The Runge0Lenz Vector (continued)

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The Runge-Lenz Vector (continued)

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I. SYNOPSIS

We continue our discussion of the Runge-Lenz vector in a quantum mechanical context. The traditional form of the Runge-Lenz vector is obtained, and the commutation relations between the Runge-Lenz vector, the Hamiltonian, and the Angular Momentum are obtained using Maple.

II. INTRODUCTION

We had \[1\]

\[
\frac{\vec{A} \cdot \vec{r} + \vec{r} \cdot \vec{A}}{2} = -\frac{1}{2Ze^2\mu} \left( \vec{L}^2 + \frac{3\hbar^2}{2} \right) + |\vec{r}| \tag{2.1}\]

so that

\[
\frac{1}{2} \left( \left\{ \frac{1}{2Ze^2\mu} \left( \vec{L} \otimes \vec{p} - \vec{p} \otimes \vec{L} \right) + \vec{r} \right\} \otimes \vec{r} + \vec{r} \otimes \left\{ \frac{1}{2Ze^2\mu} \left( \vec{L} \otimes \vec{p} - \vec{p} \otimes \vec{L} \right) + \vec{r} \right\} \right)
\]

or, expanding

\[
\frac{1}{4Ze^2\mu} \left( \left( \vec{L} \otimes \vec{p} \right) \otimes \vec{r} - \left( \vec{p} \otimes \vec{L} \right) \right) \otimes \vec{r} + \vec{r} \otimes \left( \vec{L} \otimes \vec{p} - \vec{p} \otimes \vec{L} \right) - \vec{r} \otimes \left( \vec{p} \otimes \vec{L} \right) + \vec{r} \otimes \vec{r}
\]

Since \(\vec{r} \otimes \vec{r}\) and its inverse are both zero (no partial derivatives here to goof us up!), we have

\[
\frac{1}{4Ze^2\mu} \left( \left( \vec{L} \otimes \vec{p} \right) \otimes \vec{r} - \left( \vec{p} \otimes \vec{L} \right) \right) \otimes \vec{r} + \vec{r} \otimes \left( \vec{L} \otimes \vec{p} - \vec{p} \otimes \vec{L} \right) - \vec{r} \otimes \left( \vec{p} \otimes \vec{L} \right) + \vec{r} \otimes \vec{r} \tag{2.2}\]

where we need to be careful about expanding the triple cross products, since these are not ordinary vectors. We have

\[
\vec{A} \otimes (\vec{B} \otimes \vec{C}) = \vec{A} \otimes \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
B_x & B_y & B_z \\
C_x & C_y & C_z \\
\end{vmatrix} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
A_x & A_y & A_z \\
B_y C_z - B_z C_y & B_z C_x - B_x C_z & B_x C_y - B_y C_x \\
\end{vmatrix}
\]

which expands to

\[
\vec{A} \otimes (\vec{B} \otimes \vec{C}) = \hat{i} [A_y (B_z C_y - B_y C_z) - A_z (B_z C_y - B_y C_z)] + \hat{j} [A_z (B_y C_z - B_z C_y) - A_x (B_x C_y - B_y C_z)] + \hat{k} [A_x (B_z C_x - B_x C_z) - A_y (B_y C_z - B_z C_y)] \tag{2.3}\]

while

\[
(\vec{B} \otimes \vec{C}) \otimes \vec{A} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
B_x & B_y & B_z \\
C_x & C_y & C_z \\
\end{vmatrix} \otimes \vec{A} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
B_y C_z - B_z C_y & B_z C_x - B_x C_z & B_x C_y - B_y C_x \\
A_x & A_y & A_z \\
\end{vmatrix}
\]
which expands to

\[
(\vec{B} \otimes \vec{C}) \otimes \vec{A} = i \left[ (B_x C_x - B_y C_z) A_z - (B_x C_y - B_y C_x) A_y \right] \\
+ j \left[ (B_x C_y - B_y C_z) A_z - (B_y C_y - B_y C_x) A_z \right] \\
+ k \left[ (B_y C_x - B_x C_y) A_y - (B_z C_x - B_x C_z) A_x \right]
\] (2.4)

We then have (since \(r_x = x, r_y = y, \) etc.) for \(\vec{A} \otimes (\vec{B} \otimes \vec{C})\) with \(\vec{A} \rightarrow \vec{r}, \) \(\vec{B} \rightarrow \vec{L}\) and \(\vec{C} \rightarrow \vec{p}\) is

\[
\vec{r} \otimes (\vec{L} \otimes \vec{p}) = i \left[ y (L_x p_y - L_y p_x) - z (L_z p_x - L_x p_z) \right] \\
+ j \left[ z (L_y p_z - L_z p_y) - x (L_x p_y - L_y p_x) \right] \\
+ k \left[ x (L_z p_x - L_x p_z) - y (L_y p_z - L_z p_y) \right]
\] (2.5)

and with \(\vec{A} \rightarrow \vec{r}, \) \(\vec{B} \rightarrow \vec{L}\) and \(\vec{C} \rightarrow \vec{p}\) but with the order reversed, is

\[
(\vec{L} \otimes \vec{p}) \otimes \vec{r} = i \left[ (L_x p_x - L_z p_z) z - (L_x p_y - L_y p_x) y \right] \\
+ j \left[ (L_x p_y - L_y p_x) x - (L_y p_z - L_z p_y) z \right] \\
+ k \left[ (L_y p_z - L_z p_y) y - (L_z p_x - L_x p_z) x \right]
\] (2.6)

For reference sake, in the next part of this work, we repeat this table from the earlier work:

| \(|L_x, x| = 0 \) | 
| \(|L_y, y| = 0 \) | 
| \(|L_z, z| = 0 \) | 
| \(|L_x, y| = i t z \) | \(= - [L_y, x] \) | 
| \(|L_y, x| = -i t z \) | \(= L_x, y \) | 
| \(|L_z, y| = i t z \) | \(= - [L_x, x] \) | 
| \(|L_x, p_y| = i t p_z \) | 
| \(|L_x, p_z| = -i t p_y \) | 
| \(|L_x, p_z| = 0 \) |

Adding the two, as requested, we have

\[
\vec{r} \otimes (\vec{L} \otimes \vec{p}) + (\vec{L} \otimes \vec{p}) \otimes \vec{r} = \\
i \left[ y L_x p_y - y L_y p_x - z L_z p_x + z L_x p_z + (L_x p_y - L_y p_x) - L_x p_z \right] \\
+ j \left[ z L_y p_z - z L_z p_y - x L_y p_x + x L_y p_x - L_y p_z \right] \\
+ k \left[ x L_z p_x - x L_z p_z - y L_y p_z + y L_z p_x \right]
\] (2.7)

or, re-arranging

\[
\vec{r} \otimes (\vec{L} \otimes \vec{p}) + (\vec{L} \otimes \vec{p}) \otimes \vec{r} = \\
i \left[ y L_x p_y - L_x p_y y + y L_y p_x - L_y p_x y - z L_z p_x + z L_z p_x + (L_z p_y + L_y p_x) \right] \\
+ j \left[ z L_y p_z - L_y p_z y + z L_z p_y - L_z p_y y - x L_y p_x + x L_y p_x \right] \\
+ k \left[ x L_z p_x - L_z p_x x - x L_z p_x + x L_z p_x \right]
\] (2.8)

which is, taking advantage of obvious cancellations:

\[
\vec{r} \otimes (\vec{L} \otimes \vec{p}) + (\vec{L} \otimes \vec{p}) \otimes \vec{r} = \\
i \left[ y L_x p_y + z L_z p_x - L_x p_z - L_y p_y \right]
\]
which is

\[ \mathcal{r} \circ (\mathcal{L} \circ \mathcal{p}) + (\mathcal{L} \circ \mathcal{p}) \circ \mathcal{r} = \]

\[ i [L_x(p_y - p_y y + z p_z - p_z z)] + j [L_y(z p_z - p_z z + x p_x - p_x x)] + k [L_z(x p_x - p_x x + y p_y - p_y y)] \]  

Next, we need

\[ \mathcal{r} \circ (\mathcal{p} \circ \mathcal{L}) = i [p_z L_y - p_y L_z] - z (p_z L_x - p_x L_z) \]

\[ + j [p_y L_z - p_z L_y] - x (p_y L_x - p_x L_y) \]

\[ + k [p_x L_y - p_y L_x] - y (p_x L_y - p_y L_x) \]

and

\[ (\mathcal{p} \circ \mathcal{L}) \circ \mathcal{r} = \]

\[ i [(p_z L_y - p_y L_z) z - (p_x L_y - p_y L_x) y] + j [(p_y L_z - p_z L_y) x - (p_y L_z - p_z L_y) z] + k [(p_y L_z - p_z L_y) y - (p_z L_x - p_x L_z) x] \]  

which is

\[ \mathcal{r} \circ (\mathcal{p} \circ \mathcal{L}) + (\mathcal{p} \circ \mathcal{L}) \circ \mathcal{r} = \]

\[ i [y (-p_y L_z) - z (p_z L_x - p_x L_z)] + (p_z L_x - p_x L_z) z - (p_z L_y - p_y L_z)] \]

\[ + j [z (-p_y L_z) - x (p_y L_x - p_x L_y)] + (p_y L_z - p_z L_y) x - (p_y L_z - p_z L_y) z] + k [x (-p_z L_y) - y (p_y L_z) + (p_y L_y) y - (p_z L_x - p_x L_z) x] \]  

which is

\[ \mathcal{r} \circ (\mathcal{p} \circ \mathcal{L}) + (\mathcal{p} \circ \mathcal{L}) \circ \mathcal{r} = \]

\[ i [y (p_y L_y - p_z z p_y) - z (p_z L_z - p_x x p_z)] + (p_z L_z - p_x x p_z) z - (p_z L_z - p_x x p_z)] \]

\[ + j [z (p_z L_z - p_x x p_z) - x (p_x L_x - p_y y x)] + (p_x L_x - p_y y x) x - (p_x L_x - p_y y x) z] \]

\[ + k [x (p_x L_x - p_y y x) - y (p_y L_x) + (p_y L_x) y - (p_x L_x) x] \]  

This means

\[ \frac{1}{4 \mathcal{E} c^2 \mu} (\mathcal{r} \circ (\mathcal{L} \circ \mathcal{p}) + (\mathcal{L} \circ \mathcal{p}) \circ \mathcal{r}) = \]

\[ \frac{1}{4 \mathcal{E} c^2 \mu} (2 \mathcal{h} \mathcal{L}) \]  

The other term we need (the obverse?) starts with \( \mathcal{p} \circ \overline{\mathcal{L}} \) which would be
or, expanding

\[ \vec{r} \otimes (\vec{p} \otimes \vec{L}) + (\vec{p} \otimes \vec{L}) \otimes \vec{r} = \]

\[ i \left[ -yp_yyp_z + (yp_y^2 - p_y^2)p_z + yp_zp_y^2 - zp_zyp_z + zp_zyp_z - p_z^2 p_y y + p_y yp_z y - yp_z p y y \right] + j \left[ -zp_z z p_x - xp_x z p_z + xp_x z p_z + p_x z p_x x - p_x z p_x x + p_x z p_x x - p_z^2 p_x x \right] + k \left[ -xp_x z p_y + x p_y z p_x - y p_y p z x - yp_y p z x + y p_y p z x - y p_y p z x \right] \] (2.18)

or, rearranging

\[ \vec{r} \otimes (\vec{p} \otimes \vec{L}) + (\vec{p} \otimes \vec{L}) \otimes \vec{r} = \]

\[ i \left[ -yp_yyp_z + (yp_y^2 - p_y^2)p_z + yp_zp_y^2 - zp_zyp_z + zp_zyp_z - p_z^2 p_y y + p_y yp_z y - yp_z p y y \right] + j \left[ -zp_z z p_x - xp_x z p_z + xp_x z p_z + p_x z p_x x - p_x z p_x x + p_x z p_x x - p_z^2 p_x x \right] + k \left[ -xp_x z p_y + x p_y z p_x - y p_y p z x - yp_y p z x + y p_y p z x - y p_y p z x \right] \] (2.19)

or

\[ \vec{r} \otimes (\vec{p} \otimes \vec{L}) + (\vec{p} \otimes \vec{L}) \otimes \vec{r} = \]

\[ i \left[ -yp_y + p_y y^2 \right] p_x y + (yp_y^2 - p_y^2) y + (-zp_z z + p_z^2 z) y + z p_z z p_y \]

\[ + j \left[ -zp_z z p_x + z p_z z p_x + (zp_z p_z - p_z^2 p_x z) x + (xp_x z p_z - p_z^2 p_x x) z \right] + k \left[ -xp_x x + p_x x^2 + (xp_x x - p_x x^2) p_y \right] \] (2.20)

which, using the appropriate commutators, becomes

\[ \vec{r} \otimes (\vec{p} \otimes \vec{L}) + (\vec{p} \otimes \vec{L}) \otimes \vec{r} = \]

\[ i \left[ -yp_y + p_y y^2 \right] p_x y + (yp_y^2 - p_y^2) y + (-zp_z z + p_z^2 z) y + z p_z z p_y \]

\[ + j \left[ -zp_z z p_x + z p_z z p_x + (zp_z p_z - p_z^2 p_x z) x + (xp_x z p_z - p_z^2 p_x x) z \right] + k \left[ -xp_x x + p_x x^2 + (xp_x x - p_x x^2) p_y \right] \] (2.21)

which is

\[ + k \left[ -i \hbar x p_y + (xp_x x - p_x x^2) y + (yp_y^2 + p_y^2) x + (i \hbar y) y \right] \] (2.22)

or

\[ \vec{r} \otimes (\vec{p} \otimes \vec{L}) + (\vec{p} \otimes \vec{L}) \otimes \vec{r} = \]

\[ i \left[ -ih p_z y + (yp_y^2 - p_y^2) y + (ih) p_z y \right] + j \left[ -ih z p_z + (zp_z^2 - p_z^2) x + (xp_x z p_z - p_z^2 p_x x) z + (ih) z p_x \right] \] (2.23)

\[ \vec{r} \otimes (\vec{p} \otimes \vec{L}) + (\vec{p} \otimes \vec{L}) \otimes \vec{r} = \]

\[ i \left[ -ih p_z y + (yp_y^2 - p_y^2) y + (ih) p_z y \right] + j \left[ -ih z p_z + (zp_z^2 - p_z^2) x + (xp_x z p_z - p_z^2 p_x x) z + (ih) z p_x \right] \]

\[ i \left[ -ih p_z y + (ih) p_z z + (ih) p_y z \right] \] (2.24)
\[ \frac{1}{2} (\hat{A} \otimes \hat{r} + \hat{r} \otimes \hat{A}) = -\frac{\hbar}{4 \mu} \frac{1}{Z e^2} \hat{L} \]

Unfortunately, Pauli’s Equation 52 is

III.

We now assemble all our prior work into a set of commutators, sufficient to attack the Kepler problem. We start with the one already known, i.e., the angular momentum commutator rules.

We had

\[ L_z L_y - L_y L_z = [L_z, L_y] = i\hbar L_z \]

and by cyclic permutation,

\[ \begin{align*}
L_x L_y - L_y L_x &= [L_x, L_y] = i\hbar L_x \\
L_y L_z - L_z L_y &= [L_y, L_z] = i\hbar L_y \\
L_z L_x - L_x L_z &= [L_z, L_x] = i\hbar L_x
\end{align*} \]

where \( x \to y, y \to z, \) and \( z \to x \) corresponds to the cyclic permutation we are talking about.

There is a sensual formulation of these three rules expressed as a cross product, a highly stylized rendition:

\[ \vec{L} \otimes \vec{L} = i\hbar \vec{L} \]

where of course the vectors are vector operators, so that the cross product refer to vectors and their components, and therefore, since these components are operators, do not commute and therefore do not cancel.

Let’s expand the cross product to see what is happening in this condensed notation. We have

\[ \vec{L} \otimes \vec{L} = \begin{vmatrix} i & j & k \\ L_x & L_y & L_z \\ L_x & L_y & L_z \end{vmatrix} = i (L_y L_z - L_z L_y) + j (L_z L_x - L_x L_z) + k (L_x L_y - L_y L_x) \]

or

\[ \vec{L} \otimes \vec{L} = \begin{vmatrix} L_y & L_z & L_x \\ L_x & L_y & L_z \\ L_y & L_z & L_x \end{vmatrix} = i (L_y L_z - L_z L_y) + j (L_z L_x - L_x L_z) + k (L_x L_y - L_y L_x) \]

which becomes

\[ \begin{vmatrix} i & j & k \\ L_y & L_z & L_x \\ L_z & L_x & L_y \end{vmatrix} = i L_y L_z - L_z L_y + j L_z L_x - L_x L_z + k L_x L_y - L_y L_x \]

We obtain

\[ \vec{L} \otimes \vec{L} = i\hbar L_z + j\hbar L_x + k\hbar L_y \]

we then have

\[ \vec{L} \otimes \vec{L} = i\hbar \vec{L} \]

as asserted at the outset.
IV. THE RUNGE-LENZ VECTOR COMMUTATORS

A. Similar Components Commuting with Angular Momentum Components

We had (Equation 14.10 [2])

$$\vec{A} = \frac{1}{2Ze^2\mu} (\vec{L} \otimes \vec{p} - \vec{p} \otimes \vec{L}) + \hat{r} \quad (4.1)$$

(this is also Equation 13-31 in Borowitz, loc cit) as the defining equation for the operator form of the Runge-Lenz vector, and we wish now to see how this commutes with \(\vec{L}\). First, we change over to a “better” (more normal) form.

We start with \([A_i, L_i]; \forall i\) which we think of as similar components. First, we need to evaluate

$$\vec{L} \otimes \vec{p} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ L_x & L_y & L_z \\ p_x & p_y & p_z \end{vmatrix} = \begin{pmatrix} \hat{i}(L_yp_z - L_zp_y) \\ \hat{j}(L_zp_x - L_xp_z) \\ \hat{k}(L_xp_y - L_yp_x) \end{pmatrix} \quad (4.2)$$

which is

$$\vec{L} \otimes \vec{p} = \begin{pmatrix} \hat{i}((zp_x - xp_z)p_z - (xp_y - yp_x)p_y) \\ \hat{j}((xp_y - yp_x)p_x - (yp_z - zp_y)p_z) \\ \hat{k}((yp_z - zp_y)p_y - (zp_x - xp_z)p_x) \end{pmatrix} \quad (4.3)$$

In the reverse order, we had

$$\vec{p} \otimes \vec{L} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ p_x & p_y & p_z \\ yp_z - zp_y & zp_x - xp_z & xp_y - yp_z \end{vmatrix} = \begin{pmatrix} \hat{i}(p_y(xp_y - yp_x) - p_z(zp_x - xp_z)) \\ \hat{j}(p_z(yp_z - zp_y) - p_x(xp_y - yp_x)) \\ \hat{k}(p_x(zp_x - xp_z) - p_y(yp_z - zp_y)) \end{pmatrix} \quad (4.4)$$

so, combining these two we have (don’t worry, we’re not making an error here relative to earlier work)

$$\vec{L} \otimes \vec{p} + \vec{p} \otimes \vec{L} = \begin{pmatrix} \hat{i}((zp_x - xp_z)p_z - (xp_y - yp_x)p_y + \{p_y(xp_y - yp_x) - p_z(zp_x - xp_z)\}) \\ \hat{j}((xp_y - yp_x)p_x - (yp_z - zp_y)p_z + \{p_z(yp_z - zp_y) - p_x(xp_y - yp_x)\}) \\ \hat{k}((yp_z - zp_y)p_y - (zp_x - xp_z)p_x + \{p_x(zp_x - xp_z) - p_y(yp_z - zp_y)\}) \end{pmatrix} \quad (4.5)$$

which is, combining term

$$\vec{L} \otimes \vec{p} + \vec{p} \otimes \vec{L} = \begin{pmatrix} \hat{i}((zp_x - xp_z)p_z - (xp_y - yp_x)p_y + \{p_y(xp_y - yp_x) - p_z(zp_x - xp_z)\}) \\ \hat{j}((xp_y - yp_x)p_x - (yp_z - zp_y)p_z + \{p_z(yp_z - zp_y) - p_x(xp_y - yp_x)\}) \\ \hat{k}((yp_z - zp_y)p_y - (zp_x - xp_z)p_x + \{p_x(zp_x - xp_z) - p_y(yp_z - zp_y)\}) \end{pmatrix} \quad (4.6)$$

which is, upon expansion,

$$\vec{L} \otimes \vec{p} + \vec{p} \otimes \vec{L} = \begin{pmatrix} \hat{i}(zp_xp_z - xp_zp_x - xp_yp_z + yp_zp_y - yp_zp_x - p_zzp_x + xp_xp_y) \\ \hat{j}(xp_yp_x - yp_xp_y - yp_zp_y + zp_yp_z - zp_yp_x - p_xzp_x + xp_xzp_y) \\ \hat{k}(yp_zp_y - zp_yp_z - zp_xzp_x + xp_xzp_y - p_xzp_x + yp_yp_x) \end{pmatrix} \quad (4.7)$$

or

$$\vec{L} \otimes \vec{p} + \vec{p} \otimes \vec{L} = \begin{pmatrix} \hat{i}(zp_xp_z + yp_zp_y - p_zyp_y - p_zzp_x) \\ \hat{j}(xp_yp_x + zp_yp_z - p_xzp_x + xp_xzp_y) \\ \hat{k}(yp_zp_y + xp_xzp_x - p_xzp_x + yp_yp_x) \end{pmatrix} \quad (4.8)$$

or

$$\vec{L} \otimes \vec{p} + \vec{p} \otimes \vec{L} = \begin{pmatrix} \hat{i}(zp_xp_z + p_x(zp_x - p_z)) + p_x(yp_y + p_yy) \\ \hat{j}(xp_yp_x + p_y(zp_y + p_yz)) + p_y(yp_z + p_z) \\ \hat{k}(yp_zp_y + p_y(zp_y + p_yz)) + p_y(yp_x + p_xx) \end{pmatrix} \quad (4.9)$$

or

$$\vec{L} \otimes \vec{p} + \vec{p} \otimes \vec{L} = \begin{pmatrix} \hat{i}(2\hbar p_x) \\ \hat{j}(2\hbar p_y) \\ \hat{k}(2\hbar p_z) \end{pmatrix} \quad (4.10)$$

i.e.,

$$\vec{L} \otimes \vec{p} + \vec{p} \otimes \vec{L} = 2\hbar \vec{p} \quad (4.11)$$
so we can rewrite this as
\[ \vec{p} \otimes \vec{L} = -\vec{L} \otimes \vec{p} + 2 \hbar \vec{p} \]  
(4.12)
which allows us to use a slightly simpler form in future work for the Runge-Lenz vector, i.e.,
\[ \vec{A} = \frac{1}{2Ze^2\mu} \left( \vec{L} \otimes \vec{p} - \left( -\vec{L} \otimes \vec{p} + 2 \hbar \vec{p} \right) \right) + \hat{r} \]  
(4.13)
which results in our “final” version:
\[ \vec{A} = \frac{1}{Ze^2\mu} \left( \vec{L} \otimes \vec{p} - \hbar \vec{p} \right) + \hat{r} \]  
(4.14)
This is the standard form used in most discussions of the Runge-Lenz vector operator, i.e., not the original form cited and used before.

V. COMMUTATOR OF A_\mu WITH L_i, ETC.

It is a labor of love to obtain the commutators of the Runge-Lenz vector with respect to other quantum-mechanical operators needed for dealing with the hydrogen atom. The Runge-Lenz vector form used here is from Equation 4.14. What follows is a Maple session showing the commutator relationships needed to proceed.

VI. ADDENDUM

Given these commutator relations, one can now proceed on to the ladder operator solution to the H-atom’s electronic energy levels (C. W. David, Am. J. Phys., 34, 984 (1966). This was the only time that I ever encountered the “thrill of discovery” first hand. It was exhilarating. It was even cited: Blinder, S. M., J. Chem. Educ. 2001, 78, 391.
The Runge-Lenz vector is defined here:

\[ RL_1 := L_3 \times p_2 - L_2 \times p_3 - i \hbar p_1 - q_1 / \sqrt{q_1^2 + q_2^2 + q_3^2} \]
\[ RL_2 := L_1 \times p_3 - L_3 \times p_1 - i \hbar p_2 - q_2 / \sqrt{q_1^2 + q_2^2 + q_3^2} \]
\[ RL_3 := L_2 \times p_1 - L_1 \times p_2 - i \hbar p_3 - q_3 / \sqrt{q_1^2 + q_2^2 + q_3^2} \]

To show that \( L_1 \times L_2 = i \hbar L_3 \)

\[
\text{print (' Comm3D}(L_1,L_2) = ')}
\[
\text{comm1} := \text{Comm3D}(L_1,L_2);
\]

To show that \( RL_1 \times L_1 = 0 \)

\[
\text{print (' Comm3D}(RL_1,L_1) = ')}
\[
\text{comm2} := \text{Comm3D}(RL_1,L_1);
\]

To show that \( RL_1 \times RL_2 = -K L_3 \)

\[
\text{comm3} := \text{Comm3D}(RL_1,RL_2);
\]

\[
\text{print (' Comm3D}(RL_1,RL_2) = ')}
\[
\text{comm3a} := \text{simplify}((\text{simplify}((\text{comm3a}) / (i \hbar))));
\]

\[
\text{comm3b} := \text{subs}(q_1^3 = q_1 \cdot (r^2 - q_2^2 - q_3^2), \text{comm3b});
\]

\[
\text{comm3c} := \text{expand}((\text{subs}(p_3^3 = p_3 \cdot (2 \cdot (En + 1/r) - (p_1^2 + p_2^2)), \text{comm3c}));
\]

\[
\text{comm3d} := \text{algsubs}(q_1 \cdot p_2 - q_2 \cdot p_1 = L_3, \text{comm3d});
\]

\[
\text{comm3e} := \text{expand}((\text{subs}(p_3^2 = 2 \cdot (En - ((p_1^2 + p_2^2)/2 - 1/r)), \text{comm3d}));
\]

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\[
\text{Comm3D}(RL_1,RL_2) =
\]

\[
\text{comm3a} := -q_3^2 q_2 p_1 / r^3 - q_1^2 q_2 p_1 / r^3 - p_3^2 q_1 + p_2^2 q_1^3 - q_3^3 q_1^3 + p_2^2 q_2 p_1 + p_3^2 q_2 p_1
\]

\[
\text{comm3b} := -q_3^2 q_2 p_1 / r^3 - q_1^2 q_2 p_1 / r^3 - p_3^2 q_1 + p_2^2 q_1^3 - q_3^3 q_1^3 + p_2^2 q_2 p_1 + p_3^2 q_2 p_1
\]

\[
\text{comm3c} := -p_3^2 q_1^3 - 2 q_2 p_1^3 + p_2^2 q_2 p_1 + p_3^2 q_2 p_1 - p_3^2 q_1 p_2 - p_1^2 p_2 q_1 + 2 q_1 p_2 / r - q_2 p_1 / r^3
\]

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\[
\text{Comm3D}(RL_1,RL_2) =
\]

\[
\text{comm3a} := -p_3^2 q_1^3 - 2 q_2 p_1^3 + p_2^2 q_2 p_1 + p_3^2 q_2 p_1 - p_3^2 q_1 p_2 - p_1^2 p_2 q_1 + 2 q_1 p_2 / r - q_2 p_1 / r^3
\]

\[
\text{comm3b} := -p_3^2 q_1^3 - 2 q_2 p_1^3 + p_2^2 q_2 p_1 + p_3^2 q_2 p_1 - p_3^2 q_1 p_2 - p_1^2 p_2 q_1 + 2 q_1 p_2 / r - q_2 p_1 / r^3
\]

\[
\text{comm3c} := -2 L_3 En
\]

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\[ L \text{ cross } r \text{ calculation} \]
\[ vec1 := 2 I (q_2 p_3 - q_3 p_2) \hbar \]
\[ vec2 := 2 I (q_3 p_1 - q_1 p_3) \hbar \]
\[ vec3 := 2 I (q_1 p_2 - q_2 p_1) \hbar \]