October 2006

The Wave Equation for Stringed Instruments

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(Dated: October 24, 2006)

I. SYNOPSIS

How does a violin string work to make sound? Why is a guitar's sound different from a cello's sound? We will not actually discuss all this here; the first steps for understanding will be taken here, with a musical departure from our main path left to your future extracurricula readings.

A string's shape , when supported at both ends, can be described using a displacement function z(x), where x is the position along the string, $0 \leq x \leq L$, if the left support is called x=0 and the right support is located then at x=L [1].

Since the displacement of the string (z(x)) itself will change with time, we recognize that z is a function of position (x) and time, i.e., z=z(x,t). There are two variables determining the displacement, the position along the string and when one looks at that position.

The wave equation

$$\frac{\partial^2 z(x,t)}{\partial x^2} = K\frac{\partial^2 z(x,t)}{\partial t^2} \tag{1.1}$$

which we are going to obtain here, since y is a function of two variables, is by necessity a partial differential equation. A typical solution might be

$$z(x,t) = sin(x-t)$$

(where suitable constants have been suppressed, or, it might be

$$z(x,t) = sin(x)cos(t)$$

but whatever it is, it is a function of two variables, x and t! It also could be $x^{17}t^{-93}$ but it never will be!

To construct "the wave equation", Equation 1.1, we apply Newton's second law, F=ma, to a piece of the string.

Assume the string is under tension, τ and that the displacement, z, is small everywhere. We see from the accompanying figure that the tension exerted to the left and the right of the segement are not colinear (and antiparallel) but instead are slightly canted with respect to each other. On the left hand side, the vertical (downward) component of tension is $\tau_{left} = \tau \sin\theta$ while on the right one has $\tau_{right} = \tau \sin\theta\prime$. The slight imbalance between these two values gives rise to a net force (downward) which accelerates the string segment.

Now

$$\sin\theta' = \sin\theta + \frac{\partial \sin\theta}{\partial x}$$

which is, approximately,

$$\sin\theta' = \sin\theta + \frac{\partial \frac{\partial z}{\partial x}}{\partial x}dx = \sin\theta + \frac{\partial^2 z}{\partial x^2}dx$$

since the sine is approximately the slope (really the tangent, but the tangent and the sine are almost the same thing when the cosine is about 1). We see that the **net** force on the segment, the difference between the left and the right hand components (ignoring torques!, why?) for vertical force becomes

$$\tau\frac{\partial^2 z}{\partial x^2}dx$$

which equals the acceleration of the mass of the segment,

$$a = \rho dx\frac{\partial^2 z}{\partial t^2}$$

where ρ is the linear density of the string. We finally equate the two, via Newton's second law, and obtain

$$\tau\frac{\partial^2 z}{\partial x^2}dx = \rho dx\frac{\partial^2 z}{\partial t^2}$$

for an arbitrary differential length of string. Ultimately, we have the wave equation:

$$\tau\frac{\partial^2 z}{\partial x^2} = \rho\frac{\partial^2 z}{\partial t^2}$$

Traditionally, this last equation is written as

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 z}{\partial t^2}$$

where $1/v^2 = \rho/\tau$. Since ρ has units grams/cm (in c.g.s), and τ has units of dynes, the units of v are verified to be cm/sec, a proper unit for a velocity [2].

What is this velocity?

To understand where it comes from, we consider a disturbance on the string of the form z(x-vt) where z is (again) the vertical displacement of the string, x is the position along the string, and t is the time. This time, v has the units x/t so that x-vt is some kind of distance unit. Now the point of the form is that x-vt is such that if t increases, one needs to look at higher values of x to find the place where x'-vt' (read x prime - v t prime) is the same as x-vt was. Which means that the z value at x'-vt' is the same as the value at x-vt, which was the "old" value. The disturbance is "moving to the right, you have to look at larger and larger values of x as t increases to

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see the same displacement. That means the displacement itself is moving (to the right, higher x values). Now, we show that z(x-vt) is a solution of the wave equation. We have

$$\frac{\partial^2 z(x-vt)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 z(x-vt)}{\partial t^2}$$

and we know (using the chain rule)

$$\frac{\partial (x-vt)}{\partial x} = 1$$

$$\frac{\partial (x-vt)}{\partial t} = -v$$

so, we have

$$\frac{\partial^2 z(x-vt)}{\partial (x-vt)^2} = (-v)^2\frac{1}{v^2}\frac{\partial^2 z(x-vt)}{\partial (x-vt)^2}$$

which shows that z(x-vt) is a solution of the wave equation.

Now, if z(x-vt) is a solution, so is z(x+vt).

The most common disturbance (z(x,t)) is a sine or cosine, so, for a little bit, lets write

$$z_-(x-vt) = cos(x-vt)$$

and

$$z_+(x+vt) = cos(x+vt)$$

and recognize (via DeMoivre's Theorem)

$$e^{\imath\alpha} = \cos\alpha + \imath\sin\alpha$$

and

$$e^{\imath\beta} = \cos\beta + \imath\sin\beta$$

$$e^{\imath\alpha}e^{\imath\beta} = e^{\imath(\alpha+\beta)} = (\cos\beta + \imath\sin\beta)(\cos\alpha + \imath\sin\alpha)$$

so

$$\cos(\alpha+\beta) + \imath\sin(\alpha+\beta) = \cos\beta\cos\alpha + \sin\beta\sin\alpha + \imath(\cos\beta\sin\alpha + \sin\beta\cos\alpha)$$

so that, equating Real parts on both sides (of the above) to obtain a formula for the cosine of the sum of angles, we obtain finally

$$z_-(x-vt) = \cos(x)\cos(-vt) + \sin(x)\sin(-vt)$$

or, since the sine is an odd function,

$$z_-(x-vt) = \cos(x)\cos(-vt) - \sin(x)\sin(vt)$$

$$z_+(x+vt) = \cos(x)\sin(vt) + \sin(x)\sin(vt)$$

so, we have, upon adding these last two

$$z_-(x-vt) + z_+(x+vt) = \cos(x)\cos(vt) - \sin(x)\sin(vt) + \cos(x)\cos(vt) + \sin(x)\sin(vt)$$

which is

$$z_-(x-vt) + z_+(x+vt) = 2cos(x)cos(vt)$$

This is known as a standing wave, i.e., one which is variable separable. We have a solution, z(x,t) which is a product of a space part (x) and a time part (t), like f(x)g(t) rather than z(x-t) or anything like that. This is called separation of variables, and will turn out to be meaningful in partial differential equations.

II. FIGURES

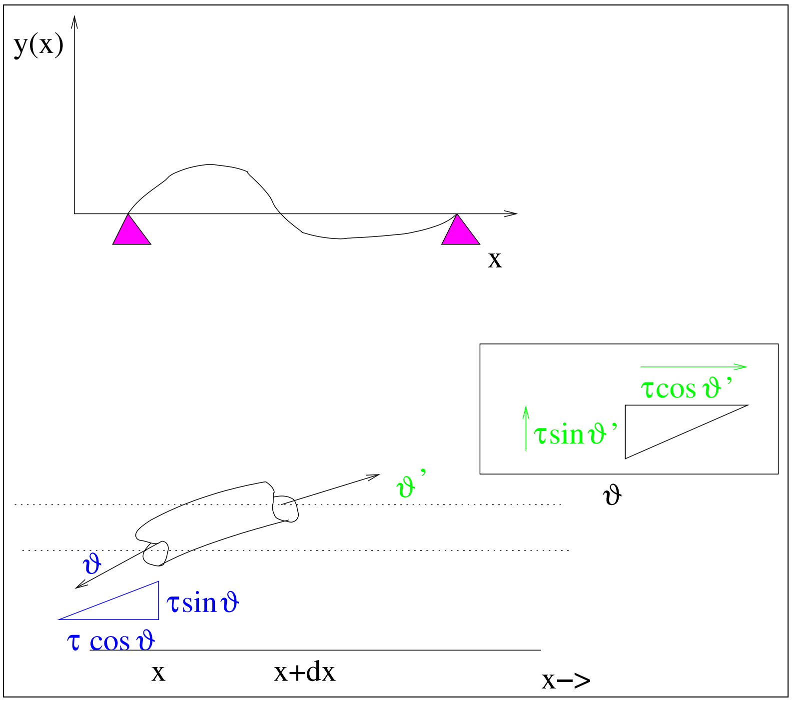

FIG. 1: A String Caught in Mid Flight

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[1] in the following, both x and t are regarded as dimensionless, i.e., x is a distance, and t is a time, but neither (temporarily) has its proper dimension, cm or sec, as example. Later, we will modify this slightly.

[2] notice that we are violating our discussion of units given before