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Pion Mass Corrections on the Lattice

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Abstract

We use the Infinite Volume Reconstruction Method to calculate the charged/neutral pion mass difference. The hadronic tensor is calculated on the lattice using a QCD+QED framework, and the mass shift is calculated with exponentially-suppressed finite volume errors. In this paper we discuss the Feynman diagrams relevant to the pion mass difference and we recapitulate the advantages of the Infinite Volume Reconstruction Method. We then discuss the extrapolation to the continuum limit $a \to 0$, and report a charged/neutral pion mass difference of $m_{\pi^+} - m_{\pi^0} = 4.52$ MeV, which is within 1.44% of the accepted value.

I Introduction

In nature, the strong force is the force responsible for the existence of hadrons. The strong force is described by the theory of quantum chromodynamics (QCD), and the calculation of hadron masses on the lattice is a primary application of QCD. Naively, one might ask why hadron masses cannot be calculated as the sum of the masses of the constituent quarks without the use of a quantum field theory such as QCD, which assigns primary importance to interactions between particles. However, this naive mass prediction process yields incorrect results, as experimentally-observed hadron masses are much greater than this process predicts. This discrepancy is removed with the inclusion of QCD, as the largest contributions to hadron masses are due to interaction energies of the substituent quarks and gluons. The usefulness of QCD in this regard suggests that the inclusion of other relevant interactions between substituent particles may prove useful as well. [Aok+17; Blu+16; Blu+10]

The interaction which provides the largest corrections to QCD results is the electromagnetic interaction, which is suppressed in the subhadronic domain by a factor of $\alpha_{\text{QED}} \approx \frac{1}{137}$ and is described by the theory of quantum electrodynamics (QED). Important motivation to include this interaction is derived from the observed mass differences between mesons of neutral electric charge and their charged counterparts, examples of which are the pion and
The inclusion of QED in QCD hadronic mass calculations (QCD+QED) is made possible by the sub-percent level precision of the lattice, which is a discretized space-time model whose extrapolation to the ”continuum limit” of non-discretized space-time has confirmed many experimental results. However, associated with lattice simulations are so-called ”finite volume errors” which are an implication not only of space-time discretization, but of the use of lattices which are finite in size. Without special consideration, these finite volume errors manifest as inversely proportional to a power of the distance between adjacent lattice points, called the ”lattice spacing” and denoted by $a$. Using the Infinite Volume Reconstruction Method (IVR) one may suppress these ”power-law” finite volume errors in favor of exponentially suppressed errors. In this paper we present the charged/neutral pion mass difference as calculated using the IVR to order $O(\alpha^{\text{QED}})$. [FJ19; Tan14]

## II Relevant Diagrams

Contributions to the charged/neutral QED pion mass splitting are derived from the following hadronic matrix element:

$$
\mathcal{H}_{\mu,\nu}^{\pm,0}(x) = \frac{Z^2}{2M_{\pi^0}} \langle \pi^{\pm,0}(0)| T[J_{\mu}\text{Local}(x) J_{\nu}\text{Local}(0)]|\pi^{\pm,0}(0)\rangle
$$

(1)

where $\pi(0)$ represents a pion with momentum 0 and $J_{\mu}\text{Local} = 2\bar{u}\gamma_{\mu}u/3 - e d\gamma_{\mu}d/3 - e s\gamma_{\mu}s/3$ is the lattice local electromagnetic current. Of the contractions which contribute to this matrix element, one yields

$$
\mathcal{H}_{\mu,\nu}^{1}(x) = \frac{\langle \text{Tr}(\gamma_{\mu}S(x; t_{\text{src}})\gamma_{5}S(t_{\text{src}}; y)\gamma_{\nu}S(y; t_{\text{snk}})\gamma_{5}S(t_{\text{snk}}; x))\rangle_{\text{QCD}}}{\langle \text{Tr}(S(t_{\text{sep}}; 0)\gamma_{5}S(0; t_{\text{sep}})\gamma_{5})\rangle_{\text{QCD}}}
$$

(2)
which is related (up to the insertion of a photon propagator) to the Feynman diagram

while the other possible contraction yields

\[ H_{\mu,\nu}^2(x) = \frac{\langle \text{Tr}(S(x; t_{\text{src}})\gamma_5 S(t_{\text{src}}; x)\gamma_\mu)\text{Tr}(S(y; t_{\text{snk}})\gamma_5 S(t_{\text{snk}}; y)\gamma_\nu) \rangle_{\text{QCD}}}{\langle \text{Tr}(S(t_{\text{sep}}; 0)\gamma_5 S(0; t_{\text{sep}})\gamma_5) \rangle_{\text{QCD}}} \] (3)

which is represented by the 'disconnected' Feynman diagram

Combining these diagrams yields the hadronic contribution to the mass shift, which is represented by

\[ H_{\mu,\nu}^\pm(x) - H_{\mu,\nu}^0(x) = \frac{e^2}{2} \left( -H_{\mu,\nu}^1(x) + H_{\mu,\nu}^2(x) \right) \] (4)

To order \( O(\alpha_{\text{QED}}) \) these two diagrams represent the only contributions to the mass shift. That contributions to the variations of the strong coupling constant \( g \), the symmetric pion mass \( m_{ud} = \frac{m_u + m_d}{2} \), and heavier quark masses are not present is due to the cancellation of isosymmetric vacuum polarization
and disconnected sea quark diagrams in the subtraction $m_{\pi^+} - m_{\pi^0}$. The cancellation of isosymmetric vacuum polarization diagrams is a general artifact of the current scheme, in which these isosymmetric variations do not contribute to first order corrections to observables which vanish in the isosymmetric theory. The cancellation of the disconnected sea quark diagrams must occur because all relevant diagrams must contain two insertions of the quark current $J_\mu = 2e\bar{u}\gamma_\mu u/3 - e\bar{d}\gamma_\mu d/3 - e\bar{s}\gamma_\mu s/3$, and must therefore be at minimum of order $\mathcal{O}(\alpha_{\text{QED}})$. As we treat $\mathcal{O}(m_d - m_u)$ corrections to be of the same perturbative order as corrections of order $\mathcal{O}(\alpha_{\text{QED}})$, terms of order $\mathcal{O}(\alpha_{\text{QED}}(m_d - m_u))$ are defined to be of order $\mathcal{O}(\alpha_{\text{QED}}^2)$ and are therefore not of our current interest. For more information see [Div+13].

III Infinite Volume Reconstruction Method

The hadron mass extraction technique referenced in the previous section, known as the Infinite Volume Reconstruction Method (IVR) as presented in [FJ19], relies on the calculation of the hadron QED self-energy for a stable hadronic state $N$ via the following euclidean-space integral

$$\Delta M = \mathcal{I} = \frac{1}{2} \int d^4x \mathcal{H}_{\mu,\nu}(x) S^\gamma_{\mu,\nu}(x), \tag{5}$$

where hadronic part $\mathcal{H}_{\mu,\nu}(x) = \mathcal{H}_{\mu,\nu}(t, \vec{x})$ is given by

$$\mathcal{H}_{\mu,\nu}(x) = \frac{1}{2M} \langle N(\vec{0})|T[J_\mu(x)J_\nu(0)]|N(\vec{0})\rangle, \tag{6}$$

where $|N(\vec{p})\rangle$ indicates a hadronic state $N$ with the mass $M$ and spatial momentum $\vec{p}$, and $S^\gamma_{\mu,\nu}$ is the photon propagator whose form is analytically known.
The IVR proposes that:

$$\mathcal{I} \approx \mathcal{I}^{(s,L)} + \mathcal{I}^{(l,L)}$$

(7)

where $$\mathcal{I}^{(s,L)}$$ (the "short distance" and "long distance" contributions, respectively) are lattice-calculable quantities given by

$$\mathcal{I}^{(s,L)} = \frac{1}{2} \int_{-t_s}^{t_s} dt \int_{-L/2}^{L/2} d^3 \vec{x} \mathcal{H}_{\mu,\nu}^L(x) S_{\gamma,\mu,\nu}^\gamma(x)$$

(8)

$$\mathcal{I}^{(l,L)} = \int_{-L/2}^{L/2} d^3 \vec{x} \mathcal{H}_{\mu,\nu}^L(t_s, \vec{x}) L_{\mu,\nu}(t_s, \vec{x})$$

(9)

where $$L_{\mu,\nu}(t_s, \vec{x})$$ is a QED weighting function, defined as:

$$L_{\mu,\nu}(t_s, \vec{x}) = \int \frac{d^3 p}{(2\pi)^3} e^{i \vec{p} \cdot \vec{x}} \int_{t_s}^{\infty} dt e^{-(E_{\vec{p}} - M)(t - t_s)}$$

(10)

where $$t_s \lesssim L$$ is a time chosen such that the all excited hadronic states with energies higher than the lowest excited state are sufficiently exponentially suppressed. The advantage of using the IVR’s QED weighting function is its exponential, rather than power-law finite volume errors, which rely on the inclusion of only the lowest energy excited hadronic state. Our use of this approximation is valid to $$\mathcal{O}(\alpha_{\text{QED}})$$ as any higher energy excited hadronic state includes more than two insertions of the vector current operator $$J_\mu$$, each of which contains a factor of $$e$$, see [FJ19].

III.I Gauge-Specific Expressions

We also present the relevant expressions for the photon propagator $$S_{\gamma,\mu,\nu}^\gamma(x)$$ and QED weighting function $$L_{\mu,\nu}(t_s, \vec{x})$$ in Feynman Gauge:
<table>
<thead>
<tr>
<th>Name</th>
<th>Lattice Volume</th>
<th>$a^{-1}$ (GeV)</th>
<th>L (fm)</th>
<th>$M_{\pi^0}$ (MeV)</th>
<th>Feynman $t_s$ (a)</th>
</tr>
</thead>
<tbody>
<tr>
<td>48I</td>
<td>$48^4 \times 96$</td>
<td>1.730(4)</td>
<td>16.4</td>
<td>135</td>
<td>24</td>
</tr>
<tr>
<td>64I</td>
<td>$64^4 \times 128$</td>
<td>2.359(7)</td>
<td>29.8</td>
<td>135</td>
<td>32</td>
</tr>
<tr>
<td>32D</td>
<td>$32^4 \times 64$</td>
<td>1.0158(40)</td>
<td>6.4</td>
<td>142</td>
<td>16</td>
</tr>
<tr>
<td>32Dfine</td>
<td>$32^4 \times 64$</td>
<td>1.378(7)</td>
<td>8.7</td>
<td>144</td>
<td>16</td>
</tr>
<tr>
<td>24D</td>
<td>$24^4 \times 64$</td>
<td>1.0158(40)</td>
<td>4.8</td>
<td>142</td>
<td>12</td>
</tr>
<tr>
<td>24DH</td>
<td>$24^4 \times 64$</td>
<td>1.0158(40)</td>
<td>4.8</td>
<td>341</td>
<td>12</td>
</tr>
</tbody>
</table>

Table 1: Ensemble Information

### III.I.1 Feynman Gauge

\[
S_{\mu,\nu}^{\gamma}(x) = \frac{\delta_{\mu,\nu}}{4\pi^2 x^2} = \frac{\delta_{\mu,\nu}}{x^2} \delta(x) \int d^4 p \, \frac{e^{ipx}}{p^2}
\]  

\[
L_{\mu,\nu}(t_s, \vec{x}) = \frac{\delta_{\mu,\nu}}{2\pi^2} \int_0^{\infty} dp \, \frac{\sin(p|\vec{x}|)}{2(p + E_p - M)|\vec{x}|} e^{-pt_s}  
\]

### IV Numerical Results

#### IV.I Main Results

Calculation of the hadronic function $H_{\mu\nu}^L(x)$ was performed on six lattices whose names and attributes are shown in Table 1. The functions $L_{\mu,\nu}(t_s, \vec{x})$ and $S_{\mu,\nu}^{\gamma}(x)$ and the integrals in Equations 5,6 were computed using the C++ language, and all plots were generated using Gnuplot. Figure 1 shows the mass shift $\Delta m_{\pi} \equiv m_{\pi^+} - m_{\pi^0}$ as a function of $t_s$ in Feynman gauge, as well as the contributions of each of the two relevant diagrams, with the left diagram labeled $H_1$ and the disconnected diagram labeled $H_2$.

Figure 2 shows the linear fit in each gauge of $\Delta m_{\pi}$ vs. $a^2$. The Feynman gauge fit yields a continuum mass (corresponding to $a = 0$) of $\Delta_F m_{\pi} = 4.56(2)$ MeV.
Figure 1: $\Delta m_\pi$ vs. $t_s$ as calculated using the 48I (top), 64I, 32D, 32Dfine, 24D and 24DH (bottom) lattices, including the contributions of both $H_1$ and $H_2$ (left), $H_1$ (middle), and $H_2$ (right). In Feynman gauge both the short-distance contribution and the total contribution are shown.
Table 2: Coulomb Gauge Time Component Contributions of to the pion Mass Splitting

<table>
<thead>
<tr>
<th>Name</th>
<th>Coulomb Time Component (MeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>48I</td>
<td>1.7426(120)</td>
</tr>
<tr>
<td>64I</td>
<td>1.7859(59)</td>
</tr>
<tr>
<td>32D</td>
<td>1.6078(107)</td>
</tr>
<tr>
<td>32Dfine</td>
<td>1.6767(56)</td>
</tr>
<tr>
<td>24D</td>
<td>1.6137(54)</td>
</tr>
<tr>
<td>24DH</td>
<td>1.3253(73)</td>
</tr>
</tbody>
</table>

One may also be interested in the individual contributions of the time component of to the Coulomb gauge mass shift $\Delta m_\pi$ as the time component of the Coulomb gauge photon propagator may be intuitively interpreted as the Coulomb potential of a point charge. This information is presented in Table 2.

The Feynman gauge time component integrand, plotted as a function of the spatial separation at $t = 0$, is presented in Figure 3.

V Conclusion

We have presented two continuum limit charged/neutral pion QED mass corrections to order $\mathcal{O}(\alpha_{\text{QED}})$. In Feynman gauge the Iwasaki-DSDR ensembles yielded an initial $\Delta_{Fm_{\pi}} = 4.52(02)(05)$ MeV while the Iwasaki ensembles yielded $\Delta_{Fm_{\pi}} = 4.85(01)(01)$ MeV. To correct the continuum extrapolations for finite volume effects, we have completed a scalar QED simulation and adjusted by the appropriate corrective factor. For clarity, we have included in each first set of parentheses the corresponding adjusted statistical error, and in the second set of parentheses the systematic error estimate using scalar QED. The Iwasaki DSDR ensemble estimate value agrees to 1.44% with current experimental observation, while the Iwasaki ensemble estimate agrees to 5.52%. We have also presented the IRV Method’s short distance contribution to the mass difference for each lattice in both gauges, and we have the contribution of the
Figure 2: Feynman gauge mass shifts are shown as a function of $a^2$, where each $\Delta m_\pi$ value is taken at the corresponding optimal value of $t_s$. For each plot, the value of the mass shift at $t_s = \frac{L}{2}$ is used. The plot for the Iwasaki ensembles is shown on the top, and that for the Iwasaki-DSDR ensembles is shown on the bottom.
Figure 3: The $|\vec{x}|$-dependence of the integrand of Equation 5, 
$$\frac{1}{2} \int_0^{\vec{x}} d^3x' H^L_{\mu,\nu}(x') S^L_{\mu,\nu}(x')$$ at $t = 0$ in Coulomb gauge is shown for the lattices 48I (top left), 64I (top middle), 32D (top right), 32Dfine, (bottom left), 24D (bottom middle), 24DH (bottom right).
time component contribution in coulomb gauge as its non-hadronic part may be interpreted as derived directly from the coulomb potential of a point charge. [Len+98]

VI Acknowledgements

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References

URL: https://link.aps.org/doi/10.1103/PhysRevLett.76.3894

URL: http://www.sciencedirect.com/science/article/pii/S0370269397013373


[RW18b] Andreas Risch and Hartmut Wittig. “Towards leading isospin breaking effects in mesonic masses with open boundaries”. In: *PoS* LAT-