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The Kepler Problem (the road to Bohr)

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I. SYNOPSIS

The classical Kepler problem is treated here as a review for Chemists. Since the Bohr atom is analogous to the (circular) Kepler problem, much of our insights into atomic states comes from this model.

Since chemists forget their physics when assailed by Organic Chemistry, this review attempts to undo the damage.

II. INTRODUCTION

No problem in classical mechanics is more central to chemistry than the Kepler problem despite the scale differences. Kepler was concerned with the Sun-Earth system, and we will be concerned with the proton-electron system, but the classical mechanics will be the same, so we study both problems together (see Figure 1). Of course, we ignore the radiation that a classical proton-electron system would emit (with subsequent loss of energy) i.e., we assume that there are true stationary, non-radiating orbits in the proton-electron system. Hence, *de facto*, we are making the Bohr assumption!

We transform to the center of mass, as before, and consider the proton-electron and the Sun-Earth systems as pseudo-particles revolving about a point in space.

Contrary to discussions vis à vis the Bohr atom, the distance from the pseudo particle representing either the proton-electron system or the Sun-Earth system will not be constant with time (i.e., the motion will not be circular!) and therefore we are going to have added complications. Even though r will not be constant, ($\dot{r} \neq 0$), there are no torques in this problem because the force attracting the Sun to the Earth (and the Earth to the Sun) lies along the line joining the two, i.e., $\vec{F}$ is collinear with $\vec{r}$, so $\vec{r} \times \vec{F} = 0$.

If the torque is zero then $\vec{L}$ is constant in time. Thus

$$\frac{d\vec{L}}{dt} = \frac{d(\vec{r} \times \vec{p})}{dt}$$

which yields

$$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F}$$

Constancy of $\vec{L}$ means that the three components of $\vec{L}$ are constants; said another way, the magnitude and direction of $\vec{L}$ are constant. Since $\vec{L}$ is, by definition, perpendicular to the plane of $\vec{r}$ and $\vec{v}$ (or $\vec{p}$) we can say that whatever plane originally contains $\vec{r}$ and $\vec{v}$ will always contain both $\vec{r}$ and $\vec{v}$, for otherwise, how could $\vec{L}$ be constant and perpendicular to that particular plane? This allows us to consider the Kepler problem in a plane, worrying later (if need be) about the orientation of that plane.

We first move to the center of gravity of the two body problem, and then we treat the resultant motion in the plane defined by the orbit of the "pseudo" particle (of mass μ). We can use simple polar coordinates once we have made this double transformation.

In the x-y plane of the motion

$$m_e\ddot{x}_e = F_e cos\vartheta$$

$$m_e\ddot{y}_e = F_e sin\vartheta$$

for the earth, and

$$m_s\ddot{x}_s = F_s cos\vartheta$$

$$m_s\ddot{y}_s = F_s sin\vartheta$$

for the sun, which we rewrite using the explicit linkage between the two bodies, as

$$m_s\ddot{x}_s = -F_e cos\vartheta$$

$$m_s\ddot{y}_s = -F_e sin\vartheta$$

(we are using the subscript 'e' to Earth and the subscript 's' for Sun) where we express the force on the Sun as minus the force on the Earth, and break the vector force into x and y components.

A. Aside on center of mass

In one dimension, for two particles,

$$F_A = m_A\ddot{x}_A \;\; ; \;\; F_B = m_B\ddot{x}_B = -F_A \tag{2.1}$$

where $F_A = -F_B$ (Newton's Third). Then adding,

$$m_a\ddot{x}_A + m_B\ddot{x}_B = 0$$

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Defining the center of mass as the balance point between the two particles, i.e., the pivot point for the see-saw which children (the particles A and B) use,

$$x_{c.of.m} \equiv \frac{m_A x_A + m_B x_B}{m_A + m_B}$$

which is rearranged to

$$(m_A + m_B)x_{c.of.m} = m_A x_A + m_B x_B$$

Taking the time derivatives of both sides we obtain

$$(m_A + m_B)\ddot{x}_{c.of.m} = m_A \ddot{x}_A + m_B \ddot{x}_B$$

which is, zero, since the right hand side cancels by virtue of Equation 2.1. This means that the total mass moves at the center of mass speed, without accelerating, forever.

Defining the inter-particle distance $r = x_B - x_A$ and defining the reduced mass:

$$\frac{1}{\mu} = \frac{1}{m_A} + \frac{1}{m_B}$$

or, equivalently,

$$\mu = \frac{m_A m_B}{m_A + m_B} \tag{2.2}$$

then

$$\mu r = \mu(x_B - x_A)$$

and, taking two time derivatives

$$\mu\ddot{r} = \mu(\ddot{x}_B - \ddot{x}_A)$$

or, using Equation 2.2

$$\mu\ddot{r} = \frac{m_A m_B}{m_A + m_B}\ddot{x}_B - \frac{m_A m_B}{m_A + m_B}\ddot{x}_A$$

which is

$$\mu\ddot{r} = \frac{m_A}{m_A + m_B}F_B - \frac{m_B}{m_A + m_B}F_A$$

which, taking advantage again of Newton's third law (Equation 2.1), is

$$\mu\ddot{r} = \frac{m_A}{m_A + m_B}F_B + \frac{m_B}{m_A + m_B}F_B$$

(we could have used F_A instead, and in fact, will define $F \equiv F_B = -F_A$). Then we finally have

$$\mu\ddot{r} = \frac{m_B + m_B}{m_A + m_B}F_B = F$$

which says that the relative coordinate (r) moves in the force F, as if there was a pseudo particle (mass μ) obeying Newton's Second Law. Wonderful.

III. USING THE CENTER OF MASS SYSTEM

For the pseudo particle equivalent to the Sun-Earth system,

$$\frac{1}{\mu} = \frac{1}{m_s} + \frac{1}{m_e}$$

we have, using the notation that $F = |F_e| = |F_s|$,

$$\mu\ddot{x} = -Fcos\vartheta = \mu\frac{d^2x}{dt^2} = \mu\frac{d^2(rcos\vartheta)}{dt^2} \tag{3.1}$$

$$\mu\ddot{y} = -Fsin\vartheta = \mu\frac{d^2y}{dt^2} = \mu\frac{d^2(rsin\vartheta)}{dt^2} \tag{3.2}$$

since $x = r\cos\vartheta$ and $y = r\sin\vartheta$ are the transformation equations from polar to Cartesian coordinates.

By direct differentiation, we obtain

$$\frac{d(rcos\vartheta)}{dt} = \dot{r}cos\vartheta - rsin\vartheta\dot{\vartheta}$$

and for the second derivative,

$$\frac{d^2(rcos\vartheta)}{dt^2} = \ddot{r}cos\vartheta - 2\dot{r}sin\vartheta\dot{\vartheta} - rcos\vartheta\dot{\vartheta}^2 - rsin\vartheta\ddot{\vartheta}$$

while the equivalent terms for the first and second derivatives of $r\sin\vartheta$ gives:

$$\frac{d(rsin\vartheta)}{dt} = \dot{r}sin\vartheta + rcos\vartheta\dot{\vartheta}$$

and

$$\ddot{r}sin\vartheta + 2\dot{r}cos\vartheta\dot{\vartheta} - rsin\vartheta\dot{\vartheta}^2 + rcos\vartheta\ddot{\vartheta}$$

which can be substituted directly into Equation 3.1 to give:

$$\mu(\ddot{r}cos\vartheta - 2\dot{r}sin\vartheta\dot{\vartheta} - rcos\vartheta\dot{\vartheta}^2 - rsin\vartheta\ddot{\vartheta}) = -F(r)cos\vartheta \tag{3.3}$$

$$\mu(\ddot{r}sin - 2\dot{r}cos\vartheta\dot{\vartheta} - rsin\vartheta\dot{\vartheta}^2 - rsin\vartheta\ddot{\vartheta}) = -F(r)sin\vartheta \tag{3.4}$$

Looking at these two equations, the thought occurs that some simplification would occur if we could use the infamous $sin^2\vartheta + cos^2\vartheta = 1$ relation. Multiply the top equation (3.3) by $\cos\vartheta$ and the bottom equation (3.4) by $\sin\vartheta$ and then add the two. We obtain

$$\mu(\ddot{r} - r\dot{\vartheta}^2) = -F(r) \tag{3.5}$$

which certainly seems simple enough. Let's do it again, this time reversing the order (multiplying the top equation (3.3) by sin ϑ and the bottom (3.4) by cos ϑ and subtracting rather than adding). We obtain

$$-2\mu\dot{r}\dot{\vartheta} - r\mu\ddot{\vartheta} = 0$$

This latter equation is just a perfect derivative of a "known" quantity, i.e., the left hand side is just:

$$-\frac{1}{r}\frac{d(\mu r^2\dot{\vartheta})}{dt} = 0$$

Since the factor of (1/r) can never be zero, we see that $\mu r^2\dot{\vartheta}$ must be constant since its time derivative is zero. This result recovers for us the principle idea that in torque free situations the angular momentum is constant! Remember,

$$\ell = rp = r\mu v = r\mu r\dot{\vartheta}$$

A. (Why is the torque zero in this case?)

$$Torque = \vec{r} \otimes \vec{F}$$

and, since r is collinear with F, the cross product vanishes!

IV. RETURNING TO THE MAIN ARGUMENT

Remember that $r\dot{v} = \dot{\vartheta}$, so $\ell = r(\mu v) = rp$ for this situation. The first equation (Equation 3.5) was

$$\mu(\ddot{r} - r\dot{\vartheta}^2) = -F(r) \tag{4.1}$$

is a differential equation which must be solved. Rather than solve it ourselves (a non-trivial task[1]) we will instead prove that the solution claimed for this equation (an ellipse) is in fact the solution. To do this, we must first ask what is the equation of an ellipse (the proposed orbit) in polar coordinates. It is:

$$\frac{1}{r} = A + Bcos\vartheta$$

To refresh your memory, see Figure 3. Solving this equation for r we get

$$r = \frac{1}{A + Bcos\vartheta} \tag{4.2}$$

from which, taking one time derivative we get

$$\dot{r} = \frac{Bsin\vartheta\dot{\vartheta}}{(A + Bcos\vartheta)^2} \tag{4.3}$$

Calling the angular momentum $\ell = \mu r^2\dot{\vartheta}$, we can solve for $\dot{\vartheta}$ and obtain

$$\dot{\vartheta} = \frac{\ell}{\mu r^2}$$

which we can substitute into our expression for $\dot{r}$, (Equation 4.3.), to obtain:

$$\dot{r} = \frac{\ell Bsin\vartheta}{\mu r^2(A + Bcos\vartheta)^2}$$

and since Equation 4.2 holds, we have

$$= \frac{B\ell}{\mu}sin\vartheta$$

Taking another derivative we obtain

$$\ddot{r} = \left(\frac{B\ell}{\mu}\right)cos\vartheta\dot{\vartheta}$$

which we substitute into Equation 4.1 finding

$$\mu(\ddot{r} - r\dot{\vartheta}^2) = -F = \frac{\mu B\ell}{\mu}cos\vartheta\dot{\vartheta} - \frac{\ell^2}{\mu^2 r^3}$$

This last result may be simplified to

$$B\frac{\ell}{\mu}cos\vartheta\frac{\ell}{\mu r^2} - \frac{\ell^2}{\mu^2 r^3} = -\frac{F}{\mu}$$

i.e.,

$$Bcos\vartheta = \frac{\mu^2 r^2}{\ell^2}\left(\frac{\ell^2}{\mu^2 r^3} - \frac{F}{\mu}\right) = \frac{1}{r - A} \tag{4.4}$$

Therefore, from Equation 4.4, we have

$$\frac{\ell^2}{\mu^2 r^2}\left(\frac{1}{r - A}\right) - \frac{\ell^2}{\mu^2 r^3} = -\frac{F}{\mu}$$

which simplifies to

$$-\frac{\ell^2}{\mu^2 r^2} = -\frac{F}{\mu}$$

i.e.,

$$\frac{\ell^2}{A\mu r^2} = F$$

Remembering that ℓ is a constant, we see that F has to be inversely quadratic with r in order that the ellipse be a solution to the original differential equation. Of course, the gravitational force is just proportional to r^{-2} exactly as needed, so we "recover" the Kepler elliptical orbits. Further, since Coulomb's law also has the force proportional to r^{-2} we have a classical orbit for the electron around the proton, barring radiation loss.

Of course, we have not shown what values of A and B are legal, so there are potential problems with our solution (the ellipse) but our plausibility arguments illustrate something about the elliptical Keplerian orbits without going into the gruesome detail of actually solving the applicable differential equation!

V. A SPECIAL RELATION FOR THE KEPLER PROBLEM (*)

The Kepler problem abounds with strange and magical results [2].

The gravitational (and Coulomb) forces are written in vector notation as

$$\vec{F} = -k\frac{\vec{r}}{r^3}$$

where the right hand side could also be written as

$$-k\frac{\vec{r}_{unit}}{r^2}$$

(where r is the magnitude of $\vec{r}$), and

$$\vec{r}_{unit} \equiv \frac{\vec{r}}{r}$$

$$\vec{r}_{unit} = \frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}$$

(note that oftentimes the unit vector is designated $\hat{r}$). The unit vector form shows most clearly the inverse square form.

As we know, in the plane

$$\mu\ddot{x} = -kx/r^3 = -k\frac{cos\vartheta}{r^2}$$

$$\mu\ddot{y} = -ky/r^3 = -k\frac{sin\vartheta}{r^2}$$

Further, we know that $\dot{\vartheta} = \ell/(\mu r^2)$ from the constancy of the angular momentum.

Changing from an acceleration to a velocity [3] notation we have:

$$\mu\frac{dv(x)}{dt} = -k\frac{cos\vartheta}{r^2} \tag{5.1}$$

$$\mu\frac{dv(y)}{dt} = -k\frac{sin\vartheta}{r^2} \tag{5.2}$$

Dividing one side of of Equation 5.1 by $\dot{\vartheta}$ and the other side by $\ell/(\mu r^2)$, we obtain (and doing the same for the other equation (5.2)):

$$\mu\frac{\frac{dv(x)}{dt}}{\dot{\vartheta}} = -k\frac{\frac{cos\vartheta}{r^2}}{\frac{\ell}{\mu r^2}} \tag{5.3}$$

$$\frac{dv(y)}{d\vartheta} = -k\frac{\sin\vartheta}{\ell} \tag{5.4}$$

since

$$\frac{\frac{dv}{dt}}{\frac{d\vartheta}{dt}} = \left(\frac{\left(\frac{dv}{dt}\right)}{\dot{\vartheta}}\right) = \frac{dv}{d\vartheta}$$

Cleaning up and integrating, we obtain:

$$\int dv(x) = \int \frac{dv(x)}{d\vartheta}d\vartheta = -\frac{k}{\ell}\int cos\vartheta d\vartheta$$

i.e.,

$$v(x) = -\left(\frac{k}{\ell}\right) sin\vartheta + a\ constant$$

and, in parallel,

$$v(y) = \left(\frac{k}{\ell}\right) cos\vartheta + another\ constant$$

What does this result mean? It means that the velocity vector moves in a circle (of radius k/ℓ) illustrates how the velocity vector rotates. This final result is an obscure and little known one, which is of virtually no utility. But the calculus was quite nice, wasn't it?

VI. THE RUNGE LENZ VECTOR

Another little known fact about the Kepler problem has to do with the Runge Lenz vector. We are going to switch over to the proton-electron problem now, leaving the planets to the astronomers. In order to treat one electron atoms, a long tradition has evolved of writing the coulomb force for an isoelectronic sequence of one electron problems beginning with H, then He^+ , then Li^{2+} , Be^{3+} . etc., by defining the atomic number (the number of protons in the nucleus, as Z, and defining the unit charge on the electron Notice that a force in dynes can be achieved if one uses the charge on the electron in statcoulombs, so that $e = \sqrt{dyne \times cm}$

$$\vec{F} = -(Ze)(e)\frac{\vec{r}}{r^3} = \frac{d\vec{p}}{dt} \tag{6.1}$$

When we study the Bohr atom, we will explore in more detail this formula. For now, we note that the attractive nature of the force is explicitly designated by the minus sign, so that 'Ze' and 'e' are both positive, one the charge on the nucleus, the other the magnitude of the charge on the electron.

Newton's Law for the 'H-atom' is:

$$\mu\frac{d^2\vec{r}}{dt} = \frac{d\vec{p}}{dt} = -Ze^2\frac{\vec{r}_{unit}}{r^2}$$

We have already learned that $\vec{L}$ is a constant of the motion for this system, and now we propose to show that $\vec{A}$

$$\vec{A} = \frac{1}{\mu Ze^2}\left(L \times \vec{p}\right) + \vec{r}_{unit}$$

also is a constant of the motion. To investigate the time behavior of $\vec{A}$, take its derivative with respect to time and see what happens (The time derivative will vanish, so $\vec{A}$ will be a constant of the motion like $\vec{L}$). Let us do this in small steps, since it is not trivial. First, let's obtain the time derivative of the unit 'r' vector. Since the unit r vector is $\vec{r}/r$, where r is the magnitude of $\vec{r}$ i.e.,

$$r = \sqrt{x^2 + y^2 + z^2} = |\vec{r}|$$

$$\vec{r}_{unit} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}} = \frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}$$

we then have

$$\frac{d\vec{r}_{unit}}{dt} = \frac{\dot{x}}{r}\hat{i} + \frac{\dot{y}}{r}\hat{j} + \frac{\dot{z}}{r}\hat{k}\vec{r} + \left(x\hat{i} + y\hat{j} + z\hat{k}\right)\frac{dr^{-1}}{dt}$$

$$= \frac{\dot{x}}{r}\hat{i}+\frac{\dot{y}}{r}\hat{j}+\frac{\dot{z}}{r}\hat{k}-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\left(\frac{d\left(x^2+y^2+z^2\right)}{dt}\right)\vec{r}$$

which is

$$= \frac{\dot{x}}{r}\hat{i} + \frac{\dot{y}}{r}\hat{j} + \frac{\dot{z}}{r}\hat{k} - \left(\frac{1}{r^3}\left(x\frac{dx}{dt} + y\frac{dy}{dt} + z\frac{dz}{dt}\right)\right)\vec{r}$$

which becomes, using full vector notation

$$\frac{d\vec{r}_{unit}}{dt} = \frac{\vec{v}}{r} - (\vec{r}\cdot\vec{v})\left(\frac{\vec{r}}{r^3}\right)$$

$$\frac{d\vec{r}_{unit}}{dt} = \frac{\vec{p}}{\mu r} - (\vec{r}\cdot\vec{p})\frac{\vec{r}}{\mu r^3} \tag{6.2}$$

where we have used the definition $\vec{p} = \mu\vec{v}$

Now we are ready to ask for the time derivative of $\vec{A}$. We have

$$\frac{d\vec{A}}{dt} = \frac{1}{\mu Z e^2}\vec{L}\times\frac{d\vec{p}}{dt} + \frac{d\vec{L}}{dt}\times\frac{\vec{p}}{\mu Z e^2} + \frac{d\vec{r}_{unit}}{dt}$$

which equals (using Equation 6.1 and Equation 6.2) and we have used the fact that $\vec{L}$ is a constant of the motion so that the time derivative of $\vec{L}$ is zero (so the middle term (above) vanishes),

$$\frac{d\vec{A}}{dt} = -\vec{L}\times\frac{\vec{r}}{\mu r^3} + \frac{\vec{p}}{\mu r} - (\vec{r}\cdot\vec{p})\frac{\vec{r}}{\mu r^3} \tag{6.3}$$

The term $\vec{L}\times\vec{r}$ is, despite the typography, a triple cross product of the form

$$(\vec{r}\times\vec{p})\times\vec{r}$$

so we must take some care in expanding this term. We get

$$\vec{L} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix}$$

or, expanding:

$$= \hat{i}(yp_z - zp_y) + \hat{j}(zp_x - xp_z) + \hat{k}(xp_y - yp_x)$$

so that

$$\vec{L}\times\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ (yp_z - zp_y) & (zp_x - xp_z) & (xp_y - yp_x) \\ x & y & z \end{vmatrix}$$

which expands to

$$= \hat{i}\left\{(zp_x - xp_z)z - (xp_y - yp_x)y\right\} + \hat{j}\left\{(xp_y - yp_x)x - (yp_z - zp_y)z\right\} +$$

i.e.,

$$\hat{k}\left\{(yp_z - zp_y)y - (zp_x - xp_z)x\right\}$$

which now must be expanded and interpreted. We obtain

$$= \vec{L}\times\vec{r} = \hat{i}\left(\left\{z^2p_x - xzp_z - xyp_y\right\} + \left\{y^2p_x + (x^2p_x - x^2p_x)\right\}\right)$$
$$+\hat{j}\left(\left\{x^2p_y - xyp_x - yxp_z\right\} + \left\{z^2p_y + (y^2p_y - y^2p_y)\right\}\right)$$
$$+\hat{k}\left(\left\{y^2p_z - yzp_y - xzp_x\right\} + \left\{x^2p_z + (z^2p_z - z^2p_z)\right\}\right)$$

where we have explicitly added and subtracted selected terms to each component which will help in the interpretation, since we now have:

$$\vec{L}\times\vec{r} = \hat{i}\left\{r^2 p_x - x(xp_x + yp_y + zp_z)\right\}$$

$$+\hat{j}\left\{r^2 p_y - y(xp_x + yp_y + zp_z)\right\}$$

$$+\hat{k}\left\{r^2 p_z - z(xp_x + yp_y + zp_z)\right\}$$

which can be re-written to

$$\vec{L}\times\vec{r} = r^2\vec{p} - \vec{r}(\vec{r}\cdot\vec{p}) \tag{6.4}$$

which is certainly a lovely result (it may even be useful).

Substituting Equation 6.4 into Equation 6.3 yields

$$\frac{d\vec{A}}{dt} = -\vec{L}\times\vec{r}\frac{1}{\mu r^3} + \frac{\vec{p}}{\mu r} - (\vec{r}\cdot\vec{p})\frac{\vec{r}}{\mu r^3}$$

$$\frac{d\vec{A}}{dt} = -\left(r^2\vec{p} - \vec{r}(\vec{r}\cdot\vec{p})\right)\frac{1}{\mu r^3} + \frac{\vec{p}}{\mu r} - (\vec{r}\cdot\vec{p})\frac{\vec{r}}{\mu r^3}$$

or

$$\frac{d\vec{A}}{dt} = -\frac{[r^2\vec{p} - \vec{r}(\vec{r}\cdot\vec{p})]}{\mu r^3} + \frac{\vec{p}}{\mu r} - \vec{r}\frac{(\vec{r}\cdot\vec{p})}{\mu r^3}$$

which equals zero, as promised. Amazing!

This is the result we sought, that $\vec{A}$ has a zero time derivative, i.e., that it, $\vec{A}$, is a constant in time. Since $\vec{A}$ is a vector, we know that this is a short hand for three constants of the motion, the three components of $\vec{A}$, or said in another way, $\vec{A}$ is constant in magnitude (one constant) and direction (two constants, angles). Therefore, the Kepler problem has 7 constants of the motion, 3 for $\vec{A}$, 3 for $\vec{L}$, and the energy.

It is of some interest to ask what $\vec{A}\cdot\vec{r}$ is. We have

$$\vec{A}\cdot\vec{r} = \frac{1}{(Ze^2\mu)}(\vec{L}\times\vec{p})\cdot\vec{r} + \vec{r}_{unit}\cdot\vec{r} \equiv |\vec{A}||\vec{r}|cos(angle\ between\ \vec{A}\ and\ \vec{r})$$

Since

$$\vec{L}\times\vec{p} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ (yp_z - zp_y) & (zp_x - xp_z) & (xp_y - yp_x) \\ p_x & p_y & p_z \end{vmatrix}$$

we have upon expansion:

$$\vec{L}\times\vec{p} = \hat{i}\left\{(zp_x - xp_z)p_z - (xp_y - yp_x)p_y\right\}$$

$$+\hat{j}\left\{(xp_y - yp_x)p_x - (yp_z - zp_y)p_z\right\}$$

$$+\hat{k}\left\{(yp_z - zp_y)p_y - (zp_x - xp_z)p_x\right\}$$

so that

$$(\vec{L}\times\vec{p})\cdot\vec{r} = (zp_x - xp_z)xp_z - (xp_y - yp_x)xp_y$$

$$+(xp_y - yp_x)yp_x - (yp_z - zp_y)yp_z$$

$$+(yp_z - zp_y)zp_y - (zp_x - xp_z)zp_x$$

which equals

$$= (zp_x - xp_z)(xp_z - zp_x) + (xp_y - yp_x)(yp_x - xp_y) + (yp_z - zp_y)(zp_y - yp_z)$$

which becomes

$$= -(zp_x - xp_z)^2 - (xp_y - yp_x)^2 - (yp_x - zp_y)^2$$

so that we have a semi-final result

$$(\vec{L}\times\vec{p})\cdot\vec{r} = -\vec{L}^2 = -\ell^2$$

Remembering that $\vec{A}$ points along the y-axis (in our example) so the angles between $\vec{r}$ and $\vec{A}$ is, *de facto*, the traditional angle ϑ. We finally obtain

$$|\vec{A}||\vec{r}|\cos(angle\ between\ them) = -\frac{\ell^2}{Ze^2\mu} + r = |\vec{A}|rcos\vartheta$$

where $|\vec{A}|$ is a constant, call it 'ℵ', so that we obtain

$$\aleph rcos\vartheta = -\frac{\ell^2}{Ze^2\mu} + r$$

This allows us to solve, again, for 1/r. We obtain

$$\frac{1}{r} = \frac{\ell^2}{Ze^2\mu}(1 - \aleph cos\vartheta)$$

which recovers for us, again, the ellipse we once had. Beautiful, wasn't it?

VII. A CUTE ALTERNATIVE DERIVATION OF THE ELLIPTICAL ORBIT

For the motion in a plane (perpendicular to the $\vec{L}$ vector), we have

$$\ddot{x} = -\frac{Ze^2x}{r^3} \tag{7.1}$$

$$\ddot{y} = -\frac{Ze^2y}{r^3} \tag{7.2}$$

But we know that, from the definition of the angular momentum,

$$r^2\dot{\theta} = \frac{\ell}{\mu}$$

so

$$\frac{\ddot{x}}{r^2\dot{\theta}} = -\frac{Ze^2\mu x}{\ell r^3} \tag{7.3}$$

$$\frac{\ddot{y}}{r^2\dot{\theta}} = -\frac{Ze^2\mu y}{\ell r^3} \tag{7.4}$$

or

$$\frac{\ddot{x}}{\dot{\theta}} = -\frac{Ze^2\mu\cos\theta}{\ell} \tag{7.5}$$

$$\frac{\ddot{y}}{\dot{\theta}} = -\frac{Ze^2\mu\sin\theta}{\ell} \tag{7.6}$$

which are

$$\frac{\frac{d\dot{x}}{dt}}{\frac{d\theta}{dt}} = -\frac{Ze^2\mu\cos\theta}{\ell} \tag{7.7}$$

$$\frac{\frac{d\dot{y}}{dt}}{\frac{d\theta}{dt}} = -\frac{Ze^2\mu\sin\theta}{\ell} \tag{7.8}$$

$$\frac{d\dot{x}}{d\theta} = -\frac{Ze^2\mu\cos\theta}{\ell} \tag{7.9}$$

$$\frac{d\dot{y}}{d\theta} = -\frac{Ze^2\mu\sin\theta}{\ell} \tag{7.10}$$

which are integrable to

$$\dot{x} = -\frac{Ze^2\mu}{\ell}\sin\theta + C_1 \tag{7.11}$$

$$\dot{y} = \frac{Ze^2\mu}{\ell}\cos\theta + C_2 \tag{7.12}$$

so, since (from the Cartesian form of the definition of Angular Momentum in this case)

$$x\dot{y} - y\dot{x} = \frac{\ell}{\mu}$$

which becomes

$$r\cos\theta\frac{Ze^2\mu}{\ell}(\cos\theta + C_2) + r\sin\theta\frac{Ze^2\mu}{\ell}(\sin\theta + C_1) = \frac{\ell}{\mu}$$

we finally obtain

$$r\frac{Ze^2\mu}{\ell}(1 + \cos\theta C_2 + \sin\theta C_1) = \frac{\ell}{\mu}$$

Defining two new constants, δ and γ, via the equations

$$C_1 = \delta\sin\gamma$$
$$C_2 = \delta\cos\gamma \tag{7.13}$$

which can be inverted to

$$\delta = \sqrt{C_1^2 + C_2^2}$$
$$\gamma = \tan^{-1}\frac{C_1}{C_2} \tag{7.14}$$

which reduces our equation to

$$r\frac{Ze^2\mu}{\ell}(1 + \delta(\cos\theta\sin\gamma + \sin\theta\cos\gamma)) = \frac{\ell}{\mu}$$

which equals

$$r\frac{Ze^2\mu}{\ell}(1 + \delta\cos(\theta - \gamma)) = \frac{\ell}{\mu}$$

or

$$\frac{1}{r} = \frac{1}{Ze^2\mu}(1 + \delta\cos(\theta - \gamma))$$

which is our equation for an ellipse.

This derivation is a slight alteration of one given by R. Weinstock, Am. J. Phys., 60, 615(1992).

VIII. FIGURES

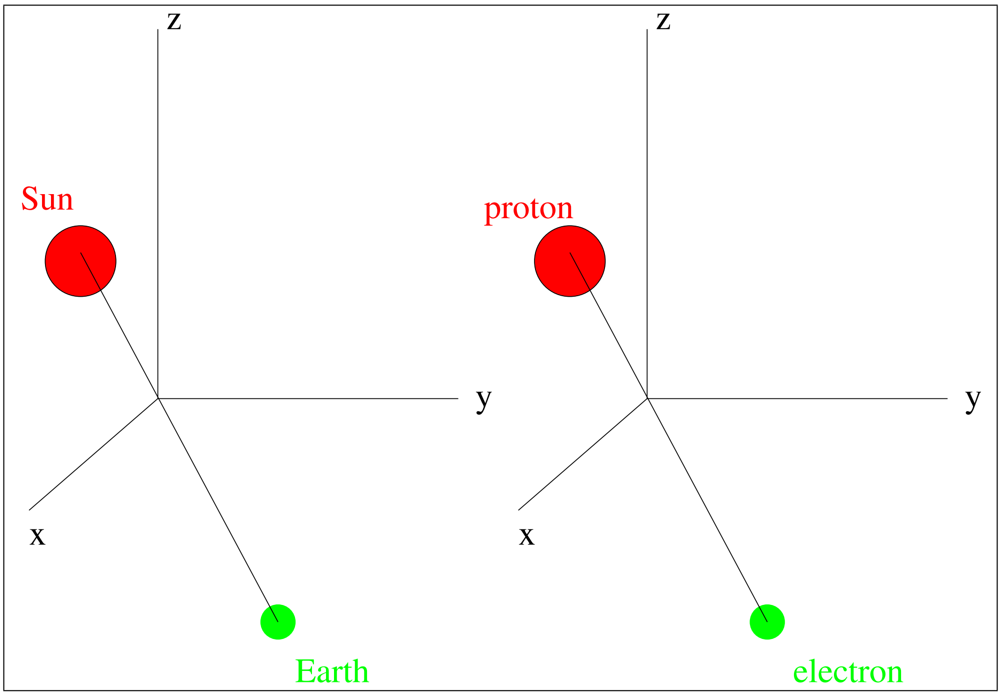

FIG. 1: The relationship between the Sun-Earth system on the one hand and the proton-electron system on the other.

[1] we would obtain $\dot{\vartheta} = \ell/(\mu r^2)$ and its square, etc..

[2] J. Milnor, American Mathematical Monthly, xx,353(1983).

[3] where we are writing v(x) instead of v_x.

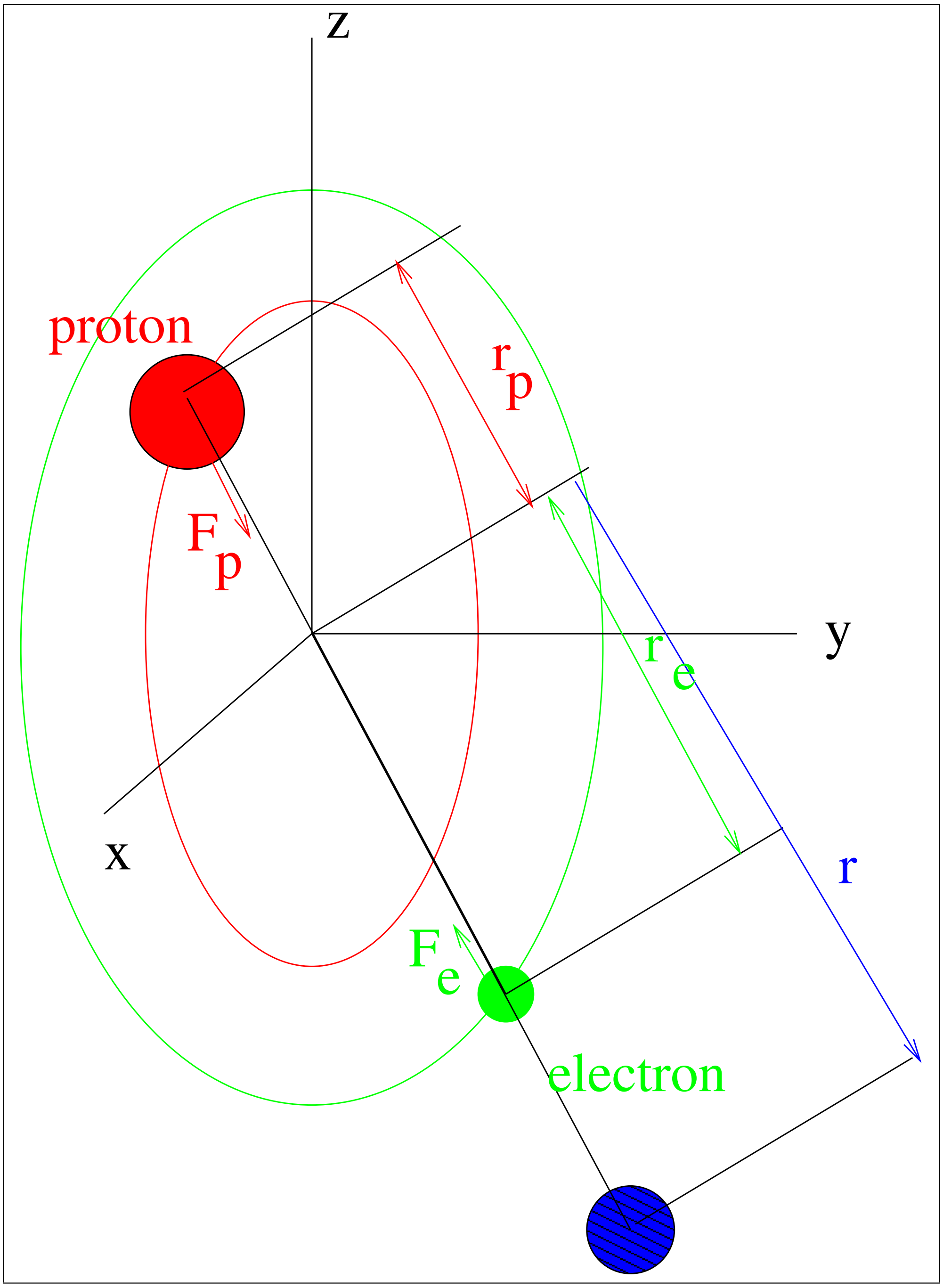

FIG. 2: The center of mass transformation. This allows us to define a polar coordinate representation for the orbit of the pseudo particle mimicking the proton-electron system.

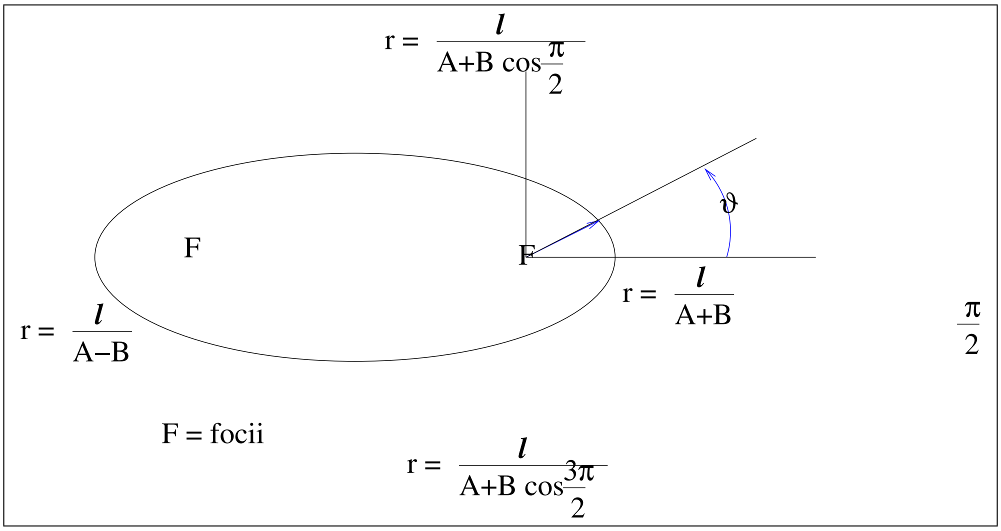

FIG. 3: The construction of an ellipse in polar coordinates.