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Liouville-Type Theorems for Higher Order Elliptic Systems

Frank Arthur, Ph.D.

University of Connecticut, 2016

ABSTRACT

We study positive solutions of the following higher order elliptic system

$$\begin{cases} (-\Delta)^m u = |x|^a v^p \\ (-\Delta)^m v = |x|^b u^q \end{cases} \text{in } \mathbb{R}^N$$

Here $p \geq 1$, $q \geq 1$, $(p,q) \neq (1,1)$.

Henon-Lane-Emden conjecture says (1.0.1) admits no positive solutions if $\frac{1+\frac{a}{N}}{p+1} + \frac{1+\frac{b}{N}}{q+1} > 1 - \frac{2m}{N}$.

When $a = b = 0$, we solve the conjecture under the additional assumption $\max\left(\frac{2m(p+1)+a+bp}{pq-1}, \frac{2m(q+1)+aq+b}{pq-1}\right) > N - 2m - 1$.

In particular, when $N = 2m+1$ or $N = 2m+2$, the conjecture hold true.

When $a > 0, b > 0$, we prove the conjecture under the additional assumptions $\max\left(\frac{2m(p+1)+a+bp}{pq-1}, \frac{2m(q+1)+aq+b}{pq-1}\right) > N - 2m - 1$.

Liouville-Type Theorems for Higher Order Elliptic Systems

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Frank Arthur

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APPROVAL PAGE

Doctor of Philosophy Dissertation

Liouville-Type Theorems for Higher Order Elliptic Systems

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Chapter 1

Introduction

In this thesis, we consider positive solutions $(u > 0,\ v > 0)$ of the following higher order Hénon-Lane-Emden type elliptic system

$$\begin{cases} (-\Delta)^m u = |x|^a v^p \\ (-\Delta)^m v = |x|^b u^q \end{cases} \text{in } \mathbb{R}^N, \tag{1.0.1}$$

where $p > 0$, $q > 0$, $a \geq 0$, $b \geq 0$ and $N \geq 3$.

We are mainly concerned with the question of nonexistence of such positive solutions.

The Hénon-Lane-Emden conjecture for polyharmonic system (1.0.1) states the following:

Conjecture 1.0.1. Let (u, v) be a pair of nonnegative solution of $(1.0.1)$. If

$$\frac{N+a}{p+1} + \frac{N+b}{q+1} > N - 2m,$$

then $u = v = 0$.

For $1 \leq N \leq 2m$, the conjecture follows directly from a growth estimate of integral of $|x|^a v^p$ and $|x|^b u^q$ on ball of radius R (Lemma 1 of [4]). We shall focus on cases $N \geq 2m+1$ in this paper. For the rest of the introduction, we shall review some known results for case $a = b = 0$ and for case when at least one of a or b is positive.

1.1 Lane-Emden System($a = b = 0$)

When $a = b = 0$. (1.0.1) reduces to the well studied Lane-Emden system

$$\begin{cases} (-\Delta)^m u = v^p \\ (-\Delta)^m v = u^q \end{cases} \text{in } \mathbb{R}^N. \tag{1.1.1}$$

The conjecture then states that the curve $\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2m}{N}$ is the dividing curve for existence and nonexistence of positive solutions of (1.1.1).

For $m = 1$, the conjecture was completely solved in the case of radial solutions [9, 14, 16]. Mitidieri [9] showed that there is no positive radial solutions to (1.1.1) below the curve $\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N}$ if $p > 1$, $q > 1$; the condition $p > 1$, $q > 1$ was later relaxed to $p > 0$, $q > 0$ by Serrin and Zou [14, 16]. Furthermore, it is proved by Serrin and Zou [16] that there are infinitely many positive radial solutions above the curve $\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N}$. Therefore $\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N}$ serves as the dividing curve for existence and nonexistence of positive radial solutions of (1.1.1).

The question for the general positive solution to (1.1.1), to the best of our knowledge, has not been completely solved yet for $n > 5$. Partial answers have been obtained over the years. Souto [18] proved nonexistence of positive C^2 solutions below curve $\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N-1}$ when p, $q > 0$. Felmer and de Figureiredo [6] showed that when

$0 < p, q \leq \frac{N+2}{N-2}$ and $(p,q) \neq \left(\frac{N+2}{N-2}, \frac{N+2}{N-2}\right)$, (1.1.1) has no positive C^2 solutions. Further evidence supporting the conjecture can be found in [10], where it is shown that there exists no positive supersolutions to (1.1.1) below the curve

$$\left\{p > 0, q > 0 : \ \frac{1}{p+1} + \frac{1}{q+1} = 1 - \frac{2}{N-2} \max\left(\frac{1}{p+1}, \frac{1}{q+1}\right)\right\}. \tag{1.1.2}$$

We refer to (1.1.2) as S curve and the hyperbola in the conjecture $\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N}$ will be referred as Sobolev's hyperbola throughout the paper. For $p > 0$ and $q > 0$ if $pq \leq 1$ or $pq > 1$ and $\max\left(\frac{2(p+1)}{pq-1}, \frac{2(q+1)}{pq-1}\right) \geq N - 2$, nonexistence of positive solutions was proved by Serrin and Zou in [15]. Direct calculation shows this is the same range of (p,q) as region below and on S curve. Furthermore, Serrin and Zou [15] showed (1.1.1) admits no positive solutions satisfying algebraic growth at infinity below the Sobolev hyperbola when $N = 3$. For the special case $\min(p,q) = 1$, the conjecture was proved by C.-S. Lin [7] . Busca and Manásevich [2] proved that if $p, q > 0$, $pq > 1$,

$$\frac{N-2}{2} \leq \min\left(\frac{2(p+1)}{pq-1}, \frac{2(q+1)}{pq-1}\right) \leq \max\left(\frac{2(p+1)}{pq-1}, \frac{2(q+1)}{pq-1}\right) < N - 2,$$

and

$$\left(\frac{2(p+1)}{pq-1}, \frac{2(q+1)}{pq-1}\right) \neq \left(\frac{N-2}{2}, \frac{N-2}{2}\right),$$

there exists no positive classical solutions to (1.1.1) . Most recently, the conjecture was fully solved in the case $N = 3$ by Poláčik, Quittner and Souplet [13] and by Souplet [17] when $N = 4$. Souplet also proved the conjecture when $N \geq 5$ under the additional assumption that $\max\left(\frac{2(p+1)}{pq-1}, \frac{2(q+1)}{pq-1}\right) > N - 3$.

Comparing to the Lane-Emden system for $m = 1$, less is known about the higher order system (1.1.1) when $m > 1$. In the single equation case, Mitidieri [9] proved that for $1 < q < \frac{N+4m}{N-4m}$, $N > 4m$, the problem

$$\begin{cases} \Delta^{2m} u = u^q \\ (-\Delta)^s u \geq 0, \ \ s = 1, 2 \cdots, 2m-1 \end{cases}$$

in $\mathbb{R}^N$ has no positive radial solution of class $C^{4m}\left(\mathbb{R}^N\right)$. For the system case, it is proved in [8, 23] that if $N \geq 3$, $N > 2m$, if $p \geq 1$, $q \geq 1$, $(p,q) \neq (1,1)$ satisfying

$$\frac{1}{p+1} + \frac{1}{q+1} > 1 - \frac{2m}{N}, \tag{1.1.3}$$

then system (1.1.1) has no positive radial solutions. For general solutions, the results in [8, 23] show that if $p, q \geq 1$, $(p,q) \neq (1,1)$ satisfies

$$\max\left(\frac{2m\,(p+1)}{pq-1}, \frac{2m\,(q+1)}{pq-1}\right) \geq N - 2m,$$

then system (1.1.1) admits no positive solutions. It is also proved in [8] that system (1.1.1) does not admit any positive solutions if

$$1 < p, q < \frac{N+2m}{N-2m}.$$

Under the additional assumptions $(-\Delta)^i u > 0, (-\Delta)^i v > 0$ for $i = 1, 2 \cdots, m-1$, Yan [23] proved system (1.1.1) admits no positive solutions if $pq \leq 1$. Most recently, Arthur, Yan and Zhao [1] proved there are no positive solutions for (1.1.1) if $p \geq 1$, $q \geq 1$, $pq > 1$ satisfies (1.1.3) when $N = 2m+1$, or $N = 2m+2$. They also proved the conjecture for same p, q under additional assumption $\max\left(\frac{2m(p+1)}{pq-1}, \frac{2m(q+1)}{pq-1}\right) > N - 2m - 1$, therefore generalized Souplet's result for $m \geq 1$.

1.2 Henon-Lane-Emden System ($a \neq 0$,or $b \neq 0$)

When $a \neq 0$ or $b \neq 0$, Liouville type theorem for (1.0.1) was first approached by Phan and Souplet [12]. Combining a measure and feedback argument with Pohozaev identity, they proved nonexistence of positive bounded solution to scalar Hénon equation

$$-\Delta u = |x|^a u^p \text{ in } \mathbb{R}^3$$

when $1 < p < 5 + 2a$ and $a > -2$, confirming the conjecture in the case $N = 3, m = 1, a = b > -2$ and $p = q > 1$. Another result confirming the conjecture in scalar case was proved by Cowan [3] where he showed nonexistence of positive bounded solutions for $m = 2$, $N = 5$ provided $1 < p < 9 + 2a$. Phan and Souplet's result was generalized to polyharmonic system (1.0.1) when $m = 1$ by Fazly and Ghoussoub [5] in dimension 3 and for $m \geq 1$ by Fazly [4] in dimension $N = 2m + 1$. Fazly also shows that (1.0.1) does not admit any positive solution (u, v) if $\max\left(\frac{2m(p+1)+a+bp}{pq-1}, \frac{2m(q+1)+aq+b}{pq-1}\right) > N - 2m$. In fact, it is pointed out in [11] that (1.0.1) does not admit any positive solution (u, v) if $\max\left(\frac{2m(p+1)+a+bp}{pq-1}, \frac{2m(q+1)+aq+b}{pq-1}\right) \geq N - 2m$ by a similar argument as in [15]. Moreover, the following theorems are proved by Phan when $m = 1$.

Theorem 1.2.1. (Theorem 1.1 [11]) Let $a, b > -2$ and $N \geq 3$. Assume $pq > 1$, $p \geq q$. Assume

$$\frac{N+a}{p+1} + \frac{N+b}{q+1} > N - 2. \tag{1.2.1}$$

Assume in addition that

$$0 \leq a - b \leq \frac{N-2}{p-q},$$

$$\max\left(\frac{2(p+1)}{pq-1}, \frac{2(q+1)}{pq-1}\right) > N - 3.$$

Then (1.0.1) with $m = 1$ has no positive solution.

Theorem 1.2.2. (Theorem 1.2 [11]) Let $a, b > -2$ and $N \geq 3$. Assume $pq > 1$, $p \geq q$. Assume (1.2.1)and

$$\frac{N}{p+1} + \frac{N}{q+1} > N - 2. \tag{1.2.2}$$

Assume in addition that

$$\max\left(\frac{2(p+1)}{pq-1}, \frac{2(q+1)}{pq-1}\right) > N - 3.$$

Then (1.0.1) with $m = 1$ has no positive solution.

For case $a < 0$, $b < 0$, Liouville type theorems for both integer and fractional Laplacian have been obtained in [19].

The thesis is organized as follows. Chapter 2 deals with $a = b = 0$. Chapter 3 discusses results for $a > 0$ or $b > 0$. Chapter 4 states some future directions.

Chapter 2

Liouville Theorem for Higher Order Lane-Emden System

In this chapter, we prove Liouville type theorem for higher order Lane-Emden System.

2.1 Preparations

When $pq > 1$, we introduce the following notation

$$\alpha = \frac{2\,(p+1)}{pq-1}, \quad \beta = \frac{2\,(q+1)}{pq-1}$$

and assume $\alpha \geq \beta$ throughout the rest of the paper. The assumption

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2m}{m}$$

can be rewritten as

$$m\alpha + m\beta > N - 2m.$$

For $w \in C\left(\mathbb{R}^N\right)$, we denote the spherical average of w by

$$\overline{w}\left(r\right) = \frac{1}{\omega_N} \int_{S^{N-1}} w\left(r, \theta\right) ds, \quad r > 0,$$

where ω_N is the area of the unit sphere S^{N-1}.

We quote the following growth estimates from [23].

Lemma 2.1. *(Lemma 3.3. [23]) If* $pq = 1$, *there is no nontrivial positive solution of* $(1.1.1)$. *If* (u, v) *is a positive solution of* $(1.1.1)$ *and* $p, q \geq 1$, *and* $pq > 1$, *there exists a positive constant* $M = M\left(p, q, n\right)$ *such that*

$$\overline{u}\left(r\right) \leq Mr^{-m\alpha}, \qquad \overline{v}\left(r\right) \leq Mr^{-m\beta} \quad \text{for} \quad r > 0. \tag{2.1.1}$$

and for $k = 1, \cdots, m-1$, $u_k = \left(-\Delta\right)^k u, v_k = \left(-\Delta\right)^k v$

$$\left(-\Delta\right)^i u > 0, \quad \left(-\Delta\right)^i v > 0, \; i = 1, 2, \cdots, m-1.$$

$$\overline{u_k}\left(r\right) \leq Mr^{-m\alpha-2k}, \qquad \overline{v_k}\left(r\right) \leq Mr^{-m\beta-2k} \quad \text{for} \quad r > 0. \tag{2.1.2}$$

Lemma 2.2. *(Lemma 3.4 [23]) Suppose that* $p, q \geq 1$ *and* (u, v) *is a positive solution of* $(1.1.1)$. *Then*

$$\int_{B_R} u^q \leq cR^{N-2m-m\beta}, \quad \int_{B_R} v^p \leq cR^{N-2m-m\alpha}, \tag{2.1.3}$$

where $c = c\left(p, q, n\right)$.

We state the following interpolation inequalities and elliptic estimates.

Lemma 2.3. *(L^p estimates on B_R) Given* $1 < k < \infty,\ R > 0,\ z \in W^{2m,k}(B_{2R})$, *then*

$$\int_{B_R} \left|D^{2m}z\right|^k \le C\left(\int_{B_{2R}} |\Delta^m z|^k + R^{-2mk}\int_{B_{2R}} |z|^k\right).$$

Proof. Lemma follows from standard elliptic L^p estimates for second order elliptic equations and interpolation inequalities. ■

Lemma 2.4. *For any* $R > 0,\ l = 1, 2, \cdots m-1,$

$$\int_{B_R} |\nabla_x u_l| \le CR\int_{B_{2R}} u_{l+1} + CR^{-1}\int_{B_{2R}} u_l.$$

Lemma 2.5. *(Sobolev inequality on S^{N-1})* $N \ge 2,\ j \ge 1,\ 1 < \mu < \lambda \le \infty.\ \mu \ne \frac{N-1}{j}$

$$\|w\|_\lambda \le C\left(\left\|D_\theta^j w\right\|_\mu + \|w_1\|\right)$$

here

$$\begin{aligned}\frac{1}{\mu} - \frac{1}{\lambda} &= \frac{j}{N-1} \quad \text{if} \quad \mu < \frac{N-1}{j}\\ \lambda &= \infty \quad \text{if}\ \mu > \frac{N-1}{j}.\end{aligned}$$

Lemma 2.6. *The following holds for* $l = 1, 2\cdots, m-1,\ k = \frac{p+1}{p},\ d = \frac{q+1}{q}$

$$\int_0^R \|u_l(r)\|_1 r^{N-1}dr \le Cr^{N-m\alpha-2l} \tag{2.1.4}$$

$$\int_0^R \|v_l(r)\|_1 r^{N-1}dr \le Cr^{N-m\beta-2l} \tag{2.1.5}$$

$$\int_0^R \|D_x u_l\|_1 r^{N-1} dr \le C r^{N-m\alpha-2l-1} \tag{2.1.6}$$

$$\int_0^R \|D_x v_l\|_1 r^{N-1} dr \le C r^{N-m\beta-2l-1} \tag{2.1.7}$$

$$\int_0^R \left\|D_x^{2m} u\right\|_k^k r^{N-1} dr \le CF(2R) \tag{2.1.8}$$

$$\int_0^R \left\|D_x^{2m} v\right\|_d^d r^{N-1} dr \le CF(2R) \tag{2.1.9}$$

$$\int_0^R \left\|D_x^{2m} u\right\|_{1+\epsilon}^{1+\epsilon} r^{N-1} dr \le CR^{N-2m-m\alpha} \tag{2.1.10}$$

$$\int_0^R \left\|D_x^{2m} v\right\|_{1+\epsilon}^{1+\epsilon} r^{N-1} dr \le CR^{N-2m-m\beta} \tag{2.1.11}$$

Here

$$F(R) = \int_{B_R} \left[v^{p+1} + u^{q+1}\right] dx.$$

Proof. $(2.1.4), (2.1.5)$ are restatements of Lemma 2.1. $(2.1.6)$ and $(2.1.7)$ follows directly from Lemma 2.1 and Lemma 2.4. To prove $(2.1.8)$, Lemma 2.3 implies

$$\begin{aligned}
\int_0^R \left\|D_x^{2m} u\right\|_k^k r^{N-1} dr &= \int_{B_R} \left|D^{2m} u\right|^k \\
&\le C\left(\int_{B_{2R}} \left|\Delta^m u\right|^k + R^{-2mk} \int_{B_{2R}} u^k\right) \\
&= C\int_{B_{2R}} v^{p+1} + R^{-2mk} \int_{B_{2R}} u^k
\end{aligned}$$

By Hölder's inequality and the fact that $k = \frac{p+1}{p} < q+1$, $F(R) \geq F(1) > 0$, $R \geq 1$

$$\begin{aligned} R^{-2mk} \int_{B_{2R}} u^k &\leq CR^{-2mk} R^{N\frac{pq-1}{p(q+1)}} \left(\int_{B_{2R}} u^{q+1} \right)^{\frac{p+1}{p(q+1)}} \\ &\leq CR^{-\frac{\chi}{p}} F(2R) \end{aligned}$$

where

$$\chi = 2m(p+1) - N\frac{pq-1}{q+1}.$$

Since

$$\frac{1}{p+1} + \frac{1}{q+1} > 1 - \frac{2m}{N},$$

we have $\chi > 0$, and (2.1.8) follows. (2.1.9) is proved similarly. Lastly we proved (2.1.10)

$$\begin{aligned} \int_0^R \left\| D_x^{2m} u \right\|_{1+\epsilon}^{1+\epsilon} r^{N-1} dr &\leq C \left(\int_{B_{2R}} |\Delta^m u|^{1+\varepsilon} + R^{-2m(1+\varepsilon)} \int_{B_{2R}} u^{1+\varepsilon} \right) \\ &= C \left(\int_{B_{2R}} v^{p(1+\varepsilon)} + R^{-2m(1+\varepsilon)} \int_{B_{2R}} u^{1+\varepsilon} \right) \\ &\leq C \left(\int_{B_{2R}} v^p + R^{-2m(1+\varepsilon)} \int_{B_{2R}} u \right) \\ &\leq C \left(R^{N-mp\beta} + R^{-2m(1+\varepsilon)} \cdot R^{N-m\alpha} \right) \\ &\leq CR^{N-mp\beta} \end{aligned}$$

Here we used the boundedness of u and v in the second inequality and the fact that

$$\alpha + 2 = p\beta.$$

(2.1.11) is proved similarly using $\beta + 2 = q\alpha$. ∎

In the rest of the section, we prove a Rellich-Pohozahav identity.

We recall the following function defined in [9]

$$R_n(u,v) = \int_\Omega \Delta^n u\,(x, \nabla v) + \Delta^n v\,(x, \nabla u)\,dx$$

where $\Omega \subset \mathbb{R}^N$, $u, v \in C^{2n}\left(\overline{\Omega}\right), n \geq 1$. If $n = 1$, we have

$$\begin{aligned} R_1(u,v) &= \int_{\partial\Omega} \left\{ \frac{\partial u}{\partial n}(x, \nabla v) + \frac{\partial v}{\partial n}(x, \nabla u) - (\nabla u, \nabla v)(x, n) \right\} ds \\ &\quad + (N-2) \int_\Omega (\nabla u, \nabla v)\,dx. \end{aligned}$$

If $n = 2$,

$$R_2(u,v) = R_1(\Delta u, v) + R_1(u, \Delta v) - B(u,v) \tag{2.1.12}$$

where

$$B(u,v) = \int_{\partial\Omega} \Delta u \Delta v\,(x, n)\,ds - N \int_\Omega \Delta u \Delta v dx. \tag{2.1.13}$$

We quote the following Lemma from [9]

Lemma 2.7. *(Lemma 2.2 in [9]) If $u, v \in C^{2n}\left(\overline{\Omega}\right)$, then for $1 \leq s \leq n-2$*

$$R_n(u,v) = \sum_{k=0}^{s} R_{n-s}\left(\Delta^k u, \Delta^{s-k} v\right) - \sum_{k=0}^{s-1} R_{n-(s+1)}\left(\Delta^{k+1} u, \Delta^{s-k} v\right). \tag{2.1.14}$$

Remark 2.8. An immediate consequence of Lemma 2.7 is the following implicit form of Rellich's identity. If $u, v \in C^{2n}\left(\overline{\Omega}\right)$, then

$$R_n(u,v) = \sum_{k=0}^{n-1} R_1\left(\Delta^k u, \Delta^{n-1-k} v\right) - \sum_{k=0}^{n-2} B\left(\Delta^k u, \Delta^{n-2-k} v\right) \tag{2.1.15}$$

Proof. Choose $s = n-2$ in (2.1.14) , taking into account of (2.1.12) and (2.1.13) , (2.1.15) follows. ■

Write

$$u^{q+1}(r) = \int_{S^{N-1}} u^{q+1}(r,\theta)\, d\theta, \quad v^{p+1}(r) = \int_{S^{N-1}} v^{p+1}(r,\theta)\, d\theta,$$

we have the following Rellich-Pohozav identity.

Lemma 2.9. *For any $a_1 + a_2 = N - 2m$, $r > 0$*

$$\begin{aligned}
& \left(\frac{N}{p+1} - a_1\right) \int_{B_r} v^{p+1}(x)\, dx + \left(\frac{N}{q+1} - a_2\right) \int_{B_r} u^{q+1}(x)\, dx \\
= \; & \frac{1}{p+1} v^{p+1}(r)\, r^N + \frac{1}{q+1} u^{q+1}(r)\, r^N \\
& - (-1)^m \{ \sum_{k=0}^{m-1} 2r^N \int_{S^{N-1}} \frac{\partial \Delta^k u}{\partial n} \cdot \frac{\partial \Delta^{m-1-k} v}{\partial n} + \frac{\partial \Delta^{m-1-k} v}{\partial n} \cdot \frac{\partial \Delta^k u}{\partial n} ds \\
& - \sum_{k=0}^{m-1} r^N \int_{S^{N-1}} \left(\nabla \Delta^k u, \nabla \Delta^{m-1-k} v\right) ds - \sum_{k=0}^{m-2} r^N \int_{S^{N-1}} \left(\Delta^{k+1} u, \Delta^{m-1-k} v\right) ds \\
& + \sum_{k=0}^{m-1} (2m - 2k - 2 + a_1)\, r^{N-1} \int_{S^{N-1}} \frac{\partial \Delta^k u}{\partial n} \Delta^{m-1-k} v ds \\
& + \sum_{k=0}^{m-1} (a_2 + 2k)\, r^{N-1} \int_{S^{N-1}} \frac{\partial \Delta^{m-1-k} v}{\partial n} \Delta^k u \; ds \}.
\end{aligned}$$

Proof. By (1.1.1) ,

$$\begin{aligned}
(-1)^m R_m(u,v) &= \int_{B_r} (-\Delta)^m u(x)(x,\nabla v) + (-\Delta)^m v(x,\nabla u)\,dx \\
&= \int_{B_r} v^p(x)(x,\nabla v) + u^q(x)(x,\nabla u)\,dx \\
&= \int_{\partial B_r} \frac{v^{p+1}}{p+1}(x,n) + \frac{u^{q+1}}{q+1}(x,n)\,ds \\
&\quad -\frac{N}{p+1}\int_{B_r} v^{p+1}(x)\,dx - \frac{N}{q+1}\int_{B_r} u^{q+1}(x)\,dx \\
&= \frac{1}{p+1}v^{p+1}(r)\,r^N + \frac{1}{q+1}u^{q+1}(r)\,r^N \\
&\quad -\frac{N}{p+1}\int_{B_r} v^{p+1}(x)\,dx - \frac{N}{q+1}\int_{B_r} u^{q+1}(x)\,dx.
\end{aligned}$$

By (2.1.15) , we have

$$\begin{aligned}
R_m(u,v) &= \sum_{k=0}^{m-1} R_1\left(\Delta^k u, \Delta^{m-1-k} v\right) - \sum_{k=0}^{m-2} B\left(\Delta^k u, \Delta^{m-2-k} v\right) \\
&= \sum_{k=0}^{m-1} \int_{\partial B_r} \frac{\partial \Delta^k u}{\partial n}\left(x, \nabla\Delta^{m-1-k} v\right) + \frac{\partial \Delta^{m-1-k} v}{\partial n}\left(x, \nabla \Delta^k u\right) dx \\
&\quad - \sum_{k=0}^{m-1} \int_{\partial B_r} \left(\nabla\Delta^k u, \nabla\Delta^{m-1-k} v\right)(x,n)\,ds \\
&\quad + (N-2)\sum_{k=0}^{m-1} \int_{B_r} \left(\nabla\Delta^k u, \nabla\Delta^{m-1-k} v\right) dx \\
&\quad - \sum_{k=0}^{m-2} \int_{\partial B_r} \left(\Delta^{k+1} u, \Delta^{m-1-k} v\right)(x,n)\,ds \\
&\quad + N\sum_{k=0}^{m-2} \int_{B_r} \left(\Delta^{k+1} u, \Delta^{m-1-k} v\right) dx. \qquad (2.1.16)
\end{aligned}$$

Since

$$\int_{B_r} \left(\nabla \Delta^k u, \nabla \Delta^{m-1-k} v\right) dx = \int_{\partial B_r} \frac{\partial \Delta^k u}{\partial n} \Delta^{m-1-k} v ds - \int_{B_r} \left(\Delta^{k+1} u, \Delta^{m-1-k} v\right) dx,$$

we can rewrite (2.1.16) as

$$\begin{aligned} R_m(u,v) \;=\; & \sum_{k=0}^{m-1} \int_{\partial B_r} \frac{\partial \Delta^k u}{\partial n} \left(x, \nabla \Delta^{m-1-k} v\right) + \frac{\partial \Delta^{m-1-k} v}{\partial n} \left(x, \nabla \Delta^k u\right) ds \\ & - \sum_{k=0}^{m-1} \int_{\partial B_r} \left(\nabla \Delta^k u, \nabla \Delta^{m-1-k} v\right)(x,n)\, ds \\ & - \sum_{k=0}^{m-2} \int_{\partial B_r} \left(\Delta^{k+1} u, \Delta^{m-1-k} v\right)(x,n)\, ds \\ & + (N-2) \sum_{k=0}^{m-1} \int_{\partial B_r} \frac{\partial \Delta^k u}{\partial n} \Delta^{m-1-k} v ds \\ & - (N-2) \sum_{k=0}^{m-1} \int_{B_r} \left(\Delta^{k+1} u, \Delta^{m-1-k} v\right) dx \\ & + N \sum_{k=0}^{m-2} \int_{B_r} \left(\Delta^{k+1} u, \Delta^{m-1-k} v\right) dx \end{aligned}$$

$$\begin{aligned} = \; & \sum_{k=0}^{m-1} \int_{\partial B_r} \frac{\partial \Delta^k u}{\partial n} \left(x, \nabla \Delta^{m-1-k} v\right) + \frac{\partial \Delta^{m-1-k} v}{\partial n} \left(x, \nabla \Delta^k u\right) ds \\ & - \sum_{k=0}^{m-1} \int_{\partial B_r} \left(\nabla \Delta^k u, \nabla \Delta^{m-1-k} v\right)(x,n)\, ds \\ & - \sum_{k=0}^{m-2} \int_{\partial B_r} \left(\Delta^{k+1} u, \Delta^{m-1-k} v\right)(x,n)\, ds + (N-2) \sum_{k=0}^{m-1} \int_{\partial B_r} \frac{\partial \Delta^k u}{\partial n} \Delta^{m-1-k} v ds \\ & - N \int_{B_r} \left(\Delta^m u, v\right) dx + 2 \sum_{k=0}^{m-1} \int_{B_r} \left(\Delta^{k+1} u, \Delta^{m-1-k} v\right) dx. \end{aligned} \tag{2.1.17}$$

Recall

$$\begin{aligned}&\int_{B_r}\left(\Delta^{k+1}u,\Delta^{m-1-k}v\right)dx-\int_{B_r}\left(\Delta^{k}u,\Delta^{m-k}v\right)dx\\ =\;&\int_{\partial B_r}[\frac{\partial\Delta^k u}{\partial n}\Delta^{m-1-k}v-\frac{\partial\Delta^{m-1-k}v}{\partial n}\Delta^k u]ds\end{aligned}$$

it then follows

$$\begin{aligned}\sum_{k=0}^{m-1}\int_{B_r}\left(\Delta^{k+1}u,\Delta^{m-1-k}v\right)dx &= m\int_{B_r}(u,\Delta^m v)+\sum_{k=0}^{m-1}\sum_{l=0}^{k}\int_{\partial B_r}\frac{\partial\Delta^l u}{\partial n}\Delta^{m-1-l}vds\\ &\quad-\sum_{k=0}^{m-1}\sum_{l=0}^{k}\int_{\partial B_r}\frac{\partial\Delta^{m-1-l}v}{\partial n}\Delta^l uds \qquad (2.1.18)\end{aligned}$$

and

$$\begin{aligned}\int_{B_r}(\Delta^m u,v)\,dx &= \int_{B_r}(u,\Delta^m v)+\sum_{l=0}^{m-1}\int_{\partial B_r}[\frac{\partial\Delta^l u}{\partial n}\Delta^{m-1-l}v\\ &\quad-\frac{\partial\Delta^{m-1-l}v}{\partial n}\Delta^l u]ds \qquad (2.1.19)\end{aligned}$$

From (2.1.18) and (2.1.19) we deduce

$$
\begin{aligned}
(N-2)\sum_{k=0}^{m-1}\int_{\partial B_r}\frac{\partial\Delta^k u}{\partial n}\Delta^{m-1-k}v ds - N\int_{B_r}(\Delta^m u, v)\,dx \\
+2\sum_{k=0}^{m-1}\int_{B_r}\left(\Delta^{k+1}u, \Delta^{m-1-k}v\right)dx = (2m-N)\int_{B_r}(u,\Delta^m v) \\
+2\sum_{k=0}^{m-1}\sum_{l=0}^{k}\int_{\partial B_r}\frac{\partial\Delta^l u}{\partial n}\Delta^{m-1-l}v - \frac{\partial\Delta^{m-1-l}v}{\partial n}\Delta^l u ds \\
+(N-2)\sum_{k=0}^{m-1}\int_{\partial B_r}\frac{\partial\Delta^k u}{\partial n}\Delta^{m-1-k}v ds - N\sum_{l=0}^{m-1}\int_{\partial B_r}[\frac{\partial\Delta^l u}{\partial n}\Delta^{m-1-l}v \\
-\frac{\partial\Delta^{m-1-l}v}{\partial n}\Delta^l u]ds = (2m-N)\int_{B_r}(u,\Delta^m v) \\
+2\sum_{l=0}^{m-1}(m-l)\int_{\partial B_r}[\frac{\partial\Delta^l u}{\partial n}\Delta^{m-1-l}v - \frac{\partial\Delta^{m-1-l}v}{\partial n}\Delta^l u]ds \\
-2\sum_{k=0}^{m-1}\int_{\partial B_r}\frac{\partial\Delta^k u}{\partial n}\Delta^{m-1-k}v ds + N\sum_{l=0}^{m-1}\int_{\partial B_r}\frac{\partial\Delta^{m-1-l}v}{\partial n}\Delta^l u ds \\
= -a_1\int_{B_r}(\Delta^m u, v) - a_2\int_{B_r}(u,\Delta^m v) \\
+a_1\sum_{l=0}^{m-1}\int_{\partial B_r}[\frac{\partial\Delta^l u}{\partial n}\Delta^{m-1-l}v - \frac{\partial\Delta^{m-1-l}v}{\partial n}\Delta^l u]ds \\
+2\sum_{l=0}^{m-1}(m-l)\int_{\partial B_r}[\frac{\partial\Delta^l u}{\partial n}\Delta^{m-1-l}v - \frac{\partial\Delta^{m-1-l}v}{\partial n}\Delta^l u]ds \\
-2\sum_{k=0}^{m-1}\int_{\partial B_r}\frac{\partial\Delta^k u}{\partial n}\Delta^{m-1-k}v ds + N\sum_{l=0}^{m-1}\int_{\partial B_r}\frac{\partial\Delta^{m-1-l}v}{\partial n}\Delta^l u ds \\
= -a_1\int_{B_r}(\Delta^m u, v) - a_2\int_{B_r}(u,\Delta^m v) \\
+\sum_{l=0}^{m-1}(2m-2l-2+a_1)\int_{\partial B_r}\frac{\partial\Delta^l u}{\partial n}\Delta^{m-1-l}v ds \\
+\sum_{l=0}^{m-2}(N-2m+2l-a_1)\int_{\partial B_r}\frac{\partial\Delta^{m-1-l}v}{\partial n}\Delta^l u ds
\end{aligned}
\tag{2.1.20}
$$

$$= -a_1 \int_{B_r} (\Delta^m u, v) - a_2 \int_{B_r} (u, \Delta^m v)$$
$$+ \sum_{l=0}^{m-1} (2m - 2l - 2 + a_1) \int_{\partial B_r} \frac{\partial \Delta^l u}{\partial n} \Delta^{m-1-l} v ds$$
$$+ \sum_{l=0}^{m-1} (a_2 + 2l) \int_{\partial B_r} \frac{\partial \Delta^{m-1-l} v}{\partial n} \Delta^l u ds, \tag{2.1.21}$$

■

conclusion follows from (2.1.17) and (2.1.21).

2.2 Main Theorem

The main theorem we prove is as follows.

Theorem 2.2.1. $N \geq 3$, $N > 2m$, if $p \geq 1$, $q \geq 1, (p, q) \neq (1, 1)$ satisfies

$$\frac{N}{p+1} + \frac{N}{q+1} > N - 2m \tag{2.2.1}$$

and

$$\max\left(\frac{2m(p+1)}{pq-1}, \frac{2m(q+1)}{pq-1}\right) > N - 2m - 1,$$

then (1.1.1) has no positive solutions of class $C^{2m}\left(\mathbb{R}^N\right)$. Moreover, when $N = 2m+1$ or $N = 2m+2$, if $p \geq 1, q \geq 1$, $(p, q) \neq (1, 1)$ satisfies (3.2.3), then (1.1.1) admits no positive solutions.

To prove this theorem we do it in two steps.

Step 1: Reduce the problem to bounded solution by showing that if (1.1.1) does not admit bounded positive solution then it does not admit any positive solution.

Step 2: We show that (1.1.1) does not admit bounded positive solutions.

2.2.1 Reduction to bounded solutions

More precisely, we prove the following Theorems regarding bounded solutions.

Theorem 2.2.2. Let $N \geq 3$, $p > 1$, $q > 1$ be fixed, and assume (1.1.1) does not admit any bounded nontrivial (nonnegative) solution in $\mathbb{R}^N$, then it does not admit any nontrivial (nonnegative) solution in $\mathbb{R}^N$, bounded or not. In particular, the conclusion holds if $\max\left(\frac{2m(p+1)}{pq-1}, \frac{2m(q+1)}{pq-1}\right) \geq N - 2m$ or if $1 < p, q < \frac{N+2m}{N-2m}$.

Theorem 2.2.3. Let $p, q > 1$. Assume (1.1.1) does not admit any bounded nontrival (nonnegative) solution in $\mathbb{R}^N$. Let $\Omega \neq \mathbb{R}^N$ be a domain of $\mathbb{R}^N$. Then there exists $C = C(N, p, q, m) > 0$ (independent of Ω and (u, v)) such that any (nonnegative) solution (u, v) of (1.1.1) in Ω satisfies

$$u(x) \leq C \operatorname{dist}^{-m\alpha}(x, \partial\Omega), \quad x \in \Omega,$$

and

$$v(x) \leq C \operatorname{dist}^{-m\beta}(x, \partial\Omega), \quad x \in \Omega.$$

If Ω is exterior domain, that is $\Omega \supset \{x \in \mathbb{R}^N : |x| > R\}$ for some $R > 0$, then it follows that

$$u(x) \leq C |x|^{-m\alpha}, \ \ |x| \geq 2R,$$

and

$$v(x) \leq C |x|^{-m\beta}, \ \ |x| \geq 2R.$$

In particular, the above conclusions hold if $\max\left(\frac{2m(p+1)}{pq-1}, \frac{2m(q+1)}{pq-1}\right) \geq N - 2m$ or if $1 < p, q < \frac{N+2m}{N-2m}$.

Proof of Theorems 2.2.2 and 2.2.3 uses idea of [13] in the case of $m = 1$, which relies on the following Doubling property Lemma and remark.

Lemma 2.10. *(Lemma 5.1 [13])Let (X,d) be a complete metric space, and let* $\emptyset \neq D \subset \Sigma \subset X$ *with* Σ *closed. Set* $\Gamma = \Sigma \backslash D$*. Finally, let* $M : D \to (0, \infty)$ *be bounded on compact subsets of* D*, and fix a real* $k > 0$*. If* $y \in D$ *is such that*

$$M(y)\, dist(y, \Gamma) > 2k$$

then there exists $x \in D$ *such that*

$$M(x)\, dist(x, \Gamma) > 2k, \quad M(x) \geq M(y),$$

and

$$M(z) \leq 2M(x) \quad \text{for all } z \in D \cap \overline{B_X}(x, kM^{-1}(x)).$$

Remark 2.11. (Remark 5.2 [13]).

(a) If $\Gamma = \emptyset$, then $\text{dist}(x, \Gamma) = \infty$.

(b) Take $X = \mathbb{R}^n$, take Ω an open subset of $\mathbb{R}^n$, put $D = \Omega$, $\Sigma = \overline{D}$; hence $\Gamma = \partial\Omega$. Then we have $\overline{B}(x, kM^{-1}(x)) \subset D$. Indeed, since D is open, implies $\text{dist}(x, D^c) = \text{dist}(x, \Gamma) > 2kM^{-1}(x)$.

Proof of Theorem 2.2.3. Assume the theorem fails. Then there exist sequences

Ω_k, (u_k, v_k), $y_k \in \Omega_k$ such that (u_k, v_k) solves (1.1.1) on Ω_k and

$$M_k := u_k^{\frac{1}{m\alpha}} + v_k^{\frac{1}{m\beta}}, \quad k = 1, 2, \cdots$$

satisfies

$$M_k(y_k) > 2k\text{dist}^{-1}(y_k, \partial\Omega_k).$$

By Lemma 2.10 and Remark 2.11, it follows that there exists $x_k \in \Omega_k$ such that

$$M_k(x_k) > 2k\text{dist}^{-1}(x_k, \partial\Omega_k)$$

and

$$M_k(z) \leq 2M_k(x_k), \qquad |z - x_k| \leq kM_k^{-1}(x_k).$$

Define rescaling of (u_k, v_k) as follows

$$\begin{aligned} \lambda_k &= M_k^{-1}(x_k) \\ \widetilde{u}_k(y) &= \lambda_k^{m\alpha} u_k(x_k + \lambda_k y), \quad \widetilde{v}_k(y) = \lambda_k^{m\beta} v_k(x_k + \lambda_k y), \quad |y| \leq k. \end{aligned}$$

Since $\alpha + 2 = p\beta$, $\beta + 2 = q\alpha$, $(\widetilde{u}_k, \widetilde{v}_k)$ satisfies

$$\begin{aligned} (-\Delta_y)^m \widetilde{u}_k(y) &= \widetilde{v}_k^p(y) \\ (-\Delta_y)^m \widetilde{v}_k(y) &= \widetilde{u}_k^q(y) \end{aligned}$$

for $|y| \leq k$. Moreover,

$$\widetilde{u}_k^{\frac{1}{m\alpha}}(0) + \widetilde{v}_k^{\frac{1}{m\beta}}(0) = 1$$

and

$$\widetilde{u}_k^{\frac{1}{m\alpha}}(y) + \widetilde{v}_k^{\frac{1}{m\beta}}(y) \le 2, \quad |y| \le k.$$

By standard elliptic L^p estimates and Sobolev embeddings, we conclude that subject to a subsequence, $(\widetilde{u}_k, \widetilde{v}_k)$ converges in $C^{2m}_{loc}(\mathbb{R}^N)$ to a (classical) solution $(\widetilde{u}, \widetilde{v})$ of (1.1.1) in $\mathbb{R}^n$. Moreover, $\widetilde{u}^{\frac{1}{m\alpha}}(0) + \widetilde{v}^{\frac{1}{m\beta}}(0) = 1$ and $\widetilde{u}^{\frac{1}{m\alpha}}(y) + \widetilde{v}^{\frac{1}{m\beta}}(y) \le 2$. i.e. $(\widetilde{u}, \widetilde{v})$ is nontrivial and bounded, contradicts to the assumptions of the theorem. In particular, if $\max\left(\frac{2m(p+1)}{pq-1}, \frac{2m(q+1)}{pq-1}\right) \ge N - 2m$ or if $1 < p, q < \frac{N+2m}{N-2m}$, Liouville theorems in [8] and [23] implies the assumptions in the theorem hold.

Proof of Theorem 2.2.2.

Assume (u, v) is a solution of (1.1.1) on $\mathbb{R}^N$ (bounded or not). Then for each $x_0 \in \mathbb{R}^N$ and $R > 0$, by applying Theorem 2.2.3 in $\Omega = B(x_0, R)$, we obtain

$$u(x_0) \le CR^{-m\alpha}, \quad v(x_0) \le CR^{-m\beta}.$$

Letting $R \to \infty$, we obtain

$$u(x_0) = v(x_0) = 0,$$

therefore

$$u \equiv v \equiv 0.$$

2.2.2 Nonexistence of bounded positive solutions

We prove system does not admit any bounded positive solutions.

We shall adapt Souplet's idea of a measure and feedback argument combined with

Rellich-Pohazahav identity [17].

$$F\left(R\right)=\int_{B_R(0)}v^{p+1}+u^{q+1}dx$$

Lemma 2.9 implies

$$F\left(R\right)\le CG_1\left(R\right)+CG_2\left(R\right)$$

where

$$G_1\left(R\right)=R^N\sum_{l=0}^{m}\int_{S^{N-1}}\left|u_l\right|\left|v_{m-l}\right|ds$$

and

$$G_2\left(R\right)=R^N\int_{S^{N-1}}\sum_{l=0}^{m-1}\left(\left|u'_{m-l-1}\right|+R^{-1}\left|u_{m-l-1}\right|\right)\left(\left|v'_l\right|+R^{-1}\left|v_l\right|\right)ds$$

Following Souplet's idea, we shall prove there exists constants C, $a>0$, $b<1$ such that

$$F\left(R\right)\le CR^{-a}F^b\left(R\right). \tag{2.2.2}$$

It then follows

$$F\left(R\right)\to 0 \quad \text{as} \quad R\to\infty,$$

which implies

$$u=v\equiv 0.$$

To prove (2.2.2) , we follow a similar procedure as [17]. We shall first estimate $G_1\left(R\right)$ and $G_2\left(R\right)$ in terms of highest derivatives of the solution (u,v) in suitable L^p spaces. Then use a feedback and measure argument to evaluate those bounds in terms of $F\left(R\right)$.

Step1. *Estimation of* $G_1\left(R\right)$ *in terms of suitable norms of* $D_x^{2m}u\left(R\right)$ *and*

$D_x^{2m} v\,(R)\,.$

Fix $l \in \{0, 1, \cdots m, \}$,

$$\begin{aligned} & \int_{S^{N-1}} |u_l|\,|v_{m-l}|\,ds \\ \leq\ & \|u_l\|_{\alpha_l}\,\|v_{m-l}\|_{\alpha_l'}\,, \end{aligned}$$

where $\frac{1}{\alpha_l} + \frac{1}{\alpha_l'} = 1$ is chosen so that

$$\begin{aligned} \frac{p}{p+1} - \frac{2m-2l}{N-1} &\leq \frac{1}{\alpha_l} \leq 1 - \frac{2m-2l}{N-1} \\ \frac{q}{q+1} - \frac{2l}{N-1} &\leq 1 - \frac{1}{\alpha_l} \leq 1 - \frac{2l}{N-1}. \end{aligned} \tag{2.2.3}$$

Such α_l exists since by assumption,

$$\frac{1}{p+1} + \frac{1}{q+1} > 1 - \frac{2m}{N} > 1 - \frac{2m}{N-1}.$$

Let

$$\begin{aligned} \frac{1}{\gamma_l} &= \frac{p}{p+1} - \frac{2m-2l}{N-1}, \frac{1}{\delta_l} = \frac{N-2m+2l-1}{N-1}, \\ \frac{1}{\omega_l} &= \frac{q}{q+1} - \frac{2l}{N-1}, \frac{1}{\psi_l} = \frac{N-2l-1}{N-1}. \end{aligned}$$

Case I: $\gamma_l > 0$, $\omega_l > 0$. By Hölder's inequality, we have

$$\begin{aligned} \|u_l\|_{\alpha_l} &\leq \|u_l\|_{\delta_l}^{\nu_{1l}}\,\|u_l\|_{\gamma_l}^{1-\nu_{1l}} \\ \|v_{m-l}\|_{\alpha_l'} &\leq \|v_{m-l}\|_{\upsilon_l}^{\nu_{2l}}\,\|v_{m-l}\|_{\omega_l}^{1-\nu_{2l}} \end{aligned} \tag{2.2.4}$$

with

$$\begin{aligned} \frac{1}{\alpha_l} &= \frac{\nu_{1l}}{\delta_l} + \frac{1-\nu_{1l}}{\gamma_l}, \\ \frac{1}{\alpha_l'} &= \frac{\nu_{2l}}{\psi_l} + \frac{1-\nu_{2l}}{\omega_l} \end{aligned}$$

Applying Lemma 2.5, we deduce

$$\begin{aligned} \|u_l\|_{\delta_l} &\leq C\left(\left\|D_\theta^{2m-2l}u_l\right\|_{1+\varepsilon} + \|u_l\|_1\right) \\ &\leq C\left(R^{2m-2l}\left\|D_x^{2m-2l}u_l\right\|_{1+\varepsilon} + \|u_l\|_1\right), \end{aligned} \tag{2.2.5}$$

$$\begin{aligned} \|u_l\|_{\gamma_l} &\leq C\left(\left\|D_\theta^{2m-2l}u_l\right\|_k + \|u_l\|_1\right) \\ &\leq C\left(R^{2m-2l}\left\|D_x^{2m-2l}u_l\right\|_k + \|u_l\|_1\right), \end{aligned} \tag{2.2.6}$$

and

$$\begin{aligned} \|v_{m-l}\|_{\psi_l} &\leq C\left(\left\|D_\theta^{2l}v_{m-l}\right\|_{1+\varepsilon} + \|v_{m-l}\|_1\right) \\ &\leq C\left(R^{2l}\left\|D_x^{2l}v_{m-l}\right\|_{1+\varepsilon} + \|v_{m-l}\|_1\right) \end{aligned} \tag{2.2.7}$$

$$\begin{aligned} \|v_{m-l}\|_{\psi_l} &\leq C\left(\left\|D_\theta^{2l}v_{m-l}\right\|_d + \|v_{m-l}\|_1\right) \\ &\leq C\left(R^{2l}\left\|D_x^{2l}v_{m-l}\right\|_d + \|v_{m-l}\|_1\right) \end{aligned} \tag{2.2.8}$$

Combining (2.2.4) , (2.2.5) , (2.2.6) , (2.2.7) and (2.2.8) , we conclude

$$\begin{aligned}
&\int_{S^{N-1}} |u_l|\,|v_{m-l}|\,ds \\
\leq\ & \|u_{l+1}\|_{\alpha_l} \|v_{m-l}\|_{\alpha_l'} \\
\leq\ & CR^{2m}\left(\left\|D_x^{2m-2l}u_l\right\|_{1+\varepsilon} + R^{-2m+2l}\|u_l\|_1\right)^{\nu_{1l}} \\
& \cdot\left(\left\|D_x^{2m-2l}u_l\right\|_k + R^{-2m+2l}\|u_l\|_1\right)^{1-\nu_{1l}} \\
& \cdot\left(\left\|D_x^{2l}v_{m-l}\right\|_{1+\varepsilon} + R^{-2l}\|v_{m-l}\|_1\right)^{\nu_{2l}} \\
& \cdot\left(\left\|D_x^{2l}v_{m-l}\right\|_d + R^{-2l}\|v_{m-l}\|_1\right)^{1-\nu_{2l}} \qquad (2.2.9)
\end{aligned}$$

Case II: Either $\gamma_l \leq 0$ or $\omega_l \leq 0$ but not both. We can take $\nu_{1l} = 1$ (if $\gamma_l \leq 0$) or $\nu_{2l} = 1$ (if $\gamma_l \leq 0$), it is easy to see that (2.2.9) still follows.

Case III: Both $\gamma_l \leq 0$ and $\omega_l \leq 0$. This is equivalent to

$$\frac{1}{p+1} > 1 - \frac{2m-2l}{N-1}$$

and

$$\frac{1}{q+1} > 1 - \frac{2l}{N-1}$$

which gives

$$\frac{1}{p+1} + \frac{1}{q+1} > 2 - \frac{2m}{N-1}$$

Contradiction to $p \geq 1$, $q \geq 1$ and $N \geq 2m+1$.

From (2.2.9) we obtain the following upper bound on $G_1(R)$.

$$\begin{aligned} G_1(R) \leq \quad & CR^{N+2m} \sum_{l=0}^{m} \{ \left(\left\| D_x^{2m-2l} u_l \right\|_{1+\varepsilon} + R^{-2m+2l} \left\| u_l \right\|_1 \right)^{\nu_{1l}} \\ & \cdot \left(\left\| D_x^{2m-2l} u_l \right\|_k + R^{-2m+2l} \left\| u_l \right\|_1 \right)^{1-\nu_{1l}} \\ & \cdot \left(\left\| D_x^{2l} v_{m-l} \right\|_{1+\varepsilon} + R^{-2l} \left\| v_{m-l} \right\|_1 \right)^{\nu_{2l}} \\ & \cdot \left(\left\| D_x^{2l} v_{m-l} \right\|_d + R^{-2l} \left\| v_{m-l} \right\|_1 \right)^{1-\nu_{2l}} \} \end{aligned} \tag{2.2.10}$$

Step 2. *Estimation of* $G_2(R)$ *in terms of sutiable norms of* $D_x^{2m} u(R)$ *and* $D_x^{2m} v(R)$.

Fix $l \in \{0, 1, 2, \cdots, m-1\}$. For $\frac{1}{\beta_l} + \frac{1}{\beta_l'} = 1$

$$\begin{aligned} & \int_{S^{N-1}} \left(\left| u_{m-l-1}' \right| + R^{-1} \left| u_{m-l-1} \right| \right) \left(\left| v_l' \right| + R^{-1} \left| v_l \right| \right) \\ \leq \quad & \left(\left\| u_{m-l-1}' \right\|_{\beta_l} + R^{-1} \left\| u_{m-l-1} \right\|_{\beta_l} \right) \left(\left\| v_l' \right\|_{\beta_l'} + R^{-1} \left\| v_l \right\|_{\beta_l'} \right). \end{aligned} \tag{2.2.11}$$

By Lemma 2.5,

$$\begin{aligned} R^{-1} \left\| u_{m-l-1} \right\|_{\beta_l} & \leq CR^{-1} \left(\left\| D_\theta u_{m-l-1} \right\|_{\beta_l} + \left\| u_{m-l-1} \right\|_1 \right) \\ & \leq C \left(\left\| D_x u_{m-l-1} \right\|_{\beta_l} + R^{-1} \left\| u_{m-l-1} \right\|_1 \right), \end{aligned} \tag{2.2.12}$$

$$\begin{aligned} R^{-1} \left\| v_l \right\|_{\beta_l'} & \leq CR^{-1} \left(\left\| D_\theta v_l \right\|_{\beta_l'} + \left\| v_l \right\|_1 \right) \\ & \leq C \left(\left\| D_x v_l \right\|_{\beta_l'} + R^{-1} \left\| v_l \right\|_1 \right). \end{aligned} \tag{2.2.13}$$

By Lemma 2.5, for $\frac{1}{\rho_l} = \frac{p}{p+1} - \frac{2l+1}{N-1}$

$$\begin{aligned}\|D_x u_{m-l-1}\|_{\rho_l} &\le C\left(\left\|D_\theta^{2l+1} D_x u_{m-l-1}\right\|_k + \|D_x u_{m-l-1}\|_1\right)\\ &\le C\left(R^{2l+1}\left\|D_x^{2l+1} D_x u_{m-l-1}\right\|_k + \|D_x u_{m-l-1}\|_1\right), \quad (2.2.14)\end{aligned}$$

and for $\frac{1}{\sigma_l} = \frac{q}{q+1} - \frac{2m-2l-1}{N-1}$

$$\begin{aligned}\|D_x v_l\|_{\sigma_l} &\le C\left(\left\|D_\theta^{2m-2l-1} D_x v_l\right\|_d + \|D_x v_l\|_1\right)\\ &\le C\left(R^{2m-2l-1}\left\|D_x^{2m-2l-1} D_x v_l\right\|_d + \|D_x v_l\|_1\right). \quad (2.2.15)\end{aligned}$$

For $\eta_l = \frac{N-1}{N-2l-2}$, $\kappa_l = \frac{N-1}{N-2m+2l}$, Lemma 2.5 implies

$$\begin{aligned}\|D_x u_{m-l-1}\|_{\eta_l} &\le C\left(\left\|D_\theta^{2l+1} D_x u_{m-l-1}\right\|_{1+\varepsilon} + \|D_x u_{m-l-1}\|_1\right)\\ &\le C\left(R^{2l+1}\left\|D_x^{2l+1} D_x u_{m-l-1}\right\|_{1+\varepsilon} + \|D_x u_{m-l-1}\|_1\right)\end{aligned}$$

and

$$\begin{aligned}\|D_x v_l\|_{\kappa_l} &\le C\left(\left\|D_\theta^{2m-2l-1} D_x v_l\right\|_{1+\varepsilon} + \|D_x v_l\|_1\right)\\ &\le C\left(R^{2m-2l-1}\left\|D_x^{2m-2l-1} D_x v_l\right\|_{1+\varepsilon} + \|D_x v_l\|_1\right)\end{aligned}$$

Assumption $\frac{1}{p+1} + \frac{1}{q+1} > 1 - \frac{2m}{N} > 1 - \frac{2m}{N-1}$ implies $\frac{1}{\rho_l} + \frac{1}{\sigma_l} < 1$. Therefore we can pick

$\beta_l = z_l \in (1, \infty)$ in (2.2.11) such that

$$\begin{aligned} \frac{p}{p+1} - \frac{2l+1}{N-1} &\leq \frac{1}{z_l} \leq 1 - \frac{2l+1}{N-1} \\ \frac{q}{q+1} - \frac{2m-2l-1}{N-1} &\leq 1 - \frac{1}{z_l} \leq 1 - \frac{2m-2l-1}{N-1}. \end{aligned} \tag{2.2.16}$$

Case I: Either $\rho_l > 0$ or $\sigma_l > 0$. Hölder inequality gives

$$\begin{aligned} \|D_x u_{m-l-1}\|_{z_l} &\leq \|D_x u_{m-l-1}\|_{\eta_l}^{\tau_1} \|D_x u_{m-l-1}\|_{\rho_l}^{1-\tau_1} \\ &\leq C \left(R^{2l+1} \left\|D_x^{2l+1} D_x u_{m-l-1}\right\|_{1+\varepsilon} + \|D_x u_{m-l-1}\|_1 \right)^{\tau_{1l}} \\ &\quad \cdot \left(R^{2l+1} \left\|D_x^{2l+1} D_x u_{m-l-1}\right\|_k + \|D_x u_{m-l-1}\|_1 \right)^{1-\tau_{1l}} \\ &= C R^{2l+1} \left(\left\|D_x^{2l+1} D_x u_{m-l-1}\right\|_{1+\varepsilon} + R^{-2l-1} \|D_x u_{m-l-1}\|_1 \right)^{\tau_{1l}} \\ &\quad \cdot \left(\left\|D_x^{2l+1} D_x u_{m-l-1}\right\|_k + R^{-2l-1} \|D_x u_{m-l-1}\|_1 \right)^{1-\tau_{1l}} \end{aligned} \tag{2.2.17}$$

where

$$\frac{1}{z_l} = \frac{\tau_{1l}}{\eta_l} + \frac{1-\tau_{1l}}{\rho_l}.$$

$$\begin{aligned} \|D_x v_l\|_{z_l'} &\leq \|D_x v_l\|_{\kappa_l}^{\tau_{2l}} \|D_x v_l\|_{\sigma_l}^{1-\tau_{2l}} \\ &\leq C \left(R^{2m-2l-1} \left\|D_x^{2m-2l-1} D_x v_l\right\|_{1+\varepsilon} + \|D_x v_l\|_1 \right)^{\tau_{2l}} \\ &\quad \cdot \left(R^{2m-2l-1} \left\|D_x^{2m-2l-1} D_x v_l\right\|_d + \|D_x v_l\|_1 \right)^{1-\tau_{2l}} \\ &= C R^{2m-2l-1} \left(\left\|D_x^{2l+1} D_x u_{m-l-1}\right\|_{1+\varepsilon} + R^{-2m+2l+1} \|D_x v_l\|_1 \right)^{\tau_{2l}} \\ &\quad \cdot \left(\left\|D_x^{2m-2l-1} D_x v_l\right\|_d + R^{-2m+2l+1} \|D_x v_l\|_1 \right)^{1-\tau_2 l} \end{aligned} \tag{2.2.18}$$

with

$$1 - \frac{1}{z_l} = \frac{1}{z_l'} = \frac{\tau_{2l}}{\kappa_l} + \frac{1-\tau_{2l}}{\sigma_l}.$$

Combining (2.2.12) , (2.2.13) , (2.2.14) , (2.2.15) , (2.2.17) , (2.2.18) we have

$$\begin{aligned}
& \int_{S^{N-1}} \left(\left|u'_{m-l-1}\right| + R^{-1}\left|u_{m-l-1}\right|\right)\left(\left|v'_l\right| + R^{-1}\left|v_l\right|\right) \\
\leq\ & \left(\left\|u'_{m-l-1}\right\|_{z_l} + R^{-1}\left\|u_{m-l-1}\right\|_{z_l}\right)\left(\left\|v'_l\right\|_{z'_l} + R^{-1}\left\|v_l\right\|_{z'_l}\right) \\
\leq\ & C\left(\left\|D_x u_{m-l-1}\right\|_{z_l} + R^{-1}\left\|u_{m-l-1}\right\|_1\right)\left(\left\|D_x v_l\right\|_{z'_l} + R^{-1}\left\|v_l\right\|_1\right) \\
\leq\ & CR^{2m}\left(\left\|D_x^{2l+1} D_x u_{m-l-1}\right\|_{1+\varepsilon} + R^{-2l-1}\left\|D_x u_{m-l-1}\right\|_1 + R^{-2l-2}\left\|u_{m-l-1}\right\|_1\right)^{\tau_{1l}} \\
& \cdot\left(\left\|D_x^{2l+1} D_x u_{m-l-1}\right\|_k + R^{-2l-1}\left\|D_x u_{m-l-1}\right\|_1 + R^{-2l-2}\left\|u_{m-l-1}\right\|_1\right)^{1-\tau_{1l}} \\
& \cdot\left(\left\|D_x^{2l+1} D_x u_{m-l-1}\right\|_{1+\varepsilon} + R^{-2m+2l+1}\left\|D_x v_l\right\|_1 + R^{-2m+2l}\left\|v_l\right\|\right)^{\tau_{2l}} \\
& \cdot\left(\left\|D_x^{2l+1} D_x u_{m-l-1}\right\|_d + R^{-2m+2l+1}\left\|D_x v_l\right\|_1 + R^{-2m+2l}\left\|v_l\right\|\right)^{1-\tau_{2l}} \qquad (2.2.19)
\end{aligned}$$

Case II: Either $\sigma_l \leq 0$ or $\rho_l \leq 0$ but not both. We can take $\tau_{1l} = 1$ (if $\rho_l \leq 0$) or $\tau_{2l} = 1$ (if $\sigma_l \leq 0$), it is easy to see that (2.2.19) still holds.

Case III: Both $\sigma_l \leq 0$ and $\rho_l \leq 0$. This is equivalent to

$$\frac{1}{p+1} > 1 - \frac{2m-2l-1}{N-1}$$

and

$$\frac{1}{q+1} > 1 - \frac{2l+1}{N-1}$$

which gives

$$\frac{1}{p+1} + \frac{1}{q+1} > 2 - \frac{2m}{N-1}$$

Contradiction to $p \geq 1$, $q \geq 1$ and $N \geq 2m+1$.

It follows from (2.2.19) that

$$
\begin{aligned}
G_2(R) &\le CR^N \sum_{l=0}^{m-1} \int_{S^{N-1}} \left(\left|u'_{m-l-1}\right| + R^{-1}\left|u_{m-l-1}\right|\right)\left(\left|v'_l\right| + R^{-1}\left|v_l\right|\right) \\
&\le CR^{N+2m} \\
&\quad \cdot \sum_{l=1}^{m-1} \{ \left(\left\|D_x^{2l+1} D_x u_{m-l-1}\right\|_{1+\varepsilon} + R^{-2l-1}\left\|D_x u_{m-l-1}\right\|_1 + R^{-2l-2}\left\|u_{m-l-1}\right\|_1\right)^{\tau_{1l}} \\
&\quad \cdot \left(\left\|D_x^{2l+1} D_x u_{m-l-1}\right\|_k + R^{-2l-1}\left\|D_x u_{m-l-1}\right\|_1 + R^{-2l-2}\left\|u_{m-l-1}\right\|_1\right)^{1-\tau_{1l}} \\
&\quad \cdot \left(\left\|D_x^{2m-2l-1} D_x v_l\right\|_{1+\varepsilon} + R^{-2m+2l+1}\left\|D_x v_l\right\|_1 + R^{-2m+2l}\left\|v_l\right\|_1\right)^{\tau_{2l}} \\
&\quad \cdot \left(\left\|D_x^{2m-2l-1} D_x v_l\right\|_d + R^{-2m+2l+1}\left\|D_x v_l\right\|_1 + R^{-2m+2l}\left\|v_l\right\|_1\right)^{1-\tau_{2l}} \} \qquad (2.2.20)
\end{aligned}
$$

Step 3. *Measure and Feedback argument.*

We first define the following set

$$\Gamma_0^1(R) = \left\{ r \in (R, 2R) : \|v(r)\|_p^p > KR^{-mp\beta} \right\}$$

$$\Gamma_0^2(R) = \left\{ r \in (R, 2R) : \|u(r)\|_q^q > KR^{-mq\alpha} \right\}$$

$$\Gamma_1(R) = \left\{ r \in (R, 2R) : \left\|D_x^{2m} u(r)\right\|_k^k > KR^{-N} F(4R) \right\}$$

$$\Gamma_2(R) = \left\{ r \in (R, 2R) : \left\|D_x^{2m} v(r)\right\|_d^d > KR^{-N} F(4R) \right\}$$

$$\Gamma_3(R) = \left\{ r \in (R, 2R) : \left\|D_x^{2m} u\right\|_{1+\varepsilon}^{1+\varepsilon} > KR^{-mp\beta} \right\}$$

$$\Gamma_4(R) = \left\{ r \in (R, 2R) : \left\|D_x^{2m} v\right\|_{1+\varepsilon}^{1+\varepsilon} > KR^{-mq\alpha} \right\}$$

For fixed $l \in \{1, 2, \cdots, m-1\}$

$$\Gamma_{5l}(R) = \left\{ r \in (R, 2R) : \left\|u_{m-l-1}(r)\right\|_1 > KR^{-m\alpha - 2(m-l-1)} \right\}$$

$$\Gamma_{6l}(R) = \left\{ r \in (R, 2R) : \|v_l(r)\|_1 > KR^{-m\beta - 2l} \right\}$$

$$\Gamma_{7l}(R) = \left\{ r \in (R, 2R) : \|D_x u_{m-l-1}(r)\|_1 > KR^{-m\alpha - 2(m-l-1) - 1} \right\}$$

$$\Gamma_{8l}(R) = \left\{ r \in (R, 2R) : \|D_x v_l(r)\|_1 > KR^{-m\beta - 2l - 1} \right\}$$

Since $\alpha + 2 = p\beta$, from Lemma 2.2 we deduce

$$cR^{N - mp\beta} \geq \int_R^{2R} \|v(r)\|_p^p r^{N-1} dr \geq |\Gamma_0(R)| KR^{-mp\beta} R^{N-1}$$

which implies

$$\left|\Gamma_0^1(R)\right| < \frac{1}{4m+8} R$$

for $K \gg 1$. Similarly, we get

$$\left|\Gamma_0^2(R)\right| < \frac{1}{4m+8} R$$

for $K \gg 1$.

To estimate $\Gamma_1(R)$, by (2.1.8) in Lemma 2.6,

$$\begin{aligned} CF(4R) &\geq \int_0^{2R} \left\|D_x^{2m} u\right\|_k^k r^{N-1} dr \\ &\geq |\Gamma_1(R)| KR^{-N} F(4R) R^{N-1} \\ &= |\Gamma_1(R)| KR^{-1} F(4R), \end{aligned}$$

From which it follows that for $K \gg 1$

$$|\Gamma_1(R)| < \frac{1}{4m+8} R.$$

Similarly we deduce from $(2.1.9), (2.1.10)$ and $(2.1.11)$ in Lemma 2.6 that

$$|\Gamma_2(R)| < \frac{1}{4m+8}R, \;\; |\Gamma_3(R)| < \frac{1}{4m+8}R, \;\; |\Gamma_4(R)| < \frac{1}{4m+8}R$$

By (2.1.4) in Lemma 2.6,

$$\begin{aligned} CR^{N-m\alpha-2(m-l-1)} &\geq \int_0^{2R} \|u_{m-l-1}\|_1 r^{N-1} dr \\ &\geq |\Gamma_{5l}(R)| KR^{-m\alpha-2(m-l-1)} R^{N-1} \end{aligned}$$

which gives

$$|\Gamma_{5l}(R)| < \frac{1}{4m+8}R$$

when $K \gg 1$ and similarly $(2.1.5), (2.1.6)$ and $(2.1.7)$ implies

$$|\Gamma_{6l}(R)| < \frac{1}{4m+8}R, \;\; |\Gamma_{7l}(R)| < \frac{1}{4m+8}R, \;\; |\Gamma_{8l}(R)| < \frac{1}{4m+8}R$$

when $K \gg 1$. In particular,

$$\Gamma(R) = (R, 2R) \setminus \{\cup_{j=1}^{2} \Gamma_0^j(R) \cup_{i=1}^{4} \Gamma_i(R) \cup_{l=1}^{m-1} \cup_{j=5}^{8} \Gamma_{jl}(R)\} \neq \emptyset.$$

Pick $\widetilde{R} \in \Gamma(R)$, by (2.2.10) together with the observation that $u_m = v^p$, $v_m = u^q$, we have

$$
\begin{aligned}
G_1\left(\widetilde{R}\right) &\leq C\widetilde{R}^N \sum_{l=0}^{m} \int_{S^{N-1}} |u_l|\,|v_{m-l}| \\
&\leq C\widetilde{R}^{N+2m} \sum_{l=0}^{m} \{\left(\left\|D_x^{2m-2l}u_l\right\|_{1+\varepsilon} + \widetilde{R}^{-2m+2l}\left\|u_l\right\|_1\right)^{\nu_{1l}} \\
&\quad \cdot \left(\left\|D_x^{2m-2l}u_l\right\|_k + \widetilde{R}^{-2m+2l}\left\|u_l\right\|_1\right)^{1-\nu_{1l}} \\
&\quad \cdot \left(\left\|D_x^{2l}v_{m-l}\right\|_{1+\varepsilon} + \widetilde{R}^{-2l}\left\|v_{m-l}\right\|_1\right)^{\nu_{2l}} \cdot \left(\left\|D_x^{2l}v_{m-l}\right\|_d + \widetilde{R}^{-2l}\left\|v_{m-l}\right\|_1\right)^{1-\nu_{2l}}\} \\
&\leq CR^{N+2m} \sum_{l=0}^{m} R^{\frac{-mp\beta\nu_{1l}}{1+\varepsilon}} \left(R^{-N}F(4R)\right)^{\frac{1-\nu_{1l}}{k}} \left(R^{-\frac{2m+m\beta}{1+\varepsilon}} + R^{-2m-m\beta}\right)^{\nu_{2l}} \\
&\quad \cdot \left(R^{-\frac{N}{d}}F(4R)^{\frac{1}{d}} + R^{-2m-m\beta}\right)^{1-\nu_{2l}} \\
&\leq CR^{-\widehat{a}}F^{\widehat{b}}(4R)
\end{aligned}
$$

with

$$
\begin{aligned}
\widehat{a} &= \widehat{a}_\varepsilon = \min_l\{-N-2m+\frac{mp\beta}{1+\varepsilon}\nu_{1l}+\frac{mq\alpha}{1+\varepsilon}\nu_{2l}+\frac{N}{k}(1-\nu_{1l})+\frac{N}{d}(1-\nu_{2l})\}, \\
\widehat{b} &= \max_l \frac{1}{k}(1-\nu_{1l})+\frac{1}{d}(1-\nu_{2l}).
\end{aligned}
$$

On the other hand, we have

$$\begin{aligned}
G_2\left(\widetilde{R}\right) &\leq CR^{N+2m} \\
&\cdot \sum_{l=1}^{m-1}\{\left(\left\|D_x^{2l+1}D_x u_{m-l-1}\right\|_{1+\varepsilon}+R^{-2l-1}\left\|D_x u_{m-l-1}\right\|_1+R^{-2l-2}\left\|u_{m-l-1}\right\|_1\right)^{\tau_{1l}} \\
&\cdot \left(\left\|D_x^{2l+1}D_x u_{m-l-1}\right\|_k+R^{-2l-1}\left\|D_x u_{m-l-1}\right\|_1+R^{-2l-2}\left\|u_{m-l-1}\right\|_1\right)^{1-\tau_{1l}} \\
&\cdot \left(\left\|D_x^{2m-2l-1}D_x v_l\right\|_{1+\varepsilon}+R^{-2m+2l+1}\left\|D_x v_l\right\|_1+R^{-2m+2l}\left\|v_l\right\|_1\right)^{\tau_{2l}} \\
&\cdot \left(\left\|D_x^{2m-2l-1}D_x v_l\right\|_d+R^{-2m+2l+1}\left\|D_x v_l\right\|_1+R^{-2m+2l}\left\|v_l\right\|_1\right)^{1-\tau_{2l}}\} \\
&\leq CR^{N+2m}\sum_{l=1}^{m-1}\{\left(\left\|D_x^{2m}u\right\|_{1+\varepsilon}+R^{-2l-1}\left\|D_x u_{m-l-1}\right\|_1+R^{-2l-2}\left\|u_{m-l-1}\right\|_1\right)^{\tau_{1l}} \\
&\cdot \left(\left\|D_x^{2m}u\right\|_k+R^{-2l-1}\left\|D_x u_{m-l-1}\right\|_1+R^{-2l-2}\left\|u_{m-l-1}\right\|_1\right)^{1-\tau_{1l}} \\
&\cdot \left(\left\|D_x^{2m}v\right\|_{1+\varepsilon}+R^{-2m+2l+1}\left\|D_x v_l\right\|_1+R^{-2m+2l}\left\|v_l\right\|_1\right)^{\tau_{2l}}\} \\
&\cdot \left(\left\|D_x^{2m}v\right\|_d+R^{-2m+2l+1}\left\|D_x v_l\right\|_1+R^{-2m+2l}\left\|v_l\right\|_1\right)^{1-\tau_{2l}}\} \\
&\leq CR^{N+2m}\sum_{l=1}^{m-1}\left(R^{-\frac{mp\beta}{1+\varepsilon}}+R^{-2l-1}R^{-m\alpha-2(m-l-1)-1}+R^{-2l-2}R^{-m\alpha-2(m-l-1)}\right)^{\tau_{1l}} \\
&\cdot \left(R^{-N/k}F^{\frac{1}{k}}\left(4R\right)+R^{-2l-1}R^{-m\alpha-2(m-l-1)-1}+R^{-2l-2}R^{-m\alpha-2(m-l-1)}\right)^{1-\tau_{1l}} \\
&\cdot \left(R^{-\frac{mq\alpha}{1+\varepsilon}}+R^{-2l-1}R^{-m\beta-2(m-l-1)-1}+R^{-2l-2}R^{-m\beta-2(m-l-1)}\right)^{\tau_{2l}} \\
&\cdot \left(R^{-N/d}F^{\frac{1}{d}}\left(4R\right)+R^{-2l-1}R^{-m\beta-2(m-l-1)-1}+R^{-2l-2}R^{-m\beta-2(m-l-1)}\right)^{1-\tau_{2l}} \\
&\leq CR^{-\overline{a}}F^{\overline{b}}\left(4R\right)
\end{aligned}$$

Here

$$\overline{a}=\overline{a}_\varepsilon=\min_l\{-N-2m+\frac{mp\beta}{1+\varepsilon}\tau_{1l}+\frac{mq\alpha}{1+\varepsilon}\tau_{2l}+\frac{N}{k}\left(1-\tau_{1l}\right)+\frac{N}{d}\left(1-\tau_{2l}\right)\}$$

and

$$\overline{b} = \max_l \frac{1}{k}\left(1-\tau_{1l}\right) + \frac{1}{d}\left(1-\tau_{2l}\right).$$

We claim that there exists a constant $M > 0$ and a sequence $R_i \to \infty$ such that

$$F\left(4R_i\right) \leq MF\left(R_i\right).$$

Otherwise for any $M > 0$, there exists R_M such that for $R \geq R_M$

$$F\left(4R\right) > MF\left(R\right).$$

Since (u, v) is bounded, we have $F\left(R\right) \leq CR^N$, $R > 0$. Thus

$$M^i F\left(R_M\right) \leq F\left(4^i R_M\right) \leq CR_M^N\left(4^N\right)^i$$

Contradiction for i large if $M > 4^N$.

Assume we have shown $a = a_\varepsilon = \min\left(\widehat{a_\varepsilon}\ ,\overline{a_\varepsilon}\right) > 0,\ b = b_\varepsilon = \max\left(\widehat{b},\ \overline{b}\right) < 1$,we have

$$\begin{aligned} F\left(R_i\right) &\leq CR^{-a}F^b\left(4R_i\right) \\ &\leq CM^bR^{-a}F^b\left(R_i\right) \end{aligned}$$

which gives

$$F\left(R_i\right) \leq CR_i^{-\frac{a}{1-b}}.$$

Letting $i \to \infty$, we deduce that

$$\int_{\mathbb{R}^n} u^{q+1} + v^{p+1} = 0,$$

hence $u = v \equiv 0$, a contradiction.

Step 4. *If* $m\alpha > N - 2m - 1$, *then* $\overline{b}, \widehat{b} < 1$ *and* $\overline{a_\varepsilon}, \widehat{a_\varepsilon} > 0$ *for* $\varepsilon \ll 1$.

First we show $\widehat{a_\varepsilon} > 0$, $\widehat{b} < 1$.

Since $\nu_{1l} = \left(\frac{1}{\delta_l} - \frac{1}{\gamma_l}\right)^{-1}\left(\frac{1}{\alpha_l} - \frac{1}{\gamma_l}\right)$, $\nu_{2l} = \left(\frac{1}{\psi_l} - \frac{1}{\omega_l}\right)^{-1}\left(\frac{1}{\alpha_l'} - \frac{1}{\omega_l}\right)$,
to show $\widehat{b} < 1$, we need to show for all l,

$$\begin{aligned} & \frac{1}{k}(1 - \nu_{1l}) + \frac{1}{d}(1 - \nu_{2l}) \\ = \;& p\widehat{A}_{1l} + q\widehat{A}_{2l} \\ = \;& p\left(\frac{N - 2m + 2l - 1}{N - 1} - \frac{1}{\alpha_l}\right) + q\left(\frac{1}{\alpha_l} - \frac{2l}{N-1}\right) < 1. \end{aligned} \tag{2.2.21}$$

Here

$$\begin{aligned} \widehat{A}_{1l} &= \frac{1}{p+1}(1 - \nu_{1l}) \\ &= \left(\frac{N - 2m + 2l - 1}{N - 1} - \frac{1}{\alpha_l}\right), \\ \widehat{A}_{2l} &= \frac{1}{q+1}(1 - \nu_{2l}) \\ &= \left(\frac{1}{\alpha_l} - \frac{2l}{N-1}\right). \end{aligned}$$

(2.2.21) is equivalent to

$$p\frac{N - 2m + 2l - 1}{N - 1} - q\frac{2l}{N - 1} - 1 < \frac{p - q}{\alpha_l}. \tag{2.2.22}$$

Recall α_l is chosen to satisfy (2.2.3). Such $\alpha_l \in (1, \infty)$ satisfying (2.2.3) and (2.2.22) exists provided

$$\max\left(\frac{p}{p+1} - \frac{2m-2l}{N-1}, \frac{2l}{N-1}\right) \leq \min\left(1 - \frac{2m-2l}{N-1}, \frac{1}{q+1} + \frac{2l}{N-1}\right) \quad (2.2.23)$$

and

$$p\frac{N-2m+2l-1}{N-1} - q\frac{2l}{N-1} - 1 < (p-q)\left(1 - \frac{2m-2l}{N-1}\right), \quad (2.2.24)$$

$$p\frac{N-2m+2l-1}{N-1} - q\frac{2l}{N-1} - 1 < (p-q)\left(\frac{1}{q+1} + \frac{2l}{N-1}\right). \quad (2.2.25)$$

(2.2.23) follows from the assumption that $N \geq 2m+1$ and

$$\frac{1}{p+1} + \frac{1}{q+1} > 1 - \frac{2m}{N} > 1 - \frac{2m}{N-1}.$$

(2.2.24) is equivalent to

$$q\frac{N-1-2m}{N-1} < 1,$$

which follows from

$$q \leq \frac{p(q+1)}{p+1} = 1 + \frac{2}{\alpha} < 1 + \frac{2m}{N-2m-1} = \frac{N-1}{N-1-2m}.$$

And (2.2.25) can be rewritten as

$$\frac{N-2m-1}{N-1}p < \frac{p+1}{q+1}$$

which is equivalent to

$$m\alpha > N - 2m - 1.$$

Finally, for each l

$$\begin{aligned}
\widehat{a_{0l}} &= -N - 2m + mp\beta\nu_{1l} + mq\alpha\nu_{2l} + \frac{N}{k}\left(1-\nu_{1l}\right) + \frac{N}{d}\left(1-\nu_{2l}\right) \\
&= 2m - N + m\alpha + m\beta + \left(N - \frac{2m\left(p+1\right)\left(q+1\right)}{pq-1}\right)\left(p\widehat{A}_{1l} + q\widehat{A}_{2l}\right) \\
&= \left(2m - N + m\alpha + m\beta\right)\left(1 - p\widehat{A}_{1l} - q\widehat{A}_{2l}\right) \\
&> 0.
\end{aligned}$$

It then follows $\widehat{a_0} > 0$, thus $\widehat{a_\varepsilon} > 0$ for $\varepsilon \ll 1$.

Secondly we show $\overline{a_\varepsilon} > 0$, $\overline{b} < 1$. This can be shown in a similar way as $\widehat{a_\varepsilon}, \widehat{b}$. We write all details for readers' convenience. Since $\tau_{1l} = \left(\frac{1}{\eta_l} - \frac{1}{\rho_l}\right)^{-1}\left(\frac{1}{z_l} - \frac{1}{\rho_l}\right)$, $\tau_{2l} = \left(\frac{1}{\kappa_l} - \frac{1}{\sigma_l}\right)^{-1}\left(\frac{1}{z_l'} - \frac{1}{\sigma_l}\right)$, to show $\overline{b} < 1$, we need to show for all l,

$$\begin{aligned}
& \frac{1}{k}\left(1-\tau_{1l}\right) + \frac{1}{d}\left(1-\tau_{2l}\right) \\
= \; & pA_{1l} + qA_{2l} \\
= \; & p\left(\frac{N-2l-2}{N-1} - \frac{1}{z_l}\right) + q\left(\frac{1}{z_l} - \frac{2m-2l-1}{N-1}\right) < 1. \qquad (2.2.26)
\end{aligned}$$

Here we used

$$\begin{aligned} A_{1l} &= \frac{1}{p+1}\left(1-\tau_{1l}\right) \\ &= \left(\frac{N-2l-2}{N-1}-\frac{1}{z_l}\right), \\ A_{2l} &= \frac{1}{q+1}\left(1-\tau_{2l}\right) \\ &= \left(\frac{1}{z_l}-\frac{2m-2l-1}{N-1}\right). \end{aligned}$$

(2.2.26) is equivalent to

$$p\frac{N-2l-2}{N-1}-q\frac{2m-2l-1}{N-1}-1<\frac{p-q}{z_l}. \tag{2.2.27}$$

Recall z_l is chosen to satisfy (2.2.16). Such $z_l \in (1,\infty)$ satisfying (2.2.16) and (2.2.27) exists provided

$$\max\left(\frac{p}{p+1}-\frac{2l+1}{N-1},\frac{2m-2l-1}{N-1}\right)\leq \min\left(1-\frac{2l+1}{N-1},\frac{1}{q+1}+\frac{2m-2l-1}{N-1}\right) \tag{2.2.28}$$

and

$$p\frac{N-2l-2}{N-1}-q\frac{2m-2l-1}{N-1}-1<(p-q)\left(1-\frac{2l+1}{N-1}\right), \tag{2.2.29}$$

$$p\frac{N-2l-2}{N-1}-q\frac{2m-2l-1}{N-1}-1<(p-q)\left(\frac{1}{q+1}+\frac{2m-2l-1}{N-1}\right). \tag{2.2.30}$$

(2.2.28) follows from

$$\frac{1}{p+1}+\frac{1}{q+1}>1-\frac{2m}{N}>1-\frac{2m}{N-1}.$$

(2.2.29) is equivalent to

$$q\frac{N-1-2m}{N-1}<1,$$

which follows from

$$q\leq\frac{p\left(q+1\right)}{p+1}=1+\frac{2}{\alpha}<1+\frac{2m}{N-2m-1}=\frac{N-1}{N-1-2m}.$$

And lastly (2.2.30) can be rewritten as

$$\frac{N-2m-1}{N-1}p<\frac{p+1}{q+1}$$

which is equivalent to

$$m\alpha>N-2m-1.$$

Finally, for each l

$$\begin{aligned}\overline{a_{0l}} &= -2m-N+mp\beta\left(1-\left(p+1\right)A_{1l}\right)+mq\alpha\left(1-\left(q+1\right)A_{2l}\right)+NpA_{1l}+NqA_{2l}\\ &= 2m-N+m\alpha+m\beta+\left(N-\frac{2m\left(p+1\right)\left(q+1\right)}{pq-1}\right)\left(pA_{1l}+qA_{2l}\right)\\ &= \left(2m-N+m\alpha+m\beta\right)\left(1-pA_{1l}-qA_{2l}\right)\\ &> 0.\end{aligned}$$

It then follows $\overline{a_0}>0$, thus $\overline{a_\varepsilon}>0$ for $\varepsilon\ll1$.

Remark 2.12. Note Lemma 3.2 implies when $pq > 1$, $N \leq 2m$, (1.1.1) does not admit any positive solutions. In particular, this implies the following equation

$$(-\Delta)^m u = u^p$$

admit no positive solutions if $N \leq 2m$, $p > 1$.

Chapter 3

Liouville Theorem for Higher Order Henon-Lane-Emden System

In this chapter, we prove Liouville type theorem for higher order Henon-Lane-Emden System.

3.1 Preparations

When $pq > 1$,we introduce the following notations

$$\alpha = \frac{2m\,(p+1) + a + bp}{pq-1}, \quad \beta = \frac{2m\,(q+1) + aq + b}{pq-1}$$

and assume $\alpha \geq \beta$ throughout the rest of the paper. The assumption

$$\frac{1+\frac{a}{N}}{p+1} + \frac{1+\frac{b}{N}}{q+1} > \frac{N-2m}{N}$$

can be rewritten as

$$\alpha + \beta > N - 2m.$$

For $w \in C\left(\mathbb{R}^N\right)$, we denote the spherical average of w by

$$\overline{w}\left(r\right) = \frac{1}{\omega_N} \int_{S^{N-1}} w\left(r, \theta\right) ds, \quad r > 0,$$

where ω_N is the area of the unit sphere S^{N-1}.

We have the following growth estimates.

Lemma 3.1. *If $pq = 1$, there is no positive solution of $(1.0.1)$. If (u, v) is a positive solution of $(1.0.1)$ and $p, q \geq 1$, and $pq > 1$, there exists a positive constant $M = M\left(p, q, n\right)$ such that*

$$\overline{u}\left(r\right) \leq M r^{-\alpha}, \qquad \overline{v}\left(r\right) \leq M r^{-\beta} \quad \text{for} \quad r > 0. \tag{3.1.1}$$

and for $k = 1, \cdots, m-1$, $u_k = \left(-\Delta\right)^k u, v_k = \left(-\Delta\right)^k v$, we have

$$\left(-\Delta\right)^i u > 0, \quad \left(-\Delta\right)^i v > 0, \; i = 1, 2, \cdots, m-1.$$

$$\overline{u_k}\left(r\right) \leq M r^{-\alpha-2k}, \qquad \overline{v_k}\left(r\right) \leq M r^{-\beta-2k} \quad \text{for} \quad r > 0. \tag{3.1.2}$$

Proof. Lemma follows from the same argument as in proof of Lemma 3.3 in [23]. ■

The following growth estimates was proved in [4].

Lemma 3.2. *(Lemma 1 in [4]) Suppose that $p, q \geq 1$ and (u, v) is a positive solution* *of* (1.0.1) . *Then*

$$\int_{B_R} |x|^b u^q \leq cR^{N-2m-\beta}, \quad \int_{B_R} |x|^a v^p \leq cR^{N-2m-\alpha}, \tag{3.1.3}$$

where $c = c(p,q,n)$.

As a direct corollary of Lemma 3.2, we have the following nonexistence result for (1.0.1) . This was pointed out in [11] We write down the details for readers' convenience.

Corollary 3.1.1. If $p, q \geq 1$ and $\max(\alpha, \beta) \geq N - 2m$, (1.0.1) does not admit any positive solution.

Proof. We only need to prove case $\max(\alpha, \beta) = N - 2m$. Without loss of generality, we can assume $\alpha \geq \beta$. Recall that for $w > 0$, $\Delta w \leq 0$, we have

$$w(x) \geq c|x|^{2-N} \quad \text{for } |x| \geq 1.$$

Since

$$-\Delta u_{k-1} = u_k,$$

it follows from Lemma 2.7 of [15] that

$$\overline{u_{k-1}} \geq cr^2 \overline{u_k}, \quad k = 1, \cdots, m-1.$$

Iteration then gives

$$\overline{u}(r) \geq r^{2m-N} \text{ for } r \geq 1.$$

Applying Lemma 2.7 of [15] to $\overline{v_k}$ for $k = 0, 1, \cdots m-1$ yields

$$\overline{v}\left(r\right) \geq Cr^{2m+b}\overline{u^q} \geq Cr^{2m+b}\overline{u}^q \geq Cr^{2m+b-(N-2m)q}.$$

Therefore by (3.1.3)

$$\begin{aligned} C &\geq \int_{B_R} |x|^a v^p \geq \int_0^R r^{N-1+a}\overline{v}^p \geq \int_1^R r^{N-1+a+2mp+bp-(N-2m)pq} \\ &= \int_1^R r^{-1} dr = \ln R \end{aligned} \tag{3.1.4}$$

The first equality in (3.1.4) follows from assumption on α and identity

$$N - 1 + a + 2mp + bp - (N-2m)pq = -1 + (pq-1)(\alpha - N + 2m) = -1.$$

Letting R goes to infinity in (3.1.4), contradiction. ■

We state the following interpolation inequalities and elliptic estimates.

Lemma 3.3. *(L^p estimates on B_R) Given $1 < k < \infty$, $R > 0$, $z \in W^{2m,k}(B_{2R})$, then*

$$\int_{B_R \setminus B_{\frac{R}{2}}} \left|D^{2m} z\right|^k \leq C\left(\int_{B_{2R}\setminus B_{\frac{R}{4}}} |\Delta^m z|^k + R^{-2mk}\int_{B_{2R}\setminus B_{\frac{R}{4}}} |z|^k\right).$$

Proof. Lemma follows from standard elliptic L^p estimates for second order elliptic equations and interpolation inequalities. ■

We can prove the following growth estimates for u, v and their derivatives.

Lemma 3.4. *Let*

$$\begin{aligned} k &= \frac{2m(p+1)(q+1)+a(q+1)+b(p+1)}{2mp(q+1)+a+bp}, \\ d &= \frac{2m(p+1)(q+1)+a(q+1)+b(p+1)}{2mq(p+1)+aq+b}. \end{aligned}$$

If bounded solution pair (u,v) *of* (1.0.1)*satisfies the following decay assumptions*

$$u(x) \leq C|x|^{-\alpha}, \; v(x) \leq C|x|^{-\beta} \quad \textit{for } |x| \geq 1, \tag{3.1.5}$$

then the following estimates hold for $l = 1, 2\cdots, m-1$,

$$\int_0^R \|u_l(r)\|_1 r^{N-1} dr \leq C r^{N-\alpha-2l}, \tag{3.1.6}$$

$$\int_0^R \|v_l(r)\|_1 r^{N-1} dr \leq C r^{N-\beta-2l}, \tag{3.1.7}$$

$$\int_0^R \|D_x u_l\|_1 r^{N-1} dr \leq C r^{N-\alpha-2l-1}, \tag{3.1.8}$$

$$\int_0^R \|D_x v_l\|_1 r^{N-1} dr \leq C r^{N-\beta-2l-1}, \tag{3.1.9}$$

$$\int_{\frac{R}{2}}^R \left\|D_x^{2m} u\right\|_k^k r^{N-1} dr \leq CF(2R), \tag{3.1.10}$$

$$\int_{\frac{R}{2}}^R \left\|D_x^{2m} v\right\|_d^d r^{N-1} dr \leq CF(2R), \tag{3.1.11}$$

$$\int_0^R \left\|D_x^{2m} u\right\|_{1+\epsilon}^{1+\epsilon} r^{N-1} dr \leq C R^{N-2m-\alpha+a\epsilon}, \tag{3.1.12}$$

$$\int_0^R \left\| D_x^{2m} v \right\|_{1+\epsilon}^{1+\epsilon} r^{N-1} dr \le C R^{N-2m-\beta+b\epsilon}. \tag{3.1.13}$$

Here

$$F(R) = \int_{B_R} \left[|x|^a v^{p+1} + |x|^b u^{q+1} \right] dx.$$

Proof. $(3.1.6), (3.1.7)$ are restatements of Lemma 3.1. $(3.1.8)$ and $(3.1.9)$ follows directly from Lemma 3.1 and Lemma 2.4. To prove $(3.1.10)$, Lemma 3.3 implies

$$\begin{aligned}
\int_{R/2}^R \left\| D_x^{2m} u \right\|_k^k r^{N-1} dr &= \int_{B_R \setminus B_{R/2}} \left| D^{2m} u \right|^k \\
&\le C \left(\int_{B_{2R} \setminus B_{R/4}} |\Delta^m u|^k + R^{-2mk} \int_{B_{2R} \setminus B_{R/4}} u^k \right) \\
&= C \left(\int_{B_{2R} \setminus B_{R/4}} |x|^{ak} v^{pk} + R^{-2mk} \int_{B_{2R} \setminus B_{R/4}} u^k \right) \\
&\le C \left(\int_{B_{2R}} |x|^a v^{p+1} + R^{-2mk} \int_{B_{2R} \setminus B_{R/4}} u^k \right).
\end{aligned}$$

Here we used growth assumption $(3.1.5)$ and identity

$$\frac{a(k-1)}{pk - (p+1)} = \beta.$$

Since $pq > 1$, it follows $\frac{p+1}{p} < q+1$ therefore $\frac{p+1}{p} < k < q+1$. By Hölder's inequality and the fact that $F(R) \geq F(1) > 0$, $R \geq 1$, we obtain

$$\begin{aligned} R^{-2mk} \int_{B_{2R} \setminus B_{R/4}} u^k &\leq CR^{-2mk} \left(\int_{B_{2R}} |x|^b u^{q+1} \right)^{\frac{k}{q+1}} \left(\int_{B_{2R} \setminus B_{R/4}} |x|^{-\frac{bk}{q+1-k}} \right)^{1-\frac{k}{q+1}} \\ &\leq CR^{-2mk} F(2R)^{\frac{k}{q+1}} R^{\frac{N(q+1-k)-bk)}{q+1}} \\ &\leq CR^{\chi k} F(2R) (F(1))^{\frac{k}{q+1}-1} \\ &\leq CR^{\chi k} F(2R), \end{aligned}$$

where

$$\chi = -2m - \frac{N+b}{q+1} + \frac{N}{k}.$$

We can write

$$\begin{aligned} &\chi \left[2m(p+1)(q+1) + a(q+1) + b(p+1)\right] \\ = \ &2m(pq-1)\left[N - 2m - \alpha - \beta\right] + b(p+1)\left[N - 2m - \frac{N+a}{p+1} - \frac{N+b}{q+1}\right] \end{aligned}$$

Since

$$\frac{1+\frac{a}{N}}{p+1} + \frac{1+\frac{b}{N}}{q+1} > 1 - \frac{2m}{N},$$

we have $\chi < 0$, and (3.1.10) follows. (3.1.11) is proved similarly by using (3.1.5) and

$$\frac{b(d-1)}{qd-(q+1)} = \alpha$$

Lastly we prove (3.1.12) .

$$\begin{aligned}
\int_0^R \left\| D_x^{2m} u \right\|_{1+\epsilon}^{1+\epsilon} r^{N-1} dr &\le C\left(\int_{B_{2R}} \left|\Delta^m u\right|^{1+\varepsilon} + R^{-2m(1+\varepsilon)} \int_{B_{2R}} u^{1+\varepsilon} \right) \\
&= C\left(\int_{B_{2R}} |x|^{a(1+\epsilon)} v^{p(1+\varepsilon)} + R^{-2m(1+\varepsilon)} \int_{B_{2R}} u^{1+\varepsilon} \right) \\
&\le C\left(R^{a\epsilon} \int_{B_{2R}} |x|^a v^p + R^{-2m(1+\varepsilon)} \int_{B_{2R}} u \right) \\
&\le C\left(R^{N-2m-\alpha+a\epsilon} + R^{-2m(1+\varepsilon)} \cdot R^{N-\alpha} \right) \\
&\le CR^{N-2m-\alpha+a\epsilon}.
\end{aligned}$$

■

We have the following Rellich-Pohozav identity.

Lemma 3.5. *For any $a_1 + a_2 = N - 2m$, $r > 0$*

$$\begin{aligned}
& \left(\frac{N+a}{p+1} - a_1 \right) \int_{B_r} |x|^a v^{p+1} dx + \left(\frac{N+b}{q+1} - a_2 \right) \int_{B_r} |x|^b u^{q+1} dx \\
= \; & \frac{1}{p+1} v^{p+1}(r) r^{N+a} + \frac{1}{q+1} u^{q+1}(r) r^{N+b} \\
& - (-1)^m \left\{ \sum_{k=0}^{m-1} 2r^N \int_{S^{N-1}} \frac{\partial \Delta^k u}{\partial n} \cdot \frac{\partial \Delta^{m-1-k} v}{\partial n} ds \right. \\
& - \sum_{k=0}^{m-1} r^N \int_{S^{N-1}} \left(\nabla \Delta^k u, \nabla \Delta^{m-1-k} v \right) ds \\
& - \sum_{k=0}^{m-2} r^N \int_{S^{N-1}} \left(\Delta^{k+1} u, \Delta^{m-1-k} v \right) ds \\
& + \sum_{k=0}^{m-1} (2m - 2k - 2 + a_1) r^{N-1} \int_{S^{N-1}} \frac{\partial \Delta^k u}{\partial n} \Delta^{m-1-k} v ds \\
& \left. + \sum_{l=0}^{m-1} (a_2 + 2k) r^{N-1} \int_{S^{N-1}} \frac{\partial \Delta^{m-1-k} v}{\partial n} \Delta^k u \; ds \right\}.
\end{aligned}$$

Proof. A similar Rellich-Pohozaev identity can be found in [4]. For purpose of later estimates, we prefer to write our Rellich-Pohozaev identity with a slightly different boundary terms on the right hand side. By (1.0.1)

$$\begin{aligned}
(-1)^m R_m(u,v) &= \int_{B_r} (-\Delta)^m u(x)(x,\nabla v) + (-\Delta)^m v(x,\nabla u)\,dx \\
&= \int_{B_r} v^p(x)(x,\nabla v) + u^q(x)(x,\nabla u)\,dx \\
&= \int_{\partial B_r} \frac{v^{p+1}}{p+1}|x|^a(x,n) + \frac{u^{q+1}}{q+1}|x|^b(x,n)\,ds \\
&\quad -\frac{N+a}{p+1}\int_{B_r}|x|^a v^{p+1}dx - \frac{N+b}{q+1}\int_{B_r}|x|^b u^{q+1}dx \\
&= \frac{1}{p+1}v^{p+1}(r)\,r^{N+a} + \frac{1}{q+1}u^{q+1}(r)\,r^{N+b} \\
&\quad -\frac{N+a}{p+1}\int_{B_r}|x|^a v^{p+1}dx - \frac{N+b}{q+1}\int_{B_r}|x|^b u^{q+1}dx.
\end{aligned}$$

To finish the proof, we follow the same argument as in proof of Lemma 2.8 in [1] to estimate $R_m(u,v)$ using (2.1.15) and integration by parts. ■

3.2 Main theorem

Our main result in this chapter is as follows.

Theorem 3.2.1. $N \geq 3$, $N > 2m$, $a \geq 0$, $b \geq 0$, assume $p \geq 1$, $q \geq 1$, $(p,q) \neq (1,1)$. If

$$\frac{N+a}{p+1} + \frac{N+b}{q+1} > N-2m \tag{3.2.1}$$

and

$$\max\left(\frac{2m(p+1)+a+bp}{pq-1}, \frac{2m(q+1)+aq+b}{pq-1}\right) > N-2m-1,$$

the problem (1.0.1) has no positive solutions of class $C^{2m}\left(\mathbb{R}^N\right)$ which satisfies slow decay assumptions

$$u(x) \leq C \min\left(|x|^{-\alpha}, 1\right),\ v(x) \leq C \min\left(|x|^{-\beta}, 1\right) \tag{3.2.2}$$

Moreover, when $N = 2m+1$ or $2m+2$, if $p \geq 1, q \geq 1$, $(p,q) \neq (1,1)$ satisfies $(3.2.1)$, $(1.0.1)$ admits no positive solutions satisfying $(3.2.2)$.

Under stronger assumptions on p, q, we can remove the decay assumptions on (u, v).

Theorem 3.2.2. *$N \geq 3$, $N > 2m$, if $p \geq 1$, $q \geq 1, (p,q) \neq (1,1)$ satisfies*

$$\frac{N}{p+1} + \frac{N}{q+1} > N - 2m \tag{3.2.3}$$

and

$$\max\left(\frac{2m(p+1)}{pq-1}, \frac{2m(q+1)}{pq-1}\right) > N - 2m - 1,$$

then (1.0.1) has no positive solutions of class $C^{2m}\left(\mathbb{R}^N\right)$. Moreover, when $N = 2m+1$ or $N = 2m+2$, if $p \geq 1, q \geq 1$, $(p,q) \neq (1,1)$ satisfies (3.2.3), then (1.0.1) admits no positive solutions.

We prove Theorem (3.2.1) in two steps as in chapter 2.

3.2.1 Reduction to bounded solutions

In this subsection, we show if (1.1.1) does not admit bounded positive solutions, then (1.0.1) with same p, q does not admit positive solutions with slow decay.

More precisely, we prove the following Theorem. From this theorem, Theorem (3.2.2) follows.

Theorem 3.2.3. Let $N \geq 3, p > 1, q > 1$ be fixed, and assume (1.1.1) does not admit any bounded nontrivial (nonnegative) solution in $\mathbb{R}^N$, then (1.0.1) with same p, q does not admit any notrivial (nonnegative) solution in $\mathbb{R}^N$, bounded or not. In particular, the conclusion holds if $N = 2m+1$, or $2m+2$ and $\frac{2m(p+1)}{pq-1} + \frac{2m(q+1)}{pq-1} > N - 2m$. The conclusion also holds when $N > 2m+2$ and p, q satisfies $\frac{2m(p+1)}{pq-1} + \frac{2m(q+1)}{pq-1} > N - 2m$ and $\max\left(\frac{2m(p+1)}{pq-1}, \frac{2m(q+1)}{pq-1}\right) > N - 2m - 1$.

We shall follow a similar proof as in section 2.2.1

We first prove the following Lemma.

Lemma 3.6. *Let $\delta \in (0,1]$. Let $c_i \in C^{\delta}\left(\overline{B_1}\right)$ satisfy*

$$\|c_i\|_{C^{\delta}}\left(\overline{B_1}\right) \leq C_1 \quad \text{and } c_i(x) \geq C_2, \quad x \in \overline{B_1}, i = 1, 2$$

for some positive constants C_1, C_2. Assume (1.1.1) does not admit any bounded positive solutions. There exists a constant C, depending only on $\delta, C_1, C_2, p, q, N$, such that any nonnegative solutions (u, v) of

$$\begin{cases} (-\Delta)^{2m} u = c_1(x) v^p \\ (-\Delta)^{2m} v = c_2(x) u^q \end{cases} \quad x \in \overline{B_1} \tag{3.2.4}$$

with same p, q satisfies

$$u(x) \leq C\left(1 + dist^{-\gamma}(x, \partial B_1)\right), \ x \in B_1$$

and

$$v(x) \leq C\left(1 + dist^{-\sigma}(x, \partial B_1)\right), \ x \in B_1.$$

Here $\gamma = \frac{2m(p+1)}{pq-1}$, $\sigma = \frac{2m(q+1)}{pq-1}$.

Proof of Lemma 3.6. Assume the Lemma fails. Then there exist sequences (u_k, v_k), $y_k \in B_1$ such that (u_k, v_k) solves (3.2.4) on B_1 and

$$M_k := u_k^{\frac{1}{\gamma}} + v_k^{\frac{1}{\sigma}}, \quad k = 1, 2, \cdots$$

satisfies

$$M_k(y_k) > 2k\left(1 + \text{dist}^{-1}(y_k, \partial B_1)\right).$$

By Lemma 2.10 and Remark 2.11, it follows that there exists $x_k \in B_1$ such that

$$M_k(x_k) \geq M_k(y_k) > 2k$$

and

$$M_k(z) \leq 2M_k(x_k), \quad \text{for } |z - x_k| \leq kM_k^{-1}(x_k).$$

Define rescaling of (u_k, v_k) as follows

$$\begin{aligned} \lambda_k &= M_k^{-1}(x_k) \\ \widetilde{u}_k(y) &= \lambda_k^{\gamma} u_k(x_k + \lambda_k y), \quad \widetilde{v}_k(y) = \lambda_k^{\sigma} v_k(x_k + \lambda_k y), \quad |y| \leq k. \end{aligned}$$

We then have $\lambda_k \to 0$ and $(\widetilde{u}_{k,}\widetilde{v}_k)$ satisfies

$$\begin{aligned}(-\Delta_y)^m \widetilde{u}_k(y) &= \widetilde{c}_{1k}(y)\widetilde{v}_k^p(y) \\ (-\Delta_y)^m \widetilde{v}_k(y) &= \widetilde{c}_{2k}(y)\widetilde{u}_k^q(y)\end{aligned}$$

for $|y| \le k$. Here

$$\widetilde{c}_{ik}(y) = c_i(x_k + \lambda_k y), \quad i = 1, 2$$

satisfies $C_2 \le \widetilde{c}_{ik}(y) \le C_1$ and for each $R > 0$, $k \ge k_0(R)$

$$|\widetilde{c}_{ik}(y) - \widetilde{c}_{ik}(z)| \le C_1 |\lambda_k(y-z)|^\delta \le C_1 |(y-z)|^\delta \quad \text{for } |y|, |z| \le R. \tag{3.2.5}$$

By Ascoli-Arzelá theorem, there exists $\widetilde{c}_i$ in $c(\mathbb{R}^N)$ with $\widetilde{c} \ge C_2$ such that $\widetilde{c}_{ik} \to \widetilde{c}_i$ in $C_{loc}(\mathbb{R}^N)$ subject to a subsequence. Since $\lambda_k \to 0$, (3.2.5) implies limit functions $\widetilde{c}_i$ are actually constants. We write the limit constants as l_1, l_2. Moreover, By standard elliptic L^p estimates and Sobolev embeddings, we conclude that subject to a subsequence, $(\widetilde{u}_{k,}\widetilde{v}_k)$ converges in $C^{2m}_{loc}(\mathbb{R}^N)$ to a (classical) solution $(\widetilde{u}_,\widetilde{v})$ of

$$\begin{aligned}(-\Delta_y)^m \widetilde{u}(y) &= l_1 \widetilde{v}^p(y) \\ (-\Delta_y)^m \widetilde{v}(y) &= l_2 \widetilde{u}^q(y)\end{aligned} \tag{3.2.6}$$

in $\mathbb{R}^N$. Since

$$\widetilde{u}_k^{\frac{1}{\gamma}}(0) + \widetilde{v}_k^{\frac{1}{\sigma}}(0) = 1$$

and

$$\widetilde{u}_k^{\frac{1}{\gamma}}(y) + \widetilde{v}_k^{\frac{1}{\sigma}}(y) \le 2, \text{ when } \; |y| \le k.$$

We have $\widetilde{u}^{\frac{1}{\gamma}}(0)+\widetilde{v}^{\frac{1}{\sigma}}(0)=1$ and $\widetilde{u}^{\frac{1}{\gamma}}(y)+\widetilde{v}^{\frac{1}{\sigma}}(y)\leq 2$. i.e. $(\widetilde{u},\widetilde{v})$ is nontrivial and bounded solution of (3.2.6), contradicting the assumption for (1.1.1). In particular, Liouville theorems for (1.1.1) implies the assumption holds when $N=2m+1$, or $2m+2$ and $\frac{2m(p+1)}{pq-1}+\frac{2m(q+1)}{pq-1}>N-2m$. The assumption of this Lemma also holds when $N>2m+2$ and p,q satisfies $\frac{2m(p+1)}{pq-1}+\frac{2m(q+1)}{pq-1}>N-2m$ and $\max\left(\frac{2m(p+1)}{pq-1},\frac{2m(q+1)}{pq-1}\right)>N-2m-1$.

Lemma 3.7. *Assume* (1.1.1) *does not admit any bounded nontrivial nonegative solution in* $\mathbb{R}^N$*. There exists a constant* $C=C(N,p,q,a,b)>0$ *(independent of* Ω *and* (u,v)*) such that the following holds.*

i) Any nonnegative solution of (1.0.1) in $\Omega=\left\{x\in R^N:0<|x|<\rho\right\}$ satisfies

$$u(x)\leq C|x|^{-\alpha}\quad\text{and } v(x)\leq C|x|^{-\beta}\;,\;\;0<|x|<\frac{\rho}{2}.$$

ii) Any nonnegative solution of (1.0.1) in $\Omega=\left\{x\in R^N:|x|>\rho\right\}$ satisfies

$$u(x)\leq C|x|^{-\alpha}\text{ and } v(x)\leq C|x|^{-\beta},\qquad |x|>2\rho.$$

Proof. Assume either $\Omega=\left\{x\in R^N:0<|x|<\rho\right\}$ and $0<|x_0|<\frac{\rho}{2}$ or $\Omega=\left\{x\in R^N:0<|x|<\rho\right\}$ and $|x|>2\rho$. Let $R=\frac{|x_0|}{2}$, it then follows

$$\frac{|x_0|}{2}<|x_0+Ry|<\frac{3|x_0|}{2}\text{ for } y\in B_1.$$

So $x_0 + Ry \in \Omega$ in either case. Define

$$U(y) = R^{\alpha} u(x_0 + Ry), \quad V(y) = R^{\beta} v(x_0 + Ry).$$

Then for $y \in B_1$, (U, V) is a solution to

$$\begin{cases} (-\Delta)^{2m} U = c(y)^a V^p(y) \\ (-\Delta)^{2m} V = c(y)^b U^q(y) \end{cases}$$

with $c(y) = \left|y + \frac{x_0}{R}\right|$. Recall that $\left|y + \frac{x_0}{R}\right| \in [1, 3]$ for $y \in B_1$ and $\|c(y)\|_{C^1} \leq C$. Apply Lemma 3.6 we yield

$$U(0) + V(0) \leq C.$$

From which it follows

$$u(x_0) < CR^{-\alpha}, \; v(x_0) < CR^{-\beta},$$

the conclusion then follows. ■

Proof of Theorem 3.2.3.

Assume (u, v) is a solution of (1.0.1) on $\mathbb{R}^N$ (bounded or not). Then for each $x_0 \in \mathbb{R}^N$ and $R > 0$, by applying Lemma 3.7 in $\Omega = B(x_{0,}R)$, we obtain

$$u(x_0) \leq CR^{-\alpha}, \quad v(x_0) \leq CR^{-\beta}.$$

Letting $R \to \infty$, we obtain

$$u(x_0) = v(x_0) = 0,$$

therefore

$$u \equiv v \equiv 0.$$

3.2.2 Nonexistence of bounded positive solutions with slow decays

Proof of Theorem 3.2.1.

In this section, we focus our attention to bounded positive solutions and prove Theorem 3.2.1.

We shall adapt Souplet's idea [17] of a measure and feedback argument combined with Rellich-Pohazaev identity. Lemma 3.5 implies

$$F(R) \leq CG_1(R) + CG_2(R),$$

where

$$G_1(R) = R^N \sum_{l=0}^{m} \int_{S^{N-1}} |u_l| \, |v_{m-l}| \, ds$$

and

$$G_2(R) = R^N \int_{S^{N-1}} \sum_{l=0}^{m-1} \left(\left| u'_{m-l-1} \right| + R^{-1} \left| u_{m-l-1} \right| \right) \left(\left| v'_l \right| + R^{-1} \left| v_l \right| \right) ds.$$

Following Souplet's idea, we shall prove there exist constants C, $a > 0$, $b < 1$ such that

$$F(R) \leq CR^{-a} F^b(R). \tag{3.2.7}$$

It then follows

$$F(R) \to 0 \quad \text{as} \quad R \to \infty,$$

which implies

$$u = v \equiv 0.$$

To prove (3.2.7) , we follow a similar procedure as [17]. We shall first estimate $G_1(R)$ and $G_2(R)$ in terms of highest derivatives of the solution (u, v) and (u, v) in suitable L^p spaces. Then use a feedback and measure argument to evaluate those bounds in terms of $F(R)$.

Step1. *Estimation of* $G_1(R)$ *in terms of suitable norms of* $D_x^{2m}u(R)$ *and* $D_x^{2m}v(R)$.

Fix $l \in \{0, 1, \cdots m\}$, Hölder's inequality gives

$$\begin{aligned} &\int_{S^{N-1}} |u_l| \, |v_{m-l}| \, ds \\ \leq\ & \|u_l\|_{\alpha_l} \|v_{m-l}\|_{\alpha_l'}, \end{aligned}$$

where $\frac{1}{\alpha_l} + \frac{1}{\alpha_l'} = 1$ is chosen so that

$$\begin{aligned} \frac{1}{k} - \frac{2m-2l}{N-1} &\leq \frac{1}{\alpha_l} \leq 1 - \frac{2m-2l}{N-1}, \\ \frac{1}{d} - \frac{2l}{N-1} &\leq 1 - \frac{1}{\alpha_l} \leq 1 - \frac{2l}{N-1}. \end{aligned} \tag{3.2.8}$$

Here

$$\begin{aligned} \frac{1}{k} &= \frac{2mp(q+1) + a + bp}{2m(p+1)(q+1) + a(q+1) + b(p+1)}, \\ \frac{1}{d} &= \frac{2mq(p+1) + aq + b}{2m(p+1)(q+1) + a(q+1) + b(p+1)}. \end{aligned}$$

Such α_l exists since by assumption,

$$\frac{N+a}{p+1}+\frac{N+b}{q+1}>N-2m>N-1-2m.$$

Let

$$\begin{aligned}\frac{1}{\gamma_l} &= \frac{1}{k}-\frac{2m-2l}{N-1}, \frac{1}{\delta_l}=\frac{N-2m+2l-1}{N-1},\\ \frac{1}{\omega_l} &= \frac{1}{d}-\frac{2l}{N-1}, \frac{1}{\psi_l}=\frac{N-2l-1}{N-1}.\end{aligned}$$

Case I: $\gamma_l > 0$, $\omega_l > 0$.

By Hölder's inequality, we have

$$\begin{aligned}\|u_l\|_{\alpha_l} &\le \|u_l\|_{\delta_l}^{\nu_{1l}}\|u_l\|_{\gamma_l}^{1-\nu_{1l}},\\ \|v_{m-l}\|_{\alpha_l'} &\le \|v_{m-l}\|_{\psi_l}^{\nu_{2l}}\|v_{m-l}\|_{\omega_l}^{1-\nu_{2l}},\end{aligned} \tag{3.2.9}$$

with

$$\begin{aligned}\frac{1}{\alpha_l} &= \frac{\nu_{1l}}{\delta_l}+\frac{1-\nu_{1l}}{\gamma_l},\\ \frac{1}{\alpha_l'} &= \frac{\nu_{2l}}{\psi_l}+\frac{1-\nu_{2l}}{\omega_l}.\end{aligned}$$

Applying Lemma 2.5, we deduce

$$\begin{aligned}\|u_l\|_{\delta_l} &\le C\left(\left\|D_\theta^{2m-2l}u_l\right\|_{1+\varepsilon}+\|u_l\|_1\right) \qquad (3.2.10)\\ &\le C\left(R^{2m-2l}\left\|D_x^{2m-2l}u_l\right\|_{1+\varepsilon}+\|u_l\|_1\right),\end{aligned}$$

$$\begin{aligned} \|u_l\|_{\gamma_l} &\leq C\left(\left\|D_\theta^{2m-2l}u_l\right\|_k + \|u_l\|_1\right) \\ &\leq C\left(R^{2m-2l}\left\|D_x^{2m-2l}u_l\right\|_k + \|u_l\|_1\right), \end{aligned} \tag{3.2.11}$$

and

$$\begin{aligned} \|v_{m-l}\|_{\psi_l} &\leq C\left(\left\|D_\theta^{2l}v_{m-l}\right\|_{1+\varepsilon} + \|v_{m-l}\|_1\right) \\ &\leq C\left(R^{2l}\left\|D_x^{2l}v_{m-l}\right\|_{1+\varepsilon} + \|v_{m-l}\|_1\right), \end{aligned} \tag{3.2.12}$$

$$\begin{aligned} \|v_{m-l}\|_{\omega_l} &\leq C\left(\left\|D_\theta^{2l}v_{m-l}\right\|_d + \|v_{m-l}\|_1\right) \\ &\leq C\left(R^{2l}\left\|D_x^{2l}v_{m-l}\right\|_d + \|v_{m-l}\|_1\right). \end{aligned} \tag{3.2.13}$$

Combining $(3.2.9), (3.2.10), (3.2.11), (3.2.12)$ and $(3.2.13)$, we conclude

$$\begin{aligned} &\int_{S^{N-1}} |u_l|\,|v_{m-l}|\,ds \\ \leq\ & \|u_l\|_{\alpha_l}\|v_{m-l}\|_{\alpha_l'} \\ \leq\ & CR^{2m}\left(\left\|D_x^{2m-2l}u_l\right\|_{1+\varepsilon} + R^{-2m+2l}\|u_l\|_1\right)^{\nu_{1l}} \\ & \cdot\left(\left\|D_x^{2m-2l}u_l\right\|_k + R^{-2m+2l}\|u_l\|_1\right)^{1-\nu_{1l}} \\ & \cdot\left(\left\|D_x^{2l}v_{m-l}\right\|_{1+\varepsilon} + R^{-2l}\|v_{m-l}\|_1\right)^{\nu_{2l}} \\ & \cdot\left(\left\|D_x^{2l}v_{m-l}\right\|_d + R^{-2l}\|v_{m-l}\|_1\right)^{1-\nu_{2l}}. \end{aligned} \tag{3.2.14}$$

Case II: Either $\gamma_l \leq 0$ or $\omega_l \leq 0$ but not both. We can take $\nu_{1l} = 1$ (if $\gamma_l \leq 0$) or $\nu_{2l} = 1$ (if $\omega_l \leq 0$), it is easy to see that (3.2.14) still follows.

Case III: Both $\gamma_l \leq 0$ and $\omega_l \leq 0$. This is equivalent to

$$\frac{2m(q+1)+aq+b}{2m(p+1)(q+1)+a(q+1)+b(p+1)} > 1-\frac{2m-2l}{N-1}$$

and

$$\frac{2m(p+1)+a+bp}{2m(p+1)(q+1)+a(q+1)+b(p+1)} > 1-\frac{2l}{N-1},$$

which gives

$$\frac{2m(p+2+q)+a(q+1)+b(p+1)}{2m(p+1)(q+1)+a(q+1)+b(p+1)} > 2-\frac{2m}{N-1}.$$

Contradiction to $pq > 1$ and $N \geq 2m+1$.

From (3.2.14) we obtain the following upper bound on $G_1(R)$.

$$\begin{aligned} G_1(R) \leq\ & CR^{N+2m}\sum_{l=0}^{m}\Big\{\left(\left\|D_x^{2m-2l}u_l\right\|_{1+\varepsilon}+R^{-2m+2l}\left\|u_l\right\|_1\right)^{\nu_{1l}} \\ & \cdot\left(\left\|D_x^{2m-2l}u_l\right\|_k+R^{-2m+2l}\left\|u_l\right\|_1\right)^{1-\nu_{1l}} \\ & \cdot\left(\left\|D_x^{2l}v_{m-l}\right\|_{1+\varepsilon}+R^{-2l}\left\|v_{m-l}\right\|_1\right)^{\nu_{2l}} \\ & \cdot\left(\left\|D_x^{2l}v_{m-l}\right\|_d+R^{-2l}\left\|v_{m-l}\right\|_1\right)^{1-\nu_{2l}}\Big\}. \end{aligned} \tag{3.2.15}$$

Step 2. *Estimation of $G_2(R)$ in terms of sutiable norms of $D_x^{2m}u(R)$ and $D_x^{2m}v(R)$.*

Fix $l \in \{0,1,2,\cdots,m-1\}$. For $\frac{1}{\beta_l}+\frac{1}{\beta_l'}=1$,

$$\begin{aligned} & \int_{S^{N-1}}\left(\left|u_{m-l-1}'\right|+R^{-1}\left|u_{m-l-1}\right|\right)\left(\left|v_l'\right|+R^{-1}\left|v_l\right|\right) \\ \leq\ & \left(\left\|u_{m-l-1}'\right\|_{\beta_l}+R^{-1}\left\|u_{m-l-1}\right\|_{\beta_l}\right)\left(\left\|v_l'\right\|_{\beta_l'}+R^{-1}\left\|v_l\right\|_{\beta_l'}\right). \end{aligned} \tag{3.2.16}$$

By Lemma 2.5 and Hölder inequality,

$$\begin{aligned} R^{-1}\left\|u_{m-l-1}\right\|_{\beta_l} &\leq CR^{-1}\left(\left\|D_\theta u_{m-l-1}\right\|_{\beta_l}+\left\|u_{m-l-1}\right\|_1\right) \\ &\leq C\left(\left\|D_x u_{m-l-1}\right\|_{\beta_l}+R^{-1}\left\|u_{m-l-1}\right\|_1\right), \end{aligned} \tag{3.2.17}$$

$$\begin{aligned} R^{-1}\left\|v_l\right\|_{\beta_l'} &\leq CR^{-1}\left(\left\|D_\theta v_l\right\|_{\beta_l'}+\left\|v_l\right\|_1\right) \\ &\leq C\left(\left\|D_x v_l\right\|_{\beta_l'}+R^{-1}\left\|v_l\right\|_1\right). \end{aligned} \tag{3.2.18}$$

By Lemma 2.5 for $\frac{1}{\rho_l}=\frac{1}{k}-\frac{2l+1}{N-1}$

$$\begin{aligned} \left\|D_x u_{m-l-1}\right\|_{\rho_l} &\leq C\left(\left\|D_\theta^{2l+1}D_x u_{m-l-1}\right\|_k+\left\|D_x u_{m-l-1}\right\|_1\right) \\ &\leq C\left(R^{2l+1}\left\|D_x^{2l+1}D_x u_{m-l-1}\right\|_k+\left\|D_x u_{m-l-1}\right\|_1\right), \end{aligned} \tag{3.2.19}$$

and for $\frac{1}{\sigma_l}=\frac{1}{d}-\frac{2m-2l-1}{N-1}$

$$\begin{aligned} \left\|D_x v_l\right\|_{\sigma_l} &\leq C\left(\left\|D_\theta^{2m-2l-1}D_x v_l\right\|_d+\left\|D_x v_l\right\|_1\right) \\ &\leq C\left(R^{2m-2l-1}\left\|D_x^{2m-2l-1}D_x v_l\right\|_d+\left\|D_x v_l\right\|_1\right). \end{aligned} \tag{3.2.20}$$

For $\eta_l=\frac{N-1}{N-2l-2}$, $\kappa_l=\frac{N-1}{N-2m+2l}$, Lemma 2.5 implies

$$\begin{aligned} \left\|D_x u_{m-l-1}\right\|_{\eta_l} &\leq C\left(\left\|D_\theta^{2l+1}D_x u_{m-l-1}\right\|_{1+\varepsilon}+\left\|D_x u_{m-l-1}\right\|_1\right) \\ &\leq C\left(R^{2l+1}\left\|D_x^{2l+1}D_x u_{m-l-1}\right\|_{1+\varepsilon}+\left\|D_x u_{m-l-1}\right\|_1\right) \end{aligned}$$

and

$$\begin{aligned}
\|D_x v_l\|_{\kappa_l} &\leq C\left(\left\|D_\theta^{2m-2l-1} D_x v_l\right\|_{1+\varepsilon} + \|D_x v_l\|_1\right) \\
&\leq C\left(R^{2m-2l-1}\left\|D_x^{2m-2l-1} D_x v_l\right\|_{1+\varepsilon} + \|D_x v_l\|_1\right).
\end{aligned}$$

Assumption $\frac{N+a}{p+1} + \frac{N+b}{q+1} > N - 2m$ implies $\frac{1}{\rho_l} + \frac{1}{\sigma_l} < 1$.

Therefore we can pick $\beta_l = z_l \in (1, \infty)$ in (3.2.16) such that

$$\begin{aligned}
\frac{1}{k} - \frac{2l+1}{N-1} &\leq \frac{1}{z_l} \leq 1 - \frac{2l+1}{N-1}, \\
\frac{1}{d} - \frac{2m-2l-1}{N-1} &\leq 1 - \frac{1}{z_l} \leq 1 - \frac{2m-2l-1}{N-1}.
\end{aligned} \tag{3.2.21}$$

Case I: $\rho_l > 0$, $\sigma_l > 0$. Hölder's inequality gives

$$\begin{aligned}
\|D_x u_{m-l-1}\|_{z_l} &\leq \|D_x u_{m-l-1}\|_{\eta_l}^{\tau_{1l}} \|D_x u_{m-l-1}\|_{\rho_l}^{1-\tau_{1l}} \\
&\leq C\left(R^{2l+1}\left\|D_x^{2l+1} D_x u_{m-l-1}\right\|_{1+\varepsilon} + \|D_x u_{m-l-1}\|_1\right)^{\tau_{1l}} \\
&\quad \cdot \left(R^{2l+1}\left\|D_x^{2l+1} D_x u_{m-l-1}\right\|_k + \|D_x u_{m-l-1}\|_1\right)^{1-\tau_{1l}} \\
&= CR^{2l+1}\left(\left\|D_x^{2l+1} D_x u_{m-l-1}\right\|_{1+\varepsilon} + R^{-2l-1}\|D_x u_{m-l-1}\|_1\right)^{\tau_{1l}} \\
&\quad \cdot \left(\left\|D_x^{2l+1} D_x u_{m-l-1}\right\|_k + R^{-2l-1}\|D_x u_{m-l-1}\|_1\right)^{1-\tau_{1l}},
\end{aligned} \tag{3.2.22}$$

where

$$\frac{1}{z_l} = \frac{\tau_{1l}}{\eta_l} + \frac{1-\tau_{1l}}{\rho_l},$$

and

$$\begin{aligned}
\|D_x v_l\|_{z_l'} &\leq \|D_x v_l\|_{\kappa_l}^{\tau_{2l}} \|D_x v_l\|_{\sigma_l}^{1-\tau_{2l}} \\
&\leq C\left(R^{2m-2l-1}\left\|D_x^{2m-2l-1}D_x v_l\right\|_{1+\varepsilon}+\|D_x v_l\|_1\right)^{\tau_{2l}} \\
&\quad \cdot \left(R^{2m-2l-1}\left\|D_x^{2m-2l-1}D_x v_l\right\|_d+\|D_x v_l\|_1\right)^{1-\tau_{2l}} \\
&= CR^{2m-2l-1}\left(\left\|D_x^{2l+1}D_x u_{m-l-1}\right\|_{1+\varepsilon}+R^{-2m+2l+1}\|D_x v_l\|_1\right)^{\tau_{2l}} \\
&\quad \cdot \left(\left\|D_x^{2m-2l-1}D_x v_l\right\|_d+R^{-2m+2l+1}\|D_x v_l\|_1\right)^{1-\tau_{2l}}, \qquad (3.2.23)
\end{aligned}$$

with

$$1-\frac{1}{z_l}=\frac{1}{z_l'}=\frac{\tau_{2l}}{\kappa_l}+\frac{1-\tau_{2l}}{\sigma_l}.$$

Combining $(3.2.17), (3.2.18), (3.2.19), (3.2.20), (3.2.22), (3.2.23)$ we have

$$\begin{aligned}
&\int_{S^{N-1}}\left(\left|u_{m-l-1}'\right|+R^{-1}\left|u_{m-l-1}\right|\right)\left(\left|v_l'\right|+R^{-1}\left|v_l\right|\right) \\
\leq\ &\left(\left\|u_{m-l-1}'\right\|_{z_l}+R^{-1}\left\|u_{m-l-1}\right\|_{z_l}\right)\left(\left\|v_l'\right\|_{z_l'}+R^{-1}\left\|v_l\right\|_{z_l'}\right) \\
\leq\ &C\left(\left\|D_x u_{m-l-1}\right\|_{z_l}+R^{-1}\left\|u_{m-l-1}\right\|_1\right)\left(\left\|D_x v_l\right\|_{z_l'}+R^{-1}\left\|v_l\right\|_1\right) \\
\leq\ &CR^{2m}\left(\left\|D_x^{2l+1}D_x u_{m-l-1}\right\|_{1+\varepsilon}+R^{-2l-1}\left\|D_x u_{m-l-1}\right\|_1+R^{-2l-2}\left\|u_{m-l-1}\right\|_1\right)^{\tau_{1l}} \\
&\cdot\left(\left\|D_x^{2l+1}D_x u_{m-l-1}\right\|_k+R^{-2l-1}\left\|D_x u_{m-l-1}\right\|_1+R^{-2l-2}\left\|u_{m-l-1}\right\|_1\right)^{1-\tau_{1l}} \\
&\cdot\left(\left\|D_x^{2l+1}D_x u_{m-l-1}\right\|_{1+\varepsilon}+R^{-2m+2l+1}\left\|D_x v_l\right\|_1+R^{-2m+2l}\left\|v_l\right\|\right)^{\tau_{2l}} \\
&\cdot\left(\left\|D_x^{2l+1}D_x u_{m-l-1}\right\|_d+R^{-2m+2l+1}\left\|D_x v_l\right\|_1+R^{-2m+2l}\left\|v_l\right\|\right)^{1-\tau_{2l}}. \qquad (3.2.24)
\end{aligned}$$

Case II: $\sigma_l \leq 0$ or $\rho_l \leq 0$ but not both. We can take $\tau_{1l}=1$ (if $\rho_l \leq 0$) or $\tau_{2l}=1$ (if $\sigma_l \leq 0$), it is easy to see that (3.2.24) still holds.

Case III: Both $\sigma_l \leq 0$ and $\rho_l \leq 0$. This is equivalent to

$$\frac{2m(q+1)+aq+b}{2m(p+1)(q+1)+a(q+1)+b(p+1)} > 1 - \frac{2m-2l-1}{N-1}$$

and

$$\frac{2m(p+1)+a+bp}{2m(p+1)(q+1)+a(q+1)+b(p+1)} > 1 - \frac{2l+1}{N-1},$$

which gives

$$\frac{2m(p+2+q)+a(q+1)+b(q+1)}{2m(p+1)(q+1)+a(q+1)+b(p+1)} > 2 - \frac{2m}{N-1}.$$

Contradiction to $pq > 1$ and $N \geq 2m+1$.

It follows from (3.2.24) that

$$\begin{aligned} G_2(R) &\leq CR^N \sum_{l=0}^{m-1} \int_{S^{N-1}} \left(\left|u'_{m-l-1}\right| + R^{-1}\left|u_{m-l-1}\right|\right)\left(\left|v'_l\right| + R^{-1}\left|v_l\right|\right) \\ &\leq CR^{N+2m} \\ &\quad \cdot \sum_{l=1}^{m-1} \left\{ \left(\left\| D_x^{2l+1} D_x u_{m-l-1} \right\|_{1+\varepsilon} + R^{-2l-1} \left\| D_x u_{m-l-1} \right\|_1 + R^{-2l-2} \left\| u_{m-l-1} \right\|_1 \right)^{\tau_{1l}} \right. \\ &\quad \cdot \left(\left\| D_x^{2l+1} D_x u_{m-l-1} \right\|_k + R^{-2l-1} \left\| D_x u_{m-l-1} \right\|_1 + R^{-2l-2} \left\| u_{m-l-1} \right\|_1 \right)^{1-\tau_{1l}} \\ &\quad \cdot \left(\left\| D_x^{2m-2l-1} D_x v_l \right\|_{1+\varepsilon} + R^{-2m+2l+1} \left\| D_x v_l \right\|_1 + R^{-2m+2l} \left\| v_l \right\|_1 \right)^{\tau_{2l}} \\ &\quad \left. \cdot \left(\left\| D_x^{2m-2l-1} D_x v_l \right\|_d + R^{-2m+2l+1} \left\| D_x v_l \right\|_1 + R^{-2m+2l} \left\| v_l \right\|_1 \right)^{1-\tau_{2l}} \right\}. \end{aligned} \quad (3.2.25)$$

Step 3. *Measure and Feedback argument.*

We first define the following set

$$\Gamma_0^1(R) = \left\{ r \in (R, 2R) : \|v(r)\|_p^p > KR^{-2m-\alpha-a} \right\},$$

$$\Gamma_0^2(R) = \left\{ r \in (R, 2R) : \|u(r)\|_q^q > KR^{-2m-\beta-b} \right\},$$

$$\Gamma_1(R) = \left\{ r \in (R, 2R) : \left\|D_x^{2m} u(r)\right\|_k^k > KR^{-N} F(4R) \right\},$$

$$\Gamma_2(R) = \left\{ r \in (R, 2R) : \left\|D_x^{2m} v(r)\right\|_d^d > KR^{-N} F(4R) \right\},$$

$$\Gamma_3(R) = \left\{ r \in (R, 2R) : \left\|D_x^{2m} u\right\|_{1+\varepsilon}^{1+\varepsilon} > KR^{-2m-\alpha+a\epsilon} \right\},$$

$$\Gamma_4(R) = \left\{ r \in (R, 2R) : \left\|D_x^{2m} v\right\|_{1+\varepsilon}^{1+\varepsilon} > KR^{-2m-\beta+b\epsilon} \right\}.$$

For fixed $l \in \{1, 2, \cdots, m-1\}$

$$\Gamma_{5l}(R) = \left\{ r \in (R, 2R) : \|u_{m-l-1}(r)\|_1 > KR^{-\alpha-2(m-l-1)} \right\},$$

$$\Gamma_{6l}(R) = \left\{ r \in (R, 2R) : \|v_l(r)\|_1 > KR^{-\beta-2l} \right\},$$

$$\Gamma_{7l}(R) = \left\{ r \in (R, 2R) : \|D_x u_{m-l-1}(r)\|_1 > KR^{-\alpha-2(m-l-1)-1} \right\},$$

$$\Gamma_{8l}(R) = \left\{ r \in (R, 2R) : \|D_x v_l(r)\|_1 > KR^{-\beta-2l-1} \right\}.$$

From Lemma 3.2 we deduce

$$cR^{N-2m-\alpha} \geq \int_R^{2R} r^a \|v(r)\|_p^p r^{N-1} dr \geq \left|\Gamma_0^1(R)\right| KR^{-2m-\alpha} R^{N-1},$$

which implies

$$\left|\Gamma_0^1(R)\right| < \frac{1}{4m+8} R$$

for $K \gg 1$. Similarly, we get

$$\left|\Gamma_0^2(R)\right| < \frac{1}{4m+8} R$$

for $K \gg 1$.

To estimate $\Gamma_1(R)$, by (3.1.10) in Lemma 3.4,

$$\begin{aligned} CF(4R) &\geq \int_R^{2R} \left\| D_x^{2m} u \right\|_k^k r^{N-1} dr \\ &\geq |\Gamma_1(R)| \, K R^{-N} F(4R) \, R^{N-1} \\ &= |\Gamma_1(R)| \, K R^{-1} F(4R), \end{aligned}$$

From which it follows that for $K \gg 1$

$$|\Gamma_1(R)| < \frac{1}{4m+8} R.$$

Similarly we deduce from (3.1.11), (3.1.12) and (3.1.13) in Lemma 3.4 that

$$|\Gamma_2(R)| < \frac{1}{4m+8} R, \ \ |\Gamma_3(R)| < \frac{1}{4m+8} R, \ \ |\Gamma_4(R)| < \frac{1}{4m+8} R.$$

By (3.1.6) in Lemma 3.4,

$$\begin{aligned} CR^{N-\alpha-2(m-l-1)} &\geq \int_0^{2R} \left\| u_{m-l-1} \right\|_1 r^{N-1} dr \\ &\geq |\Gamma_{5l}(R)| \, K R^{-\alpha-2(m-l-1)} R^{N-1}, \end{aligned}$$

which gives

$$|\Gamma_{5l}(R)| < \frac{1}{4m+8} R$$

when $K \gg 1$ and similarly (3.1.7), (3.1.8) and (3.1.9) implies

$$|\Gamma_{6l}(R)| < \frac{1}{4m+8} R, \ \ |\Gamma_{7l}(R)| < \frac{1}{4m+8} R, \ \ |\Gamma_{8l}(R)| < \frac{1}{4m+8} R$$

when $K \gg 1$. In particular, when $K \gg 1$,

$$\Gamma(R) = (R, 2R) \setminus \left\{ \cup_{j=1}^{2} \Gamma_0^j(R) \cup_{i=1}^{4} \Gamma_i(R) \cup_{l=1}^{m-1} \cup_{j=5}^{8} \Gamma_{jl}(R) \right\} \neq \emptyset.$$

Pick $\widetilde{R} \in \Gamma(R)$, by (3.2.15) together with the observation that $u_m = |x|^a v^p$, $v_m = |x|^b u^q$, we have

$$\begin{aligned}
G_1\left(\widetilde{R}\right) &\leq C\widetilde{R}^N \sum_{l=0}^{m} \int_{S^{N-1}} |u_l| \, |v_{m-l}| \\
&\leq C\widetilde{R}^{N+2m} \sum_{l=0}^{m} \left\{ \left(\left\| D_x^{2m-2l} u_l \right\|_{1+\varepsilon} + \widetilde{R}^{-2m+2l} \left\| u_l \right\|_1 \right)^{\nu_{1l}} \right. \\
&\quad \cdot \left(\left\| D_x^{2m-2l} u_l \right\|_k + \widetilde{R}^{-2m+2l} \left\| u_l \right\|_1 \right)^{1-\nu_{1l}} \\
&\quad \cdot \left(\left\| D_x^{2l} v_{m-l} \right\|_{1+\varepsilon} + \widetilde{R}^{-2l} \left\| v_{m-l} \right\|_1 \right)^{\nu_{2l}} \cdot \left(\left\| D_x^{2l} v_{m-l} \right\|_d + \widetilde{R}^{-2l} \left\| v_{m-l} \right\|_1 \right)^{1-\nu_{2l}} \Bigg\} \\
&\leq CR^{N+2m} \sum_{l=0}^{m} \{ R^{\frac{a\varepsilon-(2m+\alpha)}{1+\varepsilon}\nu_{1l}} \left(R^{-N} F(4R) \right)^{\frac{1-\nu_{1l}}{k}} \left(R^{\frac{b\varepsilon-(2m+\beta)}{1+\varepsilon}} + R^{-2m-\beta} \right)^{\nu_{2l}} \\
&\quad \cdot \left. \left(R^{-\frac{N}{d}} F(4R)^{\frac{1}{d}} + R^{-2m-\beta} \right)^{1-\nu_{2l}} \right\} \\
&\leq CR^{-\widehat{a}} F^{\widehat{b}}(4R),
\end{aligned}$$

with

$$\begin{aligned}
\widehat{a} &= \widehat{a}_\varepsilon = \min_l \{ -N - 2m + \frac{2m+\alpha-a\varepsilon}{1+\varepsilon}\nu_{1l} + \frac{2m+\beta-b\varepsilon}{1+\varepsilon}\nu_{2l} + \frac{N}{k}(1-\nu_{1l}) \\
&\quad + \frac{N}{d}(1-\nu_{2l}) \} \\
\widehat{b} &= \max_l \frac{1}{k}(1-\nu_{1l}) + \frac{1}{d}(1-\nu_{2l}).
\end{aligned}$$

On the other hand, we have

$$
\begin{aligned}
G_2\left(\widetilde{R}\right) \leq\ & CR^{N+2m} \\
& \cdot \sum_{l=1}^{m-1} \Big\{ \left(\left\| D_x^{2l+1} D_x u_{m-l-1} \right\|_{1+\varepsilon} + R^{-2l-1} \left\| D_x u_{m-l-1} \right\|_1 + R^{-2l-2} \left\| u_{m-l-1} \right\|_1 \right)^{\tau_{1l}} \\
& \cdot \left(\left\| D_x^{2l+1} D_x u_{m-l-1} \right\|_k + R^{-2l-1} \left\| D_x u_{m-l-1} \right\|_1 + R^{-2l-2} \left\| u_{m-l-1} \right\|_1 \right)^{1-\tau_{1l}} \\
& \cdot \left(\left\| D_x^{2m-2l-1} D_x v_l \right\|_{1+\varepsilon} + R^{-2m+2l+1} \left\| D_x v_l \right\|_1 + R^{-2m+2l} \left\| v_l \right\|_1 \right)^{\tau_{2l}} \\
& \cdot \left(\left\| D_x^{2m-2l-1} D_x v_l \right\|_d + R^{-2m+2l+1} \left\| D_x v_l \right\|_1 + R^{-2m+2l} \left\| v_l \right\|_1 \right)^{1-\tau_{2l}} \Big\} \\
\leq\ & CR^{N+2m} \\
& \cdot \sum_{l=1}^{m-1} \Big\{ \left(\left\| D_x^{2m} u \right\|_{1+\varepsilon} + R^{-2l-1} \left\| D_x u_{m-l-1} \right\|_1 + R^{-2l-2} \left\| u_{m-l-1} \right\|_1 \right)^{\tau_{1l}} \\
& \cdot \left(\left\| D_x^{2m} u \right\|_k + R^{-2l-1} \left\| D_x u_{m-l-1} \right\|_1 + R^{-2l-2} \left\| u_{m-l-1} \right\|_1 \right)^{1-\tau_{1l}} \\
& \cdot \left(\left\| D_x^{2m} v \right\|_{1+\varepsilon} + R^{-2m+2l+1} \left\| D_x v_l \right\|_1 + R^{-2m+2l} \left\| v_l \right\|_1 \right)^{\tau_{2l}} \\
& \cdot \left(\left\| D_x^{2m} v \right\|_d + R^{-2m+2l+1} \left\| D_x v_l \right\|_1 + R^{-2m+2l} \left\| v_l \right\|_1 \right)^{1-\tau_{2l}} \Big\} \\
\leq\ & CR^{N+2m} \\
& \cdot \sum_{l=1}^{m-1} \Big\{ \left(R^{\frac{a\varepsilon-(2m+\alpha)}{1+\varepsilon}} + R^{-2l-1} R^{-\alpha-2(m-l-1)-1} + R^{-2l-2} R^{-\alpha-2(m-l-1)} \right)^{\tau_{1l}} \\
& \cdot \left(R^{-N/k} F^{\frac{1}{k}}(4R) + R^{-2l-1} R^{-\alpha-2(m-l-1)-1} + R^{-2l-2} R^{-\alpha-2(m-l-1)} \right)^{1-\tau_{1l}} \\
& \cdot \left(R^{\frac{b\varepsilon-(2m+\beta)}{1+\varepsilon}} + R^{-2l-1} R^{-\beta-2(m-l-1)-1} + R^{-2l-2} R^{-\beta-2(m-l-1)} \right)^{\tau_{2l}} \\
& \cdot \left(R^{-N/d} F^{\frac{1}{d}}(4R) + R^{-2l-1} R^{-\beta-2(m-l-1)-1} + R^{-2l-2} R^{-\beta-2(m-l-1)} \right)^{1-\tau_{2l}} \Big\} \\
\leq\ & CR^{-\overline{a}} F^{\overline{b}}(4R).
\end{aligned}
$$

Here

$$
\overline{a} = \overline{a}_\varepsilon = \min_l \{ -N - 2m + \frac{2m+\alpha-a\varepsilon}{1+\varepsilon}\tau_{1l} + \frac{2m+\beta-b\varepsilon}{1+\varepsilon}\tau_{2l} + \frac{N}{k}(1-\tau_{1l}) + \frac{N}{d}(1-\tau_{2l}) \}
$$

and

$$\overline{b} = \max_{l} \frac{1}{k} (1 - \tau_{1l}) + \frac{1}{d} (1 - \tau_{2l}) .$$

We claim that there exists a constant $M > 0$ and a sequence $R_i \to \infty$ such that

$$F (4R_i) \leq MF (R_i) .$$

Otherwise for any $M > 0$, there exists R_M such that for $R \geq R_M$

$$F (4R) > MF (R) .$$

Since (u, v) is bounded, we have $F (R) \leq CR^N$, $R > 0$. Thus

$$M^i F (R_M) \leq F \left(4^i R_M\right) \leq CR_M^N \left(4^N\right)^i .$$

Contradiction for i large if $M > 4^N$.

Assume we have shown $a = a_\varepsilon = \min (\widehat{a_\varepsilon} , \overline{a_\varepsilon}) > 0$, $b = b_\varepsilon = \max \left(\widehat{b},\ \overline{b}\right) < 1$, we have

$$\begin{aligned} F (R_i) &\leq CR^{-a} F^b (4R_i) \\ &\leq CM^b R^{-a} F^b (R_i) , \end{aligned}$$

which gives

$$F (R_i) \leq CR_i^{-\frac{a}{1-b}}.$$

Letting $i \to \infty$, we deduce that

$$\int_{\mathbb{R}^n} \left[|x|^a u^{q+1} + |x|^b v^{p+1} \right] dx = 0,$$

hence $u = v \equiv 0$, a contradiction.

Step 4. *If* $\alpha > N - 2m - 1$, *then* $\overline{b}, \widehat{b} < 1$ *and* $\overline{a_\varepsilon}, \widehat{a_\varepsilon} > 0$ *for* $\varepsilon \ll 1$.

First we show $\widehat{a_\varepsilon} > 0$, $\widehat{b} < 1$.

Since $\nu_{1l} = \left(\frac{1}{\delta_l} - \frac{1}{\gamma_l}\right)^{-1} \left(\frac{1}{\alpha_l} - \frac{1}{\gamma_l}\right)$ and $\nu_{2l} = \left(\frac{1}{\psi_l} - \frac{1}{\omega_l}\right)^{-1} \left(\frac{1}{\alpha_l'} - \frac{1}{\omega_l}\right)$,
to show $\widehat{b} < 1$, we need to show for all l,

$$\begin{aligned} & \frac{1}{k}(1 - \nu_{1l}) + \frac{1}{d}(1 - \nu_{2l}) \\ = \;& \widetilde{p}\widehat{A}_{1l} + \widetilde{q}\widehat{A}_{2l} \\ = \;& \widetilde{p}\left(\frac{N - 2m + 2l - 1}{N - 1} - \frac{1}{\alpha_l}\right) + \widetilde{q}\left(\frac{1}{\alpha_l} - \frac{2l}{N - 1}\right) < 1. \end{aligned} \tag{3.2.26}$$

Here

$$\widetilde{p} = \frac{2mp(q+1) + a + bp}{2m(q+1) + aq + b},\ \widetilde{q} = \frac{2mq(p+1) + aq + b}{2m(p+1) + a + bp}.$$

It then follows that

$$k = \frac{\widetilde{p} + 1}{\widetilde{p}},\ d = \frac{\widetilde{q} + 1}{\widetilde{q}}.$$

And

$$\alpha \geq \beta$$

implies

$$\widetilde{p} \geq \widetilde{q}.$$

We have

$$\frac{1}{\widetilde{q}+1} = \frac{2m\,(p+1) + a + bp}{2m\,(p+1)\,(q+1) + a\,(q+1) + b\,(p+1)}.$$

$$\begin{aligned}
\widehat{A}_{1l} &= \frac{1}{\widetilde{p}+1}\,(1-\nu_{1l}) \\
&= \left(\frac{N-2m+2l-1}{N-1} - \frac{1}{\alpha_l}\right), \\
\widehat{A}_{2l} &= \frac{1}{\widetilde{q}+1}\,(1-\nu_{2l}) \\
&= \left(\frac{1}{\alpha_l} - \frac{2l}{N-1}\right).
\end{aligned}$$

(3.2.26) is equivalent to

$$\widetilde{p}\frac{N-2m+2l-1}{N-1} - \widetilde{q}\frac{2l}{N-1} - 1 < \frac{\widetilde{p}-\widetilde{q}}{\alpha_l}. \tag{3.2.27}$$

Recall α_l is chosen to satisfy (3.2.8). Such $\alpha_l \in (1,\infty)$ satisfying (3.2.8) and (3.2.27) exists provided

$$\max\left(\frac{1}{k} - \frac{2m-2l}{N-1}, \frac{2l}{N-1}\right) \leq \min\left(1 - \frac{2m-2l}{N-1}, \frac{1}{\widetilde{q}+1} + \frac{2l}{N-1}\right) \tag{3.2.28}$$

and

$$\widetilde{p}\frac{N-2m+2l-1}{N-1} - \widetilde{q}\frac{2l}{N-1} - 1 < (\widetilde{p}-\widetilde{q})\left(1 - \frac{2m-2l}{N-1}\right), \tag{3.2.29}$$

$$\widetilde{p}\frac{N-2m+2l-1}{N-1} - \widetilde{q}\frac{2l}{N-1} - 1 < (\widetilde{p}-\widetilde{q})\left(\frac{1}{\widetilde{q}+1} + \frac{2l}{N-1}\right). \tag{3.2.30}$$

(3.2.28) follows from the assumption that $N \geq 2m+1$ and

$$\frac{1+\frac{a}{N}}{p+1}+\frac{1+\frac{b}{N}}{q+1} > 1-\frac{2m}{N} > 1-\frac{2m}{N-1}.$$

(3.2.29) is equivalent to

$$\widetilde{q}\frac{N-1-2m}{N-1} < 1,$$

which follows from

$$\widetilde{q} \leq \frac{\widetilde{p}\,(\widetilde{q}+1)}{\widetilde{p}+1} = 1+\frac{2m}{\alpha} < 1+\frac{2m}{N-2m-1} = \frac{N-1}{N-1-2m}.$$

And (3.2.30) can be rewritten as

$$\frac{N-2m-1}{N-1}\widetilde{p} < \frac{\widetilde{p}+1}{\widetilde{q}+1},$$

which is equivalent to

$$\alpha > N-2m-1.$$

Finally, since

$$(2m+\alpha)\,(k-1)=\beta, \quad (2m+\beta)\,(d-1)=\alpha,$$

we can write for each l

$$\begin{aligned}
\widehat{a_{0l}} &= -N-2m+(2m+\alpha)\,\nu_{1l}+(2m+\beta)\,\nu_{2l}+\frac{N}{k}\,(1-\nu_{1l})+\frac{N}{d}\,(1-\nu_{2l}) \\
&= 2m-N+\alpha+\beta+(N-2m-\alpha-\beta)\left(\widetilde{p}\widehat{A}_{1l}+\widetilde{q}\widehat{A}_{2l}\right) \\
&= (2m-N+\alpha+\beta)\left(1-\widetilde{p}\widehat{A}_{1l}-\widetilde{q}\widehat{A}_{2l}\right) \\
&> 0.
\end{aligned}$$

It then follows $\widehat{a_0} > 0$, thus $\widehat{a_\varepsilon} > 0$ for $\varepsilon \ll 1$.

Secondly we show $\overline{a_\varepsilon} > 0$, $\overline{b} < 1$. This can be shown in a similar way as $\widehat{a_\varepsilon}, \widehat{b}$. We write all details for readers' convenience. Since $\tau_{1l} = \left(\frac{1}{\eta_l} - \frac{1}{\rho_l}\right)^{-1}\left(\frac{1}{z_l} - \frac{1}{\rho_l}\right)$, $\tau_{2l} = \left(\frac{1}{\kappa_l} - \frac{1}{\sigma_l}\right)^{-1}\left(\frac{1}{z_l'} - \frac{1}{\sigma_l}\right)$, to show $\overline{b} < 1$, we need to show for all l,

$$\begin{aligned}
& \frac{1}{k}(1-\tau_{1l}) + \frac{1}{d}(1-\tau_{2l}) \\
= \;& \widetilde{p}A_{1l} + \widetilde{q}A_{2l} \\
= \;& \widetilde{p}\left(\frac{N-2l-2}{N-1} - \frac{1}{z_l}\right) + \widetilde{q}\left(\frac{1}{z_l} - \frac{2m-2l-1}{N-1}\right) < 1. \qquad (3.2.31)
\end{aligned}$$

Here we used

$$\begin{aligned}
A_{1l} &= \frac{1}{\widetilde{p}+1}(1-\tau_{1l}) \\
&= \left(\frac{N-2l-2}{N-1} - \frac{1}{z_l}\right), \\
A_{2l} &= \frac{1}{\widetilde{q}+1}(1-\tau_{2l}) \\
&= \left(\frac{1}{z_l} - \frac{2m-2l-1}{N-1}\right).
\end{aligned}$$

(3.2.31) is equivalent to

$$\widetilde{p}\frac{N-2l-2}{N-1} - \widetilde{q}\frac{2m-2l-1}{N-1} - 1 < \frac{\widetilde{p}-\widetilde{q}}{z_l}. \qquad (3.2.32)$$

Recall z_l is chosen to satisfy (3.2.21). Such $z_l \in (1, \infty)$ satisfying (3.2.21) and

(3.2.32) exists provided

$$\max\left(\frac{\widetilde{p}}{\widetilde{p}+1}-\frac{2l+1}{N-1},\frac{2m-2l-1}{N-1}\right)\le\min\left(1-\frac{2l+1}{N-1},\frac{1}{\widetilde{q}+1}+\frac{2m-2l-1}{N-1}\right) \tag{3.2.33}$$

and

$$\widetilde{p}\frac{N-2l-2}{N-1}-\widetilde{q}\frac{2m-2l-1}{N-1}-1<(\widetilde{p}-\widetilde{q})\left(1-\frac{2l+1}{N-1}\right), \tag{3.2.34}$$

$$\widetilde{p}\frac{N-2l-2}{N-1}-\widetilde{q}\frac{2m-2l-1}{N-1}-1<(\widetilde{p}-\widetilde{q})\left(\frac{1}{q+1}+\frac{2m-2l-1}{N-1}\right). \tag{3.2.35}$$

(3.2.33) follows from

$$\frac{1+\frac{a}{N}}{p+1}+\frac{1+\frac{b}{N}}{q+1}>1-\frac{2m}{N}>1-\frac{2m}{N-1}.$$

(3.2.34) is equivalent to

$$\widetilde{q}\frac{N-1-2m}{N-1}<1,$$

which follows from

$$\widetilde{q}\le\frac{\widetilde{p}\,(\widetilde{q}+1)}{\widetilde{p}+1}=1+\frac{2m}{\alpha}<1+\frac{2m}{N-2m-1}=\frac{N-1}{N-1-2m}.$$

And lastly (3.2.35) can be rewritten as

$$\frac{N-2m-1}{N-1}\widetilde{p}<\frac{\widetilde{p}+1}{\widetilde{q}+1},$$

which is equivalent to

$$\alpha > N - 2m - 1.$$

Finally, for each l

$$\begin{aligned}
\overline{a_{0l}} &= -2m - N + (2m+\alpha)\left(1 - (\widetilde{p}+1)\, A_{1l}\right) + (2m+\beta)\left(1 - (\widetilde{q}+1)\, A_{2l}\right) \\
&\quad + N\widetilde{p}A_{1l} + N\widetilde{q}A_{2l} \\
&= 2m - N + \alpha + \beta + (N - 2m - \alpha - \beta)\left(\widetilde{p}A_{1l} + \widetilde{q}A_{2l}\right) \\
&= (2m - N + \alpha + \beta)\left(1 - \widetilde{p}A_{1l} - \widetilde{q}A_{2l}\right) \\
&> 0.
\end{aligned}$$

It then follows $\overline{a_0} > 0$, thus $\overline{a_\varepsilon} > 0$ for $\varepsilon \ll 1$.

Chapter 4

Conclusion

4.1 Future Directions

My future research for the higher order elliptic systems are in two folds. Firstly, to remove the extra condition.

$$\max\left(\frac{2m\,(p+1)+a+bp}{pq-1}, \frac{2m\,(q+1)+aq+b}{pq-1}\right) > N-2m-1.$$

This of course requires new ideas. I will start with simple cases $a=b=0$ and try to see if there is additional identity that can be used in combination with measure and feedback argument.

Secondly, I would like to extend my result to the study of the Liouville type theorem for integral systems.

$$\begin{cases} u(x) = \int_{\mathbb{R}^N} \dfrac{v^p(y)}{|x-y|^{N-\alpha}} \\ v(x) = \int_{\mathbb{R}^N} \dfrac{u^q(y)}{|x-y|^{N-\alpha}} \end{cases} \text{in } \mathbb{R}^N, \tag{4.1.1}$$

Where $p > 0, q > 0$ and $N \geq 3$, $0 \leq \alpha \leq N$. The corresponding differential equation to the integral systems (4.1.1) is equivalent to the higher order Lane-Emden system

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u = v^p \\ (-\Delta)^{\frac{\alpha}{2}} v = u^q \end{cases} \text{in } \mathbb{R}^N\{0\}, \tag{4.1.2}$$

where $\alpha = 2m$ and $m \in N$. I would be mainly concern with for what values p and q does (4.1.1) admit no positive solutions. Chen and Li [22] showed the following. Under the integrability conditions $v \in L^{p_1}(R^N)$ and $u \in L^{q_1}(R^N)$ where $p_1 = \frac{n(pq-1)}{2(q+1)}$ and $q_1 = \frac{n(pq-1)}{2(p+1)}$, positive solutions are radial. In particular, their result solves the conjecture for Lane-Emden system when $\alpha = 2$ under the integrability assumption. Note their integrability assumption is mainly used in deriving radial symmetry via moving plane method for integral system. I would like to see if I can drop the integrability assumption.

Thirdly, I would like to extend my result to the study of the Hardy-Sobolev type integral systems

$$\begin{cases} u(x) = \int_{\mathbb{R}^N} \dfrac{v^p(y)}{|x-y|^{N-\alpha}|y|^{\sigma_1}} \mathrm{d}y \\ v(x) = \int_{\mathbb{R}^N} \dfrac{u^q(y)}{|x-y|^{N-\alpha}|y|^{\sigma_2}} \mathrm{d}y \end{cases} \text{in } \mathbb{R}^N, \tag{4.1.3}$$

Where $p > 0, q > 0$ and $N \geq 3$, $0 \leq \alpha \leq N$, $\sigma_1 \geq 0$, $\sigma_2 \geq 0$. Its corresponding system of differential equation is

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u = \frac{v^p}{|x|^{\sigma_1}} \\ (-\Delta)^{\frac{\alpha}{2}} v = \frac{u^q}{|x|^{\sigma_2}} \end{cases} \text{ in } \mathbb{R}^N\{0\}, \tag{4.1.4}$$

If $\alpha = 2m$ and $\sigma_i \in \mathbb{R}^N$ system (4.1.4) reduces to the Henon-Lane-Emden system (1.0.1). Villavert [20] showed the following. Let $p, q > 0$ and $\alpha \in [2, n)$ and $\sigma_1, \sigma_2 \in (-\infty, \alpha)$. Then the system of integral equations (4.1.3) has no positive radial solutions. In particular his results solved the conjecture for the radial solutions for the Henon-Lane-Emden system. Villavert solved the conjecture using decay estimates for radial solutions and integral forms of a Pohozaev type identity. I would like to see if I can use integral form of some type of identities combined with the measure and feedback argument to prove the general case.

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