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# Marriage Matching: A Conjecture of Donald Knuth

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**Marriage Matching: A Conjecture of Donald Knuth**

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## **Abstract**

Variations of the Gale-Shapley algorithm have been used and studied extensively in real world markets. Examples include matching medical residents with residency programs, the kidney exchange program and matching college students with on-campus housing. The performance of the Gale-Shapley marriage matching algorithm (1962) has been studied extensively in the special case of men's and women's preferences random. We drop the assumption that women's preferences are random and show that  $E_n/n \ln n \rightarrow 1$ , where  $E_n$  is the expected number of proposals made when the men-propose Gale-Shapley algorithm is used to match  $n$  men with  $n$  women. This establishes in spirit a conjecture of Donald Knuth (1976, 1997) of thirty years standing. Under the same assumptions, we also establish bounds on the expected ranking by a woman of her assigned mate. Bounds on men's rankings of their assigned mates follow directly from the conjecture.

**Journal of Economic Literature Classification:** C78, D63, D70

**Keywords:** Two-Sided Matching, Gale-Shapley algorithm

## 1. Introduction.

One basic line of research in the field of two-sided matching problems is concerned with the performance of the Gale-Shapley algorithm (1962). In particular, when the Gale-Shapley algorithm is used to match  $n$  men with  $n$  women, how many proposals will be made and what will an individual's ranking of his or her assigned mate be? This paper addresses these questions in the context of men's preferences random and women's preferences arbitrary. That case is important as a transitional case halfway between the well-studied case in which all preferences are random and the more general setting of men's and women's preferences arbitrary.

Before we address these questions about proposals and rankings in the transitional case, we need to describe the marriage matching problem and the Gale-Shapley algorithm. Given  $n$  men,  $n$  women and for each individual a preference ranking of the  $n$  members of the opposite sex, the problem is to find a stable matching into  $n$  couples, each consisting of a man and a woman. A matching is *stable* if there do not exist a man and a woman such that each prefers the other to his assigned mate. We will call the  $n$  ranking order lists of each group (men or women) a preference profile.

As shown by Gale and Shapley (1962), the Gale-Shapley algorithm always produces a stable matching. We will actually work with the McVitie-Wilson version (1971) of the Gale-Shapley algorithm which produces a stable matching in  $n$  rounds. In round 1 the first man proposes to his most preferred woman. She tentatively accepts and round 1 ends. In round  $i > 1$ , the  $i$ th man proposes to his most preferred woman. If she has not been proposed to before, she tentatively accepts and round  $i$  ends. Otherwise she tentatively accepts man  $i$  if she prefers him to her current match, or rejects him otherwise. Then the unmatched man, either man  $i$  or the man he displaced, proposes to his most preferred woman among those who have not yet rejected him. Round  $i$  continues in this manner until some woman receives her first proposal. The  $n$  tentative matches that exist after round  $n$  make up the final matching. The McVitie-Wilson algorithm makes exactly the same proposals as the Gale-Shapley algorithm, and yields the same stable matching. It is easier to work with for our purposes since the proposals are made sequentially rather than

in large batches.

In all that follows we will be discussing the number of proposals made and men's and women's rankings of their mates when the men-propose Gale-Shapley algorithm is used to match  $n$  men with  $n$  women. Wilson (1972) proved the following:

**Proposition 1.** *(Wilson, 1972) For any profile of women's preferences, if men's preferences are generated randomly, then the expected number of proposals is bounded above by  $n(1 + \frac{1}{2} + \dots + \frac{1}{n})$ .*

The main result of this paper, Proposition 3 stated below, concerns a lower bound for the expected number of proposals when men's preferences are random and women's preferences arbitrary, but first we review some results proven under the stronger hypotheses of random preferences for both men and women. Knuth (1997) describes the following result, which establishes a lower bound for the expected number of proposals when preferences are random, as the most important result in his book on marriage matching.

**Proposition 2.** *(Knuth, 1997) If both men's and women's preferences are random, then the expected number of proposals is bounded **below** by  $n(1 + \frac{1}{2} + \dots + \frac{1}{n}) - K \ln^4 n$  for some constant  $K$ .*

Under the hypotheses of Proposition 2, Pittel (1989) strengthened the conclusion of Proposition 2, showing that as  $n$  increases the number of proposals divided by  $n \ln n$  rapidly approaches, in probability, 1.

But what can be said about the performance of the Gale-Shapley algorithm when the assumption that preferences are random is dropped? In particular what can be said, beyond the conclusion of Proposition 1, in the transitional case presented in the hypotheses of Proposition 1—men's preferences random, women's arbitrary? This case is of interest because it is a step towards the more realistic scenario in which men's preferences as well as women's would be expected to exhibit some degree of positive correlation. It is probably due to Knuth's recognition of the importance of this transitional case that the following conjecture is the only conjecture placed in the body of the text in his book.

It also appears in the problem section. It combines a Proposition 2-like conclusion with the weaker hypotheses of Proposition 1—men’s preferences random, women’s preferences arbitrary and fixed.

**Conjecture.** (*Knuth, 1997*) *For any profile of women’s preferences, if men’s preferences are random, then the expected number of proposals is bounded below by*

$$(n + 1)\left(1 + \frac{1}{2} + \dots + \frac{1}{n}\right) - n.$$

Knuth’s conjecture can be motivated as follows. If women’s preferences are identical and men’s random, Knuth (1997) proved that the expected number of proposals is  $(n + 1)\left(1 + \frac{1}{2} + \dots + \frac{1}{n}\right) - n$ . Any profile of women’s preferences other than preferences identical is less highly correlated. Then one would expect the matching algorithm to produce more exchanges of men accomplished by more proposals. That said, the conjecture is perhaps overly optimistic in form. If true, it would constitute a best possible result, since it posits a lower bound that holds for all  $n$  and is attainable when women’s preferences are identical. In contrast, Knuth’s own result (Proposition 2 above), although proven under stronger hypotheses than those of the conjecture, does not establish an attainable lower bound, or even a specific bound since  $K$  is unspecified.

The following proposition establishes Knuth’s conjecture in spirit, in that it implies that, for any  $\epsilon > 0$ , the product of  $1 - \epsilon$  and the conjectured lower bound is in fact a lower bound, for sufficiently large  $n$ .

**Proposition 3.** *If for each  $n > 0$   $P_n$  is a given preference profile for  $n$  women and  $E_n$  is the expected number of proposals when  $n$  men with random preferences are matched with  $n$  women with the given preference profile  $P_n$ , then*

$$E_n / \left( (n + 1)\left(1 + \frac{1}{2} + \dots + \frac{1}{n}\right) - n \right) \rightarrow 1$$

and equivalently

$$E_n / n \ln n \rightarrow 1$$

Now let  $R_n$  be the expected sum of the men’s rankings of their mates under the scenario governing Proposition 3. Since the number of proposals a man makes is *equal* to

his ranking of his assigned mate, we have the following.

**Corollary.**  $R_n/n \ln n \rightarrow 1$ .

We also establish attainable bounds on the expected ranking by an arbitrary man of his assigned mate.

**Proposition 4.** *If  $n$  men with random preferences are matched with  $n$  women with arbitrary preferences, the expected ranking by a man of his assigned mate is bounded above by  $\frac{n+1}{2}$  and below by 1, and these bounds are attainable.*

In addition to the theoretical results of Propositions 1, 3 and 4 above, there is another study of marriage matching with non-random lists using computer simulations due to Caldarelli and Copocci (2001), who introduce correlation into preference profiles, then run the Gale-Shapley algorithm. They find that preferred men tend to be matched with preferred women, and that there is less difference in satisfaction between proposers and proposees than when lists are random.

For an overview of the matching mechanism literature, see Roth and Sotomayer (1990). Most papers in the two-sided matching literature focus on problems in more complex real-world markets, for example Roth, Sönmez and Ünver (2004) and Chen and Sönmez (2002); or on strategic considerations, that is, strategic reporting of preferences, for example, Roth and Vande Vate (1991) and Demange, Gale and Sotomayer (1987).

The paper is organized as follows. Section 2 contains preliminaries. Propositions 3 and 4 are proven in Section 3. Section 4 establishes bounds on the expected rankings by women of their assigned mates. Section 5 contains concluding remarks.

## 2. Preliminaries.

In the introduction, the description of the marriage matching problem, the definition of stable matching, the description of the McVitie-Wilson version of the Gale-Shapley algorithm and the statement of the results required little or no notation. For Sections 3 and 4 we require the following.

In the marriage matching problem, men in the set  $M = \{m_1, m_2, \dots, m_n\}$  are to be matched with women in  $W = \{w_1, w_2, \dots, w_n\}$ . Each individual has an ordered list of preferences over the  $n$  members of the opposite sex. A *matching* of  $M$  with  $W$  is a function  $\mu: M \cup W \rightarrow M \cup W$  such that  $\mu(M) \subseteq W$ ,  $\mu(W) \subseteq M$  and for  $m \in M$ ,  $w \in W$ ,  $\mu(m) = w$  if and only if  $\mu(w) = m$ . For  $m \in M$  and  $w \in W$ ,  $r_w(m)$  is the rank of  $m$  by  $w$ , which takes on values from 1, for  $w$ 's most preferred man, to  $n$  for her least preferred man.

A *random variable*  $X_n$  is a function from the set  $MP$  of men's preferences profiles to the nonnegative reals,  $X_n: MP \rightarrow \mathfrak{R}^+$ . The expected value of a random variable is

$$E(X_n) = \sum_{t \in MP} \text{Prob}(t) X_n(t) = \left( \sum_{t \in MP} X_n(t) \right) / (n!)^n,$$

reflecting the assumption in this paper that men's preferences are random, that is, chosen from  $MP$  with the uniform probability distribution. In Section 1,  $E_n$ , the expected number of proposals and  $R_n$ , the expected sum of men's rankings of their mates, are examples of expected values of random variables. Also,  $E(X_n: X_n > a)$  is the expected value of  $X_n$  given that  $X_n > a$ ; and the probability of the event  $X_n = a$  will be written  $\text{Prob}(X_n = a)$ .

If  $(a_n)$  is a real valued sequence then

$$\limsup a_n = \lim_{n \rightarrow +\infty} LUB\{a_n, a_{n+1}, \dots\} \text{ and } \liminf a_n = \lim_{n \rightarrow +\infty} GLB\{a_n, a_{n+1}, \dots\}$$

where  $LUB$  is the least upper bound and  $GLB$  is the greatest lower bound. If  $\limsup a_n = \liminf a_n$ , then  $\lim_{n \rightarrow +\infty} a_n$  exists and is equal to  $\limsup a_n$ .

### 3. Proofs of Propositions 3 and 4.

#### Proof of Proposition 3.

Recall the hypotheses of Proposition 3: for each  $n > 0$   $P_n$  is a given preference profile of  $n$  women and  $E_n$  is the expected number of proposals when  $n$  men with random preferences are matched by the McVitie-Wilson version of the Gale-Shapley algorithm to  $n$  women with the given preference profile  $P_n$ .

Proposition 1 implies  $\limsup E_n / ((n+1)(1 + \frac{1}{2} + \dots + \frac{1}{n}) - n) \leq 1$ , so that it remains to show  $\liminf E_n / ((n+1)(1 + \frac{1}{2} + \dots + \frac{1}{n}) - n) \geq 1$  or equivalently  $\liminf E_n / n \ln n \geq 1$ .



For the remainder of the proof we consider a run of the McVitie-Wilson algorithm in which the hypotheses of Proposition 3 hold, the men propose in random order and the women are numbered in the order in which they receive their first proposals. It will help to think of a woman receiving a number *when* she receives her first proposal.

We need the following simple lemma, whose statement and proof are closely related to the statement of Proposition 1 and its proof by Wilson (1972).

**Lemma 1.** *The expected number of proposals made to  $w_j$  during the first  $L$  rounds satisfies*

$$E(\#\text{proposals to } w_j) \leq \begin{cases} 0 & \text{if } 1 \leq L < j \\ 1 + \frac{1}{n-j} + \frac{1}{n-j-1} + \dots + \frac{1}{n-L+1} & \text{if } j \leq L \leq n \end{cases}$$

In the special cases  $L = j$ ,  $1 + \frac{1}{n-j} + \frac{1}{n-j-1} + \dots + \frac{1}{n-j+1}$  is to be read as 1.

*Proof.* The conclusion holds for  $L \leq j$ , since the  $j$ th round ends when  $w_j$  receives her first proposal. Assume men have no memory, so that men choose their next proposal by drawing numbered balls from an urn with replacement. This assumption can only increase the number of proposals to  $w_j$  in the first  $L$  rounds. Under this assumption, the expected number of proposals to  $w_j$  in round  $l$ ,  $j + 1 \leq l \leq L$ , is at least

$$\begin{aligned} & \sum_{i=1}^{+\infty} i \left( \frac{1}{(n-l+2)} \right)^i \left( \frac{n-l+1}{n-l+2} \right) \\ &= (n-l+1) \left( \sum_{s=1}^{+\infty} \frac{1}{(n-l+2)^s} \right) \left( \sum_{i=1}^{+\infty} \frac{1}{(n-l+2)^i} \right) \end{aligned}$$

which, summing geometric series, is

$$(n-l+1) \left( \frac{1}{n-l+1} \right) \left( \frac{1}{n-l+1} \right) = \frac{1}{n-l+1}$$

Summing over rounds  $j + 1$  to  $L$  completes the proof. ■

The following lemma says that the expected value of the multiplicative inverse of a random variable that takes on two positive values is greater than or equal to the multiplicative inverse of the expected value of the random variable.

**Lemma 2.** If  $x > 0$ ,  $p, q, a, b \geq 0$  and  $p + q = 1$ , then  $p\left(\frac{1}{x+a}\right) + q\left(\frac{1}{x+b}\right) \geq \frac{1}{x+pa+qb}$

*Proof.*

$$\frac{p}{x+a} + \frac{q}{x+b} - \frac{1}{x+pa+qb}$$

simplifies to  $\frac{pq(a-b)^2}{(x+a)(x+b)(x+pa+qb)} \geq 0$ . ■

Now we can begin to build a lower bound for the expected number of proposals.

**Lemma 3.** The expected number of proposals made to  $w_j$ ,  $1 \leq j \leq n - 1$ , during all  $n$  rounds satisfies

$$E(\# \text{ proposals to } w_j) \geq \sum_{k=j+1}^n \frac{1}{n - k + 2 + \frac{1}{n-j} + \frac{1}{n-j-1} + \dots + \frac{1}{n-k+2}}$$

*Proof.* Fix  $j$ ,  $1 \leq j \leq n - 1$ . For  $k > j$ , define *short round  $k$*  to begin when round  $k$  begins and to end when either

1) round  $k$  ends (that is, when an unmatched woman is proposed to)

or

2) a woman who is matched with a man who has proposed to  $w_j$  is proposed to (one such woman is  $w_j$ ).

Suppose exactly  $r$  men have proposed to  $w_j$  before round  $k$ . Then  $1 \leq r \leq k - 1$  and

$$E(\# \text{ proposals to } w_j \text{ in round } k) \geq$$

$$E(\# \text{ proposals to } w_j \text{ in short round } k) \geq \frac{1}{n-k+1+r} \tag{1}$$

Inequality (1) holds since the man who starts round  $k$  is as likely to—and any man displaced without ending short round  $k$  is at least as likely to—propose to  $w_j$  as to any of the other  $n - k + r$  women who would end short round  $k$ .

For each  $r$ , let  $p_r$  be the probability that exactly  $r$  men have proposed to  $w_j$  before round  $k$  begins. Then the following inequalities hold by (1), Lemma 2 applied repeatedly

and Lemma 1 with  $L = k - 1$ , respectively.

$$\begin{aligned}
E(\# \text{ proposals to } w_j \text{ in round } k) &\geq \sum_{r=1}^{k-1} p_r \times \frac{1}{n - k + 1 + r} \\
&\geq 1/(n - k + 1 + \sum_{r=1}^{k-1} p_r r) \\
&\geq \frac{1}{n - k + 2 + \frac{1}{n-j} + \frac{1}{n-j-1} + \dots + \frac{1}{n-k+2}}
\end{aligned}$$

Summing over rounds  $j + 1$  to  $n$  completes the proof. ■

We can now prove Proposition 3. Summing the inequalities in the conclusion of Lemma 3 over  $w_1, w_2, \dots, w_{n-1}$ , the expected number of proposals  $E_n$  satisfies

$$E_n \geq \sum_{j=1}^{n-1} \sum_{k=j+1}^n \frac{1}{n - k + 2 + \frac{1}{n-j} + \frac{1}{n-j-1} + \dots + \frac{1}{n-k+2}}$$

On the graph of  $y = 1/x$ , a comparison of the sum  $\frac{1}{n-j} + \frac{1}{n-j-1} + \dots + \frac{1}{n-k+2}$  and the integral  $\int_1^n \frac{dx}{x}$  yields

$$E_n \geq \sum_{j=1}^{n-1} \sum_{k=j+1}^n \frac{1}{n - k + 2 + \ln n}$$

On the graph of  $y = 1/x$ , a comparison of the sum  $\sum_{k=j+1}^n \frac{1}{n-k+2+\ln n}$  and the integral  $\int_{2+\ln n}^{n-j+2+\ln n} \frac{dx}{x}$  yields

$$\begin{aligned}
E_n &\geq \sum_{j=1}^{n-1} (\ln(n - j + 2 + \ln n) - \ln(2 + \ln n)) \\
&\geq \sum_{j=1}^{n-1} \ln(n - j + 2) - (n - 1) \ln(2 + \ln n) \\
&= \sum_{j=3}^{n+1} \ln j - (n - 1) \ln(2 + \ln n)
\end{aligned}$$

Furthermore,  $\lim_{n \rightarrow +\infty} \frac{(n-1) \ln(2+\ln n)}{n \ln n} = 0$  by L'Hopital's rule, and a comparison on the graph of  $y = \ln x$  of the sum  $\sum_{j=3}^{n+1} \ln j$  and the integral  $\int_2^{n+1} \ln x dx$  yields

$$\frac{\sum_{j=3}^{n+1} \ln j}{n \ln n} \geq \frac{\int_2^{n+1} \ln x dx}{n \ln n} = \frac{(x \ln x - x)|_2^{n+1}}{n \ln n} = \frac{(n+1) \ln(n+1) - n+1 - 2 \ln 2 + 2}{n \ln n} \rightarrow 1.$$

Therefore  $\liminf \frac{E_n}{n \ln n} \geq 1$ , which completes the proof of Proposition 3. ■

#### Proof of Proposition 4.

Clearly  $E(r_m(\mu(m))) \geq 1$ . Also,  $r_m(\mu(m)) = 1$  if  $m$  is the first choice of every woman, since  $m$ 's first proposal will be accepted. Thus, 1 is an attainable upper bound.

To show that  $E(r_m(\mu(m))) \leq (n+1)/2$ , consider first the McVitie-Wilson version of the women-propose Gale-Shapley algorithm. Since men's preferences are random,  $m$ 's first proposal will be from a woman whose expected rank by  $m$  is  $(n+1)/2$ . In the men-propose Gale-Shapley matching,  $m$  does no worse than his first proposal in the women-propose Gale-Shapley algorithm. Therefore  $E(r_m(\mu(m))) \leq (n+1)/2$ .

Now suppose women's preferences are identical and  $m$  is the last choice of every woman. The sum of women's expected rankings of their assigned mates is  $n(n+1)/2$  for any matching algorithm, since for every matching regardless of men's preferences, some woman is matched with the first ranked man, some woman is matched with the second-ranked man, etc. Since for every men's preferences profile each woman does at least as well in the women-propose matching as in the men-propose matching and for both matchings the expected sum of women's rankings of their assigned mates is  $n(n+1)/2$ , each woman does exactly as well. Therefore in every instance the women-propose matching and the men-propose matching are identical. Also, in the women-propose Gale-Shapley matching,  $m$ 's expected ranking of his assigned mate is  $(n+1)/2$ , since his first proposal is the last proposal made. Therefore also in the men-propose matching  $E(r_m(\mu(m))) = (n+1)/2$ . The upper bound of Proposition 4 is attainable. ■

#### 4. Women's Rankings of their Assigned Mates.

In this section, as above, the men-propose Gale-Shapley algorithm is used to match  $n$  men with  $n$  women. Then we have

**Proposition 5.** *For any women's preference profile, if  $w \in W$  and men's preferences are random, then*

$$E(r_w(\mu(w))) \leq \frac{n+1}{2}$$

*Proof:* Fix a women's preference profile and  $w \in W$  and assume men's preferences are random. By relabeling the men we assume  $r_w(m_i) = i$  for  $1 \leq i \leq n$ . For  $k \in \{1, 2, \dots, n-1\}$

$$E(r_w(\mu(w)): r_w(\mu(w)) > k-1) = p_k k + (1-p_k)E(r_w(\mu(w)): r_w(\mu(w)) > k) \quad (2)$$

where  $p_k$  is the probability that  $m_k$  proposes to  $w$  during a run of the Gale-Shapley algorithm, given that  $m_1, m_2, \dots, m_{k-1}$  do not.

During a run of the Gale-Shapley algorithm,  $m_k$  must propose to some woman other than  $\mu(m_1), \mu(m_2), \dots, \mu(m_{k-1})$ . Under the assumption  $r_w(\mu(w)) > k-1$ , none of these  $k-1$  women is  $w$ . Therefore under the assumption  $r_w(\mu(w)) > k-1$ , the event that  $m_k$ 's first proposal to a woman other than  $\mu(m_1), \mu(m_2), \dots, \mu(m_{k-1})$  is to  $w$  has probability  $\frac{1}{n-(k-1)}$ . It follows that  $p_k \geq \frac{1}{n-k+1}$ , so that if  $A > k$ , then

$$p_k k + (1-p_k)A \leq \left(\frac{1}{n-k+1}\right)k + \left(\frac{n-k}{n-k+1}\right)A \quad (3)$$

By (2) and (3) with  $A = E(r_w(\mu(w)): r_w(\mu(w)) > k)$ , for  $k = \{1, 2, \dots, n-1\}$ ,

$$E(r_w(\mu(w)): r_w(\mu(w)) > k-1) \leq \left(\frac{1}{n-k+1}\right)k + \left(\frac{n-k}{n-k+1}\right)E(r_w(\mu(w)): r_w(\mu(w)) > k) \quad (4)$$

Applying (4) successively with  $k = 1, 2, \dots, n-1$ ,

$$\begin{aligned} E(r_w(\mu(w))) &= E(r_w(\mu(w)): r_w(\mu(w)) > 0) \\ &\leq \frac{1}{n} + \frac{n-1}{n} E(r_w(\mu(w)): r_w(\mu(w)) > 1) \\ &\leq \frac{1}{n} + \frac{n-1}{n} \left( \frac{2}{n-1} + \frac{n-2}{n-1} E(r_w(\mu(w)): r_w(\mu(w)) > 2) \right) \\ &= \frac{1}{n} + \frac{2}{n} + \frac{n-2}{n} E(r_w(\mu(w)): r_w(\mu(w)) > 2) \\ &\leq \frac{1}{n} + \frac{2}{n} + \frac{3}{n} + \frac{n-3}{n} E(r_w(\mu(w)): r_w(\mu(w)) > 3) \\ &\quad \cdot \\ &\quad \cdot \\ &\quad \cdot \\ &\leq \frac{1}{n} + \frac{2}{n} + \frac{3}{n} + \dots + \frac{n-1}{n} + \frac{1}{n} E(r_w(\mu(w)): r_w(\mu(w)) > n-1) \\ &= \frac{1}{n} + \frac{2}{n} + \frac{3}{n} + \dots + \frac{n-1}{n} + \frac{n}{n} \\ &= \frac{1}{n} \left( \frac{n(n+1)}{2} \right) = \frac{n+1}{2}. \quad \blacksquare \end{aligned}$$

The following corollary is immediate.

**Corollary to Proposition 5.** *Under the hypotheses of Proposition 5, the expected sum of women’s rankings of their assigned mates is less than or equal to  $\frac{n(n+1)}{2}$ .*

Notice that when women’s preferences are identical, the men-propose Gale-Shapley algorithm (in fact any matching) matches one woman with the first ranked man, one woman with the second ranked man, etc. Then the sum of women’s rankings of their mates is  $1 + 2 + \dots + n = \frac{n(n+1)}{2}$ . Therefore the upper bound of the corollary is attainable. The assumption that women’s preferences are identical allows us to invoke symmetry and conclude  $E(r_w(\mu(w))) = \frac{n+1}{2}$  for every  $w \in W$ . In other words the upper bound of Proposition 5 is not only attainable but attainable by all  $n$  women simultaneously.

Finally, we establish a lower bound for a woman’s expected ranking of her assigned mate under the men-propose Gale-Shapley algorithm.

**Proposition 6.** *For any women’s preference profile, if  $w \in W$  and men’s preferences are random, then*

$$E(r_w(\mu(w))) \geq \left(\frac{n+1}{n}\right) \left(\frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n+1}\right)$$

*Proof.* Fix women’s preferences,  $w \in W$  and assume men’s preferences are random. Relabel the men so that  $r_w(m_i) = n - i + 1$  for  $1 \leq i \leq n - 1$  and run the McVitie-Wilson version of the Gale-Shapley algorithm with  $m_i$  beginning round  $i$ .

For  $1 \leq k \leq n$ ,

$$\text{Prob}(r_w(\mu(w)) \geq n - k + 1) \geq \frac{k}{n(n-k+1)} \tag{5}$$

since 1)  $r_w(\mu(w)) \geq n - k + 1$  if  $w$  is proposed to in the first  $k$  rounds and  $w$  is not proposed to in the last  $n - k$  rounds; 2)  $\frac{k}{n}$  is the probability that  $w$  is proposed to in the first  $k$  rounds; 3) the probability that  $w$  is not proposed to in the last  $n - k$  rounds is greater than or equal to  $\frac{1}{n-k+1}$ . (Notice that the word “if ” in 1) cannot in general be replaced by “if and only if,” and that the words “greater than or equal to” in 3) cannot in general be replaced by “equal to,” since during the last  $n - k$  rounds there may be times when a man who has proposed to  $w$  in the first  $k$  rounds is proposing.)

$$\begin{aligned}
& \text{Then } E(r_w(\mu(w))) = \sum_{k=1}^n k \text{ Prob}(r_w(\mu(w)) = k) = \sum_{k=1}^n \text{Prob}(r_w(\mu(w)) \geq k) \\
& = \sum_{k=1}^n \text{Prob}(r_w(\mu(w)) \geq n - k + 1) \text{ so that by (5), } E(r_w(\mu(w))) \geq \sum_{k=1}^n \frac{k}{n(n-k+1)} \\
& = \frac{1}{n} \sum_{k=1}^n \frac{n-k+1}{k} = \frac{1}{n} (\sum_{k=1}^n \frac{n+1}{k} - \sum_{k=1}^n \frac{k}{k}) = \left(\frac{n+1}{n}\right) \left(\frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n+1}\right). \quad \blacksquare
\end{aligned}$$

Consider the preference profile in which  $r_w(m_i) = n - i + 1$  for all  $i$  and  $r_{w'}(m_i) = i$  for all  $i$ , all  $w' \in W - \{w\}$ . For this example, the argument justifying (5) yields  $\text{Prob}(r_w(\mu(w)) \geq n - k + 1) = \frac{k}{n(n-k+1)}$ , since the “if ” in 1) can be replaced by “if and only if” and “greater than or equal to” in 3) can be replaced by “equal to”. Therefore  $E(r_w(\mu(w))) = \left(\frac{n+1}{n}\right) \left(\frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n+1}\right)$ . In other words, the lower bound of Proposition 6 is attained by  $w$  if the other women’s preferences are exactly the reverse of hers.

The situation here is not like the situation surrounding Propositions 4 and 5, in that for the following corollary to Proposition 6, it does not seem likely that the lower bound is attainable.

**Corollary to Proposition 6.** *Under the hypotheses of Proposition 5, the expected sum of women’s rankings of their mates is greater than or equal to  $(n + 1) \left(\frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n+1}\right)$ .*

## 5. Concluding Remarks.

For the men-propose Gale-Shapley algorithm with men’s preferences random and women’s arbitrary, we have established attainable upper and lower bounds for the expected ranking by a man of his assigned mate; for the expected sum of men’s rankings of their assigned mates; and for the expected ranking by a woman of her assigned mate. We have also established an attainable upper bound for the expected sum of women’s rankings of their mates. The question of an attainable lower bound remains open.

## References

1. Caldarelli G. and A. Capocci, “Beauty and distance in the stable matching problem,” *Physica A* **300** (2001), 325-331.
2. Chen, Yan and Tayfun Sönmez, “Improving Efficiency of On-Campus Housing: An Experimental Study,” *American Economic Review* **92** (2002), 1669-1686.
3. Demange, Gabrielle, David Gale and Marilda Sotomayor, “A Further Note on the Stable Matching Problem,” *Discrete Applied Mathematics* **16** (1987), 217-22.
4. Gale, David and Lloyd S. Shapley, “College Admissions and the Stability of Marriage,” *American Mathematical Monthly* **69** (1962), 9-15.
5. Knuth, Donald E., *Mariages Stables et leurs relations avec d’ autres problèmes combinatoires*, Les Presses de l’ Université de Montréal, Montréal, (1976).
6. Knuth, Donald E., *Stable marriage and its relation to other combinatorial problems*, CRM Proceedings and Lecture Notes, Vol. 10 American Mathematical Society, (1997).
7. McVitie, D. G and L. B. Wilson, “Three procedures for the stable matching problem,” *ACM Algorithm 411. Comm. of the ACM* **14** (1971), 491-492.
8. Pittel, Boris, “The average number of stable matchings,” *Society for Industrial and Applied Mathematics* **2** (1989), 530-549.
9. Roth, Alvin E., Tayfun Sönmez and M. Utku Ünver, “Kidney Exchange,” *Quarterly Journal of Economics* **19** (2004), 457-488.
10. Roth, Alvin E. and John H. Vande Vate, “Incentives in Two-Sided Matching with Random Stable Mechanisms,” *Economic Theory* **1** (1991), 31-44.
11. Roth, Alvin E. and Marilda A. Oliveira Sotomayor, *Two-Sided matching: a study in game theoretic modeling and analysis*, Econometric Society Monographs, No. 18 Cambridge University Press, (1990).



12. Wilson, L. B. , “An analysis of the marriage matching assignment algorithm,” *BIT* **12** (1972), 569-75.