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Infinitesimal Rotations

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I. SYNOPSIS

In earlier readings,
http://digitalcommons.uconn.edu/chem_educ/22,
finite rotations and their non-commutative algebra were discussed. In this piece, the infinitesimal rotations used to describe the kinematics of rigid bodies is introduced.

II. INTRODUCTION

When one considers infinitesimal rotations, one is beginning the discussion of the kinematics of rotation usually for spatially extended bodies.

Consider the rotation about the z-axis, and consider that we do a finite rotation first. Then we would have

$$R_z = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.1}$$

For a $\theta = 90$ degree rotation, this would yield

$$R_z = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which if it operated on the row vector

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \tag{2.2}$$

would yield

$$\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \leftarrow R_z \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \tag{2.3}$$

What does this mean? When looking at the R_z rotation, in the top of Figure 1, we see that in the new coordinate system, the coordinates of the point have changed as shown in the two relevant equations (above), i.e., vector 2.2 $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ transforms into vector given in Equation 2.3 $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ Remember, to see what is going on you have to position your eye along the rotation axis (the positive part), and look into the origin.

We now repeat the procedure about the x-axis, *the new x-axis*, i.e.,

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \tag{2.4}$$

as can be seen in the second part of Figure 1. That leaves us only with the y-rotation,

$$R_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$$

It takes some head turning to see these coordinates in Figure 1.

III. NON-COMMUTATIVITY TO COMMUTATIVITY

We know that finite rotations are not commutative, i.e., that if we did a rotation about the z axis and then about the x axis, the result would be different from doing the x axis rotation first, then the z axis rotation. This non-commutativity is *not true* when we deal with infinitesimal rotations, amazing as that seems. To begin the discussion we first specialize to infinitesimal rotations.

Let's assume that θ is an infinitesimal, i.e., almost zero. Then since $\cos 0 = 1$ and, to first order, $\sin\theta \rightarrow \theta$ for θ very, very small. We then transform Equation 2.1 into its infinitesimal form. Let's write this out in components:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & d\theta & 0 \\ -d\theta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

which becomes

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & d\theta & 0 \\ -d\theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\} \otimes \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

i.e.,

$$\begin{aligned} x' &= x + y d\theta \\ y' &= -x d\theta + y \\ z &= z \end{aligned}$$

Typeset by REVTEX

This means,

$$\frac{x' - x}{dt} = y\frac{d\theta}{dt}$$

$$\frac{y' - y}{dt} = -x\frac{d\theta}{dt}$$

Now

$$\frac{d\theta}{dt} = \omega_z$$

an angular velocity of rotation.

$$\frac{dx}{dt} = \omega_z y \tag{3.1}$$

$$\frac{dy}{dt} = -\omega_z x \tag{3.2}$$

$$\tag{3.3}$$

Equation 3.1 is the $\hat{\imath}$ component, while Equation 3.2 is the $\hat{\jmath}$ component. Of course, z is not changing, i.e., has no time derivative. If we attempt to write this is vector notation, we could write

$$\frac{d\vec{r}}{dt} = \vec{\omega}_z \times \vec{r} = \vec{v} \tag{3.4}$$

$$\vec{\omega}_z \times \vec{r} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 0 & \omega_z \\ x & y & z \end{vmatrix} = \hat{\jmath}\omega_z x - \hat{\imath}\omega_z y$$

$\vec{v}$ is the linear velocity of a point during the rotation. What we see here is that the signs are wrong. All right, not wrong, just inconsistent. If one wants to use the form of Equation 3.4 then one is forced to change the direction of rotation used in Equation 2.1. This corresponds to changing our point of view. Instead of rotating the coordinate system, we are rotating the particle.

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & -d\theta & 0 \\ d\theta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

which means

$$\frac{dx}{dt} = -\omega_z y \tag{3.5}$$

$$\frac{dy}{dt} = \omega_z x \tag{3.6}$$

$$\tag{3.7}$$

which is now consistent with Equation 3.4

IV. WHY ARE WE NOW COMMUTATIVE?

Why are these infinitesimal rotations commutative? Take as an example R_x and R_y Reversing the direction of rotation, reconsider the rotation about the x-axis, and consider that we do a finite rotation first. Then we would have

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \\ 0 & -\sin\psi & \cos\psi \end{pmatrix}$$

and now let's assume that ψ is an infinitesimal, i.e., almost zero. Then

$$R_x \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & d\psi \\ 0 & -d\psi & 1 \end{pmatrix}$$

In components:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & d\psi \\ 0 & -d\psi & 1 \end{pmatrix} \otimes \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

i.e.,

$$x' = x$$
$$y' = +z d\psi + y$$
$$z' = z - y d\psi$$

Continuing for the $\hat{k}$ component

$$\frac{z' - z}{dt} = -y\frac{d\psi}{dt}$$

and for the $\hat{\jmath}$ component

$$\frac{y' - y}{dt} = z\frac{d\psi}{dt}$$

Defining

$$\frac{d\psi}{dt} \equiv \omega_x$$

an angular velocity of rotation.

$$\hat{\jmath}\left(\frac{dy}{dt} = \omega_x z\right)$$
$$\hat{k}\left(\frac{dz}{dt} = -\omega_x y\right)$$

where, of course, x is not changing, i.e., has no time derivative. If we attempt to write this in vector notation, we could write

$$\frac{d\vec{r}}{dt} = \vec{\omega}_x \times \vec{r} = \vec{v}$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \omega_x & 0 & 0 \\ x & y & z \end{vmatrix} = -\hat{\jmath}\omega_x z + \hat{k}\omega_z y$$

$\vec{v}$ is, again, the linear velocity of a point during the rotation. However, the signs are wrong. We have run into the same difficulty here which we had before, which we address in the same manner as before, i.e., we need to use the reversed angle if we are going to keep the vector cross product notation.

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -d\psi \\ 0 & d\psi & 1 \end{pmatrix} \otimes \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Lastly, consider the rotation about the y-axis, and consider that we do a finite rotation first. Then we would have

$$R_y = \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix}$$

and now let's assume that ϕ is an infinitesimal, i.e., almost zero. Then

$$R_y \to \begin{pmatrix} 1 & 0 & d\phi \\ 0 & 1 & 0 \\ -d\phi & 0 & 1 \end{pmatrix}$$

In components:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & d\phi \\ 0 & 1 & 0 \\ -d\phi & 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

i.e.,

$$\begin{aligned} y' &= y \\ x' &= +zd\phi + x \\ z' &= z - xd\phi \end{aligned}$$

This means,

$$\frac{z' - z}{dt} = -x\frac{d\phi}{dt}$$

$$\frac{x' - x}{dt} = z\frac{d\phi}{dt}$$

At last, we have

$$\frac{d\phi}{dt} = \omega_y$$

an angular velocity of rotation.

$$\begin{aligned} \hat{i}\left(\frac{dx}{dt} = \omega_y z\right) \\ \hat{k}\left(\frac{dz}{dt} = -\omega_y x\right) \end{aligned}$$

where, of course, y is not changing, i.e., has no time derivative. If we attempt to write this is vector notation, we could write

$$\frac{d\vec{r}}{dt} = \vec{\omega}_y \times \vec{r} = \vec{v}$$

$$vec\omega_y \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & \omega_y & 0 \\ x & y & z \end{vmatrix}$$

Notice, that this last one is in consistent notation, compared to the others.

For a composite rotation, about a generalized axis, one would have

$$\tilde{\Omega} = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ +\omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix}$$

This is known as an antisymmetric matrix.

We also attach a "vector" to the rotation, a "pseudovector", which we will call (as before) $\tilde{\omega}$.

$$\tilde{\omega} = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

Using this composite rotation matrix, we have

$$\vec{v} = \tilde{\omega} \otimes \vec{r}$$

$$\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_x & \omega_y & \omega_z \\ x & y & z \end{vmatrix}$$

i.e.,

$$\vec{v} = \hat{i}\left(\omega_y z - \omega_z y\right) + \hat{j}\left(\omega_z x - \omega_x z\right) + \hat{k}\left(\omega_x y - \omega_y x\right)$$

so the kinetic energy would be

$$\frac{1}{2}\mu\vec{v}\cdot\vec{v} = \frac{1}{2}\mu\vec{v}\cdot(\vec{\omega}\otimes\vec{r}) = \frac{1}{2}\mu\left(\tilde{\omega}\otimes\vec{r}\right)\cdot\left(\tilde{\omega}\otimes\vec{r}\right)$$

which would be

$$\frac{1}{2}\mu\left(\omega_z^2y^2+\omega_y^2z^2-2\omega_z\omega_yyz+\omega_z^2x^2+\omega_x^2z^2-2\omega_z\omega_xxz+\omega_y^2x^2+\omega_x^2y^2-2\omega_y\omega_xxy\right)$$

i.e., $\vec{v}\cdot\vec{v}$ would be

$$\vec{v}\cdot\vec{v}=(\omega_yz-\omega_zy)^2+(\omega_zx-\omega_xz)^2+(\omega_xy-\omega_yx)^2$$

We would then have, expanding,

$$\vec{v}\cdot\vec{v}=(\omega_y^2z^2+\omega_z^2y^2)-2(\omega_yz\omega_zy)+(\omega_z^2x^2+\omega_x^2z^2)-2(\omega_zx\omega_zz)+(\omega_x^2y^2+\omega_y^2x^2)-2(\omega_xy\omega_yx)$$

which can be rearranged to appear as

$$\vec{v}\cdot\vec{v}=\omega_y^2z^2+\omega_y^2x^2+\omega_z^2x^2+\omega_z^2y^2+\omega_x^2y^2+\omega_x^2z^2-2(\omega_xy\omega_yx)-2\omega_yz\omega_zy-2\omega_zx\omega_zz$$

to which we add *and subtract*

$$\omega_x^2x^2+\omega_y^2y^2+\omega_z^2z^2$$

to obtain

$$KE=\frac{1}{2}\mu\left(r^2(\omega_x^2+\omega_y^2+\omega_z^2)-(\omega_xx+\omega_yy+\omega_zz)^2\right)$$

i.e.,

$$KE=\frac{1}{2}\mu\left(r^2\vec{\omega}\cdot\vec{\omega}-(\vec{\omega}\cdot\vec{r})^2\right)$$

V. RIGID BODY KINEMATICS

Consider a rigid body rotating about an arbitrary (in space) axis. The angular velocity is

$$\vec{v}=\vec{\omega}\otimes\vec{r}$$

If the point has mass m, then the linear momentum of the point during the rotation is

$$m\vec{v}=m\vec{\omega}\otimes\vec{r}$$

The angular momentum of the point is

$$\vec{L}=\vec{r}\otimes\vec{p}$$

which is

$$\vec{L}=\vec{r}\otimes m\left(\vec{\omega}\otimes\vec{r}\right)$$

i.e.,

$$\vec{L}=m\vec{r}\otimes\Big((\omega_yz-\omega_zy)\,\hat{i}+((\omega_zx-\omega_xz)\,\hat{j}+((\omega_xy-\omega_yx)\,\hat{k}\Big)$$

which expands, in components, to be

$$\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ x & y & z\\ (\omega_yz-\omega_zy) & (\omega_zx-\omega_xz) & (\omega_xy-\omega_yx)\end{vmatrix}\tag{5.1}$$

which is

$$\begin{aligned}\frac{\vec{L}}{m}=&\,(+y((\omega_xy-\omega_yx)-z((\omega_zx-\omega_xz))\,\hat{i}+\\ &(+z((\omega_yz-\omega_zy)-x((\omega_xy-\omega_yx))\,\hat{j}+\\ &(+x((\omega_zx-\omega_xz)-y((\omega_yz-\omega_zy))\,\hat{k}\end{aligned}\tag{5.2}$$

which is

$$\begin{aligned}=&\,\hat{i}\left(\omega_xy^2-\omega_yxy-\omega_zxz+\omega_xz^2\right)+\\ &\hat{j}\left(\omega_yz^2-\omega_zyz-\omega_xxy+\omega_yx^2\right)+\\ &\hat{k}\left(\omega_zx^2-\omega_xzx-\omega_yzy-\omega_zy^2\right)\end{aligned}\tag{5.3}$$

Rearranging,

$$\begin{aligned}=&\,\hat{i}\left(\omega_x(y^2+z^2)-\omega_yxy-\omega_zxz\right)+\\ &\hat{j}\left(\omega_y(z^2+x^2)-\omega_zyz-\omega_xxy\right)+\\ &\hat{k}\left(\omega_z(x^2+y^2)-\omega_xzx-\omega_yzy\right)\end{aligned}\tag{5.4}$$

$$\begin{aligned}=&\,\hat{i}\left(\omega_x(x^2+y^2+z^2)-\omega_yxy-\omega_zxz-\omega_xx^2\right)+\\ &\hat{j}\left(\omega_y(x^2+y^2+z^2)-\omega_zyz-\omega_xxy-\omega_yy^2\right)+\\ &\hat{k}\left(\omega_z(x^2+y^2+z^2)-\omega_xzx-\omega_yzy-\omega_zz^2\right)\end{aligned}\tag{5.5}$$

which is

$$\frac{\vec{L}}{m}=r^2\vec{\omega}-(\vec{\omega}\cdot\vec{r})\vec{r}\tag{5.6}$$

from which we obtain

$$2KE=\frac{\vec{L}}{m}\cdot\omega=\left(r^2\vec{\omega}-(\vec{\omega}\cdot\vec{r})\vec{r}\right)\cdot\omega$$

The equation 5.6 can be rewritten as

$$\begin{aligned}\frac{\vec{L}}{m}=&\,r^2(\omega_x\hat{i}+\omega_y\hat{j}+\omega_z\hat{k})\\ &-(\omega_xx+\omega_yy+\omega_zz)(x\hat{i}+y\hat{j}+z\hat{k})\end{aligned}\tag{5.7}$$

i.e.,

$$\frac{\vec{L}}{m} = \omega_x \hat{i} r^2 + \omega_y \hat{j} r^2 + \omega_z \hat{k} r^2 - \omega_x x x \hat{i} - \omega_y y y \hat{j} - \omega_z z z \hat{k}$$
$$- \omega_x x y \hat{j} - \omega_y x z \hat{k} - \omega_x y x \hat{i} - \omega_y y z \hat{k} - \omega_x x z \hat{i} - \omega_y y z \hat{j} \tag{5.8}$$

which appears to be

$$\hat{i}(\omega_x (y^2 + z^2) - \omega_y xy - \omega_z xz) + \hat{j}(\omega_y (x^2 + z^2) - \omega_x xy - \omega_z yz) + \hat{k}(\omega_z (x^2 + y^2) - \omega_x xz - \omega_y yz)$$

It is tempting to define an object which allows us to write

$$\vec{L} = m \underbrace{I} \otimes \vec{\omega}$$

which looks like a matrix, smells like a matrix, but in truth is actually a tensor, the tensor of the moment of inertia. It looks something like:

$$\vec{L} = m \underbrace{\begin{pmatrix} r^2 - x^2 & -xy & -xz \\ -yx & r^2 - y^2 & -yz \\ -zx & -zy & r^2 - z^2 \end{pmatrix}} \otimes \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \tag{5.9}$$

The difference between a normal (for us) rotation matrix and this construct is that a rotation matrix carries coordinates into coordinates, i.e., the elements of the matrix have no units. Here, on the other hand, the angular momentum on the left hand side, has different units than the $\vec{\omega}$ vector, so the construct must have units which makes the equation itself "true".

VI. EXTENDING THE ARGUMENT TO MORE THAN ONE POINT PARTICLE

Consider that this derivation has been conducted for one particle rotating about a point. For more than one mass point (the situation considered when considering molecules), rotating about the common center of gravity, we will have one such equation for each mass

$$\vec{L}_i = m_i \underbrace{I_i} \otimes \vec{\omega}$$

assuming they are all rotating together (rigid body rotations, which is our main interest here). For the assembly, we then have

$$\vec{L} = \sum_i \vec{L}_i = \sum_i \left(m_i \underbrace{I_i} \right) \otimes \vec{\omega}$$

$$\vec{L} = \begin{pmatrix} \sum_i m_i (r_i^2 - x_i^2) & -\sum_i m_i x_i y_i & -\sum_i m_i x_i z_i \\ -\sum_i m_i y_i x_i & \sum_i m_i (r_i^2 - y_i^2) & -\sum_i m_i y_i z_i \\ -\sum_i m_i z_i x_i & -\sum_i m_i z_i y_i & \sum_i m_i (r_i^2 - z_i^2) \end{pmatrix} \otimes \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \tag{6.1}$$

We see that we've obtained the tensor of the moment of inertia explicitly.

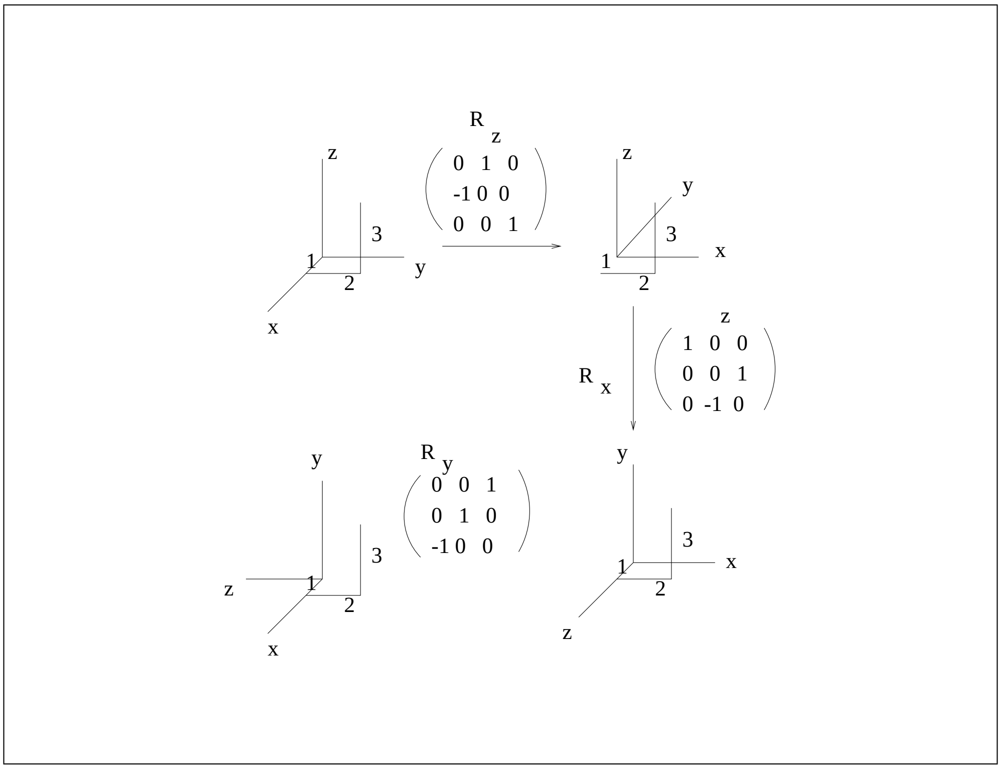

FIG. 1: Sequential Rotations