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EXPLORING THE SPECTRUM OF THE LAPLACIAN ON $3N$ -GASKETS

MAXWELL MARGENOT

ABSTRACT. The Laplacian operator is a central object of fractal analysis. It has been shown that the Laplacian on a $3N$ -Gasket can be found using the method of spectral decimation. The $3N$ -Gasket is a family of fractals closely related to the construction of the Sierpinski Gasket, also known as the 3-Gasket. For the purpose of this paper we will explore the method of spectral decimation in finding the Laplacian on several different cases of $3N$ -Gaskets.

1. INTRODUCTION

In this paper we shall be using the method of spectral decimation in order to calculate the spectrum of the Laplacian of several examples of $3N$ -Gaskets. The Laplacian is an important linear operator that is often used to describe physical phenomena such as electric potentials or heat and fluid flows.

2. PRELIMINARIES

For the purpose of the calculations in this paper, we will require some unfamiliar definitions. First, we shall be using Chebyshev polynomials. Chebyshev polynomials can be expressed both in a recursive and an explicit fashion. We will be using Chebyshev polynomials of the first kind and of the second kind.

Definition 2.1 (Chebyshev Polynomials). Chebyshev polynomials of the first kind can be defined recursively as follows:

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad T_0(x) = 1, \quad T_1(x) = x.$$

The explicit form of $T_n(x)$ is as follows:

$$T_n(x) = \cos(n \cos^{-1}(x))$$

Chebyshev polynomials of the second kind can be defined recursively as follows:

$$U_{n+1} = 2xU_n(x) - U_{n-1}(x), \quad U_0(x) = 1, \quad U_1(x) = 2x$$

The explicit form of $U_N(x)$ is as follows:

$$U_n(x) = \frac{\sin((n+1) \cos^{-1}(x))}{\sin(\cos^{-1}(x))}$$

These definitions are required to simplify the functions that we use to derive the spectrum of a given $3N$ -Gasket. The first of these is the function $R(z)$. $R(z)$ is defined as the spectral decimation function.

Theorem 2.2. [1] *On the $3N$ -Gasket, the function $R(z)$ is rational and takes the form*

$$(2.1) \quad R(z) = \begin{cases} \frac{(z-1)\sqrt{z}U_{N-1}(\sqrt{z})(2T_N(1-2z)+2U_{N-1}(1-2z)+1)}{T_N(\sqrt{z})} & \text{if } N \text{ is even} \\ \frac{\sqrt{z}T_N(\sqrt{z})(2T_N(1-2z)+2U_{N-1}(1-2z)+1)}{U_{N-1}(\sqrt{z})} & \text{if } N \text{ is odd} \end{cases}$$

The singularities of $R(z)$ are all poles and are of the form

$$(2.2) \quad \gamma_k = \begin{cases} \cos^2\left(\frac{(m-\frac{1}{2})\pi}{N}\right) : & \text{if } N \text{ is even} \\ \cos^2\left(\frac{m\pi}{N}\right) : & \text{if } N \text{ is odd} \end{cases}$$

These functions are important because they allow us to find the spectrum of the Dirichlet boundary conditions of a fractal, signified by D . We use the Dirichlet boundary conditions to help us find the spectrum of Laplacian of the n -th level approximation of a fractal, specified as Δ_n . The n -th level approximation can be thought of as the number of iterations that the fractal has gone through. With each increase in n , we run the pattern through an iterative mapping function, further compounding it. As $n \rightarrow \infty$, we arrive at the fractal itself.

With these tools in hand, we can now find the spectrum of D .

Theorem 2.3. [1] *The spectrum of D is the union of the set of zeros of $R(z)$, the set of poles of $\phi(z)$, and $\frac{3}{2}$. Specifically, if N is even,*

$$(2.3) \quad \sigma(D) = \left\{\frac{3}{2}\right\} \cup \left\{\cos^2\left(\frac{m\pi}{N}\right) : m = 1, \dots, \frac{N}{2}\right\}$$

$$(2.4) \quad \cup \{z : T_N(\sqrt{z}) - 2(z-1)\sqrt{z}U_{N-1}(\sqrt{z}) = 0\}$$

$$(2.5) \quad \cup \{z : 2T_N(1-2z) + 2U_{N-1}(1-2z) + 1 = 0\}$$

and if N is odd,

$$(2.6) \quad \sigma(D) = \left\{\frac{3}{2}\right\} \cup \left\{\cos^2\left(\frac{(m-\frac{1}{2})\pi}{N}\right) : m = 1, \dots, \frac{N-1}{2}\right\}$$

$$(2.7) \quad \cup \{z : U_{N-1}(\sqrt{z}) - 2\sqrt{z}T_N(\sqrt{z}) = 0\}$$

$$(2.8) \quad \cup \{z : 2T_N(1-2z) + 2U_{N-1}(1-2z) + 1 = 0\}$$

Theorem 2.4. [1] *The normalized limiting distribution of eigenvalues is a pure point measure κ with the set of atoms*

$$\begin{aligned} \sigma(\Delta) &= \left\{\frac{3}{2}\right\} \cup \left(\bigcup_{k=0}^{\infty} R^{-k} \left(\left\{\sin^2\left(\frac{m\pi}{3N}\right) : m = 0, \dots, 3N-1 \text{ and } 3 \nmid m\right\}\right.\right. \\ &\quad \cup \{z : T_N(\sqrt{z}) - 2(z-1)\sqrt{z}U_{N-1}(\sqrt{z}) = 0\} \\ &\quad \left.\left.\cup \left\{\cos^2\left(\frac{m\pi}{N}\right) : m = 1, \dots, \frac{N}{2}\right\}\right)\right) \end{aligned}$$

for N even and

$$\begin{aligned} \sigma(\Delta) &= \left\{\frac{3}{2}\right\} \cup \left(\bigcup_{k=0}^{\infty} R^{-k} \left(\left\{\sin^2\left(\frac{m\pi}{3N}\right) : m = 0, \dots, 3N-1 \text{ and } 3 \nmid m\right\}\right.\right. \\ &\quad \cup \{z : T_N(\sqrt{z}) - 2(z-1)\sqrt{z}U_{N-1}(\sqrt{z}) = 0\} \\ &\quad \left.\left.\cup \left\{\cos^2\left(\frac{m\pi}{N}\right) : m = 1, \dots, \frac{N-1}{2}\right\}\right)\right) \end{aligned}$$

for N odd, where $R^{-k}(z)$ denotes the k -th inverse composition.

It is worth noting that these equations for the entire spectrum are entirely impractical for physical applications. Because the spectra of these $3N$ -Gaskets are infinite, we cannot easily use them to construct any computer models of physical processes modeled by the Laplacian.

In order to create models for more practical use, we will instead use approximations of $3N$ -Gaskets. A fractal can be divided into several different levels, signifying how many times they have been compounded upon themselves. For example

Rather than calculating the infinite set of eigenvalues for a fractal, we can simply find the spectrum of a finite approximation of the fractal, of one of its levels. This can be done using the following theorem.

Proposition 2.5. [1] *The spectrum of the Laplacian on the n -th level of a $3N$ -Gasket can be found by calculating*

$$\begin{aligned} \sigma(\Delta_n) = & \sigma(\Delta_0) \cup \left(\bigcup_{m=0}^{n-1} R^{-m} (\sigma(\Delta_1) - \sigma(\Delta_0)) \right) \\ & \cup \left(\bigcup_{m=0}^{n-2} R^{-m} (A \cup \{z : T_N(\sqrt{z}) - 2(z-1)\sqrt{z}U_{N-1}(\sqrt{z}) = 0\}) \right), \end{aligned}$$

Where

$$A = \left\{ \cos^2 \left(\frac{(m - \frac{1}{2})\pi}{N} \right) : m = 1, \dots, \frac{N}{2} \right\}$$

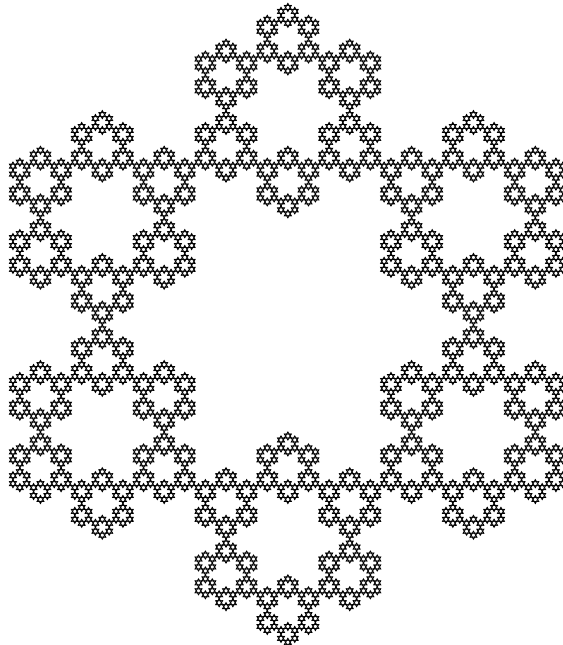
for N even and

$$A = \left\{ \cos^2 \left(\frac{m\pi}{N} \right) : m = 1, \dots, \frac{N-1}{2} \right\}$$

for N odd.

3. THE 6-GASKET

First, we will examine the 6-Gasket.



For the 6-Gasket, we know that $N = 2$. We use Theorem 2.4 with $N = 2$ and see that

$$\begin{aligned} \sigma(\Delta) &= \left\{ \frac{3}{2} \right\} \cup \left(\bigcup_{k=0}^{\infty} R_{-k} \left(\left\{ \sin^2 \left(\frac{m\pi}{6} \right) : m = 0, \dots, 5 \text{ and } 3 \nmid m \right\} \right) \right) \\ &\cup \{ z : T_2(\sqrt{z}) - 2(z-1)\sqrt{z}U_1(\sqrt{z}) = 0 \} \\ &\cup \left\{ \cos^2 \left(\frac{m\pi}{2} \right) : m = 1 \right\} \end{aligned}$$

We can see from this that our first few eigenvalues are as follows. Note that we use decimal approximations for many eigenvalues. Many of them are irrational and in order to get them to display in Mathematica, we were required to approximate them.

$$\left\{ 0, 1, \frac{3}{2}, \frac{1}{4}, \frac{3}{4}, 1.10355, 1.17111, \dots \right\}$$

But as we mentioned before, this is an infinite set and is thus unfeasible to use for any engineering or computer-based applications which may have need of these sorts of calculations. To this end we will calculate the eigenvalues for the $n = 3$ level of our fractal.

For this we use Proposition 2.5. So

$$\begin{aligned} \sigma(\Delta_3) &= \sigma(\Delta_0) \cup \left(\bigcup_{m=0}^2 R^{-m}(\sigma(\Delta_1) - \sigma(\Delta_0)) \right) \\ &\cup \left(\bigcup_{m=0}^1 R^{-m} \left(\left\{ \cos^2 \left(\frac{(m-\frac{1}{2})\pi}{2} \right) : m = 1 \right\} \right) \right) \\ &\cup \{ z : T_2(\sqrt{z}) - 2(z-1)\sqrt{z}U_1(\sqrt{z}) = 0 \}. \end{aligned}$$

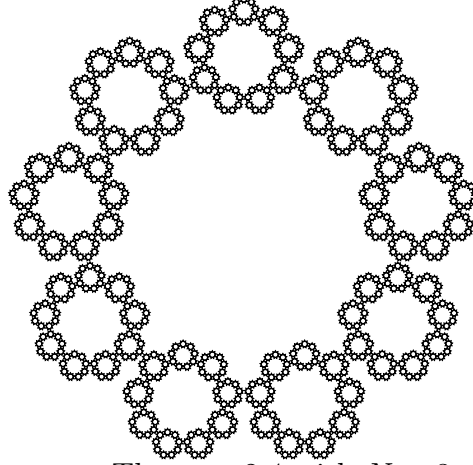
When we combine this with Mathematica we get the following values for the spectrum of this level 3 approximation:

$$\begin{aligned} &\left\{ 0, \frac{1}{4}, \frac{3}{4}, 1, \frac{3}{2}, 1.17111, 0.924223, 0.38597, 0.0187009, 1.24104, 0.838496, 0.357455, 0.0630091, \right. \\ &1.26661, 0.805548, 0.336944, 0.0908988, 1.28219, 1.25925, 1.19411, 1.11161, 0.991323, \\ &0.896816, 0.815093, 0.785192, 0.395727, 0.379382, 0.343809, 0.318473, 0.114142, 0.0818446, \\ &0.0296878, 0.00134014, 1.28821, 1.25054, 1.18965, 1.12714, 0.974327, 0.902199, 0.826335, \\ &0.777302, 0.393982, 0.380823, 0.350853, 0.309244, 0.125244, 0.0722724, 0.0273292, \\ &0.00455087, 1.29036, 1.24707, 1.18633, 1.13534, 0.965202, 0.906178, 0.830793, 0.774475, \\ &0.392856, 0.38184, 0.353384, 0.305526, 0.129636, 0.0687562, 0.0256512, 0.00659826, 1.15938, \\ &0.937874, 0.38862, 0.0141237, 1.21056, 0.876735, 0.373265, 0.0394411, 1.29388, 0.769853, \\ &0.298845, 0.137419 \} \end{aligned}$$

The multiplicities of these eigenvalues can be found using the theorems in Section 6 and the source code used to calculate these values can be found in Section 7.

4. THE 9-GASKET

Next, we will examine the spectrum of the 9-Gasket.



For the 9-Gasket, $N = 3$, so we use Theorem 2.4 with $N = 3$ and see that

$$\begin{aligned} \sigma(\Delta) &= \left\{ \frac{3}{2} \right\} \cup \left(\bigcup_{k=0}^{\infty} R_{-k} \left(\left\{ \sin^2 \left(\frac{m\pi}{9} \right) : m = 0, \dots, 8 \text{ and } 3 \nmid m \right\} \right) \right) \\ &\cup \left\{ z : T_3(\sqrt{z}) - 2(z-1)\sqrt{z}U_2(\sqrt{z}) = 0 \right\} \\ &\cup \left\{ \cos^2 \left(\frac{m\pi}{3} \right) : m = 1 \right\} \end{aligned}$$

We will skip calculating the spectrum of the fractal itself and go right to calculating the spectrum of an approximation. If we want the $n = 3$ level approximation, we will use the following formula:

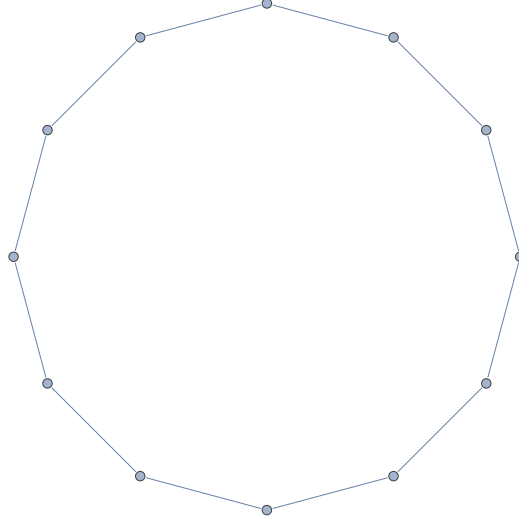
$$\begin{aligned} \sigma(\Delta_3) &= \sigma(\Delta_0) \cup \left(\bigcup_{m=0}^2 R^{-m} (\sigma(\Delta_1) - \sigma(\Delta_0)) \right) \\ &\cup \left(\bigcup_{m=0}^1 R^{-m} \left(\left\{ \cos^2 \left(\frac{m\pi}{2} \right) : m = 1 \right\} \right. \right. \\ &\left. \left. \cup \left\{ z : T_3(\sqrt{z}) - 2(z-1)\sqrt{z}U_2(\sqrt{z}) = 0 \right\} \right) \right). \end{aligned}$$

So we use our Mathematica notebook from Section 7 and get that our approximation is:

$$\begin{aligned} &\left\{ 0, \frac{3}{4}, \frac{3}{2}, \frac{1}{2} \left(1 + \cos \left[\frac{\pi}{9} \right] \right), \frac{1}{2} \left(1 - \cos \left[\frac{2\pi}{9} \right] \right), 1.19437, 0.867245, 0.474284, 0.181182, \right. \\ &0.0329143, 1.18187, 0.895831, 0.454329, 0.172248, 0.0457238, 1.22345, 0.773517, 0.550533, \\ &0.198071, 0.00442796, 1.21085, 0.821221, 0.509956, 0.191341, 0.0166343, 1.22678, 1.22084, \\ &1.20805, 1.18792, 1.16694, 0.925434, 0.88252, 0.829968, 0.784915, 0.757083, 0.565199, \\ &0.540555, 0.502883, 0.463383, 0.435853, 0.199712, 0.196745, 0.189731, 0.176718, 0.159713, \\ &0.0620621, 0.0394612, 0.0193663, 0.00694467, 0.00122639, 0.175532, 0.160551, 0.0610179, \\ &0.0411502, 0.0184649, 0.00659058, 0.00170771, 1.22789, 1.22014, 1.20445, 1.19311, 1.16479, \\ &0.929378, 0.870323, 0.840576, 0.787787, 0.75098, 0.570725, 0.538067, 0.49448, 0.472046, \\ &0.43358, 0.200245, 0.196385, 0.187592, 0.180334, 0.157678, 0.0645754, 0.0341826, \\ &0.0228988, 0.00761756, 0.00016413, 0.178543, 0.158576, 0.0634706, 0.0368222, 0.021001, \\ &0.00734887, 0.000617952, 1.16277, 0.933013, 0.431527, 0.155699, 0.0669873, 1.20685, \\ &0.83358, 0.5, 0.189026, 0.0205417, 1.21797, 0.796346, 0.530718, 0.195247, 0.00971654, \\ &1.22806, 0.75, 0.571616, 0.200327, 0., \frac{1}{2} \left(1 - \sin \left[\frac{\pi}{18} \right] \right) \left. \right\} \end{aligned}$$

5. THE 12-GASKET

The 12-Gasket is a little bit more difficult to represent on paper so we will only include the first-level approximation of it.



As with the other $3N$ -Gasket fractals, the 12-Gasket is shown by compounding this shape similarly to the 3-Gasket shown above. We cannot come up with more appealing-looking higher levels of the 12-Gasket because while the other two fractals we have examined are finitely ramified, the 12-Gasket is infinitely ramified.

For the 12-Gasket we have $N = 4$, so we use Theorem 2.4 with $N = 4$ and see that

$$\begin{aligned} \sigma(\Delta) &= \left\{ \frac{3}{2} \right\} \cup \left(\bigcup_{k=0}^{\infty} R_{-k} \left(\left\{ \sin^2 \left(\frac{m\pi}{6} \right) : m = 0, \dots, 11 \text{ and } 3 \nmid m \right\} \right) \right) \\ &\cup \left\{ z : T_4(\sqrt{z}) - 2(z-1)\sqrt{z}U_3(\sqrt{z}) = 0 \right\} \\ &\cup \left\{ \cos^2 \left(\frac{m\pi}{4} \right) : m = 1, 2 \right\} \end{aligned}$$

However, again we only want the $n = 3$ level approximation of the 12-Gasket. We can find this with the following formula:

$$\begin{aligned} \sigma(\Delta_3) &= \sigma(\Delta_0) \cup \left(\bigcup_{m=0}^2 R^{-m}(\sigma(\Delta_1) - \sigma(\Delta_0)) \right) \\ &\cup \left(\bigcup_{m=0}^1 R^{-m} \left(\left\{ \cos^2 \left(\frac{(m-\frac{1}{2})\pi}{2} \right) : m = 1, 2 \right\} \right) \right) \\ &\cup \left\{ z : T_4(\sqrt{z}) - 2(z-1)\sqrt{z}U_3(\sqrt{z}) = 0 \right\}. \end{aligned}$$

Again, we use our calculations from Section 7 and get that our approximation is:

$$\begin{aligned} &\left\{ 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{3}{2}, \frac{1}{2} \left(1 - \frac{\sqrt{3}}{2} \right), \frac{1}{2} \left(1 + \frac{\sqrt{3}}{2} \right), 0.0136441, 0.00809672, 0.00274436, \right. \\ &0.000135756, 0.0145154, 0.00753501, 0.00266268, 0.00028757, 0.0221067, 0.0153495, \\ &0.00706775, 0.00255823, 0.000461785, 0.0162122, 0.00666032, 0.00241791, \\ &0.000670591, 0.0292112, 0.022637, 0.0129243, 0.00861629, 0.00279388, 0.0000350466, \\ &0.0276152, 0.0218769, 0.0159742, 0.00676474, 0.00246023, 0.000610109, 1.20154, \\ &0.987758, 0.817697, 0.535693, 0.333262, 0.118084, 0.00596671, 1.20712, 0.975962, \end{aligned}$$

0.812415, 0.565622, 0.311606, 0.114646, 0.0126236, 1.21228, 0.964559, 0.805447,
 0.593803, 0.293428, 0.110243, 0.0202427, 1.21707, 0.953554, 0.795787, 0.62247,
 0.277455, 0.104316, 0.0293461, 1.19712, 0.99667, 0.820817, 0.510588, 0.353102,
 0.120166, 0.00154161, 1.21582, 0.956461, 0.79873, 0.614613, 0.281559, 0.106105,
 0.0267124, 1.19726, 0.996389, 0.820727, 0.511442, 0.352403, 0.120106, 0.00167446,
 1.19907, 0.992769, 0.819527, 0.522013, 0.343892, 0.119301, 0.00342251, 1.21131,
 0.96674, 0.806968, 0.588366, 0.296755, 0.111193, 0.0186709, 1.2143, 0.95995, 0.801858,
 0.605493, 0.286588, 0.108021, 0.0237877, 1.22114, 0.944003, 0.782871, 0.651115,
 0.264414, 0.0965572, 0.0398964}

6. MULTIPLICITIES

It is important to note that while we have of the individual eigenvalues, we do not yet know their multiplicities. For this, we use two separate theorems. First, we cover the multiplicity of $\frac{3}{2}$.

Proposition 6.1. [1] *This is a continuation of Proposition 2.5.*

(i) For any $n \geq 0$, we have $\text{mult}_n(\frac{3}{2}) = \frac{3N+(3N-2)(3N)^n}{3N-1}$.

(ii) For any $n \geq 1$ and $0 \leq m < n - 1$, $R^m(z) = \sin^2(\frac{k\pi}{3N}) \in \sigma(\Delta_1)$ where $3 \nmid k$,

$$\text{mult}_n(z) = \frac{3N + (3N - 2)(3N)^{n-m-1}}{3N - 1}$$

(iii) For any $n \geq 1$ and $0 \leq m < n - 1$, $R^m(z) = \sin^2(\frac{k\pi}{3N}) \in \sigma(\Delta_1)$ where $3|k$,

$$\text{mult}_n(z) = (3N)^{n-m-1} + 1$$

(iv) For any z in (2.4) and (2.7) and $n \geq 0$, $\text{mult}_n(z) = 0$.

(v) For any z with $R^m(z)$ in (2.5), $n \geq 0$, and $0 \leq m < n - 2$, we have

$$\text{mult}_n(z) = \frac{(3N)^{n-m-1} - 1}{3N - 1}$$

This proposition allows us to find the multiplicities of each individual eigenvalue contained within our level approximations for the fractal, giving us a more complete view of the spectrum.

We can also calculate the multiplicities of the eigenvalues involved with the full fractal using the following proposition

Proposition 6.2. [2]

1. If $z \notin E(M_0, M)$, then

$$\text{mult}_n(z) = \text{mult}_{n-1}(R(z)),$$

and every corresponding eigenfunction at depth n is an extension of an eigenfunction at depth $n - 1$.

2. If $z \notin \sigma(D)$, $\phi(z) = 0$ and $R(z)$ has a removable singularity at z , then

$$\text{mult}_n(z) = \dim_{n-1},$$

and every corresponding eigenfunction at depth n is localized.

3. If $z \notin \sigma(D)$, both $\phi(z)$ and $\phi(z)R(z)$ have poles at z , $R(z)$ has a removable singularity at z , and $\frac{d}{dz}R(z) \neq 0$, then

$$\text{mult}_n(z) = m^{n-1}\text{mult}_D(z) - \dim_{n-1} + \text{mult}_{n-1}(R(z)),$$

4. If $z \in \sigma(D)$, but $\phi(z)$ and $\phi(z)R(z)$ do not have poles at z , and $\phi(z) \neq 0$, then

$$\text{mult}_n(z) = m^{n-1}\text{mult}_D(z) + \text{mult}_{n-1}(R(z)).$$

In this case $m^{n-1}\text{mult}_D(z)$ linearly independent eigenfunctions are localized and $\text{mult}_{n-1}(R(z))$ more linearly independent eigenfunctions are extensions of corresponding eigenfunction at depth $n - 1$.

5. If $z \in \sigma(D)$, but $\phi(z)$ and $\phi(z)R(z)$ do not have poles at z , and $\phi(z) = 0$, then

$$\text{mult}_n(z) = m^{n-1}\text{mult}_D(z) + \text{mult}_{n-1}(R(z)) + \dim_{n-1}$$

provided $R(z)$ has a removable singularity at z . In this case there are $m^{n-1}\text{mult}_D(z) + \dim_{n-1}$ localized and $\text{mult}_{n-1}(R(z))$ non-localized corresponding eigenfunctions at depth n .

6. If $z \in \sigma(D)$, both $\phi(z)$ and $\phi(z)R(z)$ have poles at z , $R(z)$ has a removable singularity at z , and $\frac{d}{dz}R(z) = 0$, then

$$\text{mult}_n(z) = \text{mult}_{n-1}(R(z)),$$

provided there are no corresponding eigenfunctions at depth n that vanish on V_{n-1} . In general we have

$$\text{mult}_n(z) = m^{n-1}\text{mult}_D(z) - \dim_{n-1} + 2\text{mult}_{n-1}(R(z)).$$

7. If $z \notin \sigma(D)$, $\phi(z) = 0$, and $R(z)$ has a pole z , then $\text{mult}_n(z) = 0$ and z is not an eigenvalue.
8. If $z \in \sigma(D)$, but $\phi(z)$ and $\phi(z)R(z)$ do not have poles at z , $\phi(z) = 0$, and $R(z)$ has a pole z , then

$$\text{mult}_n(z) = m^{n-1}\text{mult}_D(z)$$

and every corresponding eigenfunction at depth n vanishes on V_{n-1} .

We can see from these theorems that that the majority of the eigenvalues for any given level will be $\frac{3}{2}$. Of course, the simpler formula in Proposition 6.1 is what we use to calculate the multiplicities of the eigenvalues in the approximation.

7. CALCULATIONS

We include here the source code of the Mathematica notebook we used to calculate the level 3 spectrum approximations for the 6, 9, and 12-Gaskets:

(* Now we begin calculations of the k -th level approximation. n and k are positive integers

with n being the type of gasket that you want and k being the level of approximation *)
 $k =$;

$n =$;

Part1 = $k - 1$;

Part2 = $k - 2$;

(* These are our inputs to calculate the spectrum. Part1 and Part2 are for the two separate unions in the formula listed in Theorem 2.5 *)

$T[n, x] := \text{ChebyshevT}[n, x]$;

$U[n_, x_] := \text{ChebyshevU}[n, x];$

(* We define the Chebyshev Polynomials of the first and second kind, respectively *)

$R1[n_, z_] := \frac{(z-1)\text{Sqrt}[z]U[n-1, \text{Sqrt}[z]](2T[n, 1-2z]+2U[n-1, 1-2z]+1)}{T[n, \text{Sqrt}[z]];}$

$R2[n_, z_] := \frac{\text{Sqrt}[z]T[n, \text{Sqrt}[z]](2T[n, 1-2z]+2U[n-1, 1-2z]+1)}{U[n-1, \text{Sqrt}[z]};}$

(* Here we define the odd and even formulas for the spectral decimation function, R1 and R2 respectively. *)

$\text{RNGasket}[x_] := \text{If}[\text{EvenQ}[n], R1[n, x], R2[n, x]];$

(* We will solve for the spectrum of the k -th level approximation on the $3N$ -Gasket where n represents N and d_0 is the spectrum of Δ_0 *)

(* This is for the first union *)

$d0 = \{0, 3/2\};$

$d1 = \text{Union} \left[\{3/2\}, \text{Table} \left[\frac{1 - \text{Cos} \left[\frac{2t \cdot \text{Pi}}{3n} \right]}{2}, \{t, 0, 3n - 1\} \right] \right];$

$d\text{Fin} = \text{Complement}[d1, d0];$

$\text{PreRvals1} = \{ \};$

$\text{If}[\text{Part1} < 1,$

$\text{PreRvals1} = \text{Table}[\text{NSolve}[\text{RNGasket}[x] == d\text{Fin}[[t]], x], \{t, 1, \text{Length}[d\text{Fin}]\}],$

$\text{PreRvals1} = \text{Table}[\text{Table}[\text{NSolve}[\text{Nest}[\text{RNGasket}, x, t] == d\text{Fin}[[p]], x],$

$\{p, 1, \text{Length}[d\text{Fin}]\}], \{t, 1, \text{Part1}\}$

$];$

(* This nested If-statement takes care of the first union in the theorem *)

$\text{Rvals1} = \text{Replace}[x, \text{PreRvals1}];$

(* This is for the second union *)

$A = \text{If} \left[\text{EvenQ}[n], \text{Table} \left[\text{Cos} \left[\frac{(t-1/2)\text{Pi}}{n} \right]^2, \{t, 1, n/2\} \right], \right.$

$\left. \text{Table} \left[\text{Cos} \left[\frac{t \cdot \text{Pi}}{n} \right]^2, \{t, 1, \frac{n-1}{2}\} \right] \right];$

$\text{Rset} = \text{Replace}[z, \text{NSolve}[T[n, \text{Sqrt}[z]] - 2(z-1)\text{Sqrt}[z]U[n-1, \text{Sqrt}[z]] == 0];$

Rinput = $A \cup \text{Rset}$;

If[Part2 < 0, PreRvals2 = {}],

If[Part2 < 1,

PreRvals2 = Table[NSolve[RNGasket[x] == Rinput[[t]], x], {t, 1, Length[Rinput]}],

PreRvals2 = Table[Table[NSolve[Nest[RNGasket, x, t] == Rinput[[p]], x],

{p, 1, Length[Rinput]}], {t, 1, Part2}]

]

];

(* And this nested If-statement takes care of the second union in the theorem. Note that is Part2 < 0, we do not output anything and, similarly to the first union, if Part2 == 0,

RNGasket becomes the identity function *)

Rvals2 = Replace[x, PreRvals2];

PreRvals2 = Table[NSolve[RNGasket[x] == Rinput[[t]], x], {t, 1, Length[Rinput]}];

Rvals2 = Replace[x, PreRvals2];

(* The following code should get all the appropriate outputs. *)

Spectrum = Flatten[d0 \cup dFin \cup Rvals1 \cup Rvals2]

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