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Abstract

Based on an order-theoretic approach, we derive sufficient conditions for the existence, characterization, and computation of Markovian equilibrium decision processes and stationary Markov equilibrium on minimal state spaces for a large class of stochastic overlapping generations models. In contrast to all previous work, we consider reduced-form stochastic production technologies that allow for a broad set of equilibrium distortions such as public policy distortions, social security, monetary equilibrium, and production nonconvexities. Our order-based methods are constructive, and we provide monotone iterative algorithms for computing extremal stationary Markov equilibrium decision processes and equilibrium invariant distributions, while avoiding many of the problems associated with the existence of indeterminacies that have been well-documented in previous work. We provide important results for existence of Markov equilibria for the case where capital income is not increasing in the aggregate stock. Finally, we conclude with examples common in macroeconomics such as models with fiat money and social security. We also show how some of our results extend to settings with unbounded state spaces.

Journal of Economic Literature Classification: C62, E13, O41

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1 Introduction

Building on the seminal work of Samuelson [38] and Diamond [21], the overlapping generations (OLG) model has become a workhorse in applied dynamic general equilibrium theory. Numerous versions of this model have been used to study a wide array of issues including policy topics in intergenerational risk sharing and social security, human capital formation and public education, economic growth and infrastructure or environmental degradation, macroeconomic fluctuations, public finance, and monetary economics. In most of these applications researchers have focused on Markovian equilibrium. Unfortunately, because of the complicated structure of Markovian equilibrium for OLG models, a majority of this applied work has appealed to numerical methods to characterized both the quantitative and qualitative predictions of the models.

To apply such a methodology rigorously requires researchers to have access to theoretical results that provide a sharp characterization of the Markovian equilibrium sought to be constructed numerically; existing theoretical work on these matters has, however, focused on purely topological methods that are difficult to relate to the actual numerical methods used. In addition, such topological approaches are generally not very useful to answer theoretical issues such as monotone comparison results for the set of Markovian equilibrium with respect to the space of economies, or the characterization of many important properties of equilibrium decision processes and equilibrium price systems. This precarious state of understanding is even apparent when looking at the literature on Markovian equilibrium in the very simplest class of OLG models, i.e., the two-period model with a large number of a identical agents born each period, one perishable consumption good each period, and with production using capital and labor.

This paper takes a novel approach to address some of these issues in the benchline two-period OLG model with stochastic production (either classical or nonclassical). In particular, we provide *constructive* methods for characterizing a class of Markovian equilibrium decision processes (MEDP) and their induced stationary Markovian equilibrium (SME). In contrast to many existing topological methods, our approach avoids the important problems of multiplicities and indeterminacies noted in Wang [44][45]. In that respect, a critical component to our approach is the way we handle the parameter space for the construction of MEDP: We use the "little k, big K" formulation standard in recursive competitive equilibrium theory, which we combine with an order-based fixed point theorem to identify classes of MEDP and SME that are tractable. In that sense, one can view our approach as a monotone method for constructing *particular* MEDP and SME within a much larger class of equilibrium, this larger class including many complicated and potentially unstable Markovian equilibrium.

There are a number of important results on the existence and characterization of Markovian equilibrium in OLG models with production. Galor and Ryder[25] obtained some of the first results concerning the existence, uniqueness, and stability of steady state for a deterministic setup with classical production. The work of Galor and Ryder [25] has been extended to a class of OLG models with stochastic "classical" production (essentially, a production setting that is identical to that used in the work of Brock and Mirman [9]) in an important series of papers by Wang [44] [45] and Hauenschild [27]. For economies with identical and independently distributed (iid) production shocks and classical production , Wang[44] obtains sufficient conditions for the existence of a globally unique and stable non-trivial stationary Markovian equilibrium using a topological approach. As Wang [45] noted though, for economies without capital income monotonicity problems arise when constructing the SME because of multiple equilibria and equilibrium indeterminacies. Many of the results in Wang [44] for stochastic production with iid shocks have been recently extended by Hauenschild [27] to economies with pay-as-you-go financed social security. Finally in Wang [45], the author provides some existence results along the lines of Duffie et. al. [23] for economies with stochastic classical production with Markov shocks. A key implication of Wang's work (along with the storyline in the paper of Kubler and Polemarchis [30]) is that when sunspots and multiplicities are considered, the state space that is used to construct SME can potentially get very large.[1]

In this paper we provide two sets of powerful new results on Markov equilibrium for the simple stochastic OLG model with *classical or non-classical* production and iid shocks: (i) under capital income monotonicity, an assumption often used in the existing literature, our fixed point arguments are directly tied to computations. We prove that the unique MEDP can be constructed as the limit of a globally stable isotone successive approximation technique, and we also show how to construct extremal SME for a large collection of production technologies (e.g., classical and nonclassical production); (ii) when capital monotonicity is not satisfied, we provide the first results on the existence of semi-continuous monotone Markovian equilibrium. We show that there is a complete lattice of upper-semi continuous and lower-semi continuous isotone MEDP, and we also provide a catalog of successive approximation schemes converging (in both order and topology) to extremal SME for any MEDP that is only assumed to be measurable. Our method differs from the "correspondence" approach advocated in Wang [44] and Duffie & al. [23] and focuses on invariant distributions as in Stokey et. al. [40] and Hopenhayn and Prescott [28], as opposed to ergodic distributions in Wang [44]. More importantly, it is a constructive approach to computing extremal SME, and therefore permits some

[1] There are numerous pioneering papers in OLGs models which we have not mentioned directly in our remarks. Aside from the paper by Diamond, see Balasko and Shell [3][4], Okuno and Zilcha [35], Zilcha [46], Dechert and Yamamoto [20], Demange and Laroque [17][18], and Chattopadhyay and Gottardi [10], and Barbie, Hagedorn, and Kaul [5]). Many of these papers discuss the important question of how to define dynamic efficiency in an OLG environment, and often these papers have a much more complicated structure (e.g., many assets each period and incomplete markets, many good each period, etc.)

We note that although some of these papers study economies with or without stochastic production, none of this related work addresses the questions addressed in this paper (i.e., the construction and usefulness of monotone methods in the study of MEDPs and SME in OLG models production subjected to *iid* shocks from both a theoretical and numerical point of view).

comparative statics results with respect to the set of economies.

It is important to realize that *all* of the results concerning MEDP and SME in the existing literature for models with stochastic production have been obtained in settings where: (i) the economy is endowed with a very simple form of behavioral heterogeneity (namely each generation lives for two-periods and there is a large number of a single type of agent born each generation), (ii) there is a simple set of goods and assets available each period (usually a single aggregate perishable consumption good), and (ii) there is a single asset that agents can access to save (namely capital). There are, however, some recent results on OLG models with very general commodity spaces, many types of agents in each generation, and many assets are related to this paper, for instance in Kubler and Polemarchis [30]. Kubler and Polemarchis [30] provide some very interesting negative results concerning the existence of Markovian equilibrium on minimal state spaces for economies with multiple types of agents born each period, multiple commodities, and many assets. Minimal state spaces are state spaces consisting only of current period state variables, and the existence of Markovian equilibrium on such state spaces in stochastic OLG models is a difficult question.

The economies considered in the work of Kubler and Polemarchis are much more complicated than the environments considered in this paper (and in all the existing work on stochastic OLG models with production). In this sense, we are trying to develop techniques and basic results for simple OLG models with stochastic production that we feel have a chance to be generalized to some more general settings. Given the recent positive results on existence of MEDP using Abreau, Pierce, and Stachetti (APS) approach in Miao and Santos [31], we believe our methods have the potential to be integrated with this APS approach to address more complicated versions of our model.[2]

Finally, we should mention that the present paper is related to an emerging literature on monotone and mixed-monotone recursive methods starting with the pioneering work of Coleman [11][12], and continuing with the papers by Greenwood and Huffman [26], Datta, Mirman, and Reffett [13], Morand and Reffett [33], and Datta et. al [15]. Resulting from these papers are two crucial methodological points. First, the way the parameter/state space is handled matters greatly to the construction of particular MEDP (and their implied SME). Second, relating numerical solutions to theoretical fixed point arguments is done by developing collections of monotone iterative procedures converging to actual fixed points for the economies under consideration. It should be noted that none of the techniques and results developed in these papers directly apply to OLG models because both the space of candidate MEDP and the nonlinear fixed point operator studied need to be tailored to the particular economies considered.

[2] Using the ASP approach, Mia and Santos prove existence of MEDP in the space of *all* measurable mappings. Such general result comes at the expense of losing important characterization of MEDP that prove useful in constructing associated SME, and these theoretical results remain to be tied to numerical implementations. See Reffett [36] for a discussion of how Miao and Santo's APS method relate to the methods developed in this paper.

The paper is organized as follows. In section two, we detail the class of economies under consideration and provide some preliminary results. In section three, we study the set of Markovian equilibrium investment decisions and present algorithms to construct extremal Markovian equilibrium investment decisions. In section four we address the existence and construction of extremal stationary Markov equilibrium. In section five we discuss applications of our approach and results to models studied in the literature.

2 Setup and preliminary results

We consider a generalization of the simple two-period stochastic OLG model described in Wang [44] by allowing for nonconvexities in production and for various forms of public policy distortions (e.g., nonclassical production). Agents are assumed to have preferences represented by a lifetime utility function $u(c_1, c_2)$ where we take $c = (c_1, c_2)$ to be in the commodity space $X \times X \subset \mathbb{R}^2_+$.[3] The production of the unique consumption/capital good is assumed to be constant returns to scale in the *private* inputs capital and labor and to also depend on the realization of a random variable. Although we allow for nonconvexities in the aggregate production set, firms operate at zero profit. This setting is typical of the literature on infinite horizon nonoptimal economies (see for instance, Coleman [11]), and may be taken as the reduced form for a number of economies with frictions, as discussed for example in Greenwood and Huffman [26].

2.1 Assumptions on the economic primitives

We now discuss some basic assumptions on preferences, technologies, and information that will be maintained throughout the paper. Given the symmetric (stationary) structure of household preferences over time, we will not distinguish between households born in periods $t = \{0, 1, 2, ...\}$.

Assumption 1. The utility function $u : C \to \mathbb{R}$ is:

I. twice continuously differentiable,

II. strictly increasing and strictly concave,

III. such that $\lim_{c\to 0} u_1(c, .) = \lim_{c\to 0} u_2(., c) = +\infty$,

IV. such that $u_{12} \geq 0$.

Assumption 1 is standard, with Inada conditions imply interiority of consumption solutions. We allow for non-time separability in lifetime consumption, although a special case of constant discounting occurs when $u(c_1, c_2) = U(c_1) + \beta U(c_2)$ where $\beta \in]0, 1[$. Assumption 1.IV requires that the consumption goods in the first and second period of an agent's life be weak complements.

Turning to the description of production, we first discuss the uncertainty associated with production returns. As in Wang [44] and Hauenschild [27], we

[3] We will maintain the following notations for subsets of the real line and/or their Cartesian product that contain positive (or nonegative) numbers: $\mathbb{R}_+ = \{x \in \mathbb{R},\ x \geq 0\}$, $\mathbb{R}^*_+ = \{x \in \mathbb{R},\ x > 0\}$ and $\mathbb{R}^2_+ = \mathbb{R}_+ \times \mathbb{R}_+$ (cartesian product).

assume that production shocks come in the form of a collection of *iid* random variables defined on a compact support.

Assumption 2. The random variable z_t follows an iid process characterized by the probability measure denoted G, whose support is the compact set $Z = [z_{\min}, z_{\max}] \subset \mathbb{R}$ with $z_{\max} > z_{\min} > 0$.

The non-classical production function is denoted by $F(k, n, \overline{K}, \overline{N}, z)$ where the variables $\overline{K}$ and $\overline{N}$ represent social inputs in the form of aggregate per capita capital stock and labor supply. We assume constant returns to scale in private capital and labor inputs, respectively denoted k and n, but this formulation allows for nonconvex aggregate production set and for private and social returns to differ. It is important to note that, following standard arguments in Greenwood and Huffman [26], Datta et al [13], Morand and Reffett [33], this specification of the production function can be considered the "reduced form" production of a broad set of economies with (i) production nonconvexities in social returns but constant returns to scale in private returns (as in, for instance, Romer [37]), (ii) public policy such as state contingent income tax (e.g., capital and/or wage income taxes) and social security, (iii) valued fiat currency, and (iv) monopolistic competition.

Anticipating $n = 1 = \overline{N}$ in equilibrium (since households do not value leisure), we state some of our assumptions in terms of this restriction on equilibrium labor supply.[4]

Assumption 3. The production function $F(k, n, \overline{K}, \overline{N}, z) : \mathbb{R}_+ \times [0, 1] \times \mathbb{R}_+ \times [0, 1] \times Z \to \mathbb{R}_+$ is:

I. twice continuously differentiable in its first two arguments, and continuous in all arguments;

II. increasing in all arguments, strictly increasing and strictly concave in its first two arguments;

IIIa. such that $r(k, z) = F_1(k, 1, k, 1, z)$ is decreasing and continuous in k, and that $\lim_{k \to 0} r(k, z) = +\infty$;

IIIb. such that $w(k, z) = F_2(k, 1, k, 1, z)$ is increasing and continuous in k and $\lim_{k \to 0} w(k, z) = 0$;

IV. such that there exists a maximal sustainable capital stock $k_{\max}$ (i.e., $\forall k \geq k_{\max}$ and $\forall z \in Z$ $F(k, 1, k, 1, z) \leq k_{\max}$, and $\forall k \leq k_{\max}$, $\exists z \in Z$, $F(k, 1, k, 1, z) \geq k_{\max}$).

Assumption 3 is standard in the literature on nonoptimal stochastic growth (e.g., see Coleman [11] and Greenwood and Huffman [26]). In particular, IV implies that the set of feasible capital stock can be restricted to be in the compact interval $X = [0, k_{\max}]$ (as long as we place the initial date zero capital stocks $k_0 = \bar{K}_0$ in X), and also places restrictions on the amount of nonconvexity we can allow (as well as an upper bound on the capital stock, of course). In the rest of this paper $\mathcal{B}(X)$ will denote the Borel subsets of X.

Finally, we will make a simplifying assumptions that will lead to additional properties of the MEDP. This assumption is not critical in any of our most

[4] Stachurski [39] studies the interesting case of threshold externalities in an OLG model. We do not consider this case in this paper.

important results, and, when needed, we will discuss the ways to relaxing it.

Assumption 3'. Both $r(k, z)$ and $w(k, z)$ are continuous in z for all k.

2.2 Some results from lattice theory

This paper uses many tools and concepts of lattice theory, a brief overview of which can be found in Appendix A. There are two complete lattices of interest for this paper. The first is the interval order $H = [0, w]$ in the set of isotone bounded functions (and some subsets of H of semicontinuous functions) endowed with the pointwise order, in which we will look for MEDP in Section 3 of the paper. The second is the set of probability measures defined on a compact interval of $\mathbb{R}$, endowed the stochastic order, in which we study SME in Section 4.

To construct the first complete lattice, we endow the set $S = X \times Z$ with the pointwise partial order $\leq$ (and the usual topology), and for an isotone and continuous function $w : S \to \mathbb{R}_+$, we consider the following sets:

(a). $H = \{h : S \to \mathbb{R}^+, \forall s \in S \; 0 \leq h(s) \leq m(s), \, h \text{ isotone}\}$

(b). $E_x^u = \{h \in H, \, h \text{ upper semicontinuous in } x \text{ for each } z \in Z\}$ and $E_x^l = \{h \in H, \, h \text{ lower semicontinuous in } x \text{ for each } z \in Z\}$

(c). $E_z^u = \{h \in H, \, h \text{ upper semicontinuous in } z \text{ for each } x \in X\}$ and $E_z^l = \{h \in H, \, h \text{ lower semicontinuous in } z \text{ for each } x \in X\}$.

Recalling that in a complete lattice, the greatest (least) element is the unique maximal (resp. minimal) element, we have the following important result.

Proposition 1 *The sets H, E_x^u, E_z^u, E_x^l, E_z^l endowed with the pointwise order $\leq$ are complete lattices with maximal element w and minimal element 0.*

Proof. For any $D \subset H$, the lower and upper envelopes of D are increasing elements, hence:

$$\forall D \subset H \text{ and } \forall s \in S, \;\; \wedge_H D(s) = \inf_{h \in D}\{h(s)\} \text{ and } \vee_H D(s) = \sup_{h \in D}\{h(s)\}.$$

The lower envelope of a family of upper semicontinuous (usc) functions is usc (see, for instance Aliprantis and Border, 1999), thus:

$$\forall D \subset E_x^u \text{ and } \forall s \in S, \;\; \wedge_{E_x^u} D(s) = \inf_{h \in D}\{h(s)\},$$

and $(E_x^u, \leq)$ has a (unique) maximal element w. By Theorem 29 in Appendix A, $(E_x^u, \leq)$ is a complete lattice, and so is $(E_z^u, \leq)$ by a similar argument. Notice also that:

$$\forall D \subset E_x^u \text{ and } \forall s \in S, \;\; \vee_{E_x^u} D(s) = \inf_{s<t}\{\sup_{h \in D}\{h(t)\}\}.$$

$(E_x^l, \leq)$ and $(E_z^l, \leq)$ are complete lattices by a similar argument. Also:

$$\forall D \subset E_x^l \text{ and } \forall s \in S, \;\; \vee_{E_x^l} D(s) = \sup_{h \in D}\{h(s)\},$$

and,

$$\forall D \subset E_x^l \text{ and } \forall s \in S, \ \wedge_{E_x^l} D(s) = \sup_{t<s}\{\inf_{h\in D}\{h(t)\}\}.$$

Finally, it should be noted that if D is an increasing (resp. decreasing) sequence $\{h_n\}_{n\in\mathbb{N}}$, then:

$$\sup_{h_n\in D}\{h_n(s)\} = \lim_{n\to\infty} h_n(s) \text{ (resp. } \inf_{h_n\in D}\{h_n(s)\} = \lim_{n\to\infty} h_n(s)).$$

$\blacksquare$.

The second lattice of interest is the set $\Lambda(X,\mathcal{B}(X))$ of probability measures defined on the measurable space $(X,\mathcal{B}(X))$ endowed with the stochastic order $\geq_s$ defined as follows:

$$\mu \geq_s \mu' \text{ if } \int_X f(k)\mu(dk) \geq \int_X f(k)\mu'(dk),$$

for every increasing, and bounded function $f : X \to \mathbb{R}_+$, in which case we say that μ stochastically dominates μ'.

Proposition 2 $(\Lambda(X,\mathcal{B}(X)),\geq_s)$ *is a complete lattice with minimal and maximal elements* δ_0 *and* $\delta_{k_{\max}}$.

Proof. It is easy to show that the set $\mathbb{D}(X)$ of functions $F : X \to [0,1]$, that are increasing, upper semicontinuous, and satisfy $F(b) = 1$, is a complete lattice when endowed with the pointwise order. $\mathbb{D}(X)$ has maximal and minimal elements (respectively, the function $F(k) = 1$ for all $k \in X$, and the function $G(k) = 1$ if $k = b$ otherwise $G(k) = 0$), and is in fact the set of probability distributions over the compact set K. It is well-known that to any probability measure $\mu \in \Lambda(X,\mathcal{B}(X))$ corresponds a unique distribution function $F_\mu \in \mathbb{D}(X)$ and vice versa, and $\mu \geq_s \mu'$ is equivalent to $F_\mu \leq F_{\mu'}$ (see, for instance, Stokey, et. al. [40]).[5] $(\Lambda(X,\mathcal{B}(X)),\geq_s)$ is thus isomorphic to $(\mathbb{D}(X),\leq)$, and is therefore a complete lattice with minimal element the singular probability measure δ_0, and maximal element the singular probability measure $\delta_{k_{\max}}$.$\blacksquare$

The space $\Lambda(X,\mathcal{B}(X))$ is also endowed with the weak topology under which a sequence of probability measures $\{\mu_n\}_{n\in\mathbb{N}}$ in $\Lambda(X,\mathcal{B}(X))$ is said to weakly converges to $\mu \in \Lambda(X,\mathcal{B}(X))$ if for all continuous functions $f : X \to \mathbb{R}$:

$$\lim_{n\to\infty} \int_X f(k)\mu_n(dk) = \int_X f(k)\mu'(dk), \tag{CV}$$

in which case we write $\mu_n \Longrightarrow_{+\infty} \mu$, and call μ the weak limit of the sequence $\{\mu_n\}_{n\in\mathbb{N}}$. Finally, an interesting property of monotone sequences $\{\mu_n\}_{n\in\mathbb{N}}$ in

[5] This is not true if $X \subset \mathbb{R}^l$ with $l \geq 2$, and this is one fundamental reason why the argument in this paper cannot be trivially generalized to economies with Markov shocks. See more on this at the end of Section 4 of the paper.

$(\Lambda(X, \mathcal{B}(X)), \geq_s)$ which follows from the isomorphism between $(\Lambda(X, \mathcal{B}(X)), \geq_s$ $)$ and $(\mathbb{D}(X), \geq)$ should be noted: If $\mu_0 \leq_s \mu_1 \leq_s ... \leq_s \mu_n \leq_s ...$ then:

$$\mu_n \Rightarrow \mu = \vee\{\mu_n\}_{n\in\mathbb{N}}.$$

Similarly, if $\mu_0 \geq_s \mu_1 \geq_s ... \geq_s \mu_n \geq_s \mu_{n+1} \geq_s ...$ then:

$$\mu_n \Rightarrow \mu = \wedge\{\mu_n\}_{n\in\mathbb{N}}.$$

2.3 An order-theoretic fixed point theorem

The proofs of existence of MEDP and of SME in this paper are based on an extension of Tarski's fixed point theorem. This important theorem combines the isotonicity of a map $F : P \to P$ with the completeness of the underlying lattice $(P, \geq)$ to prove the existence of a complete lattice of fixed point of F.[6] Although Tarski's fixed point theorem is not constructive, we show below (in a result related to Theorem 4.2 in Dugundji and Granas [22]), that the additional property of order continuity of F leads to algorithms to compute the extremal fixed points of F. We first define order continuity as follows:

Definition 3 *Let $(P, \geq)$ be a poset. A function $F : (P, \geq) \to (P, \geq)$ is order continuous if for any countable chain $C \subset P$ such that $\vee C$ and $\wedge C$ both exist,*

$$\vee\{F(C)\} = F(\vee C) \text{ and } \wedge\{F(C)\} = F(\wedge C).$$

It is important to note that order continuity implies isotonicity since $u \leq v$ implies $\vee\{F(u), F(v)\} = F(\vee\{u, v\}) = F(v)$, and thus $F(u) \leq F(v)$. The important property of order continuity implies that successive iterations on F starting from extremal elements will converge to actual (extremal) fixed points when indexed on the natural numbers.

Theorem 4 *Let $(P, \geq)$ be a complete lattice with maximal element $p_{\max}$ and minimal element $p_{\min}$.*

(a). If $F : (P, \geq) \to (P, \geq)$ is isotone, then the set of fixed points of F is a non-empty complete lattice with maximal and minimal elements.

(b). If $F : (P, \geq) \to (P, \geq)$ is order continuous and there exists $a \in P$ such that $F(a) \geq a$, then $\vee\{F^n(a)\}_{n\in\mathbb{N}}$ is the minimal fixed point of F in the order interval $[a, p_{\max}]$.

(c). If $F : (P, \geq) \to (P, \geq)$ is order continuous and there exists b such that $b \geq F(b)$, then $\wedge\{F^n(b)\}_{n\in\mathbb{N}}$ is the maximal fixed point in order interval $[p_{\min}, b]$.

Proof. (a). This is essentially Tarski's fixed point theorem (see Tarski [42]). Consider the set $Q = \{x \in P,\ x \leq F(x)\}$. Since $p_{\min} \in Q$, it is nonempty. Consider a chain C in Q, and $u = \vee C$. Then $c \leq u$ for all $c \in C$, so that by isotonicity of F, $c \leq F(c) \leq F(u)$ for all $c \in C$, which

[6] $p \in P$ is a *fixed point* of the mapping $F : P \to P$ if $F(p) = p$.

implies that $F(u) \geq u$. Thus $u \in C$, and every chain in Q has an upper bound. By Zorn Lemma, Q has a maximal element, which we denote q. Since $q \leq F(q)$, $F(q) \leq F^2(q)$ so $F(q) \in Q$, which implies that $F(q) = q$, and q is clearly the maximal fixed point in P since any fixed point must belong to Q. Considering $Q = \{x \in P,\, F(x) \leq x\}$ and following a symmetric argument proves the existence of a minimal fixed point.

(b). Suppose $F : (P, \geq) \to (P, \geq)$ is order continuous. Since $F(a) \geq a$ and F isotone, $\forall n \in N$, $F^{n+1}(a) \geq F^n(a)$, and $\{F^n(a)\}_{n\in\mathbb{N}}$ is a countable chain. P is a complete lattice so $\vee\{F^n(a)\}_{n\in\mathbb{N}}$ exists, and if F is order continuous, $F(\vee\{F^n(a)\}_{n\in\mathbb{N}}) = \vee\{F(a)\} = \vee\{F^n(a)\}_{n\in\mathbb{N}}$ so that $\vee\{F^n(a)\}_{n\in\mathbb{N}}$ is a fixed point of F. Consider any $d \in P$ such that $F(d) = d$. Since $d \geq a$ and F is isotone, it is easy to see that $\forall n \in N$, $d \geq F^n(a)$ which implies that d is an upper bound of $\{F^n(a)\}_{n\in\mathbb{N}}$. Thus $d \geq \vee\{F^n(a)\}_{n\in\mathbb{N}}$ and $\vee\{F^n(a)\}_{n\in\mathbb{N}}$ is thus the least fixed point of F in $[a, p_{\max}]$, and thus the unique minimal fixed point.

(c). Follows a similar argument to that in (b).■

The reader will notice that the hypothesis of order continuity in (b) and (c) can be weaken to that of order continuity along monotone recursive F-sequences, that is, sequences of the form $\{x, F(x), ..., F^n(x), ...\}$ where either $x \leq F(x)$ or $x \geq F(x)$ and as long as F is isotone. We state this important property in the a corollary.

Corollary 5 *Results (b) and (c) of the preceding theorem hold with $F : (P, \geq) \to (P, \geq)$ order continuous along monotone F-sequences and isotone.*

It is important to note that order continuity along monotone recursive $A-$sequences is distinct from the traditional notion of order continuity. One key difference is that order continuity along monotone recursive A-sequences does not imply that the operator is isotone. Consider for instance $A : [0, 1] \to [0, 1]$ such that $A(x) = 1 - x$. The only monotone recursive A-sequence is $\{1/2,\, 1/2,\, 1/2, ...\}$ and A is obviously continuous along (the only) monotone recursive A-sequence, but not isotone.

3 Existence and construction of MEDP

In this section we develop a new Euler-equation method for OLG models with non-classical stochastic production. Although related to the method used to study infinitely-lived agent models in the literature (a nonlinear operator is defined implicitly from the Euler equation, and its fixed points are the MEDP), our approach is clearly distinct from it[7]. The nonlinear operator is isotone and maps a complete lattice of candidate equilibrium policies into itself; existence of a fixed point then follows from a direct application of Tarski's fixed point theorem, and the construction of extremal fixed points relies on the order continuity property of the operator.

[7] The reader can verify that the nonlinear operator developed for instance in instance in the infinite horizon models of Greenwood and Huffman[26], Datta & al [13], or Morand and Reffett [33] is different than the one in this paper.

We show that when capital income *along equilibrium paths* is increasing in capital stock, optimal consumption and optimal investment in equilibrium are both increasing in the aggregate capital stock, which implies that the unique MEDP must be continuous. When capital income is not assumed to be increasing along equilibrium paths, our nonlinear operator delivers complete lattices of *semi-continuous* isotone MEDP, and we do not have uniqueness.

3.1 Existence of MEDP

Consider the maximization problem of a typical young agent in period t who earns the competitive wage w_t and must decide what amount to consume immediately and what amount to save for future consumption. Returns on labor and capital are obtained from the firms' optimization problem in each period, that is $w(k,z) = F_2(k,1,k,1,z)$ and $r(k,z) = F_1(k,1,k,1,z)$. To make his decisions, the agent postulates a law of motion $k' = h(k,z)$ for physical stock which he uses to compute the competitive expected return on his capital investment. Thus, for a given $(k,z) \in X^* \times Z$ and $k' = h(k,z)$, the agent seeks to solve:

$$\max_{y \in [0,w(k,z)]} \int_Z u(w(k,z) - y, r(k',z')y)G(dz').$$

A MEDP is a function h that coincides pointwise with the optimal investment policy y^* solving the maximization problem above, that is $\forall (k,z) \in X \times Z$, $h(k,z) = y^*(k,z)$, but we exclude the trivial law of motion $h \equiv 0$. Recall that the Euler equation associated with the agent's maximization problem is:

$$\begin{aligned} &\int_Z u_1(w(k,z) - y, r(h(k,z),z')y)G(dz') \\ = &\\ &\int_Z u_2(w(k,z) - y, r(h(k,z),z')y)r(h(k,z),z')G(dz'), \end{aligned}$$

so we define a MEDP as follows:

Definition 6 *A MEDP is a function $h \in H$ such that, for all $(k,z) \in X^* \times Z$, $0 < h(k,z)$ and:*

$$\begin{aligned} &\int_Z u_1(w(k,z) - h(k,z), r(h(k,z),z')h(k,z))G(dz') \\ = &\\ &\int_Z u_2(w(k,z) - h(k,z), r(h(k,z),z')h(k,z))r(h(k,z),z')G(dz'). \end{aligned} \qquad \text{(E)}$$

and $h(0,z) = 0$ for all $z \in Z$.

We use the Euler equation to defined the nonlinear operator A as follows:

Definition 7 *Define the operator A as follows:*

(i) For any $(k, z) \in X \times Z$ *such that* $h(k, z) = 0$, $Ah(k, z) = 0$.

(ii) For any $(k, z) \in X^* \times Z$ *such that* $h(k, z) > 0$, $Ah(k, z)$ *is defined as the unique solution* y *to:*

$$\int_Z [u_1(w(k,z) - y, r(h(k,z), z')y) - u_2(w(k,z) - y, r(h(k,z), z')y) r(y, z')] G(dz'). \tag{E'}$$

The next proposition establishes some important properties of A.

Proposition 8 *Under Assumptions 1, 2, 3 A is an isotone self map on* $(H, \leq)$ *(resp.* $(E_x^u, \leq)$ *and* $(E_x^l, \leq)$, *). Under Assumptions 1, 2, 3, 3' A is an isotone self map on* $(E_z^u, \leq)$ *(reps.* $(E_z^l, \leq)$*).*

Proof. It is easy to see that for all $h \in H$, Ah is increasing in (k, z) and $Ah \in [0, w]$ so that A maps H into itself, and that A is isotone in h. Next, for $h \in E_x^u$ we prove that Ah is right continuous at every $k \in [0, k_{\max}[$, which implies that Ah is usc in k since it is increasing. Suppose that there exists $\widetilde{k}$ in $[0, k_{\max}[$ where Ah is not right continuous, i.e., that there exists $\Delta > 0$ such that:

$$\lim_{k_n \to \widetilde{k}^+} Ah(k_n, z) = Ah(\widetilde{k}, z) + \Delta,$$

where $k_n \to \widetilde{k}^+$ denote convergence of the sequence $\{k_n\}_{n \in \mathbb{N}}$ from the right[8] (i.e., from above). By definition of Ah, $\forall k_n$, $n \in N$, $\forall z \in K$:

$$\begin{aligned} & \int_Z u_1(w(k_n,z) - Ah(k_n,z), r(h(k_n,z),z')Ah(k_n,z)) G(dz') \\ = & \\ & \int_Z u_2(w(k_n,z) - Ah(k_n,z), r(h(k_n,z),z')Ah(k_n,z)) r(Ah(k_n,z),z') G(dz') \end{aligned}$$

By hypothesis, h is increasing and usc and therefore continuous to the right at $\widetilde{k}$, so

$$\lim_{k_n \to \widetilde{k}^+} h(k_n, z) = h(\widetilde{k}, z).$$

By continuity of u_1, u_2,and r, letting k_n converge to $\widetilde{k}$ from the right, we have:

$$\begin{aligned} & \int_Z u_1(w(\widetilde{k},z) - Ah(\widetilde{k},z) - \Delta, r(h(\widetilde{k},z),z')(Ah(\widetilde{k},\bar{z}) + \Delta)) G(dz') \\ = & \\ & \int_Z u_2(w(\widetilde{k},z) - Ah(\widetilde{k},z) - \Delta, r(h(\widetilde{k},z),z')(Ah(\widetilde{k},z) + \Delta)) r(Ah(k,z) + \Delta, z') G(dz') \end{aligned}$$

[8] Since $\{k_n\}_{n \in N}$ is a decreasing sequence and the function Ah is increasing in k, $\{Ah(k_n, z)\}_{n \in N}$ is a decreasing (and bounded) sequence, and therefore convergent, so the expression $\lim_{k_n \to \tilde{k}^+} Ah(k_n, z)$ is legitimate.

But $\Delta \neq 0$ contradicts the uniqueness of the solution to (E') given $(\widetilde{k}, z)$. It must therefore be that $\Delta = 0$, which proves that Ah is right continuous at any $\widetilde{k} \in [0, k_{\max}[$, and thus upper semicontinuous. The operator A thus maps E_x^u into itself.

Next, for $h \in E_x^l$ substitute $k_n \to \widetilde{k}^-$ (for any $\widetilde{k} \in]0, k_{\max}]$) and $\Delta < 0$ in the previous proof to prove that $Ah \in E_x^l$ (since an increasing function that is continuous from the left is lsc), so A maps E_x^l into itself. Consequently, if h is continuous in k for each z, Ah is continuous in k for each z.

Suppose now that Assumption 3' holds. By a similar argument, if h is usc (resp. lsc) in z, similar arguments prove that Ah is usc (resp. lsc) in z, so that A maps E_z^l (resp. E_z^u) into itself. Consequently, if h is continuous in z for each k, then Ah is continuous in z for each k as well.■.

We are now prepared to state and prove our first major proposition concerning the existence of fixed points of A in H, as well as to characterize the lattice structure of this fixed point set. The proposition is a direct application of our fixed point theorem (Theorem 4) in section 2.

Proposition 9 *The set of fixed points of A in $(H, \leq)$ (resp. $(E_x^u, \leq)$, $(E_z^u, \leq)$, $(E_x^{l\prime}, \leq)$, $(E_z^l, \leq)$) is a non-empty complete lattice with minimal and maximal elements.*

3.2 Construction of the extremal MEDP by successive approximations

Equation (E) defining MEDP is a functional equation, and the investigation of numerical solutions through successive approximations for these types of equation is generally a complex task (see the pioneering work of Kantorovich ([29]) on that subject). In the case of (E), the isotonicity and order continuity along monotone recursive $A-$sequences of the operator A are sufficient to produce algorithms approximating the extremal MEDP via successive iterations.

An additional complication arises from our decision to exclude 0 from the set of MEDP, but we provide below a set of sufficient conditions for the existence of a strictly positive minimal fixed point of A, which is by definition the minimal MEDP (in H). Then, through the application of Theorem 4 of Section 2, we construct the minimal MEDP in H as the pointwise limit of a particular increasing sequence of functions. The maximal MEDP in H is obtained as the pointwise limit of a decreasing sequence of functions in a symmetric fashion. Also, since increasing functions on a compact domain are almost everywhere continuous, we show that it is a matter of simply altering extremal MEDP at most at a countable number of points to construct the extremal semicontinuous MEDP.

3.2.1 Order continuity of A

Critical to our construction by successive approximation is the property of order continuity along monotone recursive A-sequences of the operator A, although we actually prove a stronger result which we state in the proposition below.

Proposition 10 *Under Assumptions 1,2 and 3 $A : (H, \geq) \rightarrow (H, \geq)$ is order continuous along any monotone sequences.*

Proof. Recall that:

$$\forall D \subset H \text{ and } \forall s \in S, \ \wedge_H D(s) = \inf_{h \in D}\{h(s)\} \text{ and } \vee_H D(s) = \sup_{h \in D}\{h(s)\},$$

so we need to prove that for an increasing sequence $\{g_n\}_{n \in \mathbb{N}}$ in $(H, \leq)$,

$$\sup(\{Ag_n(k,z)\}_{n \in \mathbb{N}}) = A(\sup\{g_n(k,z)\}_{n \in \mathbb{N}}),$$

and the corresponding property for a decreasing sequence.

Consider then the increasing sequence $g_0 \leq g_1 \leq ... \leq g_i \leq ...$ in H. For all $(k,z) \in X * Z$, the sequence of real numbers $\{g_n(k,z)\}_{n \in \mathbb{N}}$ is increasing and bounded above (by $w(k,z)$), which implies that $\lim_{n \to \infty} g_n(k,z) = \sup\{g_n(k,z)\}_{n \in \mathbb{N}}$. For the same reason $\lim_{n \to \infty} Ag_n(k,z) = \sup\{Ag_n(k,z)\}_{n \in \mathbb{N}}$. By definition, for all $n \in \mathbb{N}$, and all $(k,z) \in K^* \times Z$:

$$\begin{array}{ll} & \int_Z u_1(w(k,z) - Ag_n(k,z), r(g_n(k,z), z')Ag_n(k,z))G(dz') \\ = & \\ & \int_Z u_2(w(k,z) - Ag_n(k,z), r(g_n(k,z), z')Ag_n(k,z))r(Ag_n(k,z), z')G(dz') \end{array}$$

The functions u_1 are u_2 continuous (Assumption 1), r is continuous in its first argument (Assumption 3), hence taking limits when n goes to infinity, we have:

$$\begin{array}{ll} & \int_Z u_1(w(k,z) - \sup\{Ag_n(k,z)\}_{n \in \mathbb{N}}, r(\sup\{g_n(k,z)\}_{n \in \mathbb{N}}, z') \sup\{Ag_n(k,z)\}_{n \in \mathbb{N}})G(dz') \\ = & \\ & \int_Z u_2(w(k,z) - \sup\{Ag_n(k,z)\}_{n \in \mathbb{N}}, r(\sup\{g_n(k,z)\}_{n \in \mathbb{N}}, z') \\ & \sup\{Ag_n(k,z)\}_{n \in \mathbb{N}})r(\sup\{Ag_n(k,z)\}_{n \in \mathbb{N}}, z')G(dz'), \end{array}$$

which implies that $A(\sup\{g_n(k,z)\}_{n \in \mathbb{N}}) = \sup\{Ag_n(k,z)\}_{n \in \mathbb{N}}$. A symmetric argument can easily be made for a decreasing sequence $\{g_n\}_{n \in \mathbb{N}}$ noting that in this case, the sequences of real numbers $\{g_n(k,z)\}_{n \in \mathbb{N}}$ is decreasing and bounded below by 0, therefore $\lim_{n \to \infty} g_n(k,z) = \inf\{g_n(k,z)\}_{n \in \mathbb{N}}$.■

With order continuity along monotone sequences of the operator A now established, we turn next to the computation of extremal MEDP. Because of additional difficulties associated with avoiding 0 as MEDP, we consider the question of computing minimal and maximal MEDP separately.

3.2.2 Minimal MEDP

Our definition requires a MEDP to be a strictly positive function, that is a function $h_0 : X \times Z \to K$ such that:

$$\forall (k,z) \in X^* \times Z,\ h_0(k,z) > 0.$$

Since 0 is the minimal fixed point of A, we present now sufficient conditions for the existence of a strictly positive minimal fixed point of A in $(H, \leq)$ in the form of Assumption 4 below.

Assumption 4. $\lim_{k \to 0^+} r(k, z_{\max})k = 0$ and $\forall c_1 > 0, \lim_{c_2 \to 0} u_2(c_1, c_2) = \infty$.

Proposition 11 *Under assumption 4, there exists $h_0 \in H$ such that (a) for all $(k,z) \in X^* \times Z$, $Ah_0(k,z) > h_0(k,z) > 0$, and (b) for all $(k,z) \in X^* \times Z$, $0 < x < h_0(k,z)$ implies $Ax > x$. (c) In addition, h_0 can be chosen to be lower semicontinuous in k for all z and constant in z for all k (and therefore continuous and increasing in z for all k).*

Proof. See Appendix B.■

It is a direct consequence of the previous proposition that A must have a fixed point greater than h_0 (since the isotone operator A then maps the complete lattice $[h_0, w]$ into itself), but also that there cannot be any other strictly positive fixed point in H smaller than h_0. By Theorem 4 in Section 2, the minimal MEDP in H must therefore be $\vee_H \{A^n h_0\}_{n \in \mathbb{N}}$. We formalize this result in the following proposition.

Proposition 12 *Under Assumption 1, 2, 3, 3', 4 we have the following results:*

(a). $\forall n \in \mathbb{N}$, $A^n h_0$ is lsc in k for all z and continuous in z for all k.

(b). $h_{\min} = \vee_H \{A^n h_0\}_{n \in \mathbb{N}}$ is the minimal MEDP in $(H, \leq)$, and is $\mathcal{B}(S)$-measurable.

(c). $h_{\min} = \vee_H \{A^n h_0\}_{n \in \mathbb{N}}$ is the minimal MEDP in $(E_x^l, \leq)$ and in $(E_z^l, \leq)$.

Proof. (a) Since h_0 is lsc in k for each z and continuous in z for each k, $\forall n \in \mathbb{N}$ the functions $A^n h_0$ have these same properties, and they are therefore all $\mathcal{B}(S)$-measurable functions (as Caratheodory functions, continuous in z and measurable -since increasing- in k). (b). It follows from the fixed point theorem of Section 2 that $\vee_H \{A^n h_0\}_{n \in \mathbb{N}}$ is the minimal fixed point in the order interval $[h_0, w] \subset H$. Note that:

$$h_{\min}(k,z) = \vee_H \{A^n h_0\}_{n \in \mathbb{N}}(k,z) = \lim_{n \to \infty} A^n h_0(k,z) = \sup\{A^n h_0(k,z)\}_{n \in \mathbb{N}}$$

Next, consider $g \in H$ with $Ag = g$ and suppose that there exists $(k,z) \in X^* * Z$ with $0 < g(k,z) \leq \vee_H \{A^n h_0\}_{n \in \mathbb{N}}(k,z)$. By Part(b) of the previous proposition, $Ag(k,z) > g(k,z)$ which contradicts the hypothesis that g is a fixed point. $\vee_H \{A^n h_0\}_{n \in \mathbb{N}}$ is thus the minimal strictly positive fixed point of

A in H. As the pointwise limit of a sequence of $\mathcal{B}(S)$-measurable functions, it is $\mathcal{B}(S)$-measurable as well. (c) $\vee_H\{A^n h_0\}_{n\in\mathbb{N}}$ is the upper envelope of a family of functions lsc in k and continuous in z, and is thus lsc in k and lsc in z. It is thus the minimal fixed point of A in E_x^l and in E_z^l.■

Remark: The main role of Assumption 3' in the previous proposition is to guarantee that $h_{\min}$ is $\mathcal{B}(S)$-measurable. Indeed, Assumption 3' is sufficient for the preservation of the continuity in z for all k of h_0 under the operator A, so that all functions $A^n h_0$ are continuous in z for all k. Given that all these functions are increasing in k, they are then also $\mathcal{B}(S)$-measurable, which implies that $h_{\min}$ is also $\mathcal{B}(S)$-measurable. There are, however, other ways to prove the $\mathcal{B}(S)$-measurability of $h_{\min}$ without relying on Assumption 3'. One way is to start the iterations on A with a function that is continuous in k and increasing in z, smaller than h_0 in proposition ?? but strictly greater than 0 on $X^* \times Z$. Since h_0 is lsc in k, increasing in z, and strictly greater than 0, it is always possible (albeit tedious) to construct such function. Then all the successive A-iterates of the initial function are continuous in k and increasing in z and therefore $\mathcal{B}(S)$-measurable. The $\mathcal{B}(S)$-measurability of the minimal MEDP in $(E_x^l, \leq)$ then follows.

An important corollary to this theorem that we will use in the sequel is as follows (by the previous remark, Assumption 3' is not necessary for the Corollary to hold):

Corollary 13 *Under Assumption 1, 2, 3, 4 the function $g : X \times Z \to K$ such that for all $(k,z) \in X \times Z$,*

$$g(k,z) = \inf_{k'>k}\{\sup\{A^n h_0(k',z)\}_{n\in\mathbb{N}}\} = \inf_{k'>k}\{\vee_H\{A^n h_0\}_{n\in\mathbb{N}}(k',z)\}$$

is the minimal MEDP in $(E_x^u, \leq)$, and is $\mathcal{B}(S)$-measurable.

Proof. By construction $g \in E_x^u$, g and $\vee_H\{A^n h_0\}_{n\in\mathbb{N}}$ differ at most at the discontinuity points of $\vee_H\{A^n h_0\}_{n\in\mathbb{N}}$ (g is therefore $\mathcal{B}(S)$-measurable), and g is the smallest usc (in k) function greater than $\vee_H\{A^n h_0\}_{n\in\mathbb{N}}$. In addition, since $\vee_H\{A^n h_0\}_{n\in\mathbb{N}}$ is increasing and lower semicontinuous in k, for any $(k,z) \in X \times Z$, $g(k,z) = \lim_{k'\to k^+} \vee_H\{A^n h_0\}_{n\in\mathbb{N}}(k',z)$. For any $(k,z) \in [0,, k_{\max}[\times Z$, and for all $k' > k$, by definition of $q = \vee_H\{A^n h_0\}_{n\in\mathbb{N}}$:

$$\begin{aligned} & \int_Z u_1(w(k',z) - q(k',z), r(q(k',z),z')q(k',z))G(dz') \\ = & \\ & \int_Z u_2(w(k',z) - q(k',z), r(q(k',z),z')q(k',z))r(q(k',z),z')G(dz'). \end{aligned}$$

Both functions u_1 and u_2 are continuous and r is continuous in its first argument (Assumption 3) so taking limits when $k' \to k^+$ on both sides of the previous equality implies that:

$$
\begin{array}{c}
\int_Z u_1(w(k,z)-g(k,z), r(g(k,z),z')g(k,z))G(dz') \\
= \\
\int_Z u_2(w(k,z)-g(k,z), r(g(k,z),z')g(k,z))r(g(k,z),z')G(dz'),
\end{array}
$$

which proves that, $Ag(k,z) = g(k,z)$.$\blacksquare$

A symmetric argument to that of the above proof, but exploiting the continuity of r with respect to its second argument (Assumption 3') easily leads to the following additional important corollary:

Corollary 14 *Under Assumption 1, 2, 3, 3', 4 the function $g : X \times Z \to K$ such that for all $(k,z) \in X \times Z$:*

$$g(k,z) = \inf_{z'>z} \{\sup\{A^n h_0(k,z')\}_{n\in\mathbb{N}}\} = \inf_{z'>z} \{\vee_H \{A^n h_0\}_{n\in\mathbb{N}}(k,z')\}$$

is the minimal MEDP in $(E^u_z, \leq)$, and is $\mathcal{B}(S)$-measurable.

3.2.3 Maximal MEDP

The computation of maximal MEDP is similar to that of the minimal MEDP, with the additional feature that iterations on A begin with the maximal element w of H which is assumed to be continuous in k and z (Assumption 3 and 3'), so that continuity is preserved at each iteration.

Proposition 15 *Under Assumption 1, 2, 3, 3' we have the following results:*

(a). $\forall n \in \mathbb{N}$, $A^n w$ is a continuous function on $X \times Z$.

(b). $h_{\max} = \wedge_H \{A^n w\}_{n\in\mathbb{N}}$ is the maximal MEDP in $(H, \leq)$, and is $\mathcal{B}(S)$-measurable.

(c). $h_{\max}$ is the maximal MEDP in $(E^u_x, \leq)$ and $(E^u_z, \leq)$.

Proof. (a) Since w is continuous in k and in z, for all $n \in \mathbb{N}$, all functions $A^n w$ have the same property (since the image by A of a continuous function is a continuous function). Thus, the functions $A^n w$ are $\mathcal{B}(S)$-measurable. (b). Follows directly from Theorem 4 in Section 2. Note that, for any $(k,z) \in X * Z$, the sequence of real numbers $\{A^n w(k,z)\}_{n\in\mathbb{N}}$ is decreasing and bounded below, hence convergent, so that:

$$h_{\max}(k,z) = \wedge_H \{A^n w\}_{n\in\mathbb{N}}(k,z) = \lim_{n\to\infty} A^n w(k,z) = \inf\{A^n w(k,z)\}_{n\in\mathbb{N}}.$$

Since $\wedge_H \{A^n w\}_{n\in\mathbb{N}}$ is the pointwise limit of a sequence of $\mathcal{B}(S)$-measurable functions, it is $\mathcal{B}(S)$-measurable. (c) By (a) above $\wedge_H \{A^n w\}_{n\in\mathbb{N}}$ is the lower envelope of a family of continuous functions, and is at least usc in k and in z. Consequently, $\wedge_H \{A^n w\}_{n\in\mathbb{N}}$ is the maximal fixed point of A in E^u_x and E^u_z.$\blacksquare$

Remark: The reader will note that, absent the hypothesis of continuity of w in z, $A^n w$ is still continuous in k but not necessarily in z. It is however increasing in z, and therefore $\mathcal{B}(S)$-measurable. As a result (b) still holds, and $h_{\max}$ is the maximal MEDP in $(E_x^u, \leq)$.

As in the case of the minimal MEDP, we now have the following corollary concerning the maximal measurable MEDP (by the previous remark, Assumption 3' is not necessary for the corollary to hold):

Corollary 16 *Under Assumption 1, 2, 3 the function $g : X \times Z \to X$ such that for all $(k, z) \in X^* \times Z$,:*

$$g(k, z) = \sup_{k' < k} \{\wedge_H \{A^n h_0\}_{n \in \mathbb{N}}(k', z)\} \text{ and } g(0, z) = 0$$

is the maximal MEDP in $(E_x^l, \leq)$ and is $\mathcal{B}(S)$-measurable.

Proof. By construction $g \in E_x^l$, g and $\wedge_H \{A^n h_0\}_{n \in \mathbb{N}}$ differ at most at the discontinuity points of $\wedge_H \{A^n h_0\}_{n \in \mathbb{N}}$ (and thus g is $\mathcal{B}(S)$-measurable as well), and g is the greater lsc (in k) function smaller than $\wedge_H \{A^n h_0\}_{n \in \mathbb{N}}$. In addition, since $\wedge_H \{A^n h_0\}_{n \in \mathbb{N}}$ is increasing and lower semicontinuous in k, for any $(k, z) \in X \times Z$, $g(k, z) = \lim_{k' \to k^-} \wedge_H \{A^n h_0\}_{n \in \mathbb{N}}(k', z)$. For any $(k, z) \in X^* \times Z$, and for all $k' < k$, by definition of $p = \wedge_H \{A^n h_0\}_{n \in \mathbb{N}}$,

$$\begin{aligned} & \int_Z u_1(w(k', z) - p(k', z), r(p(k', z), z')p(k', z))G(dz') \\ = & \\ & \int_Z u_2(w(k', z) - p(k', z), r(p(k', z), z')p(k', z))r(p(k', z), z')G(dz'). \end{aligned}$$

Both u_1 and u_2 are continuous and r is continuous in its first argument (Assumption 3) so taking limits when $k' \to k^-$ on both sides of the previous equality implies that:

$$\begin{aligned} & \int_Z u_1(w(k, z) - g(k, z), r(g(k, z), z')g(k, z))G(dz') \\ = & \\ & \int_Z u_2(w(k, z) - g(k, z), r(g(k, z), z')g(k, z))r(g(k, z), z')G(dz'), \end{aligned}$$

which proves that, $Ag(k, z) = g(k, z)$.■

Again, the following result can easily be established through a slight modification of the above proof (and relying on the continuity of r in its second argument postulated in Assumption 3').

Corollary 17 *Under Assumption 1, 2, 3, 3' the function $g : X \times Z \to X$ such that for all $(k, z) \in X \times Z \setminus \{z_{\min}\}$:*

$$g(k, z) = \sup_{z' < z} \{\wedge_H \{A^n h_0\}_{n \in \mathbb{N}}(k, z')\}$$

is the maximal MEDP in E_z^l, and is $\mathcal{B}(S)$-measurable.

3.2.4 Comparative statics results

One critical advantage of using a monotone approach to the construction of MEDP is the possibility to consider comparative statics questions on the space of economies. The comparative statics results we obtain are closely related to the "strong set order" comparative statics obtained in Veinott ([41], Chapter 10, Theorem 1) and Topkis ([43], Theorem 2.5.2). Our result rely on the key isotonicity property of A when parameterized as a function of the primitive data of production, and we focus in our discussion on ordered perturbations of production that imply ordered changes in the wage process. Denoting A_w be the operator for a given wage w in the definition of the A, we first show that pointwise ordered changes in the wage rate imply pointwise changes in $A_w h$ for any h :

Proposition 18 *The operator A is increasing in w in the following sense:*

$$\text{For all } w' \geq w, \forall h \in H, \; A_{w'}h \geq A_w h,$$

where all inequalities are in the pointwise order.

Proof. For $w' \geq w$, and for all $(k, z) \in X^* \times Z$ and $h \in H$:

$$\begin{aligned}
& \int_Z u_1(w'(k,z) - A_w h(k,z), r(h(k,z), z')A_w h(k,z))G(dz') \\
\leq & \\
& \int_Z u_1(w(k,z) - A_w h(k,z), r(h(k,z), z')A_w h(k,z))G(dz') \\
= & \\
& \int_Z u_2(w(k,z) - A_w h(k,z), r(g(k,z), z')A_w h(k,z))r(A_w h(k,z), z')G(dz') \\
\leq & \\
& \int_Z u_2(w'(k,z) - A_w h(k,z), r(g(k,z), z')A_w h(k,z))r(A_w h(k,z), z')G(dz').
\end{aligned}$$

Summarizing:

$$\begin{aligned}
& \int_Z u_1(w'(k,z) - A_w h(k,z), r(h(k,z), z')A_w h(k,z))G(dz') \\
\leq & \\
& \int_Z u_2(w'(k,z) - A_w h(k,z), r(g(k,z), z')A_w h(k,z))r(A_w h(k,z), z')G(dz'),
\end{aligned}$$

which implies that $A_{w'}h(k,z) \geq A_w h(k,z)$.■

From this monotonicity property of A, we can now obtain the following important *equilibrium* comparative static implication for the set of MEDP in H :

Proposition 19 *The maximal and minimal MEDP in $(H, \leq)$ (resp. $(E_x^u, \leq)$, $(E_z^u, \leq)$, $(E_x^l, \leq)$ ($E_z^l, \leq)$) are increasing in w.*

Proof. For any w', $A_{w'}w' \leq w'$, and since $A_{w'}$ is increasing in w', we have:

$$w' \geq w \text{ implies that } A_w w \leq A_{w'}w' \leq w'.$$

Suppose that there exists $n > 1$ such that:

$$A_w^n w \leq A_{w'}^n w'. \tag{R1}$$

Then:

$$A_w^{n+1} w = A_w(A_w^n w) \leq A_{w'}(A_w^n w) \leq A_{w'}(A_{w'}^n w') = A_{w'}^{n+1} w' \leq A_{w'}w',$$

where the first inequality results from A_w being increasing in w, the second and the third from $A_{w'}$ being isotone. Thus recursively, (R1) is true for all $n \geq 1$, and, consequently,

$$\wedge_H \{A_w^n w\}_{n \in \mathbb{N}} \leq \wedge_H \{A_{w'}^n w'\}_{n \in \mathbb{N}}.$$

which proves that the maximal MEDP in $(H, \leq)$ is increasing in w. It is then easy to prove that this same result holds in $(E_x^u, \leq)$, $(E_z^u, \leq)$, $(E_x^l, \leq)$ and ($E_z^l, \leq)$.

Next, for a given w, construct the function h_0 as in Appendix B. For $w' \geq w$:

$$A_{w'}h_0 \geq A_w h_0 \ (\geq h_0),$$

which recursively implies that, for all n :

$$A_{w'}^n h_0 \geq A_w^n h_0,$$

and thus that:

$$\vee_H \{A_w^n h_0\}_{n \in \mathbb{N}} \leq \vee_H \{A_{w'}^n h_0\}_{n \in \mathbb{N}}.$$

It is easy to see that $\vee_H \{A_{w'}^n h_0\}_{n \in \mathbb{N}}$ is the minimal MEDP (since $A_{w'}h_0 \geq h_0$ and for all $0 < e < h_0(k, z)$, $A_{w'}e \geq A_w e > e$), which proves that the minimal MEDP in $(H, \leq)$ is increasing in w, and the result also holds in $(E_x^u, \leq)$, $(E_z^u, \leq)$, $(E_x^l, \leq)$ and ($E_z^l, \leq)$.■

This analysis is not restricted to comparative statics questions on the space of production functions: Indeed, one can see that for particular parametrization of the household utility functions (say u), monotonicity of the operator A_u could be obtained, which will then lead to strong set order perturbations of the set of MEDP. We finally remark that by a standard argument, ordered perturbations in the set of MEDP in the strong set order will generate ordered perturbations in the set of SME constructed in the next section of this paper (where the partial order on the space of limiting distributions is first order stochastic dominance).[9]

[9] See Hopenhayn and Prescott [28] and Mirman, Morand, and Reffett [32] for discussion of such comparative statics statements on the set of SME.

3.3 Uniqueness of MEDP under capital income monotonicity

Under capital income monotonicity (and Assumption 4), we prove that there exists a unique MEDP h^*. h^* is thus both minimal and maximal MEDP, and by the previous results above, it is therefore both upper semicontinuous and lower semicontinuous in (k, z), and therefore continuous in (k, z). In addition, we prove that the corresponding Markovian equilibrium consumption decision policy is also continuous in (k, z).[10]

Proposition 20 *Under Assumption 4, if for all $z \in Z$, $r(y, z)y$ is increasing in y (an hypothesis we call "capital income monotonicity") then there exists a unique MEDP h^* in H. The corresponding Markovian equilibrium consumption policy, $w - h^*$ is also increasing in (k, z), which implies that both h^* and $w - h^*$ are continuous.*

Proof: Under capital income monotonicity, for all $(k, z) \in X^* \times Z$, it is easy to see that the following equation in y:

$$\begin{array}{c} \int_Z u_1(w(k,z) - y, r(y,z')y)G(dz') \\ = \\ \int_Z u_2(w(k,z) - y, r(y,z')y)r(y,z')G(dz'). \end{array}$$

has a unique solution. Note that the function h^* is therefore the maximal and minimal MEDP, and thus usc and lsc in k, i.e., continuous in k. By definition, for all $(k, z) \in K^* \times Z$:

$$\begin{array}{c} \int_Z u_1(w(k,z) - h^*(k,z), r(h^*(k,z),z')h^*(k,z))G(dz') \\ = \\ \int_Z u_2(w(k,z) - h^*(k,z), r(h^*(k,z),z')h^*(k,z))r(h^*(k,z),z')G(dz'). \end{array} \tag{E''}$$

Suppose there exists $(k, z) \in K^* \times Z$ such that $w(k, z) - h^*(k, z)$ decreases with an increase in k. Then, for all $z' \in Z$, the expression:

$$u_1(w(k,z) - h^*(k,z), r(h^*(k,z),z')h^*(k,z))$$

increases with k under the assumption of capital income monotonicity, and given that $h^*(k, z)$ is increasing in k, $u_{12} \geq 0$ and $u_{11} \leq 0$. However, for all $z' \in Z$, the expression:

$$u_2(w(k,z) - h^*(k,z), r(h^*(k,z),z')h^*(k,z))r(h^*(k,z),z')$$

[10] A careful reading of our argument in the paper shows that under capital income isotonicity consumption and investment are in fact locally Lipschitz continuous (since elements of an equicontinuous space of functions whose gradient fields are all bounded by the variation in the wage rate in equilibrium). It is important to note this when considering numerical implementations of our methods since Lipschitz continuous functions can be approximated with greater accuracy and convergence rates than merely continuous functions.

necessarily decreases with an increase in k. Thus LHS and RHS in equation (E”) above move in opposite direction when k increases, which is impossible. As a result, $w(k,z) - h^*(k,z)$ must be increasing in k. The same argument works to show that $w(k,z) - h^*(k,z)$ must be increasing in z. Finally, under the assumption that $w(k,z)$ is continuous in (k,z), if both the equilibrium investment and the equilibrium consumption policies are increasing in (k,z), they both necessarily must be continuous in (k,z).■

It is important to note that the condition $r(k,z)k$ increasing in k is not necessary for uniqueness of MEDP. Consider, for instance, preferences represented by:

$$\ln(c_t) + \ln(c_{t+1}).$$

The maximization problem of an agent is:

$$\max_{y\in[0,w(k,z)]} \left\{ \ln(w(k,z) - y) + \int_Z \ln(r(h(k,z),z')y)G(dz') \right\},$$

and the corresponding first order condition is:

$$(w(k,z) - y) = y.$$

Thus, irrespective of the production function, there exists a unique Markovian equilibrium decision policy (the function $h = .5w$).■

4 Existence and construction of stationary Markov equilibria

In this paper we define a stationary Markov equilibrium as an invariant distribution, in line with the work of Hopenhayn and Prescott [28] and Futia[24]), and in contrast to Wang[44][45] who follows the path of Duffie & al.[23]. Our approach exploits the constructive fixed point theorem of Section 2 (Theorem 4): For any MEDP, we propose algorithms converging to extremal invariant distributions corresponding to this particular MEDP. Also, we require the SME to be a probability distribution that is ”non-trivial” in the sense that we require the limiting distribution to be distinct from the ”zero” distribution. In that sense, our work is consistent with the notion of SME used in for example Brock and Mirman [9].

The assumption of iid shocks implies that an economy in any period t is fully characterized by a probability measure $\mu_t \in \Lambda(X, \mathcal{B}(X))$ defined over the endogenous state space X. In contrast, when exogenous shocks are persistent, for instance when shocks follow a first order Markov process, the measure μ_t characterizing an economy in period t belongs to $\Lambda(S, \mathcal{B}(S))$ as it is defined over the whole state space $S = X \times Z$. This means that proofs of existence, characterization, and construction of extremal SME are significantly more complicated, in part because $\Lambda(X \times Z, \mathcal{B}(X \times Z))$ endowed with the stochastic order is no longer a complete lattice, although it is a countable chain complete lattice. For this reason, we thoroughly address the case of persistent Markov shocks in a separate paper,[11] although we state and sketch the proof an important result at the end of this section.

In this section, we first define a SME as an invariant probability measure in $\Lambda(X, \mathcal{B}(X))$ of an operator associated with a $\mathcal{B}(S)$-measurable MEDP, but we exclude the trivial singular measure δ_0 (all mass at $k = 0$) from the set of SME. Next, we use our fixed point theorem of Section 2 (Theorem 4) to show existence of SME and to construct algorithms converging to extremal SME through successive monotone iterations.

4.1 Definition of stationary Markov equilibrium

Recall that any $\mathcal{B}(S)$-measurable MEDP $h \in H$ induces a Markov process for the capital stock represented by the transition function P_h defined as follows:

$$\forall A \in \mathcal{B}(K),\ P_h(k, A) = \Pr\{h(k, z) \in A\} = \lambda(\{z \in Z,\ h(k, z) \in A\}).$$

That is, $P_h(k, A)$ is the probability that the capital stock is in the set A one period after being equal to k.[12] If we denote by μ_t the probability measure associated with the random variable k_t, then μ_{t+1} is defined by applying the operator $T_h^* : (\Lambda(X, \mathcal{B}(X)), \geq_s) \to (\Lambda(X, \mathcal{B}(X)), \geq_s)$ to μ_t in the following manner::

$$\forall B \in \mathcal{B}(K),\ \mu_{t+1}(B) = T_h^* \mu_{t+1}(B) = \int_K P_h(k, B)\mu_t(dk). \tag{M1}$$

Thus, $T_h^* \mu_{t+1}(B)$ is the probability that the next period capital stock k lies in the set B if the current period capital stock is drawn according to the probability measure μ_t.

Definition 21 *Given a $\mathcal{B}(S)$-measurable MEDP h, a stationary Markov equilibrium (SME) is a probability measure $\mu \in \Lambda(X, \mathcal{B}(X))$* ***distinct from*** *$\delta_0$ such that:*

$$\forall B \in \mathcal{B}(K),\ \mu(B) = T_h^* \mu(B) = \int_K P_h(k, B)\mu(dk).$$

That is, if the current period capital is distributed according to the probability measure μ, then next period capital is also distributed according to the probability measure μ while all agents follow the MEDP h, and the probability measure μ is not the trivial singular measure δ_0.

[11] Another issue in OLG models with Markov shocks is the additional restrictions needed to prove existence of isotone MEDP.

[12] The $\mathcal{B}(S)$-measurability of h implies that P_h is indeed a transition function since for each $k \in X$, $P_h(k, .)$ is a probability measure, and for each A, $P_h(., A)$ is a measurable function.

4.2 Properties of the operator T_h^*

A SME as defined immediately above is simply a non-trivial fixed point of T_h^*. We show next that the operator T_h^* and its domain have just the right properties required to apply our fixed point theorem of Section 2. In the rest of this section we will consider a $\mathcal{B}(S)$-measurable MEDP h in H (so that h is increasing).

Proposition 22 *The transition function P_h is increasing. Consequently, T_h^* is isotone on $(\Lambda(X, \mathcal{B}(X)), \geq_s)$.*

Proof. P_h is said to be increasing if for all functions $f : X \to \mathbb{R}_+$, bounded, measurable and increasing, the function $T_h f$ defined as:

$$T_h f(k) = \int_X f(k')P_h(k, dk'),$$

is increasing. Recall that λ is the probability measure over the over the exogenous shocks and that $h(k, z)$ is increasing in k. Thus, for any $k_1 \geq k_2$ and any function $f : X \to \mathbb{R}_+$, bounded and increasing (and thus measurable):

$$\begin{array}{rcl} \int_X f(k')P_h(k_1, dk') & = & \int_Z f(h(k_1, z))\lambda(dz) \\ & \geq & \\ \int_Z f(h(k_2, z))\lambda(dz) & = & \int_X f(k')P_h(k_2, dk'), \end{array}$$

which proves that the function $T_h f$ defined as:

$$T_h f(k) = \int_X f(k')P_h(k, dk'),$$

is increasing, i.e., that P_h is increasing. Next, consider any $\mu' \geq_s \mu$ and any $f : X \to \mathbb{R}_+$, bounded, measurable and increasing:[13]

$$\langle f, T_h^* \mu' \rangle = \langle T_h f, \mu' \rangle \geq \langle T_h f, \mu \rangle = \langle f, T_h^* \mu \rangle$$

which proves that $T_h^* \mu' \geq_s T_h^* \mu$, i.e., that T_h^* is isotone.■

Recall that to obtain extremal invariant distributions via successive approximation, a sufficient condition is the order continuity along recursive monotone T_h^*-sequences of the operator T_h^*. We prove next that if h is continuous, then P_h has the Feller property and T_h^* is weakly continuous and therefore order continuity along every monotone sequence.

Proposition 23 *If $h : X \times Z \to X$ is continuous, then T_h^* is order continuous along monotone sequences.*

[13] We use here the standard notation:

$$\langle f, \mu \rangle = \int_X f(k)\mu(dk)$$

Proof. Recall that T_h^* is order continuous along monotone sequences if for any sequence of probability measures $\{\mu_n\}_{n\in\mathbb{N}}$ in $\Lambda(X, \mathcal{B}(X))$ satisfying $\mu_i \leq \mu_{i+1}$ (resp. $\mu_i \geq \mu_{i+1}$):

$$T_h^*(\vee\{\mu_n\}_{n\in\mathbb{N}}) = \vee\{T_h^*(\mu_n)\}_{n\in\mathbb{N}} \text{ (resp. } T_h^*(\wedge\{\mu_n\}_{n\in\mathbb{N}}) = \wedge\{T_h^*(\mu_n)\}_{n\in\mathbb{N}}).$$

Consider any $f : X \to \mathbb{R}$ bounded and continuous. For any $k \in X$, and any sequence $\{k_n\}_{n\in\mathbb{N}}$ in X converging to k ::

$$\lim_{n\to\infty} T_h f(k_n) = \lim_{n\to\infty} \int_Z f(h(k_n, z))\lambda(dz) = \int_Z f(h(k, z))\lambda(dz) = T_h f(k)$$

by uniform continuity of foh on the compact domain $X \times Z$, which proves that $T_h f$ is a continuous function (it is also clearly bounded since both f and h are bounded). Consider an increasing sequence $\{\mu_n\}_{n\in\mathbb{N}}$, and $\mu = \vee\{\mu_n\}_{n\in\mathbb{N}}$ its weak limit.[14] Since $T_h f : X \to \mathbb{R}$ is bounded and continuous:

$$\lim_{n\to\infty} \langle f, T_h^*\mu_n\rangle = \lim_{n\to\infty} \langle T_h f, \mu_n\rangle = \langle T_h f, \mu\rangle = \langle f, T_h^*\mu\rangle,$$

that is, $T_h^*(\mu_n) \Rightarrow T_h^*(\mu)$, which implies that T_h^* is order continuous along monotone sequences since $\{T_h^*(\mu_n)\}_{n\in\mathbb{N}}$ is an increasing sequence so that $T_h^*(\mu_n) \Rightarrow \vee\{T_h^*(\mu_n)\}_{n\in\mathbb{N}}$ and by uniqueness of the weak limit, $\vee\{T_h^*(\mu_n)\}_{n\in\mathbb{N}} = T_h^*(\vee\{\mu_n\}_{n\in\mathbb{N}})$. A symmetric argument holds for decreasing sequences.

Finally, we also prove another property of the adjoint operator T_h^* which is critical for establishing comparative statics results.

Proposition 24 *The operator T_h^* is isotone in h. That is:*

$$h' \geq h \text{ implies that } \forall\mu \in \Lambda(X, \mathcal{B}(X)),\ T_{h'}^*\mu \geq_s T_h^*\mu.$$

Proof. Consider $f : X \to \mathbb{R}$ nonnegative, increasing and bounded. Because f is increasing,

$$h' \geq h \text{ implies that } \forall(k, z) \in X \times Z,\ f(h'(k, z)) \geq f(h(k, z),$$

and therefore:

$$T_{h'} f(k) = \int_Z f(h'(k, z))\lambda(dz) \geq \int_Z f(h(k, z))\lambda(dz) = T_h f(k) \text{ for all } k \in X$$

Consequently:

$$\begin{aligned} \langle f, T_{h'}^*\mu\rangle &= \langle T_{h'} f, \mu\rangle = \int_X T_{h'} f(k)\mu(dk) \\ &\geq \\ \int_X T_h f(k)\mu(dk) &= \langle T_h f, \mu\rangle = \langle f, T_h^*\mu\rangle. \end{aligned}$$

■

[14] As noted immediately after Proposition 2 in Section 2 above, all monotone sequences weakly converge.

4.3 Existence of SME under capital income monotonicity

Under capital income monotonicity, the unique MEDP h^* is isotone and continuous in (k, z). The isotonicity and order continuity along monotone sequences demonstrated in Propositions 23 and 24 above imply the following important result concerning the set of fixed points of the operator $T^*_{h^*}$.

Proposition 25 *Under capital income monotonicity, denoting h^* the unique MEDP, the set of fixed points of $T^*_{h^*}$ is a non-empty complete lattice with maximal and minimal elements, respectively $\wedge\{T^{*n}_{h^*}\delta_{k_{\max}}\}_{n\in\mathbb{N}}$ and $\vee\{T^*_{h^*}\delta_0\}_{n\in\mathbb{N}}$.*

Proof. $(\Lambda(X, \mathcal{B}(X)), \geq_s)$ is a complete lattice and $T^*_{h^*}$ is isotone so the set of fixed points is a nonempty complete lattice. Under capital income monotonicity, the unique MEDP h is continuous and T^*_h is order continuous. A direct application of Theorem 4 of Section 2 shows then that the maximal and minimal fixed point are, respectively $\wedge\{T^{*n}_h\delta_{k_{\max}}\}_{n\in\mathbb{N}}$ and $\vee\{T^*_h\delta_0\}_{n\in\mathbb{N}} = \delta_0$.■

Since our definition of SME excludes δ_0, the previous result does not automatically imply that there exists a SME for the Markov process induced by h^*. Indeed, suppose for instance that:

$$\forall(k, z) \in X^* \times Z,\ 0 < h^*(k, z) < k.$$

It is then easy to see that given any initial distribution of capital stock, in the long run the capital stock will be 0. That is, the only fixed point of $T^*_{h^*}$ is δ_0, and the set of SME is therefore empty. An obvious case when this happens is when:

$$\forall(k, z) \in X^* \times Z,\ w(k, z) < k.$$

One can think of various sufficient conditions under which the set of SME is non-empty but it would be most useful to express any such set of conditions in terms of restrictions on the primitives of the problem, and this is what we do next.

Specifically, Assumption 5 below states sufficient conditions under which there exists an increasing function $h_0 \in H$ such that (a) $\forall k \in [0, k_0] \subset X$ and $\forall z \in Z$, $h_0(k, z) \geq k$, and (b) A maps h_0 strictly up (i.e., for all $(k, z) \in X^* \times Z$, $Ah_0(k, z) > h_0(k, z)$). The existence of h_0 implies that the isotone operator A maps the interval $[h_0, w]$ (a complete lattice when endowed with the pointwise order) into itself, so that A must have a fixed point in this interval. Since under the assumption of capital income monotonicity, the fixed point h^* of A in H is unique it must be that:

$$\forall k \in [0, k_0] \text{ and } \forall z \in Z,\ h^*(k, z) > h_0(k, z) > k.$$

Given this property of h^*, we show that there exist a fixed point of $T^*_{h^*}$ that is distinct from δ_0. The argument is the following: Consider any measure μ_0 with support in $[0, k_0]$ and distinct from δ_0 (we write $\mu_0 >_s \delta_0$).. Since h^* maps up strictly every point in $[0, k_0]$, μ_0 is mapped up strictly by the operator

$T^*_{h^*}$. By isotonicity of $T^*_{h^*}$ the sequence $\{T^{*n}_{h^*}\mu_0\}_{n\in\mathbb{N}}$ is increasing, and by order continuity along monotone sequences of $T^*_{h^*}$ it weakly converges to a fixed point of $T^*_{h^*}$. Clearly by construction this fixed point is strictly greater than δ_0. The rest of this section formalizes this argument.

Assumption 5: Assume that:

I. There exists a right neighborhood Δ of 0 such that for all $k \in \Delta$ and all $z \in Z$, $w(k, z) \geq k$.

II. The following inequality holds:

$$\lim_{k\to 0^+} u_1(w(k, z_{\min}) - k, r(k, z_{\max})k) < \lim_{k\to 0^+} u_2(w(k, z_{\min}) - k, r(k, z_{\max})k)r(k, z_{\min}).$$

Note that for log separable utility, 5.II is equivalent to:

$$\lim_{k\to 0^+} (w(k, z_{\min})/k) > 2,$$

and with the traditional Cobb-Douglass production function with multiplicative shocks, it is trivially satisfied and so is 5.I. For a polynomial utility of the form $u(c_1, c_2) = (c_1)^{\eta_1}(c_2)^{\eta_2}$ the condition is equivalent to:

$$\lim_{k\to 0^+} (w(k, z_{\min})/k) > [1 + \frac{\eta_1 r(k, z_{\max})}{\eta_2 r(k, z_{\min})}],$$

also trivially satisfied with Cobb-Douglass production and multiplicative shocks.

We can now state a key proposition that extends the uniqueness result in Coleman [12] and Morand and Reffett [33] obtained for infinite horizon economies to the present class of OLG models under assumption 5.

Proposition 26 *Under Assumption 5, the set of SME corresponding to the unique MEDP h^* is a non-empty complete lattice. The maximal SME is $\wedge\{T^{*n}_{h^*}\delta_{k_{\max}}\}_{n\in\mathbb{N}}$, and there exists $k_0 \in X$ such that the minimal SME is $\vee\{T^{*n}_{h^*}\delta_{k'}\}_{n\in\mathbb{N}}$ for any $0 < k' \leq k_0$.*

Proof: The proof is in two parts. Part 1 establishes the existence of h_0 that is mapped up strictly by the operator A, and Part 2 shows the existence of a probability measure μ_0 that is mapped up $T^*_{h^*}$, where h^* is the unique MEDP.

Part 1. By continuity of all functions in k, the inequality in Assumption 5 must be satisfied in a right neighborhood of 0. That is, there exists of $\Theta =]0, k_0] \subset \Delta$ such that, $\forall k \in \Theta$:

$$u_1(w(k, z_{\min}) - k, r(k, z_{\max})k) < u_2(w(k, z_{\min}) - k, r(k, z_{\max})k)r(k, z_{\min}).$$

Consequently, $\forall k \in \Theta =]0, k_0]$:

$$\begin{aligned}
& \int_Z u_1(w(k,z) - k, r(k,z')k) G(dz') \\
\leq & \\
& u_1(w(k, z_{\min}) - k, r(k, z_{\max})k) \\
< & \\
& u_2(w(k, z_{\min}) - k, r(k, z_{\max})k) r(k, z_{\min}) \\
\leq & \\
& \int_Z u_2(w(k,z) - k, r(k,z')k) r(k,z') G(dz').
\end{aligned}$$

Next, consider the function $h_0 : X \times Z \to X$ defined as:

$$h_0(k,z) = \begin{array}{c} 0 \text{ if } k = 0,\ z \in Z \\ k \text{ if } 0 < k \leq k_0,\ z \in Z \\ k_0 \text{ if } k \geq k_0,\ z \in Z \end{array}.$$

We prove now that $Ah_0 > h_0$. First, consider $0 < k \leq k_0$, $z \in Z$, and suppose that $Ah_0(k,z) \leq h_0(k,z) = k$. Then:

$$\begin{aligned}
& \int_Z u_1(w(k,z) - k, r(k,z')k) G(dz') \\
< & \\
& \int_Z u_2(w(k,z) - k, r(k,z')k) r(k,z') G(dz') \\
\leq & \\
& \int_Z u_2(w(k,z) - Ah_0(k,z), r(k,z') Ah_0(k,z)) r(Ah_0(k,z), z') G(dz'),
\end{aligned}$$

where the first inequality stems from the result just above, and the second from $u_{22} \leq 0$, $u_{12} \geq 0$ and r decreasing in its first argument. By definition of Ah_0, this last expression is equal to:

$$\int_Z u_1(w(k,z) - Ah_0(k,z), r(k,z') Ah_0(k,z)) G(dz').$$

Thus, we have $Ah_0(k,z) \leq k$ and:

$$\begin{aligned}
& \int_Z u_1(w(k,z) - k, r(k,z')k) G(dz') \\
< & \\
& \int_Z u_1(w(k,z) - Ah_0(k,z), r(k,z') Ah_0(k,z)) G(dz').
\end{aligned}$$

which contradicts the hypothesis that $u_{11} \leq 0$ and $u_{12} \geq 0$. It must therefore be that for all $k \in]0, k_0]$ and all $z \in Z$, $Ah_0(k,z) > h_0(k,z) = k$, that is A maps

h_0 strictly up at least in the interval $]0, k_0]$. Finally, for $k > k_0$, since Ah_0 is increasing in its first argument:

$$Ah_0(k, z) \geq Ah_0(k_0, z) > h_0(k_0, z) = k_0 = h_0(k, z).$$

We have therefore proven that A maps h_0 up (strictly). The order interval $[h_0, w] \subset H$ is a complete lattice when endowed with the pointwise order, and by isotonicity of A, there must exists a fixed point of A in that interval. Under capital income isotonicity, $h^* \in [h_0, w]$.

Part 2. Consider any probability measure in $(\Lambda(X, \mathcal{B}(X)), \geq_s)$ but with support in the compact interval $[0, k_0]$ and distinct from δ_0. We show next that $T^*_{h^*}\mu_0 \geq_s \mu_0$. Consider any $f : X \to \mathbb{R}_+$ measurable, increasing and bounded, we have:

$$\int [\int f(k')P_{h^*}(k, dk')]\mu_0(dk) = \int [\int f(h^*(k, z))\lambda(dz)]\mu_0(dk)$$
$$=$$
$$\int_{[0,k_0]} [\int_Z f(h^*(k, z))\lambda(dz)]\mu_0(dk) + \int_{[k_0,k_{\max}]} [\int f(h^*(k, z))\lambda(dz)]\mu_0(dk)$$
$$\geq$$
$$\int_{[0,k_0]} f(k)\mu_0(dk)$$

since $h^*(k, z) > k$ on $[0, k_0]$. Note that if f is strictly positive on $[0, k_0]$ then the last inequality is strict.

We have just demonstrated that $T^*_{h^*}\mu_0 \geq_s \mu_0$ and that $T^*_{h^*}\mu_0$ is distinct from μ_0, so we write $T^*_{h^*}\mu_0 >_s \mu_0$ ($>_s \delta_0$). By order continuity along any monotone sequence of $T^*_{h^*}$, necessarily the increasing sequence $\{T^{*n}_{h^*}\mu_0\}_{n\in\mathbb{N}}$ converges weakly to a fixed point of $T^*_{h^*}$ strictly greater than δ_0. In addition, it is easy to see that there cannot be any fixed point of $T^*_{h^*}$ with support in $[0, k_0]$ other than δ_0 so that the minimal non-trivial (i.e., distinct from δ_0) fixed point of $T^*_{h^*}$, which is by definition the minimal SME, can be constructed as the limit of the sequence $\{T^{*n}_{h^*}\mu_0\}_{n\in\mathbb{N}}$, where $\mu_0 = \delta_{k'}$ for any $0 < k' \leq k_0$. This completes the proof that the set of SME is the non-empty complete lattice of fixed points of $T^*_{h^*}$ minus δ_0, and that the maximal SME and minimal SME can be obtained as claimed.■

4.4 SME without capital income monotonicity

Recall that in the most general case (i.e., without the assumption of capital monotonicity) there exists a nonempty complete lattice of MEDP in H, as well as nonempty complete lattices of semicontinuous functions in H (Proposition 9) and that the minimal and maximal MEDP are $\mathcal{B}(S)$-measurable but not necessarily continuous (Propositions 12 and 15, and their corollaries). For any continuous MEDP h in H, the isotone operator T^*_h is order continuous along monotone and a result similar to that of the previous subsection clearly applies:

There exists a complete lattice of SME associated with h, and maximal and minimal SME can be constructed.

Continuity of h, however, is not necessary for T_h^* to be order continuous along recursive monotone T_h^*-sequences. In fact, we prove in a companion paper addressing isotone recursive methods in OLG models with Markov shocks that if the transition function Q characterizing the Markov shocks is increasing and satisfies Doeblin's condition (D), then the $\mathcal{B}(S)$-measurability of the isotone MEDP h is sufficient to establish that T_h^* is order continuous along recursive monotone T_h^*-sequences. While we refer the reader to our companion paper (Morand and Reffett[34]) for a detailed proof, we give an overview of the argument before stating our result. It is important to note that Doeblin's condition imposes very minimal restrictions on iid shocks.

As discussed earlier in the paper, the assumption of Markov shocks implies that we manipulate probability measures defined on the state space $S = X \times Z$, a significant difference from our analysis so far. Recall that a Markov transition function Q satisfies Doeblin's condition (D) is there exists $\gamma \in \Lambda(Z, B(Z))$ and $\varepsilon > 0$ such that:

$$\forall B \in \mathcal{B}(Z),\ \gamma(B) \leq \varepsilon \text{ implies that } \forall z \in Z,\ Q(z, B) \leq 1 - \varepsilon.$$

In Morand and Reffett, we show that if Q satisfies Doeblin's condition (D), then the transition function P_h corresponding to any $B(S)$-measurable MEDP h and defined by:

$$\forall A \times B \in \mathcal{B}(S),\ P_h(x, z; A, B) = \{ \begin{array}{c} Q(z, B) \text{ if } h(x, z) \in A \\ 0 \text{ otherwise.} \end{array}$$

also satisfies Doeblin's condition (D). Consequently, by Theorem 11.9 in Stokey & al., the n-average of any recursive T_h^*-sequence converges in the total variation norm, and therefore weakly converges, to a fixed point of the isotone T_h^*. Next, we show that the poset $(\Lambda(S, B(S)), \leq_s)$ is countable chain complete and that any monotone recursive T_h^*-sequence weakly converges. By uniqueness of the limit, the weak limit of such sequence is also the limit of the average n-sequence, and is a fixed point of T_h^*. This precisely proves that T_h^* is order continuous along recursive monotone T_h^*-sequences, and an application of Theorem 4 gives the following important result.

Proposition 27 *Under Assumptions 1, 2, 3, 3', 4, 5 and if shocks satisfy Doeblin's condition (D), for any $\mathcal{B}(S)$-measurable MEDP h in H, there exists a non-empty set of SME with maximal and minimal elements respectively given by $\gamma_{\max}(h) = \wedge\{T_h^{*n}\delta_{(k_{\max}, z_{\max})}\}_{n \in \mathbb{N}}$ and $\gamma_{\min}(h) = \vee\{T_h^{*n}\mu_0\}_{n \in \mathbb{N}}$, where $\mu_0 = \delta_{(k', z_{\min})}$ for any $0 < k' \leq k_0$, k_0 constructed from Assumption 5.*

Finally, for economies satisfying Assumption 4, by Proposition 12 and 15 in the previous section of the paper, there exist minimal and maximal MEDP $h_{\min}$ and $h_{\max}$ in H, and both are $\mathcal{B}(S)$-measurable. Necessarily, any other

MEDP h in H satisfies $h_{\min} \leq h \leq h_{\max}$. By the comparative statics result of Proposition 24 above,

$$T^*_{h_{\min}}\mu_0 \leq T^*_h \mu_0,$$

and recursively,

$$\gamma_{\min}(h_{\min}) = \vee\{T^{*n}_{h_{\min}}\mu_0\}_{n\in N} \leq \vee\{T^{*n}_h \mu_0\}_{n\in N} = \gamma_{\min}(h).$$

By a similar argument:

$$\gamma_{\max}(h_{\max}) = \wedge\{T^{*n}_{h_{\max}}\delta_{(k_{\max},z_{\max})}\}_{n\in\mathbb{N}} \geq \wedge\{T^{*n}_h \delta_{(k_{\max},z_{\max})}\}_{n\in\mathbb{N}} = \gamma_{\max}(h),$$

and this proves that $\gamma_{\max}(h_{\max})$ and $\gamma_{\min}(h_{\min})$ are the greatest and least SME, respectively. We state this very general result in the last proposition of this paper.

Proposition 28 *Under Assumptions 4 and 5, and when shocks satisfy Doeblin's condition (D), the set of SME is nonempty and there exist a greatest and a least SME, respectively* $\gamma_{\max}(h_{\max}) = \wedge\{T^{*n}_{h_{\max}}\delta_{(k_{\max},z_{\max})}\}_{n\in\mathbb{N}}$ *and* $\gamma_{\min}(h_{\min}) = \vee\{T^{*n}_{h_{\min}}\mu_0\}_{n\in N}$ *where* $\mu_0 = \delta_{(k',z_{\min})}$ *for any* $0 < k' \leq k_0$, k_0 *constructed from Assumption 5.*

5 Applications

In the last section of the paper we present some applications of our results to models that have been studied extensively in the literature. The first example emphasizes how the results can be applied in settings where the reduced form production function can represent an economy with an equilibrium distortion generated by trading frictions such as valued fiat money. In the second example we show how our results can be specialized to cover the cases of social security that have been studied in the literature. Finally, we also show that some of our results concerning the existence and construction of MEDP can be extended to cases where the state space is not necessarily compact (i.e., the case of endogenous growth).

5.1 Example 1. Fiat money (Stokey & al. 1989).

This first example is the overlapping generation model with fiat money of Stokey & al.[40] (See Ch. 17) presented here to illustrate the construction of an isotone operator whose fixed points precisely satisfy the first-order condition associated with the consumer's maximization problem. Following Stockey & al., an equilibrium is a function $n : X \to \mathbb{R}_+$satisfying the following condition:

$$n(x)H'(n(x)) = \int x'n(x')V'(n(x'))Q(x, dx').$$

Denoting $G(s) = sH'(s)$ and $m(x) = xn(x)$, we rewrite this equation as:

$$G(m(x)/x) = \int m(x)V'(m(x))Q(x, dx'), \tag{A1}$$

so that a Markovian equilibrium policy is a function $m(x)$ satisfying (A1). Consider the complete lattice $(E, \leq)$ of functions $m : X = [a, b] \to \mathbb{R}_+$ such that m is increasing and $0 \leq m \leq bL$.[15] For each $m \in E$ and $x \in X$, consider then the following equation in y:

$$G(y/x) = \int m(x')V'(m(x'))Q(x, dx'),$$

Under Assumption 17.1 the solution, which we denote $y^* = Am(x)$ is unique, and this solution is increasing in x under the assumption that Q is a weakly continuous increasing transition function. Furthermore, the mapping $A : E \to E$ is increasing in m under the additional restriction on the preferences that $-yV'(y)/v'(y) \leq 1$. For any increasing (decreasing) sequence of functions $\{m_n(x)\}_{n \in N}$ converging pointwise to $m(x)$, by the Monotone Convergence Theorem:

$$\lim_{n\to\infty} \int m_n(x')V'(m_n(x'))Q(x, dx') = \int m(x')V'(m(x'))Q(x, dx'),$$

which implies that for an increasing sequence:

$$\sup_{n \in N} \int m_n(x')V'(m_n(x'))Q(x, dx') = \int m(x')V'(m(x'))Q(x, dx'),$$

and for a decreasing sequence:

$$\inf_{n \in N} \int m_n(x')V'(m_n(x'))Q(x, dx') = \int m(x')V'(m(x'))Q(x, dx'),$$

which establishes that A is order continuous.

As a consequence, there exists a complete lattice of functions satisfying $Am(x) = m(x)$ for all $x \in X$, i.e., there exists a complete lattice of Markovian equilibrium decision policy. Given that the top element of E is the (constant) function bL, the function $A^n bL$ converges pointwise to the maximal Markovian equilibrium policy in E which we denote $m_{\max}$. Since $A0 = 0$, we need to prove that 0 is not the only one fixed point, which we do by showing the existence of a strictly positive element $m_0 \in E$ that is mapped up by the operator A. As a result, $m_{\max}$ must necessarily be a strictly positive fixed point.

Under the assumption that $\lim_{s\to 0+} G(s) = \lim_{s\to 0+} G'(s) = 0$, since $V'(a) > 0$, there exists $\alpha_0 < \min(1, L)$ such that:

$$G(\alpha_0) < \alpha_0[aV'(a)].$$

[15] Clearly $(E, \leq)$ is a complete lattice with $\vee$ and $\wedge$ being the pointwise sup and inf respectively.

Consider $m_0(x) = \alpha_0 x$ (recall that $\alpha_0 < L$ hence $m_0 \leq bL$). For all $x \in [a, b]$, we have:

$$G(m_0(x)/x) = G(\alpha_0) < \alpha_0[aV'(a)] \leq \alpha_0[aV'(\alpha_0 a)] = \int \alpha_0 aV'(\alpha_0 a)Q(x, dx').$$

where the last inequality above rests on the assumption of concavity of V. Since $x \geq a$, $m_0(x) = \alpha_0 x \geq \alpha_0 a$ and:

$$\int \alpha_0 aV'(\alpha_0 a)Q(x, dx') \leq \int m_0(x)V'(m_0(x))Q(x, dx'),$$

and therefore

$$G(m_0(x)/x) < \int m_0(x)V'(m_0(x))Q(x, dx') \text{ for all } x \text{ in } [a, b],$$

which implies that $Am_0(x) > m_0(x)$ for all $x \in [a, b]$.

5.2 Example 2. Social security (Hauenschild 2002).

Our second example shows that the results of Hauenschild [27] that incorporates a social security system in the overlapping generation model of Wang [44] can easily be derived from our setup. This example thus illustrates the power of monotone methods to generate (weak) comparative statics results. Recall that in Hauenschild [27], a Markovian equilibrium investment policy is a function h satisfying the following condition:

$$\begin{aligned} & \int_Z u_1((1-\tau)w(k,z) - h(k,z), r(h(k,z),z')h(k,z) + \tau w(h(k,z),z'))G(dz') \\ = & \\ & \int_Z u_2((1-\tau)w(k,z) - h(k,z), r(h(k,z),z')h(k,z) + \tau w(h(k,z),z')) \\ & r(h(k,z),z')G(dz'). \end{aligned} \tag{B1}$$

Consider the following equation in y :

$$\begin{aligned} & \int_Z u_1((1-\tau)w(k,z) - y, r(h(k,z),z')y + \tau w(y,z'))G(dz') \\ = & \\ & \int_Z u_2((1-\tau)w(k,z) - y, r(h(k,z),z')y + \tau w(y,z'))r(y,z')G(dz'). \end{aligned}$$

For any $(k, z) \in X \times Z$ and $h \in E$, denote $y^* = Ah(k, z)$ the unique solution to this equation. It is easy to see that, in addition to being an order continuous isotone operator mapping E into itself, A is also isotone in $-\tau$. Consequently, an increase in τ generates a decrease (in the pointwise order) of the extremal Markovian equilibrium investment policies $h_{\tau,\max}$ and $h_{\tau,\min}$.

Next, recall that any equilibrium investment policy h induces a Markov process for the capital stock defined by the following transition function P_h:

$$\text{For all } A \in \mathcal{B}(X),\ P_h(k, A) = \Pr\{h(k, z) \in A\} = \lambda(\{z \in Z,\ h(k, z) \in A\}).$$

Consider two Markovian equilibrium policies $h' \geq h$ and their respective transition functions $P_{h'}$ and P_h. For any $k \in X$ and any function $f : X \to \mathbb{R}_+$ bounded, measurable and increasing:

$$\int f(k')P_{h'}(k, dk') = \int f(h'(k, z))\lambda(dz) \geq \int f(h(k, z))\lambda(dz) = \int f(k')P_h(k, dk').$$

Thus, for any $\mu \in \Lambda(X, B(X))$:

$$\begin{array}{rcl} \int f(k')T^*_{h'}\mu(dk') & = & \int [\int f(k')P_{h'}(k, dk')]\mu(dk) \\ & \geq & \\ \int [\int f(k')P_h(k, dk')]\mu(dk) & = & \int f(k')T^*_h\mu(dk'), \end{array}$$

which establishes that $T^*_{h'}\mu \geq T^*_h\mu$. Thus the natural ordering on the set of taxes τ induces an ordering by stochastic dominance of the corresponding extremal stationary Markov equilibria in the following way:

$$\tau' \geq \tau \text{ implies } h_{\tau,\max} \geq h_{\tau',\max} \text{ implies } \lim_{n\to\infty} T^{*n}_{\tau}\delta_{k\max} \geq_s \lim_{n\to\infty} T^{*n}_{\tau'}\delta_{k\max}.$$

5.3 Example 3. Endogenous growth (Romer 1996).

This example and the next show that our results apply to a large class of models with unbounded growth and nonconvex technologies. Consider the production technology $f(k, K) = zk^{\alpha}K^{\beta}$ with $0 < \alpha <$ and $0 < \alpha + \beta < 1$. Notice that $kf_1(k, K)$ is increasing in k, hence there is unique Markovian equilibrium investment policy h satisfying the following condition (derived from the first order condition in which the equilibrium restriction $k = K$ has been imposed):

$$\begin{array}{l} \int_Z u_1((1-\alpha)zk^{\alpha+\beta} - h(k, z), \alpha z' h^{\alpha-1+\beta}(k, z)h(k, z))G(dz') \\ = \\ \int_Z u_2((1-\alpha)zk^{\alpha+\beta} - h(k, z), \alpha z' h^{\alpha-1+\beta}(k, z)h(k, z))\alpha z' h^{\alpha-1+\beta}(k, z)G(dz'). \end{array} \tag{C1}$$

If we consider the equation in y :

$$\begin{array}{l} \int_Z u_1((1-\alpha)zk^{\alpha+\beta} - y, \alpha z' h^{\alpha-1+\beta}(k, z)y)G(dz') \\ = \\ \int_Z u_2((1-\alpha)zk^{\alpha+\beta} - y, \alpha z' h^{\alpha-1+\beta}(k, z)y)\alpha z' y^{\alpha-1+\beta}G(dz'), \end{array}$$

and define the operator A as in the paper. Following our analysis, the unique Markovian equilibrium investment policy can be obtained as the pointwise limit of the sequence of functions $\{A^n h\}_{n=1}^{\infty}$ where $h(k,z) = (1-\tau)zk^{\alpha+\beta}$. In the case $\alpha + \beta = 1$ the first order condition is:

$$\begin{aligned} &\int_Z u_1((1-\alpha)zk - y, \alpha z' y)G(dz') \\ = \\ &\int_Z u_2((1-\alpha)zk - y, \alpha z' y)\alpha z' G(dz'), \end{aligned} \tag{C2}$$

growth is unbounded (i.e., $X = \mathbb{R}_+$), and the unique Markovian equilibrium investment policy is obtained directly from solving (C2).

6 Appendix A. Elements of Lattice theory

Recall that a *partial order* $\leq$ on a set X is a reflexive, transitive, and antisymmetric relation. An *upper (resp. lower) bound* of $A \subset X$ is an element u (resp. v) such that $\forall x \in A$, $u \geq x$ (resp. $v \leq x$). A chain C is a subset of X that can be linearly ordered, i.e. any two pairs of elements in the set $p, p' \in P$ are ordered. If there is a point x^u (respectively, x^l) such that x^u is the least element in the subset of upper bounds of $B \subset X$ (respectively, the greatest element in the subset of lower bounds of $B \subset X$), we say x^u (respectively, x^i) is the *supremum* (respectively, *infimum)* of B. Clearly if they exist, both the supremum (or, sup) and infimum (or, inf) of any subset must be unique. We say X is a *lattice* if for any two elements x and x' in X, X is closed under the operation of infimum in X , denoted $x \wedge x'$, and supremum in X, denoted $x \vee x'$.The former is referred to as "the meet", while the latter is referred to as "the join" of the two points, $x, x' \in X$. A subset B of X is a *sublattice* of X if it contains the sup and the inf (with respect to X) of any pair of points in B. A lattice is *complete* if any subset B of X has a least upper bound $\vee B$ and a greatest lower bound $\wedge B$ in B. If every chain $C \subset X$ is complete, then X is referred to as a *chain complete poset (or equivalent, a complete partially ordered set* or *CPO).* A set C is *countable* if it is either finite or there is a bijection from the natural numbers onto C. If every chain $C \subset X$ is countable and complete, then X is referred to as a *countable chain complete poset*.

To show that a partially ordered set is a complete lattice sometimes requires much less work that the definition of completeness would have us believe.

Theorem 29 *(Davey and Priestley [16]). A non-empty poset $(P, \leq)$ is a complete lattice if and only if P has a top (resp. bottom) element and for any $P' \subset P$, $\wedge_P P'$ (resp. $\vee_P P'$) exists (in P).*

7 Appendix B. Proof of existence of a strictly positive MEDP

Lemma 30 *Under Assumption 4, for all $k \in X^*$, there exists a right neighborhood $\Omega =]0, \overline{k}]$ with $0 < \overline{k} \leq w(k, z_{\min})$ and $M > 0$ such that, for all $x \in \Omega$,*

$$u_2(w(k, z_{\min}) - x, r(x, z_{\max})x) > M.$$

Proof. If $\lim_{x \to 0^+} r(x, z_{\max})x = 0$ then for all $k \in X^*$:

$$\lim_{x \to 0 \text{ and } x \in]0, w(k, z_{\min})[} u_2(w(k, z_{\min}) - x, r(x, z_{\max})x) = u_2(w(k, z_{\min}), \lim_{x \to 0^+} r(x, z_{\max})x) = \infty.$$

The expression $u_2(w(k, z_{\min}) - x, r(x, z_{\max})x)$ can therefore made arbitrarily large in a right neighborhood of 0, and the existence of Ω thus follows.■

Lemma 31 *For all $k \in X^*$ and $z \in Z$, there exists $h_0(k, z) \in]0, w(k, z)[$ such that:*

$$\begin{aligned} & \int_Z u_1(w(k, z) - h_0(k, z), r(h_0(k, z), z')h_0(k, z))G(dz') \qquad \textit{(E0)} \\ < & \\ & \int_Z u_2(w(k, z) - h_0(k, z), r(h_0(k, z), z')h_0(k, z))r(h_0(k, z), z')G(dz'). \end{aligned}$$

In addition, h_0 can be chosen to be increasing in k for each z, constant in z (and therefore continuous and increasing in z) for each k.

Proof. Fix $k \in X^*$. For all $z \in Z$:

$$\begin{aligned} & \lim_{x \to 0^+} \int_Z u_1(w(k, z) - x, r(x, z')x)G(dz') \\ = & \\ & \int u_1(w(k, z), 0)G(dz') \\ \leq & \\ & u_1(w(k, z_{\min}), 0). \end{aligned}$$

Thus there exists a right neighborhood of 0, denoted $\Psi =]0, \overline{x}]$, such that, for all $x \in \Psi$:

$$\begin{aligned} & \int_Z u_1(w(k, z) - x, r(x, z')x)G(dz') \\ < & \\ & .5u_1(w(k, z_{\min}), 0). \end{aligned}$$

Next, for $x \in \Omega$:

$$\int_Z u_2(w(k,z)-x, r(x,z')x) r(x,z') G(dz')$$
$$\geq$$
$$\int_Z u_2(w(k,z_{\min})-x, r(x,z_{\max})x) r(x,z') G(dz')$$
$$\geq$$
$$\int_Z M r(x,z') G(dz'),$$

where the first inequality stems from $u_{12} \geq 0$ and u_2 decreasing, and the second from the Lemma above. This last expression can be made arbitrarily large, independently of z, by choosing x in Ω sufficiently close to 0. That is, it is always possible to choose x^* sufficiently small in $\Omega \cap \Psi$ so that:

$$\int_Z M r(x^*,z') F(dz') \geq .5 u_1(w(k,z_{\min}),0). \tag{E1}$$

Pick such an x^* and set $\delta_0(k,z) = x^*$ for all $z \in Z$. By construction, any $x \in]0, \delta_0(k,z)]$ satisfies:

$$\int_Z u_1(w(k,z)-x, r(x,z')x) G(dz')$$
$$<$$
$$.5 u_1(w(k,z_{\min}),0)$$
$$\leq$$
$$\int_Z M r(x,z') G(dz')$$
$$\leq$$
$$\int_Z u_2(w(k,z)-x, r(x,z')x) r(x,z') G(dz').$$

That is, by construction, we have, for all $x \in]0, \delta_0(k,z)]$:

$$\int_Z u_1(w(k,z)-x, r(x,z')x) G(dz') < \int_Z u_2(w(k,z)-x, r(x,z')x) r(x,z') G(dz'). \tag{E2}$$

We repeat the same operation for each k in X^*, thus constructing a function $\delta_0 : X \times Z \to X$, setting $\delta_0(0,z) = 0$. By construction, for each $k \in X$, $\delta_0(k,z)$ is constant in z, and therefore increasing in z. In addition, any function smaller (pointwise) than $\delta_0(k,z)$ also satisfies (E2). In particular, the function $p_0 : X \times Z \to X$ defined as:

$$p_0(k,z) = \min_{k' \geq k} \{\delta_0(k',z)\}.$$

satisfies (E2), is increasing in k for all z, and constant in z for all k (and thus continuous in z for all k). Finally, the function h_0 defined as follows:

$$h_0(k,z) = \left\{ \begin{array}{c} \sup_{0<k'<k} p_0(k',z) \text{ for } (k,z) \in X^* \times Z \\ 0 \text{ for } k=0,\ z \in Z \end{array} \right\}$$

is smaller than p_0 (and therefore than δ_0, hence it satisfies (E2)), increasing in k for all z, constant in z for all k, and lower semicontinuous in k for all z.■

Proposition 32 *$\forall (k,z) \in X^* \times Z$, $Ah_0(k,z) > h_0(k,z) > 0$.*

Proof. $h_0(k,z) > 0$ by construction. Suppose that there exists $k \in X^*$ and $z \in Z$ such that $Ah_0(k,z) \leq h_0(k,z)$. Then:

$$\begin{aligned} & \int_Z u_1(w(k,z) - h_0(k,z), r(h_0(k,z),z')h_0(k,z))G(dz') \\ < & \\ & \int_Z u_2(w(k,z) - h_0(k,z), r(h_0(k,z),z')h_0(k,z))r(h_0(k,z),z')G(dz') \\ \leq & \\ & \int_Z u_2(w(k,z) - Ah_0(k,z), r(h_0(k,z),z')Ah_0(k,z))r(Ah_0(k,z),z')G(dz'), \end{aligned}$$

where the first inequality stems from $(E2)$ and the second from $u_{22} \leq 0$, $u_{12} \geq 0$ and r decreasing in its first argument. By definition of Ah_0, this last expression is equal to:

$$\int_Z u_1(w(k,z) - Ah_0(k,z), r(h_0(k,z),z')Ah_0(k,z))G(dz').$$

Summarizing, we have:

$$\begin{aligned} & \int_Z u_1(w(k,z) - h_0(k,z), r(h_0(k,z),z')h_0(k,z))G(dz') \\ < & \\ & \int_Z u_1(w(k,z) - Ah_0(k,z), r(h_0(k,z),z')Ah_0(k,z))G(dz'). \end{aligned}$$

which is contradicted by the hypothesis that $u_{11} \leq 0$ and $u_{12} \geq 0$. Thus, necessarily, $Ah_0(k,z) > h_0(k,z)$ and A maps h_0 strictly up.■

Proposition 33 *$\forall (k,z) \in X^* \times Z$, $h_0(k,z) > x > 0$ implies that $Ax > x$.*

Proof. By (E2), for all $0 < x < \delta_0(k,z)$ $(\leq h_0(k,z))$:

$$\begin{aligned} & \int_Z u_1(w(k,z) - x, r(x,z')x)G(dz') \\ < & \\ & \int_Z u_2(w(k,z) - x, r(x,z')x)r(x,z')G(dz'), \end{aligned}$$

and the same argument to that in the previous proposition applies directly, and shows that $Ax > x$.■

8 REFERENCES

References

[1] Aliprantis, C. and K. Border. 1999. *Infinite Dimensional Analysis: A Hitchhiker's Guide*. Springer Verlag Press.

[2] Abreu, D., D. Pearce, and E. Stachetti. 1990. Toward a theory of repeated games with discounting, *Econometrica,* 58, 1041-1063.

[3] Balasko, Y. and K. Shell. 1980. The overlapping generations model, I: the case of pure exchange without money. *Journal of Economic Theory,* 23, 281-306.

[4] Balasko, Y. and K. Shell. 1980. The overlapping generations model, II: the case of pure exchange with money. *Journal of Economic Theory,* 24, 112-142.

[5] Barbie, M., M Hagedorn, and A. Kaul. 2000. Dynamic efficiency and Pareto optimality in a stochastic OLG model with production and social security. MS. U. of Bonn.

[6] Barro, Robert J., 1990. Government Spending in a Simple Model of Endogenous Growth, *Journal of Political Economy* 98, 103-125.

[7] Birkhoff, G. 1967. *Lattice Theory.* AMS Press.

[8] Boldrin, M. and A. Rustichini. 1994. Growth and indeterminacy in dynamic models with externalities, *Econometrica,* 62, 323-342.

[9] Brock, W. and L. Mirman. 1972. Optimal growth and uncertainty: The discounted case, *Journal of Economic Theory*, 4, 479-513.

[10] Chattopadhyay, S. and P. Gottardi. 1999. Stochastic OLG models, market structure, and optimality. *Journal of Economic Theory,* 89, 21-67.

[11] Coleman, W.J. 1991. Equilibrium in a production economy with an income tax, *Econometrica*, 59, 1091-1104.

[12] Coleman, W. J., II. 2000. Uniqueness of an equilibrium in infinite-horizon economies subject to taxes and externalities, *Journal of Economic Theory* 95, 71-78.

[13] Datta, M., Mirman, L., and K. Reffett. 2002. Existence and uniqueness of equilibrium in distorted dynamic economies with capital and labor, *Journal of Economic Theory*, 103, 377-410.

[14] Datta, M., Mirman, L., Morand, O., and K. Reffett. 2002. Monotone methods for Markovian equilibrium in dynamic economies, *Annals of Operations Research*, 114, 117-144.

[15] Datta, M., Mirman, L., Morand, O., and K. Reffett. 2004. Markovian equilibrium in infinite horizon economies with incomplete markets and public policy, forthcoming, *Journal of Mathematical Economics*.

[16] Davey, B. and H. Priestley. 1990. *Introduction to Lattices and Order*. Cambridge Press.

[17] Demange, G. and G. Laroque. 1999. Social security and demographic shocks. *Econometrica,* 67, 527-542.

[18] Demange, G. and G. Laroque. 2000. Social security, optimality, and equilibria in a stochastic OLG economy. *Journal of Public Economic Theory,* 2, 1-23.

[19] Geanokoplos, J. and H. Polemarchakis. 1991. Overlapping generations models. in *Handbook of Mathematical Economics, vol 4.* ed: Hildenbrand, Werner, and Sonnenschein. 1899-1960.

[20] Dechert, W. and K. Yamamoto. Asset valuation and production efficiency in an OLG models with production shocks. *Review of Economic Studies,* 59, 389-405.

[21] Diamond, P. 1965. National debt in a neoclassical growth model, *American Economic Review*, 55, 1126-1150.

[22] Dugundji, J. and A. Granas. 2003. *Fixed Point Theory*. Springer-Verlag, New York Inc.

[23] Duffie, D., Geanakoplos, J.D., Mas-Colell, A. McLennann, A. 1994. Stationary markov equilibria, *Econometrica* 62, 745-782.

[24] Futia, Carl A. 1982. Invariant distributions and the limiting behavior of Markovian economic models, *Econometrica* 50 (2), 377-408.

[25] Galor, O. and Ryder, H. 1992. Existence, uniqueness and stability of equilibrium in an overlapping-generation model with productive capital, *Journal of Economic Theory*, 49, 360-375.

[26] Greenwood, J., and G. Huffman.1995. On the existence of nonoptimal equilibria in dynamic stochastic economies, *Journal of Economic Theory*, 65, 611-623.

[27] Hausenchild, N. 2002. Capital accumulation in a stochastic overlapping generation model with social security, *Journal of Economic Theory*, 106, 201-216.

[28] Hopenhayn, H. and E. Prescott. 1992. Stochastic monotonicity and stationary distributions for dynamic economies, *Econometrica*, 60(6), 1387-1406.

[29] Kantorovitch, L. 1939. The method of successive approximation for functional equations, *Acta Math*. 71, 63-97.

[30] Kubler, F. and H. Polemarchakis. 2004. Stationary Markov Equilibria for overlapping generations, *Economic Theory*, 1-22.

[31] Miao, J. and M. Santos. 2005. Existence and computation of Markov equilibria for dynamic non-optimal economies. MS. Arizona State University.

[32] Mirman, L., O. Morand, and K. Reffett. 2004. A Qualitative approach to Markovian equilibrium in infinite horizon economies with capital. MS.

[33] Morand, O. and K. Reffett. 2003. Existence and uniqueness of equilibrium in nonoptimal unbounded infinite horizon economies, *Journal of Monetary Economics*, 50, 1351-1373.

[34] Morand, O. and K. Reffett. 2005. Stationary Markov Equilibria in Overlapping generations models with Stochastic Production. MS. Arizona State University.

[35] Okuno, M. and I. Zilcha. 1980. On the efficiency of competitive equilibrium in infinite horizon monetary economies. *Review of Economic Studies,* 47, 797-807.

[36] Reffett, K. 2005. Mixed Monotone Recursive Methods. MS. Arizona State University.

[37] Romer, P. 1986. Increasing Returns and long-run growth, *Journal of Political Economy*, 94, 1002-1037.

[38] Samuelson, P.A. 1958. An exact consumption-loans model of interest with or without the social contrivance of money, *Journal of Political Economy*, 66. 467-482

[39] Stachurski, J. 2003. Stochastic growth with increasing returns: stability and path dependence, *Studies in Nonlinear Dynamics and Econometrics,* 7 (2).

[40] Stokey, N., R.E. Lucas and E. Prescott. 1989. *Recursive methods in economics dynamics*. Harvard Press.

[41] Veinott 1992, *Lattice Programming: Qualitative Optimization and Equilibria.* MS. Stanford University.

[42] Tarski, A. 1955. A lattice-theoretical fixpoint theorem and its applications, *Pacific Journal of Mathematics*, 5, 285-309.

[43] Topkis, D. 1998. *Supermodularity and Complementarity.* Princeton Press.

[44] Wang, Y. 1993. Stationary equilibria in an overlapping generations economy with stochastic production, *Journal of Economic Theory*, 61, 423-435.

[45] Wang, Y. 1994. Stationary Markov equilibrium in an OLG model with correlated shocks, *International Economic Review*, 35(3), 731-744.

[46] Zilcha, I. 1990. Dynamic efficiency in OLG models stochastic production. *Journal of Economic Theory*, 52, 364-379.