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Alternative Formulations for Angular Momentum Operators, Cartesian and Spherical Polar Forms

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I. CARTESIAN AND SPHERICAL POLAR FORMS

It is of value to inspect the angular momentum operator in terms of angles rather than Cartesian coördinates. Remember that

$$x = r\sin\vartheta\cos\varphi$$

$$y = r\sin\vartheta\sin\varphi$$

$$z = r\cos\vartheta$$

and

$$\vec{L} = \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{pmatrix} = \hat{i}(yp_z - zp_y) + \hat{j}(zp_x - xp_z) + \hat{k}(xp_y - yp_x) \tag{1.1}$$

We start with a feast of partial derivatives:

$$\left(\frac{\partial r}{\partial x}\right)_{y,z} = \sin\vartheta\cos\varphi \tag{1.2}$$

$$\left(\frac{\partial r}{\partial y}\right)_{x,z} = \sin\vartheta\sin\varphi \tag{1.3}$$

$$\left(\frac{\partial r}{\partial z}\right)_{x,y} = \cos\vartheta \tag{1.4}$$

$$\left(\frac{\partial \vartheta}{\partial x}\right)_{y,z} = \frac{\cos\vartheta\cos\varphi}{r} \tag{1.5}$$

$$\left(\frac{\partial \vartheta}{\partial y}\right)_{x,z} = \frac{\cos\vartheta\sin\varphi}{r} \tag{1.6}$$

$$\left(\frac{\partial \vartheta}{\partial z}\right)_{x,y} = -\frac{\sin\vartheta}{r} \tag{1.7}$$

and finally

$$\left(\frac{\partial \varphi}{\partial x}\right)_{y,z} = -\frac{\sin\varphi}{r\sin\vartheta} \tag{1.8}$$

$$\left(\frac{\partial \varphi}{\partial y}\right)_{x,z} = \frac{\cos\varphi}{r\sin\vartheta} \tag{1.9}$$

$$\left(\frac{\partial \varphi}{\partial z}\right)_{x,y} = 0 \tag{1.10}$$

which we employ on the defined x-component of the angular momentum, thus

$$L_x \equiv yp_z - zp_y = -\imath\hbar\sin\vartheta\sin\varphi\frac{\partial}{\partial z} - \left(-\imath\hbar\cos\vartheta\frac{\partial}{\partial y}\right)$$

where

$$\frac{\partial}{\partial z} = \left(\frac{\partial r}{\partial z}\right)_{x,y}\frac{\partial}{\partial r} + \left(\frac{\partial \vartheta}{\partial z}\right)_{x,y}\frac{\partial}{\partial \vartheta} + \left(\frac{\partial \varphi}{\partial z}\right)_{x,y}\frac{\partial}{\partial \varphi}$$

$$\frac{\partial}{\partial z} = (\cos\vartheta)\frac{\partial}{\partial r} + \left(\frac{\sin\vartheta}{r}\right)\frac{\partial}{\partial \vartheta} + (0)\frac{\partial}{\partial \varphi} \tag{1.11}$$

or

$$\frac{\partial}{\partial z} = \cos\vartheta\frac{\partial}{\partial r} + \left(\frac{\sin\vartheta}{r}\right)\frac{\partial}{\partial \vartheta} \tag{1.12}$$

and

$$\frac{\partial}{\partial y} = \left(\frac{\partial r}{\partial y}\right)_{x,z}\frac{\partial}{\partial r} + \left(\frac{\partial \vartheta}{\partial y}\right)_{x,z}\frac{\partial}{\partial \vartheta} + \left(\frac{\partial \varphi}{\partial y}\right)_{x,z}\frac{\partial}{\partial \varphi}$$

$$\frac{\partial}{\partial y} = (\sin\vartheta\sin\varphi)\frac{\partial}{\partial r} + \left(\frac{\cos\vartheta\sin\varphi}{r}\right)\frac{\partial}{\partial \vartheta} + \left(\frac{\cos\varphi}{r\sin\vartheta}\right)\frac{\partial}{\partial \varphi} \tag{1.13}$$

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and

$$\frac{\partial}{\partial x} = \left(\frac{\partial r}{\partial x}\right)_{y,z} \frac{\partial}{\partial r} + \left(\frac{\partial \vartheta}{\partial x}\right)_{y,z} \frac{\partial}{\partial \vartheta} + \left(\frac{\partial \varphi}{\partial x}\right)_{y,z} \frac{\partial}{\partial \varphi}$$

$$\frac{\partial}{\partial x} = (\sin\vartheta\cos\varphi)\frac{\partial}{\partial r} + \left(\frac{\cos\vartheta\cos\varphi}{r}\right)\frac{\partial}{\partial \vartheta} + \left(-\frac{\sin\varphi}{r\sin\vartheta}\right)\frac{\partial}{\partial \varphi} \tag{1.14}$$

so,

$$L_x \equiv yp_z - zp_y = -\imath\hbar r\sin\vartheta\sin\varphi\left((\cos\vartheta)\frac{\partial}{\partial r} + \left(-\frac{\sin\vartheta}{r}\right)\frac{\partial}{\partial\vartheta}\right) - \left(-\imath\hbar r\cos\vartheta\left((\sin\vartheta\sin\varphi)\frac{\partial}{\partial r} + \left(\frac{\cos\vartheta\sin\varphi}{r}\right)\frac{\partial}{\partial\vartheta} + \left(\frac{\cos\varphi}{r\sin\vartheta}\right)\frac{\partial}{\partial\varphi}\right)\right) \tag{1.15}$$

At constant r, the partial with respect to r looses meaning, and one has

$$\frac{L_x}{-\imath\hbar} = r\sin\vartheta\sin\varphi\left(+\left(-\frac{\sin\vartheta}{r}\right)\frac{\partial}{\partial\vartheta}\right) - r\cos\vartheta\left(\left(\frac{\cos\vartheta\sin\varphi}{r}\right)\frac{\partial}{\partial\vartheta} + \left(\frac{\cos\varphi}{r\sin\vartheta}\right)\frac{\partial}{\partial\varphi}\right) \tag{1.16}$$

which leads to (combining the ϑ partial derivative terms and cancelling the r terms)

$$L_x = -\imath\hbar\left(\sin\varphi\frac{\partial}{\partial\vartheta} + \frac{\cos\vartheta\cos\varphi}{\sin\vartheta}\frac{\partial}{\partial\varphi}\right) \tag{1.17}$$

and

$$L_y \equiv zp_x - xp_z = -\imath\hbar r\cos\vartheta\left(\sin\vartheta\cos\varphi\frac{\partial}{\partial r} + \left(\frac{\cos\vartheta\cos\varphi}{r}\right)\frac{\partial}{\partial\vartheta} + \left(-\frac{\sin\varphi}{r\sin\vartheta}\right)\frac{\partial}{\partial\varphi}\right) - \imath\hbar r\sin\vartheta\cos\varphi\left(\cos\vartheta\frac{\partial}{\partial r} + \left(\frac{\sin\vartheta}{r}\right)\frac{\partial}{\partial\vartheta} + (0)\frac{\partial}{\partial\varphi}\right) \tag{1.18}$$

which becomes

$$L_y = -\imath\hbar\left(\left(\frac{r\cos^2\vartheta\cos\varphi}{r}\right)\frac{\partial}{\partial\vartheta} + \left(-\frac{r cos\vartheta\sin\varphi}{r\sin\vartheta}\right)\frac{\partial}{\partial\varphi}\right) - \imath\hbar\left(\left(\frac{r\sin^2\vartheta\cos\varphi}{r}\right)\frac{\partial}{\partial\vartheta}\right) \tag{1.19}$$

so that

$$L_y = -\imath\hbar\left(\cos\varphi\frac{\partial}{\partial\vartheta} - \frac{\cos\vartheta\sin\varphi}{\sin\vartheta}\frac{\partial}{\partial\varphi}\right) \tag{1.20}$$

Finally, for L_z we have

$$l_z = xp_y - yp_x$$

which is

$$-\hbar r\left(\sin\vartheta\cos\varphi\frac{\partial}{\partial y} - \sin\vartheta\sin\varphi\frac{\partial}{\partial x}\right) \tag{1.21}$$

which becomes

$$-\hbar r\left(\sin\vartheta\cos\varphi(\sin\vartheta\sin\varphi)\frac{\partial}{\partial r} + \left(\frac{\cos\vartheta\sin\varphi}{r}\right)\frac{\partial}{\partial\vartheta} + \left(\frac{\cos\varphi}{r\sin\vartheta}\right)\frac{\partial}{\partial\varphi} - \sin\vartheta\sin\varphi(\sin\vartheta\cos\varphi)\frac{\partial}{\partial r} + \left(\frac{\cos\vartheta\cos\varphi}{r}\right)\frac{\partial}{\partial\vartheta} + \left(-\frac{\sin\varphi}{r\sin\vartheta}\right)\frac{\partial}{\partial\varphi}\right) \tag{1.22}$$

$$L_z = -\imath\hbar\frac{\partial}{\partial\varphi} \tag{1.23}$$

II.

We need to form L^2, i.e.,

$$L^2 = L_x \cdot L_x + L_y \cdot L_y + L_z \cdot L_z \tag{2.1}$$

$$L_x^2 = -\hbar^2\left(\sin\varphi\frac{\partial}{\partial\vartheta} + \frac{\cos\vartheta\cos\varphi}{\sin\vartheta}\frac{\partial}{\partial\varphi}\right)\left(\sin\varphi\frac{\partial}{\partial\vartheta} + \frac{\cos\vartheta\cos\varphi}{\sin\vartheta}\frac{\partial}{\partial\varphi}\right) \tag{2.2}$$

$$L_y^2 = -\hbar^2\left(\cos\varphi\frac{\partial}{\partial\vartheta} - \frac{\cos\vartheta\sin\varphi}{\sin\vartheta}\frac{\partial}{\partial\varphi}\right)\left(\cos\varphi\frac{\partial}{\partial\vartheta} - \frac{\cos\vartheta\sin\varphi}{\sin\vartheta}\frac{\partial}{\partial\varphi}\right) \tag{2.3}$$

$$L_z^2 = -\hbar^2\frac{\partial\frac{\partial}{\partial\varphi}}{\partial\varphi} \tag{2.4}$$

For L_x^2 we have, upon expansion

$$\begin{aligned}
\frac{L_x^2}{-\hbar^2} = \sin\varphi\frac{\partial\sin\varphi\frac{\partial}{\partial\vartheta}}{\partial\vartheta} &\Rightarrow \sin^2\varphi\frac{\partial^2}{\partial\vartheta^2} \\
+\frac{\cos\vartheta\cos\varphi}{\sin\vartheta}\frac{\partial\sin\varphi\frac{\partial}{\partial\vartheta}}{\partial\varphi} &\Rightarrow +\cos^2\varphi\cot\vartheta\frac{\partial}{\partial\vartheta} + \cot\vartheta\cos\varphi\sin\varphi\frac{\partial^2}{\partial\vartheta\partial\varphi} \\
\sin\varphi\frac{\partial\frac{\cos\vartheta\cos\varphi}{\sin\vartheta}\frac{\partial}{\partial\varphi}}{\partial\vartheta} &\Rightarrow -\sin\varphi\cos\varphi\frac{\partial}{\partial\varphi} - \frac{\cos^2\vartheta\cos\varphi\sin\varphi}{\sin^2\vartheta}\frac{\partial}{\partial\varphi} + \frac{\cos\vartheta\cos\varphi\sin\varphi}{\sin\vartheta}\frac{\partial^2}{\partial\varphi\partial\vartheta} \\
+\frac{\cos\vartheta\cos\varphi}{\sin\vartheta}\frac{\partial\frac{\cos\vartheta\cos\varphi}{\sin\vartheta}\frac{\partial}{\partial\varphi}}{\partial\varphi} &\Rightarrow -\frac{\cos^2\vartheta\cos\varphi\sin\varphi}{\sin^2\vartheta}\frac{\partial}{\partial\varphi} + \frac{\cos^2\vartheta\cos^2\varphi}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}
\end{aligned} \tag{2.5}$$

For L_y we have

$$\begin{aligned}
\frac{L_y^2}{-\hbar^2} = \cos\varphi\frac{\partial\cos\varphi\frac{\partial}{\partial\vartheta}}{\partial\vartheta} &\Rightarrow \cos^2\varphi\frac{\partial^2}{\partial\vartheta^2} \\
-\frac{\cos\vartheta\sin\varphi}{\sin\vartheta}\frac{\partial\cos\varphi\frac{\partial}{\partial\vartheta}}{\partial\varphi} &\Rightarrow +\cot\vartheta\sin^2\varphi\frac{\partial}{\partial\vartheta} - \frac{\cos\vartheta\sin\varphi}{\sin\vartheta}\cos\varphi\frac{\partial^2}{\partial\vartheta\partial\varphi} \\
-\cos\varphi\frac{\partial\frac{\cos\vartheta\sin\varphi}{\sin\vartheta}\frac{\partial}{\partial\varphi}}{\partial\vartheta} &\Rightarrow +\cos\varphi\sin\varphi\frac{\partial}{\partial\varphi} + \cos\varphi\frac{\cos^2\vartheta\sin\varphi}{\sin^2\vartheta}\frac{\partial}{\partial\varphi} - \cos\varphi\frac{\cos\vartheta\sin\varphi}{\sin\vartheta}\frac{\partial^2}{\partial\varphi\partial\vartheta} \\
+\frac{\cos\vartheta\sin\varphi}{\sin\vartheta}\frac{\partial\frac{\cos\vartheta\sin\varphi}{\sin\vartheta}\frac{\partial}{\partial\varphi}}{\partial\varphi} &\Rightarrow +\frac{\cos^2\vartheta\cos\varphi\sin\varphi}{\sin^2\vartheta}\frac{\partial}{\partial\varphi} + \frac{\cos^2\vartheta\sin^2\varphi}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}
\end{aligned} \tag{2.6}$$

and, of course

$$\frac{L_y^2}{-\hbar^2} = \frac{\partial^2}{\partial\varphi^2} \tag{2.7}$$

It follows, adding Equations 2.5, 2.6 and 2.7, that

$$\sin^2\varphi\frac{\partial^2}{\partial\vartheta^2}$$

$$
\begin{aligned}
&+\cos^2\varphi\cot\vartheta\frac{\partial}{\partial\vartheta}+\cot\vartheta\cos\varphi\sin\varphi\frac{\partial^2}{\partial\vartheta\partial\varphi}\\
&-\sin\varphi\cos\varphi\frac{\partial}{\partial\varphi}-\frac{\cos^2\vartheta\cos\varphi\sin\varphi}{\sin^2\vartheta}\frac{\partial}{\partial\varphi}+\frac{\cos\vartheta\cos\varphi\sin\varphi}{\sin\vartheta}\frac{\partial^2}{\partial\varphi\partial\vartheta}\\
&-\frac{\cos^2\vartheta\cos\varphi\sin\varphi}{\sin^2\vartheta}\frac{\partial}{\partial\varphi}\\
&+\frac{\cos^2\vartheta\cos^2\varphi}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}\\
&+\cos^2\varphi\frac{\partial}{\partial\vartheta}\\
&+\cot\vartheta\sin^2\varphi\frac{\partial}{\partial\vartheta}-\frac{\cos\vartheta\sin\varphi}{\sin\vartheta}\cos\varphi\frac{\partial^2}{\partial\vartheta\partial\varphi}\\
&+\cos\varphi\sin\varphi\frac{\partial}{\partial\varphi}+\cos\varphi\cot^2\vartheta\sin\varphi\frac{\partial}{\partial\varphi}-\cos\varphi\cot\vartheta\sin\varphi\frac{\partial^2}{\partial\varphi\partial\vartheta}\\
&+\frac{\cos^2\vartheta\cos\varphi\sin\varphi}{\sin^2\vartheta}\frac{\partial}{\partial\varphi}+\frac{\cos^2\vartheta\sin^2\varphi}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}\\
&+\frac{\partial^2}{\partial\varphi^2}
\end{aligned}
\tag{2.8}
$$

which, gathering terms for clarity becomes

$$
\begin{aligned}
&\left\{\sin^2\varphi+\cos^2\varphi\right\}\frac{\partial^2}{\partial\vartheta^2}\\
&+\left\{\cos^2\varphi\cot\vartheta+\cot\vartheta\sin^2\varphi\right\}\frac{\partial}{\partial\vartheta}\\
&+\left\{-2\frac{\cos^2\vartheta\cos\varphi\sin\varphi}{\sin^2\vartheta}+\cos\varphi\sin\varphi+2\frac{\cos^2\vartheta\cos\varphi\sin\varphi}{\sin^2\vartheta}-\sin\varphi\cos\varphi\right\}\frac{\partial}{\partial\varphi}\\
&+\left\{\cot\vartheta\cos\varphi\sin\varphi+\frac{\cos\vartheta\cos\varphi\sin\varphi}{\sin\vartheta}-\frac{\cos\vartheta\sin\varphi}{\sin\vartheta}\cos\varphi-\cos\varphi\cot\vartheta\sin\varphi\right\}\frac{\partial^2}{\partial\varphi\partial\vartheta}\\
&+\left\{\frac{\cos^2\vartheta\cos^2\varphi}{\sin^2\vartheta}+\frac{\cos^2\vartheta\sin^2\varphi}{\sin^2\vartheta}+1\right\}\frac{\partial^2}{\partial\varphi^2}
\end{aligned}
\tag{2.9}
$$

and finally

$$
\begin{aligned}
&\frac{\partial^2}{\partial\vartheta^2}\\
&\cot\vartheta\frac{\partial}{\partial\vartheta}\\
&zero\frac{\partial}{\partial\varphi}\\
&zero\frac{\partial^2}{\partial\varphi\partial\vartheta}\\
&+\left\{\frac{\cos^2\vartheta+\sin^2\vartheta}{\sin^2\vartheta}+1\right\}\frac{\partial^2}{\partial\varphi^2}
\end{aligned}
\tag{2.10}
$$

which is, ultimately

$$
L^2=-\hbar^2\left(\frac{1}{\sin^2\vartheta}\left[\sin\vartheta\frac{\partial\sin\vartheta\frac{\partial}{\partial\vartheta}}{\partial\vartheta}+\frac{\partial^2}{\partial\varphi^2}\right]\right)
\tag{2.11}
$$

We see that the operator associated with Legendre's Equation (multiplied by a constant) has emerged, meaning that Legendre Polynomials and angular momentum are intimately asssociated together.