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Schwarzschild Spacetime and
Friedmann-Lemaître-Robertson-Walker Cosmology

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Abstract

The advent of General Relativity via Einstein's field equations revolutionized our understanding of gravity in our solar system and universe. The idea of General Relativity posits that gravity is entirely due to the geometry of the universe – that is, the mass distribution throughout the universe results in the “curving” of spacetime, which gives us the physics we see on a large scale. In the framework of General Relativity, we find that the universe behaves differently than was predicted in the model of gravitation developed by Newton. We will derive the general relativistic model for a simple system near a large mass, typically a star, which is spherical, static, and vacuum – the result originally derived by Schwarzschild. The Schwarzschild model will be shown to explain the failure of Newtonian dynamics to predict the perihelion advance of the orbit of Mercury. Additionally, a model of the universe as a homogeneous, isotropic, perfect fluid made of particles that are galaxies (or galactic clusters or superclusters) will be developed under the conditions of the Einstein equation. This model, called the Friedmann-Lemaître-Robertson-Walker (FLRW) model, will yield an upper bound for the age of the universe.

1 Geometry

Einstein originally developed special relativity independently of a strictly geometric framework. He was introduced to the geometric construction he would later use by Minkowski, and it has retained Minkowski's name to be called Minkowski spacetime. The geometric construction produces a model of the universe as a four-dimensional manifold, or a "spacetime", described by a single temporal coordinate together with three spatial coordinates. It is, perhaps, the simplest to understand, as it is a spacetime with zero curvature. In addition, we must equip our space with an object called the metric tensor, which will be established in this section. We can distinguish this spacetime from four-dimensional Euclidean space, \mathbb{R}^4 , by the properties of this metric tensor and its corresponding line element,

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2, \quad (1)$$

where we observe that the time coordinate has opposite sign relative to the spatial coordinates. This is called the metric signature, written $(-+++)$. Minkowski spacetime is thus denoted \mathbb{R}_1^4 , to represent the single coordinate having opposite sign. Through this description of space, we will construct the properties of a Semi-Riemannian manifold (or Pseudo-Riemannian manifold).

1.1 Tensors

We will call the ring of differentiable functions which map M to \mathbb{R} on our manifold $\mathcal{F}(M)$. The vector fields which are a module over this ring form $\mathcal{V}(M)$, and the one-forms which are the dual module over this ring form $\mathcal{V}^*(M)$. This has the result that if $\theta \in \mathcal{V}^*(M)$ and $V \in \mathcal{V}(M)$, then there exists $f \in \mathcal{F}(M)$ such that

$$\theta(V) = V(\theta) = f. \quad (2)$$

The manifold is additionally equipped with a metric tensor, introduced later, which becomes an inner product, $\langle \cdot, \cdot \rangle$, when restricted to the tangent space of the manifold at a point. The inner product maps vectors in the tangent space to numbers, and hence the metric tensor maps vector fields on the manifold to differentiable functions.

Tensors are function valued operators which have r "contravariant slots" – those accepting one-forms – and s "covariant" slots – those accepting vector fields. Additionally, these slots are each linear with respect to $\mathcal{F}(M)$. We call such a tensor an (r, s) tensor and denote an (r, s) tensor T by T_s^r . We call the value $r + s$ the "rank" of the tensor. This notation means that $T : \theta^1 \times \dots \times \theta^r \times V_1 \times \dots \times V_s \rightarrow \mathcal{F}(M)$. Notice that a $(1, 0)$ tensor has a single slot to take in a one form and output a function. If we apply equation (2), we see that a vector field applied to a one form outputs a function in the same manner, so we conclude that a vector field is a $(1, 0)$ tensor. Similarly, we draw the conclusion that a one-form is a $(0, 1)$ tensor. In fact, all rank 1 tensors can be realized this way. Finally, a $(0, 0)$ tensor has no slots for one-forms or vector fields, but still must output a function, and is therefore a function.

Choosing a coordinate system, say $\xi = (x^1, x^2, \dots, x^m)$, on the spacetime defines its directional vectors. Let the basis vectors for this coordinate system be $\partial_1, \partial_2, \dots, \partial_m$. These ∂_i , for $1 \leq i \leq m$, as well as all other vectors, are manifestations of directional derivatives. For example, given a function f the directional derivative of f in the direction of the vector x is $x(f)$. From this, we define df , the exterior derivative of f , such that

$$df(x) = x(f). \quad (3)$$

This directly implies then that

$$dx^i(\partial_j) = \partial_j(x^i) = \delta_j^i \quad (4)$$

which makes the dx^i the dual basis to the ∂_i .

1.2 Metric Tensor and Levi-Civita Connection

One tensor that will be of great importance throughout this paper is the metric tensor, g . The metric tensor is a symmetric $(0, 2)$ tensor. Defining a sign function s_i such that $s_t = -1$ and for $i \neq t$ $s_i = 1$, we define the Minkowski metric tensor by

$$g = \sum_{i,j} s_i dx^i dx^j. \quad (5)$$

A general metric g is given by

$$g = \sum_{i,j} g_{ij} dx^i dx^j. \quad (6)$$

Additionally, since our inner product is non-degenerate, and due to the fact that g is symmetric, we know that g_{ij} must be symmetric and invertible. Thus we denote g^{ij} to be the inverse such that $g^{ij}g_{ij}$ gives the trace of the identity matrix (i.e. the dimension of our spacetime).

Now we can define the notions of spacelike, timelike, and null vectors in our manifold. A spacelike vector v_s has the property that $\langle v_s, v_s \rangle > 0$ or $v_s \equiv 0$, a timelike vector v_t has the property that $\langle v_t, v_t \rangle < 0$, and a null vector v_0 has the property that $\langle v_0, v_0 \rangle = 0$ and $v_0 \neq 0$.

In order to analyze the rate of change of one vector field in the direction of another, we will establish the Levi-Civita connection on our vector fields. In order to accomplish this, we must first establish that the Lie bracket operation $[A, B]$ for two operators A and B is

$$[A, B] = AB - BA.$$

When operators A and B are such that $[A, B] = 0$, A and B are said to “commute”. An affine connection, ∇ , is defined as follows.

Definition 1. An affine connection is a function $\nabla : \mathcal{V}(M) \times \mathcal{V}(M) \rightarrow \mathcal{V}(M)$, such that for $V, W \in \mathcal{V}(M)$,

- (i) $\nabla_V W$ is $\mathcal{F}(M)$ -linear in V ;
- (ii) $\nabla_V W$ is \mathbb{R} -linear in W ; and
- (iii) for all $f \in \mathcal{F}(M)$, $\nabla_V(fW) = (Vf)W + f\nabla_V W$.

To obtain the Levi-Civita connection, we additionally require that the connection is torsion free and metric compatible, or

- (iv) $[V, W] = \nabla_V W - \nabla_W V$ and
- (v) for $X \in \mathcal{V}(M)$, $X\langle V, W \rangle = \langle \nabla_X V, W \rangle - \langle V, \nabla_X W \rangle$,

respectively. It is important to note that ∇ is *not* a tensor. To see this, identify the connection with its potential tensor format: $\nabla_V W = \nabla(V, W)$. Were it a tensor, we would require linearity over $\mathcal{F}(M)$ in both slots; in particular, we require for all $f \in \mathcal{F}(M)$ that

$$\nabla(V, fW) = f\nabla(V, W). \quad (7)$$

By our definition, however, we see that

$$\nabla_V(fW) = \nabla(V, fW) = (Vf)W + f\nabla(V, W). \quad (8)$$

The first summand, $(Vf)W$, implies that the connection is *not* linear over $\mathcal{F}(M)$ in the second slot.

In the interest of applying the Levi-Civita connection to a manifold given a coordinate basis, we first define the Christoffel symbols, Γ_{ij}^k , given the metric tensor on our manifold, as follows

$$\nabla_{\partial_i}(\partial_j) = \sum_k \Gamma_{ij}^k \partial_k. \quad (9)$$

It is now useful to note for computation that due to (iv) and (v), we have [4]

$$\Gamma_{ij}^k = \frac{1}{2} \sum_m g^{km} \left[\frac{\partial g_{jm}}{\partial x^i} + \frac{\partial g_{im}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^m} \right]. \quad (10)$$

Finally, we note that $[\partial_i, \partial_j] = 0$, so $\Gamma_{ij}^k = \Gamma_{ji}^k$ by (iv).

These definitions allow us to speak of curves which have zero acceleration in the manifold.

Definition 2. A curve $\gamma = (x^1(t), x^2(t), \dots, x^n(t))$ that satisfies

$$\frac{d^2(x^k)}{dt^2} + \sum_{i,j} \Gamma_{ij}^k \frac{dx^i}{dt} \frac{dx^j}{dt} = 0. \quad (11)$$

is called a geodesic.

This is important because Einstein concluded that freefall, or motion due solely to gravity, was motion along the geodesics of a curved spacetime.

1.3 Curvature

There are three types of curvature which will interest us: Riemann curvature, Ricci curvature, and Scalar curvature. The first curvature tensor is the Riemann curvature tensor, $R : \mathcal{V}(M) \times \mathcal{V}(M) \times \mathcal{V}(M) \rightarrow \mathcal{V}(M)$, given by

Definition 3. For $V, W, X \in \mathcal{V}(M)$,

$$R_{VW}X = \nabla_{[V,W]}X - [\nabla_V, \nabla_W]X. \quad (12)$$

For a coordinate system x^1, \dots, x^n , with basis vectors $\partial_1, \dots, \partial_n$, the components R_{ijkl}^i of the Riemann curvature tensor are given by

$$R_{\partial_k \partial_i \partial_j} = \sum_i R_{ijkl}^i \partial_i. \quad (13)$$

The above can be used to show that R_{ijkl}^i is given by

$$R_{ijkl}^i = \Gamma_{kj,l}^i - \Gamma_{lj,k}^i + \Gamma_{lm}^i \Gamma_{kj}^m - \Gamma_{km}^i \Gamma_{lj}^m \quad (14)$$

utilizing the Einstein summation convention and the notation “, k ”, which signifies a partial derivative with respect to the coordinate direction k [4]. The Ricci curvature, Ric_{ij} is given by contraction of the Riemann curvature on the first upper index and the third lower index, i.e.

$$\text{Ric}_{ij} = R_{ijm}^m. \quad (15)$$

As such, equation (14) yields that the Ricci tensor components are

$$\text{Ric}_{ij} = \Gamma_{ji,n}^n - \Gamma_{ni,j}^n + \Gamma_{nm}^n \Gamma_{ji}^m - \Gamma_{jm}^n \Gamma_{ni}^m \quad (16)$$

where we sum over all n and m .

Finally, the Scalar curvature is the trace of the Ricci curvature. That is,

$$S = \sum g^{ij} \text{Ric}_{ij}. \quad (17)$$

1.4 Einstein Equation

We conclude this section by presenting the fundamental results that were produced by Einstein in his publication on general relativity [3]. The Einstein curvature tensor, G_{ij} , is given by

$$G_{ij} = \text{Ric}_{ij} - \frac{1}{2} S g_{ij}. \quad (18)$$

The Einstein equation equates the Einstein curvature tensor and the mass distribution of the system, described by the “stress-energy tensor”, T_{ij} . The Einstein equation is given by

$$G_{ij} = 8\pi T_{ij}. \quad (19)$$

The study of the above equation is the theory of General Relativity. This theory is comprised of a few basic ideas. First, the spacetime that our universe lies in is curved by the presence of mass. This curvature of spacetime results in changing the geodesics, since it inherently changes the metric. Furthermore, free-fall in our spacetime occurs along the geodesics. That is, the phenomenon of gravity is simply the motion of bodies along the geodesics in the spacetime. Finally, although this is outside the scope of this paper, we find that light waves also must travel along the geodesics of the spacetime, which results in the “bending” of light around massive bodies.

2 Schwarzschild Spacetime

It is of great interest how spacetime curves around systems that often occur in nature. Consider a star alone in space which has no net charge and is not rotating. The star and the spacetime around it are spherically symmetric. If it is alone, the space around it is a vacuum and contains no other matter. If it has no net charge and is not rotating, we call it static, as it has no extra motion other than that of the chosen inertial frame. Analyzing these properties of our system, we will derive the metric that will model the gravity produced by our spherically symmetric, static star in vacuum.

2.1 General Schwarzschild Metric

Throughout this section, we will be using geometrized units where $G = 1$ and $c = 1$, where G is Newton's gravitational constant and c is the speed of light. Generally, for a spherically symmetric, static spacetime, expressed in polar areal coordinates, the metric will be of the form [5]

$$g = -\alpha^2(r)dt_0^2 + a^2(r)dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\varphi^2). \quad (20)$$

We should now consider the fact that there is no mass outside the radius of our star. Thus our stress energy tensor is given by

$$T_{ij} = 0. \quad (21)$$

Utilizing this result in equation (19), then substituting into equation (18), we find that

$$\text{Ric}_{ij} - \frac{1}{2}Sg_{ij} = 0. \quad (22)$$

If we take the trace of both sides of this equation, we have the following

$$\text{trace}(\text{Ric}_{ij} - \frac{1}{2}Sg_{ij}) = \text{trace}(\text{Ric}_{ij}) - \frac{1}{2}S \text{trace}(g_{ij}). \quad (23)$$

The trace of the Ricci curvature is simply the scalar curvature, as noted in section 1. The trace of the metric is given by adding over all components of g_{ij} when multiplied with g^{ij} . That is, in our 4-dimensional spacetime,

$$\begin{aligned} \text{trace}(g_{ij}) &= g^{ij}g_{ij} \\ &= \delta_0^0 + \delta_1^1 + \delta_2^2 + \delta_3^3 \\ &= 4. \end{aligned} \quad (24)$$

From these two results, we see that equation (23) becomes

$$S - 2S = 0, \quad (25)$$

and hence, $S = 0$. This result then implies, by equation (22), that

$$\text{Ric}_{ij} = 0. \quad (26)$$

Combining this result with equation (16), we will evaluate the Christoffel symbols to obtain expressions for $\alpha(r)$ and $a(r)$. We begin by noting that all of the mixed Ricci curvature terms, Ric_{ij} where $i \neq j$, are 0 (Appendix A). This leaves us with only $\text{Ric}_{t_0t_0}$, Ric_{rr} , and $\text{Ric}_{\theta\theta} = \text{Ric}_{\varphi\varphi}$. Computing the Christoffel symbols for each of these we find

$$\text{Ric}_{t_0t_0} = -\frac{\alpha\alpha'a'}{a^3} + \frac{\alpha\alpha''}{a^2} + \frac{2\alpha\alpha'}{ra^2} \quad (27)$$

$$\text{Ric}_{rr} = -\frac{\alpha''}{\alpha} + \frac{\alpha'a'}{\alpha a} + \frac{2a'}{ra} \quad (28)$$

$$\text{Ric}_{\theta\theta} = r\frac{\alpha'}{a^2\alpha} - 1 + \frac{1}{a^2} - r\frac{a'}{a^3}. \quad (29)$$

Each of these must be 0, since the total Ricci curvature is 0. From these equations, we derive a relationship between α and a as follows.

$$\begin{aligned}
\frac{\alpha^2}{a^2} \text{Ric}_{rr} + \text{Ric}_{t_0 t_0} &= 0 \\
\frac{\alpha^2}{a^2} \text{Ric}_{rr} &= -\text{Ric}_{t_0 t_0} \\
-\frac{\alpha\alpha''}{a^2} + \frac{\alpha\alpha'a'}{a^3} + \frac{2\alpha^2 a'}{ra^3} &= \frac{\alpha\alpha'a'}{a^3} - \frac{\alpha\alpha''}{a^2} - \frac{2\alpha\alpha'}{ra^2} \\
\frac{2\alpha^2 a'}{ra^3} &= -\frac{2\alpha\alpha'}{ra^2} \\
2\alpha^2 a' &= -2\alpha\alpha'a
\end{aligned} \tag{30}$$

and finally, we have the separated differential equation

$$\frac{a'}{a} = -\frac{\alpha'}{\alpha}. \tag{31}$$

We solve this differential equation by noting that

$$\frac{a'}{a} = \frac{d}{dr} \ln(a). \tag{32}$$

Thus by integrating both sides we find

$$-\ln(a) + C_1 = \ln(\alpha) \tag{33}$$

or, by taking the exponential of both sides, and defining $C = \exp(C_1)$,

$$\alpha = \frac{C}{a}. \tag{34}$$

Then

$$g = -\left(\frac{C}{a}\right)^2 dt_0^2 + a^2 dr^2 + r^2 (d\theta^2 + \sin^2(\theta)d\varphi^2). \tag{35}$$

By redefining $t = Ct_0$, we have

$$g = -\left(\frac{1}{a}\right)^2 dt^2 + (a^2)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2(\theta)d\varphi^2). \tag{36}$$

We proceed by substituting the identity in equation (31) into $\text{Ric}_{\theta\theta}$. This gives us

$$\begin{aligned}
\text{Ric}_{\theta\theta} &= 0 \\
-2r\frac{a'}{a} - a^2 + 1 &= 0 \\
2ra' &= a - a^3 \\
\frac{a'}{a - a^3} &= \frac{1}{2r}
\end{aligned} \tag{37}$$

We separate the left hand side using the method of partial fractions. From there, we proceed by separation of variables:

$$da \left(\frac{1}{a} + \frac{1}{2(1-a)} - \frac{1}{2(1+a)} \right) = \frac{dr}{2r}, \quad (38)$$

which by integration we obtain

$$\begin{aligned} 2 \ln(a) - \ln(1-a) - \ln(1+a) &= \ln(r) + C' \\ \ln \left(\frac{a^2}{1-a^2} \right) &= \ln(r) + C' \end{aligned} \quad (39)$$

Exponentiating both sides, and relabeling $C'' = \exp(C')$, yields

$$\begin{aligned} \frac{a^2}{1-a^2} &= C'' r \\ a^2(1+C''r) &= C'' r \\ a^2 &= \frac{C'' r}{1+C'' r} \\ \frac{1}{a^2} &= \frac{1+C'' r}{C'' r} \\ \frac{1}{a^2} &= 1 + \frac{1}{C'' r}. \end{aligned} \quad (40)$$

Finally, let $C = \frac{1}{C''}$ to obtain

$$\left(\frac{1}{a^2} \right) = 1 + \frac{C}{r}. \quad (41)$$

This gives us that the Schwarzschild metric is

$$g = - \left(1 + \frac{C}{r} \right) dt^2 + \left(1 + \frac{C}{r} \right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\varphi^2). \quad (42)$$

2.2 Schwarzschild Observer

The gravitational theory given by Newton holds up very well in the case where two masses were sufficiently far apart from one another and of a large size. Due to this, we expect that the gravitational acceleration for large values of distance, r , should reduce to the Newtonian case. To accomplish this, we will use what is called a Schwarzschild observer. This “observer” is placed within a stellar system, far away from the sun, and is assumed to be moving forward in time while remaining stationary in space. Thus it must be accelerating directly away from the star in this system. We will measure, using the metric, what the acceleration of this observer is, and compare this to the Newtonian case, which will give us a constraint on the constant C in the Schwarzschild metric.

We parameterize the motion based in the proper time of the observer, τ . Thus we have that the observer must be following the curve

$$\gamma(\tau) = (t(\tau), 0, 0, 0). \quad (43)$$

In order to find the acceleration of this curve, we define the unit tangent vector and then take the covariant derivative of the tangent vector in the direction of the curve. Let U be this unit tangent vector, then

$$U = \frac{1}{\sqrt{1 + \frac{C}{r}}} \partial_t. \quad (44)$$

To condense the computations ahead, let

$$q = \frac{1}{\sqrt{1 + \frac{C}{r}}}. \quad (45)$$

Now we will find the acceleration and set it equal to the expected Newtonian acceleration,

$$\nabla_U U = \frac{M}{r^2} \partial_r \quad (46)$$

where M is the mass of the star. Based on our definition of U and the properties of the derivative, we evaluate it as follows.

$$\begin{aligned} \nabla_U U &= \nabla_{q\partial_t} q\partial_t \\ \nabla_U U &= q\nabla_{\partial_t} q\partial_t \\ \nabla_U U &= q[\partial_t(q)\partial_t + q\nabla_{\partial_t}\partial_t]. \end{aligned} \quad (47)$$

Since q does not depend on t , we then have that

$$\nabla_U U = q^2 \nabla_{\partial_t} \partial_t. \quad (48)$$

We see from Appendix A that Γ_{tt}^k is zero unless $k = r$. Then we have that

$$\begin{aligned} \nabla_U U &= q^2 \Gamma_{tt}^k \partial_k \\ \nabla_U U &= \frac{1}{1 + \frac{C}{r}} \left[\frac{1}{2} g^{rr} \left(-\frac{\partial}{\partial r} g_{tt} \right) \right] \partial_r \\ \nabla_U U &= \frac{1}{1 + \frac{C}{r}} \left[\frac{1}{2} \left(1 + \frac{C}{r} \right) \left(\frac{-C}{r^2} \right) \right] \partial_r. \end{aligned} \quad (49)$$

This finally gives us the result that

$$\nabla_U U = \frac{-C}{2r^2} \partial_r. \quad (50)$$

Thus using equation (46) and equation (50), we find

$$\frac{-C}{2r^2} \partial_r = \frac{M}{r^2} \partial_r. \quad (51)$$

So we now find that

$$C = -2M. \quad (52)$$

Combining the metric, equation (42), with the constant, equation (52), we find that the Schwarzschild metric for a static, spherically symmetric star in vacuum is

$$g = - \left(1 - \frac{2M}{r} \right) dt^2 + \left(1 - \frac{2M}{r} \right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2(\theta) d\phi^2) \quad (53)$$

where M is the mass of the star.

2.3 Perihelion Advance of Mercury

When observing the orbits of the planets, it has been observed that their orbits precess due to gravitational pull of the other large bodies of the solar system [2]. For 7 of the planets, it has been found that Newtonian gravity explains almost exactly the precession of the orbit. However, when observing Mercury's orbit, it was found that the closest approach Mercury makes to the Sun, called the perihelion, advances by an extra 43 arc seconds per Earth century [2]. We will see that this discrepancy can be resolved by treating gravity as geodesic travel under the Schwarzschild metric.

We utilize the geodesic equation to find the geodesics under the Schwarzschild geometry. The geodesic equation (in our orthogonal coordinate system) [4] is

$$\frac{d}{ds} \left(g_{kk} \frac{dx^k}{ds} \right) = \frac{1}{2} \sum_{i=0}^3 \frac{\partial g_{ii}}{\partial x^k} \left(\frac{dx^i}{ds} \right)^2, \quad (54)$$

where our coordinate system has components $x^0 = t, x^1 = r, x^2 = \theta$, and $x^3 = \varphi$, and we identify the variable $s = \tau$ where τ is the proper time of an observer following a geodesic. When $k = 0$, we find

$$\frac{d}{d\tau} \left(g_{tt} \frac{dt}{d\tau} \right) = \frac{1}{2} \left[\frac{\partial g_{tt}}{\partial t} \left(\frac{dt}{d\tau} \right)^2 + \frac{\partial g_{rr}}{\partial t} \left(\frac{dr}{d\tau} \right)^2 + \frac{\partial g_{\theta\theta}}{\partial t} \left(\frac{d\theta}{d\tau} \right)^2 + \frac{\partial g_{\varphi\varphi}}{\partial t} \left(\frac{d\varphi}{d\tau} \right)^2 \right]. \quad (55)$$

However, from the Schwarzschild metric, we see that none of the components of the metric depend on t . Thus the right hand side is constant in t , and goes to 0. This gives us

$$\frac{d}{d\tau} \left[\left(1 - \frac{2M}{r} \right) \frac{dt}{d\tau} \right] = 0. \quad (56)$$

Integrating with respect to τ gives

$$\left(1 - \frac{2M}{r} \right) \frac{dt}{d\tau} = E \quad (57)$$

where E is the constant we get from integrating 0. We will obtain another 0 on the right hand side when we let $k = 3$, since none of the components of the metric depend on φ .

$$\begin{aligned} \frac{d}{d\tau} \left(g_{\varphi\varphi} \frac{d\varphi}{d\tau} \right) &= \frac{1}{2} \left[\frac{\partial g_{tt}}{\partial \varphi} \left(\frac{dt}{d\tau} \right)^2 + \frac{\partial g_{rr}}{\partial \varphi} \left(\frac{dr}{d\tau} \right)^2 + \frac{\partial g_{\theta\theta}}{\partial \varphi} \left(\frac{d\theta}{d\tau} \right)^2 + \frac{\partial g_{\varphi\varphi}}{\partial \varphi} \left(\frac{d\varphi}{d\tau} \right)^2 \right] \\ &= 0. \end{aligned} \quad (58)$$

If we integrate as before with respect to τ , we have

$$r^2 \sin^2(\theta) \frac{d\varphi}{d\tau} = L \quad (59)$$

where L is the constant we obtain from integration on the right. We identify E with the energy of the satellite and L with the angular momentum of the satellite. Finally, when $k = 2$, we find that

$$\begin{aligned} \frac{d}{d\tau} \left(g_{\theta\theta} \frac{d\theta}{d\tau} \right) &= \frac{1}{2} \left[\frac{\partial g_{tt}}{\partial \theta} \left(\frac{dt}{d\tau} \right)^2 + \frac{\partial g_{rr}}{\partial \theta} \left(\frac{dr}{d\tau} \right)^2 + \frac{\partial g_{\theta\theta}}{\partial \theta} \left(\frac{d\theta}{d\tau} \right)^2 + \frac{\partial g_{\varphi\varphi}}{\partial \theta} \left(\frac{d\varphi}{d\tau} \right)^2 \right] \\ 2r \frac{dr}{d\tau} \frac{d\theta}{d\tau} + r^2 \frac{d^2\theta}{d\tau^2} &= \frac{1}{2} \left[2r^2 \sin(\theta) \cos(\theta) \left(\frac{d\varphi}{d\tau} \right)^2 \right] \end{aligned} \quad (60)$$

Substituting in equation (59), we have

$$2r \frac{dr}{d\tau} \frac{d\theta}{d\tau} + r^2 \frac{d^2\theta}{d\tau^2} = r^2 \sin(\theta) \cos(\theta) \left(\frac{L}{r^2 \sin^2(\theta)} \right)^2$$

$$\frac{2}{r} \frac{dr}{d\tau} \frac{d\theta}{d\tau} + \frac{d^2\theta}{d\tau^2} = \frac{L^2 \cos(\theta)}{r^4 \sin^3(\theta)}. \quad (61)$$

We find another constraint on our system when we assume that the orbit is initially equatorial, i.e. for the function of angle with respect to time $\theta(t)$, $\theta(0) = \frac{\pi}{2}$ and $\frac{d\theta}{d\tau}(0) = 0$. From these conditions, we see that $\theta(\tau) = \frac{\pi}{2}$ is a solution. We know that this solution is unique, since this is an ordinary differential equation. Thus, we see that an orbit which is initially equatorial will remain equatorial. Therefore θ is constantly $\frac{\pi}{2}$. Evaluating equation (59) under this constraint gives us

$$r^2 \frac{d\varphi}{d\tau} = L. \quad (62)$$

Note that

$$\gamma' = \left(\frac{dt}{d\tau} \right) \partial_t + \left(\frac{dr}{d\tau} \right) \partial_r + \left(\frac{d\theta}{d\tau} \right) \partial_\theta + \left(\frac{d\varphi}{d\tau} \right) \partial_\varphi. \quad (63)$$

Under our constraints from equation (62), equation (57), and the constancy of θ , we have

$$\gamma' = \frac{E}{1 - \frac{2M}{r}} \partial_t + \frac{dr}{d\tau} \partial_r + \frac{L}{r^2} \partial_\varphi \quad (64)$$

This is a normalized timelike geodesic, therefore taking the inner product of this geodesic with itself we have

$$\langle \gamma, \gamma \rangle = -1 = \frac{E^2}{\left(1 - \frac{2M}{r}\right)^2} \left[- \left(1 - \frac{2M}{r}\right) \right] + \left(\frac{dr}{d\tau} \right)^2 \left(\frac{1}{\left(1 - \frac{2M}{r}\right)^2} \right) + \frac{L^2}{r^4} r^2. \quad (65)$$

So we obtain

$$E^2 = \left(\frac{dr}{d\tau} \right)^2 + \left(1 - \frac{2M}{r}\right) \left(1 + \frac{L^2}{r^2}\right) \quad (66)$$

From here, we must find what $\frac{dr}{d\tau}$ is in terms of φ . By doing this, we will be able to find r as a function of φ , which is the angle in the plane of Mercury's orbit. We accomplish this via the chain rule. This gives us the transformation

$$\begin{aligned} \frac{dr}{d\tau} &= \frac{dr}{d\varphi} \frac{d\varphi}{d\tau} \\ &= \frac{dr}{d\varphi} \frac{L}{r^2} \end{aligned} \quad (67)$$

Substituting this into equation (66), we have

$$E^2 = L^2 \left(\frac{1}{r^2} \frac{dr}{d\varphi} \right)^2 + \left(1 - \frac{2M}{r}\right) \left(1 + \frac{L^2}{r^2}\right). \quad (68)$$

Substituting $u = \frac{1}{r}$, we arrive at

$$E^2 = L^2 \left(\frac{du}{d\varphi} \right)^2 + (1 - 2Mu) (1 + L^2 u^2). \quad (69)$$

Differentiating both sides with respect to u , we have

$$0 = L^2 \frac{d^2 u}{d\varphi^2} + L^2 u - 2L^2 M u^2 - M - M L^2 u^2. \quad (70)$$

After some manipulation, we arrive at

$$\frac{d^2 u}{d\varphi^2} + u = \frac{M}{L^2} + 3M u^2. \quad (71)$$

This orbital equation differs from the classical orbit equation by the term $3M u^2$ [4]. If we solve the classical case of the orbital equation, we obtain the solution v as

$$v = \left(\frac{M}{L^2} \right) (1 + \varepsilon \cos(\varphi)) \quad (72)$$

where ε is the eccentricity of the orbit and perihelion occurs at $\varphi = 0$. We know that $\varphi = 0$ is a critical point of the orbit, and hence the term $\sin(\varphi)$ of the homogeneous solution must go away. Furthermore, since $\varphi = 0$ is a perihelion and hence a minimum of r , $\varphi = 0$ is a maximum of v , so $\varepsilon > 0$. Using this solution in the correction term gives us the differential equation

$$\frac{d^2 u}{d\varphi^2} + u = \frac{M}{L^2} + 3M v^2. \quad (73)$$

We solve equation (73) in Appendix A and find

$$u = \frac{M}{L^2} (1 + \varepsilon \cos(\varphi)) + \frac{3M^3}{L^4} \left(1 + \varepsilon \varphi \sin(\varphi) + \frac{\varepsilon^2}{2} - \frac{\varepsilon^2}{6} \cos(2\varphi) \right) \quad (74)$$

Differentiating equation (74) gives us

$$\frac{du}{d\varphi} = -\frac{M\varepsilon}{L^2} \sin(\varphi) + \frac{3M^3}{L^4} \left(\frac{\varepsilon^2}{3} \sin(2\varphi) + \varepsilon \varphi \cos(\varphi) + \varepsilon \sin(\varphi) \right). \quad (75)$$

Setting this derivative to 0 gives us a critical point of the orbit, e.g. the perihelion. The next perihelion will occur at $\varphi = 2\pi + \delta$, where δ is the small angle we want to find. We approximate this using the dominant terms, $-\frac{M\varepsilon}{L^2} \sin(\varphi)$ and $\frac{3M^3}{L^4} \varepsilon \varphi \cos(\varphi)$, as follows

$$0 \approx -\frac{M\varepsilon}{L^2} \sin(2\pi + \delta) + \frac{3M^3 \varepsilon}{L^4} (2\pi + \delta) \cos(2\pi + \delta). \quad (76)$$

Since \sin and \cos are 2π -periodic functions, we obtain that

$$\frac{\sin(\delta)}{\cos(\delta)} = \frac{3M^2}{L^2} (2\pi + \delta). \quad (77)$$

Using the small angle approximation that $\tan(\delta) \approx \delta$, we find

$$\delta \approx \frac{6\pi M^2}{L^2 - 3M^2}. \quad (78)$$

Using the equation $Ma(1 - \varepsilon^2) = L^2$, which is derived in the appendix of [4] in order to relate the angular momentum to the eccentricity, ε , and the semi-major axis, a , we finally obtain

$$\delta \approx \frac{6\pi M}{a(1 - \varepsilon^2) - 3M}. \quad (79)$$

In our normalized units, we can express mass values as distances. Dimensionally, Newton's gravitational constant, G , has units of $\text{cm}^3\text{g}^{-1}\text{s}^{-2}$, and the speed of light has units of cm s^{-1} . Explicitly, throughout this paper we have set

$$\frac{G}{c^2} = 1. \quad (80)$$

From this, we find a conversion factor as follows

$$\begin{aligned} \frac{6.67 \times 10^{-8} \text{ cm}^3\text{g}^{-1}\text{s}^{-2}}{(2.998 \times 10^{10})^2 \text{ cm}^2 \text{ s}^{-2}} &= 1 \\ 7.43 \times 10^{-29} \text{ cm g}^{-1} &= 1 \\ 1 \text{ cm} &= 1.346 \times 10^{28} \text{ g}. \end{aligned} \quad (81)$$

Since we have that mass is equivalent to distance up to this constant, we can convert the mass of the sun to centimeters [4]

$$\begin{aligned} M &= \frac{1.99 \times 10^{33} \text{ g}}{1.346 \times 10^{28} \text{ g cm}^{-1}} \\ M &= 1.48 \times 10^5 \text{ cm}. \end{aligned} \quad (82)$$

As such we have the values required to calculate the perihelion advance

$$\begin{aligned} M &= 1.48 \times 10^5 \text{ cm} \\ \varepsilon &= 0.206 \\ a &= 5.79 \times 10^{12} \text{ cm} \end{aligned} \quad (83)$$

where M is the mass of the Sun, ε is the eccentricity of the orbit of Mercury, and a is the semi-major axis of the orbit of Mercury [4]. This gives us

$$\delta \approx 5.03 \times 10^{-7} \text{ rad/ revolution}. \quad (84)$$

Given approximately 2.06×10^5 arcseconds per revolution and 4.152 revolutions per Earth-year for Mercury, we have the perihelion advance per century (= 100 years) as

$$\delta \approx 43 \text{ arcseconds/Earth century}. \quad (85)$$

This relativistic correction matches the missing angle that was observed in the perihelion advance of Mercury [2].

This calculation is some of the first evidence that Einstein's theory of General Relativity is correct at the scale of our universe. Accompanying evidence included the observation of light "bending" around the Sun during a solar eclipse, which was also due to the geodesic travel of objects through the universe.

3 FLRW Cosmology

Friedmann-Lemaître-Robertson-Walker (FLRW) Cosmology gives us a model for the evolution of the universe as a whole. By looking at the sky, we notice a few things with respect to galaxies or galaxy clusters. We see that it does not matter what direction we point our telescopes – we still see approximately the same distribution of galaxy clusters. In addition, the galaxy clusters are seemingly distributed uniformly throughout our visible universe. So there is no preferred vector in our universe along which more galactic clusters form, and there is a uniform distribution of the galaxy clusters.

These properties, called respectively isotropy and homogeneity, give us restrictions on the cosmology of our universe. First, isotropy allows us to conclude that the universe is symmetric with respect to any vector in space. Also, it is shown in [9] that isotropy requires that the Riemann curvature has the form

$$R_{abcd} = Kh_{c[a}h_{b]d}, \quad (86)$$

where K is some function, h_{ij} is the spatial metric, and the notation $A_{[ij]}$ is the antisymmetrization of A_{ij} . Using the Second Bianchi Identity,

$$\nabla_{[e}R_{ab]cd} = 0 \quad (87)$$

and substituting equation (86), we have [9]

$$(\nabla_{[e}K)h_{|c|a}h_{b]d} = 0. \quad (88)$$

By permuting, we find

$$\begin{aligned} (\nabla_{[e}K)h_{|c|a}h_{b]d} &= (\nabla_{[a}K)h_{|c|b}h_{e]d} \\ &= (\nabla_{[b}K)h_{|c|e}h_{a]d} \\ &= -(\nabla_{[e}K)h_{|c|b}h_{a]d} \\ &= -(\nabla_{[a}K)h_{|c|e}h_{b]d} \\ &= -(\nabla_{[b}K)h_{|c|a}h_{e]d}. \end{aligned} \quad (89)$$

We can construct a normal coordinate system which has orthonormal basis vectors at a point. Since this holds at any point, we can require that all of the off-diagonal h_{ij} are zero at a point p . Thus for the left hand side to be nonzero at p , we must have $c = a$ and $d = b$. When we apply this, all of the right hand side terms vanish. This leaves us with

$$(\nabla_{[e}K)h_{|a|a}h_{b]b} = 0. \quad (90)$$

Which has as its only nontrivial element

$$(\nabla_e K)h_{aa}h_{bb} = 0. \quad (91)$$

Since $h_{aa} \neq 0$ and $h_{bb} \neq 0$, we find

$$\nabla_e K = 0. \quad (92)$$

at p . But this can be done at any point, so $\nabla_e K \equiv 0$. This implies that K is a constant. It has been shown that two spaces of constant curvature having the same dimension, metric signature, and K value are locally isometric [9]. The values when $K > 0$ are realized by 3-spheres, when $K = 0$

are realized by flat space, and when $K < 0$ are realized by 3-hyperboloids. Thus we can restrict the metrics we look at to these. This can be simplified even further. For a parameterization of a 3-sphere,

$$ds^2 = r^2 [d\psi^2 + \sin^2(\psi) (d\theta^2 + \sin^2(\theta)d\varphi^2)] \quad (93)$$

we can factor out the r^2 and absorb it into the $a^2(t)$ term which we use below. For a parameterization of a 3-hyperboloid,

$$ds^2 = r^2 [d\psi^2 + \sinh^2(\psi) (d\theta^2 + \sin^2(\theta)d\varphi^2)], \quad (94)$$

we can factor out the r^2 term once again and absorb it into the $a^2(t)$ coefficient. Thus we only have to worry about the sign of the curvature (without the magnitude). This leaves us with the three cases of curvature sign: $-1, 0$ and 1 . These correspond to hyperbolic space, flat space, and spherical space, respectively. Thus the metric on this space has three possibilities:

$$g = -dt^2 + a^2(t) \begin{cases} d\psi^2 + \sinh^2(\psi) [d\theta^2 + \sin^2(\theta)d\varphi^2] \\ dx^2 + dy^2 + dz^2 \\ d\psi^2 + \sin^2(\psi) [d\theta^2 + \sin^2(\theta)d\varphi^2]. \end{cases} \quad (95)$$

Following from the homogeneity and isotropy, a natural choice of model is a perfect fluid, composed of particles that are the galaxy clusters mentioned above, to model the system. The perfect fluid model gives us the form of our stress-energy tensor,

$$T_{ij} = \rho u_i u_j + P(g_{ij} + u_i u_j), \quad (96)$$

where $u^i \partial_i = \partial_t$ is the tangent vector of the world lines of one of the galaxies. This makes t the proper time as measured by the particles of our fluid (i.e. galaxy clusters). From this identification, we see that $u_i = -dt$, and thus that $u_i u_j = dt^2$. Note that we will express the total metric, g_{ij} , as a sum of the spatial and temporal pieces separately, i.e.

$$g_{ij} = -u_i u_j + h_{ij}. \quad (97)$$

Evaluating equation (96) under this condition, we have

$$T_{ij} = \rho u_i u_j + P h_{ij}. \quad (98)$$

This tells us that the Einstein equation separates over time and space, equation (99) and equation (100) respectively, for the FLRW conditions:

$$\text{Ric}_{tt} - \frac{1}{2} S g_{tt} = 8\pi\rho \quad (99)$$

$$\text{Ric}_{ij} - \frac{1}{2} S g_{ij} = 8\pi P h_{ij}. \quad (100)$$

For flat space, when $K = 0$, we find that

$$\begin{aligned} \text{Ric}_{tt} &= -\frac{3\ddot{a}}{a} \\ \text{Ric}_{xx} = \text{Ric}_{yy} = \text{Ric}_{zz} &= 2\dot{a}^2 + a\ddot{a}. \end{aligned} \quad (101)$$

Since the scalar curvature is the sum over these, we obtain

$$\begin{aligned} S &= -\left(-\frac{3\ddot{a}}{a}\right) + \frac{1}{a^2} (3(2\dot{a}^2 + a\ddot{a})) \\ &= 6\frac{\ddot{a}}{a} + 6\frac{\dot{a}^2}{a^2}. \end{aligned} \tag{102}$$

Using this result, the temporal Einstein equation is

$$\begin{aligned} G_{tt} &= \text{Ric}_{tt} - \frac{1}{2}Sg_{tt} \\ &= -\frac{3\ddot{a}}{a} + 3\frac{\ddot{a}}{a} + 3\frac{\dot{a}^2}{a^2} \\ &= 3\frac{\dot{a}^2}{a^2} = 8\pi\rho. \end{aligned} \tag{103}$$

And the spatial Einstein equation is, for $i \neq t$,

$$\begin{aligned} G_{ii} &= \text{Ric}_{ii} - \frac{1}{2}Sg_{ii} \\ &= \text{Ric}_{ii} - \frac{1}{2}Sa^2 \\ &= 2\dot{a}^2 + a\ddot{a} - 3a\ddot{a} - 3\dot{a}^2 \\ &= -\dot{a}^2 - 2a\ddot{a} \end{aligned} \tag{104}$$

Since $h_{ii} = a^2$, we have

$$\begin{aligned} -h_{ii} \left(\frac{\dot{a}^2}{a^2} + 2\frac{\ddot{a}}{a} \right) &= 8\pi Ph_{ii} \\ -\frac{\dot{a}^2}{a^2} - 2\frac{\ddot{a}}{a} &= 8\pi P \end{aligned} \tag{105}$$

From equation (103), we see equation (105) becomes

$$\frac{\ddot{a}}{a} = -\frac{4}{3}\pi(\rho + 3P). \tag{106}$$

The generalized versions of these equations are given in Appendix B, equation (172) and equation (173), reproduced here for curvature sign k (where $k = 0$ is flat, $k = 1$ is spherical, and $k = -1$ is hyperbolic),

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{8}{3}\pi\rho \tag{107}$$

$$\frac{\ddot{a}}{a} = -\frac{4}{3}\pi(\rho + 3P). \tag{108}$$

Since ρ is the density of the dust in space and P is the pressure, we know that any sum of the two must remain nonnegative at all times. Therefore, the right hand side of this equation is always negative. From this, we know the graph of the expansion coefficient versus time must be concave down. Furthermore, from equation (107), we see that \dot{a} is positive in the flat case ($k = 0$) and the

hyperboloid case ($k = -1$). Thus the zero of the tangent line to the curve is an upper bound for the time t at which $a(t) = 0$ in these cases.

To accomplish this estimation, suppose we are given two isotropic observers who are a distance r apart from one another, and we will analyze the rate of change of r . Hence

$$\begin{aligned} v &:= \frac{dr}{dt} \\ &= \frac{dr}{da} \frac{da}{dt}. \end{aligned} \tag{109}$$

Letting $r^2 = a^2 r_0^2$ for some constant r_0 , we have that

$$\begin{aligned} a^2 r_0^2 &= r^2 \\ 2ar_0^2 &= 2r \frac{dr}{da} \\ \frac{ar_0^2}{r} &= \frac{dr}{da} \\ \frac{a r^2}{r a^2} &= \frac{dr}{da} \\ \frac{r}{a} &= \frac{dr}{da}. \end{aligned} \tag{110}$$

Thus we have

$$v = \frac{r}{a} \frac{da}{dt} = r \frac{\dot{a}}{a}. \tag{111}$$

We call the value $\frac{\dot{a}}{a}$ in equation (111) Hubble's constant, H . This constant can be experimentally obtained via observations of the rate of expansion of the universe relative to distance between two galaxies in the universe. A recent such measurement [1] has given us the "best fit" value that

$$H = 68.14 \text{ km s}^{-1} \text{ Mpc}^{-1}. \tag{112}$$

Following from equation (111), we see that

$$\frac{v}{r} = H. \tag{113}$$

We can approximate a linearly by

$$a \approx a(0) + \dot{a}(0)t. \tag{114}$$

To find the zero of this line, we have

$$\begin{aligned} 0 &= a(0) + \dot{a}(0)t \\ -\dot{a}(0)t &= a(0) \\ t &= -\frac{a(0)}{\dot{a}(0)}. \end{aligned} \tag{115}$$

And from equation (111), we know then that the initial time is

$$t = -\frac{a(0)}{\dot{a}(0)} = -\frac{r}{v}. \tag{116}$$

Thus based on our model, we estimate an upper bound for the amount of time which has elapsed:

$$\frac{r}{v} = t_{\text{universe}} \leq \frac{1}{H} = 4.528 \times 10^{17} \text{seconds.} \quad (117)$$

Converting this value to Earth years gives us the upper bound on the age of the universe as

$$t_{\text{universe}} \leq 14.36 \text{ billion years.} \quad (118)$$

4 Developments

Recently, experiments have been published, such as [6], [7], and [8], which suggest that the Einstein equation should actually be

$$G_{ij} + \Lambda g_{ij} = 8\pi T_{ij} \quad (119)$$

where Λ is called the ‘‘cosmological constant’’. This constant would be a very small, positive parameter, and hence would only contribute to intergalactic interactions of bodies in the universe. Furthermore, adding this constant to the Einstein equation accounts for the accelerating expansion of the universe as observed in [6], [7], and [8].

Adding this modification to our theory in the flat case, we find for the temporal Einstein equation that

$$\begin{aligned} G_{tt} + \Lambda g_{tt} &= \text{Ric}_{tt} + \frac{1}{2}S + \Lambda g_{tt} \\ &= 3\frac{\dot{a}^2}{a^2} - \Lambda = 8\pi\rho \end{aligned} \quad (120)$$

And for the spatial Einstein equation we have

$$\begin{aligned} G_{ii} + \Lambda g_{ii} &= \frac{g_{ii}}{a^2} (-\dot{a}^2 - 2a\ddot{a}) + \Lambda g_{ii} \\ &= \frac{g_{ii}}{a^2} (-\dot{a}^2 - 2a\ddot{a} + \Lambda a^2) \\ &= 8\pi P h_{ii}. \end{aligned} \quad (121)$$

Applying equation (120), with $i \neq t$, we have

$$g_{ii} \left(-\frac{8}{3}\pi\rho - \frac{\Lambda}{3} - 2\frac{\ddot{a}}{a} + \Lambda \right) = 8\pi P h_{ii}. \quad (122)$$

Which reduces to

$$\frac{\ddot{a}}{a} = -\frac{4}{3}\pi(\rho + 3P) + \frac{\Lambda}{3}. \quad (123)$$

Previously, in equation (108), we were able to conclude that the curve for our parameter a was concave down, allowing our estimation of an upper bound for the age of the universe. However, in equation (123), we see that if Λ is sufficiently large, the right hand side is not necessarily negative. This means the graph could have any concavity. Specifically, given enough time, ρ and P will become small enough that Λ dominates. As such we cannot conclude that our method for bounding the universe age was correct under the condition of an accelerating universe.

Appendix A

The calculations required for section 2 are produced here. The necessary objects used in deriving the Schwarzschild metric are given first, followed by the computations required for calculating the perihelion advance of Mercury's orbit.

Ricci Curvature

The computations to obtain the Ricci curvature components in the Schwarzschild metric are given here. We will replace t_0 with t throughout this computation, since the notation is not crucial for this part. Note that for $i \neq j$, the terms $g_{ij} = 0$. This is straightforward from the metric,

$$ds^2 = -\alpha^2(r)dt^2 + a^2(r)dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\varphi^2), \quad (124)$$

since none of the components are coupled (e.g. $dt d\theta$, $dr d\varphi$). Therefore the only remaining Christoffel symbols are those where for Γ_{ij}^k , i, j , and k are equal pairwise. Additionally, the only Christoffel symbol which survives under $i = j = k$ is Γ_{rr}^r , since no other metric component g_{ii} depends on x^i other than when $i = r$. We can now evaluate the Christoffel symbols.

$$\begin{aligned} \Gamma_{rt}^t = \Gamma_{tr}^t &= \frac{1}{2}g^{tt} \left[\frac{\partial g_{tr}}{\partial t} + \frac{\partial g_{tt}}{\partial r} - \frac{\partial g_{rt}}{\partial t} \right] \\ &= \frac{1}{2} \left(-\frac{1}{\alpha^2} \right) \left[0 - \frac{\partial}{\partial r}(\alpha^2) - 0 \right] \\ &= -\frac{1}{2\alpha^2} [-2\alpha\alpha'] \\ &= \frac{\alpha'}{\alpha}. \end{aligned} \quad (125)$$

Similarly, we find

$$\begin{aligned} \Gamma_{rr}^r &= \frac{a'}{a} & \Gamma_{tt}^r &= \frac{\alpha\alpha'}{a^2} & \Gamma_{\theta\theta}^r &= -\frac{r}{a^2} \\ \Gamma_{\varphi\varphi}^r &= -\frac{r \sin^2(\theta)}{a^2} & \Gamma_{\theta r}^\theta &= \Gamma_{r\theta}^\theta = \frac{1}{r} & \Gamma_{\varphi\varphi}^\theta &= -\sin(\theta) \cos(\theta) \\ \Gamma_{\varphi r}^\varphi &= \Gamma_{r\varphi}^\varphi = \frac{1}{r} & \Gamma_{\varphi\theta}^\varphi &= \Gamma_{\theta\varphi}^\varphi = \cot(\theta). \end{aligned}$$

By equation (16), we find that the Ricci components are

$$\begin{aligned} \text{Ric}_{tt} &= \Gamma_{tt,n}^n - \Gamma_{nt,t}^n + \Gamma_{nm}^n \Gamma_{tt}^m - \Gamma_{tm}^n \Gamma_{nt}^m \\ &= \Gamma_{tt,r}^r + \Gamma_{rr}^r \Gamma_{tt}^r + \Gamma_{\theta r}^\theta \Gamma_{tt}^r + \Gamma_{\varphi r}^\varphi \Gamma_{tt}^r - \Gamma_{tt}^r \Gamma_{tr}^t \\ &= \frac{\partial}{\partial r} \left(\frac{\alpha\alpha'}{a^2} \right) + \left(\frac{a'}{a} \right) \left(\frac{\alpha\alpha'}{a^2} \right) + \left(\frac{1}{r} \right) \left(\frac{\alpha\alpha'}{a^2} \right) \\ &\quad + \left(\frac{1}{r} \right) \left(\frac{\alpha\alpha'}{a^2} \right) - \left(\frac{\alpha\alpha'}{a^2} \right) \left(\frac{\alpha'}{\alpha} \right) \\ &= \frac{\alpha\alpha'' + \alpha'^2}{a^2} - \frac{\alpha\alpha'a'}{a^3} + 2\frac{\alpha\alpha'}{ra^2} - \frac{\alpha'^2}{a^2} \\ &= -\frac{\alpha\alpha'a'}{a^3} + \frac{\alpha\alpha''}{a^2} + \frac{2\alpha\alpha'}{ra^2} \end{aligned} \quad (126)$$

$$\begin{aligned}
\text{Ric}_{rr} &= \Gamma_{rr,n}^n - \Gamma_{nr,r}^n + \Gamma_{nm}^n \Gamma_{rr}^m - \Gamma_{rm}^n \Gamma_{nr}^m \\
&= -\Gamma_{tr,r}^t - \Gamma_{\theta r,r}^\theta - \Gamma_{\varphi r,r}^\varphi + \Gamma_{tr}^t \Gamma_{rr}^r + \Gamma_{\theta r}^\theta \Gamma_{rr}^r \\
&\quad + \Gamma_{\varphi r}^\varphi \Gamma_{rr}^r - \Gamma_{rt}^t \Gamma_{rt}^t - \Gamma_{r\theta}^\theta \Gamma_{r\theta}^\theta - \Gamma_{r\varphi}^\varphi \Gamma_{r\varphi}^\varphi \\
&= -\frac{\partial}{\partial r} \left(\frac{\alpha'}{\alpha} \right) - \frac{\partial}{\partial r} \left(\frac{1}{r} \right) - \frac{\partial}{\partial r} \left(\frac{1}{r} \right) \\
&\quad + \left(\frac{\alpha'}{\alpha} \right) \left(\frac{a'}{a} \right) + \left(\frac{1}{r} \right) \left(\frac{a'}{a} \right) + \left(\frac{1}{r} \right) \left(\frac{a'}{a} \right) \\
&\quad - \left(\frac{\alpha'}{\alpha} \right) \left(\frac{\alpha'}{\alpha} \right) - \left(\frac{1}{r} \right) \left(\frac{1}{r} \right) - \left(\frac{1}{r} \right) \left(\frac{1}{r} \right) \\
&= -\frac{\alpha''}{\alpha} + \frac{2}{r^2} + \frac{\alpha' a'}{\alpha a} + \frac{2a'}{ra} - \frac{2}{r^2} \\
&= -\frac{\alpha''}{\alpha} + \frac{\alpha' a'}{\alpha a} + \frac{2a'}{ra}
\end{aligned} \tag{127}$$

$$\begin{aligned}
\text{Ric}_{\theta\theta} &= \Gamma_{\theta\theta,n}^n - \Gamma_{n\theta,\theta}^n + \Gamma_{nm}^n \Gamma_{\theta\theta}^m - \Gamma_{\theta m}^n \Gamma_{n\theta}^m \\
&= \Gamma_{\theta\theta,r}^r - \Gamma_{\varphi\theta,\theta}^\varphi + \Gamma_{tr}^t \Gamma_{\theta\theta}^r + \Gamma_{rr}^r \Gamma_{\theta\theta}^r \\
&\quad + \Gamma_{\varphi r}^\varphi \Gamma_{\theta\theta}^r - \Gamma_{\theta\theta}^r \Gamma_{\theta r}^\theta - \Gamma_{\theta\varphi}^\varphi \Gamma_{\theta\varphi}^\varphi
\end{aligned}$$

$$\begin{aligned}
&= \frac{\partial}{\partial r} \left(-\frac{r}{a^2} \right) - \frac{\partial}{\partial \theta} (\cot(\theta)) + \left(\frac{\alpha'}{\alpha} \right) \left(-\frac{r}{a^2} \right) \\
&\quad + \left(\frac{a'}{a} \right) \left(-\frac{r}{a^2} \right) + \left(\frac{1}{r} \right) \left(-\frac{r}{a^2} \right) \\
&\quad - \left(-\frac{r}{a^2} \right) \left(\frac{1}{r} \right) - \cot^2(\theta) \\
\text{Ric}_{\theta\theta} &= 1 - \frac{1}{a^2} + \frac{ra'}{a^3} - \frac{r\alpha'}{a^2\alpha}. \tag{128}
\end{aligned}$$

By an almost identical computation to $\text{Ric}_{\theta\theta}$, we find

$$\text{Ric}_{\varphi\varphi} = \sin^2(\theta) \text{Ric}_{\theta\theta}. \tag{129}$$

These give us our values for equation (27), equation (28), and equation (29), respectively (up to the change of $t \rightarrow t_0$).

Since each component must be equal to zero if the total Ricci tensor is zero, we can multiply each of these by some factor. Thus, multiplying Ric_{tt} by $-\frac{a^2}{\alpha^2}$, Ric_{rr} by -1 , and $\text{Ric}_{\theta\theta}$ by $-a^2$, we have the system of equations as

$$0 = \frac{\alpha'a'}{\alpha a} - \frac{\alpha''}{\alpha} - \frac{2\alpha'}{r\alpha} \tag{130}$$

$$0 = \frac{\alpha''}{\alpha} - \frac{\alpha'a'}{\alpha a} - \frac{2a'}{ra} \tag{131}$$

$$0 = 1 - a^2 - \frac{ra'}{a} + \frac{r\alpha'}{\alpha}. \tag{132}$$

Perihelion Advance

When solving for the perihelion advance of Mercury, we arrived at equation (73), reproduced here

$$\frac{d^2 u}{d\varphi^2} + u = \frac{M}{L^2} + 3Mv^2, \tag{133}$$

where $v = \left(\frac{M}{L^2}\right) (1 + \varepsilon \cos(\varphi))$. If we substitute in this identity, we obtain

$$\frac{d^2 u}{d\varphi^2} + u = \frac{M}{L^2} + \frac{3M^3}{L^4} + \frac{6M^3}{L^4} \varepsilon \cos(\varphi) + \frac{3M^3}{L^4} \varepsilon^2 \cos(\varphi)^2. \tag{134}$$

First, we will separate the right hand side into three parts,

$$\begin{aligned}
f_1(\varphi) &:= \frac{M}{L^2} \\
f_2(\varphi) &:= \frac{3M^3}{L^4} + \frac{6M^3}{L^4} \varepsilon \cos(\varphi) + \frac{3M^3}{L^4} \varepsilon^2 \cos(\varphi)^2.
\end{aligned} \tag{135}$$

A solution satisfying the equation with forcing function $f_1(\varphi)$ is the solution from classical physics, i.e.

$$u_1 = \frac{M}{L^2} (1 + \varepsilon \cos(\varphi)). \tag{136}$$

From here on, we solve using the method of undetermined coefficients, where we split up each of the terms in the forcing function ($f_2(\varphi)$). First, we have

$$\frac{d^2u}{d\varphi^2} + u = \frac{3M^3}{L^4}. \quad (137)$$

A particular solution for this has the form

$$y_1(\varphi) = a \quad (138)$$

where a is a constant. Next, we solve

$$\frac{d^2u}{d\varphi^2} + u = \frac{6M^3}{L^4}\varepsilon \cos(\varphi). \quad (139)$$

For this, we guess that the solution is of the order $b \cos(\varphi) + c \sin(\varphi)$, and we multiply this by φ to account for the factors of $\cos(\varphi)$ and $\sin(\varphi)$ in the homogeneous solution. So, we obtain

$$y_2(\varphi) = \varphi(b \cos(\varphi) + c \sin(\varphi)) \quad (140)$$

where b and c are constants. Finally we have

$$\frac{d^2u}{d\varphi^2} + u = \frac{3M^3}{L^4}\varepsilon^2 \cos(\varphi)^2. \quad (141)$$

Using the identity $\cos^2(\varphi) = \frac{1}{2} + \frac{\cos(2\varphi)}{2}$, we obtain

$$\frac{d^2u}{d\varphi^2} + u = \frac{3M^3}{2L^4}\varepsilon^2 + \frac{3M^3}{2L^4}\varepsilon^2 \cos(2\varphi). \quad (142)$$

We guess that our solution has the form

$$y_3(\varphi) = d + e \cos(2\varphi) + f \sin(2\varphi). \quad (143)$$

Combining these solutions into one, we have

$$y = y_1 + y_2 + y_3 = a + b\varphi \cos(\varphi) + c\varphi \sin(\varphi) + d + e \cos(2\varphi) + f \sin(2\varphi). \quad (144)$$

Using our differential equation, $\frac{d^2y}{d\varphi^2} + y = f_2(\varphi)$, we have

$$\begin{aligned} & a + b\varphi \cos(\varphi) + c\varphi \sin(\varphi) - b\varphi \cos(\varphi) - b \sin(\varphi) - b \sin(\varphi) - c\varphi \sin(\varphi) + c \cos(\varphi) \\ & + c \cos(\varphi) + d + e \cos(2\varphi) + f \sin(2\varphi) - 4e \cos(2\varphi) - 4f \sin(2\varphi) \\ & = \frac{3M^3}{L^4} + \frac{6M^3}{L^4}\varepsilon \cos(\varphi) + \frac{3M^3}{2L^4}\varepsilon^2 + \frac{3M^3}{2L^4}\varepsilon^2 \cos(2\varphi) \end{aligned} \quad (145)$$

Simplifying, we obtain

$$\begin{aligned} & a - 2b \sin(\varphi) + 2c \cos(\varphi) + d - 3e \cos(2\varphi) - 3f \sin(2\varphi) \\ & = \frac{3M^3}{L^4} + \frac{6M^3}{L^4}\varepsilon \cos(\varphi) + \frac{3M^3}{2L^4}\varepsilon^2 + \frac{3M^3}{2L^4}\varepsilon^2 \cos(2\varphi). \end{aligned} \quad (146)$$

From here, we group together like terms to get a system of equations,

$$\begin{aligned}
a &= \frac{3M^3}{L^4} \\
-2b &= 0 \\
2c &= \frac{6M^3}{L^4} \varepsilon \\
d &= \frac{3M^3}{2L^4} \varepsilon^2 \\
-3e &= \frac{3M^3}{2L^4} \varepsilon^2 \\
-3f &= 0.
\end{aligned} \tag{147}$$

This system reduces to

$$\begin{aligned}
a &= \frac{3M^3}{L^4} \\
b &= 0 \\
c &= \frac{3M^3}{L^4} \varepsilon \\
d &= \frac{3M^3}{2L^4} \varepsilon^2 \\
e &= -\frac{M^3}{2L^4} \varepsilon^2 \\
f &= 0.
\end{aligned} \tag{148}$$

Thus our solution for $f_2(\varphi)$ is

$$u_2(\varphi) = \frac{3M^3}{L^4} + \frac{3M^3}{L^4} \varepsilon \varphi \sin(\varphi) + \frac{3M^3}{2L^4} \varepsilon^2 - \frac{M^3}{2L^4} \varepsilon^2 \cos(2\varphi) \tag{149}$$

This gives us our total solution as

$$u = \frac{M}{L^2} (1 + \varepsilon \cos(\varphi)) + \frac{3M^3}{L^4} \left(1 + \varepsilon \varphi \sin(\varphi) + \frac{\varepsilon^2}{2} - \frac{\varepsilon^2}{6} \cos(2\varphi) \right), \tag{150}$$

which is equation (74).

Appendix B

The spherical and hyperbolic FLRW metrics are analyzed under the conditions of the Einstein equation. We use the formulation of the Christoffel symbols and the Ricci tensor in order to obtain an equation for the parameter $a(t)$.

FLRW Metric on a 3-Sphere

The spherical FLRW metric, when the curvature is $k = 1$, is

$$ds^2 = -dt^2 + a^2(t) [d\psi^2 + \sin^2(\psi) (d\theta^2 + \sin^2(\theta)d\varphi^2)]. \quad (151)$$

Computing the Christoffel symbols, by equation (10), we first remark that any term g^{ij} where $i \neq j$ is identically 0 in this metric. Continuing, for Γ_{jk}^i with i, j , and k distinct we find that the metric components would be mixed, i.e. we would compute derivatives of the form $\frac{\partial g_{ij}}{\partial x^k}$. However, for $i \neq j$, $g_{ij} \equiv 0$. Thus any Christoffel symbol for which i, j , and k are distinct is taken to be 0. Furthermore, any Christoffel symbol for which $j = k$ and x^j, x^k are independent of x^i (i.e. $\frac{\partial}{\partial x^i} g_{jj} = 0$) can be taken to be 0. Computationally, we see

$$\begin{aligned} \Gamma_{jj}^i &= \frac{1}{2} g^{ii} \left[\frac{\partial g_{ij}}{\partial x^j} + \frac{\partial g_{ij}}{\partial x^j} - \frac{\partial g_{jj}}{\partial x^i} \right] \\ &= \frac{1}{2} g^{ii} \left[-\frac{\partial g_{jj}}{\partial x^i} \right] \\ &= 0. \end{aligned} \quad (152)$$

Thus

$$\Gamma_{tt}^t = \Gamma_{\psi\psi}^\psi = \Gamma_{\theta\theta}^\theta = \Gamma_{\varphi\varphi}^\varphi = \Gamma_{tt}^\psi = \Gamma_{tt}^\theta = \Gamma_{tt}^\varphi = \Gamma_{\psi\psi}^\theta = \Gamma_{\psi\psi}^\varphi = \Gamma_{\theta\theta}^\varphi = 0.$$

Finally, for Γ_{ij}^i , if the metric component g_{ii} is independent of the coordinate corresponding to the index j (x^j), we get 0 by a similar argument. Thus

$$\Gamma_{t\psi}^t = \Gamma_{t\theta}^t = \Gamma_{t\varphi}^t = \Gamma_{\psi\theta}^\psi = \Gamma_{\psi\varphi}^\psi = \Gamma_{\theta\varphi}^\theta = 0.$$

This leaves us with 12 nonzero Christoffel symbols, which we will now evaluate.

$$\begin{aligned} \Gamma_{\psi\psi}^t &= \frac{1}{2} g^{tt} \left[\frac{\partial g_{t\psi}}{\partial \psi} + \frac{\partial g_{\psi t}}{\partial \psi} - \frac{\partial g_{\varphi\varphi}}{\partial t} \right] \\ &= -\frac{1}{2} \left[0 + 0 - \frac{\partial}{\partial t} (a^2(t)) \right] \\ &= \frac{1}{2} [2a\dot{a}] \\ &= a\dot{a}. \end{aligned} \quad (153)$$

Similarly, we find

$$\begin{aligned} \Gamma_{\theta\theta}^t &= a\dot{a} \sin^2(\psi), & \Gamma_{\varphi\varphi}^t &= a\dot{a} \sin^2(\psi) \sin^2(\theta), & \Gamma_{\theta\theta}^\psi &= -\sin(\psi) \cos(\psi), \\ \Gamma_{\varphi\varphi}^\psi &= -\sin(\psi) \cos(\psi) \sin^2(\theta), & \Gamma_{\varphi\varphi}^\theta &= -\sin(\theta) \cos(\theta), & \Gamma_{t\psi}^\psi &= \Gamma_{\psi t}^\psi = \frac{\dot{a}}{a}, \\ \Gamma_{t\theta}^\theta &= \Gamma_{\theta t}^\theta = \frac{\dot{a}}{a}, & \Gamma_{t\varphi}^\varphi &= \Gamma_{\varphi t}^\varphi = \frac{\dot{a}}{a}, & \Gamma_{\psi\theta}^\theta &= \Gamma_{\theta\psi}^\theta = \cot(\psi), \\ \Gamma_{\psi\varphi}^\varphi &= \Gamma_{\varphi\psi}^\varphi = \cot(\psi), & \Gamma_{\theta\varphi}^\varphi &= \Gamma_{\varphi\theta}^\varphi = \cot(\theta). \end{aligned}$$

Having the Christoffel symbols allows us to find the components of the Ricci tensor, via equation (16). Hence,

$$\begin{aligned}
\text{Ric}_{tt} &= \Gamma_{tt,n}^n - \Gamma_{nt,t}^n + \Gamma_{nm}^n \Gamma_{tt}^m - \Gamma_{tm}^n \Gamma_{nt}^m \\
&= -\Gamma_{\psi t,t}^\psi - \Gamma_{\theta t,t}^\theta - \Gamma_{\varphi t,t}^\varphi - \Gamma_{t\psi}^\psi \Gamma_{\psi t}^\psi - \Gamma_{t\theta}^\theta \Gamma_{\theta t}^\theta - \Gamma_{t\varphi}^\varphi \Gamma_{\varphi t}^\varphi \\
&= -\frac{\partial}{\partial t} \left(\frac{\dot{a}}{a} \right) - \frac{\partial}{\partial t} \left(\frac{\dot{a}}{a} \right) - \frac{\partial}{\partial t} \left(\frac{\dot{a}}{a} \right) - \left(\frac{\dot{a}}{a} \right)^2 - \left(\frac{\dot{a}}{a} \right)^2 - \left(\frac{\dot{a}}{a} \right)^2 \\
&= -3 \left[-\left(\frac{\dot{a}}{a} \right)^2 + \frac{\ddot{a}}{a} \right] - 3 \left(\frac{\dot{a}}{a} \right)^2 \\
&= 3 \left(\frac{\dot{a}}{a} \right)^2 - 3 \left(\frac{\ddot{a}}{a} \right) - 3 \left(\frac{\dot{a}}{a} \right)^2 \\
&= -3 \frac{\ddot{a}}{a}
\end{aligned} \tag{154}$$

$$\begin{aligned}
\text{Ric}_{\psi\psi} &= \Gamma_{\psi\psi,n}^n - \Gamma_{n\psi,\psi}^n + \Gamma_{nm}^n \Gamma_{\psi\psi}^m - \Gamma_{\psi m}^n \Gamma_{n\psi}^m \\
&= \Gamma_{\psi\psi,t}^t - \Gamma_{\theta\psi,\psi}^\theta - \Gamma_{\varphi\psi,\psi}^\varphi + \Gamma_{\psi t}^\psi \Gamma_{\psi\psi}^t + \Gamma_{\theta t}^\theta \Gamma_{\psi\psi}^t + \Gamma_{\varphi t}^\varphi \Gamma_{\psi\psi}^t \\
&\quad - \Gamma_{\psi\psi}^t \Gamma_{t\psi}^\psi - \Gamma_{\psi t}^\psi \Gamma_{\psi\psi}^t - \Gamma_{\psi\theta}^\theta \Gamma_{\theta\psi}^\theta - \Gamma_{\psi\varphi}^\varphi \Gamma_{\varphi\psi}^\varphi \\
&= \frac{\partial}{\partial t} (a\dot{a}) - \frac{\partial}{\partial \psi} (\cot(\psi)) - \frac{\partial}{\partial \psi} (\cot(\psi)) + \left(\frac{\dot{a}}{a} \right) (a\dot{a}) + \left(\frac{\dot{a}}{a} \right) (a\dot{a}) \\
&\quad + \left(\frac{\dot{a}}{a} \right) (a\dot{a}) - (a\dot{a}) \left(\frac{\dot{a}}{a} \right) - \left(\frac{\dot{a}}{a} \right) (a\dot{a}) - 2 \cot^2(\psi) \\
&= \dot{a}^2 + a\ddot{a} + 2 \csc^2(\psi) + \dot{a}^2 - 2 \cot^2(\psi) \\
&= 2\dot{a}^2 + a\ddot{a} + 2 (\csc^2(\psi) - \cot^2(\psi)) \\
&= 2\dot{a}^2 + a\ddot{a} + 2
\end{aligned} \tag{155}$$

The following computations are excessively long and follow the same routine as above, so we omit the messy details and just produce the result. Note that these values are exactly the same with the respective multiplication by $\sin^2(\psi)$ and $\sin^2(\psi) \sin^2(\theta)$.

$$\begin{aligned}
\text{Ric}_{\theta\theta} &= \Gamma_{\theta\theta,n}^n - \Gamma_{n\theta,\theta}^n + \Gamma_{nm}^n \Gamma_{\theta\theta}^m - \Gamma_{\theta m}^n \Gamma_{n\theta}^m \\
&= \Gamma_{\theta\theta,t}^t + \Gamma_{\theta\theta,\psi}^\psi - \Gamma_{\varphi\theta,\theta}^\varphi + \Gamma_{\psi t}^\psi \Gamma_{\theta\theta}^t + \Gamma_{\theta t}^\theta \Gamma_{\theta\theta}^t + \Gamma_{\theta\psi}^\theta \Gamma_{\theta\theta}^\psi + \Gamma_{\varphi t}^\varphi \Gamma_{\theta\theta}^t \\
&\quad + \Gamma_{\varphi\psi}^\varphi \Gamma_{\theta\theta}^\psi - \Gamma_{\theta\theta}^t \Gamma_{t\theta}^\theta - \Gamma_{\theta\theta}^\psi \Gamma_{\psi\theta}^\theta - \Gamma_{\theta t}^\theta \Gamma_{\theta\theta}^t - \Gamma_{\theta\psi}^\theta \Gamma_{\theta\theta}^\psi - \Gamma_{\theta\varphi}^\varphi \Gamma_{\varphi\theta}^\varphi \\
&= \sin^2(\psi) (2\dot{a}^2 + a\ddot{a} + 2)
\end{aligned} \tag{156}$$

$$\begin{aligned}
\text{Ric}_{\varphi\varphi} &= \Gamma_{\varphi\varphi,n}^n - \Gamma_{n\varphi,\varphi}^n + \Gamma_{nm}^n \Gamma_{\varphi\varphi}^m - \Gamma_{\varphi m}^n \Gamma_{n\varphi}^m \\
&= \Gamma_{\varphi\varphi,t}^t + \Gamma_{\varphi\varphi,\psi}^\psi + \Gamma_{\varphi\varphi,\theta}^\theta + \Gamma_{\psi t}^\psi \Gamma_{\varphi\varphi}^t + \Gamma_{\theta t}^\theta \Gamma_{\varphi\varphi}^t + \Gamma_{\theta\psi}^\theta \Gamma_{\varphi\varphi}^\psi + \Gamma_{\varphi t}^\varphi \Gamma_{\varphi\varphi}^t \\
&\quad + \Gamma_{\varphi\psi}^\varphi \Gamma_{\varphi\varphi}^\psi + \Gamma_{\varphi\theta}^\varphi \Gamma_{\varphi\varphi}^\theta - \Gamma_{\varphi\varphi}^t \Gamma_{t\varphi}^\varphi - \Gamma_{\varphi\varphi}^\psi \Gamma_{\psi\varphi}^\psi \\
&\quad - \Gamma_{\varphi\varphi}^\theta \Gamma_{\theta\varphi}^\theta - \Gamma_{\varphi t}^\varphi \Gamma_{t\varphi}^t - \Gamma_{\varphi\psi}^\varphi \Gamma_{\psi\varphi}^\psi - \Gamma_{\varphi\theta}^\varphi \Gamma_{\theta\varphi}^\theta \\
&= \sin^2(\psi) \sin^2(\theta) (2\dot{a}^2 + a\ddot{a} + 2).
\end{aligned} \tag{157}$$

Continuing in the same manner as above, we obtain for the scalar curvature

$$\begin{aligned} S &= -\left(-\frac{3\ddot{a}}{a}\right) + \frac{1}{a^2}(3(2\dot{a}^2 + a\ddot{a} + 2)) \\ &= 6\frac{\ddot{a}}{a} + 6\frac{\dot{a}^2}{a^2} + 6\frac{1}{a^2}. \end{aligned} \quad (158)$$

Using this result, the temporal Einstein equation is

$$\begin{aligned} G_{tt} &= \text{Ric}_{tt} - \frac{1}{2}Sg_{tt} \\ &= -\frac{3\ddot{a}}{a} + 3\frac{\ddot{a}}{a} + 3\frac{\dot{a}^2}{a^2} + 3\frac{1}{a^2} \\ &= 3\frac{\dot{a}^2}{a^2} + 3\frac{1}{a^2} = 8\pi\rho. \end{aligned} \quad (159)$$

And the spatial Einstein equation, where $i \neq t$, is

$$\begin{aligned} G_{ii} &= \text{Ric}_{ii} - \frac{1}{2}Sg_{ii} \\ G_{ii} &= \frac{g_{ii}}{a^2}(2\dot{a}^2 + a\ddot{a} + 2 - 3a\ddot{a} - 3\dot{a}^2 - 3) \\ &= \frac{g_{ii}}{a^2}(-\dot{a}^2 - 2a\ddot{a} - 1). \end{aligned} \quad (160)$$

Thus we have

$$g_{ii} \left(-\frac{\dot{a}^2}{a^2} - 2\frac{\ddot{a}}{a} - \frac{1}{a^2} \right) = 8\pi P h_{ii} \quad (161)$$

where $i \neq t$. Combining this with equation (159) and equation (97), we obtain our system

$$\frac{\ddot{a}}{a} = -\frac{4}{3}\pi(\rho + 3P) \quad (162)$$

$$\frac{\dot{a}^2}{a^2} + \frac{1}{a^2} = \frac{8}{3}\pi\rho. \quad (163)$$

FLRW Metric on a 3-Hyperboloid

Finally, we solve for the hyperbolic case, using the hyperbolic FLRW metric, where $k = -1$,

$$ds^2 = -dt^2 + a(t)^2 [d\psi^2 + \sinh^2(\psi) (d\theta^2 + \sin^2(\theta)d\varphi^2)], \quad (164)$$

we find the Christoffel symbols are

$$\begin{aligned} \Gamma_{\psi\psi}^t &= a\dot{a}, & \Gamma_{\theta\theta}^t &= a\dot{a} \sinh^2(\psi), & \Gamma_{\varphi\varphi}^t &= a\dot{a} \sinh^2(\psi) \sin^2(\theta), \\ \Gamma_{\theta\theta}^\psi &= -\sinh(\psi) \cosh(\psi), & \Gamma_{\varphi\varphi}^\psi &= -\sinh(\psi) \cosh(\psi) \sin^2(\theta), & \Gamma_{\theta\varphi}^\theta &= -\sin(\theta) \cos(\theta), \\ \Gamma_{t\psi}^\psi &= \Gamma_{\psi t}^\psi = \frac{\dot{a}}{a}, & \Gamma_{t\theta}^\theta &= \Gamma_{\theta t}^\theta = \frac{\dot{a}}{a}, & \Gamma_{t\varphi}^\varphi &= \Gamma_{\varphi t}^\varphi = \frac{\dot{a}}{a}, \\ \Gamma_{\psi\theta}^\theta &= \Gamma_{\theta\psi}^\theta = \coth(\psi), & \Gamma_{\psi\varphi}^\varphi &= \Gamma_{\varphi\psi}^\varphi = \coth(\psi), & \Gamma_{\theta\varphi}^\varphi &= \Gamma_{\varphi\theta}^\varphi = \cot(\theta). \end{aligned}$$

Using these, we find the Ricci tensor components are

$$\begin{aligned}
\text{Ric}_{tt} &= -3\frac{\ddot{a}}{a} \\
\text{Ric}_{\psi\psi} &= 2\dot{a}^2 + a\ddot{a} - 2 \\
\text{Ric}_{\theta\theta} &= \sinh^2(\psi) (2\dot{a}^2 + a\ddot{a} - 2) \\
\text{Ric}_{\varphi\varphi} &= \sinh^2(\psi) \sin^2(\theta) (2\dot{a}^2 + a\ddot{a} - 2).
\end{aligned} \tag{165}$$

Hence for Scalar curvature we have

$$\begin{aligned}
S &= -\left(-\frac{3\ddot{a}}{a}\right) + \frac{1}{a^2} (3(2\dot{a}^2 + a\ddot{a} - 2)) \\
&= 6\frac{\ddot{a}}{a} + 6\frac{\dot{a}^2}{a^2} - 6\frac{1}{a^2}.
\end{aligned} \tag{166}$$

Using this result, the temporal Einstein equation is

$$\begin{aligned}
G_{tt} &= \text{Ric}_{tt} - \frac{1}{2}Sg_{tt} \\
&= -\frac{3\ddot{a}}{a} + 3\frac{\ddot{a}}{a} + 3\frac{\dot{a}^2}{a^2} - 3\frac{1}{a^2} \\
&= 3\frac{\dot{a}^2}{a^2} - 3\frac{1}{a^2} = 8\pi\rho.
\end{aligned} \tag{167}$$

And the spatial Einstein equation, where $i \neq t$, is

$$\begin{aligned}
G_{ii} &= \text{Ric}_{ii} - \frac{1}{2}Sg_{ii} \\
G_{ii} &= \frac{g_{ii}}{a^2} (2\dot{a}^2 + a\ddot{a} - 2 - 3a\ddot{a} - 3\dot{a}^2 + 3) \\
&= \frac{g_{ii}}{a^2} (-\dot{a}^2 - 2a\ddot{a} - 1).
\end{aligned} \tag{168}$$

Thus we have

$$-g_{ii} \left(\frac{\dot{a}^2}{a^2} - 2\frac{\ddot{a}}{a} + \frac{1}{a^2} \right) = 8\pi P h_{ii} \tag{169}$$

where $i \neq t$. Combining this with equation (167) and equation (97), we obtain our system

$$\frac{\ddot{a}}{a} = -\frac{4}{3}\pi(\rho + 3P) \tag{170}$$

$$3\frac{\dot{a}^2}{a^2} - 3\frac{1}{a^2} = 8\pi\rho \tag{171}$$

We use equation (103), equation (106), equation (162), equation (163), equation (170), and equation (171) to obtain the general equations for the FLRW metric with curvature sign $k = -1, 0, 1$

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{8}{3}\pi\rho \tag{172}$$

$$\frac{\ddot{a}}{a} = -\frac{4}{3}\pi(\rho + 3P). \tag{173}$$

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