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Frobenius Solution for Legendre's Equation, Rodrigue's Formula and Normalization

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Frobenius Method for Legendre Polynomials, Rodrique's formula and Normalization

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I. SYNOPSIS

The Frobenius Solution is illustrated by application to the Legendre Differential Equation. Rodrigue's formula is obtained as well as an explicit formula of the normalization constants.

II. FROBENIUS SOLUTION TO LEGENDRE'S EQUATION

We start a Frobenius solution without worrying about the technical details of the indicial equation, and just assert that the proposed solution *Ansatz* will be

$$S_{\ell,0}(\mu) = y(\mu) = \sum_{n=0}^{\infty} c_n \mu^n = c_0 + c_1\mu + c_2\mu^2 + \cdots \quad (1)$$

(starting the $n = 0$, interested readers can see the indicial equation argument in vertually every formal locale) which we substitute into Legendre's differential equation, viz,

$$\frac{\partial\left((1-\mu^2)\frac{\partial y(\mu)}{\partial\mu}\right)}{\partial\mu} = -\ell(\ell+1)y(\mu) \quad (2)$$

we obtain

$$\frac{\partial\left(\frac{\partial y(\mu)}{\partial\mu}\right)}{\partial\mu} - \frac{\partial\left(\mu^2\frac{\partial y(\mu)}{\partial\mu}\right)}{\partial\mu} + \ell(\ell+1)y(\mu) = 0 \quad (3)$$

$$\frac{\partial^2 y(\mu)}{\partial\mu^2} - \mu^2\frac{\partial^2 y(\mu)}{\partial\mu^2} - 2\mu\frac{\partial y(\mu)}{\partial\mu} + \ell(\ell+1)y(\mu) = 0 \quad (4)$$

so that when we feed the *Ansatz* into this differential equation (Equation 1) we obtain

$$\begin{aligned}
\frac{\partial^2 y(\mu)}{\partial\mu^2} &\to (2)(1)c_2 + (3)(2)c_3\mu + (4)(3)c_4\mu^2 + (5)(4)c_5\mu^3 + \cdots \\
-\mu^2\frac{\partial^2 y(\mu)}{\partial\mu^2} &\to -\mu^2\left((2)(1)c_2 + (3)(2)c_3\mu + (4)(3)c_4\mu^2 + (5)(4)c_5\mu^3 + \cdots\right) \\
-2\mu\frac{\partial y(\mu)}{\partial\mu} &\to -2\mu\left(c_1 + (2)c_2\mu + (3)c_3\mu^2 + (4)c_4\mu^3 + \cdots\right) \\
+\ell(\ell+1)y(\mu) &\to +\ell(\ell+1)\left(c_0 + c_1\mu + c_2\mu^2 + c_3\mu^3 + \cdots\right) = 0
\end{aligned} \quad (5)$$

which, in standard Frobenius form, we separately equate to zero (power by power)

$$\begin{aligned}
(2)(1)c_2 + (3)(2)c_3\mu + (4)(3)c_4\mu^2 + (5)(4)c_5\mu^3 \cdots \\
-(2)(1)c_2\mu^2 - (3)(2)c_3\mu^3 - (4)(3)c_4\mu^4 - (5)(4)c_5\mu^5 - \cdots \\
-2c_1\mu - 2(2)c_2\mu^2 - 2(3)c_3\mu^3 - 2(4)c_4\mu^4 - \cdots \\
+\ell(\ell+1)c_0 + \ell(\ell+1)c_1\mu + \ell(\ell+1)c_2\mu^2 + \ell(\ell+1)c_3\mu^3 + \cdots = 0
\end{aligned} \quad (6)$$

to achieve the appropriate recursion relationships. Note that there is an even and an odd set, based on starting with either c_0 or c_1, which correspond to the two arbitrary constants associated with a second order differential equation. We obtain the separately equal to zero equations:

$$\begin{aligned}
(2)(1)c_2 + \ell(\ell+1)c_0 = 0 \\
+(3)(2)c_3\mu - 2c_1\mu + \ell(\ell+1)c_1\mu = 0 \\
+(4)(3)c_4\mu^2 - (2)(1)c_2\mu^2 - 2(2)c_2\mu^2 + \ell(\ell+1)c_2\mu^2 = 0 \\
(5)(4)c_5\mu^3 - (3)(2)c_3\mu^3 - 2(3)c_3\mu^3 + \ell(\ell+1)c_3\mu^3 = 0
\end{aligned}$$

Typeset by REVTEX

$$-(4)(3)c_4\mu^4 - 2(4)c_4\mu^5 + \cdots\ etc = 0 \quad (7)$$

which we re-write as

$$\begin{aligned}(2)(1)c_2 + \ell(\ell+1)c_0 &= 0\\ +(3)(2)c_3 - 2c_1 + \ell(\ell+1)c_1 &= 0\\ +(4)(3)c_4 - (2)(1)c_2 - 2(2)c_2 + \ell(\ell+1)c_2 &= 0\\ (5)(4)c_5 - (3)(2)c_3 - 2(3)c_3 + \ell(\ell+1)c_3 &= 0\\ -(4)(3)c_4\mu^4 - 2(4)c_4\mu^5\ etc + \cdots &= 0 \end{aligned} \quad (8)$$

and, upon cleaning up the equations, obtain

$$\begin{aligned}(2)(1)c_2 &= -\ell(\ell+1)c_0\\ +(3)(2)c_3 &= (2 - \ell(\ell+1))c_1\\ +(4)(3)c_4 &= ((2)(1) + 2(2) - \ell(\ell+1))c_2\\ (5)(4)c_5&((3)(2) + 2(3) - \ell(\ell+1))c_3\\ -(4)(3)c_4\mu^4 - 2(4)c_4\mu^5\ etc + \cdots &= 0\end{aligned}$$

Solving the equations using "earlier" results, from top to bottom, we have

$$\begin{aligned}c_2 &= -\frac{\ell(\ell+1)}{2!}c_0\\ c_3 &= -\frac{(2\ell + (\ell+1))}{(3)(2)}c_1\\ c_4 &= -\frac{(-(2)(1) - 2(2) + \ell(\ell+1))}{(4)(3)}c_2\\ c_5 &= -\frac{((3)(2) - 2(3) + \ell(\ell+1))}{(5)(4)}c_3\\ &etc.\end{aligned}$$

which cleans up to

$$\begin{aligned}c_2 &= -\frac{\ell(\ell+1)}{2!}c_0\\ c_3 &= -\frac{(2 + \ell(\ell+1))}{(3)(2)}c_1\\ c_4 &= -\left(\frac{(-(2)(1) - 2(2) + \ell(\ell+1))}{(4)(3)}\right)\left(-\frac{\ell(\ell+1)}{2!}\right)c_0\\ c_5 &= -\left(\frac{((3)(2) - 2(3) + \ell(\ell+1))}{(5)(4)}\right)\left(-\frac{(2 + \ell(\ell+1))}{(3)(2)}\right)c_1\\ &etc.\end{aligned}$$

so $s_{\ell,0}(\mu) = f_1c_0 + f_2c_1$ where f_1 and f_2 are power series based on the above set of coëfficients. For an even series, declare $c_1 = 0$ and choose an ℓ value which truncates the power series into a polynomial. Do the opposite for an odd solution.

These polynomials are the Legendre polynomials, about which much is written in a wide variety of locations (including this compendium of educational papers).

III. RODRIQUE'S FORMULA

Rodrique's formula is

$$S_{\ell,0} \to P_\ell(\mu) = \frac{1}{2^\ell \ell!}\frac{d^\ell(\mu^2-1)^\ell}{d\mu^\ell}$$

where Legendre's Equation is

$$(1-\mu^2)\frac{P_\ell(\mu)}{d\mu^2} - 2\mu\frac{dP_\ell}{d\mu} + \ell(\ell+1)P_\ell(\mu) = 0$$

To show this, we start by defining

$$g_\ell \equiv (\mu^2-1)^\ell$$

and find that

$$\frac{dg_\ell}{d\mu} = 2\mu\ell(\mu^2-1)^{\ell-1}$$

and

$$\frac{d^2g_\ell}{d\mu^2} = 2\ell(\mu^2-1)^{\ell-1} + 4\mu^2\ell(\ell-1)(\mu^2-1)^{\ell-2}$$

We now form (construct)

$$\begin{aligned}(1-\mu^2)\frac{d^2g_\ell}{d\mu^2} &= -2\ell(\mu^2-1)^\ell - 4\mu^2\ell(\ell-1)(\mu^2-1)^{\ell-1}\\ 2(\ell-1)\mu\frac{dg_\ell}{d\mu} &= 4\mu^2\ell(\ell-1)(\mu^2-1)^{\ell-1}\\ +2\ell g_\ell &= 2\ell(\mu^2-1)^\ell\end{aligned}$$

The r.h.s. of this equation set adds up to zero, and one obtains on the left:

$$A(\mu) = (1-\mu^2)\frac{d^2g_\ell}{d\mu^2} + 2(\ell-1)\mu\frac{dg_\ell}{d\mu} + 2\ell g_\ell = 0 \quad (9)$$

Defining the l.h.s of this equation as $A(\mu)$, we form

$$\frac{dA(\mu)}{d\mu} = \left[(1-\mu^2)\frac{d^3}{d\mu^3} - 2\mu\frac{d^2}{d\mu^2} + 2(\ell-1)\mu\frac{d^2}{d\mu^2} + 2(\ell-1)\frac{d}{d\mu} + 2\ell\frac{d}{d\mu}\right]g_\ell$$

$$= \left[(1-\mu^2)\frac{d^2}{d\mu^2} - (2\ell-4)\mu\frac{d}{d\mu} + (4\ell-2)\mu\right]\frac{dg_\ell}{d\mu}$$

and, doing it again

$$\frac{d^2A(\mu)}{d\mu^2} = \left[(1-\mu^2)\frac{d^2}{d\mu^2} - (2\ell-6)\mu\frac{d}{d\mu} + (6\ell-6)\mu\right]\frac{d^2g_\ell}{d\mu^2}$$

and once more for the radio audience:

$$\frac{d^3A(\mu)}{d\mu^3} = \left[(1-\mu^2)\frac{d^2}{d\mu^2} - (2\ell-8)\mu\frac{d}{d\mu} + (8\ell-12)\mu\right]\frac{d^3g_\ell}{d\mu^3}$$

Continuing, one notices that the changing coëfficients are regular in their appearance, so that the following table, which summarizes the pattern of coëfficients,

$\kappa = 1$	$2\ell - 4 = 2\ell - 2 * (\kappa + 1)$	$4\ell - 2 = (3 + \kappa)\ell - 2 * \kappa$	$=6$ if $\ell = 1$
$\kappa = 2$	$2\ell - 6 = 2\ell - 2 * (\kappa + 1)$	$6\ell - 6 = (3 + \kappa)\ell - 2 * \kappa$	$= 6$ if $\ell = 2$
$\kappa = 3$	$2\ell - 8 = 2\ell - 2 * (\kappa + 1)$	$8\ell - 12 = (3 + \kappa)\ell - 2 * \kappa$	$= 12$ if $\ell = 3$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\kappa = \ell$	$2\ell - 2 * (\ell + 1) = 2$	$8\ell - 12 = (3 + \ell)\ell - 2 * \ell$	$= \ell(\ell + 1)$

leads to generalization by which one finally obtains

$$\frac{\partial^\ell A(x)}{\partial \mu^\ell} = \left[(1-\mu^2)\frac{\partial^2}{\partial \mu^2} - 2\mu\frac{\partial}{\partial \mu} + \ell(\ell+1)\right]\frac{g_\ell(\mu)}{d\mu^\ell} \tag{10}$$

so, defining a normalization constant (and the Legendre polynomial of order ℓ)

$$\frac{d^\ell g_\ell}{d\mu^\ell} \equiv N_\ell P_\ell(\mu)$$

with N_ℓ constant, then

$$P_\ell(\mu) = \frac{1}{N_\ell}\frac{\partial d^\ell g_\ell}{d\mu^\ell} = \frac{1}{N_\ell}\frac{d^\ell(\mu^2-1)^\ell}{d\mu^\ell}$$

Here,

$$2^\ell \ell!$$

is chosen for N_ℓ's value to make the normalization automatic.