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Computable Linear Orders and Turing Reductions

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Computable Linear Orders and Turing Reductions

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Introduction

Introduction

0.1 Computability Theory

The main focus of this thesis is to measure the complexity of a variety of relations on computable linear orders. To do this measurement, we will use two reducibilities. A set $A \subset \mathbb{N}$ is *Turing reducible* to a set B if $A = \phi_e^B$ meaning that there is an oracle machine that computes the characteristic function of A using oracle B. We denote this as $A \leq_T B$. During our proofs, this basically means that we can ask our set B questions, specifically whether or not a number is in B. A is *many-one reducible* or *m-reducible* to B if there is a computable function f such that $x \in A$ if and only if $f(x) \in B$. We denote this as $A \leq_m B$. Recall that if $A \leq_m B$, then $A \leq_T B$, but not conversely. We will give a concrete example of when the converse fails in this thesis.

Through this thesis, we will use standard notation from computability theory as found in Robert I. Soare's *Recursively Enumerable Sets and Degrees* or Hartley Rogers, Jr.'s *Theory of Recursive Functions and Effective Computability*. We use ϕ_0, ϕ_1, ϕ_2,...to denote the standard list of partial computable functions and W_0, W_1, W_2,...to denote their domains. Recall that the sets W_e are called computable enumerable or c.e. sets. We utilize several familiar index sets: K, Fin, and Inf, formally defined as:

(i) $K = \{x \mid \phi_x(x)$ converges $\}$

(ii) $Fin = \{x \mid W_x$ is finite $\}$

(iii) $Inf = \{x \mid W_x$ is infinite $\}$

These sets live in a hierarchy which is defined by quantifier complexity. We define the Σ_n^0 and Π_n^0 sets in the following way.

Definition: Let A be a set.

(i) A is in Σ_0^0 or Π_0^0 if and only if A is computable.

(ii) For $n \geq 1$, A is in Σ_n^0 if there is a computable $R(x, y_1, y_2, ..., y_n)$ such that

$$x \in A \text{ if and only if } (\exists y_1)(\forall y_2)(\exists y_3)...(Qy_n)R(x, y_1, y_2, ..., y_n).$$

(iii) For $n \geq 1$, A is in Π_n^0 if there is a computable $R(x, y_1, y_2, ..., y_n)$ such that

$$x \in A \text{ if and only if } (\forall y_1)(\exists y_2)(\forall y_3)...(Qy_n)R(x, y_1, y_2, ..., y_n).$$

By fully writing out their definitions, we have that $K \in \Sigma_1^0$, $Fin \in \Sigma_2^0$, and $Inf \in \Pi_2^0$. $A \in \Sigma_n^0(\Pi_n^0)$ is $\Sigma_n^0(\Pi_n^0)$-complete if for any arbitrary set $B \in \Sigma_n^0(\Pi_n^0)$, we have $B \leq_m A$. We will use the facts that K is Σ_1^0-complete, Fin is Σ_2^0-complete, and Inf is Π_2^0-complete.

0.2 Linear Orders

A linear order is a pair $(\mathcal{D}, \leq_{\mathcal{D}})$ where $\mathcal{D}$ is a set and $\leq_{\mathcal{D}}$ is a binary relation on $\mathcal{D}$ which is reflexive, transitive, and anti-symmetric. For this thesis, we will work with countable (and often computable) linear orders and will typically assume that $\mathcal{D} = \mathbb{N}$. Our standard notation for a linear order will be $\mathcal{L} = (\mathbb{N}, \leq_{\mathcal{L}})$. We say that $\mathcal{L} = (\mathbb{N}, \leq_{\mathcal{L}})$ is computable if and only if the binary relation $\leq_{\mathcal{L}}$ is computable.

We will use several classical notions from the theory of linear orders. Specifically, we need the following definitions.

- A linear order $\mathcal{L}$ is <u>discrete</u> if every element $a \in \mathcal{L}$ has an immediate successor and an immediate predecessor unless a is the least or greatest element. If a is the least element of $\mathcal{L}$, we require a to have an immediate successor and if a is the greatest element of $\mathcal{L}$, then we require a to have an immediate predecessor. Note that every finite linear order is discrete. An interval $(a, b) \subset \mathcal{L}$ is discrete if the ordering given by $\leq_{\mathcal{L}}$ restricted to (a, b) is discrete.

- A linear order $\mathcal{L}$ is <u>dense</u> if $\mathcal{L}$ is isomorphic to the usual order on $\mathbb{Q}$. (Recall that we assume our orderings are countable.) As above, we say an interval (a, b) in $\mathcal{L}$ is dense if the order given by $\leq_{\mathcal{L}}$ restricted to (a, b) is dense.

- Let $\mathcal{L}$ be a linear order and let $a \in \mathcal{L}$. We say <u>b is in the same block as a</u> if the interval $[a, b]$ (if $a \leq_{\mathcal{L}} b$) or $[b, a]$ (if $b \leq_{\mathcal{L}} a$) is finite. The <u>block of a</u> is the set of elements b such that b is in the same block as a.

Let $\mathcal{L}$ be a computable linear order. The following are ordering relations we will examine:

(i) $FinBl_{\mathcal{L}} = \{c \mid c \text{ is in a finite block in } \mathcal{L}\}$

(ii) $Den_{\mathcal{L}} = \{\langle b, c\rangle \mid (b, c) \text{ is dense in } \mathcal{L}\}$

(iii) $Dis_{\mathcal{L}} = \{\langle b, c\rangle \mid (b, c) \text{ is discrete in } \mathcal{L}\}$

We can calculate the complexity of each of these relations as follows. $Den_{\mathcal{L}} \in \Pi^0_2$ because $\langle b, c\rangle \in Den_{\mathcal{L}}$ if and only if

$$b <_{\mathcal{L}} c \wedge \forall x, y \in (b, c)\, [x <_{\mathcal{L}} y \rightarrow \exists\, z\, (x <_{\mathcal{L}} z <_{\mathcal{L}} y)]$$
$$\wedge\ \forall\, x \in (b, c)[\exists\, z(b <_{\mathcal{L}} z <_{\mathcal{L}} x) \wedge \exists\, u(x <_{\mathcal{L}} u <_{\mathcal{L}} c)].$$

To analyze the complexity of $FinBl_{\mathcal{L}}$ and $Den_{\mathcal{L}}$, we first consider the immediate predecessor relation $Pred_{\mathcal{L}}(x, y)$ and the immediate successor relation $Succ_{\mathcal{L}}(x, y)$. $Pred_{\mathcal{L}}(x, y)$ holds if and only if

$$x <_{\mathcal{L}} y \wedge \neg\exists\, z(x <_{\mathcal{L}} z <_{\mathcal{L}} y)$$

and hence is Π^0_1. $Succ_{\mathcal{L}}(x, y)$ holds if and only if

$$y <_{\mathcal{L}} x \wedge \neg\exists\, z(y <_{\mathcal{L}} z <_{\mathcal{L}} x)$$

and hence is also Π^0_1. We can now show that $Dis_{\mathcal{L}} \in \Pi^0_3$ because $\langle b, c\rangle \in Dis_{\mathcal{L}}$ if and only if

$$b <_{\mathcal{L}} c \wedge \forall x \in (b,c)\, \exists\, u, v(Pred_{\mathcal{L}}(u,x) \wedge Succ_{\mathcal{L}}(v,x))$$

Finally, to analyze $FinBl_{\mathcal{L}}$, we also need the complexity of the limit from below relation, $LimBelow_{\mathcal{L}}(x)$ and the limit from above relation, $LimAbove_{\mathcal{L}}(x)$. $LimBelow_{\mathcal{L}}(x)$ holds if and only if

$$\forall\, y\, (y <_{\mathcal{L}} x \rightarrow \exists\, z(y <_{\mathcal{L}} z <_{\mathcal{L}} x))$$

and hence is Π^0_2. $LimAbove_{\mathcal{L}}(x)$ holds if and only if

$$\forall\, y\, (x <_{\mathcal{L}} y \rightarrow \exists\, z(x <_{\mathcal{L}} z <_{\mathcal{L}} y))$$

and hence is also Π^0_2. Note the following subtlety of these definitions. If $\mathcal{L}$ has a least element a, then $LimBelow_{\mathcal{L}}(a)$ holds since we do not require that there is a $y <_{\mathcal{L}} x$ in the definition of $LimBelow_{\mathcal{L}}(x)$. Similarly, if $\mathcal{L}$ has a greatest element a, then $LimAbove_{\mathcal{L}}(a)$ holds. This aspect of these definitions will make the definition of $FinBl_{\mathcal{L}}(x)$ more compact.

Now, we have that $FinBl_{\mathcal{L}}(x) \in \Sigma^0_3$ because $FinBl_{\mathcal{L}}(x)$ holds if and only if

$$\exists\, y\, (y = \langle x_1, ..., x_n\rangle \wedge \exists\, i \leq n(x_i = x) \wedge$$
$$\forall i \leq n\, (Succ_{\mathcal{L}}(x_{i+1}, x_i)) \wedge LimBelow_{\mathcal{L}}(x_1) \wedge LimAbove_{\mathcal{L}}(x_n))$$

In Chapter 1, we will show that each of these relations is complete for some computable $\mathcal{L}$.

Theorem: There are computable linear orders $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{L}_3$ such that $Den_{\mathcal{L}_1}$ is Π^0_2-complete, $Dis_{\mathcal{L}_2}$ is Π^0_3-complete, and $FinBl_{\mathcal{L}_3}$ is Σ^0_3-complete.

In Chapters 2 and 3, we will consider the complexity of $Den_{\mathcal{L}}$, $Dis_{\mathcal{L}}$, and $Block_{\mathcal{L}}$ when we fix the element b to be the least element in $\mathcal{L}$. Specifically, if $\mathcal{L}$ is a computable linear order and $b \in \mathcal{L}$, then we define

(i) $Den_{\mathcal{L}}(b) = \{c \mid (b,c) \text{ is dense in } \mathcal{L}\}$

(ii) $Dis_{\mathcal{L}}(b) = \{c \mid (b,c) \text{ is discrete in } \mathcal{L}\}$

(iii) $Block_{\mathcal{L}}(b) = \{c \mid c \text{ is in the same block as } b\}$

We show that we can code both $0'$ and $0''$ into these relations by a Turing reduction when b is the least element of $\mathcal{L}$.

Theorem: There are computable linear orders $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{L}_3$ with b denoting the least element in each order such that $0'' \leq_T Den_{\mathcal{L}_1}(b)$, $0'' \leq_T Dis_{\mathcal{L}_2}(b)$, and $0'' \leq_T Block_{\mathcal{L}_3}(b)$.

Before starting the main results of this thesis, we present a theorem originally due to Carl Jockusch which has not appeared in print. This theorem shows that for the $Block_{\mathcal{L}}(b)$ relation, we cannot improve our Turing reduction in the previous theorem to an m-reduction.

Theorem: If $\mathcal{L}$ is a computable linear order and B is a block in $\mathcal{L}$ with least element $b \in B$, then $K \not\leq_m B$.

Proof: Let $\mathcal{L}$ be a computable linear order and let B be a block in $\mathcal{L}$ with least element $b \in B$. We want to show that $K \not\leq_m B$.

By way of contradiction, assume that $K \leq_m B$ and fix a computable function f such that $n \in K$ if and only if $f(n) \in B$. Without loss of generality, we assume that for all $n, b <_{\mathcal{L}} f(n)$.

We define two partial computable functions $g(x, y)$ and $h(x, y)$ using the s-m-n theorem:

$$\phi_{g(x,y)}(u) = \begin{cases} \uparrow & \text{if } f(x) <_{\mathcal{L}} f(y) \\ 0 & \text{if } f(x) \geq_{\mathcal{L}} f(y) \end{cases}$$

$$\phi_{h(x,y)}(u) = \begin{cases} 0 & \text{if } f(x) <_{\mathcal{L}} f(y) \\ \uparrow & \text{if } f(x) \geq_{\mathcal{L}} f(y) \end{cases}$$

We want to apply the following theorem which is an adjustment to the regular recursion theorem found in Hartley Rogers, Jr.'s *Theory of Recursive Functions and Effective Computability*.

Double Recursive Theorem: For any recursive functions g and h, there exist m and n such that $\phi_m = \phi_{g(m,n)}$ and $\phi_n = \phi_{h(m,n)}$.

By this theorem, fix some n and m such that $\phi_n = \phi_{g(m,n)}$ and $\phi_m = \phi_{h(m,n)}$. Using our function f, the relationship between point placement in our linear order can be broken down into three possibilities. We are using the placement of points in $\mathcal{L}$ and in particular, whether the points are included in the block B, to create a contradiction.

(i) $f(n) = f(m)$

If $f(n) = f(m)$, then we know that

$$\phi_n(n) = \phi_{g(n,m)}(n) \downarrow= 0 \Rightarrow \phi_n(n) \downarrow= 0$$

So, $n \in K$ which implies that $f(n) \in B$. On the other hand, we also know that

$$\phi_m(m) = \phi_{h(n,m)}(m) \uparrow \Rightarrow \phi_m(m) \uparrow.$$

So, $m \notin K$ which implies that $f(m) \notin B$. Thus, we have a contradiction since $f(n) = f(m)$, but $f(n)$ is in the block and $f(m)$ is not in the block.

(ii) $f(n) <_{\mathcal{L}} f(m)$

If $f(n) <_{\mathcal{L}} f(m)$, then we know that

$$\phi_n(n) = \phi_{g(n,m)}(n) \uparrow \Rightarrow \phi_n(n) \uparrow$$

So, $n \notin K$ which implies that $f(n) \notin B$. On the other hand, we also know that

$$\phi_m(m) = \phi_{h(n,m)}(m) \downarrow = 0 \Rightarrow \phi_m(m) = 0.$$

So, $m \in K$ which implies that $f(m) \in B$. Thus, we have a contradiction since $b <_{\mathcal{L}} f(n) <_{\mathcal{L}} f(m)$, but $f(n)$ is not in the block and $f(m)$ is in the block.

(iii) $f(m) <_{\mathcal{L}} f(n)$ If $f(m) <_{\mathcal{L}} f(n)$, then we know that

$$\phi_n(n) = \phi_{g(n,m)}(n) \downarrow = 0 \Rightarrow \phi_n(n) = 0$$

So, $n \in K$ which implies that $f(n) \in B$. On the other hand, we also know that

$$\phi_m(m) = \phi_{h(n,m)}(m) \uparrow \Rightarrow \phi_m(m) \uparrow.$$

So, $m \notin K$ which implies that $f(m) \notin B$. Thus, we have a contradiction since $b <_{\mathcal{L}} f(m) <_{\mathcal{L}} f(n)$, but $f(n)$ is in the block and $f(m)$ is not in the block.

Thus, we know that the relation between the computable function f and our linear order derives a contradiction in each case. Therefore, $K \not\leq_m B$.

Chapter 1

Completeness

In this chapter, we prove the completeness results stated in the Introduction. Recall that if $\mathcal{L}$ is a computable linear order, then $Den_{\mathcal{L}} \in \Pi^0_2$, $Dis_{\mathcal{L}} \in \Pi^0_3$, and $FinBl_{\mathcal{L}} \in \Sigma^0_3$. We show that in each case, we can construct a computable $\mathcal{L}$ for which the relation is complete at the given level of the arithmetic hierarchy.

1.1 Dense

Theorem: There is a computable linear order $\mathcal{L}$ for which
$Den_{\mathcal{L}} = \{\langle b, c\rangle \mid (b, c) \text{ is dense in } \mathcal{L}\}$ is Π^0_2-complete.

Proof: Since $Inf = \{e \mid W_e \text{ is infinite}\}$ is Π^0_2-complete, it suffices to build a computable linear order $\mathcal{L}$ such that $Inf \leq_m Den_{\mathcal{L}}$. To accomplish this reduction, we use pairs of witness points b_n and c_n, and we make the interval (b_n, c_n) dense if and only if W_n is infinite. The requirements are the following:

$$R_n : n \in Inf \text{ if and only if } (b_n, c_n) \text{ is dense in } \mathcal{L}.$$

Construction:

Stage 0: Set down the set of even numbers in their usual order and label the numbers in pairs as b_n and c_n. Each pair b_i and c_i is associated to the domain W_i.

$$b_0 <_{\mathcal{L}} c_0 <_{\mathcal{L}} b_1 <_{\mathcal{L}} c_1 <_{\mathcal{L}} ... <_{\mathcal{L}} b_n <_{\mathcal{L}} c_n <_{\mathcal{L}} ...$$

Stage s+1: At stage s+1,we examine each W_n, for $n \leq s$, to see if it receives a new element at stage $s + 1$. Since the requirements for each W_n are applied to each separate interval, we can treat each such requirement individually. Note that this means that there is no injury in this construction.

Case I: Assume a new element enters W_n. We need to make progress towards making the interval (b_n, c_n) dense. To accomplish this, suppose the interval (b_n, c_n) currently contains m many points and appears as

$$b_n <_{\mathcal{L}} z_m <_{\mathcal{L}} z_{m-1} <_{\mathcal{L}} ... <_{\mathcal{L}} z_1 <_{\mathcal{L}} c_n$$

Let $y_n^1, ...y_n^{m+1}$ be the $m + 1$ least unused odd numbers. Place the odd numbers into the interval (b_n, c_n) between each current pair of successor points as follows:

$$b_n <_{\mathcal{L}} y_n^{m+1} <_{\mathcal{L}} z_m <_{\mathcal{L}} y_n^m <_{\mathcal{L}} z_{m-1} <_{\mathcal{L}} ... <_{\mathcal{L}} y_n^2 <_{\mathcal{L}} z_1 <_{\mathcal{L}} y_n^1 <_{\mathcal{L}} c_n$$

In later constructions, we will describe this process of adding a new point between each pair of current successors in $[b_n, c_n]$ as *partially densifying the interval* (b_n, c_n).

Case II: Assume no new elements are enumerated into W_n. We leave the interval (b_n, c_n) as it is and do not add points towards densifying the interval.

Verification:

(i) $n \in Inf$ if and only if (b_n, c_n) is dense

We know that $n \in Inf$ if and only if W_n is an infinite domain. This is true if and only if we enumerate infinitely many points into W_n. When a point enumerates into W_n, we place points into (b_n, c_n). This process, when done infinitely often, creates a dense interval. Thus, if there are infinitely many points in W_n, then (b_n, c_n) is dense.

We know that $n \notin Inf$ if and only if W_n is finite. This is true if and only if only a finite number of points are enumerated in W_n which implies that only finitely many points were placed into (b_n, c_n). Thus, if $n \notin Inf$, then (b_n, c_n) is finite and specifically, is not dense.

(ii) Effective Construction

The construction is effective because there are only a finite number of things done at each stage for each W_n. The points we place into each $[b_n, c_n]$ are all odd and thus, there will never be a lack of available numbers as there are only finitely many odds used at each stage. Also, since there are indices W_n for which W_n is infinite, the construction will use all of the odd numbers. Hence the domain of L is $\mathbb{N}$, which is computable. This implies that to compare i and j in order, we need just run the construction until both elements appear and compare where they land in $\mathcal{L}$.

1.2 Discrete

Theorem: There is a computable linear order $\mathcal{L}$ such that

$Dis_{\mathcal{L}} = \{\langle b, c\rangle \mid [b, c] \text{ is discrete in } \mathcal{L}\}$ is Π^0_3-complete.

Proof: Let X be a Π^0_3-complete set. We need to build a computable linear order $\mathcal{L}$ such that $X \leq_m Dis_{\mathcal{L}}$. Since $Fin = \{e \mid W_e \text{ is finite }\}$ is Σ^0_2-complete, we can fix a computable function $f(x, n)$ such that

$$n \in X \text{ if and only if } \forall x(W_{f(x,n)} \text{ is finite })$$

To accomplish this, we will use pairs of witness points b_n and c_n to meet the following requirements.

$$R_n : \forall x(W_{f(x,n)} \text{ finite) if and only if } (b_n, c_n) \text{ is discrete in } \mathcal{L}.$$

Construction:

Stage 0: Effectively partition the even numbers into infinitely many infinite sets X, P_0, P_1, P_2, ...We will use these sets of numbers to put down a basic structure for our computable order $\mathcal{L}$ that will be filled in at later stages.

To define this basic structure, first place the numbers in X in $\mathcal{L}$ in their usual order and label them as follows:

$$b_0 <_{\mathcal{L}} c_0 <_{\mathcal{L}} b_1 <_{\mathcal{L}} c_1 <_{\mathcal{L}} b_2 <_{\mathcal{L}} c_2 <_{\mathcal{L}} ...$$

For each n, we will place the numbers in P_n into the interval (b_n, c_n) in their usual order so our ordering $\mathcal{L}$ at stage 0 looks like:

$$b_0 <_{\mathcal{L}} \text{points in } P_0 <_{\mathcal{L}} c_0 <_{\mathcal{L}} b_1 <_{\mathcal{L}} \text{points in } P_1 <_{\mathcal{L}} c_1 <_{\mathcal{L}} ...$$

Label the points in each P_n in groups of three as follows:

$$u_n^0 <_{\mathcal{L}} d_n^0 <_{\mathcal{L}} v_n^0 <_{\mathcal{L}} u_n^1 <_{\mathcal{L}} d_n^1 <_{\mathcal{L}} v_n^1 <_{\mathcal{L}} ...$$

Therefore, our order $\mathcal{L}$ at stage 0 looks like:

$$b_0 <_{\mathcal{L}} u_0^0 <_{\mathcal{L}} d_0^0 <_{\mathcal{L}} v_0^0 <_{\mathcal{L}} u_0^1 <_{\mathcal{L}} d_0^1 <_{\mathcal{L}} v_0^1 <_{\mathcal{L}} ...c_0 <_{\mathcal{L}} b_1 <_{\mathcal{L}} u_1^0 <_{\mathcal{L}} d_1^0 <_{\mathcal{L}} v_1^0 <_{\mathcal{L}}$$
$$... <_{\mathcal{L}} c_1 <_{\mathcal{L}} b_2 <_{\mathcal{L}} ...$$

Stage s+1: At stage s+1,we let each requirement R_n for $n \leq s$ act in turn. Since R_n will work only in the interval (b_n, c_n), we can treat each requirement individually and there is no injury in this construction.

Action for R_n: For each $x \leq s$, we check if $W_{f(x,n)}$ has received a new element.

Case I: Assume $W_{f(x,n)}$ has received a new element. We need to make progress towards making the interval (b_n, c_n) not discrete. Let y and z be the least unused odd numbers. We add y and z to $\mathcal{L}$ as the immediate predecessor and successor of d_n^x as follows:

$$b_n <_{\mathcal{L}} ... <_{\mathcal{L}} u_n^x <_{\mathcal{L}} \text{finite} <_{\mathcal{L}} y <_{\mathcal{L}} d_n^x <_{\mathcal{L}} z <_{\mathcal{L}} \text{finite} <_{\mathcal{L}} v_n^x <_{\mathcal{L}} ... <_{\mathcal{L}} c_n$$

Notice that if we add a new predecessor and successor for d_n^x infinitely often, then d_n^x becomes a limit point and (b_n, c_n) is not discrete. However, if we add only finitely many such points, the interval (u_n^x, v_n^x) will be finite.

Case II: Assume $W_{f(x,n)}$ had not received a new element. We leave the interval (b_n, c_n) looking discrete, so we do not add any new points and we move on to $x + 1$.

Verification:

(i) $n \in X$ if and only if (b_n, c_n) is discrete

First, suppose $n \notin X$. In this case, we can fix an x such that $W_{f(x,n)}$ is infinite. Since $W_{f(x,n)}$ receives a new element at infinitely many stages, we add a new successor and predecessor to d_n^x infinitely often. Therefore, d_n^x is a limit point and has neither an immediate predecessor nor an immediate successor in $\mathcal{L}$. Since $d_n^x \in (b_n, c_n)$, the interval (b_n, c_n) is not discrete in $\mathcal{L}$.

On the other hand, suppose $n \in X$. In this case, each set $W_{f(x,n)}$ is finite and hence each interval (u_n^x, v_n^x) is finite. Thus the interval (b_n, c_n) in $\mathcal{L}$ looks like

$$u_n^0 <_{\mathcal{L}} \text{finite} <_{\mathcal{L}} v_n^0 <_{\mathcal{L}} u_n^1 <_{\mathcal{L}} \text{finite} <_{\mathcal{L}} v_n^1 <_{\mathcal{L}} \ldots$$

Since (b_n, c_n) has order type $\mathbb{N}$, it is discrete.

(ii) Effective Construction

The construction is effective because there are only a finite number of things done at each stage for each W_n. The points we place into each (b_n, c_n) are all odd and thus, there will never be a lack of available numbers as there are only finitely many odds used at each stage. Also, since there are numbers $n \notin X$, and hence sets $W_{f(x,n)}$ which are infinite, the construction will use all of the odd numbers. Hence the domain of L is $\mathbb{N}$, which is computable. This implies that to compare i and j in order, we need just run the construction until both elements appear and compare where they land in L.

1.3 Finite Block

Theorem: There is a computable linear order $\mathcal{L}$ such that

$FinBl_{\mathcal{L}} = \{c \mid c \text{ is in a finite block in } \mathcal{L}\}$ is Σ^0_3-complete.

Proof: Let X be a Σ^0_3-complete set. We need to build a computable linear order $\mathcal{L}$ such that $X \leq_m FinBl_{\mathcal{L}}$. Since Inf is Π^0_2-complete, we can fix a computable function $f(n, x)$ such that

$$n \in X \text{ if and only if } \exists\, x\ (W_{f(x,n)} \text{ is infinite}).$$

To build $\mathcal{L}$, we will use witness points c_n and meet the requirements:

$$R_n : \exists x (W_{f(x,n)} \text{ is infinite}) \text{ if and only if } c_n \in FinBl_{\mathcal{L}}$$

Construction:

Stage 0: Effectively partition the even numbers into infinitely many infinite sets X, P_0, P_1, P_2, ...We will use these sets of numbers to put down a basic structure for our computable order $\mathcal{L}$ that will be filled in at later stages.

To define this basic structure, first place the numbers in X in $\mathcal{L}$ in their usual order and label them as follows:

$$c_0 <_{\mathcal{L}} c_1 <_{\mathcal{L}} c_2 <_{\mathcal{L}} \ldots$$

For each n, we will place the numbers in P_n around c_n and order them in order type $\mathbb{Z}$ with labels as follows:

$$... <_{\mathcal{L}} b_0^1 <_{\mathcal{L}} b_0^0 <_{\mathcal{L}} c_0 <_{\mathcal{L}} d_0^0 <_{\mathcal{L}} d_0^1 <_{\mathcal{L}} ... <_{\mathcal{L}} b_1^1 <_{\mathcal{L}} b_1^0 <_{\mathcal{L}} c_1 <_{\mathcal{L}} d_1^0 <_{\mathcal{L}} d_1^1 <_{\mathcal{L}} ...$$

Stage s+1: At stage s+1, we let each requirement R_n for $n \leq s$ act in turn. Since R_n will act within the part of $\mathcal{L}$ defined by the P_n points, we can treat each requirement individually and there is no injury in this construction.

Action for R_n: For each $x \leq s$, we check if $W_{f(x,n)}$ has received a new element.

Case I: Assume $W_{f(x,n)}$ does receive a new element. We need to make progress towards making c_n a member of a finite block. So, let z_1 and z_2 be the two least unused odd numbers. Place z_1 into $\mathcal{L}$ as the immediate predecessor of d_n^{x-1} (or c_n if $x = 0$) and z_2 into $\mathcal{L}$ as the immediate successor of d_n^{x-1} (or c_n if $x = 0$). The order looks like:

$$... <_{\mathcal{L}} b_n^x <_{\mathcal{L}} \text{finite}$$

$$<_{\mathcal{L}} z_1 <_{\mathcal{L}} b_n^{x-1} <_{\mathcal{L}} ... <_{\mathcal{L}} c_n <_{\mathcal{L}} ... <_{\mathcal{L}} d_n^{x-1} <_{\mathcal{L}} z_2 <_{\mathcal{L}}\text{finite}<_{\mathcal{L}} d_n^x <_{\mathcal{L}} ...$$

Notice that if b_n^{x-1} and d_n^{x-1} receive new predecessors and successors infinitely often, then they become limit points from below and above respectively, and the block containing c_n cannot extend beyond $[b_n^{x-1}, d_n^{x-1}]$.

Case II: Assume $W_{f(x,n)}$ did not receive a new element. We do nothing in this case and do not add any new points to $\mathcal{L}$. Proceed to $x + 1$.

Verification:

(i) $n \in X$ if and only if c_n is in a finite block

First, suppose $n \in X$. We can fix x such that $W_{f(x,n)}$ is infinite. Assume we have fixed the least such x. Since b_n^{x-1} receives infinitely many new predecessors, it is a limit point from below. Similarly, d_n^{n-1} is a limit point from above. (If $x = 0$, then c_n is a limit point from below and above, and hence is in a block of size 1. We continue assuming $x \neq 0$). Thus, our order around c_n looks like

$$... <_{\mathcal{L}} b_n^{x-1} <_{\mathcal{L}} \text{finite} <_{\mathcal{L}} b_n^0 <_{\mathcal{L}} \text{finite} <_{\mathcal{L}} c_n <_{\mathcal{L}} \text{finite}$$
$$<_{\mathcal{L}} d_n^0 <_{\mathcal{L}}\text{finite}<_{\mathcal{L}} d_n^{x-1} <_{\mathcal{L}} ...$$

The interval $[b_n^{x-1}, d_n^{x-1}]$ is finite and constitutes the block containing c_n. Therefore, c_n is a finite block.

On the other hand, suppose $n \notin X$. In this case, $W_{f(x,n)}$ is finite for all x and hence each interval of the form $[b_n^x, b_n^{x-1}]$, $[b_n^0, c_n]$, $[c_n, d_n^0]$, and $[d_n^{x-1}, d_n^x]$ is finite. Therefore, the block containing c_n has order type $\mathbb{Z}$ and is infinite.

(ii) Effective Construction

The construction is effective because there are only a finite number of things done at each stage. The points we place into each set of even numbers around c_n are all odd and thus, there will never be a lack of available numbers as there are only finitely many odds used at each stage. Also, since there are numbers $n \in X$ and hence infinite sets, the construction will use all of the odd numbers. Hence the domain of L is $\mathbb{N}$, which is computable. This implies that to compare i and j in order, we need just run the construction until both elements appear and compare where they land in L.

Chapter 2

Discrete

The main goal of this chapter is to construct a computable linear order $\mathcal{L}$ with a least element b such that

$$0'' \leq_T Dis_{\mathcal{L}}(b) = \{c \mid (b, c) \text{ is discrete in } \mathcal{L}\}$$

$$\text{and } 0'' \leq_T Block_{\mathcal{L}}(b) = \{c \mid [b, c] \text{ is finite in } \mathcal{L}\}$$

We first give a simpler construction coding $0'$ instead of $0''$ and then we show how to modify this construction to code $0''$.

2.1 Construction I

Recall: We define an interval as discrete if every element has a successor and predecessor except if the interval has a least or greatest element. If the interval has a least element, the least element will not have a predecessor and if the interval has a greatest element, then the greatest element will not have a successor. In particular, finite intervals are discrete and we will utilize that part of the definition in this proof.

<u>Theorem</u>: There is a computable linear order $\mathcal{L}$ with least element b such that $0' \leq_T Dis_{\mathcal{L}}(b) = \{c \mid (b, c) \text{ is discrete in } \mathcal{L}\}$.

Proof: In order to prove this theorem, we want to build a computable linear order $\mathcal{L}$ around a least element b such that the interval (b, x_n) is discrete if and only if $n \in K$. We have the following requirements:

$$R_n : n \in K \text{ if and only if } x_n \in Dis_{\mathcal{L}}(b)$$

with ordering $R_0 < R_1 < R_2 < ...$

The basic strategy for a single requirement R_0 is to put down a pair of points l_0 and x_0 such that

$$b <_{\mathcal{L}} l_0 <_{\mathcal{L}} x_0.$$

Our goal is to do one of two things in the interval (l_0, x_0) depending on whether $0 \in K$ or not. If $0 \notin K$, then we want to make the open interval (l_0, x_0) isomorphic to ω^*. This action makes l_0 into a limit point from above and hence, makes (b, x_0) not discrete because l_0 has no successor. If $0 \in K$, then we want to make $[l_0, x_0]$ finite which makes (l_0, x_0) discrete. In the context of a single requirement, this also makes (b, x_0) finite and thus, discrete.

To accomplish this goal, at each stage s, we check whether $0 \in K_s$. If not, then we add a new least point in the interval (l_0, x_0).

$$b <_{\mathcal{L}} l_0 <_{\mathcal{L}} \text{new point} <_{\mathcal{L}} z_k <_{\mathcal{L}} ... <_{\mathcal{L}} z_1 <_{\mathcal{L}} z_0 <_{\mathcal{L}} x_0$$

In this case, we regard R_0 as a building state requirement and in the general construction, we will be taking the B outcome (for building).

On the other hand, if $0 \in K_s$, then we want to stop building our ω^*-chain and restrain the interval $[l_0, x_0]$ from ever growing again. We regard R_0 as a restraining state requirement. In the general construction, we will be taking the R outcome (for restraining).

To handle a second requirement R_1, we need a second pair of witness points $l_1 <_{\mathcal{L}} x_1$. The placement of these points depends on the action of R_0. As long as R_0 is in the building state, we are working under the assumption that $[l_0, x_0]$ will not be discrete in the limit and therefore we can put any points we want into the interval (b, l_0). Thus, we place the points l_1 and x_1 as follows:

$$b <_{\mathcal{L}} l_1 <_{\mathcal{L}} x_1 <_{\mathcal{L}} l_0 <_{\mathcal{L}} x_0$$

The requirement R_1 now works exactly as R_0 did. As long as $1 \notin K_s$, R_1 continues to add points to (l_1, x_1) towards making this interval isomorphic to ω^*. If $1 \in K_s$, then R_1 restrains $[l_1, x_1]$ by not allowing any additional points to enter this interval.

However, consider what happens if R_0 changes to the restraining state. In this case, R_0 freezes the finite size of $[l_0, x_0]$ and wants to also make sure that (b, x_0) is finite. Therefore, R_1 needs to stop adding points in its current interval (l_1, x_1)

since these points are added into the interval $[b, l_0]$.

In this situation, R_1 adds new witness points l_1^* and x_1^* and places them such that

$$b <_{\mathcal{L}} l_1 <_{\mathcal{L}} \text{Finite} <_{\mathcal{L}} x_1 <_{\mathcal{L}} l_0 <_{\mathcal{L}} \text{Finite} <_{\mathcal{L}} x_0 <_{\mathcal{L}} l_1^* <_{\mathcal{L}} x_1^*.$$

R_1 can now proceed as before using the interval (l_1^*, x_1^*). Notice that if $0 \in K$ and $1 \in K$, then R_0 makes $[l_0, x_0]$ finite and makes $[b, l_0]$ finite (by forcing R_1 to stop using witnesses l_0 and x_0). R_1 also makes $[l_1^*, x_1^*]$ finite. Thus, (b, x_0) and (b, x_1^*) are both finite (and hence discrete), winning R_0 and R_1.

Notice that with two requirements, we need to know the outcome at R_0 in order to know which interval in $\mathcal{L}$ codes information about whether $1 \in K$. To use $\{c \mid (b, c) \text{ is discrete in } \mathcal{L}\}$ to compute K, we proceed as follows. First, we need to ask if (b, x_0) is discrete. If the interval is not discrete, then we know that $0 \notin K$ and that the witness pair for R_1 is (l_1, x_1). So, we ask if (b, x_1) is discrete. If so, then $1 \notin K$ and if not, then $1 \in K$.

On the other hand, if (b, x_0) is discrete, then we know that $0 \in K$. So, at some finite point in the construction, we switched our witness pair for R_1 to (l_1^*, x_1^*). Therefore, to determine if $1 \in K$, we need to ask if (b, x_1^*) is discrete. If it is discrete, then $1 \in K$ and if it is not discrete $1 \notin K$.

The witness x_2 is set down based upon the restrictions of the higher priority requirements R_0 and R_1.

- If $0 \in K$ and $1 \in K$, then x_2 is set down such that $b <_{\mathcal{L}} x_0 <_{\mathcal{L}} x_1 <_{\mathcal{L}} x_2$.
- If $0 \in K$ and $1 \notin K$, then x_2 is set down such that $b <_{\mathcal{L}} x_0 <_{\mathcal{L}} x_2 <_{\mathcal{L}} x_1$.
- If $0 \notin K$ and $1 \in K$, then x_2 is set down such that $b <_{\mathcal{L}} x_1 <_{\mathcal{L}} x_2 <_{\mathcal{L}} x_0$.
- If $0 \notin K$ and $1 \notin K$, then x_2 is set down such that $b <_{\mathcal{L}} x_2 <_{\mathcal{L}} x_1 <_{\mathcal{L}} x_0$.

The rest of the witnesses are set down based upon the higher priority requirements.

Notice that, as described above for R_0 and R_1, in order to determine which interval in $\mathcal{L}$ codes information about whether $2 \in K$, we need to know the outcomes for R_0 and R_1. The answer to the question of whether (b, x_0) is discrete tells us which witness pair for R_1 codes the information about whether $1 \in K$. Once we know which witness pair codes this information, we can ask a discreteness question to determine which witness pair for R_2 codes information about whether $2 \in K$. In general, to determine which witness pair codes information about whether $n \in K$, we will have to use discreteness questions to determine the correct witness pairs for $0, 1, ..., n-1$. This process illustrates why our reduction is a Turing reduction as opposed to an m-reduction.

We will be setting up a tree of strategies $T = \{R, B\}^{<\omega}$ such that $R <_L B$. We want to indicate that the order determined by the tree will be represented by L as opposed to $\mathcal{L}$ which refers to the actual linear order. The basic universal strategy is to stay in a build state until n is enumerated into K. During this time, we will be building an ω^*-chain between the witness pair x_n and l_n. When n is enumerated into K, we want to switch to the restrain strategy which will not allow any new points to be introduced in the interval $[l_n, x_n]$.

It remains to describe where a strategy $\alpha \in T$ places its witness points l_α and x_α when it is first eligible to act. To describe this placement, we treat the pair l_α and x_α as a single entity w_α and write

$$w_\alpha <_{\mathcal{L}} w_\beta$$

as an abbreviation for $l_\alpha <_{\mathcal{L}} x_\alpha <_{\mathcal{L}} l_\beta <_{\mathcal{L}} x_\beta$. Our method of adding points (as described below) will ensure that the intervals (l_α, x_α) and (l_β, x_β) are always disjoint. For distinct strategies α and β, we place $w_\alpha <_{\mathcal{L}} w_\beta$ if and only if either

- $\beta <_L \alpha$ (β is to the left of α in the tree of strategies)
- or $\alpha \subseteq \beta$ and $\beta(|\alpha|) = R$ (β extends $\alpha * R$)
- or $\beta \subseteq \alpha$ and $\alpha(|\beta|) = B$ (α extends $\beta * B$).

Local Action for α for R_e :

(i) When R_α is first eligible to act, place a new pair of witnesses $l_\alpha <_{\mathcal{L}} x_\alpha$ in $\mathcal{L}$ as described above.

(ii) If $e \notin K_s$, then add a new least point into the interval (l_α, x_α) and take outcome $\alpha * B$.

(iii) If $e \in K_s$, do not add any points to (l_α, x_α) and take outcome $\alpha * R$.

Note two properties of the placement of points in our linear order $\mathcal{L}$. First, only α is allowed to put points in the interval (l_α, x_α). This protects our interval against other witnesses encroaching on its territory. Second, when l_α and x_α are placed, the interval contains no l_β and x_β points. This serves the same purpose as the previous restriction in that it preserves the previous intervals.

Construction

Stage 0: We begin with the empty set. So, we need to set down point b.

Stage s+1: Follow the path down the tree of strategy to level s as directed by the action of the strategies eligible to act. When we reach level $s+1$, end the stage.

Verification:

(i) True Path

The true path in our tree of strategies is the leftmost path visited infinitely often. Notice that if α is on the true path, then either α always takes outcome $\alpha * B$ or at some stage s, α switches to outcome $\alpha * R$ and always takes $\alpha * R$ at all future stages. Therefore, as the construction proceeds, the paths taken only move left and a node α at level n is on the true path if and only if α is eventually on the path at every stage past some stage s.

(ii) Lemma 2.1.1: Let α be on the true path. If $\alpha * R$ is on the true path, then (b, x_α) is finite and hence discrete.

Proof: Fix t such that for all $s \geq t$, $\alpha * R$ is on the path at stage s. To prove this lemma, we need to consider the ways that a point could possibly enter the interval (b, x_α) after stage t. There are two possibilities:

(a) α places points into $[l_\alpha, x_\alpha]$ after stage t.

If we have taken the outcome R at stage t, then we know that n entered K by stage t. By the construction, there is no possibility for points to enter $[l_\alpha, x_\alpha]$ because we cannot return to the building outcome.

(b) Another strategy β places points into (b, x_α). In this case, we must have $w_\beta <_{\mathcal{L}} w_\alpha$. Consider the ways in which this could happen.

- If $\beta \subseteq \alpha$, then $w_\beta <_{\mathcal{L}} w_\alpha$ means $\alpha(|\beta|) = R$ and hence $\beta * R \subseteq \alpha$. Therefore, $\beta * R$ is on the true path and is on the current path at all stages $s \geq t$. By the action at β, β does not add any points to $[l_\beta, x_\beta]$.

- If $\alpha \subseteq \beta$, then $w_\beta <_{\mathcal{L}} w_\alpha$ means $\beta(|\alpha|) = B$ and hence $\alpha * B \subseteq \beta$. However, $\alpha * B$ is never on the path after state t, so β is never eligible to act after stage t. Therefore, β cannot add new points to $[l_\beta, x_\beta]$.

- If α and β are incomparable, then $w_\beta <_{\mathcal{L}} w_\alpha$ means $\alpha <_L \beta$. Since our path only moves left and α is on the true path, β is never eligible to act after stage t and never adds any new points to $\mathcal{L}$ after stage t.

In all cases, we see that no strategy $\beta \neq \alpha$ can add new points to (b, x_α) after stage t.

(iii) Lemma 2.1.2: Let α be on the true path. If $\alpha * B$ is on the true path, then (b, x_α) is not discrete.

Proof: If $\alpha * B$ is on the true path, then it is eligible to act infinitely often and it adds points to make (l_α, x_α) isomorphic to ω^*. Therefore, $l_\alpha \in (b, x_\alpha)$ and l_α has no immediate successor. Hence, (b, x_α) is not discrete.

(iv) Lemma 2.1.3: Let α be the R_e strategy on the true path. Then, the following are equivalent:

- (b, x_α) is discrete.
- (b, x_α) is finite.
- $e \in K$
- $\alpha * R$ is on the true path.

Proof: If $e \in K$, then by our local action for α, $\alpha * R$ is on the true path. By Lemma 2.1.1, (b, x_α) is discrete and finite. If $e \notin K$, then by our local action for α, $\alpha * B$ is on the true path. By Lemma 2.1.2, (b, x_α) is not discrete and hence infinite.

(v) Lemma 2.1.4: $0' \leq_T Dis_{\mathcal{L}}(b) = \{c \mid (b, c) \text{ is discrete in } \mathcal{L}\}$

Proof: We define a function $f : \mathbb{N} \to T$ by setting $f(e) = \alpha$ if α is the R_e strategy on the true path. Notice that f is computable from $Dis_{\mathcal{L}}(b)$ since $f(0)$ is the unique R_0 strategy and by Lemma 2.1.3,

$$f(e+1) = \begin{cases} f(e) * B & \text{if } (b, x_{f(e)}) \text{ is not discrete} \\ f(e) * R & \text{if } (b, x_{f(e)}) \text{ is discrete} \end{cases}$$

We can compute $0'$ from f using Lemma 2.1.3 since

$$n \in K \text{ if and only if } (b, x_{f(n)}) \text{ is discrete}$$

and thus $n \in K$ if and only if $f(n+1) = f(n) * R$. Therefore, we have $K \leq_T f$ and $f \leq_T Dis_{\mathcal{L}}(b)$, so $K \leq_T Dis_{\mathcal{L}}(b)$.

(vi) Effective Construction

The construction is effective because there are only a finite number of things done at each stage for each requirement R_n. Also, since there are $n \notin K$, we will build at least one infinite ω^*-chain, the construction will use all of the natural numbers. Hence the domain of L is $\mathbb{N}$, which is computable. This implies that to compare i and j in order, we need just run the construction until both elements appear and compare where they land in L.

Corollary: $0' \leq_T Block_{\mathcal{L}}(b) = \{c \mid [b, c] \text{ is finite in } \mathcal{L}\}$

Proof: By Lemma 2.1.3, if α is an R_e strategy on the true path, then (b, x_α) is discrete if and only if (b, x_α) is finite. Therefore, we could equivalently define the function f in Lemma 2.1.4 by

$$f(e+1) = \begin{cases} f(e) * B & \text{if } (b, x_{f(e)}) \text{ is infinite} \\ f(e) * R & \text{if } (b, x_{f(e)}) \text{ is finite} \end{cases}$$

Thus, $f \leq_T Block_{\mathcal{L}}(b) = \{c \mid [b, c] \text{ is finite in } \mathcal{L}\}$ and hence $K \leq_T Block_{\mathcal{L}}(b) = \{c \mid [b, c] \text{ is finite in } \mathcal{L}\}$.

2.2 Construction II

Theorem: There is a computable linear order $\mathcal{L}$ with least element b such that $0'' \leq_T Dis_{\mathcal{L}}(b) = \{c \mid (b, c) \text{ is discrete in } \mathcal{L}\}$.

Proof: In order to prove this theorem, we want to build a computable linear order $\mathcal{L}$ around a least element b such that the interval (b, x_n) is not discrete if and only if $n \in Inf$ where $Inf = \{e \mid W_e \text{ is infinite}\}$. We have the following requirements:

$$R_n : n \in Inf \text{ if and only if } (b, x_n) \notin Dis_{\mathcal{L}}(b)$$

with ordering $R_0 < R_1 < R_2 < ...$

The basic strategy for a single requirement R_0 is similar to the strategy in the $0'$ construction. We want to put down a pair of points l_0 and x_0 such that

$$b <_{\mathcal{L}} l_0 <_{\mathcal{L}} x_0.$$

Our goal is to do one of two things in the interval $[l_0, x_0]$ depending on whether $0 \in Inf$ or not. If $0 \in Inf$, then we want to make the open interval (l_0, x_0) isomorphic to ω^*. This action makes l_0 into a limit point from above and hence, makes (b, x_0) not discrete because l_0 has no successor. If $0 \notin Inf$, then we want to make $[l_0, x_0]$ finite which makes $[l_0, x_0]$ discrete. In the context of a single requirement, this also makes (b, x_0) finite and thus, discrete.

To accomplish this goal, at each stage s, we check whether W_0 had received a new point. If so, then we add a new least point in the interval (l_0, x_0).

$$b <_{\mathcal{L}} l_0 <_{\mathcal{L}} \text{new point} <_{\mathcal{L}} z_k <_{\mathcal{L}} ... <_{\mathcal{L}} z_1 <_{\mathcal{L}} z_0 <_{\mathcal{L}} x_0$$

In this case, we regard R_0 as a building state requirement and in the general construction, we will be taking the B outcome (for building).

On the other hand, if W_0 has not received a new point, then we want to stop (at least temporarily) building our ω^*-chain and restrain the interval $[l_0, x_0]$ from growing. We regard R_0 as a restraining state requirement. In the general construction, we will be taking the R outcome (for restraining).

We will be setting up a tree of strategies $T = \{B, R\}^{<\omega}$ such that $B <_L R$. Notice that we have switched from $R <_L B$ in the $0'$ construction to $B <_L R$ in the $0''$ construction. In the $0''$ construction, it is possible to take both the B and R outcomes infinitely often. For, example, we would do this if there are infinitely many stages at which W_n gets a new element and infinitely many stages at which W_n does not get a new element. In this case, the true outcome is the B outcome since W_n is infinite. In order to have the true path be the leftmost path visited infinitely often, we need $B <_L R$ for the $0''$ construction.

The basic universal strategy is to stay in a restrain state until W_n adds a new point. In a restrained stage, we will not allow any new points to be introduced in the interval $[l_n, x_n]$. When W_n grows, we want to switch to the build strategy and add a single point towards building a copy of ω^* in (l_n, x_n). If we take the outcome B infinitely often, then (l_n, x_n) will grow to a copy of ω^*.

It remains to describe where a strategy $\alpha \in T$ places its witness points l_α and x_α when it is first eligible to act. To describe this placement, we treat the pair l_α and x_α as a single entity w_α and write

$$w_\alpha <_{\mathcal{L}} w_\beta$$

as an abbreviation for $l_\alpha <_{\mathcal{L}} x_\alpha <_{\mathcal{L}} l_\beta <_{\mathcal{L}} x_\beta$. Our method of adding points (as described below) will ensure that the intervals (l_α, x_α) and (l_β, x_β) are always disjoint. For distinct strategies α and β, we place $w_\alpha <_{\mathcal{L}} w_\beta$ if and only if either

- $\alpha <_L \beta$ (α is to the left of β in the tree of strategies)
- or $\alpha \subseteq \beta$ and $\beta(|\alpha|) = R$ (β extends $\alpha * R$)
- or $\beta \subseteq \alpha$ and $\alpha(|\beta|) = B$ (α extends $\beta * B$).

Local Action for α for R_e :

(i) When R_α is first eligible to act, place a new pair of witnesses $l_\alpha <_{\mathcal{L}} x_\alpha$ in $\mathcal{L}$.

Let $\hat{s}$ be the last stage at which α was eligible to act (with $\hat{s} = 0$ if this is the first time α is eligible to act).

(ii) If $W_{e,s} \neq W_{e,\hat{s}}$, then add a new least point into the interval (l_α, x_α) and take outcome $\alpha * B$.

(iii) If $W_{e,s} = W_{e,\hat{s}}$, do not add any points to (l_α, x_α) and take outcome $\alpha * R$.

Note two properties of the placement of points in our linear order $\mathcal{L}$. First, only α is allowed to put points in the interval (l_α, x_α). This protects our interval against other witnesses encroaching on its territory. Second, when l_α and x_α are placed, the interval contains no l_β and x_β points. This serves the same purpose as the previous restriction in that it preserves the previous intervals.

Construction

Stage 0: We begin with the empty set. So, we need to set down point b.

Stage s+1: Follow the path down the tree of strategy to level s as directed by the action of the strategies eligible to act. When we reach level $s + 1$, end the stage.

Verification:

(i) True Path

First let s be an α-stage if α is eligible to act at stage s. The true path in our tree of strategies is the leftmost path visited infinitely often. Assume

α is on the true path. If there are infinitely many α-stages when we take outcome $\alpha * B$, then $\alpha * B$ is on the true path. Otherwise, there exists a stage t such that for all α-stages after t, we take $\alpha * R$ and $\alpha * R$ is on the true path. Note that if α is on the true path, then there exists only finitely many stages s when the true path is to the left of α.

(ii) <u>Lemma 2.2.1</u>: Let α be on the true path. If $\alpha * R$ is on the true path, then (b, x_α) is finite and hence discrete.

Proof: Fix a stage t such that for all $s \geq t$, the path is not to the left of $\alpha * R$. To prove this lemma, we need to consider the ways that a point could enter the interval (b, x_α) after stage t.

(a) α places points in $[l_\alpha, x_\alpha]$ after stage t.

If α places a point in $[l_\alpha, x_\alpha]$, then it takes outcome $\alpha * B$. Since $\alpha * B <_L \alpha * R$ and the path is never left of $\alpha * R$ after stage t, α cannot place any points in $[l_\alpha, x_\alpha]$ after stage t.

(b) Another strategy β places points in $[b, x_\alpha]$. In this case, we must have $w_\beta <_\mathcal{L} w_\alpha$. Consider the ways this could happen.

(i) If $\beta \subseteq \alpha$, then $w_\beta <_\mathcal{L} w_\alpha$ means $\alpha(|\beta|) = R$ and hence $\beta * R \subseteq \alpha$. If β adds points to $\mathcal{L}$, it takes outcome $\beta * B$ which is left of $\alpha * R$. Therefore, β adds no more points after stage t.

(ii) If $\alpha \subseteq \beta$, then $w_\beta <_\mathcal{L} w_\alpha$ means $\beta(|\alpha|) = B$ and hence $\alpha * B \subseteq \beta$. Since α takes outcome R at all α-stages after t, β is never eligible to act after stage t and hence does not add any points after stage t.

(iii) If α and β are incomparable, then $w_\beta <_{\mathcal{L}} w_\alpha$ means $\beta <_L \alpha$. However, the path is never to the left of $\alpha * R$ after stage t and therefore, never to the left of α after stage t. Hence, β cannot add points after stage t.

In all cases, we see that no strategy $\beta \neq \alpha$ can add new points to (b, x_α) after stage t.

(iii) <u>Lemma 2.2.2</u>: Let α be on the true path. If $\alpha * B$ is on the true path, then (b, x_α) is not discrete.

Proof: If $\alpha * B$ is on the true path, then it is eligible to act infinitely often and it adds points to make (l_α, x_α) isomorphic to ω^*. Therefore, $l_\alpha \in (b, x_\alpha)$ and l_α has no immediate successor. Hence, (b, x_α) is not discrete.

(iv) <u>Lemma 2.2.3</u>: Let α be the R_e strategy on the true path. Then the following are equivalent:

- (b, x_α) is discrete.
- (b, x_α) is finite.
- $e \notin Inf$
- $\alpha * R$ is on the true path.

Proof: If $e \notin Inf$, then by our local action for α, $\alpha * R$ is on the true path. By Lemma 2.2.1, (b, x_α) is discrete and finite. If $e \in Inf$, then by our local action for α, $\alpha * B$ is on the true path. By Lemma 2.2.2, (b, x_α) is not discrete and hence infinite.

(v) Lemma 2.2.4: $0'' \leq_T Dis_{\mathcal{L}}(b) = \{c \mid (b, c) \text{ is discrete in } \mathcal{L}\}$

Proof: We define a function $f : \mathbb{N} \to T$ by setting $f(e) = \alpha$ if α is the R_e strategy on the true path. Notice that f is computable from $Dis_{\mathcal{L}}(b)$ since $f(0)$ is the unique R_0 strategy and by Lemma 2.2.3,

$$f(e+1) = \begin{cases} f(e) * B & \text{if } (b, x_{f(e)}) \text{ is not discrete} \\ f(e) * R & \text{if } (b, x_{f(e)}) \text{ is discrete} \end{cases}$$

We can compute $0''$ from f using Lemma 2.2.3 since

$$n \in Inf \text{ if and only if } (b, x_{f(n)}) \text{ is not discrete}$$

and hence $n \in Inf$ if and only if $f(n+1) = f(n) * B$. Therefore, we have $Inf \leq_T f$ and $f \leq_T Dis_{\mathcal{L}}(b)$, so $Inf \leq_T Dis_{\mathcal{L}}(b)$.

(vi) Effective Construction

The construction is effective because there are only a finite number of things done at each stage for each requirement R_n. Also, since there are $n \in Inf$, we will build at least one infinite ω^*-chain, the construction will use all of the natural numbers. Hence the domain of L is $\mathbb{N}$, which is computable. This implies that to compare i and j in order, we need just run the construction until both elements appear and compare where they land in L.

Corollary: $0'' \leq_T Block_{\mathcal{L}}(b)$

Proof: By Lemma 2.2.3, if α is an R_e strategy on the true path, then (b, x_α) is discrete if and only if (b, x_α) is finite. Therefore, we could equivalently define the function f in Lemma 3.2.4 by

$$f(e+1) = \begin{cases} f(e) * B & \text{if } (b, x_{f(e)}) \text{ is infinite} \\ f(e) * R & \text{if } (b, x_{f(e)}) \text{ is finite} \end{cases}$$

Thus, $f \leq_T Block_{\mathcal{L}}(b)$ and hence $0'' \leq_T Block_{\mathcal{L}}(b)$.

Chapter 3
Dense

In this chapter, we construct a computable linear order $\mathcal{L}$ with a least element b such that

$$0'' \leq_T Den_{\mathcal{L}}(b) = \{c \mid (b, c) \text{ is dense in } \mathcal{L}\}.$$

We first give a simpler construction coding $0'$ instead of $0''$.

3.1 Construction I

Recall: We define an interval as dense if it is isomorphic to $\mathbb{Q}$.

Theorem: There is a computable linear order $\mathcal{L}$ with least element b such that $0' \leq_T Den_{\mathcal{L}}(b) = \{c \mid (b, c) \text{ is dense in } \mathcal{L}\}$.

Proof: In order to prove this theorem, we want to build a computable linear order $\mathcal{L}$ around a least element b such that the interval (b, x_n) is dense if and only if $n \notin K$. We have the following requirements:

$$R_n : n \notin K \text{ if and only if } (b, x_n) \in Den_{\mathcal{L}}(b)$$

with ordering $R_0 < R_1 < R_2 < ...$

The basic strategy for a single requirement R_0 is to put down a point x_0 such that

$$b <_{\mathcal{L}} x_0.$$

Our goal is to do one of two things in the interval (b, x_0) depending on whether $0 \in K$ or not. If $0 \notin K$, then we want to make the open interval (b, x_0) isomorphic to $\mathbb{Q}$. This action makes (b, x_0) dense. If $0 \in K$, then we want to make (b, x_0) not dense.

To accomplish this goal, at each stage s, we check whether $0 \in K_s$. If not, then we add new points between each point in the interval (b, x_0).

$$b <_{\mathcal{L}} \text{new} <_{\mathcal{L}} w_k <_{\mathcal{L}} \text{new} <_{\mathcal{L}} ... <_{\mathcal{L}} \text{new} <_{\mathcal{L}} w_1 <_{\mathcal{L}} \text{new} <_{\mathcal{L}} w_0 <_{\mathcal{L}} \text{new} <_{\mathcal{L}} x_0$$

In this case, we regard R_0 as a building state requirement and in the general construction, we will be taking the B outcome (for building). Since we will repeat this process many times, we introduce the following terminology. Let (u, v) be a finite interval in $\mathcal{L}$ at stage s. To partially densify (u, v) means to add a new least element and a new greatest element to this open interval and to add one new point between each pair of points in (u, v) which are currently successors. Notice that if a fixed interval (u, v) is partially densified infinitely often, then (u, v) has order type $\mathbb{Q}$.

On the other hand, if $0 \in K_s$, then we want to stop building our $\mathbb{Q}$ and restrain the interval (b, x_0) from becoming dense. To do this, we want to add two points $b < z_0 < y_0 < x_0$ as immediate predecessors of x_0 and not allow any points to enter interval (z_0, x_0). If we maintain this restraint, then z_0 will be an immediate predecessor of y_0 and hence (b, x_0) will not be dense. We regard R_0 as a restraining state requirement. In the general construction, we will be taking the R outcome (for restraining).

To handle a second requirement R_1, we need a witness point x_1. The placement of this points depends on the action of R_0. As long as R_0 is in the building state, we are working under the assumption that (b, x_0) will be dense in the limit and therefore, we want to protect the interval (b, x_0). Thus, we place the point x_1 as follows:

$$b <_{\mathcal{L}} x_0 <_{\mathcal{L}} x_1$$

The requirement R_1 now works exactly as R_0 did. As long as $1 \notin K_s$, R_1 continues to partially densify (b, x_1), making this interval isomorphic to $\mathbb{Q}$. Notice that if $0 \notin K$ and $1 \notin K$, then the action of R_1 towards making (b, x_1) isomorphic to $\mathbb{Q}$ does not injure the action of R_0 towards making (b, x_0) isomorphic to $\mathbb{Q}$. If $1 \in K_s$, then R_1 restrains (b, x_1) by not allowing this interval to become dense by placing two points $x_0 <_{\mathcal{L}} z_1 <_{\mathcal{L}} y_1 <_{\mathcal{L}} x_1$ as immediate predecessors of x_1 and not allowing any points to enter between $[z_1, x_1]$. Since we have $x_0 <_{\mathcal{L}} z_1 <_{\mathcal{L}} y_1$, the requirement R_0 can make (b, x_0) dense while the requirement R_1 can make (b, x_1) not dense by making z_1 and immediate predecessor of y_1.

However, consider what happens if R_0 changes to the restraining state. In this case, R_0 adds two points $b < z_0 < y_0 < x_0$ as immediate predecessors of x_0 and does not allow any points to enter between (z_0, y_0) which will make sure that (b, x_0) is not dense. Therefore, R_1 needs to stop partially densifying its current interval (b, x_1) since this action adds points in the interval (z_0, y_0).

In this situation, R_1 adds a new witness point (or chooses one in the interval (b, z_0)) x_1^* and places it such that

$$b <_{\mathcal{L}} x_1^* <_{\mathcal{L}} z_0 <_{\mathcal{L}} y_0 <_{\mathcal{L}} x_0 <_{\mathcal{L}} \text{finite} <_{\mathcal{L}} x_1.$$

R_1 can now proceed as before using the interval (b, x_1^*). Notice that if $0 \in K$ and $1 \in K$, then R_0 makes (b, x_0) not dense with the witnesses $z_0 <_{\mathcal{L}} y_0$ and R_1 makes (b, x_1) not dense with witnesses $z_1^* <_{\mathcal{L}} y_1^*$. Thus, (b, x_0) and (b, x_1^*) are both not dense, winning R_0 and R_1.

Notice that with two requirements, we need to know the outcome of R_0 in order to know which interval in $\mathcal{L}$ codes information about whether $1 \in K$. To use

$\{c \mid (b, c) \text{ is dense}\}$ to compute K, we proceed as follows. First, we need to ask if (b, x_0) is dense. If the interval is dense, then we know that $0 \notin K$ and that the witness for R_1 is x_1. So, we ask if (b, x_1) is dense. If so, then $1 \notin K$ and if not, then $1 \in K$.

On the other hand, if (b, x_0) is not dense, then we know that $0 \in K$. So, at some finite point in the construction, we switched our witness for R_1 to x_1^*. Therefore, to determine if $1 \in K$, we need to ask if (b, x_1^*) is dense. If it is dense, then $1 \notin K$ and if it is not dense, $1 \in K$.

The witness x_2 is set down based upon the restrictions of the higher priority requirements R_0 and R_1.

- If $0 \notin K$ and $1 \notin K$, then x_2 is set down such that $b <_{\mathcal{L}} x_0 <_{\mathcal{L}} x_1 <_{\mathcal{L}} x_2$.
- If $0 \notin K$ and $1 \in K$, then x_2 is set down such that $b <_{\mathcal{L}} x_0 <_{\mathcal{L}} x_2 <_{\mathcal{L}} x_1$.
- If $0 \in K$ and $1 \notin K$, then x_2 is set down such that $b <_{\mathcal{L}} x_1 <_{\mathcal{L}} x_2 <_{\mathcal{L}} x_0$.
- If $0 \in K$ and $1 \in K$, then x_2 is set down such that $b <_{\mathcal{L}} x_2 <_{\mathcal{L}} x_1 <_{\mathcal{L}} x_0$.

The rest of the witnesses are set down based upon the higher priority requirements.

Notice that, as described above for R_0 and R_1, in order to determine which interval in $\mathcal{L}$ codes information about whether $2 \in K$, we need to know the outcomes for R_0 and R_1. The answer to the question of whether (b, x_0) is dense tells us which witness for R_1 codes the information about whether $1 \in K$. Once we know which witness codes this information, we can ask a denseness question to determine which witness for R_2 codes information about whether $2 \in K$. In general, to determine which witness codes information about whether $n \in K$, we will have to use denseness questions to determine the correct witness for $0, 1, ..., n-1$. This process illustrates why our reduction is a Turing reduction as opposed to an m-reduction.

We will be setting up a tree of strategies $T = \{R, B\}^{<\omega}$ such that $R <_L B$. The basic universal strategy is to stay in a build state until n is enumerated into K. During this time, we will be building $\mathbb{Q}$ between the witness x_n and b. When n is enumerated into K, we want to switch to the restrain strategy which will protect the interval (b, x_n) from being dense by inserting $z_n < y_n < x_n$ as immediate predecessors of x_n and restraining any new elements from entering (z_n, y_n).

It remains to describe where α places its witness point x_α when it is first eligible to act. To describe this placement, we treat the point x_α as an entity w_α (to keep a similar notation as in the Discrete Construction Proofs) and write

$$w_\alpha <_{\mathcal{L}} w_\beta$$

as a notation for $x_\alpha <_{\mathcal{L}} x_\beta$. For distinct strategies α and β, we place $w_\alpha <_{\mathcal{L}} w_\beta$ if and only if either

- $\alpha <_L \beta$ (α is to the left of β in the tree of strategies)
- or $\alpha \subseteq \beta$ and $\beta(|\alpha|) = B$ (β extends $\alpha * B$)
- or $\beta \subseteq \alpha$ and $\alpha(|\beta|) = R$ (α extends $\beta * R$).

Local Action for α for R_e :

(i) When R_α is first eligible to act, place a new witness x_α in $\mathcal{L}$ or choose an existing point that satisfies the ordering conditions above.

(ii) If $e \notin K_s$, then partially densify the interval (b, x_α) and take outcome $\alpha * B$.

(iii) If s is the least stage such that $e \in K_s$, add z_α and y_α as immediate predecessors of x_α and take outcome $\alpha * R$. If z_α and y_α have already been added, just take outcome $\alpha * R$.

Note a property of the placement of points in our linear order $\mathcal{L}$. Only α is allowed to place z_α and y_α into (b, x_α). This protects our interval against other witnesses encroaching on its territory and making it dense.

Construction

Stage 0: We begin with the empty set. So, we need to set down point b.

Stage s+1: Follow the path down the tree of strategies to level s as directed by the action of the strategies eligible to act. When we reach level $s + 1$, end the stage.

Verification:

(i) True Path

The true path in our tree of strategies is the leftmost path visited infinitely often. Notice that if α is on the true path, then either α always takes outcome $\alpha * B$ or at some stage s, α switches to outcome $\alpha * R$ and always takes $\alpha * R$ at all future stages. Therefore, as the construction proceeds, the paths taken only move left and a node α at level n is on the true path if and only if α is eventually on the path at every stage past some state s.

(ii) Lemma 3.1.1: Let α be on the true path. If $\alpha * R$ is on the true path, then (b, x_α) is not dense.

Proof: Fix t such that $\alpha * R$ is first on the true path at stage t and hence is on the path at stage s for all $s \geq t$. At stage t, α places the points y_α and z_α such that $z_\alpha <_{\mathcal{L}} y_\alpha <_{\mathcal{L}} x_\alpha$ and $(z_\alpha, x_\alpha) = \{y_\alpha\}$. To show that (b, x_α) is not dense, it suffices to show that no strategy can add points to $[z_\alpha, x_\alpha]$ after stage t. There are two possibilities:

(a) α places points into $[z_\alpha, x_\alpha]$ after stage t.

If we have taken the outcome R at stage t, then we know that n entered K by stage t. By the construction, there is no possibility for points to enter $[z_\alpha, x_\alpha]$ because we cannot return to the building outcome.

(b) Another strategy β places points into $[z_\alpha, x_\alpha]$. In this case, we must have $w_\alpha <_{\mathcal{L}} w_\beta$. Consider the ways in which this could happen.

- If $\beta \subseteq \alpha$, then $w_\alpha <_{\mathcal{L}} w_\beta$ means $\alpha(|\beta|) = R$ and hence $\beta * R \subseteq \alpha$. Therefore, $\beta * R$ is on the true path and is on the current path at all stages $s \geq t$. By the action at β, β does not add any new points to $\mathcal{L}$.
- If $\alpha \subseteq \beta$, then $w_\alpha <_{\mathcal{L}} w_\beta$ means $\beta(|\alpha|) = B$ and hence $\alpha * B \subseteq \beta$. However, $\alpha * B$ is never on the path after stage t, so β is never eligible to act after stage t. Therefore, β cannot add new points to $\mathcal{L}$.
- If α and β are incomparable, then $w_\alpha <_{\mathcal{L}} w_\beta$ means $\alpha <_L \beta$. Since our path only moves left and α is on the true path, β is never eligible to act after stage t and never adds any new points to $\mathcal{L}$ after stage t.

In all cases, we see that no strategy $\beta \neq \alpha$ can add new points to $[z_\alpha, x_\alpha]$ after stage t.

(iii) Lemma 3.1.2: Let α be on the true path. If $\alpha * B$ is on the true path, then (b, x_α) is dense.

Proof: If $\alpha * B$ is on the true path, then it is eligible to act infinitely often and it adds points to make (b, x_α) isomorphic to $\mathbb{Q}$. Hence, (b, x_α) is dense.

(iv) Lemma 3.1.3: Let α be an R_e strategy on the true path. Then, (b, x_α) is dense if and only if $e \notin K$ if and only if $\alpha * B$ is on the true path.

Proof: If $e \notin K$, then by our local action for α, $\alpha * B$ is on the true path. By the Lemma 3.1.2, (b, x_α) is dense. If $e \in K$, then $\alpha * R$ is on the true path and by Lemma 3.1.1, (b, x_α) is not dense.

(v) Lemma 3.1.4: $0' \leq_T Den_{\mathcal{L}}(b) = \{c \mid (b, c) \text{ is dense in } \mathcal{L}\}$

Proof: We define a function $f : \mathbb{N} \to T$ by setting $f(e) = \alpha$ if α is the R_e strategy on the true path. Notice that f is computable from $Den_{\mathcal{L}}(b)$ since $f(0)$ is the unique R_0 strategy and by Lemma 3.1.3,

$$f(e+1) = \begin{cases} f(e) * B & \text{if } (b, x_{f(e)}) \text{ is dense} \\ \\ f(e) * R & \text{if } (b, x_{f(e)}) \text{ is not dense} \end{cases}$$

We can compute $0'$ from f using Lemma 3.1.3 since

$$n \in K \text{ if and only if } (b, x_{f(n)}) \text{ is not dense.}$$

and hence $n \in K$ if and only if $f(n+1) = f(n) * R$. Therefore, we have $K \leq_T f$ and $f \leq_T Dis$, so $K \leq_T Dis$.

(vi) Effective Construction

The construction is effective because there are only a finite number of things done at each stage for each requirement R_n. Also, since there are $n \notin K$, we will build at least one infinite $\mathbb{Q}$, the construction will use all of the natural numbers. Hence the domain of L is $\mathbb{N}$, which is computable. This implies that to compare i and j in order, we need just run the construction until both elements appear and compare where they land in L.

3.2 Construction II

Theorem: There is a computable linear order $\mathcal{L}$ with least element b such that $0'' \leq_T Den_{\mathcal{L}}(b) = \{c \mid (b, c) \text{ is dense in } \mathcal{L}\}$.

Proof: In order to prove this theorem, we want to build a computable linear order $\mathcal{L}$ around a least element b such that the interval (b, x_n) is dense if and only if $n \in Inf$ where $Inf = \{e \mid W_e \text{ is infinite}\}$. We have the following requirements:

$$R_n : n \in Inf \text{ if and only if } (b, x_n) \in Den_{\mathcal{L}}$$

with ordering $R_0 < R_1 < R_2 < ...$

The basic strategy for a single requirement R_0 is similar to the strategy in the $0'$ construction. We want to put down an x_0 such that

$$b <_{\mathcal{L}} x_0.$$

Our goal is to do one of two things in the interval (b, x_0) depending on whether $0 \in Inf$ or not. If $0 \in Inf$, then we want to make the open interval (b, x_0) isomorphic to $\mathbb{Q}$. This action makes (b, x_0) dense. If $0 \notin Inf$, then we want to make (b, x_0) not dense.

To accomplish this goal, at each stage s, we check whether W_0 adds a new point. If so, then we partially densify (b, x_0) by adding new points in the interval (b, x_0).

$$b <_{\mathcal{L}} \text{new} <_{\mathcal{L}} w_k <_{\mathcal{L}} \text{new} <_{\mathcal{L}} ... <_{\mathcal{L}} \text{new} <_{\mathcal{L}} w_1 <_{\mathcal{L}} \text{new} <_{\mathcal{L}} w_0 <_{\mathcal{L}} \text{new} <_{\mathcal{L}} x_0$$

In this case, we regard R_0 as a building state requirement and in the general construction, we will be taking the B outcome (for building).

On the other hand, if W_0 does not add a new point, then we want to stop building (at least temporarily) our copy of $\mathbb{Q}$ and restrain the interval (b, x_0) from densifying. To do this, we want to add two points $b < z_0 < y_0 < x_0$ as immediate predecessors of x_0 and not allow any points to enter between $[z_0, x_0]$. We regard R_0 as a restraining state requirement. In the general construction, we will be taking the R outcome (for restraining). However, if we see W_0 get a new element at a later stage, we initialize z_0 and y_0 in the sense that we forget that these points had any special significance and we regard the parameters y_0 and z_0 as undefined. When we partially densify (b, x_0), we treat the points formally labeled by y_0 and z_0 as any other points in (b, x_0) and add a new point between them.

We will be setting up a tree of strategies $T = \{B, R\}^{<\omega}$ such that $B <_L R$. The basic universal strategy is to stay in a restrain state until W_n adds a new point. In the restrained stage, we will not allow any new points to be introduced in the interval $[z_n, x_n]$. When W_n grows, we want to switch to the build strategy. In this strategy, we will forget about any y_n and z_n designation and continue to build a copy of $\mathbb{Q}$ between b and x_n.

It remains to describe where α places its witness point x_α when it is first eligible to act. To describe this placement, we treat x_α as an entity w_α (to keep with previous notation) and write

$$w_\alpha <_{\mathcal{L}} w_\beta$$

as notation for $x_\alpha <_{\mathcal{L}} x_\beta$. For distinct strategies α and β, we place $w_\alpha <_{\mathcal{L}} w_\beta$ if and only if either

- $\beta <_L \alpha$ (β is to the left of α in the tree of strategies)
- or $\alpha \subseteq \beta$ and $\beta(|\alpha|) = B$ (β extends $\alpha * B$)
- or $\beta \subseteq \alpha$ and $\alpha(|\beta|) = R$ (α extends $\beta * R$).

Local Action for α for R_e :

(i) When R_α is first eligible to act, place a new witness x_α in $\mathcal{L}$ (or choose a point x_α in $\mathcal{L}$ satisfying the order conditions above).

Let $\hat{s}$ be the last stage at which α was eligible to act (with $\hat{s} = 0$ if this is the first time α is eligible to act).

(ii) If $W_{e,s} \neq W_{e,\hat{s}}$, then initialize y_α and z_α (if they are defined), partially densify (b, x_α), and take outcome $\alpha * B$.

(iii) If $W_{e,s} = W_{e,\hat{s}}$, add points z_α and y_α as immediate predecessors of x_α (unless they are already defined) and take outcome $\alpha * R$.

Note a property of the placement of points in our linear order $\mathcal{L}$: only α is allowed to place z_α and y_α into the interval (b, x_α). This protects our interval against other witnesses encroaching on its territory.

<u>Construction</u>

Stage 0: We begin with the empty set. So, we need to set down point b.

Stage s+1: Follow the path down the tree of strategy to level s as directed by the action of the strategies eligible to act. When we reach level $s+1$, end the stage.

<u>Verification</u>:

(i) True Path

First let s be an α-stage if α is eligible to act at stage s. The true path in our tree of strategies is the leftmost path visited infinitely often. If α is on the true path, then either there are infinitely many α-stages when we take $\alpha * B$ and $\alpha * B$ is on the true path, or there exists a stage t such that for all α-stages after t, we take $\alpha * R$ and $\alpha * R$ is on the true path. Note that if α is on the true path, then there exists only finitely many stages s when the true path is to the left of α.

(ii) Lemma 3.2.1: Let α be on the true path. If $\alpha * R$ is on the true path, then (b, x_α) is not dense.

Proof: Fix the least stage t such that $\alpha * R$ is on the path at stage t and the path is never to the left of $\alpha * R$ after t. At stage t, α defines y_α and z_α and places them so that $z_\alpha <_{\mathcal{L}} y_\alpha <_{\mathcal{L}} x_\alpha$ and $(z_\alpha, x_\alpha) = \{y_\alpha\}$. Since α never takes outcome B after stage t, these witnesses y_α and z_α are never initialized by α. Therefore they remain defined forever. To prove that (b, x_α) is not dense, it suffices to show that no strategy can add points to $[z_\alpha, x_\alpha]$ after stage t. There are two possibilities.

(a) α places points in (z_α, x_α) after stage t.

If α places a point in $[z_\alpha, x_\alpha]$, then it takes outcome $\alpha * B$. Since $\alpha * B <_L \alpha * R$ and the path is never left of $\alpha * R$ after stage t, α cannot place any points in $[z_\alpha, x_\alpha]$ after stage t.

(b) Another strategy β places points in $[z_\alpha, x_\alpha]$. In this case, we must have $w_\alpha <_{\mathcal{L}} w_\beta$. Consider the ways this could happen.

(i) If $\beta \subseteq \alpha$, then $w_\alpha <_{\mathcal{L}} w_\beta$ means $\alpha(|\beta|) = R$ and hence $\beta * R \subseteq \alpha$. If β adds points to $\mathcal{L}$, it takes outcome $\beta * B$ which is left of $\alpha * R$. Therefore, β adds no more points after stage t.

(ii) If $\alpha \subseteq \beta$, then $w_\alpha <_{\mathcal{L}} w_\beta$ means $\beta(|\alpha|) = B$ and hence $\alpha * B \subseteq \beta$. Since α takes outcome R at all α-stages after t, β is never eligible to act after stage t and hence does not add any points after stage t.

(iii) If α and β are incomparable, then $w_\alpha <_{\mathcal{L}} w_\beta$ means $\beta <_L \alpha$. However, the path is never to the left of $\alpha * R$ after stage t and therefore, never to the left of α after stage t. Hence, β cannot add points after stage t.

In all cases, we see that no strategy $\beta \neq \alpha$ can add new points to $[z_\alpha, x_\alpha]$ after stage t.

(iii) Lemma 3.2.2: Let α be on the true path. If $\alpha * B$ is on the true path, then (b, x_α) is dense.

Proof: If $\alpha * B$ is on the true path, then it is eligible to act infinitely often and it adds points to make (b, x_α) isomorphic to $\mathbb{Q}$. Hence, (b, x_α) is dense.

(iv) Lemma 3.2.3: Let α be an R_e strategy on the true path. Then, (b, x_α) is not dense if and only if $e \in Inf$ if and only if $\alpha * R$ is on the true path.

Proof: If $e \notin Inf$, then by our local action for α, $\alpha * R$ is on the true path. By the Lemma 3.2.1, (b, x_α) is not dense. If $e \in Inf$, then $\alpha * B$ is on the true path and by Lemma 3.2.2, (b, x_α) is dense.

(v) Lemma 3.2.4: $0'' \leq_T Den_{\mathcal{L}}(b) = \{c \mid (b, c) \text{ is dense in } \mathcal{L}\}$

Proof: We define a function $f : \mathbb{N} \to T$ by setting $f(e) = \alpha$ if α is the R_e strategy on the true path. Notice that f is computable from $Den_{\mathcal{L}}(b)$ since $f(0)$ is the unique R_0 strategy and by Lemma 3.2.3,

$$f(e+1) = \begin{cases} f(e) * B & \text{if } (b, x_{f(e)}) \text{ is dense} \\ f(e) * R & \text{if } (b, x_{f(e)}) \text{ is not dense} \end{cases}$$

We can compute $0''$ from f using Lemma 3.2.3 since

$$n \in Inf \text{ if and only if } (b, x_{f(n)}) \text{ is dense.}$$

and hence $n \in Inf$ if and only if $f(n+1) = f(n) * B$. Therefore, we have $Inf \leq_T f$ and $f \leq_T Den_{\mathcal{L}}(b)$, so $Inf \leq_T Den_{\mathcal{L}}(b)$.

(vi) Effective Construction

The construction is effective because there are only a finite number of things done at each stage for each requirement R_n. Also, since there are $n \in Inf$, we will build at least one infinite $\mathbb{Q}$, the construction will use all of the natural numbers. Hence the domain of L is $\mathbb{N}$, which is computable. This implies that to compare i and j in order, we need just run the construction until both elements appear and compare where they land in L.

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