'\left(r\right)e_{r}^{i}$ and $(\nabla W^{0})^{2}=W_{0}'(r)^{2}e_{r}^{i}e_{r}^{i}=W_{0}'(r)^{2}$. This occurs because $e_{r}^{i}e_{r}^{i}=\vec{e_{r}}^{2}=1$.

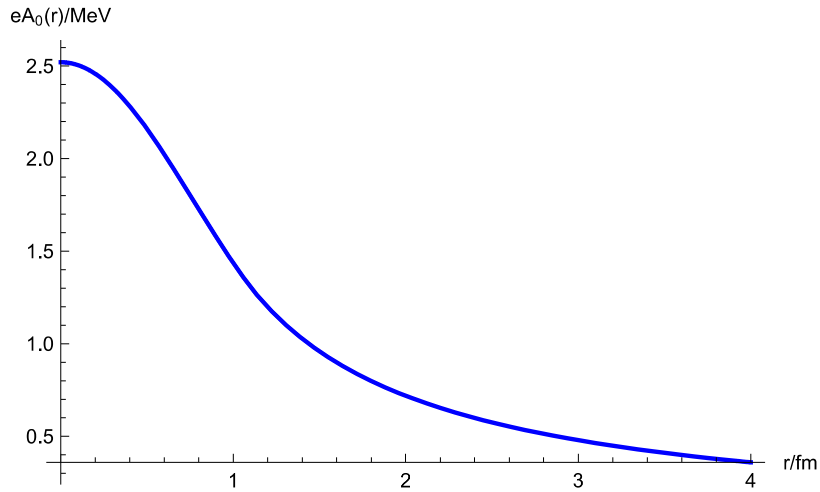

Figure 2: The photon field, $eA_0(r)$, vs. the radius r.

In Figure 2, we show how the photon field (scaled with the electric charge e), eA_0, changes at different points in the bag. Since the mass of the photon is equal to zero, we see that the long range of the electromagnetic potential is exactly $\frac{1}{r}$, for $r > R$. This means that the range of the electric forces extends far beyond the radius $R = 1.05$ fm of the bag. However, the electromagnetic interaction is relatively weak, and $eA_0\left(r\right)$ does not exceed 2.5 MeV in the center of the proton.

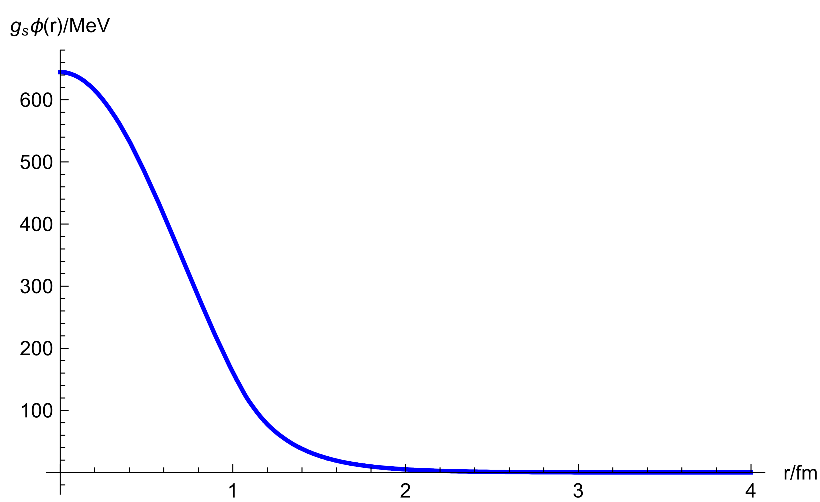

Figure 3: The scaled scalar field, $g_s\phi\left(r\right)$, vs. radius r.

In Figure 3, we display the scalar field ϕ scaled with a charge g_s at different points in the bag. We clearly see that the potential energy associated with this field is much larger than the electrostatic potential $eA_0\left(r\right)$. This is because this field is associated with nuclear forces which are much stronger than electric forces. We see that $g_s\phi\left(r\right)$ is larger than 600 MeV in the center of the proton. Since the field $\phi_s\left(r\right)$ is massive, its range is also much shorter because the field is proprotional to $\frac{e^{-m_s r}}{r}$ for $r > R$ outside the bag.

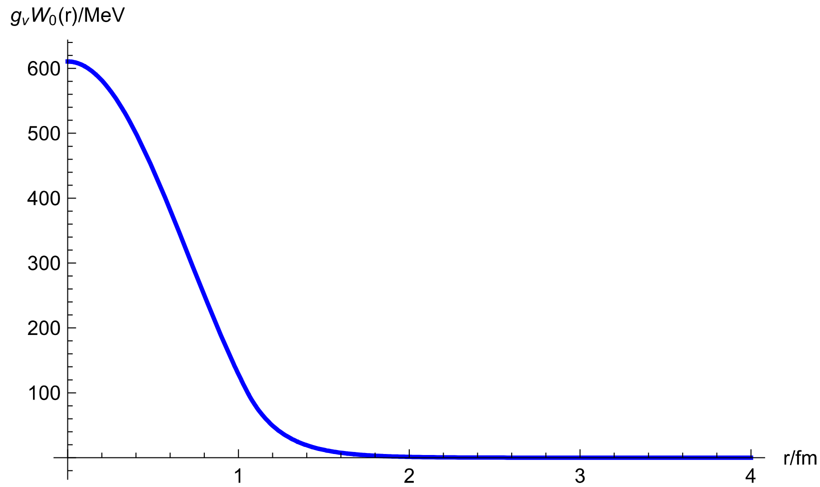

Figure 4: The scaled vector field, $g_w W_0\left(r\right)$, vs. the radius r.

In Figure 4, we show the vector field, $W_0\left(r\right)$, scaled with a charge of g_w throughout the bag. $g_w W_0\left(r\right)$ is very large, as this field is also associated with the strong force. $g_w W_0\left(r\right)$ is somewhat smaller than $g_s\phi\left(r\right)$ in the center of the proton. Additionally, its range $\sim \frac{e^{-m_w r}}{r}$, is shorter than the range of the field ϕ_s because $m_w > m_s$.

The Figures 1-4 have already been shown in Ref. [2]. Our results shown in this section reproduce the results of Ref. [2]. This is an important test of the numerical calculations which were made using Mathematica.

13 Energy-Momentum Tensor Density

In this section we use the model of Ref. [2] to present new results on the energy-momentum tensor densities. The results shown in this section represent original research and have not been shown in literature before.

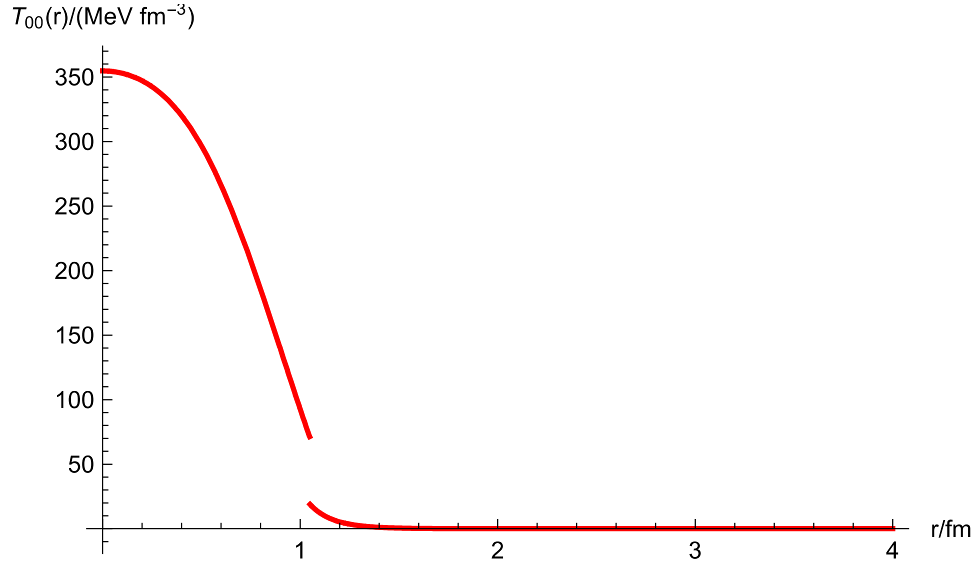

Figure 5: The zero-zero component of the energy-momentum tensor, T^{00}, vs. the radius r.

In Figure 5, we show the energy density $T^{00}(r)$, i.e., the zero-zero component of the energy-momentum tensor plotted against the radius. Parallel to Figure 1, there is a discontinuity at 1.05 fm due to the discontinuity of the matter distribution. If we integrate the energy density over the volume we reproduce equation (116).

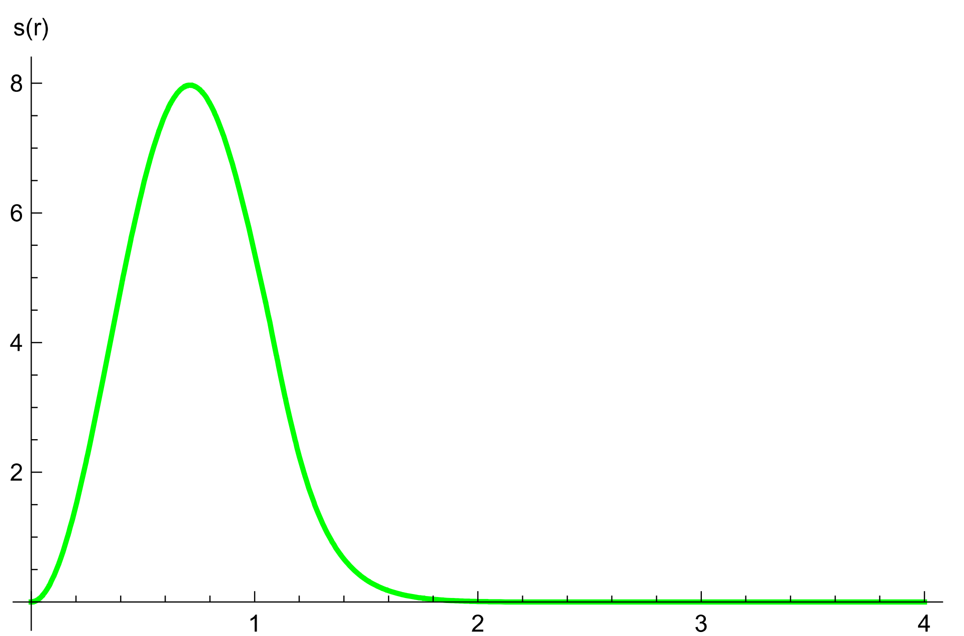

Figure 6: The shear force, $s\left(r\right)$, vs. the radius r.

Figure 6 shows the shear force, $s\left(r\right)$ vs. the radius, r. If our finite-size particle had a “sharp” edge at the radius R, then the shear force would be proportional to a delta function concentrated at R. However, our finite-size particle is much more diffuse and doesn’t have a “sharp edge.” Instead, our result looks like a “very diffuse” delta function. We can characterize the size of particle in terms of the charge radius, which is defined as:

$$r_p^2 = \int d^3r\ r^2 \rho\left(r\right)$$

Numerically, $r_p = 0.714$ fm which corresponds to about the size of a proton. In Figure 6, $s\left(r\right)$ peaks at 0.710 fm, which is very close to our calculated value.

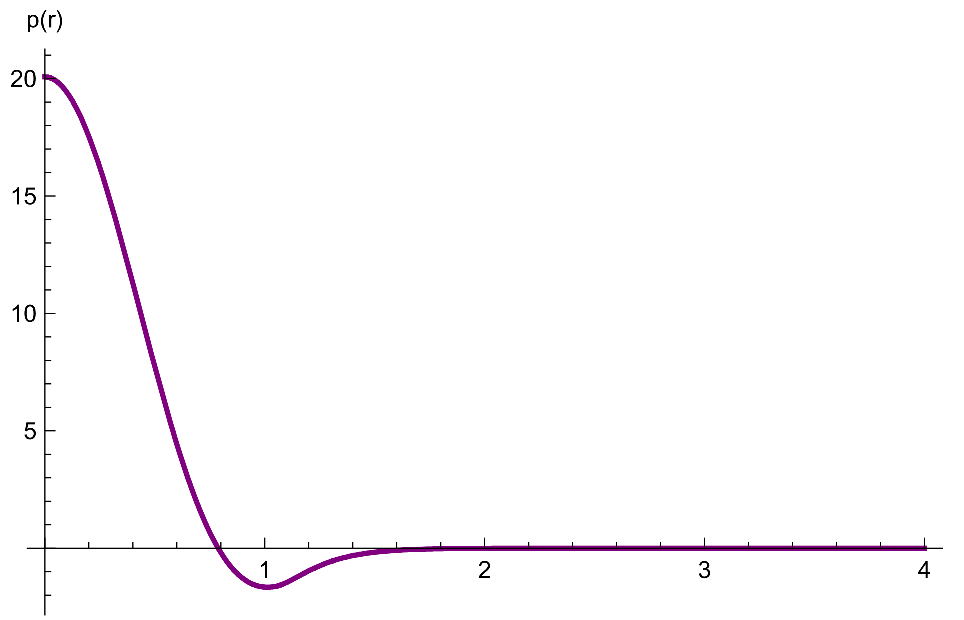

Figure 7: The pressure distribution, $p\left(r\right)$, vs. the radius r.

In Figure 7, we plot the pressure distribution $p(r)$ vs. the radius r. The pressure is positive in the inner region, which corresponds to repulsive forces oriented towards the outside. At $r = 0.9\,\text{fm}$, the pressure changes sign, and becomes negative, reaching a minimum at $r = 1.01\,\text{fm}$. The negative sign means that attractive forces are being directed towards the inside. The attractive and repulsive forces compensate each other exactly according to the integral in Eq.(101), which constitutes a strong test of the numerical calculation.

Some comments regarding the size of the forces are in order. In our model, the pressure in the center of the proton is about $20\,\text{MeV}/\text{fm}^3$. This is about an order of magnitude less than the forces between quarks inside the proton in the chiral quark soliton model [5]. The reason is as follows. The forces in the model of Ref. [5] are the strong forces acting between the quarks. In contrast to this, the strong forces (the massive scalar and vector fields) used in the model are modeled using nuclear physics phenomenology. These "residual nuclear forces" are about a factor of ten or more weaker than the strong forces inside the proton in more realistic models. This is in fact what we observe. Notice that the very first experimental indication for the size of the strong forces in the center of the nucleon also yields a roughly ten times larger value for the pressure in the center of the nucleon [3].

One difference between the results of this thesis and previous works like [5], for instance, is that here we have explicitly included the electric forces. Since the photon mass is zero, the electric potential, $A_0(r)$, is approximately $\frac{1}{r}$ at large distances. Even though the electric forces are much weaker than the strong forces in the center of the proton, at large distances ($r > 4$ fm), they dominate over the short range, exponentially suppressed, strong forces due to the scalar and vector fields. This feature was not encountered before, and deserves future study.

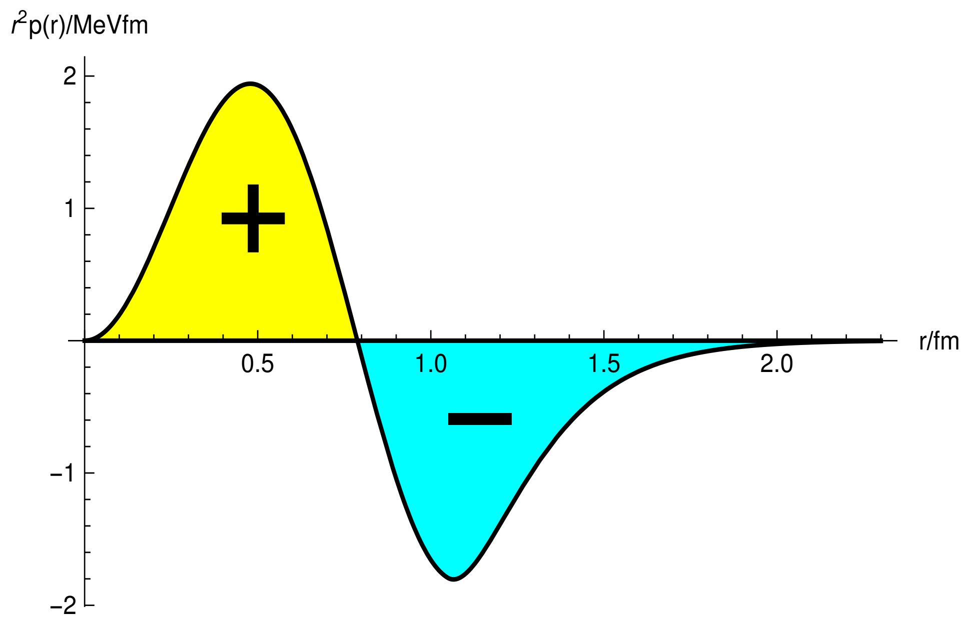

Figure 8: Results showing the stability of the proton: $r^2 p(r)$ vs. r.

Figure 8 shows how the integral relation in Equation (111) is satisfied. Since $\int drr^2p(r) = 0$, the pressure distribution holds true. The areas above and below the x-axis are exactly equal but have opposite signs. This figure shows how the stability of the proton is realized in this model. We remark that our numerical results for $s(r)$ and $p(r)$ also satisfy the differential equation (110). The compliance of the classical model with the requirements in Equations (111) and (110) shows that the energy-momentum tensor is conserved in this model.

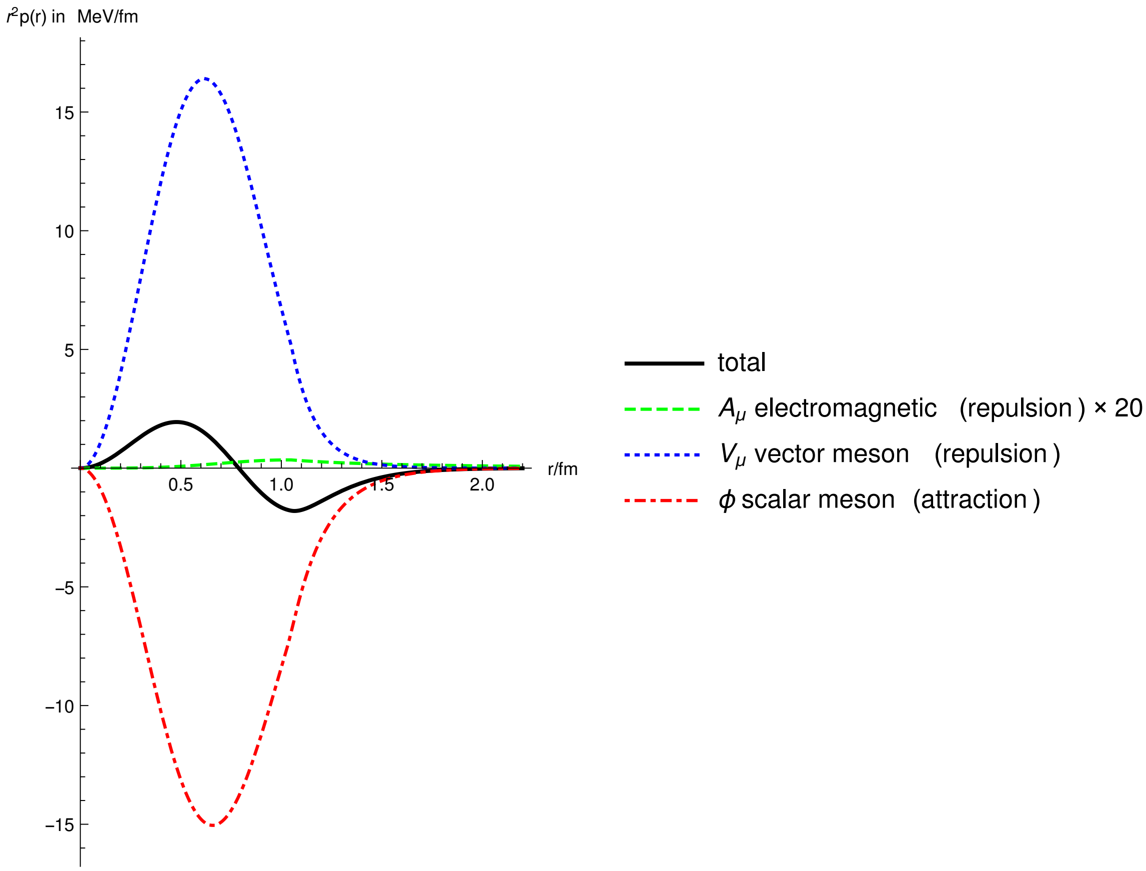

Figure 9: Contributions from $A_\mu, V_\mu,$ and ϕ to the stability of the proton.

It is instructive to see how the forces from the different fields contribute to the stability of the proton in this model. In Figure 9 we show again the result from Figure 8 as a solid black line. The contribution of the electric forces is shown as a green dashed line. This contribution is very small and multiplied by a factor of 20 for better visibility. This contribution is always positive, which reflects the fact that the positive dust particles repel each other and would disperse to infinity if they were not bound. The binding force is due to the scalar field which is depicted by the red dashed line. The contribution of the scalar field is always negative. The binding force due to the scalar field is more than two orders of magnitude stronger than the electric forces. Clearly, the electric repulsion by itself is not sufficient to stabilize the proton. This is achieved by the massive vector field which provides the necessary strong repulsive forces. Notice that there is no contribution to the pressure from the dust particles themselves. This is because they feel no forces (i.e. are in static equilibrium) due to Eq.(62).

14 The D-Term

It would be insightful to compute the D-term of the nucleon. The D-term is the value of the form factor $D(t)$ at zero momentum transfer $t = 0$, and is the "last unknown global property of the nucleon" [1] mentioned in the Introduction. Interestingly, the D-term is undefined in this model. The reason is as follows.

One difference between the results of this thesis and previous works like [5], for instance, is that here we have explicitly included the electric forces. Since the photon mass is zero, the electric potential, $A_0(r)$, is proportional to $1/r$ at large distances. Even though the electric forces are much weaker than the strong forces in the center of the proton, at large distances ($r > 2.4fm$), they dominate over the short range, exponentially suppressed, strong forces due to the scalar and vector fields.

As a consequence, at a large r, the pressure and shear force distributions are dominated by electromagnetic forces, as we can see from the Eqs. (118, 119) for $p(r)$ and $s(r)$. Since $A_0(r) \sim 1/r$ at very large $r > 4fm$, the pressure is positive and behaves like $p(r) \sim 1/r^4$ while $s(r)$ becomes negative and is also proportional to $1/r^4$. (The pressure distribution has a second node at $r = 2.39\,\text{fm}$, which is however hardly visible in Figure 7. Similarly, $s(r)$ has a node at $r = 2.14\,\text{fm}$ which is also not visible in Figure 6).

These features were not encountered before, because so far nobody has studied the effects of long range fields like the electromagnetic forces on the stability of the proton. In fact, the proton is still perfectly stable in this model. But so far in all models, the pressure was always negative at large distances, and the shear forces were always positive. The long range of the electromagnetic forces changes this picture here.

In particular, the D-term does not exist in this model for the following reason. The D-term is defined in terms of $s(r)$ and $p(r)$ as: $D = m \int d^3r r^2 p(r) = -4/15m \int d^3r r^2 s(r)$. But $p(r)$ and $s(r)$ at large distances behave as $1/r^4$ such that the integrals diverge logarithmically. We make this observation here in a classical field theoretical model of the nucleon. It would be interesting to investigate the role of long-range electromagnetic forces in a quantum system. This topic deserves further study.

15 Conclusions

The goal of this thesis was to compute the energy-momentum tensor in a classical model. The model consists of a swarm of particles carrying an electric charge e and interacting with the photon field A_μ. The particles also carry a charge of g_s with which they interact with a massive scalar field, and a charge g_ν with which they couple to a massive vector field W_μ. This model was proposed in Ref. [2]. After introducing the relativistic notation, formulating electromagnetism in terms of the four-vector potential A_μ, and motivating the equations of motion for massive vector and scalar fields, the model was reviewed and numerical results for the fields from Ref. [2] were reproduced.

The model was applied to the description of the energy momentum tensor of the proton. The obtained results for the pressure are in qualitative agreement with results from quantum mechanical models of the proton such as Ref. [5] and the recent experimental findings presented in the Nature article [3]. One interesting finding from this thesis is that it is actually not possible to calculate the D-term using this classical model due to the long range of the electric potential. Since the model studied is classical, it represents a simple way to study the pressure inside a proton, which leads to qualitatively accurate results but is limited in its predictive reliability. Further studies should be done to investigate the effects of the long-range forces on the stability of particles and the D-term. It would also be interesting to see whether this model can be fit to represent other particle types.

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