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Abstract

In a two-market Bertrand duopoly,each of two firms chooses one of two markets and a price in that market. All four choices are made simultaneously. In a two-market Cournot duopoly, the firms choose quantities rather than prices.It is well known that in the one-market case the threat of price undercutting means that Bertrand equilibrium prices and profits will be lower and quantities higher than Cournot equilibrium prices, profits and quantities.We find a quite different consequence of price undercutting in two-market duopoly. In the two-market case the threat of price undercutting means that Bertrand equilibria are in continuous mixed strategies, while every Cournot duopoly has an equilibrium in pure strategies, or in strategies that are pure in each market.

Journal of Economic Literature Classification: C72, D43

Keywords: Cournot, Bertrand, Two-Market Duopoly

1. Introduction

Two firms are licensed to sell a new product in Denver and Seattle. Each firm has sufficient resources to enter only one market. Each firm chooses a city and sets a price, and each firm must make both decisions without knowing the city chosen or the price set by the other firm. In other words, all four choices are made simultaneously. This game will be referred to as two-market Bertrand duopoly or two-market duopoly with price competition.

In an alternative scenario, the two markets are distinguished by product rather than location. Two firms are capable of producing lawn mowers or golf carts. Each firm chooses a product and sets a price. Again, all four choices are made simultaneously.

In this paper I investigate the equilibrium structure of two-market duopoly with price competition, as in the above scenarios, and with quantity competition. In each case, the equilibrium structure is shaped by the fact that a firm must set a price (or choose a quantity) without knowing whether it will be a monopolist or a duopolist in its chosen market.

It is well known that in one-market duopoly with price competition, the threat of price undercutting drives profits to zero. In two-market duopoly with price competition, it seems likely that equilibrium profits should be nonzero. Therefore, we should search for equilibria among continuous mixed strategy profiles, since mixed strategy profiles with finite support (including pure strategies) are suseptible to price undercutting. We will in fact construct symmetric continuous mixed-strategy equilibria for all two-market duopolies with price competition.

On the other hand we will show that every two-market duopoly with quantity competition has an equilibrium in pure strategies or in strategies that assign probability one to a union of two quantities, one in one market and one in the other. We will call such strategies semi-pure.

We will also give an example of a two-market duopoly such that, for the proper selection of equilibria, Bertrand equilibrium profits exceed Cournot equilibrium profits.

The literature on price and quantity competition has to a large extent focused on two-stage games that model markets in which capacity, product, or location decisions must be made before prices or quantities are set. In the seminal paper by Kreps and Scheinkman(1983) firms first simultaneously choose capacities and then in the second stage choose prices simultaneously. Allen et al. (2000) consider a two-stage game in which the firms choose capacities sequentially and then compete in prices. In Elberfeld and Wolfstetter (1999) firms first decide whether to enter a market or not and then compete in prices in stage two. In papers by Gersbach and Schmutzler (1999) and Hamilton, Thisse and Weskamp (1989) firms choose location and then a pricing game ensues. There are also studies in which each firm's location or choice of product characteristic is determined exogenously (as in d'Aspremont,Gabszewicz and Thisse (1979) and Salop (1979))and then a pricing or quantity game is played.

In our two-market duopoly games, it is not assumed that product or location decisions must be made before prices or quantities are set. Consider the example alluded to briefly above in which each firm chooses to produce lawn mowers or golf carts. Because of the similarity of the two products, it would not be necessary to commit to one or the other far in advance of the date when lawn mowers or golf carts start rolling off the assembly line. On the other hand, if the choice were between lawn mowers and sports cars, it might be necessary to choose a product years in advance of setting prices or quantities and a two-stage game would be an appropriate model. In the other example alluded to above, firms locate in Denver or Seattle. If it is necessary for each firm to set up production facilities in the city it chooses (for example if the firms are opening sushi restaurants) it may need to commit to a city far in advance of setting prices or quantities and a two-stage game would be an appropriate model. On the other hand, if the firms plan to sell software produced in Arizona, there is no need to commit to a city far in advance of setting prices or quantities and our two-market duopoly games would be appropriate models.

Janssen and Rasmusen (2002) study profits generated by price competition in a single market which each firm enters with an exogenously determined probability α. Their construction of a symmetric, continuous Nash equilibrium for Bertrand duopoly with uncertain entry into a single market proved useful to us *. It made possible considerable

* I would like to thank Wieland Müller for suggesting that Janssen and Rasmusen (2002) might be of use in this paper

abridgement of an earlier version of our construction of a symmetric, continuous Nash equilibrium in two-market Bertrand duopoly (see Proposition 2 below). Their construction was adaptable to the two-market case because in a symmetric Nash equilibrium for two-market Bertrand duopoly each firm enters market 1 with some positive (though not exogenously determined) probability α, and each firm enters market 2 with positive probability $1 - \alpha$. The results of Janssen and Rasmusen are potentially useful not only when entry is uncertain for each firm, but, as they point out, in other scenarios, for example when consumers are imperfectly informed about entry or about prices.

The classical mixed strategy equilibrium existence results of Glicksberg (1952), Dasgupta and Maskin (1986) and Simon (1987) do not apply directly to our two-market duopoly games which have strategy sets that are neither compact nor connected. Of course, even if it proved possible to adapt classical equilibrium existence theorems to two-market duopoly games, these theorems would not provide specific equilibria, as in the proof of Proposition 2 below, nor would they guarantee semi-pure-strategy equilibria as in the proofs of Propositions 4 and 5.

This paper is organized as follows. Section 2 introduces a notation for mixed strategies in two-market duopoly games. In Section 3 a symmetric, continuous mixed-strategy equilibrium is constructed for an arbitrary two-market Bertrand duopoly game. Section 4 contains pure-strategy and semi-pure-strategy equilibrium existence theorems for two-market Cournot duopoly games. An example illustrates the theorems. Section 5 compares Bertrand and Cournot equilibrium profits.

2. Definitions

A two-market duopoly game is a two-player, one-period game of complete information. Regardless of whether the players compete in prices or quantities, the strategy set for each player is the disjoint union of two copies of $[0, +\infty)$, the non-negative reals. One copy of $[0, +\infty)$ represents possible prices (or quantities) in one market, and the other copy of $[0, +\infty)$ represents possible prices (or quantities) in the other. Lower case letters will be used for prices and quantities in market 1, upper case letters for prices and quantities in market 2.

In the quantity competition version of two-market duopoly with inverse demand functions $p(q)$ in market 1 and $P(Q)$ in market 2, we will, for example, write $\pi_1(q_1, q_2) = q_1 p(q_1 + q_2)$ as the payoff to firm 1 if firm one chooses market 1 and quantity q_1 and firm 2 chooses market 1 and quantity q_2. (Notice that in order to simplify the analysis we are assuming zero costs.) Similarly, $\pi_1(q, Q) = qp(q)$ denotes the payoff to firm 1 if firm 1 chooses market 1 and quantity q and firm 2 chooses market 2 and quantity Q.

In the price competition version of two-market duopoly with demand functions $q(p)$ in market 1 and $Q(P)$ in market 2,

$$\Pi_1(P_1, P_2) = \begin{cases} 0 & \text{if } P_2 < P_1; \\ P_1 Q(P_1) & \text{if } P_1 < P_2; \\ P_1 Q(P_1)/2 & \text{if } P_1 = P_2. \end{cases}$$

denotes the payoff to firm 1 if firm 1 chooses market 2 and sets price P_1, and firm 2 chooses market 2 and sets price P_2.

These three examples should be enough to explain the notation. Of course with this notation $\pi_1(4, 5)$ is meaningless since it could indicate $\pi_1(q_1, q_2)$ with $q_1 = 4$, $q_2 = 5$, or $\pi_1(q, Q)$ with $q = 4$, $Q = 5$, etc.

A mixed strategy for two-market duopoly with price competition is a real valued function F on the strategy space such that

$$F \text{ is nondecreasing on both copies of } [0, +\infty) \tag{1}$$

$$F \text{ is continuous from the right on both copies of } [0, +\infty) \tag{2}$$

$$0 \leq F(p) \leq F(P) \text{ for every market 1 price } p \text{ and market 2 price } P \tag{3}$$

$$\lim_{P \to +\infty} F(P) = 1 \tag{4}$$

Then F determines a firm i mixed strategy as follows. Set

$$\text{Probability } (i \text{ chooses market 1 and sets } p_i \leq p) = F(p)$$

$$\text{Probability } (i \text{ chooses market 2 and sets } P_i \leq P) = F(P) - \lim_{p \to +\infty} F(p)$$

Similarly, a function F that satisfies (1)-(4) with quantities in place of price defines a firm i mixed strategy for two-market duopoly with quantity competition.

3. Price Competition.

Each of our two equilibrium existence results for two-market duopoly with price competition will require assumptions from the following list of assumptions involving the nonnegative demand functions $q(p)$ and $Q(P)$ defined for $p \geq 0$ and $P \geq 0$.

A1. *$q(p)$ and $Q(P)$ are continuous from the right.*

A2. *$q(p)$ and $Q(P)$ are continuous from the left.*

A3. *There exist $b, B > 0$ such that*

$pq(p)$ is nondecreasing on $[0, b]$ and nonincreasing on $[b, +\infty)$

$PQ(P)$ is nondecreasing on $[0, B]$ and nonincreasing on $[B, +\infty)$

A4. $q(b)Q(B) \neq 0$.

Recall that costs have been assumed to be zero so that $pq(p)$ and $PQ(P)$ are profit functions in market 1 and market 2, $bq(b)$ is the monopoly profit in market 1 and $BQ(B)$ is the monopoly profit in market 2.

Proposition 1. *Under assumptions A2, A3 and A4, duopoly with price competition has a pure-strategy Nash equilibrium if and only if market 1 monopoly profit equals market 2 monopoly profit.*

Proof. Suppose $bq(b) = BQ(B)$. Then $p_1 = b, P_2 = B$ is a pure-strategy Nash equilibrium.

Suppose $bq(b) \neq BQ(B)$. Without loss of generality assume $BQ(B) > bq(b)$.

If firms 1 and 2 choose market 1 and set prices p_1 and p_2 respectively, then firm 1 can increase its payoff by changing its strategy to $P_1 = B$.

Suppose the firms choose different markets. Without loss of generality assume firm 1 chooses market 1 and sets price p_1, and firm 2 chooses market 2 and sets price P_2. If $P_2Q(P_2) < BQ(B)$, then firm 2 can increase its payoff by changing its strategy from P_2 to B. If $P_2Q(P_2) = BQ(B)$ then firm 1 can increase its payoff by changing its strategy from

p_1 to $P_1 = P_2 - \epsilon$, where $\epsilon > 0$ is small enough so that $P_1Q(P_1) > bq(b)$. There is such an ϵ because $BQ(B) > bq(b)$ and A2 holds.

Finally, suppose firms 1 and 2 choose market 2 and set prices P_1 and P_2 respectively. Without loss of generality, assume $P_1 \geq P_2$. If $P_1 > P_2$ or if $P_1 = P_2$ and $P_2Q(P_2) = 0$, then $\pi_1(P_1, P_2) = 0$ so that, by A4, firm 1 can increase its payoff by changing its strategy to $p_1 = b$.

If $P_1 = P_2$ and $P_2Q(P_2) > 0$, then $\pi_1(P_1, P_2) = P_2Q(P_2)/2$ so that firm 1 can increase its payoff by changing its strategy to $P_2 - \epsilon$ where $\epsilon > 0$ is small enough so that $(P_2 - \epsilon)Q(P_2 - \epsilon) > P_2Q(P_2)/2$. Such an ϵ exists because $P_2Q(P_2) > P_2Q(P_2)/2$ and A2 holds. ■

Proposition 2. *Under A1-A4 every two-market duopoly with price competition has a symmetric, continuous mixed-strategy Nash equilibrium.*

Our starting point in constructing an equilibrium is a result in Janssen and Rasmusen (2002, equation (22)). They consider a single market pricing game in which each of two firms knows that its opponent will enter the market with probability α, $0 < \alpha < 1$. They find an equilibrium in which each firm enters the market with probability α and chooses a price according to the cumulative distribution.

$$F(p) = \begin{cases} 0 & \text{if } p \leq a; \\ (1/\alpha)(1 - (1-\alpha)p_m q(p_m)/pq(p)) & \text{if } a < p \leq b; \\ 1 & \text{if } p > b. \end{cases}$$

Here b is as defined in A3 above and a will be defined below.

Since in a continuous mixed strategy equilibrium of our two-market Bertrand duopoly each firm will enter market 1 with some positive probability and will enter market 2 with some positive probability, we find a symmetric equilibrium that in each market takes on the same general form as the cumulative distribution above. This saves us the trouble of setting up and solving differential equations to find a continuous equilibrium. Of course our mixed strategy must be defined over two markets. In addition, we do not have an exogenous probability α of entry. Instead the probability of entry into each market and therefore the explicit form of the equilibrium for two-market duopoly with price competition is determined by the requirement that in playing against F, firm i is indifferent between points in the support of F and therefore indifferent between markets.

After proving Proposition 2, we will determine the probability of entry into each market and each firm's profit in equilibrium.

Proof.

Let

$c = bq(b)BQ(B)/(bq(b) + BQ(B))$

$a = min\{p\colon\ pq(p) = c\}$

$A = min\{P\colon\ PQ(P) = c\}$

$$F(p) = \begin{cases} 0 & \text{if } 0 \le p \le a; \\ 1 - c/pq(p) & \text{if } a \le p \le b; \\ 1 - c/bq(b) & \text{if } b \le p < +\infty. \end{cases}$$

$$F(P) = \begin{cases} 1 - c/bq(b) & \text{if } 0 \le P \le A; \\ 1 - c/bq(b) + 1 - c/PQ(P) & \text{if } A \le P \le B; \\ 1 & \text{if } B \le P < +\infty. \end{cases}$$

By A4, c is well defined and $c > 0$. By A1 and A2, a and A are well defined and positive.

By A1-A4, F is well defined and continuous. By A3, F is non-decreasing in both markets; that is (1) holds. By A1, (2) holds. Clearly (3) and (4) hold. Since (1)-(4) hold, F is a mixed strategy.

To show that (F, F) is a Nash equilibrium, notice that for all p

$$\begin{aligned} \pi_1(p, F) &= pq(p)(F(b) - F(p) + 1 - F(b)) \\ &= pq(p)(1 - F(p)) \end{aligned}$$

$$= \begin{cases} pq(p)(1 - F(a)) & \text{if } 0 \le p \le a; \\ pq(p)(1 - F(p)) & \text{if } a \le p \le b; \\ pq(p)(1 - F(b)) & \text{if } b \le p < +\infty. \end{cases}$$

$$= \begin{cases} cpq(p)/aq(a) & \text{if } 0 \le p \le a; \\ c & \text{if } a \le p \le b; \\ cpq(p)/bq(b) & \text{if } b \le p < +\infty. \end{cases}$$

Similarly,

$$\pi_1(P,F) = \begin{cases} cPQ(P)/AQ(A) & \text{if } 0 \le P \le A; \\ c & \text{if } A \le P \le B; \\ cPQ(P)/BQ(B) & \text{if } B \le P < +\infty. \end{cases}$$

Applying A3 one can see that $\pi_1(p,F)$ attains a maximum value of c on $[a,b]$ and $\pi_1(P,F)$ achieves a maximum of c on $[A,B]$. From the definition of F it is clear that if firm 1 plays F, the probability it chooses market 1 but prices outside of $[a,b]$ is zero as is the probability that it chooses market 2 but prices outside of $[A,B]$. Therefore F is a best response to F. In other words, (F,F) is a Nash equilibrium. ■

If firm i employs strategy F, then the probability that firm i chooses market 1 is

$$F(b) = 1 - c/bq(b) = bq(b)/(bq(b) + BQ(B))$$

and the probability that firm i chooses market 2 is

$$1 - F(b) = BQ(B)/(b(q(b) + BQ(B)).$$

These probabilities depend only on the ratio of the monopoly profits in the two markets.

Now suppose both firms employ strategy F and market 1 monopoly profits are much greater than those in market 2. Then $F(b) = bq(b)/(bq(b) + BQ(B)) \approx 1$ and $\pi_1(F,F) = c = bq(b)BQ(B)/(bq(b) + BQ(B)) \approx BQ(B)$. In other words, both firms almost certainly enter market 1, but each firms' profit is approximately equal to the monopoly profit in market 2.

Next suppose both firms employ strategy F and market 1 monopoly profits are approximately equal to market 2 monopoly profits. Then $F(b) = bq(b)/(bq(b)+BQ(B)) \approx 1/2$ and $\pi_1(F,F) = bq(b)BQ(B)/(bq(b) + BQ(B)) \approx bq(b)/2$. When monopoly profits are exactly equal, $\pi_1(F,F) = bq(b)/2$ while for the pure-strategy equilibrium (b,B), $\pi_1(b,B) = bq(b)$. In the special case when there exists a pure-strategy equilibrium, the pure-strategy equilibrium outcome strictly Pareto dominates the mixed-strategy equilibrium outcome.

In two-market duopoly with price competition, there is an incentive for a firm to price high in order to maximize profits if the firm's opponent chooses the other market. There is also an incentive to price low in order to undercut and to avoid being undercut. The mixed-strategy equilibrium accommodates both incentives. On the other hand the special case pure-strategy equilibrium (b, B) eliminates the incentive to price low, but selection of a pure-strategy equilibrium requires choosing between two equilibria (b, B) and (B, b) which are indistinguishable in the sense that the two markets yield identical monopoly profits, so that the mixed-strategy equilibrium may be considered a focal point, since it is the unique mixed-strategy equilibrium.

4. Quantity Competition.

Throughout this section, the following assumptions on the market 1 and market 2 inverse demand functions $p(q)$ and $P(Q)$ hold: there are market one quantities q_D and q_M and market 2 quantities Q_D and Q_M such that (q_D, q_D) and (Q_D, Q_D) are the unique pure strategy Nash equilibria for market 1 duopoly with quantity competition and market 2 duopoly with quantity competition, respectively; q_M and Q_M are the unique profit maximizing quantities for a market 1 monopolist and a market 2 monopolist, respectively; and duopoly equilibrium profits are positive in both markets (we are not assuming $p(q) \geq 0$ for all q or $P(Q) \geq 0$ for all Q).

Proposition 3. *Two -market duopoly with quantity competition possesses a pure-strategy Nash equilibrium if and only if there is no q_0 such that $\pi_1(q_D, q_D) < \pi(Q_M) < \pi_1(q_0, q_M)$ and no Q_0 such that $\pi_1(Q_D, Q_D) < \pi(q_M) < \pi_1(Q_0, Q_M)$.*

Proof. To prove necessity, suppose there is a q_0 such that $\pi_1(q_D, q_D) < \pi(Q_M) < \pi_1(q_0, q_M)$. The only possible pure-strategy Nash equilibria are (q_D, q_D), (Q_D, Q_D), (q_M, Q_M) and (Q_M, q_m).

First, $\pi_1(q_D, q_D) < \pi(Q_M) = \pi_1(Q_M, q_D)$.

Second, $\pi_1(Q_D, Q_D) < \pi_1(Q_D, Q_D) + \pi_2(Q_D, Q_D) \leq \pi(Q_M) < \pi_1(q_0, q_M) < \pi(q_M) = \pi_1(q_M, Q_D)$.

Third, $\pi_2(q_M, Q_M) = \pi(Q_M) < \pi_1(q_0, q_M) = \pi_2(q_M, q_0)$.

Finally, $\pi_1(Q_M, q_M) = \pi(Q_M) < \pi_1(q_0, q_M)$.

Therefore there is no pure-strategy Nash equilibrium. Symmetrically, there is no pure-strategy equilibrium if there is a Q_0 such that $\pi_1(Q_D, Q_D) < \pi(q_M) < \pi_1(Q_0, Q_M)$.

To prove sufficiency, suppose there is no q_0 such that $\pi_1(q_D, q_D) < \pi(Q_M) < \pi_1(q_0, q_M)$ and no Q_0 such that $\pi_1(Q_D, Q_D) < \pi(q_M) < \pi_1(Q_0, Q_M)$. Then $\pi_1(q_D, q_D) \geq \pi(Q_M)$ or $\pi_1(Q_D, Q_D) \geq \pi(q_M)$ or both $\pi(Q_M) \geq \pi_1(q, q_M)$ for all q and $\pi(q_M) \geq \pi_1(Q, Q_M)$ for all Q.

If $\pi_1(q_D, q_D) \geq \pi(Q_M)$, then (q_D, q_D) is a Nash equilibrium, since $\pi_1(q_D, q_D) \geq \pi(Q_M)$ means neither player can gain by a unilateral move to market 2, and neither player can gain by a unilateral move within market 1 by the definition of q_D.

Similarly, if $\pi(Q_D, Q_D) \geq \pi(q_M)$, then (Q_D, Q_D) is a Nash equilibrium.

Finally, if $\pi(Q_M) \geq \pi_1(q, q_M)$ for all q and $\pi(q_M) \geq \pi(Q, Q_M)$ for all Q, then (q_M, Q_M) is a Nash equilibrium, since

$$\pi_1(q_M, Q_M) = \pi(q_M) \geq \pi_1(Q, Q_M) \text{ for all } Q$$

$$\pi_1(q_M, Q_M) = \pi(q_M) \geq \pi(q) = \pi_1(q, Q_M) \text{ for all } q$$

$$\pi_2(q_M, Q_M) = \pi(Q_M) \geq \pi_1(q, q_M) = \pi_2(q_M, q) \text{ for all } q$$

$$\pi_2(q_M, Q_M) = \pi(Q_M) \geq \pi(Q) = \pi_2(q_M, Q) \text{ for all } Q. \quad \blacksquare$$

It is easy to construct examples that satisfy the hypotheses of Proposition 3 and therefore possess pure-strategy Nash equilibria. For example, if $\pi(q_M) = \pi(Q_M)$, then $\pi(q_M) = \pi(Q_M) \geq \pi_1(Q, Q_M)$ for all Q and $\pi(Q_M) = \pi(q_M) \geq \pi_1(q, q_M)$ for all q.

The following two-market duopoly game with quantity competition fails to satisfy the hypotheses of Proposition 3 and therefore has no pure-strategy Nash equilibrium, but does satisfy the hypotheses of Proposition 4 below and therefore has a semi-pure mixed-strategy equilibrium.

Example 1. Let $p(q) = (q+1)^{-3}$ and $P(Q) = (1-Q)/5$. To maximize $\pi(Q) = QP(Q)$, set $\pi'(Q) = P(Q) + QP'(Q) = 0$. Then $Q_M = -P(Q_M)/P'(Q_M) = 1 - Q_M$. Solving, $Q_M = 1/2$.

To find an equilibrium (Q_D, Q_D) of market 2 duopoly, set

$$\partial \pi_1(Q_1, Q_2)/\partial Q_1 = P(Q_1 + Q_2) + Q_1 P'(Q_1 + Q_2) = 0$$
$$\partial \pi_2(Q_1, Q_2)/\partial Q_2 = P(Q_1 + Q_2) + Q_2 P'(Q_1 + Q_2) = 0$$

Then $Q_{D1} = Q_{D2} = -P(Q_{D1} + Q_{D2})/P'(Q_{D1} + Q_{D2})$ or $Q_D = 1 - 2Q_D$. Solving, $Q_D = 1/3$.

Notice that $\pi(Q_M) > 0$ and $\pi_1(Q_D, Q_D) > 0$ as required for Proposition 3.

In market 1, $q_M = -p(q_M)/p'(q_M) = (q_M + 1)^{-3}/3(q_M + 1)^{-4} = (q_M + 1)/3$. Solving, $q_M = 1/2$

For market 1 duopoly, $q_D = -p(2q_D)/p'(2q_D) = (2q_D + 1)/3$. Solving, $q_D = 1$.

Then

$$\pi_1(q_D, q_D) = q_D(2q_D + 1)^{-3} = 3^{-3} = 1/27$$
$$\pi(Q_M) = (1/2)(1/2)/5 = 1/20$$
$$\pi_1(q_M, q_M) = (1/2)2^{-3} = 1/16$$

so that $\pi_1(q_D, q_D) < \pi(Q_M) < \pi_1(q_M, q_M)$ and by Proposition 3 Example 1 has no pure-strategy Nash equilibrium.

Example 1 will also be used to illustrate Proposition 4, a mixed-strategy equilibrium existence result. The hypotheses of Proposition 4 will include A5, a continuity assumption and A6, which generalizes the assumption of existence and uniqueness of q_D, q_M, Q_D and Q_M made at the beginning of this section.

A5. *$p(q)$ and $P(Q)$ are continuous on* $[0, +\infty)$.

If $a \in [0, 1]$, q is a market 1 quantity and Q is a market 2 quantity, let $m(a, q, Q)$ denote the semi-pure strategy that plays q with probability a and Q with probability $1 - a$.

A6. *There exist bounded functions q^*, Q^*: $[0, 1] \to [0, +\infty)$ such that for $a \in [0, 1]$, $q^*(a)$ and $Q^*(a)$ are the unique market 1 and market 2 quantities such that*

$$\pi_1(q^*(a), m(a, q^*(a), Q^*(a)) = \max_q \pi_1(q, m(a, q^*(a), Q^*(a)))$$
$$\pi_1(Q^*(a), m(a, q^*(a), Q^*(a)) = \max_Q \pi_1(Q, m(a, q^*(a), Q^*(a)))$$

Notice that when $a = 0$, A6 asserts the existence and uniqueness, of q_M and Q_D, and when $a = 1$, A6 asserts the existence and uniqueness of q_D and Q_M.

Assumption A6 may seem extravagant, which criticism would translate into a criticism of any proposition containing A6 in its hypothesis as weak. However, after stating and proving a result using assumption A6, we will show that A6 follows from a simple convexity assumption.

Proposition 4. *If a two-market duopoly game with quantity competition satisfies A5, A6,* $\pi(Q_M) > \pi_1(q_D, q_D)$, *and* $\pi(q_M) > \pi_1(Q_D, Q_D)$, *then it possesses a symmetric, semi-pure strategy equilibrium.*

Proof. Suppose q^* and Q^* are as in A6. If the existence of $a^* \in [0, 1]$ such that

$$\pi_1(q^*(a^*), m(a^*, q^*(a^*), Q^*(a^*))) = \pi_1(Q^*(a^*), m(a^*, q^*(a^*), Q^*(a^*)))$$

can be established, then by this equation and the two equations in A6 with $a = a^*$, $(m(a^*, q^*(a^*), Q^*(a^*)), m(a^*, q^*(a^*), Q^*(a^*)))$ will be a Nash equilibrium.

Since $q^*(0)$ maximizes $\pi_1(q, m(0, q^*(0), Q^*(0))) = \pi(q)$, $q^*(0) = q_M$. Since $Q^*(0)$ maximizes $\pi_1(Q, m(0, q^*(0), Q^*(0))) = \pi_1(Q, Q^*(0)) = \pi_2(Q^*(0), Q)$, $Q^*(0) = Q_D$. Therefore

$$\pi_1(q^*(0), m(0, q^*(0), Q^*(0))) - \pi_1(Q^*(0), m(0, q^*(0), Q^*(0))) = \pi(q_M) - \pi_1(Q_D, Q_D) > 0 \tag{5}$$

The inequality is from the hypotheses of Proposition 4.

By a similar argument

$$\pi_1(q^*(1), m(1, q^*(1), Q^*(1))) - \pi_1(Q^*(1), m(1, q^*(1), Q^*(1))) = \pi_1(q_D, q_D) - \pi(Q_m) < 0 \tag{6}$$

To see that $q^*(a)$ is a continuous function of a, suppose $a_0 \in [0, 1]$ and (a_n) is a sequence in $[0, 1]$ such that $a_n \to a_0$. Since by A6 $(q^*(a_n))$ is a bounded sequence, it has a convergent subsequence $(q^*(a_{n_k}))$. Let $q_0 = lim_{k\to+\infty} q^*(a_{n_k})$. Since $q^*(a_{n_k}) \to q_0$, $q^*(a_{n_k})$ maximizes $\pi_1(q, m(a_{n_k}, q^*(a_{n_k}), Q^*(a_{n_k})))$, and $\pi_1(q, m(a, \bar{q}, \bar{Q}))$ is continuous in q, a and $\bar{q}$ and constant in $\bar{Q}$, it must be that q_0 maximizes $\pi_1(q, m(a_0, q_0, Q^*(a_0)))$. By the uniqueness provision of A6, $q_0 = q^*(a_0)$. In summary, every sequence $(q^*(a_n))$ such that $a_n \to a_0$ has a subsequence that converges to $q^*(a_0)$. Therefore $q^*(a)$ is a continuous function of a.

Similarly $Q^*(a)$ is a continuous function of a.

Therefore $\pi_1(q^*(a), m(a, q^*(a), Q^*(a))) - \pi_1(Q^*(a), m(a, q^*(a), Q^*(a)))$ is a continuous function of a. By (5), (6) and the Intermediate Value Theorem, there is an a^* such that

$$\pi_1(q^*(a^*), m(a^*, q^*(a^*), Q^*(a^*))) = \pi_1(Q^*(a^*), m(a^*, q^*(a^*), Q^*(a^*))). \quad \blacksquare$$

We will see in the proof of Proposition 5 that A6 holds if the following simple convexity assumptions hold.

A7. *$\pi_1(q_1, q_2)$ and $\pi_1(Q_1, Q_2)$ are twice continously differentiable, $\partial^2\pi_1(q_1, q_2)/\partial q_1^2 < 0$, $\partial^2\pi_1(q_1, q_2)/\partial q_2 \partial q_1 < 0$, $\partial^2\pi_1(Q_1, Q_2)/\partial Q_1^2 < 0$ and $\partial^2\pi_1(Q_1, Q_2)/\partial Q_2 \partial Q_1 < 0$.*

Proposition 5. *If a two-market duopoly game with quantity competition satisfies A5, A7, $\pi(Q_M) > \pi_1(q_D, q_D)$ and $\pi(q_M) > \pi_1(Q_D, Q_D)$, then it possesses a symmetric, semi-pure-strategy equilibrium.*

Proof. Fix a, q_2 and Q_2. Since $\pi(q_1) = \pi_1(q_1, q)$ when $q = 0$, by A7 $d^2\pi(q_1)/dq_1^2 < 0$. Since q_M exists (assumed at the beginning of this section) and $d^2\pi(q_1)/dq_1^2 < 0$, there exists q_0 such that $q_1 \geq q_0$ implies $\pi(q_1) < 0$. Since $d^2\pi(q_1)/dq_1^2 < 0$, $d\pi(q_1)/dq_1 < 0$ for $q_1 > q_M$ and therefore $p(q_1)$ is decreasing for $q_1 > q_M$. Therefore $\pi_1(q_1, q_2) \leq \pi(q_1)$ for $q_1 > q_M$ so that $\pi_1(q_1, q_2) < 0$ for $q_1 \geq q_0$. Therefore, $\pi_1(q_1, m(a, q_2, Q_2)) < 0$ for $q_1 \geq q_0$. Since $\partial^2\pi_1(q_1, q_2)/\partial q_1^2 < 0$ and $d^2\pi(q_1)/dq_1^2 < 0$, $\partial^2\pi_1(q_1, m(a, q_2, Q_2))/\partial q_1^2 < 0$. Since $\partial^2\pi_1(q_1, m(a, q_2, Q_2))/\partial q_1^2 < 0$ and $\pi_1(q, m(a, q_2, Q_2)) < 0$ if $q_1 > q_0$, $\pi_1(q, m(a, q_2, Q_2))$ attains a maximum at a unique real $r(q_2)$, player 1's best response to $m(a, q_2, Q_2)$. Now allow q_2 to vary. By the continuity of $\partial\pi_1(q_1, q_2)/\partial q_1$ assumed in A7, $r(q_2)$ is a continuous function of q_2. If $a = 0$ then $r(q_2) = q_M$ for all q_2. To see that $r(q_2)$ is differentiable when $a \neq 0$ and $r(q_2) \neq 0$, let $F(q_1, q_2) = \partial\pi_1(q_1, m(a, q_2, Q_2))/\partial q_1$. Then for all q_2 with $r(q_2) > 0$

$$F(r(q_2), q_2) = 0 \tag{7}$$

For small $h \neq 0$

$$[F(r(q_2+h), q_2+h) - F(r(q_2), q_2)]/h =$$

$$(F(r(q_2+h), q_2+h) - F(r(q_2), q_2+h))/h + (F(r(q_2), q_2+h) - F(r(q_2), q_2))/h = 0 \quad (8)$$

For small h the second term of (8) is approximately

$$F_2(r(q_2), q_2) = a\partial^2\pi_1(q_1, q_2)/\partial q_2 \partial q_1 \Big|_{q_1 = r(q_2)} < 0 \text{ by A7.}$$

Therefore the first term of (8) is non-zero so that $r(q_2+h) \neq r(q_2)$ and we can write the first term of (8) as

$$\begin{aligned}\big[(F(r(q_2+h), q_2+h) - F(r(q_2), q_2+h))/(r(q_2+h) - r(q_2))\big]\big[(r(q_2+h) - r(q_2))/h\big] \\ \to -F_2(r(q_2), q_2).\end{aligned} \quad (9)$$

Since r is continuous, the Mean Value Theorem and the continuity of $\partial^2\pi_1(q_1, q_2)/\partial q_1^2$ can be used to prove that the first factor on the left hand side of (9) converges to $F_1(r(q_2), q_2)$ as $h \to 0$. Therefore the second factor converges; that is, $r'(q_2)$ exists and

$$r'(q_2) = -F_2(r(q_2), q_2)/F_1(r(q_2), q_2) =$$

$$-a\partial^2\pi_1(q_1, q_2)/\partial q_2 \partial q_1 \Big|_{q_1 = r(q_2)} \Big/ (a\partial^2\pi_1(q_1, q_2)/\partial q_1^2 \Big|_{q_1 = r(q_2)} + (1-a)\pi''(r(q_2)))$$

Therefore, if $a \neq 0$ and $r(q_2) \neq 0$ then $r'(q_2) < 0$. If $a = 0$, $r(q_2) = q_M$ for all q_2. In either case the graph of $q_1 = r(q_2)$ intersects the graph of $q_1 = q_2$ in a single point $q^*(a) \leq q_0$. We then have $q^*(a) = r(q^*(a))$ so that $q^*(a)$ is a bounded function of a and for each $a \in [0, 1]$ $q^*(a)$ is the unique market 1 quantity such that

$$\pi_1(q^*(a), m(a, q^*(a), Q^*(a))) = \max_q \pi_1(q, m(a, q^*(a), Q^*(a))).$$

Thus half of A6 is satisfied.

By a similar argument in market 2, the other half of A6 holds, and we can invoke Proposition 4 which has the same conclusion as Proposition 5. ∎

Example 1 continued. Recall that we have already shown that Example 1 satisfies the conditions specified at the beginning of this section, the existence and uniqueness of q_D, q_M, Q_D and Q_M. We also showed that $\pi_1(q_D, q_D) < \pi(Q_M) < \pi_1(q_M, q_M)$ from which is follows that $\pi(Q_M) > \pi_1(q_D, q_D)$ and $\pi(q_M) > \pi_1(q_M, q_M) > \pi(Q_M) > \pi_1(q_D, q_D)$. Since $p(q) = (q+1)^{-3}$ and $P(Q) = (1-Q)/5$, Example 1 satisfies A5. It is easy to show that $P(Q)$ satisfies A7. For $Q_1, Q_2 \in [0, +\infty)$, $\pi_1(Q_1, Q_2) = (Q_1 - Q_1^2 - Q_1 Q_2)/5$, $\partial\pi_1(Q_1, Q_2)/\partial Q_1 = (1 - 2Q_1 - Q_2)/5$, $\partial^2\pi_1(Q_1, Q_2)/\partial Q_1^2 = -2/5 < 0$ and $\partial^2\pi_1(Q_1, Q_2)/\partial Q_2 \partial Q_1 = -1/5 < 0$.

Unfortunately $p(q) = (q + 1)^{-3}$ does not satisfy A7, nor does any nonnegative price function with a q_M. We will instead show that $p(q)$ satisfies A6.

Step 1. For $\bar{q} \in [0, +\infty)$ and $a \in [0, 1]$, there is a unique value of q that maximizes $\pi_1(q, m(a, \bar{q}, Q^*(a)))$. With a and $\bar{q}$ fixed, differentiate $\pi_1(q, m(a, \bar{q}, Q^*(a))) =$
$aq(q + \bar{q} + 1)^{-3} + (1 - a)q(q + 1)^{-3}$ and set equal to zero

$$d\pi_1(q, m(a, \bar{q}, Q^*(a)))/dq = a(q+\bar{q}+1)^{-4}(-2q+\bar{q}+1)+(1-a)(q+1)^{-4}(-2q+1) = 0 \quad (10)$$

Then $d\pi_1(q, m(a, \bar{q}, Q^*(a)))/dq$ is a continuous function of q on $[0, +\infty)$. It is positive if $q < 1/2$ and negative if $q > 1/2 + \bar{q}/2$. By the Intermediate Value Theorem, there exists $q^*(a, m(a, \bar{q}, Q^*(a))) \in [1/2, 1/2 + \bar{q}/2]$ such that (10) holds for $q = q^*(a, m(a, \bar{q}, Q(a^*)))$.

Since $d^2\pi_1(q, m(a, \bar{q}, Q(a^*)))/dq^2 = a(q+\bar{q}+1)^{-5}(6q-6\bar{q}-6)+(1-a)(q+1)^{-5}(6q-6) < 0$ if $q < 1+\bar{q}$ and $d\pi_1(q, m(a, \bar{q}, Q^*(a)))/dq$ is negative if $q > 1/2+\bar{q}/2$, $q^*(a, m(a, \bar{q}, Q^*(a)))$ is the unique solution to (10) and the unique value of q that maximizes $\pi_1(q, m(a, \bar{q}, Q^*(a)))$.

Step 2. For $a \in [0, 1]$, there exists $q^*(a)$ such that

$$\pi_1(q^*(a), m(a, q^*(a), Q^*(a))) = \max_q \pi_1(q, m(a, q^*(a), Q^*(a)))$$

In (10) replace q^* by q:

$$a(1 - q)(2q + 1)^{-4} + (1 - a)(1 - 2q)(q + 1)^{-4} = 0 \quad (11)$$

Let $f(q)$ denote the left side of (11). Then $f(q)$ is continuous on $[0, +\infty)$, $f(q) > 0$ if $q < 1/2$ and $f(q) < 0$ if $q > 1$. By the Intermediate Value Theorem there exists $q^*(a) \in [1/2, 1]$

such that $f(q^*(a)) = 0$. Therefore $q = q^*(a)$ satisfies (10) when $\bar{q} = q^*(a)$. As was shown in Step 1, this implies that

$$\pi_1(q^*(a), m(a, q^*(a), Q^*(a))) = \max_q \pi_1(q, m(a, q^*(a), Q^*(a))).$$

We know from Step 1 that there is no $q' \neq q^*(a)$ that maximizes $\pi_1(q, m(a, q^*(a), Q^*(a)))$, but is there a $q' \neq q^*(a)$ that maximizes $\pi_1(q, m(a, q', Q^*(a))$?

Step 3. For $a \in [0, 1]$ there is a *unique* $q^*(a)$ such that

$$\pi_1(q^*(a), m(a, q^*(a), Q^*(a))) = \max_q \pi_1(q, m(a, q^*(a), Q^*(a))).$$

Since $f'(q) = a(2q + 1)^{-5}(6q - 6) + (1 - a)(q + 1)^{-5}(6q - 6) < 0$ if $q < 1$, $f(q) > 0$ if $q < 1/2$ and $f(q) < 0$ if $q > 1$, there is a unique $q^*(a)$ that satisfies (11) and therefore a unique $q^*(a)$ such that

$$\pi_1(q^*(a), m(a, q^*(a), Q^*(a))) = \max_q \pi_1(q, m(a, q^*(a), Q^*(a)))$$

Finally, in Step 2 it was shown that for $a \in [0, 1]$, $q^*(a) \leq 1$. Therefore $p(q)$ satisfies A6. Since $P(Q)$ was shown to satisfy A7, by the proof of Proposition 5 it also satisfies A6. By Proposition 4, Example 1 possesses a symmetric, semi-pure strategy equilibrium.

The following Proposition combines Propositions 3,4 and 5.

Proposition 6. *If a two-market duopoly game with quantity competition satisfies A5 and each market satisfies A6 or A7, then it possesses a symmetric Nash equilibrium in pure or semi-pure strategies.*

Proof. If there is a q_0 such that $\pi_1(q_D, q_D) < \pi(Q_M) < \pi_1(q_0, q_M)$, then $\pi(Q_M) > \pi_1(q_D, q_D)$ and $\pi(q_M) \geq \pi_1(q_0, q_M) > \pi(Q_M) \geq \pi_1(Q_D, Q_D)$. Apply Propositions 4 and/or 5. If there is a Q_0 such that $\pi_1(Q_D, Q_D) < \pi(q_M) < \pi_1(Q_0, Q_M)$, Propositions 4 and/or 5 apply by a similar argument. If there is no such q_0 and no such Q_0, then apply Proposition 3. ∎

5. Profit Levels.

How do two-market Bertrand equilibrium profits compare with two-market Cournot equilibrium profits? We start with an example.

Example 2. $q(p) = 1 - p$ and $Q(P) = 1 - P$. There are unique monopoly profit maximizing prices in both markets. They are $p_M = 1/2$ and $P_M = 1/2$. Then (p_M, P_M) is a pure-strategy Nash equilibrium for two-market Bertrand duopoly, and equilibrium profits are $\pi_i(p_M, P_M) = 1/4$ for $i = 1, 2$.

For two-market Cournot duopoly we have $p(q) = 1 - q$ and $P(Q) = 1 - Q$. Then $q_M = 1/2$, $q_D = 1/3$, $Q_M = 1/2$, $Q_D = 1/3$ as was seen in the discussion of Example 1. Then $\pi(q_M) = 1/4 > 1/9 = \pi_1(q_D, q_D)$ and $\pi(Q_M) > \pi_1(Q_D, Q_D)$ so that by Proposition 6 there is a semi-pure strategy Nash equilibrium of two-market Cournot duopoly. Let's find it. Since the two markets have identical demand functions, let's try $a = 1/2$.

$\pi_1(q, m(1/2, q^*(1/2), Q^*(1/2))) = (1/2)\pi(q) + (1/2)\pi_1(q, q^*(1/2))$

$= (1/2)(q - q^2) + (1/2)(q - q^2 - qq^*(1/2))$

$= q - q^2 - (1/2)qq^*(1/2).$

$d\pi_1(q, m(1/2, q^*(1/2), Q^*(1/2)))/dq = 1 - 2q - (1/2)q^*(1/2) = 0.$

Setting $q = q^*(1/2)$, $2 - 5q^*(1/2) = 0$. Solving, $q^*(1/2) = 2/5 = Q^*(1/2)$. In equilibrium Cournot profits are $\pi_1(m(1/2, 2/5, 2/5), m(1/2, 2/5, 2/5)) = \pi_1(q, m(1/2, 2/5, 2/5))$ for $q = 2/5 = (1/2)(2/5)(1 - 2/5) + (1/2)(2/5)(1 - 4/5) = 3/25 + 1/25 = 4/25$. We have found a Bertrand equilibrium and a Cournot equilibrium such that Bertrand equilibrium profits are greater than Cournot equilibrium profits. Of course we have started with the special case when a pure-strategy Bertrand equilibrium exists and compared the better of the two Bertrand equilibria (there is also a Proposition 2 equilibrium with profit 1/8 for each firm) with the worse of two Cournot equilibria (there is also a pure-strategy equilibrium (q_M, Q_M) with profit 1/4 for each firm).

There remains the question of whether it can happen that the worst Bertrand equilibrium profits are greater than the best Cournot equilibrium profits. We discuss one case.

Comparison: A pure-strategy Cournot equilibrium and the Bertrand equilibrium of Proposition 2. As we saw in the proof of Proposition 3, Cournot profits are $\pi(q_M), \pi(Q_M)$, $\pi_1(q_D, q_D) \geq \pi(Q_M)$ or $\pi_1(Q_D, Q_D) \geq \pi(q_M)$. As we saw in the comments following the proof of Proposition 2, Bertrand profits are $\pi(q_M)\pi(Q_M)/(\pi(q_M) + \pi(Q_M)) < min\{\pi(q_M), \pi(Q_M)\}$. Bertrand equilibrium profits are lower.

The relationship between Bertrand and Cournot profits for a continuous Bertrand equilibrium and Cournot equilibrium composed of semi-pure strategies remains an open question.

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