Sobolev Inequalities and Riemannian Manifolds

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Sobolev Inequalities
and Riemannian Manifolds

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Abstract

Sobolev inequalities, named after Sergei Lvovich Sobolev, relate norms in Sobolev spaces and give insight to how Sobolev spaces are embedded within each other. This thesis begins with an overview of Lebesgue and Sobolev spaces, leading into an introduction to Sobolev inequalities. Soon thereafter, we consider the behavior of Sobolev inequalities on Riemannian manifolds. We discuss how Sobolev inequalities are used to construct isoperimetric inequalities and bound volume growth, and how Sobolev inequalities imply families of other Sobolev inequalities. We then delve into the usefulness of Sobolev inequalities in determining the geometry of a manifold, such as how they can be used to bound a manifold’s number of ends. We conclude with examples of real-world applications Sobolev inequalities.

1 Introduction to Sobolev Spaces

We will begin by introducing Sobolev spaces, and to do so we start with Lebesgue spaces.

Let $p \in [1, \infty)$. The space of functions which are Lebesgue integrable on a set $\Omega$ $p$-many times is denoted

$$L^p(\Omega) = \{ f : \int_\Omega |f|^p(x)dx < \infty \}.$$ 

These **Lebesgue spaces** have the norm

$$||f||_{L^p(\Omega)} = \left( \int_\Omega |f|^p(x)dx \right)^{1/p}. $$

If $p = \infty$, the space $L^\infty(\Omega)$ is the space of all functions such that $|f(x)| < \infty$ “almost everywhere” on $\Omega$, which is to say that $|f(x)|$ can be infinite on only a set of measure zero. Here, the norm is the essential supremum of $|f(x)|$ for $x \in \Omega$ [3].

The elements of a Lebesgue space are equivalence classes, and functions are considered equivalent if they differ only on a set of Lebesgue measure zero. Lebesgue spaces are complete normed spaces.

Let $k \in \mathbb{N} \cup \{0\}$ and $p \in [1, \infty]$. The **Sobolev Space** $W^{k,p}(\Omega)$ is defined by

$$W^{k,p}(\Omega) = \{ u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \ \forall \alpha \text{ with } |\alpha| \leq k \}. $$

1
This space has the norm

\[ ||u||_{W^{k,p}(\Omega)} := \sum_{|\alpha| \leq k} ||D^\alpha u||_{L^p(\Omega)} \]

as discussed in [3].

For a function \( u \) to be included in a Sobolev space, all derivatives \( D^\alpha u \) must exist and must be represented by an element of \( L^p(\Omega) \). Note that if \( k = 0 \), the space \( W^{0,p}(\Omega) = L^p(\Omega) \) identically, as the restriction on the derivatives of each element is gone.

Sobolev spaces are vector spaces, as elements can be added together and multiplied by scalars, with the resultant functions being elements of the Sobolev space. In addition, the Sobolev space \( W_0^{k,p}(\Omega) \) is the completion of \( C_0^\infty(\Omega) \) (the space of infinitely differentiable functions with compact support) with respect to the norm of \( W^{k,p}(\Omega) \) [3].

Equipped with the definition of Sobolev spaces, we can intuitively embed Sobolev spaces in one another: let \( \Omega \subset \mathbb{R}^d \) be a domain with \( p \in [1, \infty) \) and \( k \leq m \). Then, \( W^{m,p}(\Omega) \subset W^{k,p}(\Omega) \). This is clear because if a function has \( m \) derivatives represented in \( L^p(\Omega) \), then it must have \( k \) derivatives in \( L^p(\Omega) \). Alternatively, for \( k \geq 0 \) and \( p, q \in [1, \infty] \) with \( q > p \) it is clear that \( W^{k,q}(\Omega) \subset W^{k,p}(\Omega) \): if we can integrate \( q \)—many times, we can integrate \( p \)—many times.

Now, we will consider the **Sobolev Embedding Theorem**. Consider \( W^{k,p}(\mathbb{R}^n) \) with \( k \in \mathbb{N}^+ \), \( p \in [1, \infty) \). If we have two real numbers \( l, q \) such that \( k > l \), \( 1 \leq p < q < \infty \), and

\[
\frac{1}{p} - \frac{k}{n} = \frac{1}{q} - \frac{l}{n},
\]

then \( W^{k,p}(\mathbb{R}^n) \subseteq W^{l,q}(\mathbb{R}^n) \), and this embedding is continuous.
2 Introduction to Sobolev Inequalities

When we are working with smooth functions $f : \mathbb{R} \rightarrow \mathbb{R}$ with compact support, we have the inequality
\[
|f(x)| \leq \frac{1}{2} \int_{-\infty}^{\infty} |f'(t)| \, dt.
\]
This inequality bounds the value of $|f(x)|$ using its derivative. As we move up to multidimensional spaces, the notions of absolute value and the derivative become more complicated, and we cannot use this inequality to bound our functions. This brings us to the topic of Sobolev inequalities: how can we construct an inequality that bounds the norm of a function using partial derivatives?

While working in $\mathbb{R}^n$, let $p \in [1, n)$. The Sobolev conjugate of $p$ is
\[
p^* := \frac{np}{n - p} \in (p, \infty).
\]

**Theorem 2.1:** When $n \geq 2$, the Sobolev inequality in $\mathbb{R}^n$ states
\[
\left( \int_{\mathbb{R}^n} |f(x)|^{p^*} \, dx \right)^{1/p^*} \leq C(n, p) \left( \int_{\mathbb{R}^n} |\nabla f(x)|^p \, dx \right)^{1/p}
\]  
(1)

for all smooth functions $f$ with compact support, for any $p \in [1, n)$. Note that $C(n, p)$ is a constant that depends on $n$ and $p$ [1].

Note that this inequality can be written as $||f||_{L^{p^*}(\mathbb{R}^n)} \leq C||\nabla f||_{L^p(\mathbb{R}^n)}$, and moving forward we will notate these norms as $||f||_{p^*} \leq C||\nabla f||_p$.

To prove Theorem 2.1, we will illustrate Gagliardo and Nirenberg’s proof of the Sobolev inequality in $\mathbb{R}^n$ for $p = 1$, and then expand upon that proof to show this inequality holds for all real $p \in [1, n)$. Note that $\mu = \mu_n$ denotes the Lebesgue measure in $\mathbb{R}^n$.

**Proposition 2.2:** When $n \geq 2$,
\[
\left( \int_{\mathbb{R}^n} |f(x)|^{1^*} \, dx \right)^{1/1^*} \leq C(n, 1) \left( \int_{\mathbb{R}^n} |\nabla f(x)|^{1} \, dx \right)^{1/1}
\]

for all smooth functions $f$ with compact support. Note that $C(n, 1)$ is a constant that depends on $n$.

**Proof.** Let $1 \leq p, p' \leq \infty$ with $1 = \frac{1}{p} + \frac{1}{p'}$. For $f \in L^p(\mu), g \in L^{p'}(\mu)$, Hölder’s inequality states that for positive measure $\mu$,
\[
\left| \int f g d\mu \right| \leq ||f||_p ||g||_{p'}.
\]
By induction, for \( f_i \in L^{p_i}, 1 \leq i \leq k, 1 \leq p_i \leq \infty, \frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_k} = 1, \) we see

\[
\left| \int f_1 f_2 \ldots f_k d\mu \right| \leq \prod_{i=1}^{k} \|f_i\|_{p_i}.
\]  

(2)

Fix \( f \in C^\infty_0(\mathbb{R}^n) \): a smooth function with compact support that. By the inequality

\[
|f(x)| \leq \frac{1}{2} \int_{-\infty}^{\infty} |f'(t)|dt,
\]

for any \( x = (x_1, \ldots, x_n) \) and any integer \( i \in [1, n] \) we observe

\[
|f(x)| \leq \frac{1}{2} \int_{-\infty}^{\infty} |\partial_{x_i} f(x_1, x_2, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n)|dt.
\]  

(3)

Fix

\[
F_i(x) = \int_{-\infty}^{\infty} |\partial_{x_i} f(x_1, x_2, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n)|dt,
\]

which depends on \( n - 1 \) coordinates of \( x \) with the integration variable replacing the \( i^{th} \) coordinate. Additionally, fix

\[
F_{i,m}(x) = \begin{cases} 
\int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} |\partial_{x_i} f(x)|dx_1 \ldots dx_m & \text{when } i \leq m \\
\int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} F_i(x)dx_1 \ldots dx_m & \text{when } i > m
\end{cases}
\]

which depends on \( n - m \) or \( n - m - 1 \) variables, for all integers \( m \in [1, n] \). Now, by (3) we have the inequality

\[
|f|^n \leq \frac{1}{2^n} F_1 \ldots F_n.
\]

Raising both sides of this equality to the power of \( 1/(n - 1) \), we see

\[
|f|^{n/(n-1)} \leq \frac{1}{2^n/(n-1)} (F_1 \ldots F_n)^{1/(n-1)}.
\]

Setting \( k = n - 1, p_1 = p_2 = \ldots = p_{n-1} = n - 1, \) we see \( \frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_k} = (n - 1) \frac{1}{n - 1} = 1. \)

By (2) with induction on \( m \leq n \) we see

\[
\int \ldots \int |f(x)|^{n/(n-1)} dx_1 \ldots dx_m \leq \frac{1}{2^{n/(n-1)}} (F_{1,m}(x) \ldots F_{n,m}(x))^{1/(n-1)}.
\]

When \( n = m, \) we see

\[
||f||_{n/(n-1)} \leq (1/2) \left( \prod_{i=1}^{n} ||\partial_{x_i} f||_1 \right)^{1/n}.
\]

Note that both sides were raised to the power of \( (n - 1)/n \) when taking the norm in \( L^{n/(n-1)}(\mathbb{R}^n) \). Because \( (\prod_1^n a_i)^{1/n} \leq \frac{1}{n} \sum_1^n a_i \) for \( a_i > 0 \) and \( n \in \mathbb{Z} \), and \( \sum_1^n |\partial_x f| \leq \sqrt{n} |\nabla f|, \) this becomes

\[
||f||_{n/(n-1)} \leq \frac{1}{2n} \sum_{i=1}^{n} ||\partial_{x_i} f(x)||_1 dx \leq \frac{1}{2\sqrt{n}} ||\nabla f||_1.
\]

This proves Proposition 2.2, the Sobolev Inequality in \( \mathbb{R}^n \) for \( p = 1. \) \[1\]
Next, we are going to build off of the above proof to show that Theorem 2.1 (1) holds for \( p \geq 1 \).

**Proposition 2.3:** If Theorem 2.1, the Sobolev Inequality in \( \mathbb{R}^n \), holds for \( p = 1 \), then it holds for all \( p \in [1, n) \).

**Proof.** Knowing that (1) holds for \( p = 1 \), or that

\[
\forall f \in C^\infty_0(\mathbb{R}^n) \quad ||f||_{n/(n-1)} \leq \frac{1}{2\sqrt{n}}||\nabla f||_1, \tag{4}
\]

fix a \( p > 1 \). For any \( \alpha > 1 \) and function \( f \in C^\infty_0(\mathbb{R}^n) \), \( |f|^\alpha \) is \( C^1_0 \) and satisfies

\[
|\nabla |f|^\alpha| = \alpha |f|^\alpha - 1 |\nabla f|.
\]

The function \( |f|^\alpha \) can be approximated by a sequence of smooth functions \( (f_i) \in C_0 \) such that \( \nabla f_i \to \nabla |f|^\alpha \). This allows us to substitute \( |f|^\alpha \) into (4):

\[
||f||_{\alpha n/(n-1)}^\alpha \leq \frac{1}{2\sqrt{n}} \alpha \int |f(x)|^{\alpha-1} |\nabla f(x)| dx \\
\leq \frac{1}{2\sqrt{n}} \alpha \left( \int |f(x)|^{(\alpha-1)p'} dx \right)^{1/p'} \left( \int |\nabla f(x)|^p dx \right)^{1/p}
\]

with \( 1/p + 1/p' = 1 \). Setting \( \alpha = \frac{(n-1)p}{n(p-1)} \) implies

\[
||f||_{\alpha n/(n-1)}^\alpha \leq \frac{1}{2\sqrt{n}} \frac{(n-1)p}{n(p-1)} \frac{(n-1)p}{n-1} ||f||_{np/(n-1)} ||\nabla f||_p.
\]

To simplify, divide both sides by \( ||f||_{np/(n-1)} \). We see that

\[
\frac{(n-1)p}{n(p-1)} - \frac{n(p-1)}{n-1} = \frac{np - np + n}{n(p-1)} = 1,
\]

and so

\[
||f||_{p^*} \leq \frac{1}{2\sqrt{n}} \frac{(n-1)p}{n-1} ||\nabla f||_p.
\]

We have thereby proven a Sobolev inequality in \( \mathbb{R}^n \) for \( p > 1 \): for any integer \( n \geq 2 \) and \( p \in [1, n) \), then

\[
\forall f \in C^\infty_0(\mathbb{R}^n) \quad ||f||_{p^*} \leq \frac{np - p}{2(n-p)\sqrt{n}} ||\nabla f||_p.
\]

\( \square \)
We have shown the Sobolev Inequality in $\mathbb{R}^n$ holds for $p \in [1, n)$, and thus have proven Theorem 2.1 [1]. Alternatively, in the case that $n < p \leq \infty$, we have the H"older continuity estimate:

$$\sup_{x,y \in \mathbb{R}^n} \left\{ \frac{|f(x) - f(y)|}{|x - y|^{1-n/p}} \right\} \leq C \left( \int_{\mathbb{R}^n} |\nabla f(x)|^p dx \right)^{1/p}$$

for smooth functions $f$ with compact support, for some constant $C$ that depends only on $n$ and $p$.

We can generalize Sobolev inequalities to subsets of $\mathbb{R}^n$: let’s consider the general Sobolev inequality for $k < \frac{n}{p}$. Let $\Omega \subset \mathbb{R}^n$ be Lipschitz, with $k \in \{1, ..., n-1\}$ and $p \in [1, \frac{n}{k})$. Let

$$\frac{1}{q} = \frac{1}{p} - \frac{k}{n} \implies q = \frac{np}{n - kp} > p.$$

Then, $W^{k,p}(\Omega) \subset L^q(\Omega)$ and

$$\|u\|_{L^q(\Omega)} \leq C \|u\|_{W^{k,p}(\Omega)}$$

for some constant $C$ that depends on $k, p, n$, and $\Omega$. It is clear that $L^q \subset L^p$, but this inequality tells us that $W^{k,p} \subset L^q \subset L^p$.

The Gagliardo-Nirenberg-Sobolev inequality states for $p \in [1, n)$,

$$\|u\|_{L^p(\mathbb{R}^n)} \leq C \|Du\|_{L^p(\mathbb{R}^n)}$$

for $u \in C^1_0(\mathbb{R}^n)$.

Various Sobolev inequalities are related to one another; for example, there are equivalences between “weak” and “strong” Sobolev inequalities. Consider the Nash inequality on a Riemannian manifold $M$:

$$\forall f \in C^\infty_0(M), \quad \|f\|_2^{1+2/v} \leq C \|\nabla f\|_2 \|f\|_1^{2/v}$$

this inequality is equivalent to the Sobolev inequality

$$\forall f \in C^\infty_0(M), \quad \|f\|_{2v/(v-2)} \leq C \|\nabla f\|_2.$$

We will examine more examples of inequality equivalences in further detail in Section 3.
3 Sobolev Inequalities on Manifolds

Sobolev inequalities were prevalent during the development of analysis on manifolds because Fourier analysis and other tools in Euclidean space are no longer available on manifolds. On manifolds, the original question of which inequalities are true or not is expanded into a search for necessary and sufficient conditions for Sobolev inequalities to hold true.

Not every manifold admits global Sobolev inequalities; a workaround is to use families of local Sobolev inequalities. For example, we can construct a Sobolev inequality locally on a ball; for a ball \( B = B(x, r) \) on a complete Riemannian manifold, there is a constant \( C(B) \) such that for smooth functions \( f \) with compact support in \( B \)

\[
\left( \int_B |f|^q d\mu \right)^{2/q} \leq \frac{C(B)v^2}{\mu(B)^2/v} \int_B (|\nabla f|^2 + r^{-2}|f|^2) du
\]

for constants \( q, v > 2 \) such that \( \frac{1}{q} = \frac{1}{2} - \frac{1}{v} \).

3.1 Isoperimetric Inequalities

Geometric properties of manifolds can be determined using isoperimetric inequalities, inequalities that relate the measure of a set’s boundary to the set’s measure or volume. For some background, consider \( \mathbb{R}^2 \). A circle is the 2-dimensional or planar shape that maximizes area \( (A) \) in a fixed perimeter \( (\text{circumference } C) \). These two values are related by the equation

\[
4\pi A = C^2
\]

derived from \( C = 2\pi r, A = \pi r^2 \). Thus, the isoperimetric inequality in \( \mathbb{R}^2 \) tells us that for a closed curve of length \( L \), with \( A \) the area of the region enclosed by \( L \),

\[
L^2 \geq 4\pi A.
\]

Let \( \Omega \) be a bounded subset of a Riemannian manifold \( M \), with a smooth boundary \( \partial \Omega \). We want to find the maximal volume that can be enclosed in a hypersurface of a fixed \((n - 1)\)-measure. Using the same intuition as the \( \mathbb{R}^2 \) example, in Euclidean space an \( n \)–dimensional sphere or ball maximizes enclosed volume with a fixed surface area. So, if \( M \) is \( n \)–dimensional Euclidean space, we can derive an isoperimetric inequality from the equation that defines the volume of a sphere in \( n \) dimensions.

Now we will consider the isoperimetric inequality for a Riemannian manifold that may not be in Euclidean space. Let \( M \) be an \( n \)–dimensional Riemannian manifold. If \( M \) satisfies

\[
\mu_n(\Omega)^{1-1/v} \leq C(M, v)\mu_{n-1}(\partial \Omega)
\]  

(5)
for any $\Omega \subset M$ with a smooth boundary for some $v \geq n$, $C(M,v) > 0$, then

$$V(x,r) \geq \frac{r^v}{(vC(M,v))^v}$$

(6)

with $V(x,r)$ denoting the volume of a ball with radius $r$ [1]. This is because $\partial_r V(x,r)$ is the $(n-1)$—dimensional Riemannian volume of the ball $B(x,r)$’s boundary: in general, the volume of a sphere in $n$ dimensions is

$$V(x,r) = C_n \pi^{[n/2]} r^n$$

for some constant $C_n$ [2]. Meanwhile, this sphere’s “surface area” can be found by the formula

$$A_n = C_n \pi^{[n/2]} nr^{n-1},$$

which is equivalent to $\partial_r (V(x,r))$ [2].

We see that

$$V(x,r)^{1-1/v} \leq C(M,v) \partial_r V(x,r)$$

$$\implies V(x,r)^{1-1/v} \leq C(M,v) V(x,r)n \frac{1}{r}$$

$$\implies V(x,r)^{-1/v} \leq C(M,v)n \frac{1}{r}$$

$$\implies V(x,r)^{1/v} \geq \frac{r}{nC(M,v)}$$

$$\implies V(x,r) \geq \frac{r^v}{n^v C(M,v)^v} \geq \frac{r^v}{v^v C(M,v)^v}$$

because $v \geq n$.

**Theorem 3.1:** A manifold $M$ satisfies the conditions in (5) for some $v, C(M,v) > 0$ if and only if

$$\forall f \in C_0^\infty(M), \quad ||f||_{v/(v-1)} \leq C(M,v) ||\nabla f||_1.$$ 

This is an $(L^1,v)$—Sobolev inequality [1].

**Definition 3.2:** If we fix $p,v$ such that $1 \leq p < v$, a Riemannian manifold $M$ satisfies an $(L^p,v)$—Sobolev inequality if there exists a constant $C(M,p,v)$ that satisfies

$$\forall f \in C_0^\infty(M), \quad ||f||_{ps/(p-v)} \leq C(M,p,v) ||\nabla f||_p.$$ 

Now, we have a definition of the $(L^p,v)$—Sobolev inequality. Let us consider the following theorem, which states that $(L^p,v)$—Sobolev inequalities imply other inequalities of this type.
**Theorem 3.3:** If $M$ satisfies an $(L^p, v)$-Sobolev inequality, then $M$ satisfies $(L^q, v)$-Sobolev inequalities for all $q \in [p, v)$.

**Proof.** Let $\gamma > 1$ and use the $(L^p, v)$ inequality on $|f|^{\gamma}$:

$$||f||_{\gamma pv/(v-p)}^{\gamma} \leq \gamma C(M, p, v) \left( \int_M |f|^{p(\gamma-1)} |\nabla f|^p d\mu \right)^{1/p}.$$

By the Hölder inequality, we see

$$\int |f|^{p(\gamma-1)} |\nabla f|^p \leq |||f|^{p(\gamma-1)}||_{q/(q-p)} |||\nabla f|^p||_{q/p}$$

which implies

$$\left( \int_M |f|^{p(\gamma-1)} |\nabla f|^p d\mu \right)^{1/p} \leq \left( \int_M |f|^{pq(\gamma-1)/(q-p)} d\mu \right)^{1/p-1/q} \left( \int_M |\nabla f|^q d\mu \right)^{1/q}.$$

Using these two inequalities, we see

$$||f||_{\gamma pv/(v-p)}^{\gamma} \leq \gamma C(M, p, v) \left( \int_M |f|^{pq(\gamma-1)/(q-p)} d\mu \right)^{1/p-1/q} \left( \int_M |\nabla f|^q d\mu \right)^{1/q}.$$

and if we select $\gamma = \frac{q(n-p)}{p(n-q)}$ we find $\gamma - 1 = n(q-p)/p(n-q)$, implying

$$||f||_{q*} \leq \frac{q(n-p)}{p(n-q)} C(M, p, v) ||\nabla f||_q,$$

an $(L^q, v)$-Sobolev inequality [1].

### 3.2 Bounding Volume Growth with Sobolev Inequalities

Let $M$ be a manifold that satisfies an $(L^p, v)$-Sobolev Inequality:

$$\forall f \in C_0^\infty(M), \quad ||f||_{pv/(v-p)} \leq C(M, p, v) ||\nabla f||_p.$$

Let $q = vp/(v-p) \implies \frac{1}{q} = \frac{1}{p} - \frac{1}{v}$, and let $r, s \in (0, \infty)$ and $\theta \in [0, 1]$ such that

$$\frac{1}{r} = \theta \left( \frac{1}{p} - \frac{1}{v} \right) + \frac{1-\theta}{s} = \frac{\theta}{q} + \frac{1-\theta}{s}.$$

The Hölder inequality tells us

$$||f||_r \leq ||f||_q^\theta ||f||_s^{1-\theta}$$

and because $q = vp/(v-p)$ we apply the $(L^p, v)$-Sobolev inequality above to find

$$||f||_r \leq (C(M, p, v)||\nabla f||_p)^\theta ||f||_s^{1-\theta}.$$
Thus, if \( M \) satisfies an \((L^p, v)\)-Sobolev inequality, it also satisfies the above inequality for all \( r, s \in (0, \infty) \) and \( \theta \in [0, 1] \) such that \( \frac{1}{r} = \theta \left( \frac{1}{p} - \frac{1}{s} \right) + \frac{1-\theta}{s} \) [1].

We want to use this result to generalize the isoperimetric inequality (6). In the introduction to this section, we discussed that not all manifolds admit global Sobolev inequalities, and at times we have to use local inequalities based in balls. Now, we will use the above result to place a lower bound on the volume of a ball \( B(x, r) \). We will notate the volume growth function as \( V(x, r) = \mu(B(x, r)) \), the measure of the ball of radius \( r \) centered at \( x \in M \).

The following result provides a lower bound for the volume growth \( V(x, t) \) in \( M \), as seen in [1, Theorem 3.1.5].

**Theorem 3.4:** Assume that on a manifold \( M \)

\[
\forall f \in C^\infty_0(M), \quad ||f||_r \leq (C(M, p, v)||\nabla f||_p)^\theta ||f||_s^{1-\theta}
\]

is satisfied for some \( r, s \) with \( 0 < s \leq r \leq \infty \) and some \( \theta \in (0, 1] \), such that

\[
\frac{\theta}{p} + \frac{1-\theta}{s} - \frac{1}{r} > 0.
\]

Then, for \( v \) defined by

\[
\frac{\theta}{v} = \frac{\theta}{p} + \frac{1-\theta}{s} - \frac{1}{r}
\]

we have

\[
V(x, t) \geq \frac{t^v}{2v^{2/\theta r + v/\theta}C(M, p, v)^v} = \left( \frac{t}{2^{v/\theta r + 1/\theta}C(M, p, v)} \right)^v.
\]

Particularly, if a manifold \( M \) satisfies an \((L^p, v)\)-Sobolev inequality with \( 1 \leq p < v \), then \( n \geq v \) and

\[
\inf_{x \in M} \{ t^v V(x, t) \} > 0.
\]

**Proof.** For a fixed \( x \in M \) and \( t > 0 \), consider

\[
f(y) = \max\{t - d(x, y), 0\}.
\]

Then,

\[
||f||_r \geq \frac{t}{2} V(x, t/2)^{1/r}
\]

\[
||f||_s \leq t V(x, t)^{1/s}
\]

\[
||\nabla f||_p \leq V(x, t)^{1/p}.
\]
and thus
\[ V(x, t)^{\theta/p+(1-\theta)/s} \geq \frac{1}{2} (t/C)^{\theta} V(x, t/2)^{1/r}. \]

Here, define \( v \) by
\[ \frac{\theta}{v} = \frac{\theta}{p} + \frac{1 - \theta}{s} - \frac{1}{r} \]
to find
\[ V(x, t) \geq (2C^{\theta})^{-rv/(v+\theta r)} t^{\theta rv/(v+\theta r)} V(x, t/2)^{v/(v+\theta r)}. \]

For \( a = v/(v + \theta r) \), we find
\[ V(x, t) \geq (2C^{\theta})^{-ra \theta r a} V(x, t/2)^{a} \]
\[ \geq (2C^{\theta})^{-r} \sum_{i} a^i \theta r \sum_{i} a^i 2^{-\theta r} \sum_{i} (j-1) a^i V(x, t/2)^{i \theta r}. \]

Additionally,
\[ \sum_{j=1}^{\infty} a^j = \frac{a}{1-a} = \frac{v}{\theta r}, \quad \sum_{j=1}^{\infty} (j-1) a^j = \left( \frac{a}{1-a} \right)^2 = \frac{v^2}{\theta^2 r^2}, \]
so as we allow \( i \to \infty \) we find
\[ V(x, t) \geq (2C^{\theta})^{-v/\theta} t^{v-2v^2/(\theta r)} = \frac{t^v}{2^{v/\theta+v^2/(\theta r)} C^v}. \]

\[ \square \]

### 3.3 Weak and Strong Sobolev Inequalities

In order to prove that a certain manifold satisfies a Sobolev inequality, at times we must use the equivalences between weak and strong forms of Sobolev-type inequalities that were introduced in Section 2. As demonstrated in subsection 4.1, we know that if \( M \) satisfies an \((L^p, v)\)-Sobolev inequality, then \( M \) satisfies \((L^q, v)\)-Sobolev inequalities for all \( q \in [p, v) \).

Any Sobolev inequality can be used in conjunction other inequalities, such as the Hölder inequality, to lead to “weaker” inequalities. Moving forward, we will discuss how “weaker” Sobolev inequalities imply stronger Sobolev inequalities.

In the previous subsection, we saw the inequality
\[ \forall f \in C_0^\infty(M), \quad \sup_{t>0} \{ t \mu(|f| > t)^{1/r} \} \leq (C(M, p, v)|\nabla f||_p)^\theta (|f||_\infty \mu(\text{supp}(f))^{1/s})^{1-\theta}. \]

This is called the \((S_{r,s}^{*,\theta})\) inequality, and is the weakest Sobolev inequality discussed so far. The \((S_{r,s}^{\theta})\) inequality is stronger and states
\[ \forall f \in C_0^\infty(M), \quad ||f||_r \leq (C(M, p, v)|\nabla f||_p)^\theta ||f||_s^{1-\theta}. \]
Here, $0 < r, s \leq \infty$ and $\theta \in (0, 1]$. When $\theta = 1$, we find $(S^{1}_{r,s})$ states
\[
\forall f \in C_{0}^{\infty}(M), \quad ||f||_{r} \leq C(M, p, v)||\nabla f||_{p}
\]
and $(S^{*,1}_{r,s})$ states
\[
\forall f \in C_{0}^{\infty}(M), \quad \sup_{t > 0}\{t\mu(|f| > t)^{1/r}\} \leq C(M, p, v)||\nabla f||_{p}.
\]

We will show how a weak inequality $(S^{*,0}_{r_0,s_0})$ implies a collection of stronger inequalities. First, consider the parameter $q = q(r, s, \theta) \neq 0$ that satisfies
\[
\frac{1}{r} = \theta \frac{1}{q} + \frac{1-\theta}{s}
\]
Recall that this $q$ was introduced at the beginning of subsection 4.2 and satisfies $\frac{1}{q} = \frac{1}{p} - \frac{1}{v}$. We will prove that any weak inequality $(S^{*,0}_{r_0,s_0})$ for fixed $r_0, s_0, \theta_0$ implies all stronger inequalities $(S^{\theta}_{r,s})$ for all $r, s \in (0, \infty]$ and $\theta \in (0, 1]$ with $q(r, s, \theta) = q(r_0, s_0, \theta_0)$: an inequality $(S^{*,0}_{r_0,s_0})$ for fixed $r_0, s_0, \theta_0$, and $p_0$ implies all inequalities $(S^{\theta}_{r,s})$ for all $p \geq p_0$ with $r, s, \theta, p$ satisfying
\[
\frac{1}{r} = \theta \frac{1}{q} + \frac{1-\theta}{s} \quad \text{and} \quad \frac{1}{q} = \frac{1}{p} - \frac{1}{v}.
\]

3.3.1 Equivalences when $q \in (0, \infty)$

Begin by fixing $p \in [1, \infty)$. Assume that, on a manifold $M$, $(S^{*,0}_{r_0,s_0})$ is satisfied for some $0 < r_0, s_0 \leq \infty$ and $\theta \in (0, 1]$. Let $q = q(r_0, s_0, \theta_0)$ satisfy $\frac{1}{r} = \theta \frac{1}{q} + \frac{1-\theta}{s}$ with $p \leq q < \infty$. Let $f \in C_{0}^{\infty}(M)$ be nonnegative and set
\[
f_{\rho,k} = (f - \rho^{k})_{+} \land \rho^{k}(\rho - 1)
\]
for any $\rho > 1$ and $k \in \mathbb{Z}$. Here, $u_{+} = \max\{u, 0\}$ and $u \land v = \min\{u, v\}$: $f_{\rho,k}$ reads as
\[
\min\{(\max\{f - \rho^{k}, 0\}), \rho^{k}(\rho - 1)\}.
\]

We see that $|\nabla f_{\rho,k}| \leq |\nabla f|$. Because $\{f_{\rho,k} \geq (\rho - 1)\rho^{k}\} = \{f \geq \rho^{k+1}\}$, and because $M$ satisfies $(S^{*,0}_{r_0,s_0})$ we see
\[
(\rho - 1)\rho^{k}\mu(\{f \geq \rho^{k+1}\})^{1/\rho_0} \leq (C(M, p, v)||\nabla f||_{p})^{\theta_0}((\rho - 1)\rho^{k}\mu(\{f \geq \rho^{k}\})^{1/\rho_0})^{1-\theta_0}.
\]
Set \( N(f) = \sup_k \{ \rho^k \mu(\{ f \geq \rho^k \}) \}^{1/q} \). We then find
\[
\mu(\{ f \geq \rho^{k+1} \})^{1/r_0} \leq \rho^{-k(\theta_0 + (1-\theta_0)q/s_0)} \left( \frac{C \| \nabla f \|_p}{\rho - 1} \right)^{\theta_0} N(f)^{(1-\theta_0)/s_0} \\
\leq \rho^{-kq/r_0} \left( \frac{C \| \nabla f \|_p}{\rho - 1} \right)^{\theta_0} N(f)^{(1-\theta_0)/s_0} \\
\implies N(f)^{q/r_0} \leq \rho^{q/r_0} \left( \frac{C \| \nabla f \|_p}{\rho - 1} \right)^{\theta_0} N(f)^{(1-\theta_0)/s_0} \\
\implies N(f) \leq \frac{\rho^{q/r_0}}{\rho - 1} C \| \nabla f \|_p
\]
by the definition of \( q \). This tells us
\[
\sup_{t > 0} \left\{ t \mu(\{ f \geq t \}) \right\}^{1/q} \leq \rho N(f) \leq \frac{\rho^{1+q/r_0}}{\rho - 1} C \| \nabla f \|_p.
\]
When we set \( \rho = 1 + r_0 \theta_0/q \) this gives
\[
\sup_{t > 0} \left\{ t \mu(\{ f \geq t \}) \right\}^{1/q} \leq e \left( 1 + \frac{q}{r_0 \theta_0} \right) C \| \nabla f \|_p. \tag{7}
\]
This is the “weak” form of the \((L^p, v)\)-Sobolev inequality \( \| f \|_{p v/(v-p)} \leq AC \| \nabla f \|_p \). We want to show that if \( M \) satisfies \((S^\ast \theta_0, 1 - s_0/r_0 \theta_0)\), then all inequalities \((S^\theta_{r,s})\) with \( 0 < r, s \leq \infty \), \( \theta \in (0, 1) \) with \( q(r, s, \theta) = q \) are satisfied. Particularly, we want to show that there is a constant \( A \) such that
\[
\forall f \in C^\infty_0(M), \quad \| f \|_q \leq AC \| \nabla f \|_p, \tag{8}
\]
an \((L^p, v)\)-Sobolev inequality (recall \( q = \frac{pv}{p-v} \)).

We move forward using the fact that for \( 1 \leq p < q < \infty \), if
\[
\forall f \in C^\infty_0(M), \quad \sup_{t > 0} \left\{ t \mu(\{ f \geq t \}) \right\}^{1/q} \leq C \| \nabla f \|_p
\]
then
\[
\forall f \in C^\infty_0(M), \quad \| f \|_q \leq 2(1 + q)C \| \nabla f \|_p.
\]
Using this with (7) we find the desired result (8), with \( A = 2e(1 + q) \left( 1 + \frac{q}{r_0 \theta_0} \right) \) and \( C \) the constant from \((S^\ast \theta_0)\).

### 3.3.2 Equivalences when \( q = \infty \)

Again, let us fix \( p \in [1, \infty) \). When \( q = \infty \), the relationship \( \frac{1}{r} = \frac{\theta}{q} + \frac{1-\theta}{s} \) becomes \( r = \frac{s}{1-\theta} \). Thus, \( \theta = 1 - \frac{s}{r} \). We want to show that if \((S^\ast \theta_0 \infty^{1-s_0/r_0})\) is satisfied on a manifold \( M \) for some
0 < s₀ < r₀ < ∞, then all inequalities \((S_{r,s}^{1-s/r})\) with 0 < s < r < ∞ are satisfied on M. Following the same approach as the previous sub-subsection, if \(f \in C_{0}^{\infty}(M)\) is nonnegative and we consider \(f_{\rho,k}\), we find

\[
\rho^{kr_{0}}\mu\{f \geq \rho^{k+1}\} \leq \left(\frac{C||\nabla f||_{p}}{\rho - 1}\right)^{r_{0} - s_{0}} \rho^{ks_{0}}\mu\{f \geq \rho^{k}\}.
\]

Set a real number \(t > 0\) and set \(r = r_{0} + t, s = s_{0} + t\). Multiplying both sides by \(\rho^{t}\) brings us to

\[
\rho^{kr}\mu\{f \geq \rho^{k+1}\} \leq \left(\frac{C||\nabla f||_{p}}{\rho - 1}\right)^{r_{0} - s_{0}} \rho^{ks}\mu\{f \geq \rho^{k}\}.
\]

Note that \(r_{0} - s_{0} = r - s\) because \(r_{0} = r - t, s_{0} = s - t\). If we sum over all \(k\), we see

\[
\sum_{k} \rho^{kr}\mu\{f \geq \rho^{k+1}\} \leq \left(\frac{C||\nabla f||_{p}}{\rho - 1}\right)^{r_{0} - s_{0}} \sum_{k} \rho^{ks}\mu\{f \geq \rho^{k}\}.
\]

Building off of these sums, we can find

\[
\frac{1}{\rho^{2r} - \rho^{r}}||f||_{r}^{2r} \leq \frac{1}{\rho^{2r} - \rho^{r}} \sum_{k} r \int_{r_{k+1}}^{r_{k+2}} t^{s-1}\mu\{f \geq t\}dt \leq \rho^{kr}\mu\{f \geq \rho^{k+1}\}
\]

and

\[
\sum_{k} \rho^{ks}\mu\{f \geq \rho^{k}\} \leq \frac{\rho^{s}}{\rho^{s} - 1} ||f||_{s}^{s}.
\]

Putting this together, we find for all \(t > 1, r = r_{0} + t, s = s_{0} + t\)

\[
||f||_{r}^{r} \leq \frac{\rho^{r+s}(\rho^{r} - 1)}{(\rho^{s} - 1)(\rho - 1)^{r-s}}(C||\nabla f||_{p})^{r-s} ||f||_{s}^{s}
\]

\[
\Rightarrow ||f||_{r} \leq \left(\frac{\rho^{r+s}(\rho^{r} - 1)}{(\rho^{s} - 1)(\rho - 1)^{r-s}}(C||\nabla f||_{p})^{1-s/r} ||f||_{s/r}^{s/r}\right)^{1/r}
\]

which is the definition of \((S_{r,s}^{1-s/r})\), with a constant depending on the fixed \(\rho > 1\). Thus, all inequalities \((S_{r,s}^{1-s/r})\) with \(s_{0} \leq s < r < \infty, r_{0} \leq r\) are satisfied on M if M satisfies \((S_{r_{0},s_{0}}^{s_{0}-s_{0}/r_{0}})\) for some \(0 < s_{0} < r_{0} < \infty\). Meanwhile, the Hölder inequality shows that if \(r \geq s\), \((S_{r,s}^{\theta})\) implies \((S_{r',s'}^{\theta'})\) for all \(r', s'\) such that \(s' \leq r' \leq r\) and \(s' \leq s\). Thus, if M satisfies \((S_{r_{0},s_{0}}^{s_{0}-s_{0}/r_{0}})\), M satisfies all \((S_{r,s}^{1-s/r})\) with \(0 < s < r < \infty\).

For all bounded sets \(\Omega \subset M\) and all \(r \geq 1\), if M satisfies \((S_{r_{0},s_{0}}^{s_{0}})\) for some \(0 < s_{0} < r_{0} < \infty\) then there exists a constant \(A_{1}\) such that

\[
\forall f \in C_{0}^{\infty}(\Omega), \quad ||f||_{r} \leq A_{1}(1 + r)\mu(\Omega)^{1/r}||\nabla f||_{p}.
\]

In addition, there exist constants \(\alpha > 0\) and \(A_{2}\) such that

\[
\forall f \in C_{0}^{\infty}(\Omega), \quad \int_{\Omega} e^{\alpha ||f||_{p}/||\nabla f||_{p}} d\mu \leq A_{2}\mu(\Omega)
\]
3.3.3 Equivalences when $q \in (-\infty, 0)$

Begin by fixing $p \in [1, \infty)$. Assume that on a manifold $M$, $(S^*_{r_0, s_0})$ is satisfied for some $0 < s_0 < r_0 \leq \infty$ and $\theta_0 \in (0, 1]$. Again, take $q = q(r_0, s_0, \theta_0)$ as before. With $s_0 < r_0$, we have $q \in (-\infty, 0)$. We want to show that all inequalities $(S^0_{r_0, s_0})$ with $0 < s < r \leq \infty$, $\theta \in (0, 1]$ and $q(r, s, \theta) = q(r_0, s_0, \theta_0)$ are satisfied: particularly, we want to show that there is a constant $A \in \mathbb{R}$ such that

$$\forall f \in C^\infty(M), \quad ||f||_\infty \leq A(C||\nabla f||_p)^{1/(1-s/q)}||f||_s^{1/(1-q/s)}$$

for all $s \in (0, \infty)$, with $C$ the constant from $(S^*_{r_0, s_0})$. Set a nonnegative function $f \in C^\infty_0(M)$ with $||f||_\infty \neq 0$. In addition, fix $\varepsilon > 0$ and $\rho > 1$, and define $k(f)$ to be the largest integer $k$ such that $\rho^k < ||f||_\infty$. Then, set

$$f_{p,k} = \min\{\max\{f - ||f||_\infty + \varepsilon + \rho^k, 0\}, \rho^{-k-1}(\rho - 1)\}$$

$$= (f - (||f||_\infty - \varepsilon - \rho^k))_+ \wedge \rho^{-k-1}(\rho - 1)$$

for all $k \leq k(f)$. Fix $\lambda_k = ||f||_\infty - \varepsilon - \rho^k$. Note that $f_{p,k}$ has support in $\{f \geq \lambda_k\}$ and

$$\{f_{p,k} \geq \rho^{-k-1}(\rho - 1)\} = \{f \geq \lambda_{k-1}\}.$$

Because $M$ satisfies $(S^*_{r_0, s_0})$, apply this inequality to $f_{p,k}$ to find

$$\rho^{-k-1}\mu(\{f \geq \lambda_{k-1}\})^{1/r_0} \leq \left(\frac{C||\nabla f||_p}{\rho - 1}\right)^{\theta_0} \rho^{(k-1)(1-\theta_0)}\mu(\{f \geq \lambda_k\})^{(1-\theta_0)/s_0}$$

$$\Rightarrow \mu(\{f \geq \lambda_{k-1}\})^{1/r_0} \leq \left(\frac{C||\nabla f||_p}{\rho - 1}\right)^{\theta_0} \rho^{(1-\theta_0)}\mu(\{f \geq \lambda_k\})^{(1-\theta_0)/s_0}$$

by dividing by $\rho^{k-1}$. Now, set $a_k = \rho^k \mu(\{f \geq \lambda_k\})$ and multiply by $\rho^{\delta(k-1)}$ with $r_0(1+\delta) = q$ to find

$$a_{k-1}^{1/r_0} \leq \rho^{-q(1-\theta_0)/s_0} \left(\frac{C||\nabla f||_p}{\rho - 1}\right)^{\theta_0} a_k^{(1-\theta_0)/s_0}$$

holds for all $k \leq k(f)$. Note that $a_k > 0$ for all $k \leq k(f)$, and $\lim_{k \to -\infty} a_k = \infty$. Set $a = \inf_{k \leq k(f)} a_k > 0$ to find

$$a^{1/r_0} \leq \rho^{-q(1-\theta_0)/s_0} \left(\frac{C||\nabla f||_p}{\rho - 1}\right)^{\theta_0} a^{(1-\theta_0)/s_0}$$

which implies

$$a \geq \rho^{-q^2(1-\theta_0)/(s_0\theta_0)} \left(\frac{C||\nabla f||_p}{\rho - 1}\right)^{\theta_0}$$

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Because $\lambda^s \mu(\{f \geq \lambda\}) \leq \|f\|_s^{\ast}$, we find
\[ \lambda^{-s} \rho^k \|f\|_s^\ast \geq \rho^{-q^r(1-\theta_0)/s_0\theta_0} \left( \frac{C\|\nabla f\|_{p_0}}{\rho - 1} \right)^{\theta_0}. \]

Letting $\varepsilon = 0$ and setting $k = k(f) - 1$, $\rho = 1 + \frac{1}{1 + |q|}$ we find
\[ \|f\|_{\infty} \leq 4(1 + |q|)(e^{(1-\theta_0)/s_0\theta_0} C\|\nabla f\|_{p_0})^{1/(1-s/q)} \|f\|_s^{1/(1-q/s)}. \]

Thus, all inequalities $(S_{r,s}^\theta)$ with $0 < s < r \leq \infty, \theta \in (0,1]$ with $q(r,s,\theta) = q(r_0,s_0,\theta_0)$ are satisfied on $M$.

In addition, if $M$ satisfies $(S_{r_0,s_0}^\theta)$ with $q \in (0,0)$ then there exists a constant $A$ such that
\[ \forall f \in C^\infty_0(\Omega), \|f\|_{\infty} \leq AC\mu(\Omega)^{-1/q}\|\nabla f\|_p \]
for bounded domains $\Omega \in M$, with $C$ the constant from $(S_{r_0,s_0}^\theta)$.

### 3.3.4 Equivalences under different values of $p$

In the previous sections when we examined equivalences between weak- and strong-Sobolev inequalities, we fixed $p$ and found equivalences under the same value of $q$. Allow $(S_{r,s}^\theta(p))$ and $(S_{r,s}^\theta(p))$ refer to the inequalities $(S_{r,s}^\theta)$ and $(S_{r,s}^\theta)$ as we will now vary $p \in [1, \infty)$.

Assume that on a manifold $M$, the inequality $(S_{r_0,s_0}^\theta(p_0))$ is satisfied for some $p_0 \in [1, \infty), 0 < r_0, s_0 \leq \infty, 0 \in (0,1]$. Assume that $q_0 = q(r_0,s_0,\theta_0)$ satisfies $1/q_0 < 1/p_0$, and let $v = \frac{p_0q_0}{q_0 - p_0}$ ($\frac{1}{q_0} = \frac{1}{p_0} - \frac{1}{v}$). By the results of the previous sub-sections, we know the inequalities $(S_{u_0,v_0}^\sigma(p_0))$ are satisfied for all $u_0, \sigma_0$ such that $q(p_0, u_0, \sigma_0) = q_0$.

Let $f \in C^\infty_0(M)$ and $\gamma \geq 1$. Apply some $(S_{p_0,q_0}^\sigma(p_0))$ to $|f|^\gamma$ to find
\[ \|f\|_{p_0}^{\gamma} \leq C \left( \gamma \right) \left( \int_M |f|^{(\gamma-1)p_0} \|\nabla f\|_{p_0} d\mu \right)^{\sigma_0/p_0} \|f\|_{u_0\gamma}^{(1-\sigma_0)}. \]

The Hölder inequality states
\[ \left( \int_M |f|^{(\gamma-1)p_0} d\mu \right)^{1/p_0} \leq \|f\|_{p_0\gamma}^{\gamma-1} \|g\|_{p_0\gamma}, \]
which brings us to
\[ \|f\|_{p_0}^{\gamma} \leq C \gamma^{\sigma_0/p_0} \|f\|_{p_0\gamma}^{(\gamma-1)\sigma_0} \|\nabla f\|_{p_0\gamma}^{\sigma_0} \|f\|_{u_0\gamma}^{\gamma(1-\sigma_0)} \]

If we set $p = \gamma p_0, \sigma = \frac{\sigma_0}{\sigma_0 + \gamma(1-\sigma_0)}, u = \frac{u_0}{p_0}$ we see that the inequality $(S_{p,u}^\sigma(p))$ is satisfied. In addition, we have
\[ \frac{1 - \sigma}{\sigma} = \gamma \frac{1 - \sigma_0}{\sigma_0}, \]
\[ \gamma \frac{1 - \sigma_0}{\sigma_0} \]
and that the parameter \( v \) has not changed from \((S_{\sigma_0}^{\sigma_0}(p_0))\) to \((S_{\sigma}^{\sigma}(p))\);
\[
\frac{1}{v} = \frac{1}{p_0} - \frac{1}{q_0} = \frac{1}{p} - \frac{1}{q(p, u, \sigma)}.
\]

Thus, if \( M \) satisfies \((S_{\sigma_0}^{\sigma_0}(p_0))\), \( M \) satisfies all \((S_{\sigma}^{\sigma}(p))\) such that
\[
\frac{1}{q(p, u, \sigma)} = \frac{1}{p} - \frac{1}{v}.
\]

This allows us to increase \( p \) and construct a strong Sobolev inequality. In addition, the results from the previous three sections show that even under the new value of \( p \), all inequalities \((S_{r,s}^{\theta}(p))\) with \( 0 < r, s \leq \infty, \theta \in (0, 1]\) such that \( \frac{1}{q(r,s,\theta)} = \frac{1}{p} - \frac{1}{v} \) are satisfied.

### 3.4 Pseudo-Poincaré Inequalities

A pseudo-Poincaré inequality is a tool that can be used to prove a manifold satisfies a Sobolev inequality. Here, we will provide a brief overview of what a pseudo-Poincaré inequality is, and demonstrate its ability to prove what Sobolev inequalities a manifold satisfies. In the next section, a connection between a manifold’s geometry and a pseudo-Poincaré inequality is illustrated.

For any \( f \in C_0^\infty(M) \), set
\[
f_t(x) = \frac{1}{V(x,t)} \int_{B(x,t)} f(y) d\mu(y).
\]

It is said that \( M \) satisfies a **pseudo-Poincaré inequality** in \( L^p \) if there exists a constant \( C \) such that
\[
\forall f \in C_0^\infty(M), \forall t > 0, \quad ||f - f_t||_p \leq C t ||\nabla f||_p.
\]

Note that with this definition of \( f_t \), we have \( ||f||_\infty \leq \frac{||f||_p}{V(x,t)} \). The following result provides a connection between pseudo-Poincaré inequalities and Sobolev inequalities of the \((S_{r,s}^{\theta}(p))\)–type, as seen in [1, Theorem 3.3.1].

**Theorem 3.5:** Assume that \( M \) satisfies
\[
\forall f \in C_0^\infty(M), \forall t > 0, \quad ||f - f_t||_p \leq C t ||\nabla f||_p.
\]

for some \( p_0 \in [1, \infty) \) and that
\[
\inf_{x \in M, t \geq 0} \{ t^{-v} V(x,t) \} > 0
\]

for some \( v > 0 \). Then the inequalities \((S_{r,s}^{\theta}(p))\) are satisfied on \( M \) for all \( p \geq p_0 \), and all \( 0 < r, s \leq \infty, \theta \in (0, 1] \) such that \( q(r,s,\theta) \) satisfies \( 1/q = 1/p - 1/v \).
Proof. By the result of sub-subsection 3.3.4 Equivalences under different values of p, we only have to treat the case \( p = p_0 \). Let \( M \) be a manifold that satisfies

\[
\forall f \in C_0^\infty(M), \forall t > 0, \quad ||f - f_t||_p \leq Ct||\nabla f||_p
\]

for some \( p_0 \in [1, \infty) \). There exists a constant \( c > 0 \) such that \( V(x, t) \geq (ct)^v \) for all \( x \in M, t > 0 \). Set a nonnegative \( f \in C_0^\infty(M) \) and a constant \( \lambda > 0 \). For any \( t > 0 \), write

\[
\mu(\{f \geq \lambda\}) \leq \mu(\{|f - f_t| \geq \lambda/2\}) + \mu(\{f_t \geq \lambda/2\})
\]

and set \( t \) such that \( \frac{||f||_1}{(ct)^v} = \lambda/4 \). Then, \( \mu(\{f_t \geq \lambda/2\}) = 0 \) and

\[
\mu(\{f \geq \lambda\}) \leq \mu(\{|f - f_t| \geq \lambda/2\}) \\
\leq (2/\lambda)^p||f - f_t||^p_p \\
\leq (2Ct||\nabla f||^p_p/\lambda)^p \\
= [(2^{1+2/v}C/c)||f||^{1/v}_1||\nabla f||_p^p/\lambda^{1-1/v}]^p.
\]

and so

\[
\lambda^{(vp+p)/v} \mu(\{f \geq \lambda\}) \leq \left[ \frac{2^{1+2/v}C}{c} ||f||^{1/v}_1||\nabla f||^p_p \right]^p.
\]

Raising this to the power \( \frac{v}{p + pv} \) we find

\[
\lambda^{\frac{v}{p + pv}} \mu(\{f \geq \lambda\}) \leq 4(C/c)^{v/(1+v)} ||f||^{1/(1+v)}_1||\nabla f||^v_p/(1+v),
\]

which is the inequality \( (S^{*,v/(1+v)}_{p(v+1)/v,1}(p)) \). By the results of the previous sub-subsections, this proves Theorem 3.5.

Thus, the pseudo-Poincaré inequality is useful to show what Sobolev inequalities are satisfied on a manifold. In the next section, we will see that there is a geometric sufficient condition to show that a manifold satisfies a pseudo-Poincaré inequality.
4  Sobolev Inequalities and Manifold Geometry

In this section, we are going to investigate the information about a Riemannian manifold’s geometry that is encoded in the manifold’s Sobolev-type inequalities. We will begin with some background in manifold geometry.

On a complete Riemannian manifold \((M, g)\), you can start at any point \(p \in M\) and follow a “straight” line in any direction indefinitely. The notation \((M, g)\) refers to the smooth manifold \(M\) and the Riemannian metric \(g\), a positive-definite inner product tensor that provides a notion of distance between two points of \(M\).

Manifolds are known as spaces that locally resemble Euclidean space. This conceptualization begs the question: how different is my manifold from Euclidean space? To answer this question, the Ricci curvature tensor was developed. The Ricci curvature tensor \(\mathcal{R}\) provides a measure of how different the geometry of a manifold is from the geometry of ordinary Euclidean space. Lower bounds on the Ricci tensor can provide information on the global geometry and topology of the manifold.

The Ricci curvature tensor is a symmetric two-tensor obtained by the contraction of the full curvature tensor of a manifold, and can be compared to the metric tensor \(g\). As such, bounding the Ricci tensor below by some scalar multiple of \(g\) \((\mathcal{R} \geq kg, k \in \mathbb{R})\) is useful for uncovering analytic and geometric information about a manifold [1]. An example is that if \(\mathcal{R} \geq kg\) for \(k > 0\), then the manifold \(M\) is compact. If \(\mathcal{R} \geq 0\), the volume growth on \((M, g)\) is at most Euclidean: \(\forall r > 0, \ V(x, r) \leq \Omega_n r^n\). Note that \(\Omega_n = \frac{\omega_{n-1}}{n}\) , where \(\omega_{n-1}\) is the surface measure of the unit sphere in \(\mathbb{R}^n\).

In addition, Ricci curvature can provide insight to what Sobolev inequalities a manifold satisfies. For example, [1, Theorem 3.3.8] shows that complete \(n\)-dimensional manifolds with nonnegative Ricci curvature and maximal volume growth satisfy the pseudo-Poincaré inequality:

**Theorem 4.1:** Let \((M, g)\) be a complete manifold of dimension \(n\) with Ricci curvature \(\mathcal{R} \geq 0\). Assume that there exists \(c > 0\) such that \(\forall r > 0, \ V(x, r) \geq cr^n\). Then, \(\forall r > 0, \forall f \in C^\infty_0 (M), \ ||f - f_r||_1 \leq (\Omega_n/c) r ||\nabla f||_1\).

This is the psuedo-Poincaré inequality on \(M\). Furthermore, all Sobolev inequalities \((S^\theta_{r,s}(p))\) with \(1/r = \theta (1/p = 1/n) + (1 + \theta)/s\), \(0 < r, s \leq \infty\), \(\theta \in (0, 1]\) are satisfied on \(M\). Particularly, for all \(p \in [1, n]\), \(\forall f \in C^\infty_0 (M), \ ||f||_{p^*} \leq C(n, p) ||\nabla f||_p\).
Proof. Observe that for $f \in C_0^\infty(M)$,

$$
||f - f_r||_1 = \int_M |f(x) - \frac{1}{V(x,r)} \int_{B(x,r)} f(y) dy| dx
\leq \int_M \int_M |f(x) - f(y)| \frac{1_{B(x,r)}(y)}{V(x,r)} dydx.
$$

Note that $1$ is the indicator function; $1_A(x) = 1$ if $x \in A$, and $1_A(x) = 0$ if $x \not\in A$. In the proof of this theorem, we find

$$
\int_{B(x,r)} |f(x) - f(y)| dy \leq \int_{S^{n-1}} \int_0^r \int_0^\rho |\nabla f(t,\theta)| t^{1-n} \sqrt{g}(\rho,\theta) dt \rho^{n-1} d\rho d\theta
\leq \frac{r^n}{n} \int_{S^{n-1}} \int_0^r |\nabla f(t,\theta)| t^{1-n} \sqrt{g}(t,\theta) dt d\theta
= \frac{r^n}{n} \int_{B(x,r)} \frac{\nabla f(y)}{d(x,y)^{n-1}} dy
$$

by integrating along the geodesic segment from $x$ to $y$ using polar exponential coordinates $y = (\rho,\theta)$ around $x$. The Riemannian volume element in polar coordinates is $dy = \sqrt{g}(\rho,\theta)d\rho d\theta$.

Now, we use the hypothesis that $(M,g)$ has non-negative Ricci curvature: by Bishop’s theorem, the function $s \mapsto \sqrt{g}(s,\theta)/s^{n-1}$ is non-increasing. Thus, we find

$$
\int_{B(x,r)} |f(x) - f(y)| dy \leq \int_{S^{n-1}} \int_0^r \int_0^\rho |\nabla f(t,\theta)| t^{1-n} \sqrt{g}(t,\theta) dt \rho^{n-1} d\rho d\theta
\leq \frac{r^n}{n} \int_{S^{n-1}} \int_0^r |\nabla f(t,\theta)| t^{1-n} \sqrt{g}(t,\theta) dt d\theta
= \frac{r^n}{n} \int_{B(x,r)} \frac{\nabla f(y)}{d(x,y)^{n-1}} dy
$$

Now, using this along with the hypothesis of maximal volume growth that $V(x,r) \geq cr^n$, we find

$$
\int_M \int_M |f(x) - f(y)| \frac{1_{B(x,r)}(y)}{V(x,r)} dy dx \leq \frac{1}{cn} \int_M \int_{B(x,r)} \frac{|\nabla f(y)|}{d(x,y)^{n-1}} dy dx
\leq \frac{1}{cn} \int_M |\nabla f(y)| \int_{B(y,r)} \frac{1}{d(x,y)^{n-1}} dxdy
$$

By Bishop’s theorem, $\sqrt{g}(t,\theta) \leq t^{n-1}$, so we see

$$
\int_{B(y,r)} \frac{1}{d(x,y)^{n-1}} dxdy \leq \omega_{n-1} r^n
$$

and so

$$
||f - f_r||_1 \leq \frac{\omega_{n-1}}{cn} r ||\nabla f||_1.
$$

Recalling that $\Omega_n = \omega_{n-1}/n$, this is the desired inequality, and concludes the proof of Theorem 4.1. \qed

In fact, we can construct a pseudo-Poincaré inequality on $M$ without maximal volume growth, as proven in [1, Theorem 3.3.9].

[20]
4.1 Bounding the Number of Ends

Topologically, an “end” of a manifold is a connected component of the ideal boundary of the manifold. An end of a manifold is a topologically distinct way to move to infinity on the manifold. If $M$ is a compact manifold with a boundary, the number of ends in the interior of $M$ is the number of connected components of $\partial M$. In their paper *Positive Solutions to Schrödinger Equations and Geometric Applications*, authors Munteanu, Schulze and Wang demonstrate that positive solutions to given Schrödinger equations on a manifold “must be of polynomial growth of fixed order under a suitable scaling invariant Sobolev inequality” [4]. Each end of a complete Riemannian manifold has an associated positive solution to this given Schrödinger equation; as such, we can use Sobolev inequalities to prove that a manifold’s number of ends is finite, and even estimate that number. We will discuss two theorems from this paper that demonstrate how Sobolev inequalities can be used to bound a manifold’s number of ends.

**Theorem 4.2** (cf. [4, Theorem 1.2]): Let $(M, g)$ be a complete Riemannian manifold of dimension $n \geq 3$ that satisfies the inequality

$$\forall f \in C_0^\infty(M), \quad \left( \int_M f^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \leq A \int_M (|\nabla f|^2 + \sigma f^2)$$

for $A > 0$ a constant and $\sigma \geq 0$ a continuous function. Set

$$\alpha = \limsup_{R \to \infty} \frac{1}{V(p, R)} \int_{B(p, R)} (r^2 \sigma)^{\frac{n-1}{2}} < 0$$

and

$$V_\infty = \limsup_{R \to \infty} \frac{V(p, R)}{R^n} < \infty,$$

with $p \in M$ a fixed point, and $r(x) = d(p, x)$ the distance function to $p$. Then the number of ends of $M$ is bounded above by a constant $\Gamma = \Gamma(n, A, \alpha, V_\infty)$ dependent solely on $n, A, \alpha,$ and $V_\infty$.

Recall from earlier that $B(p, R)$ denotes the ball of radius $R$ centered at $p$, and $V(p, R)$ denotes the measure of this ball. Below is a more general version of Theorem 4.2.

**Theorem 4.3** (cf. [4, Theorem 2.1]): Let $(M, g)$ be an n-dimensional complete Riemannian manifold satisfying the Sobolev inequality

$$\forall f \in C_0^\infty(M), \quad \left( \int_M |f|^{\frac{2n}{n-q}} \right)^{\frac{n-q}{n}} \leq A \int_M (|\nabla f|^q + \sigma |f|^q)$$

(9)
for some \( q \in [1, n-1] \), with \( A > 0 \) a constant and \( \sigma \geq 0 \) a continuous function. Set
\[
\alpha = \limsup_{R \to \infty} \frac{1}{V(p, R)} \int_{B(p, R)} (r^q \sigma)^{\frac{n-1}{q}}
\]
and
\[
V_\infty = \limsup_{R \to \infty} \frac{V(p, R)}{R^n}.
\]
If both \( \alpha \) and \( V_\infty \) are finite, then the number of ends of \( M \) is bounded above by a constant \( \Gamma(n, A, \alpha, V_\infty) \) dependent solely on \( n, A, \alpha, \) and \( V_\infty \).

Proof. For an end \( E \) of an \( n \)-dimensional complete Riemannian manifold \( M \), write \( E(R) = B(p, R) \cap E \). Let \( M \) have at least \( k > 1 \) ends, and take \( R \) such that
\[
B(p, 2R) \setminus B(p, R) = \bigcup_{i=1}^{k} E_i(2R) \setminus E_i(R).
\]
By the definition of \( V_\infty \), we know that \( \frac{V(p, t)}{t^n} \leq 2V_\infty \), and by the definition of \( \alpha \) we see
\[
\sum_{i=1}^{k} \int_{E_i(3R) \setminus E_i(R)} (r^q \sigma)^{\frac{n-1}{q}} \leq 2\alpha V(p, 3R)
\]
\[
\Rightarrow \sum_{i=1}^{k} \int_{E_i(3R) \setminus E_i(R)} \sigma^{\frac{n-1}{q}} \leq C_0 \frac{V(p, 3R)}{R^{n-1}}
\]
The constant \( C_0 \) is dependent on \( n, A, \alpha, \) and \( V_\infty \), as with the \( C_1, C_2 \ldots \) seen below.

Label the ends \( E_1, \ldots, E_k \) such that
\[
\left\{ \int_{E_i(3R) \setminus E_i(R)} \sigma^{\frac{n-1}{q}} \right\}_{i=1,\ldots,k}
\]
is an increasing sequence. Then, we see
\[
\int_{E_i(3R) \setminus E_i(R)} \sigma^{\frac{n-1}{q}} \leq \frac{2C_0 V(p, 3R)}{kR^{n-1}}, \quad (10)
\]
for all \( i \in [1, \lfloor \frac{k}{2} \rfloor] \). For all \( i \in \{ 1, 2, ..., \lfloor \frac{k}{2} \rfloor \} \) pick a point
\[
z_i \in \partial E_i(2R)
\]
and relabel \( E_1, \ldots, E_{\lfloor \frac{k}{2} \rfloor} \) if necessary such that
\[
\{ V(z_i, R) \}_{i=1,\ldots,\lfloor \frac{k}{2} \rfloor}
\]
is an increasing sequence. Assume for a contradiction that

\[ V(z_1, R) \geq \frac{C_1}{k} R^n \]

where \( C_1 = 3^{n+2} V_\infty \). We know that

\[ B(z_i, R) \subset E_i(3R) \setminus E_i(R) \]

and that \( \{B(z_i, R)\}_{i=1}^{\lfloor \frac{k}{2} \rfloor} \) are disjoint in \( B(p, 3R) \). By our assumption, we see

\[ V(p, 3R) \geq \sum_{i=1}^{\lfloor \frac{k}{2} \rfloor} V(z_i, R) \geq \frac{k}{2} \frac{C_1}{k} R^n \geq \frac{C_1}{3} R^n = 3 V_\infty (3R)^n. \]

However, because \( \frac{V(p,t)}{t^n} \leq 2 V_\infty \), we know that

\[ V(p, 3R) \leq 3 V_\infty (3R)^n, \]

and so this is a contradiction. Thus, we see

\[ V(z_1, R) < \frac{C_1}{k} R^n. \tag{11} \]

Let \( E = E_1, z = z_1 \), so \( z \in \partial E(2R) \) and \( V(z, R) < \frac{C_1}{k} R^n \). By (10) we know

\[ \int_{E(3R) \setminus E(R)} \sigma^{n-1} \leq \frac{C_2 V(p, 3R)}{k R^{n-1}}. \]

Let \( \gamma(t) \) be a minimizing geodesic from \( p \) to \( z \), with \( \gamma(0) = p, \gamma(2R) = z: t \in [0, 2R] \). For \( t \in [4R/3, 5R/3] \) and \( x \in \gamma(t) \), because \( d(x, z) \leq \frac{2R}{3} \) the triangle inequality implies

\[ B \left( x, \frac{R}{3} \right) \subset B(z, R). \]

Thus, we see by (11) that

\[ V \left( x, \frac{R}{3} \right) < \frac{C_1}{k} R^n \tag{12} \]

for all \( x = \gamma(t) \) for \( t \in [4R/3, 5R/3] \).

Next, authors Munteanu, Schulze, and Wang assume for a contradiction that

\[ \int_{B(x,r)} \sigma \leq \delta r^{q-1} V(x, r)^{\frac{n-q-1}{q-1}} \tag{13} \]
for \( r \in (0, R/3) \) with \( \delta > 0 \) a constant. Fix an \( r \in (0, R/3) \) and consider the cut-off function \( \phi \) with support in \( B(x, r) \) such that \( \phi = 1 \) on \( B(x, r/2) \) and \( |\nabla\phi| \leq \frac{2}{r} \). Applying the Sobolev inequality (9), we see

\[
V\left(x, \frac{r}{2}\right)^\frac{n-q}{n} \leq A\left(\frac{2^q}{r^q} V(x, r) + \int_{B(x, r)} \sigma\right).
\]

By the assumption (13), we see

\[
V\left(x, \frac{r}{2}\right)^\frac{n-q}{n} \leq A\left(\frac{2^q}{r^q} V(x, r) + \delta r^{q-1} V(x, r)^\frac{n-q-1}{n-1}\right)
\]

for any \( r \in (0, R/3) \). Assume that there exists an \( r_0 \in (0, R/3) \) such that \( V(x, r_0) \leq \delta \frac{n-1}{q} r_0^{n-q} \). This brings us to

\[
\Rightarrow V\left(x, \frac{r_0}{2}\right) \leq 2^n (A(2^q + 1))^{\frac{n}{n-q}} \delta^{\frac{n-1}{n-q}} \left(\frac{r_0}{2}\right)^n
\]

Choose \( \delta \) such that

\[
2^n (A(2^q + q))^{\frac{n}{n-q}} \delta^{\frac{n-1}{n-q}} < 1.
\]

Then, we have

\[
V\left(x, \frac{r_0}{2}\right) \leq \delta^{\frac{n-1}{q}} \left(\frac{r_0}{2}\right)^n.
\]

Thus, given the assumption that (13) holds for \( r \in (0, R/3) \), then if there exists an \( r_0 \) in this interval such that \( V(x, r_0) \leq \delta \frac{n-1}{q} r_0^{n-q} \), the above inequality holds. If \( k \) is large enough to satisfy \( \frac{3nC_1}{k} \leq \delta \frac{n-1}{q} \), (12) implies that

\[
V\left(x, \frac{R}{3}\right) < \delta^{\frac{n-1}{q}} \left(\frac{R}{3}\right)^n
\]

and we can set \( r_0 = R/3 \) to find

\[
V\left(x, \frac{R}{3 \cdot 2^m}\right) \leq \delta^{\frac{n-1}{q}} \left(\frac{R}{3 \cdot 2^m}\right)^n
\]

by induction. However, if we set \( \delta \) small enough that \( \delta^{\frac{n-1}{q}} < V(0, 1)_{\mathbb{R}^n} \), then as \( m \to \infty \) this is a contradiction. Therefore, (13) does not hold; for any \( x = \gamma(t) \) with \( t \in [4R/3, 5R/3] \) there exists an \( r_x \in (0, R/3) \) such that

\[
\int_{B(x, r_x)} \sigma > \delta r_x^{\frac{q}{n-1}} V(x, r_x)^{\frac{n-q-1}{n-1}}
\]
and the Hölder inequality brings us to
\[
\int_{B(x,r)} \sigma \leq \left( \int_{B(x,r)} \sigma^{\frac{n-1}{q}} \right)^{\frac{q}{n-1}} V(x,r)^{\frac{n-q-1}{n-1}}
\]
\[
\implies \int_{B(x,r)} \sigma^{\frac{n-1}{q}} \geq \frac{1}{C_3} r_x
\]
for any \( x = \gamma(t) \) for \( t \in [4R/3, 5R/3] \). A covering argument as in [4] implies that we can choose at most countably many disjoint balls \( \{B(x_m, r_{x_m})\}_{m \geq 1} \) with \( x_m = \gamma(t_m), t_m \in [4R/3, 5R/3] \), each satisfying the above inequality. These balls cover at least one third of the geodesic \( \gamma([4R/3, 5R/3]) \).

Thus,
\[
\sum_{m \geq 1} r_{x_m} \geq \frac{1}{3} \left( \frac{5R}{3} - \frac{4R}{3} \right) = \frac{R}{9}
\]
and so
\[
\frac{R}{9} \leq \sum_{m \geq 1} r_{x_m} \leq C_3 \sum_{m \geq 1} \int_{B(x_m, r_{x_m})} \sigma^{\frac{n-1}{q}} \leq C_3 \int_{B(z,R)} \sigma^{\frac{n-1}{q}}
\]
because \( \{B(x_m, r_{x_m})\}_{m \geq 1} \) are disjoint in \( B(z, R) \) and \( B(x, \frac{R}{3}) \subset B(z, R) \).

The authors conclude that
\[
\frac{1}{9C_3} R \leq \int_{E(3R)\setminus E(R)} \sigma^{\frac{n-1}{q}} \leq \frac{C_2 V(p,3R)}{kR^{n-1}}
\]
\[
\implies V(p,3R) \geq \frac{kR^n}{9C_3C_2} = \frac{k}{C_4} R^n.
\]
If \( k > 2V_\infty C_4 3^n \), we have
\[
V(p,3R) > 2V_\infty (3R)^n
\]
This is a contradiction of \( \frac{V(p,t)}{R^n} \leq 2V_\infty \) (from the definition of \( V_\infty \)). Thus, \( k \), the number of ends of \( M \), is bounded above above by \( 2V_\infty C_4 3^n \). This proves the theorem. \( \square \)

In addition, there is a corollary to this theorem that uses the mean curvature \( H \) of the manifold \( M \).

**Corollary 4.4:** If \( M^n \) is a complete submanifold of \( \mathbb{R}^N \) with \( n \geq 2 \), let
\[
\bar{\alpha} = \limsup_{R \to \infty} \frac{1}{V(p,R)} \int_{B(p,R)} |H|^n \]
and
\[
V_\infty = \limsup_{R \to \infty} \frac{V(p,R)}{R^n} < \infty.
\]
Then the number of ends of \( M \) is bounded above by a constant \( \Gamma \) dependent only on \( n, \bar{\alpha}, \) and \( V_\infty \).

25
5 Applications of Sobolev Inequalities on Manifolds

When physicists and mathematicians study the conduction and diffusion of heat, a heat kernel is the fundamental solution to the heat equation when operating in a specified domain with boundary conditions. The heat kernel represents the change or evolution in temperature in a region when this region’s boundary is held at a fixed temperature.

Sobolev inequalities, notably the Nash inequality, have been used to study the heat diffusion semigroup $H_t = e^{-t\Delta}$. In fact, the Nash inequality implies that $H_t$ is a bounded operator from $L^1$ to $L^\infty$ with explicit time dependent estimates. The following theorem [1, Theorem 4.1.1] was first proven by Nash and contributes to this connection between Sobolev inequalities and the heat kernel.

**Definition 5.1:** The $q$ to $p$ norm, notated $|||A|||_{q\rightarrow p}$, is defined as

$$|||A|||_{q\rightarrow p} := \max\{||Ax||_p : ||x||_q = 1\}.$$ 

Here, the $x$ represent elements of the set that the operator $A$ acts on in-context. In our context, we will be considering the heat diffusion semigroup $H_t$ as our operator, with our elements being functions $f \in D$, the domain of a Dirichlet form on $L^2(M, \mu)$.

**Theorem 5.2:** Let $Q$ be a Dirichlet form on $L^2(M, \mu)$ with domain $D$ and associated semigroup $(H_t)_{t>0}$. Assume that the Nash inequality

$$\forall f \in D, \quad ||f||^2_{L^2} \leq C Q(f, f) ||f||^4_{L^1}$$

is satisfied for some $v > 0$. Then $(H_t)_{t>0}$ is ultracontractive and

$$\forall t > 0, \quad ||H_t||_{1\rightarrow \infty} \leq (Cv/2t)^{v/2},$$

or, equivalently, $(H_t)_{t>0}$ admits a density with respect to $\mu$ satisfying

$$\forall t > 0, \quad \sup_{x,y \in M} \{h(t, x, y)\} \leq (Cv/2t)^{v/2},$$

where $h(t, x, y)$ is the non-negative smooth function known as the heat kernel associated to $V^{-1}\Delta$ on $L^2(M, d\mu)$.

Equipped with this theorem, we move forward to setting Gaussian estimates on the heat kernel. Let $M$ be a complete non-compact Riemannian manifold with Riemannian measure...
\( \mu. \) Let \( \Delta \) be the Laplacian on \( M \) and let \( h(t, x, y) \) be the heat diffusion kernel on \( M \); for each \( x \in M, h(t, x, y) = u(t, y) \) is the minimal solution of
\[
\begin{cases}
(\partial_t + \Delta)u = 0 \\
u(0, y) = \partial_x(y).
\end{cases}
\]
This means that \( h(t, x, y) \) is the kernel of the semigroup \( H_t \) associated to the Dirichlet form \( Q(f, f) = \int |\nabla f|^2d\mu \) with domain \( W^1_2(M) \): the closure of \( C^\infty_0(M) \) under the norm \( \sqrt{||f||^2 + ||\nabla f||^2} \).

For any function \( \phi \in C^\infty_0(M) \) with \( ||\nabla \phi||_\infty \leq 1 \) and any complex number \( \alpha \), consider the semigroup
\[
H_t^{\alpha, \phi} f(x) = e^{-\alpha \phi(x)} \int h(t, x, y)e^{\alpha \phi(y)}f(y)dy = e^{-\alpha \phi(x)} H_t(e^{\alpha \phi} f)(x);
\]
this is a semigroup of operators on the spaces \( L^p(M, \mu) \), and its generator is given by
\[
-A_{\alpha, \phi} f = -e^{-\alpha \phi} \Delta (e^{\alpha \phi} f).
\]
The semigroup \( (H_t^{\alpha, \phi})_{t>0} \) satisfies
\[
\forall t > 0, \quad ||H_t^{\alpha, \phi}||_{2 \to 2} \leq e^{\alpha^2 t}.
\]
In addition, if \( \alpha \) is real, then for all \( t > 0, \zeta = e^{ir} \) with \( |r| \leq \epsilon, \epsilon \in (0, \pi/4] \) we have
\[
||H_t^{\alpha, \phi}||_{2 \to 2} \leq e^{\alpha^2 (1+\epsilon)t}
\]
and there exists a constant \( C \) such that
\[
||\partial_t^k H_t^{\alpha, \phi}||_{2 \to 2} \leq C^k k! e^{\alpha^2 (1+\epsilon)t} \frac{e^{\alpha^2 (1+\epsilon)t}}{(et)^k}.
\]

If the heat diffusion semigroup \( (H_t)_{t>0} \) satisfies \( ||H_t||_{2 \to 2} \leq (C/t)^{v/4} \) for all \( t > 0 \) for some constant \( C \), then for all \( p \in [2, \infty] \)
\[
\forall t > 0, \quad ||H_t||_{p \to \infty} \leq (C/t)^{v/2p}.
\]
In addition, if there exists a constant \( C \) and \( v > 0 \) such that
\[
\forall t > 0, \quad ||H_t||_{2 \to \infty} \leq (C/t)^{v/4},
\]
then
\[
||H_t^{\alpha, \phi}||_{p \to q} \leq (C/t)^{v(1/p-1/q)/2} e^{q\alpha^2 t/2}
\]
for all $q \geq p \geq 2, t > 0, \alpha \in \mathbb{R}$ and $\phi \in C_0^\infty(M)$ with $||\nabla \phi||_\infty \leq 1$.

Now, we will consider [1, Theorem 4.2.6].

**Theorem 5.3:** Assume that a manifold $M$ satisfies the Nash inequality

$$\forall f \in C_0^\infty(M), \quad ||f||_2^{2(1+2/v)} \leq C||\nabla f||_2^2||f||_4^{4/v}. \quad (14)$$

Then, for any $\delta > 0$ there exists a finite constant $C(\delta)$ such that the kernel $h(t, x, y)$ of the heat diffusion semigroup $H_t = e^{-t\Delta}, t > 0$ satisfies

$$h(t, x, y) \leq \left(\frac{\delta}{C(\delta)}\right)^{v/2} \exp\left(\frac{d(x, y)^2}{4(1 + \delta)t}\right).$$

This theorem states that if a manifold $M$ satisfies the Nash inequality, we have an upper bound on the heat kernel of the heat diffusion semigroup $H_t$.

However, this Gaussian upper bound does not make for the best estimate due to the exponential growth. We want to find a way to refine this estimate. If $M$ is a Riemannian manifold of dimension $n$ that satisfies the Nash inequality (14) for some $v > 0$, we have seen that the volume lower bound

$$\forall t > 0, \quad V(t) \geq ct^v$$

is implied, and so $v \geq n$. If the volume growth is faster than $t^v$, this bound is not useful. So, we have the following theorem [1, Theorem 4.2.8], that has an added restriction on volume growth to provide us with bounds above and below the heat kernel.

**Theorem 5.4:** Fix $v > 0$. Assume that $M$ satisfies the Nash inequality (14) and the volume growth condition

$$\forall x \in M, \forall r > 0, \quad c_0 \leq r^{-v}V(x, r) \leq C_0.$$ 

Then the heat kernel $h(t, x, y)$ is bounded above and below on the diagonal by

$$ct^{-v/2} \leq h(t, x, x) \leq Ct^{-v/2}.$$ 

The Nash inequality, a Sobolev-type inequality, is integral to these theorems that bound the heat kernel.
References


