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The Geometry of Spacetime and its Singular Nature

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Abstract

One hundred years ago, Albert Einstein revolutionized our understanding of gravity, and thus the large-scale structure of spacetime, by implementing differential geometry as the primary medium of its description, thereby condensing the relationship between mass, energy and curvature of spacetime manifolds with the Einstein field equations (EFE), the primary component of his theory of General Relativity. In this paper, we use the language of Semi-Riemannian Geometry to examine the Schwarzschild and the Friedmann-Lemaître-Robertson-Walker metrics, which represent some of the most well-known solutions to the EFE. Our investigation of these metrics will lead us to the problem of singularities arising within them, which have mathematical meaning, but whose physical meaning at first seems dubious, due to the highly symmetric nature of the metrics. We then use techniques of causal structure on Lorentz manifolds to see how theorems due to Roger Penrose and Stephen Hawking justify that physical singularities do, in fact, occur where we guessed they would.

1 Geometry and some history

In the nineteenth century, a few mathematicians, primarily Bolyai, Lobachevskii and Gauss, began to construct settings for geometry that did not include Euclid's parallel axiom as a restriction. Towards the end of the same century, Tullio Levi-Civita and Gregorio Ricci-Curbastro laid the foundations for tensor calculus, which serves as a generalization of vector calculus to arbitrarily many indices and which allows for the formulation of physical equations without the burden of coordinates. The mingling of non-Euclidean geometries and tensor calculus ultimately gave rise to the formulation of Riemannian geometry, one of many "differential" geometries. Initially, many thought that the ideas of differential geometry were to be pondered only by mathematicians, due to the apparent Euclidean-ness of the real world. However, early in the twentieth century and with the help of Hermann Minkowski, Albert Einstein noticed that to generalize his Special Theory of Relativity, which he did not initially construct with a geometric framework, he had to use a system of geometry which permitted the curvature of spacetime wherever gravity was present, resulting in the introduction of Semi-Riemannian geometry to the vocabulary of theoretical physicists. Einstein began to think of our universe as a four-dimensional manifold, which has three dimensions of space and one dimension of time, where the temporal dimension has the opposite sign of the spatial dimensions. Thus, instead of the standard line element of Euclidean four-space $\mathbb{R}^4$, we have the line element of $\mathbb{R}^4_1$,

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2. \tag{1}$$

$\mathbb{R}^4_1$ is called "Minkowski spacetime," which is a flat Lorentz manifold. Lorentz manifolds are Semi-Riemannian manifolds with index one, i.e. where one dimension has opposite sign from the rest, and is therefore the type of manifold we are interested in for studying General Relativity. We denote the metric signature of a four-dimensional Lorentz manifold as $(-+++)$. Another convention, prevalent in physics literature, is the signature $(+---)$, but we have chosen the former. We choose units such that the Newtonian gravitational constant G and the speed of light constant c are both 1.

1.1 Tensors

Tensors are multilinear maps that help us generalize the ideas of vector fields, one-forms, and real-valued functions. Let V be a module over a ring K. The dual module of V, V^*, is the set of all K-linear functions from V to K. With this, we can define a K-multilinear function $A : V^{*r} \times V^s \to K$, called a *tensor of type* (r, s), or "rank" $r + s$, which takes r elements of V^* and s elements of V and maps them to an element of K.

If we let $\mathcal{F}(\mathcal{M})$ be the ring of real-valued functions on our manifold $\mathcal{M}$, we denote the module of vector fields over $\mathcal{F}(\mathcal{M})$ as $\mathcal{V}(\mathcal{M})$ and the module of one-form fields as $\mathcal{V}^*(\mathcal{M})$. Putting this under consideration lets us define a *tensor field*, $A : \mathcal{V}^*(\mathcal{M})^r \times \mathcal{V}(\mathcal{M})^s \to \mathcal{F}(\mathcal{M})$, which maps r one-forms and s vector fields to a real-valued function. Linearity of additivity and scaling is crucial for a multilinear map to be a tensor. For example, if T is a type $(1, 1)$ tensor, then $T(\theta, fX + gY) = fT(\theta, X) + gT(\theta, Y)$, where $\theta \in \mathcal{V}^*(\mathcal{M})$, $X, Y \in \mathcal{V}(\mathcal{M})$ and $f, g \in \mathcal{F}(\mathcal{M})$. It is also important to note that if $\theta \in \mathcal{V}^*(\mathcal{M})$ and $X \in \mathcal{V}(\mathcal{M})$, then there exists a unique $f \in \mathcal{F}(\mathcal{M})$ such that

$$\theta(X) = X(\theta) = f. \tag{2}$$

From here, we can define a coordinate system on a set $\mathcal{U} \subseteq \mathcal{M}$, let's call it $\xi = (x^1, x^2, \ldots, x^n)$, which denotes the set of its coordinate functions. We let the differential operators of this coordinate system be $\partial_1, \partial_2, \ldots, \partial_n$. All of these ∂_j and their linear combinations, namely all of the "tangent vectors" to a point on the manifold, are manifestations of directional derivatives. This leads us to define the exterior derivative for a function $f \in \mathcal{F}(\mathcal{M})$ as $df(v) = v(f)$, where $v \in \mathcal{V}(\mathcal{M})$ and $df \in \mathcal{V}^*(\mathcal{M})$. This results in the significant relationship:

$$dx^i(\partial_j) = \partial_j(dx^i) = \delta_{ij}. \tag{3}$$

Here, the ∂_j serve as a basis for the tangent space of $\mathcal{M}$ at the point p, $T_p(\mathcal{M})$, and the dx^i comprise the basis for $T^*_p(\mathcal{M})$, the dual of the tangent space. $T_p(\mathcal{M})$ is a vector space, and all properties of tensors as machines mapping elements from modules and dual modules to rings correspond to tensors mapping elements from vector spaces and their duals to fields. In particular, our tensors will be maps from $\mathcal{V}(\mathcal{M})$ and $\mathcal{V}^*(\mathcal{M})$ or $T_p(\mathcal{M})$ and $T^*_p(\mathcal{M})$ to $\mathcal{F}(\mathcal{M})$ or $\mathbb{R}$.

1.2 The Metric

Now that we've gotten comfortable with the idea of a tensor, we introduce the most significant one in geometry: *the metric tensor.*

Definition 1.1. *A metric tensor g on a smooth manifold $\mathcal{M}$ is a symmetric nondegenerate (0,2) tensor field on $\mathcal{M}$ of constant index.*

The metric, as it is called for short, is a generalization of the inner product of Euclidean space to the tangent space of a curved space which provides an infinitesimal measurement of distance. We will use two notations for the metric–either brackets or parentheses, to emphasize the metric g. So $g(u,v) = \langle u, v\rangle \in \mathbb{R}$ and $g(V,W) = \langle V, W\rangle \in \mathcal{F}(\mathcal{M})$, where $u, v \in T_p(\mathcal{M})$ and $V, W \in \mathcal{V}(\mathcal{M})$.

If $(x^1, \ldots, x^n)$ is a coordinate system on $\mathcal{U} \subseteq \mathcal{M}$, then the components of the metric tensor on $\mathcal{U}$ are $g_{ij} = \langle \partial_i, \partial_j\rangle, (1 \le i, j \le n)$. And so, the metric tensor can be written as

$$g = \sum_{i,j} g_{ij} dx^i \otimes dx^j. \tag{4}$$

So now have a succinct way of writing down the metric of Minkowski spacetime $\mathbb{R}^4_1$:

$$\eta = \sum_i \sigma_i dx^i \otimes dx^i, \tag{5}$$

where $\sigma_i = -1$ if $dx^i = dt$ and is equal to 1 otherwise. This is where we see the difference between a Riemannian metric, being strictly positive definite, and a Semi-Riemannian metric, being only nondegenerate. The introduction of this metric will now let us investigate the *causal character* of tangent vectors to $\mathcal{M}$, a central concept in relativity theory.

A tangent vector v to $\mathcal{M}$ is *timelike* if $\langle v, v\rangle < 0$, *spacelike* if $\langle v, v\rangle > 0$ or $v = 0$, and *null* if $\langle v, v\rangle = 0$ and $v \neq 0$. In our case–for Lorentz manifolds–another term for *null* is *lightlike*. This derives from the fact that the null vectors of relativity are those that travel at the speed of light; they cover an "equal quantity" of space and time as they traverse the manifold. As far as we know, light vectors (photons) are the only particles in our universe which behave in this curious way. The set of all null vectors at a point $p \in \mathcal{M}$ is called the *null cone* at that point. The particles we observe around us, including ourselves, are timelike. There has been speculation about the existence of spacelike vectors, but a more important idea for our purposes is that of a spacelike hypersurface, which has only theoretical significance, and will be discussed much later in the paper.

Before we start discussing how to understand the curvature of a manifold, it is necessary to introduce a generalization of the directional derivative to vector fields. If V and W are vector fields on a manifold $\mathcal{M}$, then we should define a new vector field $\nabla_V W$ that represents the vector rate of change of W in the V_p direction.

Definition 1.2. *An affine connection, ∇, on a Semi-Riemannian manifold $\mathcal{M}$ is a function $\nabla : \mathcal{V}(\mathcal{M}) \times \mathcal{V}(\mathcal{M}) \to \mathcal{V}(\mathcal{M})$ such that $\nabla_V W$*

(i) is $\mathcal{F}(\mathcal{M})$-linear in V,

(ii) is $\mathbb{R}$-linear in W, and

(iii) $\nabla_V(fW) = (Vf)W + f\nabla_V W$ for $f \in \mathcal{F}(\mathcal{M})$.

Here, $\nabla_V W$ is called the *covariant derivative* of W with respect to V.

The connection ∇ that satisfies $[V,W] = \nabla_V W - \nabla_W V$ and $X\langle V, W\rangle = \langle \nabla_X V, W\rangle + \langle V, \nabla_X W\rangle$ is the unique *torsion-free* connection on $\mathcal{M}$ with metric g. ∇ is called the *Levi-Civita connection.* We can define the Levi-Civita connection in terms of coordinates:

$$\nabla_{\partial_i}(\partial_j) = \Gamma^k_{ij}\partial_k. \tag{6}$$

Here, we adopt the Einstein summation convention, which simply entails dropping the summation sign in front of repeated indices. The *Christoffel symbols* are the components of the Levi-Civita connection, derived from properties of this connection being torsion-free:

$$\Gamma^k_{ij} = \frac{1}{2} g^{km} \Big(\frac{\partial g_{jm}}{\partial x^i} + \frac{\partial g_{im}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^m} \Big). \tag{7}$$

Because $[\partial_i, \partial_j] = 0$, it follows that $\nabla_{\partial_i}(\partial_j) = \nabla_{\partial_j}(\partial_i)$, and thus $\Gamma^k_{ij} = \Gamma^k_{ji}$. A vector field V is *parallel* if its covariant derivatives $\nabla_X V$ are zero for all $X \in \mathcal{V}(\mathcal{M})$. Hence we see that the Christoffel symbols measure how much coordinate vector fields deviate from being parallel.

For a curve $\gamma : I \to \mathcal{M}$, let $a \in I$ and $z \in T_{\gamma(a)}(\mathcal{M})$; then there is a unique parallel vector field Z on γ such that $Z(a) = z$. $Z(t)$ is called the *parallel transport* of z along γ. This leads us to consider curves that are parallel with respect to their own vector rate of change.

If γ is a curve parameterized as $\gamma = (x^1(t), \dots, x^k(t))$, then it is called a *geodesic* if its vector rate of change γ' is parallel. These curves satisfy the geodesic equations:

$$\frac{d^2 x^k}{dt^2} + \Gamma^k_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} = 0. \tag{8}$$

For any points $p, q \in \mathcal{M}$, the shortest distance between the two is a geodesic. In Euclidean space, geodesics are straight lines. Einstein's great insight, *the equivalence principle*, was that free-falling objects are following geodesics in the four-dimensional spacetime. If the manifold is curved in any way, then it would seem from an outside frame of reference that the free-falling particles are traversing a curved path. This explains the motion of the planets, stars, and galaxies.

1.3 Curvature

Gauss established the notion of *intrinsic curvature* of a two dimensional surface. He proved that this "Gaussian curvature" is an isometric invariant of the surface. In other words, it's independent of the fact that the surface is embedded in flat space, $\mathbb{R}^3$. The idea of intrinsic curvature starts with noting that in flat space, covariant derivatives commute, but on a curved manifold, the non-commutativity becomes apparent. So investigating the non-commutativity of covariant derivatives provides a good way of understanding curvature. Thus, we see that surfaces like cylinders or cones are only *extrinsincally curved*, not intrinsically, since they are essentially identical to the plane, up to isometry. Surfaces like the sphere, however, are intrinsically curved. Intuitively, we can not wrap a plane around a sphere without "crumpling" it, whereas we can do so with a cylinder.

Definition 1.3. *If $\mathcal{M}$ is a Semi-Riemannian manifold with Levi-Civita connection ∇, then let the function $R : \mathcal{V}(\mathcal{M}) \times \mathcal{V}(\mathcal{M}) \times \mathcal{V}(\mathcal{M}) \to \mathcal{V}(\mathcal{M})$ be given by*

$$R_{XY} Z = \nabla_{[X,Y]} Z - [\nabla_X, \nabla_Y] Z. \tag{9}$$

R is a $(1,3)$ tensor field called the Riemann curvature tensor on $\mathcal{M}$.

For a coordinate neighborhood, we have

$$R_{\partial_k \partial_l}(\partial_j) = R^i_{jkl} \partial_i, \tag{10}$$

where the Einstein summation is again implied, and the components of R are given by

$$R^i_{jkl} = \frac{\partial}{\partial x^l} \Gamma^i_{kj} - \frac{\partial}{\partial x^k} \Gamma^i_{lj} + \Gamma^i_{lm} \Gamma^m_{kj} - \Gamma^i_{km} \Gamma^m_{lj}. \tag{11}$$

Before we examine some curvature tensor contractions, we must first discuss bases for tangent spaces. For a Semi-Riemannian manifold $\mathcal{M}$ of dimension n, a set $E_1, \ldots, E_n$ of mutually orthogonal unit vector fields is called a *frame field*, assigning an orthonormal basis $e_1, \ldots, e_n$ of the tangent space at each point in $\mathcal{M}$. Thus, every vector field $V \in \mathcal{V}(\mathcal{M})$ can be written as $V = \sum \sigma_i \langle V, E_i \rangle E_i$, where $\sigma_i = \langle E_i, E_i \rangle$.

Now we can define the *Ricci curvature tensor*, a symmetric $(0, 2)$ tensor, using a frame field, as

$$\mathrm{Ric}(X, Y) = \sum_i \sigma_i \langle R_{XE_i} Y, E_i \rangle. \tag{12}$$

Using coordinates, we have

$$\mathrm{Ric}_{ik} = R_{ik} = R^j_{ijk} = \Gamma^n_{ik,n} - \Gamma^n_{in,k} + \Gamma^n_{nm}\Gamma^m_{ki} - \Gamma^n_{km}\Gamma^m_{ni}. \tag{13}$$

Here, we have used the notation "$, n$" in the subscript of the Christoffel symbols to denote a partial derivative. A covariant derivative with respect to the same variable would be written as "$; n$".

It is important to note that a flat manifold is certainly Ricci flat, but being Ricci flat does not imply that the manifold is flat in terms of the Riemann tensor. Further contraction leads us to the *scalar curvature*, the trace of the Ricci tensor, which is

$$S = g^{ik} R_{ik}. \tag{14}$$

1.4 The Einstein Field Equations

The search for a relativistic theory of gravity was partially motivated by generalizing the Poisson equation for the Newtonian gravitational potential, $\Delta\Phi = 4\pi\rho$, to include relativistic effects. Here, ρ is the mass density and Φ is the potential. The generalization of the mass density would be the *stress-energy-momentum tensor* T, a symmetric $(0, 2)$ tensor which describes the energy density, mass distribution, and pressure of a physical system. It is reasonable to conjecture that the metric g should serve as the generalization of the potential, Φ. So Einstein originally suggested the equation $\mathrm{Ric} = kT$, where k is a constant, for his relativistic theory. However, certain symmetries were required, and we have $\mathrm{div}T = 0$, while $\mathrm{div}\mathrm{Ric} = \frac{1}{2}dS$, where dS is the covariant differential of the scalar curvature.

With this in mind, we can define the *Einstein gravitational tensor*:

$$G := \mathrm{Ric} - \frac{1}{2}Sg, \tag{15}$$

a symmetric $(0, 2)$ tensor defined so that $\mathrm{div}G = 0$. Thus, we can finally introduce the governing equation of General Relativity, describing the relationship between mass, energy-momentum, and the curvature of spacetime:

$$G = 8\pi T. \tag{16}$$

The EFE are the starting point of Einstein's relativistic theory of gravitation. Various solutions to them have explained formerly mysterious phenomena, like the perihelion advance of Mercury's orbit. Some features of the solutions imply interesting and sometimes unbelievable characteristics of our universe, which have thus far provided a seemingly endless source of research and investigation for mathematicians and physicists alike.

2 Schwarzschild and Friedmann-Lemaître-Robertson-Walker Spacetimes

In this section, we will look at solutions of spacetimes describing single-star systems and models for the entire cosmos. Our analysis of these solutions will lead us to issues concerning singularities apparent in these spacetimes.

2.1 Static and Spherically Symmetric Spacetimes

We start with the spacetime around an individual star, and we want it to have the correct Newtonian limit when relativistic effects become negligible. Because the well known Newtonian gravitational field equation is time-independent, we are looking for a static metric. That is, for our spacetime $\tilde{\mathcal{M}}^4$, we want a metric $\tilde{g}$ that admits a timelike Killing vector field K which is orthogonal to a family of spacelike hypersurfaces. For any given time t_0, where $t = x^0$ is the parameter of the integral curve of $K = \partial_t$, we can choose a maximal spacelike hypersurface $\mathcal{M}^3$ that is everywhere orthogonal to ∂_t. If we let (x^1, x^2, x^3) be any coordinate system on $\mathcal{M}$, then the integral curves of ∂_t carry these coordinates to other points of $\tilde{\mathcal{M}}$. This means that we can express the metric $\tilde{g}$ of $\tilde{\mathcal{M}} = \mathbb{R} \times \mathcal{M}$ as the warped product

$$\tilde{g} = -V^2 dt^2 + g, \tag{17}$$

where g is a Riemannian metric induced by $\tilde{g}$ on $\mathcal{M}$, and $V = V(x^1, x^2, x^3)$ is called the *static potential*. Because translations along integral curves of ∂_t are isometries, V is time-independent. Hence, we have a basic layout for what a Newtonian-compatible metric looks like.

Now, if we fix a Lorentz frame $(V^{-1}\partial_t, e^1, e^2, e^3)$, we get the Ricci curvature components $\tilde{R}_{ik}$ of $\tilde{\mathcal{M}}$ in terms of the Ricci curvature components of $\mathcal{M}$ and V as

$$\tilde{R}_{00} = \frac{\Delta V}{V} \tag{18}$$

$$\tilde{R}_{0i} = 0 \tag{19}$$

$$\tilde{R}_{ik} = R_{ik} - \frac{\nabla_i \nabla_k V}{V}. \tag{20}$$

These expressions are derived in Appendix 5.2.

We can rewrite the EFE in terms of the stress-energy tensor T_{ab} and its trace T, with respect to $\tilde{g}$, as

$$\tilde{R}_{ab} = 8\pi(T_{ab} - \frac{1}{2}T\tilde{g}_{ab}). \tag{21}$$

We can also assume that T_{ab} is static, and thus can express it as

$$\begin{pmatrix} \rho & 0 \\ 0 & \tau_{ik} \end{pmatrix}, \tag{22}$$

where ρ is the mass density and τ_{ik} is the stress tensor. We let the trace of τ_{ik} with respect to g and be $\tau = \mathrm{Tr}_g \tau_{ik}$. With this in mind, we see that the overall trace of the stress-energy tensor T_{ab} with respect to $\tilde{g}$ and the fixed Lorentz frame is $T = -\rho + \tau$.

With this in mind, we refer to equation (21) and see that we have

$$\tilde{R}_{00} = \frac{\Delta V}{V} = 8\pi(T_{00} - \frac{1}{2}T\tilde{g}_{00}) \tag{23}$$

$$= 8\pi(\rho + \frac{1}{2}(-\rho + \tau)) = 4\pi(\rho + \tau), \tag{24}$$

which immediately implies that

$$\Delta V = 4\pi V(\rho + \tau). \tag{25}$$

In addition, if $i, k \neq 0$,

$$\tilde{R}_{ik} = R_{ik} - \frac{\nabla_i \nabla_k V}{V} = 8\pi(\tau_{ik} + \frac{1}{2}(\rho - \tau)g_{ik}), \tag{26}$$

which, rearranging, gives us

$$VR_{ik} - \nabla_i \nabla_k V = 4\pi V(2\tau_{ik} + (\rho - \tau)g_{ik}). \tag{27}$$

In a static vacuum, where $\rho = \tau_{ik} = 0$, equations (25) and (27) reduce to

$$\Delta V = 0 \tag{28}$$

$$VR_{ik} = \nabla_i \nabla_k V. \tag{29}$$

Taking the trace of equation (29), we get

$$VS = \Delta V \implies VS = 0, \tag{30}$$

by implication from equation (28), where S is the scalar curvature. Hence, $S = 0$.

We say that a spacetime is *spherically symmetric* if it admits $SO(3)$ as a group of isometries, with the group orbits spacelike spheres, which must necessarily have constant curvature. Requiring spherical symmetry along with the static condition imposes great restrictions on the spacetime. However, it lets us deduce that this spacetime is unique up to a local isometry.

2.2 Birkhoff's Theorem

Theorem 2.1. *All spherically symmetric solutions to the static vacuum Einstein field equations are locally isometric to the Schwarzschild solution, being*

$$ds^2 = -\Big(1 - \frac{2m}{r}\Big)dt^2 + \Big(1 - \frac{2m}{r}\Big)^{-1}dr^2 + r^2 d\xi^2, \tag{31}$$

for some $m \in \mathbb{R}$, where $d\xi^2$ is the standard line element for $\mathbb{S}^2$.

This result was one of the first major implications of Einstein's theory, and bears the most significance in terms of modeling solar systems. The proof is not difficult and bolsters a good understanding of the result, though some of the calculations can be tedious. Thus, we work it out in careful steps and with a little more detail than the source material [3].

Proof. We will begin by inspecting the metric on $\mathcal{M} \subseteq \tilde{\mathcal{M}}$. Because $\mathcal{M}$ is spherically symmetric, g must be conformally flat, and thus we choose a conformal factor so that $g = u^4\delta$. From this, we also have $g^{-1} = u^{-4}\delta$. We want to calculate the components of the Ricci tensor using equation (13) and use standard Euclidean coordinates from δ to compute this. R_{22} will be calculated explicitly, and then we will make us of symmetries to deduce R_{11} and R_{33}, which together comprise the nontrivial Ricci terms. Then

$$R_{22} = \sum_n \partial_n \Gamma^n_{22} - \sum_n \partial_2 \Gamma^n_{2n} + \sum_{mn} \Gamma^n_{nm}\Gamma^m_{22} - \sum_{mn} \Gamma^n_{2m}\Gamma^m_{n2}. \tag{32}$$

Using the Christoffel symbols derived in Appendix 5.3, we have:

$$\sum_n \partial_n \Gamma^n_{22} = \frac{2}{u^2}(u_1^2 - u_2^2 + u_3^2) + \frac{2}{u}(-u_{11} + u_{22} + u_{33}), \tag{33}$$

$$\sum_n \partial_2 \Gamma^n_{2n} = \frac{-6}{u^2}u_2^2 + \frac{6}{u}u_{22}, \tag{34}$$

$$\sum_{mn} \Gamma^n_{nm}\Gamma^m_{22} = \frac{-12}{u^2}u_1^2 + \frac{12}{u^2}u_2^2 - \frac{12}{u^2}u_3^2, \tag{35}$$

$$\sum_{mn} \Gamma^m_{2n}\Gamma^n_{2m} = \frac{-8}{u^2}u_1^2 + \frac{12}{u^2}u_2^2 - \frac{8}{u^2}u_3^2. \tag{36}$$

$$\tag{37}$$

And so

$$R_{22} = \frac{2}{u^2}\Big(- u_1^2 + 2u_2^2 - u_3^2\Big) - \frac{2}{u}\Big(u_{11} + 2u_{22} + u_{33}\Big). \tag{38}$$

By entirely symmetric calculations, we get

$$R_{11} = \frac{2}{u^2}\Big(2u_1^2 - u_2^2 - u_3^2\Big) - \frac{2}{u}\Big(2u_{11} + u_{22} + u_{33}\Big), \tag{39}$$

$$R_{33} = \frac{2}{u^2}\Big(- u_1^2 - u_2^2 + 2u_3^2\Big) - \frac{2}{u}\Big(u_{11} + u_{22} + 2u_{33}\Big). \tag{40}$$

To calculate the scalar curvature S, we must contract the sum of the Ricci terms with g^{-1}:

$$S = \sum_k g^{kk} R_{kk} = u^{-4}\Big(\sum_k R_{kk}\Big) \tag{41}$$

$$= \frac{2}{u^6}\big(2u_1^2 - 2u_1^2 + 2u_2^2 - 2u_2^2 + 2u_3^2 - 2u_3^2\big) - \frac{2}{u^5}\big(4u_{11} + 4u_{22} + 4u_{33}\big), \tag{42}$$

which all reduces nicely to

$$S = \frac{-8\Delta u}{u^5}. \tag{43}$$

Equation (30) gives us that the hypotheses of the theorem imply that $S = 0$, and thus equation (43) reduces to

$$\Delta u = 0. \tag{44}$$

Because we assumed spherical symmetry, u depends only on the radius ρ, and so the spherical Laplacian of (44) reduces to

$$\frac{1}{\rho^2}\frac{\partial}{\partial\rho}\Big(\rho^2\frac{\partial}{\partial\rho}u(\rho)\Big) = 0. \tag{45}$$

$$\implies \frac{\partial}{\partial\rho}u(\rho) = \frac{c_1}{\rho^2}. \tag{46}$$

Hence, we have that

$$u(\rho) = c_2 - \frac{c_1}{\rho} \tag{47}$$

for some $c_1, c_2 \in \mathbb{R}$.

If either c_2 or c_1 were zero, then the metric would be flat, so they must be nonzero, and by rescaling the metric, we have

$$u(\rho) = 1 + \frac{m}{2\rho}. \tag{48}$$

With this, the metric g on $\mathcal{M}$ is then

$$g = \Big(1 + \frac{m}{2\rho}\Big)^4 (d\rho^2 + \rho^2 d\xi^2). \tag{49}$$

Now, changing coordinates so that $r = \rho\big(1 + \frac{m}{2\rho}\big)^2$, the metric of the spatial part of $\tilde{\mathcal{M}}$ is

$$g = \Big(1 - \frac{2m}{r}\Big)^{-1} dr^2 + r^2 d\xi^2. \tag{50}$$

Setting $h(r) = 1 - \frac{2m}{r}$, the metric on all of $\tilde{\mathcal{M}}$ is

$$\tilde{g} = -V^2 dt^2 + h^{-1} dr^2 + r^2 d\xi^2, \tag{51}$$

considering equation (17).

To solve for V, we must use the spatial part of the metric and the Laplace-Beltrami operator, being

$$\Delta_g V = \frac{1}{\sqrt{|g|}}\partial_i\Big(\sqrt{|g|}g^{ij}\partial_j V\Big), \tag{52}$$

where $|g|$ is the absolute value of the determinant of g. In particular, $\sqrt{|g|} = h^{-1/2}r^2\sin\theta$, and so

$$\Delta_g V = \frac{h^{1/2}}{r^2\sin\theta}\partial_r\Big(h^{-1/2}r^2\sin\theta g^{rr}V'\Big) \tag{53}$$

$$= \frac{h^{1/2}}{r^2}\Big(h^{1/2}r^2V'\Big)'. \tag{54}$$

A trivial solution here would be $V = const$. This isn't very interesting, so we must consider a function V that satisfies

$$h^{1/2}r^2V' = a \tag{55}$$

for some non-zero constant a. The result of the theorem suggests we check if

$$V = \Big(1 - \frac{2m}{r}\Big)^{1/2} \tag{56}$$

satisfies (55), and indeed

$$h^{1/2}r^2V' = \Big(1 - \frac{2m}{r}\Big)^{1/2} r^2 \frac{m}{r^2}\Big(1 - \frac{2m}{r}\Big)^{-1/2} = m, \tag{57}$$

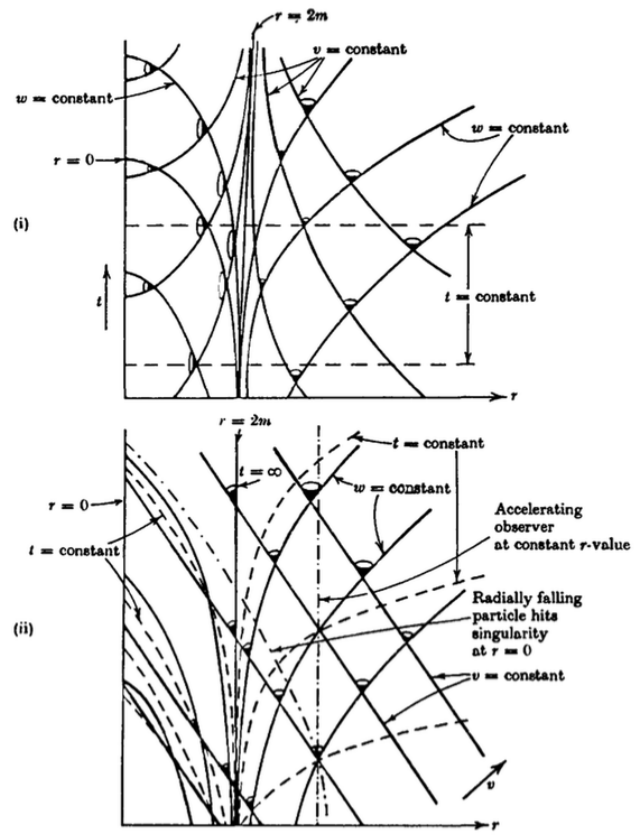
where m is simply the mass, a constant in $\mathbb{R}$.

Hence, combining the above with equation (51) implies that any spacetime adhering to the hypotheses of the theorem will have to be locally isometric to a spacetime with line element

$$ds^2 = -\Big(1 - \frac{2m}{r}\Big)dt^2 + \Big(1 - \frac{2m}{r}\Big)^{-1} dr^2 + r^2 d\xi^2, \tag{58}$$

which is the desired result. $\square$

Figure 1: Removing the $r = 2m$ singularity with a coordinate change



2.3 Singularities in Schwarzschild Spacetime

Based on a first impression of the Schwarzschild metric, one could guess that issues occur at $r = 0$ and $r = 2m$. However, a coordinate change can be cooked up that removes the singularity at $r = 2m$ [2]. The coordinates in this form comprise the Eddington-Finkelstein metric, which is as follows:

$$ds^2 = -\Big(1 - \frac{2m}{r}\Big)dv^2 + 2dvdr + r^2 d\xi^2. \tag{59}$$

On the other hand, the metric singularity at $r = 0$ is not removable by such a simple change. This suggests that there is a true physical singularity occurring there. Figure 1, from [2], illustrates this idea, where the null geodesics are shown; (i) is the Schwarzschild $t - r$ plane and (ii) is the Finkelstein $t - r$ plane.

Note that in Schwarzschild coordinates, the light cones at $r > 2m$ remain upright and point directly inwards at $r < 2m$, whereas in Finkelstein coordinates the light cones gradually begin to tilt inwards as we approach the non-removable $r = 0$ singularity. Indeed, as $r \to 0$, the curvature scalar $R^{ijkl}R_{ijkl}$ diverges like m^2/r^6 [2], further justifying that what is happening at $r = 0$ is a real singularity. This point then represents an area of causal geodesic incompleteness, and we see that although ingoing null geodesics pass through $r = 2m$ in the new coordinate system, the outgoing ones are trapped in both. This motivates the idea of a *trapped surface*, which will be discussed later in the paper. Physically, this means that once an observer passes through the event horizon of a black hole, there is no return.

2.4 Models for Cosmological Spacetimes

For dust you are, and to dust you shall return. – Torah

We view the large scale universe as an isotropic distribution of galaxies. In particular, these galaxies can be viewed as geodesic worldlines, the sum of which can be considered a dust or fluid permeating the entire cosmos, from which we can define the following properties of the associated stress-energy tensor:

Definition 2.1. *A perfect fluid on a spacetime $\mathcal{M}$ is a triple (U, ρ, p) such that:*
(i) U is a timelike future-pointing unit vector field on $\mathcal{M}$ called the flow vector field.
(ii) $\rho \in \mathcal{F}(\mathcal{M})$ is the energy density function and $p \in \mathcal{F}(\mathcal{M})$ is the pressure function.
(iii) The stress-energy tensor is then

$$T = (\rho + p)U^* \otimes U^* + pg, \tag{60}$$

where U^* is the one-form that is metrically equivalent to U.

The integral curves of U are the average worldlines of the galaxies, and for $X, Y \perp U$, we see that T satisfies $T(U,U) = \rho$, $T(X,U) = T(U,X) = 0$, and $T(X,Y) = p\langle X, Y\rangle$.

We now have a convenient formalization of how matter in the cosmos behaves, however, this definition does not necessarily imply a particular construction for the spacetime. In Newtonian physics or even in special relativity, the geometries for the cosmos are fixed; the overlying manifold is simply where the physics takes place. In general relativity, we have no a priori geometry since the physics itself determines it. However, there are some assumptions we can make from observational astronomy that have direct geometric consequences.

Most clearly, it seems to be the case that gravity attracts, on average. This is called the *timelike convergence condition*, and can be expressed as $\mathrm{Ric}(v,v) \geq 0$ for all timelike vectors v tangent to $\mathcal{M}$. In terms of the more physically tangible stress energy tensor, this is represented as $T(v,v) \geq \frac{1}{2}\mathbf{C}(T)\langle v,v\rangle$ for all timelike and null vectors v tangent to $\mathcal{M}$, where $\mathbf{C}(T)$ is the metric contraction of the stress-energy tensor T. $T(v,v)$ is the energy density, and in this form the condition is called the *strong energy condition.* We can conceive of matter configurations which violate this condition; for example, a cosmological inflationary process would do so. However, the current state of our universe seems to adhere to it, so it is not an altogether dangerous assumption. A violation of this condition would imply that our current theory of general relativity would have to be upgraded in some way.

The general form of these cosmological manifolds will be $\mathcal{M} = I \times S$, where I is a potentially infinite interval in $\mathbb{R}$ and S is a connected three dimensional manifold. For the flow vector field U, physical assumptions imply that $\langle U,U\rangle = -1$. In addition, because the relative motion of galaxies is negligible on a large scale, we have that $U \perp S(t)$, where $S(t) = t \times S$ is a hypersurface serving as the common restspace for all of the galaxies at time t. Each of these slices is thus a Riemannian spacelike hypersurface which must have constant curvature $C(t)$. With this in mind, we can define the general form of our cosmological models as follows:

Definition 2.2. *Let S be a connected three-dimensional Riemannian manifold of constant curvature $k = -1, 0,$ or 1, and let $f > 0$ by a smooth function on an open interval $I \in \mathbb{R}^1_1$. Then the warped product*

$$\mathcal{M}(k,f) = I \times_f S \tag{61}$$

is called a Robertson-Walker (or Friedmann-Lemaître) spacetime.

Based on this construction, we see that S can be fundamentally categorized as $\mathbb{H}^3, \mathbb{R}^3$, or $\mathbb{S}^3$, since it must have constant curvature $k = -1, 0$, or 1. The metric thus takes the general form:

$$ds^2 = -dt^2 + f^2(t)\Big[dr^2 + S_k^2(r)d\xi^2\Big], \tag{62}$$

where the function $S_k(r)$ is either hyperbolic, linear, or trigonometric, depending on k being $-1, 0$, or 1, respectively. Which geometry S adheres to in the physical universe is an active area of research in both observational and theoretical cosmology. The discovery of this geometry would imply incredible facts about the initial and final conditions of our universe.

With this characterization of Robertson Walker spacetimes, we'll try to deduce further properties with a few calculations.

Lemma 2.1. *If V, W are in the lift of S to $\mathcal{M}(k,f)$ and taking U to be the flow vector field, in the lift of I, we have that* $\mathrm{nor}\nabla_V W = II(V,W) = \langle V,W\rangle(f'/f)U$.

Proof.

$$\langle \nabla_V W, U\rangle = -\langle W, \nabla_V U\rangle = -\langle W, (\frac{Uf}{f})V\rangle, \tag{63}$$

given from the general form $\nabla_V X = (\frac{Xf}{f})V$. Then we have that

$$-\langle W, (\frac{Uf}{f})V\rangle = -(\frac{Uf}{f})\langle W,V\rangle = -\frac{\langle \mathrm{grad} f, U\rangle}{f}\langle V,W\rangle = \frac{f'}{f}\langle V,W\rangle. \tag{64}$$

Now that we have its components, we see that the mean curvature vector field is given as

$$II(V,W) = \frac{f'}{f}\langle V,W\rangle U. \tag{65}$$

□

This will be a very useful result later on when we will be analyzing submanifold convergence in terms of its mean curvature. Now we would like to see results for the Ricci curvature, given that it is the primary object of interest in the Einstein field equations. We will simply list the formulas given in Chapter 12 of [1], as they are a simple application of the warped product formulas in the appendix.

Lemma 2.2. *For $\mathcal{M}(k,f)$ with flow vector field $U = \partial_t$ and $X, Y \perp U$, we have*

$$\mathrm{Ric}(U,U) = -3\frac{f''}{f} \tag{66}$$

and

$$\mathrm{Ric}(X,Y) = \Big(2(\frac{f'}{f})^2 + \frac{2k}{f^2} + \frac{f''}{f}\Big)\langle X,Y\rangle. \tag{67}$$

In addition, $\mathrm{Ric}(U,X) = \mathrm{Ric}(U,Y) = 0$*, and further contraction gives us*

$$S = 6\Big((\frac{f'}{f})^2 + \frac{k}{f^2} + \frac{f''}{f}\Big). \tag{68}$$

Finally, we want to make the above calculations compatible with our definition of the stress-energy tensor. Because the Einstein equation can be rearranged as $T = \frac{1}{8\pi}\big(\mathrm{Ric} - \frac{1}{2}Sg\big)$, then a comparison with the definition of T with Lemma 2.2 quickly gives us

$$\frac{8\pi\rho}{3} = (\frac{f'}{f})^2 + \frac{k}{f^2} \tag{69}$$

$$-8\pi p = \frac{2f''}{f} + (\frac{f'}{f})^2 + \frac{k}{f^2}. \tag{70}$$

A further rearrangement stemming from a simple combination of the two above expressions gives us a relationship between the warping function and the energy density and pressure:

$$\frac{3f''}{f} = -4\pi(\rho + 3p). \tag{71}$$

In addition to these geometric results, astronomical observations have had influence on the deduction of the shape of the cosmos. For example, in 1927, Georges Lemaître theorized that the universe is expanding. Shortly thereafter in 1929, Edwin Hubble realized that the universe is indeed expanding at a rate proportional to the Hubble constant $H_0 = f'(t_0)/f(t_0) = (8 \pm 2) \times 10^{-9}$ years, where t_0 is the current time. A more accurate name for this would be the *Hubble parameter*, because further observation since 1929 has implied that the universe is expanding at an accelerating rate; however, it is sufficient for our purposes to consider it a constant. In particular, this fact implies that the restspaces $S(t)$ are expanding, and so $f' > 0$. Astronomical data has also shown that energy density is significantly higher than pressure, that is, $\rho >> p$. With all of the above, we can justify the existence of cosmological singularities.

2.5 Singular Points in the FLRW Models

To find singular points, we look at a simple result stemming from local linear approximations of the warping function f and see how this implies that f has a beginning some finite time ago.

Proposition 2.1. *Let $\mathcal{M}(k, f) = I \times_f S$. If $H_0 > 0$ for some t_0 and $\rho + p > 0$, then I has an initial endpoint $\underline{t}$ such that $t_0 - H_0^{-1} < \underline{t} < t_0$.*

Proof. Equation (71) shows us that $\rho + p > 0$ implies $f'' < 0$ and $f > 0$ everywhere up to the current time t_0. That is, the graph of f is concave down below its tangent line at t_0. This line is the graph of $F(t) = f(t_0)[1+H_0(t-t_0)]$. In particular, $F(t_0-H_0^{-1}) = f(t_0)(1-1) = 0$, and because f remains concave down, as t approaches $t_0 - H_0^{-1}$, it must be the case that f has a beginning at some point $\underline{t}$ before then. □

The future behavior of f is still under consideration in theoretical cosmology and will ultimately help us decipher if the universe continues to a big crunch or a big rip. However, the above result justifies that the past behavior of f implies the universe began a finite time ago, bounded by $H_0^{-1} = 18$ billion years. Very recent measurements of the cosmic microwave background radiation due to the Planck space observatory have pinned the age of our universe at 13.82 billion years. Thus, we see that the cosmological models due to Friedmann, Lemaître, Robertson and Walker those many decades ago have come to some fruition.

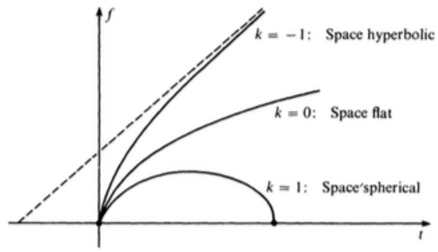
This result, however, doesn't say anything about the *state* of the universe at its beginning. If the energy density ρ approaches infinity as $t \to \underline{t}$, then we say that $\mathcal{M}(k, f)$ has a physical singularity at that time. An initial singularity is a *big bang* provided $f \to 0$ and $f' \to \infty$ as t goes to $\underline{t}$. A final singularity is called a *big crunch* if $f \to 0$ and $f' \to -\infty$ as t approaches a finite endpoint $\bar{t}$. We now present a result that generalizes the preceding proposition into including the size behavior of f, not just the finiteness of its parameter. The proof can be found in [1], but the diagram of Friedmann models neatly summarizes the intuition.

Proposition 2.2. *Assume that $\mathcal{M}(k, f)$ has only physical singularities and that I is maximal. If $H_0 > 0$ for some t_0, $\rho > 0$, and if there exist constants a, A such that $-1/3 < a \leq p/\rho \leq A$, then:*
(i) The initial singularity is a big bang.
(ii) If $k = 0$ or -1, then $I = (\underline{t}, \infty)$ and as $t \to \infty$, $f \to \infty$ and $\rho \to 0$.
(iii) If $k = 1$, then f reaches a maximum followed by a big crunch, hence I is a finite interval $(\underline{t}, \bar{t})$.

Like in the last proposition, we see that the future behavior of f is still not completely predictable. However, this result does strongly imply that the universe began in an incredibly dense, high-energy state some finite time ago. Developing theories of dark matter and dark energy suggest that a hyperbolic space is a much more likely candidate for the large scale structure of the universe than the spherical model. Additionally, further examination of the behavior of dark matter and energy could eventually imply that Einstein's general theory will have to be improved upon.

Finally, it is important to note here that all of our above assumptions about the cosmic structure contain some sort of symmetry that is mathematically feasible, but physically unrealistic. This could allow physicists to dismiss catastrophic phenomena like the big bang as a mathematical curiosity and not a real occurrence, given that the universe itself does not adhere strictly to such a high degree of symmetry. This leads us to our next section, where we observe that singularities still happen, even under weaker geometric conditions.

Figure 2: Friedmann models for f [1]



3 Hawking and Penrose Singularity Theorems

We saw in the last section that it is necessary to introduce the notion of singularities in the spacetimes described by General Relativity. It may at first seem reasonable to define a singularity as a point where the metric tensor is undefined or non-differentiable, however, we can define our spacetimes such that these points are removed, which would render them nonsingular, by this definition. In fact, it would be more appropriate to remove these points from the definition of our spacetime manifold in the first place. Hence, deducing whether a spacetime is singular consists of detecting these removed singular points. In other words, we are trying to see how our manifold is incomplete. This is done by analyzing timelike and null geodesic incompleteness, which arises from a geodesic being inextendible, yet having a finite affine parameter. Physically, this means looking at how a freely falling particle or light ray is discontinued by a "flaw" in the spacetime. To do this, we must first look at causal structure on these manifolds.

3.1 Causal Structure

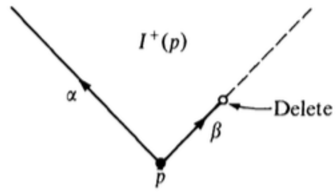
There was a young lady of Wight
Who travelled much faster than light.
She departed one day,
In a relative way,
And arrived on the previous night.

This limerick from Stephen Hawking's PhD thesis [4] offers some insight into the various paradoxes that can occur in a spacetime if we're not careful with certain conditions. The *causality relations* on a manifold are defined as $p << q$ if there exists a future-pointing timelike curve from p to q, $p < q$ if there is a future-pointing causal (timelike or lightlike) curve joining p to q, and $p \leq q$ if either $p < q$ or $p = q$.

With this in mind, for a subset $A \subseteq \mathcal{M}$, we can define $I^+(A) = \{ \ q \in \mathcal{M} : \text{there is a } p \in A \text{ with } p << q \ \}$. This is called the *chronological future* of the subset A. The *chronological past*, $I^-(A)$, is defined analogously, where $q << p$ is used in the definition, in place of $p << q$. For the same subset A, we can define $J^+(A) = \{ \ q \in \mathcal{M} : \text{there is a } p \in A \text{ with } p \leq q \ \}$, the *causal future*, and $J^-(A)$, the *causal past*, analogously. Without loss of generality, we will usually consider the chronological and causal futures of sets, because reversing time orientation is trivial in our cases.

The simplest examples for causality can be imagined in Minkowski space $\mathbb{R}^n_1$. For example, for a single point $p \in \mathbb{R}^n_1$, $I^+(p)$ is simply the future light cone beginning at p, and $J^+(p)$ is the union of the future time cone, future null cone, and p itself. Thus, we see that for $\mathbb{R}^n_1$, $I^+(p)$ is an open set with closure $J^+(p)$. In Figure 3 below, if the point was not deleted, then $I^+(p)$ would be as is and $J^+(p)$ would be $I^+(p)$ along with p and the null geodesic rays α and β.

Figure 3: Null cone with point removed [1]



It is important to note here that for an arbitrary manifold $\mathcal{M}$ and subset $A \subseteq \mathcal{M}$, $I^+(A)$ is open, whereas $J^+(A)$ is not necessarily closed. For example, let $\mathcal{M} = \mathbb{R}^2_1$, with a point on the null cone removed. Here, $I^+(p)$ is the future time cone , but $J^+(p)$ is strictly smaller than the closure of $I^+(p)$, as illustrated in the figure above.

Various causality conditions need to be discussed to acknowledge which spacetimes are physically admissible according to relativity. For example, the *chronology condition* holds if $\mathcal{M}$ contains no closed timelike curves. If we had a manifold that did not adhere to his condition, then it would be possible for an observer to leave for a trip and eventually return to the time at which they began the trip. This is a doubtful occurrence in physics, at best, so we need our manifolds to follow the chronology condition. If a manifold $\mathcal{M}$ is compact, then it contains a closed timelike curve, and so compact spacetimes are not very relevant in relativity. Analogously, a manifold adheres to the *causality condition* if there are no closed causal curves in $\mathcal{M}$.

A yet more reasonable condition to expect in physics is the *strong causality condition*, which holds at $p \in \mathcal{M}$ if given any neighborhood $\mathcal{U}$ of p, there is a neighborhood $\mathcal{V} \subset \mathcal{U}$ of p such that every causal curve segment with endpoints in $\mathcal{V}$ lies entirely in $\mathcal{U}$. In other words, causal curves that start arbitrarily close to p and leave some fixed neighborhood can never return arbitrarily close to p. Now we must consider a way to measure the separation between points in a Lorentz manifold $\mathcal{M}$.

Definition 3.1. *If $p, q \in \mathcal{M}$, the time separation from p to q is*

$$\tau(p,q) = \sup\{\ L(\alpha) : \alpha \textit{ is a future pointing causal curve segment from } p \textit{ to } q\ \}\ .$$

Here, $L(\alpha) = \int_p^q \mid \alpha'(s) \mid ds$ is the arc length of α. We can think of $\tau(p,q)$ as the proper time of a slowest trip from p to q, and as a function, it behaves best when the underlying manifold adheres to some chronology conditions. If certain chronology conditions fail, it is possible to have $\tau(p,q) = \infty$, which would be nonsense in terms of the physical analogue. As opposed to Riemannian distance d, which minimizes, τ maximizes. With this, we would like to establish sufficient conditions for the existence of a longest causal geodesic between two points in a Lorentz manifold.

Proposition 3.1. *For $p, q \in \mathcal{M}$ such that $p << q$, assume that $J(p,q) = J^+(p) \cap J^-(q)$ is compact and the strong causality condition holds on it. Then there is a causal geodesic from p to q of length $\tau(p,q)$.*

Proof. Let $\{\alpha_n\}$ be a sequence of future pointing causal curve segments from p to q such that the lengths converge to $\tau(p, q)$, which will have finite length because of the strong causality condition. Because $J(p, q)$ is the smallest set containing all future pointing causal curves from p to q, all α_n are contained within $J(p, q)$. Because $J(p, q)$ is compact and adheres to the strong causality condition, there is a causal broken geodesic γ from p to q such that $L(\gamma) = \tau(p, q)$. If γ has breaks, then there is a longer causal curve from p to q, but τ is a maximizing function, meaning that γ is unbroken. □

This proposition alludes to a nice feature of spacetimes with a very convenient and physically realistic structure. We define it as follows:

Definition 3.2. *A subset $\mathcal{U} \subseteq \mathcal{M}$ is globally hyperbolic given that the strong causality condition holds on $\mathcal{U}$, and if $p, q \in \mathcal{U}$ such that $p < q$, then $J(p, q)$ is compact and contained in $\mathcal{U}$.*

By this definition and the preceding proposition, for all $p, q \in \mathcal{U}$ such that $\mathcal{U}$ is globally hyperbolic, there is a causal geodesic joining the two points. If the entire manifold $\mathcal{M}$ is globally hyperbolic, then all sets $J^+(p), J^-(q)$, and $J(p, q)$ are closed.

Examples of globally hyperbolic spaces are Minkowski spacetime, Robertson-Walker spacetimes and Schwarzschild spacetime. Thus, it is apparent that global hyperbolicity will play an important role in our analysis of singular spacetimes. Looking at the causal behavior of subsets and submanifolds is also necessary for finding singularities in a parent manifold, so we have the following:

Definition 3.3. *A subset $\mathcal{A} \subseteq \mathcal{M}$ is achronal if $p << q$ never holds for any $p, q \in \mathcal{A}$. In other words, no timelike curve in $\mathcal{M}$ meets $\mathcal{A}$ more than once.*

The simplest example of an achronal set would be a hyperplane t constant in Minkowski space. In addition, this hyperplane has the additional feature of being *edgeless*. The *edge* of an achronal set $\mathcal{A}$ consists of all points $p \in \bar{\mathcal{A}}$ such that every neighborhood $\mathcal{U}$ of p contains a timelike curve from $I^-(p,\mathcal{U})$ to $I^+(p,\mathcal{U})$ that does not meet $\mathcal{A}$.

3.2 Spacelike Hypersurfaces

We now inspect achronal sets with a more continuous structure, viewing them as surfaces with features that help us deduce the existence of singularities in the overlying spacetime.

Definition 3.4. *A subspace $S \subset T$, where T is an n-dimensional topological manifold, is a topological hypersurface provided that for each $p \in S$, there is a neighborhood $\mathcal{U}$ of p in T and a homeomorphism ϕ of $\mathcal{U}$ onto an open set in $\mathbb{R}^n$ such that $\phi(\mathcal{U} \cap S) = \phi(\mathcal{U}) \cap \Pi$, where Π is a hyperplane in $\mathbb{R}^n$.*

An achronal set $\mathcal{A}$ is a topological hypersurface if and only if $\mathcal{A}$ and edge $\mathcal{A}$ are disjoint. Additionally, $\mathcal{A}$ being a *closed* topological hypersurface is equivalent to edge $\mathcal{A}$ being empty. A significant example of a closed topological hypersurface is $\partial J^+(p) \subset \mathbb{R}^4_1$, the null cone at a point $p \in \mathbb{R}^4_1$.

An important class of achronal hypersurfaces in Lorentzian geometry are *Cauchy hypersurfaces*, which represent an "instant in time" in the entire spacetime.

Definition 3.5. *A Cauchy hypersurface (or simply Cauchy surface) $S \subset \mathcal{M}$ is a subset that is met exactly once by every inextendible causal curve in $\mathcal{M}$.*

The null cone discussed above would then not be an example of a Cauchy hypersurface, but the hyperplanes t constant, as subsets of $\mathbb{R}^4_1$, would be. A spacetime that contains a Cauchy hypersurface is globally hyperbolic, so it's clear that not all spacetimes admit a Cauchy hypersurface. The preceding example shows that $\mathbb{R}^4_1$ admits one, but removing any one point takes away this property. The important idea behind a Cauchy hypersurface is that the initial conditions on such a surface determine observable data on all of the spacetime. This notion leads us to ask about what areas in the spacetime depend directly on a particular spacelike slice.

Definition 3.6. *If $\mathcal{A} \subseteq \mathcal{M}$ is achronal, then the future Cauchy development of $\mathcal{A}$ is the set of all points $p \in \mathcal{M}$ such that every past inextendible causal curve through p meets $\mathcal{A}$. We denote it as $D^+(\mathcal{A})$.*

In a physical sense, $D^+(\mathcal{A})$ is the part of $\mathcal{A}$'s causal future that is predictable from $\mathcal{A}$. No past intextendible curve can enter $D^+(\mathcal{A})$ without first having passed through $\mathcal{A}$. Another name for $D^+(\mathcal{A})$, more common with physicists, is the future domain of dependence of $\mathcal{A}$. Dually, $D^-(\mathcal{A})$ is defined to be the set of all points $p \in \mathcal{M}$ such that every future inextendible causal curve through p meets $\mathcal{A}$. In other words, $D^-(\mathcal{A})$, the *past Cauchy development of* $\mathcal{A}$, is the past domain of dependence of $\mathcal{A}$. Every timelike curve passing through $D^-(\mathcal{A})$ will also pass through $\mathcal{A}$. We will refer to the union of the future and past Cauchy developments, $D(A) = D^+(\mathcal{A}) \cup D^-(\mathcal{A})$, simply as the *Cauchy development of* $\mathcal{A}$.

A significant fact is that $\mathcal{A}$ is a hypersurface such that $D(\mathcal{A}) = \mathcal{M}$ if and only if $\mathcal{A}$ is a Cauchy hypersurface. This particular view of what it means to be a Cauchy hypersurface makes it clear how initial conditions on the surface define solutions of its evolution throughout the entire spacetime. We now want to prove a lemma which uses these definitions and is useful later, but we first must introduce a definition that is a useful tool in causal set theory

Definition 3.7. *Let $\{\alpha_n\}$ be an infinite sequence of future-pointing causal curves in $\mathcal{M}$, and let $\mathcal{R}$ be a convex covering of $\mathcal{M}$. A limit sequence for $\{\alpha_n\}$ relative to $\mathcal{R}$ is a finite or infinite sequence $p = p_0 < p_1 < \ldots$ in $\mathcal{M}$ such that*

(i) For each p_i, there is a subsequence $\{\alpha_n\}$ and, for each m, numbers $s_{m_0} < s_{m_1} < \ldots < s_{m_i}$ such that (a) $\lim_{m\to\infty} \alpha_m(s_{m_j}) = p_j$ for each $j \leq i$, and (b) for each $j < i$, the points p_j, p_{j+1} and the segments $\alpha_m|[s_{m_j}, s_{m_{j+1}}]$ for all m are contained in a single set $C_j \in \mathcal{R}$.

(ii) If $\{p_i\}$ is infinite, it is nonconvergent. If $\{p_i\}$ is finite, then it has more than one point and no strictly longer sequence satisfies (i).

The existence of a limit sequence is insured by two mild conditions on $\{\alpha_n\}$, namely (1): the sequence $\{\alpha_n(0)\}$ converges to a point p, and (2): there is a neighborhood of p that contains only finitely many of the α_n. If $\{p_i\}$ is a limit sequence for $\{\alpha_n\}$, we let λ_i be the future pointing causal geodesic from p_i to p_{i+1} in a convex set C_i, as in (i). Joining these together gives us a broken geodesic $\lambda = \sum \lambda_i$ called a *quasi-limit* of $\{\alpha_n\}$, which has vertices p_i. If $\{p_i\}$ is infinite, then it is nonconvergent, by definition, and so λ is future inextendible.

Lemma 3.1. *Let $\mathcal{A}$ be an achronal set. If $p \in D^+(\mathcal{A})^\circ - I^-(\mathcal{A})$, then $J^-(p) \cap D^+(\mathcal{A})$ is compact.*

Proof. Because taking $p \in \mathcal{A}$ consists of dealing with p alone, we assume that $p \in I^+(\mathcal{A}) \cap D(\mathcal{A})$. Now, we let $\{x_n\}$ be an infinite sequence in $J^-(p) \cap D^+(\mathcal{A})$ and let α_n be a past-pointing causal curve segment from p to x_n. If there is a subsequence of $\{x_n\}$ converging to p, then we are done, so we assume otherwise. Hence, there must be a past directed limit sequence $\{p_i\}$ for $\{\alpha_n\}$ starting at p. If $\{p_i\}$ is infinite, then it is nonconvergent, by the definition of a limit sequence, so for some n, there will be an $x_n \in I^-(\mathcal{A})$, which is a contradiction. Given that $\{p_i\}$ is finite, there must be some subsequence $\{x_m\}$ which converges to a point $x \in J^-(p)$. Let σ be a timelike curve from $p^+ \in D^+(\mathcal{A}) \cap I^+(p)$ to x. If σ passes through $\mathcal{A}$, then either $x \in A \subset D^+(\mathcal{A})$ or $x \in I^-(\mathcal{A})$, the latter implying that there is some $x_n \in I^-(\mathcal{A})$, which would be a contradiction, so $x \in D^+(\mathcal{A})$. If σ does not pass through $\mathcal{A}$, then we get that $x \in D^+(\mathcal{A})$. Either way, $x \in J^-(p) \cap D^+(\mathcal{A})$, meaning that $\{x_n\}$ converges to a point in $J^-(p) \cap D^+(\mathcal{A})$, which then must be closed. Because it also is bounded, it must hence be compact. □

Thus far, we have discussed the above objects in mostly topological terms, however, we have noted that they can be considered manifolds with a Lorentzian structure, where the initial surface data is smooth. Thus, we treat these hypersurfaces as submanifolds, and use results from Appendix 4.1 to deduce their properties.

Lemma 3.2. *Let S be a closed achronal spacelike hypersurface in $\mathcal{M}$. If $q \in D^+(S) - S$, then there exists a geodesic γ from S to q such that $L(\gamma) = \tau(S, q)$. Moreover, γ is normal to S and has no focal points of S before q.*

Proof. We take it as a given that because S is a closed achronal hypersurface, that $D(S)$ is globally hyperbolic. Hence, by Lemma 3.1, $J^-(q) \cap D^+(S)$ is compact. This then implies that $J^-(q) \cap S$ is compact, and because it is also globally hyperbolic, the time separation function must be continuous on it. Hence, it achieves a maximum at, say, p. This maximum must then be $\tau(p, q) = \tau(S, q)$. By Proposition 3.1, there is then a causal geodesic γ from p to q such that $L(\gamma) = \tau(S, q)$. Because $q \in D^+(S) - S$, $\tau(S, q) > 0$, meaning that γ is timelike. Hence, γ is normal to S. In addition, Proposition 5.2 in the appendix guarantees there will be no focal points along γ. □

Because the same theorem holds for $D^-(S)$, this result holds for the entire Cauchy development, $D(S)$. This leads us to look at cases of spacetimes where the entire future or past of a hypersurface cannot be deduced from the data on the surface. Previously, we mentioned that removing a point from $\mathbb{R}^4_1$ would imply it does not contain a Cauchy hypersurface, and now we generalize this with the idea of a *Cauchy horizon.*

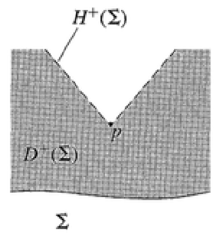
Definition 3.8. *If $\mathcal{A}$ is an achronal set, then the future Cauchy horizon is $H^+(\mathcal{A}) = \{\ p \in \overline{D^+}(\mathcal{A}) :$ $I^+(p)$ does not meet $D^+(\mathcal{A})\ \}$.*

Intuitively, $H^+(\mathcal{A})$ marks the future limit of the spacetime region that can be predicted from $\mathcal{A}$. For example, if we let $S \subset \mathbb{R}^4_1$ be the hyperplane 0 constant, then removing a point $p >> 0$ would make it so that $H^+(S) = \partial J^+(p)$. That is, events in $J^+(p)$ need not necessarily be influenced by data from S.

With $H^-(\mathcal{A})$ defined dually, we have $H(\mathcal{A}) = H^+(\mathcal{A}) \cup H^-(\mathcal{A})$. Note that for a spacelike hypersurface S, if $H(S)$ is empty, then $D(S) = \mathcal{M}$, meaning that S is a Cauchy hypersurface. The reverse implication is also true. We now state a proposition that shows the relationship between Cauchy horizons and other causal structures, along with describing generators of the set.

Proposition 3.2. *Let S be a closed acausal topological hypersurface. Then:*
(i) $H^+(S) = I^+(S) \cap \partial D^+(S) = \overline{D^+(S)} - D^+(S)$. In particular, $H^+(S)$ and S are disjoint.
(ii) $H^+(S)$, if nonempty, is a closed achronal topological hypersurface.
(iii) Starting at each $q \in H^+(S)$, there is a past-inextendible null geodesic without conjugate points that is entirely contained in $H^+(S)$. These geodesics are the generators of $H^+(S)$.

Figure 4: Deleting a point p from the domain of dependence of a spacelike surface Σ generates a Cauchy horizon [5]



The proof is a lengthy exercise in causal set theory, but inspecting a simple example provides enough intuition to understand the result, considering the definitions. In Figure 4, the generators of the future Cauchy horizon of the surface are the straight lines emanating from p that comprise $\partial J^+(p)$. It is also clear from the picture that $\partial D^+(\Sigma)$ minus $D^+(\Sigma)$ itself is the future Cauchy horizon.

With all of the causal structure we have introduced, and after analyzing conditions for the existence of causal geodesics and focal points of submanifolds, we are now prepared to examine the singularity theorems.

3.3 Hawking's Singularity Theorem

Here, we want to generalize what we discussed in the section on FLRW spacetimes. Analysis on those metrics led us to the conclusion that a singularity must exist somewhere in the finite past. However, it is easy for physicists to dismiss these singularities as simply mathematical phenomena stemming from the fact that those metrics are highly symmetric. In physical terms, even though space seems approximately isotropic, it may not be necessarily be so enough to cause catastrophic singularities in the past. In the case of the singularity theorems, we can say that a spacetime is only approximately FLRW (thus more physically realistic), but still must contain singularities under these weaker geometric conditions.

We know that a spacelike slice $t \times S$ in a Robertson-Walker spacetime is a closed topological hypersurface, and the fact that every inextendible null geodesic must meet it implies that $H(t \times S)$ is empty. Hence, this slice is a Cauchy hypersurface. In addition, observational astronomy has helped us deduce that on the spacelike slice of present time, all galaxies are diverging from one another at an accelerating rate, implying past convergence. Because we assume gravity on average attracts, that is, $\mathrm{Ric}(v, v) \geq 0$, all of this implies that past-directed timelike curves starting in the spacelike slice are incomplete. The concept of conjugate points, and thus focal points, is critical here because they determine where a timelike geodesic fails to be a local maximizer of proper time and where a null geodesic fails to remain on the future boundary of a point.

We state the results in future terms, to remain consistent with our trend. We now prove two of Hawking's singularity theorems, the first being stronger, but assuming more.

Theorem 3.1. *Suppose that* $\text{Ric}(v, v) \geq 0$ *for all timelike tangent vectors* v *to a Lorentzian manifold* $\mathcal{M}$*. Also, let* S *be a spacelike future Cauchy hypersurface with future convergence* $\mathbf{k} \geq b > 0$*. Then every future-pointing timelike curve starting in* S *must have length at most* $\frac{1}{b}$*.*

Proof. We assume that $q \in D^+(S) - S$. By Lemma 3.2, there must exist a timelike, normal geodesic γ from S to q such that it contains no focal points before q and $L(\gamma) = \tau(S, q)$. However, our assumptions imply that $\text{Ric}(\gamma', \gamma') \geq 0$, and because S has convergence $\mathbf{k} \geq b > 0$, Proposition 5.2 in the appendix implies that there will be a focal point of S along γ before q if $L(\gamma) > \frac{1}{b}$. Thus, $D^+(S) \subseteq \{ \ p \in \mathcal{M} : \tau(S,p) \leq \frac{1}{b} \ \}$. Because S is a future Cauchy hypersurface, $H^+(S)$ is empty. The definition of $H^+(S)$ and S being a future Cauchy hypersurface then directly imply that all $p \in \mathcal{M}$ are such that $I^+(p)$ meets $D^+(S)$. Because this is the case for all $p \in \mathcal{M}$, we must have that $I^+(S) \subseteq D^+(S) \subseteq \{ \ p \in \mathcal{M} : \tau(S,p) \leq \frac{1}{b} \ \}$. That is to say, every future-pointing timelike curve in S must have length at most $\frac{1}{b}$. ☐

The strongest assumption that we make here is that $\mathcal{M}$ contains a future Cauchy hypersurface. Our universe being approximately FLRW makes this a reasonable assumption to make. However, future developments in physics can disprove the validity of this assumption, so it would be nice to test for timelike incompleteness under more general circumstances.

Theorem 3.2. *Suppose that* $\text{Ric}(v, v) \geq 0$ *for all timelike tangent vectors* v *to* $\mathcal{M}$*. Additionally, let* S *be a connected, compact spacelike hypersurface with future convergence* $\mathbf{k} > 0$*. Then* $\mathcal{M}$ *is future timelike incomplete.*

Proof. Let b be the minimum value of $\mathbf{k}$ on S, which exists because S is compact. We want to prove a weaker conclusion than the previous theorem, namely, (i) that there exists an inextendible future-pointing normal geodesic γ starting at S with length bounded by $\frac{1}{b}$, which would imply that $\mathcal{M}$ is future timelike incomplete. Without loss of generality, we can assume that S is achronal. As in the previous theorem, this gives us that $D^+(S) \subseteq \{ \ p \in \mathcal{M} : \tau(S,p) \leq \frac{1}{b} \ \}$. Now, we assume that $H^+(S) \neq \varnothing$, because otherwise the previous proof holds and we are done. We also assume that (i) is false, to derive a contradiction. With this, we prove (ii) if $q \in H^+(S)$, then there exists a normal geodesic γ_v from S to q such that $L(\gamma_v) = \tau(S, q) \leq \frac{1}{b}$.

Proof. In the normal bundle of S, let B be the set of all zero vectors and future-pointing vectors v such that $|v| \leq \frac{1}{b}$. Because S is compact, B must be so as well, meaning that there exists a sequence $\{q_n\}$ in $D^+(S)$ such that $q_n \to q$. For each q_n, we know there exists a normal geodesic γ_{v_n} from S to q_n such that $L(\gamma_{v_n}) = \tau(S, q) \leq \frac{1}{b}$. Hence, for each q_n, there exists a $v_n \in B$ such that $\exp(v_n) = q_n$. Because B is compact, $\{v_n\}$ has a limit $v \in B$, and then by continuity, this means that $\{q_n\}$ has the limit $\exp(v)$, i.e. $\exp(v) = q$. In addition, we have $|v_n| \to |v| \leq \frac{1}{b}$ and $\tau(S, q_n) = |v_n|$. Because the time separation function is lower semi-continuous, $|v| \geq \tau(S, q)$. Now, because we assumed that (i) is false, this means that γ_v from S to q is defined on $[0, 1]$ with length $|v|$, i.e. $\tau(S, q) = |v| \leq \frac{1}{b}$, finishing the proof. ☐

We can now assume by Proposition 3.2 that (iii) the time separation function $p \to \tau(S, p)$ is strictly decreasing on past-pointing generators of $H^+(S)$. Let α be such a generator. Within the domain of its parameterization, assume that $s < t$. By (ii), there is a past-pointing timelike geodesic σ for $\alpha(t)$ to S such that $L(\sigma) = \tau(S, \alpha(t))$. Because α is a null curve, by definition, the causal curve $\alpha|[s, t] + \sigma$ is broken and hence can be lengthened by a fixed endpoint deformation. Thus, we have $\tau(S, \alpha(s)) > L(\alpha|[s, t] + \sigma) = L(\sigma) = L(S, \alpha(t))$. Because we assumed that (i) is false, the normal exponential map is defined on all of B, implying then that $H^+(S)$ is compact as a subset of the continuous image of B. The time separation is lower semi-continous, so its restriction to $H^+(S)$ achieves a minimum somewhere. This contradicts (iii) because there is a generator extending into the past from each point of $H^+(S)$. So $\mathcal{M}$ is future timelike incomplete. □

With this theorem, we weaken the first one significantly, but still preserve the fact that $\mathcal{M}$ is future timelike incomplete.

When the time orientation of these theorems is reversed, we recover the past singularities that arose in Propositions 2.1 and 2.2, but without the need for exact spatial isotropy. This is particularly appealing, given that recent observations of the cosmic microwave background radiation have shown it to be anisotropic. We can also weaken the assumptions slightly by requiring only that $\mathrm{Ric}(\gamma', \gamma') \geq 0$ for normal geodesics γ to S. In this case, $\mathrm{Ric}(\gamma', \gamma') \geq 0$ is equivalent to our very reasonable physical assumption that $\rho + 3p \geq 0$. From Lemma 2.1, we have that each spacelike slice has mean curvature $\frac{f'(t_0)}{f(t_0)}U$, and so the submanifold convergence is $\mathbf{k} = -\frac{f'(t_0)}{f(t_0)}$. Thus, we see that Hubble expansion at time t_0 is equivalent to $S(t_0)$ having convergence $\mathbf{k} \leq b < 0$, and if these slices are Cauchy hypersurfaces, then Theorem 3.1 holds. Deleting a closed set from $I^+(S)$ would render Theorem 3.1 inapplicable, but Theorem 3.2 will continue to hold because S remains compact.

3.4 Penrose's Singularity Theorem

The result which motivated Stephen Hawking's investigation of Big Bang singularities was that of Roger Penrose concerning "black hole" singularities. The idea is analogous to the previous theorems: under slightly less symmetric conditions, it is still the case that a collapsing star produces a singularity at $r = 0$. This is suggested by the analyses in section 2.3, but now we want to generalize those ideas by looking at a spacelike submanifold $\mathcal{P} \subseteq \mathcal{M}$ and investigating its convergence, as in the previous section.

If $\mathcal{P}$ is a spacelike submanifold of codimension ≥ 2 and if H is its mean curvature vector field, then the following are equivalent: (1) $\mathbf{k}(v) = \langle H, v\rangle > 0$ for all future-pointing null vectors v normal to $\mathcal{P}$, (2) $\mathbf{k}(w) = \langle H, w\rangle > 0$ for all future-pointing causal vectors w normal to $\mathcal{P}$, and (3) H is past-pointing timelike [1]. This motivates the following significant definition.

Definition 3.9. *A spacelike submanifold $\mathcal{P} \subseteq \mathcal{M}$ is future-converging, i.e. a trapped surface, if its mean curvature vector field H is past-pointing timelike.*

In the case of a sphere $\mathbb{S}^2(r)$ of constant time inside a Schwarzschild black hole, the mean curvature vector field is $H = -\mathrm{grad}r/r$. Then

$$\mathbf{k}(v) = \langle H, v\rangle = \langle \frac{-\mathrm{grad}r}{r}, v\rangle = \frac{-v(r)}{r}, \tag{72}$$

and so we see that H is past-pointing timelike, since v is defined to be a future pointing null vector. In particular, this means that $\mathbb{S}^2(r)$ is a trapped surface.

In terms of our language of causal set theory, we want an analogous idea of a set being "trapped."

Definition 3.10. *Let $E^+(A) = J^+(A) - I^+(A)$. If A is a closed achronal subset of $\mathcal{M}$ and if $E^+(A)$ is compact, then A is future-trapped.*

Note that $E^+(A)$ is generated by conjugate-free null geodesics. With this, we would like to see when being a trapped subset coincides with being a trapped surface; this will tell us something about the existence of singularities. First, we must introduce a lemma.

Lemma 3.3. *Assume that:*
(1) $\mathrm{Ric}(v, v) \geq 0$ *for all null tangent vectors* v *to* $\mathcal{M}$*,*
(2) $\mathcal{M}$ *is future null complete.*
Then if $\mathcal{P}$ *is a compact achronal spacelike submanifold of codimension* 2 *that is future converging, then* $\mathcal{P}$ *is future-trapped.*

The proof is contained in [1], and is completed by deducing that $E^+(\mathcal{P})$ is a compact topological manifold. We are able to hide the background calculations concerning focal points by classifying future converging submanifolds as trapped surfaces, and we will state the result in positive terms to see how dropping certain assumptions will "break" timelike completeness.

Theorem 3.3. *Assume that:*
(1) $\mathrm{Ric}(v, v) \geq 0$ *for all null tangent vectors* v *to* $\mathcal{M}$*,*
(2) $\mathcal{M}$ *contains a Cauchy hypersurface* S*,*
(3) $\mathcal{P}$ *is a future-trapped, compact achronal spacelike submanifold of codimension* 2*,*
(4) $\mathcal{M}$ *is future null complete.*
Then $E^+(\mathcal{P})$ *is a Cauchy hypersurface in* $\mathcal{M}$*.*

Proof. Because $\mathcal{M}$ contains a Cauchy hypersurface, it must be globally hyperbolic, which immediately implies that $J^+(p)$ is closed for all $p \in \mathcal{M}$. The assumption that $\mathcal{P}$ is compact then means that $J^+(\mathcal{P})$ is closed. Because $J^+(\mathcal{P})^\circ = I^+(\mathcal{P})$, we have that $E^+(\mathcal{P}) = \partial J^+(\mathcal{P})$. Hence, $E^+(\mathcal{P})$ is a topological manifold, and is thus compact by the preceding lemma.

For the given Cauchy hypersurface S, let $\rho : E^+(\mathcal{P}) \to S$ be the restriction to S of a retraction. ρ is defined to be continuous then, and since $E^+(\mathcal{P})$ is achronal, the uniqueness of integral curves implies that ρ is an injective function. Hence, ρ is a homeomorphism of $E^+(\mathcal{P})$ to an open subset of S. Because $E^+(\mathcal{P})$ is compact, $\rho(E^+(\mathcal{P}))$ must be compact, and thus closed in S. Then because S is connected, it must be the case that $\rho(E^+(\mathcal{P})) = S$. In other words, $E^+(\mathcal{P})$ and S are homeomorphic by ρ.

Now, let β be an inextendible timelike curve in $\mathcal{M}$. We must show that it meets $E^+(\mathcal{P})$. Because we assumed that $\mathcal{M}$ is time-oriented, it must be the case that β is locally an integral curve of some timelike vector field $X \in \mathcal{V}(\mathcal{M})$. Because the strong causality condition holds, this can be extended over $\mathcal{M}$ so that the retraction induced by X is a homeomorphism between $E^+(\mathcal{P})$ and S. Hence, because β meets S, it must also meet $E^+(\mathcal{P})$, meaning that it is a Cauchy hypersurface in $\mathcal{M}$. □

Corollary 3.1. *If we assume that* (1)*,* (2) *and* (3) *hold, but also that* S *is not compact, then* $\mathcal{M}$ *is future null incomplete.*

Proof. Since S is not compact, but is still homeomorphic to $E^+(\mathcal{P})$, $E^+(\mathcal{P})$ is also not compact. This means that $\mathcal{P}$ cannot be future trapped, and also that S cannot be a Cauchy hypersurface, which is a contradiction and hence $\mathcal{M}$ cannot be future null complete. □

Corollary 3.2. *If* (1)*,* (2) *and* (3) *hold, but if there is an extendible causal curve which does not pass through* $E^+(\mathcal{P})$*, then* $\mathcal{M}$ *is future null incomplete.*

Proof. Because there is an inextendible causal curve not meeting $E^+(\mathcal{P})$, $E^+(\mathcal{P})$ is not a Cauchy hypersurface. In particular, $H^+(E^+(\mathcal{P}))$ is non-empty. But $E^+(\mathcal{P})$ and S are homeomorphic by the induced retraction from $X \in \mathcal{V}(\mathcal{M})$, and so $H^+(S)$ is also non-empty. This is a contradiction with (2). Hence, $\mathcal{M}$ is future null incomplete. □

These results generalize the existence of singularities within collapsing stars, even if they do not adhere to the perfect symmetry of the Schwarzschild solution.

4 Thoughts and questions

In the further development of established theories and in the forging of new ones, it may be appropriate to consider the traceless part of the Riemann curvature tensor. This is known as the *Weyl curvature tensor*, and its components in covariant form are as follows [2]:

$$C_{ijkl} = R_{ijkl} + \frac{2}{n-2}\left(g_{i[l}R_{k]j} + g_{j[k}R_{l]i}\right) + \frac{2}{(n-2)(n-1)}Sg_{i[k}g_{l]j}, \tag{73}$$

where n is the dimension of the spacetime and the brackets in the subscripts represent $1/m!$ times the alternating sum of the enclosed indices, where m is the number of indices being permuted. The Weyl tensor represents the gravitational degrees of freedom, thus describing the tidal forces induced by a spacetime's curvature. The stress-energy tensor T_{ij} is analogous to the charge-current vector J_μ in Maxwell's electromagnetic theory. Similarly, the Weyl tensor C_{ijkl} is analogous to the field strength tensor $F_{\mu\nu}$, which is also a traceless quantity, describing the degrees of freedom of the electromagnetic field. Given that $F_{\mu\nu}$ plays a crucial role in quantum field theory, it would be apt to consider the importance of the Weyl tensor in future gravitational theory. In addition, in empty space, the spacetime curvature is entirely Weyl curvature, so we see that it is responsible for the traversal of gravitational radiation through that empty space. This is of particular interest, considering the recent confirmation of the existence of gravitational waves by LIGO.

An important use for the Weyl tensor could perhaps be in a quantized theory of gravity. For example, given that the Weyl tensor is a conformal invariant, physicist Dr. Philip Mannheim has made use of it in his theory of conformal gravity, where he describes gravity as being quantized solely as a result of it being coupled with a quantized matter source [8]. By using an action of conformal form, with the Weyl tensor, as opposed to the Einstein-Hilbert action, Dr. Mannheim justifies that the EFE as we know them are sufficient to give the Schwarzschild solution, but not necessary, allowing for a different equation of motion. In addition, he uses a *PT-symmetric* as opposed to a *Hermitian* quantum theory, to make the parts of this theory of quantized gravity come together. The struggle to unify gravity with quantum theory has been on the frontier of ongoing research among theoretical physicists and mathematicians for the past century. Whether Weyl curvature or such a refined quantum theory will play an important role in updating Einstein's theory remains to be seen, however it is most certainly the case that the insight which provides the missing link between different mathematical theories that describe the universe will come from concepts familiar to us, but that we have not yet fully considered.

Another topic in which the Weyl tensor garners attention is in Roger Penrose's *Weyl curvature hypothesis*. A characteristic feature of the Friedmann-Lemaître-Robertson-Walker spacetimes we examined earlier is that their Weyl curvature vanishes identically. So let's consider a universe evolving from an initial "Big Bang" state of uniform matter distribution. The hypothesis is that this corresponds to the primary source of curvature being Ricci curvature, whereas the Weyl curvature is effectively zero [7]. As the spacetime evolves, the initially uniform matter distribution begins to clump together sporadically, and those clumps become sources of Weyl curvature. Given enough time, in some cases, these clumps become black holes, in which the Weyl curvature diverges. In more physical terms, the initial Big Bang singularity has a very low entropy, whereas the final black hole singularities have a very high entropy. This geometric hypothesis would constrain the cosmos so that it adheres to the Second Law of Thermodynamics and closely resembles the FLRW models. This leads us to ask how the Weyl curvature tensor and other purely geometric concepts could decide further developments in gravitational theory, both in terms of explaining some mysterious large scale phenomena and in the unification of gravity with quantum theory.

A development which has attracted curiosity ever since Einstein's original 1915 publication has been that of a cosmological constant, Λ. If we assume that it is non-zero, as some recent observations might imply, then the EFE take the form:

$$G + \Lambda g = 8\pi T. \tag{74}$$

Originally, although he proposed its existence to constrain the universe to a static state, Einstein dismissed the cosmological constant quickly when it was discovered that the universe was expanding. Now, with recent observations implying an *accelerating* expansion of the universe, the cosmological constant has been brought up again as a possible explanation, along with dark energy/matter. Perhaps there is yet a more purely geometric explanation for the accelerating expansion of the universe, where the need for strange physical assumptions like that of dark energy or even the reinstatement of the cosmological constant may be deemed superfluous. Indeed, one particular idea which has strongly grabbed my attention during the writing of this thesis is the *Ricci flow*. This is a geometric flow where the metric of a manifold changes in accordance with its Ricci curvature, defined by the partial differential equation:

$$\partial_t g_{ij} = -2R_{ij}, \tag{75}$$

where the $t-$parameter does not correspond with the time coordinate of a spacetime, but rather parameterizes the individual metric components to change with t. It is generally defined in this way because it is typically done on a Riemannian (i.e. positive definite) metric. Examples of Ricci flow would be a spherical manifold shrinking away in finite time because it has positive curvature, a flat manifold remaining static because it has zero curvature and a hyperbolic manifold diverging due to its negative curvature. The concept of Ricci flow is one that I hope to do more research on, not only because of its own merit, but also to see how it could be used in gravitational theory. In particular, rigging a Lorentz metric with Ricci flow may prove interesting, given that it is not a positive definite metric. Perhaps its implementation to cosmological spacetimes could provide an explanation for phenomena we do not yet fully understand, like cosmic inflation and the accelerating expansion of the universe. Other ideas, like the *mean curvature flow*, are also of interest to me for these reasons.

One of the most relevant current issues with respect to the primary concepts introduced in this thesis are the *strong* and *weak cosmic censorship conjectures*. The intuitive idea behind the SCC conjecture, as it is called for short, is that causal geodesic incompleteness is always accompanied by divergence of the spacetime's curvature. The WCC conjecture speculates that spacetime singularities are not visible to far-away observers, i.e. there are no *naked singularities*. The most prevalent solutions to the EFE, as we have seen, are singular, so it is appropriate to characterize these singularities in some way. More formal statements of the conjectures are as follows [9]:

Conjecture 4.1 (Strong Cosmic Censorship)**.** *Globally hyperbolic spacetime solutions of the EFE cannot generically be extended as solutions past a Cauchy horizon.*

Conjecture 4.2 (Weak Cosmic Censorship)**.** *In generic asymptotically flat solutions to the EFE, singularities are contained within a black hole event horizon.*

It is important to note that the existence of spacetimes with Cauchy horizons does not disprove the SCC conjecture nor does the existence of asymptotically flat spacetimes with singularities to the causal past of far-away observers disprove the WCC, as stated in Dr. Jim Isenberg's paper [9]. This fact makes it clear exactly how difficult it can be to prove or disprove these conjectures. However, these and other issues mentioned earlier remain as some of the most pertinent and fascinating problems in mathematical relativity, providing us with enough motivation for further study.

5 Appendix

5.1 Index Form, Focal Points, and Submanifold Convergence

In this section, we introduce results from the calculus of variations on manifolds that will be useful for the Hawking and Penrose singularity theorems. This section will not be self-contained; important proofs will be covered, but many results will be taken as granted from O'Neill's Semi-Riemannian Geometry [1].

Definition 5.1. *A variation of a curve segment $\alpha : [a,b] \to M$ is a two-parameter mapping $\mathbf{x} : [a,b] \times (-\delta,\delta) \to \mathcal{M}$ such that $\alpha(u) = \mathbf{x}(u,0)$ for all $a \leq u \leq b$.*

Now, for each $v \in (-\delta,\delta)$, we let $L_{\mathbf{x}}(v)$ be the length of the longitudinal curve $u \mapsto \mathbf{x}(u,v)$. So then $L_{\mathbf{x}}(0) = L(\alpha)$. From now on, we will write $L_{\mathbf{x}}(v) = L(v)$, which won't be misleading considering the context. Our focus concerning the application of these tools will be to the second variation formula, which provides us with a link between geodesics and curvature.

We will take it as a given that a piecewise smooth curve segment α of constant speed $c > 0$ is an unbroken geodesic if and only if the first variation of arc length is zero for every fixed endpoint variation of α. The second variation, $L''(0)$, is needed only when $L'(0) = 0$, so, considering the preceding fact, we need a formula for the second variation when our curve is a geodesic. The vector field $V(u) = \mathbf{x}_v(u,0)$ gives the velocities for the transverse curves of $\mathbf{x}$ as they cross α and $A(u) = \mathbf{x}_{vv}(u,0)$ gives the accelerations. Note that if $|\alpha'| > 0$, then any vector field Y on α splits into $Y^{\mathrm{T}} + Y^{\perp}$, its tangent and normal components to α. We now introduce the general expression for the second variation, called Synge's Formula. The proof and some further implications are in [1].

Theorem 5.1. *Let $\sigma : [a,b] \to \mathcal{M}$ be a geodesic segment of speed $c > 0$ and sign ε. If $\mathbf{x}$ is a variation of σ, then*

$$L''(0) = \frac{\varepsilon}{c}\int_a^b [\langle V'^{\perp}, V'^{\perp}\rangle - \langle R_{V\sigma'}V, \sigma'\rangle]du + \frac{\varepsilon}{c}\langle \sigma', A\rangle|_a^b$$

For $p, q \in \mathcal{M}$, we denote $\Omega(p,q)$ to be the set of all piecewise smooth curve segments $\alpha : [0,b] \to \mathcal{M}$ from p to q. This space can be viewed as a manifold, and thus we have the following definition.

Definition 5.2. *If $\alpha : [a,b] \to \mathcal{M}$ is in $\Omega(p,q)$, then the tangent space $T_\alpha(\Omega)$ consists of all piecewise smooth vector fields V on α such that $V(a) = 0$ and $V(b) = 0$.*

Recall that in general manifold theory, we have that $p \in \mathcal{M}$ is considered a critical point of $f \in \mathcal{F}(\mathcal{M})$ if $v(f) = 0$ for all $v \in T_p(\mathcal{M})$. Analogously, the nonnull geodesics in Ω are the nonnull critical points of the length function L on Ω. Hence, we motivate the Ω analogue of the Hessian on a more general manifold.

Definition 5.3. *The index form I_σ of a nonnull geodesic $\sigma \in \Omega$ is the unique symmetric bilinear form $I_\sigma : T_\alpha(\Omega) \times T_\alpha(\Omega) \to \mathbb{R}$ such that if $V \in T_\alpha(\Omega)$, then $I_\sigma(V,V) = L''_{\mathbf{x}}(0)$, where $\mathbf{x}$ is any fixed endpoint variation of σ with variation vector field V.*

Thus, given Theorem 4.1, we can write the general expression for the index form for $V, W \in T_\sigma(\Omega)$ as

$$I_\sigma(V,W) = \frac{\varepsilon}{c}\int_a^b [\langle V'^{\perp}, W'^{\perp}\rangle - \langle R_{V\sigma'}W, \sigma'\rangle]du. \tag{76}$$

An important fact concerning the index form which will serve as an important reference in future results is:

Lemma 5.1. *Let σ be a nonnull geodesic of sign ε in a Semi-Riemannian manifold of dimension n and index v. Then (1): if I_σ is positive semidefinite, then $v = 0$ or $v = n$. (2): If I_σ is negative semidefinite, then either $v = 1$ and $\varepsilon = -1$ or $v = n - 1$ and $\varepsilon = 1$.*

If γ is a geodesic, a vector field Y that satisfies $Y'' = R_{Y\gamma'}(\gamma')$ is called a *Jacobi vector field.* They can be intuitively thought of as geodesic variation vector fields.

Definition 5.4. *Two points $\sigma(a), \sigma(b)$ such that $a \neq b$ on a geodesic σ are called conjugate points if there is a nonzero Jacobi field J on σ so that $J(a) = 0$ and $J(b) = 0$.*

For example, there are no conjugate points in Minkowski space $\mathbb{R}^4_1$, because geodesics never "refocus." On $S^2(r)$, for any point p, a conjugate point would simply be $-p$, after an arc length πr. In addition, p is conjugate to itself on this manifold.

Now we want to inspect a more general idea concerning conjugate points. Instead of looking at the set $\Omega(p, q)$ we described above, we look at the set $\Omega(\mathcal{P}, q)$ of all piecewise smooth curves that run from a submanifold $\mathcal{P}$ to a point q. We call $\mathcal{P}$ here the *endmanifold.* Because we will be considering the geometry of $\mathcal{P}$ as a submanifold, we must introduce the following definition describing its curvature.

Definition 5.5. *The shape tensor is an $\mathcal{F}(\mathcal{M})$-bilinear and symmetric function $II : \mathcal{V}(\mathcal{M}) \times \mathcal{V}(\mathcal{M}) \to \mathcal{V}(\mathcal{M})^\perp$ defined so that $II(V, W) = \text{nor } \overline{\nabla}_V W$, where $\overline{\nabla}$ is the Levi-Civita connection on the overlying manifold $\overline{\mathcal{M}}$.*

Using a few simple manipulations, we can use the above definition to generalize the above expression for the index form to submanifolds as

$$I_\sigma(V, W) = \frac{\varepsilon}{c} \int_a^b [\langle V'^\perp, W'^\perp \rangle - \langle R_{V\sigma'} W, \sigma' \rangle] du - \frac{\varepsilon}{c} \langle \sigma'(0), II(V(0), W(0)) \rangle \tag{77}$$

The need to generalize conjugate points from individual points to submanifolds motivates the following definition.

Definition 5.6. *Let σ be a geodesic of $\mathcal{M}$ that is normal to $\mathcal{P} \subset \mathcal{M}$, i.e. $\sigma(0) \in \mathcal{P}$ and $\sigma'(0) \perp \mathcal{P}$. Then $\sigma(r)$, where $r \neq 0$, is a focal point of $\mathcal{P}$ along σ if there is a nonzero P-Jacobi field J on σ with J(r)=0.*

The following result is a significant application of the concept of focal points. The proof is contained in [1].

Theorem 5.2. *Let $\sigma \in \Omega(\mathcal{P}, q)$ be a normal geodesic of sign ε such that the subspace $\sigma'(s)^\perp$ of $T_{\sigma(s)}\mathcal{M}$ is spacelike for all s (σ is cospacelike). Then: (1) if there are no focal points of $\mathcal{P}$ along σ, then $I_\sigma^\perp$ is definite (positive if $\varepsilon = 1$; negative if $\varepsilon = -1$), or (2) if $q = \sigma(b)$ is the only focal point of $\mathcal{P}$ along σ, then $I_\sigma^\perp$ is semi-definite, but not definite.*

With the above definitions and results, we can work through the most applicable facts with respect to the Hawking/Penrose singularity theorems. The proofs follow those in [1], but are necessary here to build the intuition for the singularity theorems.

Proposition 5.1. *Let $\mathcal{P}$ be a spacelike submanifold of a Lorentz manifold, and let σ be a $\mathcal{P}$-normal nonnull geodesic. Assume that:*
(i) $\langle \sigma'(0), II(y,y)\rangle = k > 0$ for some unit vector $y \in T_{\sigma(0)}\mathcal{P}$
(ii) $\langle R_{v\sigma'}v, \sigma'\rangle \geq 0$ for all tangent vectors $v \perp \sigma$.
Then there must exist a focal point $\sigma(r)$ of $\mathcal{P}$ along σ with $0 < r \leq \frac{1}{k}$, provided that σ is defined on this interval.

Proof. We can assume without loss of generality that $|\sigma'| = 1$. Because σ must necessarily be cospacelike, we only need to justify that $I_\sigma^\perp$ of σ on $[0, \frac{1}{k}]$ is indefinite, by Theorem 5.2. Considering Lemma 5.1, we just need to show that there is a nonzero $V \in T_\sigma^\perp(\Omega)$ such that $\varepsilon I(V,V) \leq 0$. Let us define $V(u) = (1-ku)Y(u)$, where Y is the parallel translate of y. Clearly, $V \in T_\sigma^\perp(\Omega)$. Because $V' = -kY$, and because parallel translate is defined so that $|Y| = 1$, we have:

$$\varepsilon I(V,V) = \int_0^{\frac{1}{k}} [\langle -kY, -kY\rangle - \langle R_{V\sigma'}V, \sigma'\rangle]du - \langle II(V(0), V(0)), \sigma'(0)\rangle$$

$$= \int_0^{\frac{1}{k}} [k^2 - \langle R_{V\sigma'}V, \sigma'\rangle]du - \langle II(y,y), \sigma'(0)\rangle = \int_0^{\frac{1}{k}} k^2 du - \int_0^{\frac{1}{k}} \langle R_{V\sigma'}V, \sigma'\rangle du - k.$$

The first integral then cancels with the k term at the end and we are left with $-\int_0^{\frac{1}{k}} \langle R_{V\sigma'}V, \sigma'\rangle du = -\frac{\langle R_{V\sigma'}V, \sigma'\rangle}{k}$. Because of assumption (ii), this final term must then be less than zero, and hence $\varepsilon I(V,V) \leq 0$, meaning that $I_\sigma^\perp$ of σ on $[0, \frac{1}{k}]$ is indefinite. Hence, there is a focal point $\sigma(r)$ of $\mathcal{P}$ along σ with $0 \leq r \leq \frac{1}{k}$. ☐

We can think of k in the above proposition as an initial rate of convergence. If we move $\sigma'(0)$ infinitesimally in the y direction to normal vectors v, then if $k > 0$, the geodesics γ_z are initially converging toward σ. That is, the curvature of the manifold has a direct influence on geodesic convergence. Moreover, the shape of a submanifold $\mathcal{P} \subset \mathcal{M}$ influences the existence of focal points of geodesics.

Definition 5.7. *Let $\mathcal{P}$ be a semi-Riemannian submanifold of $\mathcal{M}$ with a mean curvature vector field H. The convergence of $\mathcal{P}$ is the real valued function $\mathbf{k}$ on the normal bundle $N\mathcal{P}$ such that $\mathbf{k}(z) = \langle z, H_p\rangle$. Note here that for a spacelike hypersurface $\mathcal{P} \subset \mathcal{M}$,*

$$H_p = \frac{1}{n-1}\sum_{i=1}^{n-1} II(e_i, e_i),$$

where $e_1, \ldots, e_{n-1}$ is an orthonormal basis for $T_p\mathcal{P}$.

We use this notion of submanifold convergence to strengthen the preceding result for our purposes concerning the singularity theorems.

Proposition 5.2. *Let $\mathcal{P}$ be a spacelike hypersurface in a Lorentz manifold $\mathcal{M}$, and let σ be a geodesic normal to $\mathcal{P}$, as in the preceding result, at $p = \sigma(0)$. Assume also that:*
(i) $\langle \sigma'(0), H_p\rangle = \mathbf{k}(\sigma'(0)) > 0$
(ii) $\mathrm{Ric}(\sigma', \sigma') \geq 0$.
Then there must be a focal point $\sigma(r)$ of $\mathcal{P}$ with $0 < r \leq \frac{1}{\mathbf{k}(\sigma'(0))}$, given that σ is defined on this interval.

Proof. As in the previous proof, we let $|\sigma'| = 1$, and we also let $k = \mathbf{k}(\sigma'(0))$. Again, we are trying to show that $I_\sigma^\perp$ of σ on $[0, \frac{1}{k}]$ is indefinite. If $e_1, \ldots, e_{n-1}$ is an orthonormal basis for $T_p(\mathcal{P})$, we can parallel translate along σ to obtain a frame field $E_1, \ldots, E_{n-1}$. Analogously to the previous proof, we define $f(u) = 1 - ku$ on $[0, \frac{1}{k}]$. Thus, $fE_i \in T_\sigma(\Omega)$ and

$$\varepsilon I_\sigma(fE_i, fE_i) = \int_0^{\frac{1}{k}} [k^2 - f^2 \langle R_{E_i\sigma'} E_i, \sigma' \rangle] du - \langle II(e_i, e_i), \sigma'(0) \rangle$$

$$= k - \int_0^{\frac{1}{k}} [-f^2 \langle R_{E_i\sigma'} E_i, \sigma' \rangle] du - \langle II(e_i, e_i), \sigma'(0) \rangle.$$

By the definition of the Ricci tensor, and given that the E_i are spacelike, we can sum over all i in the above expression to get:

$$(n-1)k - \int_0^{\frac{1}{k}} f^2 \mathrm{Ric}(\sigma', \sigma') du - \langle \sigma'(0), (n-1)H_p \rangle = (n-1)k - \int_0^{\frac{1}{k}} f^2 \mathrm{Ric}(\sigma', \sigma') du - (n-1)\langle \sigma'(0), H_p \rangle$$

$$= (n-1)k - \int_0^{\frac{1}{k}} f^2 \mathrm{Ric}(\sigma', \sigma') du - (n-1) = -\int_0^{\frac{1}{k}} f^2 \mathrm{Ric}(\sigma', \sigma') du.$$

Now, because we assume initially that $\mathrm{Ric}(\sigma', \sigma') \geq 0$, it follows that $\varepsilon I_\sigma(E_i, E_i) \leq 0$ for at least one i, meaning that it is indefinite on $[0, \frac{1}{k}]$. Hence, there must be a focal point of σ somewhere on this interval. □

The above two propositions are incredibly significant in justifying the final results of section three; the convergence of spacelike hypersurfaces has a deep connection with the inextendibility of geodesics in a spacetime, which is indicative of the existence of singularities.

5.2 Warped products and their curvatures

We can take product manifolds and generalize them to a more interesting class that will be useful in our study of spacetimes.

Definition 5.8. *Suppose B and F are semi-Riemannian manifolds, where π and σ are the projections of their product manifold $B \times F$ onto B and F, respectively. The warped product $B \times_f F$ is the product manifold $B \times F$ with the metric tensor $g = \pi^*(g_B) + (f \circ \pi)^2 \sigma^*(g_F)$.*

Vectors in the lift of B are called horizontal, and vectors in the lift of F are called vertical. We see that if $f = 1$, then this is simply the product manifold of B and F. Now we state a result from [1] which follows quickly from a Riemann tensor contraction with respect to a frame field and will be useful in our analysis of both spherically symmetric and static spacetimes and cosmological spacetimes.

Proposition 5.3. *Let the warped product be $\mathcal{M} = B \times_f F$, where $\dim F \geq 1$, and let X, Y be horizontal and V, W be vertical. Ric_B is the Ricci tensor on B and Ric_F is the Ricci tensor on F. Then:*
(i) $\mathrm{Ric}(X, Y) = \mathrm{Ric}_B(X, Y) - (\frac{d}{f}) H^f(X, Y)$
(ii) $\mathrm{Ric}(X, V) = \mathrm{Ric}(Y, W) = 0$
(iii) $\mathrm{Ric}(V, W) = \mathrm{Ric}_F(V, W) - \langle V, W \rangle \Big(\frac{\Delta f}{f} + (d-1) \frac{\langle \mathrm{grad} f, \mathrm{grad} f \rangle}{f^2} \Big)$.

The use of this proposition quickly gives us the equations in 2.1. In our chosen Lorentz frame, $(V^{-1}\partial_t, e^1, e^2, e^3)$, $V^{-1}\partial_t$ is a vertical vector and the e^i are horizontal. Then we have:

$$\tilde{R}_{00} = \mathrm{Ric}_F(V^{-1}\partial_t, V^{-1}\partial_t) - \langle V^{-1}\partial_t, V^{-1}\partial_t\rangle\Big(\frac{\Delta V}{V} + (1-1)\frac{\langle \mathrm{grad}V, \mathrm{grad}V\rangle}{V^2}\Big) \tag{78}$$

$$= 0 + \Big(\frac{\Delta V}{V} + 0\Big) = \frac{\Delta V}{V}, \tag{79}$$

which follows from (iii). From (ii) we have:

$$\tilde{R}_{0i} = 0. \tag{80}$$

And finally, (i) gives us:

$$\tilde{R}_{ik} = \mathrm{Ric}(e^i, e^k) - \frac{1}{V}H^V(e^i, e^k) = R_{ik} - \frac{\nabla_i\nabla_k V}{V}. \tag{81}$$

5.3 A few calculations

In section 2.2, we need the Christoffel symbols for the metric $g = u^4\delta$ to calculate the components of the Ricci tensor. We use regular Euclidean coordinates for this calculation, even though we switch over to spherical coordinates to ultimately solve for u in Birkhoff's theorem. Note that the notation u_i represents $\frac{\partial u}{\partial x^i}$. Recall the expression for the Christoffel symbols:

$$\Gamma^k_{ij} = \frac{1}{2}g^{km}\Big(\frac{\partial g_{jm}}{\partial x^i} + \frac{\partial g_{im}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^m}\Big).$$

With this, we have:

$$\begin{aligned}
\Gamma^1_{11} &= \frac{2}{u}u_1 & \Gamma^2_{11} &= \frac{-2}{u}u_2\\
\Gamma^3_{11} &= \frac{-2}{u}u_3 & \Gamma^1_{22} &= \frac{-2}{u}u_1\\
\Gamma^2_{22} &= \frac{2}{u}u_2 & \Gamma^3_{22} &= \frac{-2}{u}u_3\\
\Gamma^1_{33} &= \frac{-2}{u}u_1 & \Gamma^2_{33} &= \frac{-2}{u}u_2\\
\Gamma^3_{33} &= \frac{2}{u}u_3 & \Gamma^1_{12} &= \frac{2}{u}u_2\\
\Gamma^1_{13} &= \frac{2}{u}u_3 & \Gamma^2_{12} &= \frac{-2}{u}u_1\\
\Gamma^3_{13} &= \frac{2}{u}u_1 & \Gamma^3_{23} &= \frac{2}{u}u_2.
\end{aligned}$$

Recall that $\Gamma^k_{ij} = \Gamma^k_{ji}$. Also, if there are no pairs among i, j and k, then $\Gamma^k_{ij} = 0$. This gives us all of the Christoffel symbols. Although these calculations are relatively simple, it is nice to see the symmetries revealed in a conformally flat metric.

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