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The Runge0Lenz Vector (continued)

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The Runge-Lenz Vector (continued)

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I. SYNOPSIS

We continue our discussion of the Runge-Lenz vector in a quantum mechanical context. The traditional form of the Runge-Lenz vector is obtained, and the commutation relations between the Runge-Lenz vector, the Hamiltonian, and the Angular Momentum are obtained using Maple.

II. INTRODUCTION

We had [1]

$$\frac{\vec{A}\cdot\vec{r}+\vec{r}\cdot\vec{A}}{2} = -\frac{1}{Ze^2\mu}\left(L^2+\frac{3\hbar^2}{2}\right)+|\vec{r}| \tag{2.1}$$

(written in slightly different form). Now we seek something of the order of $\vec{A}\otimes\vec{r}$, i.e., the symmetric form: form:

$$\frac{1}{2}\left(\vec{A}\otimes\vec{r}+\vec{r}\otimes\vec{A}\right)$$

where

$$\vec{A}\otimes\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_1 & A_2 & A_3 \\ x & y & z \end{vmatrix}$$

so that

$$\frac{1}{2}\left(\left\{\frac{1}{2Ze^2\mu}\left(\vec{L}\otimes\vec{p}-\vec{p}\otimes\vec{L}\right)+\hat{r}\right\}\otimes\vec{r}+\vec{r}\otimes\left\{\frac{1}{2Ze^2\mu}\left(\vec{L}\otimes\vec{p}-\vec{p}\otimes\vec{L}\right)+\hat{r}\right\}\right)$$

or, expanding

$$\frac{1}{4Ze^2\mu}\left(\left(\vec{L}\otimes\vec{p}\right)\otimes\vec{r}-\left(\vec{p}\otimes\vec{L}\right)\otimes\vec{r}+\hat{r}\otimes\vec{r}+\vec{r}\otimes\left(\vec{L}\otimes\vec{p}\right)-\vec{r}\otimes\left(\vec{p}\otimes\vec{L}\right)+\vec{r}\otimes\hat{r}\right)$$

Since $\vec{r}\otimes\hat{r}$ and its inverse are both zero (no partial derivatives here to goof us up!), we have

$$\frac{1}{4Ze^2\mu}\left(\overbrace{\left(\vec{L}\otimes\vec{p}\right)\otimes\vec{r}-\underbrace{\left(\vec{p}\otimes\vec{L}\right)\otimes\vec{r}}}+\overbrace{\vec{r}\otimes\left(\vec{L}\otimes\vec{p}\right)-\underbrace{\vec{r}\otimes\left(\vec{p}\otimes\vec{L}\right)}}\right) \tag{2.2}$$

where we need to be careful about expanding the triple cross products, since these are not ordinary vectors. We have

$$\vec{A}\otimes(\vec{B}\otimes\vec{C}) = \vec{A}\otimes\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_yC_z-B_zC_y & B_zC_x-B_xC_z & B_xC_y-B_yC_x \end{vmatrix}$$

which expands to

$$\begin{aligned}\vec{A}\otimes(\vec{B}\otimes\vec{C}) = &\hat{i}\left[A_y\left(B_xC_y-B_yC_x\right)-A_z\left(B_zC_x-B_xC_z\right)\right]\\ &+\hat{j}\left[A_z\left(B_yC_z-B_zC_y\right)-A_x\left(B_xC_y-B_yC_x\right)\right]\\ &+\hat{k}\left[A_x\left(B_zC_x-B_xC_z\right)-A_y\left(B_yC_z-B_zC_y\right)\right]\end{aligned} \tag{2.3}$$

while

$$(\vec{B}\otimes\vec{C})\otimes\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}\otimes\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ B_yC_z-B_zC_y & B_zC_x-B_xC_z & B_xC_y-B_yC_x \\ A_x & A_y & A_z \end{vmatrix}$$

which expands to

$$
\begin{aligned}
(\vec{B}\otimes\vec{C})\otimes\vec{A} = \hat{\imath}\,&[(B_zC_x - B_xC_z)\,A_z - (B_xC_y - B_yC_x)\,A_y] \\
+\hat{\jmath}\,&[(B_xC_y - B_yC_x)\,A_x - (B_yC_z - B_zC_y)\,A_z] \\
+\hat{k}\,&[(B_yC_z - B_zC_y)\,A_y - (B_zC_x - B_xC_z)\,A_x]
\end{aligned} \tag{2.4}
$$

We then have (since $r_x = x$, $r_y = y$, etc.) for $\vec{A}\otimes(\vec{B}\otimes\vec{C})$ with $\vec{A}\to\vec{r}$, $\vec{B}\to\vec{L}$ and $\vec{C}\to\vec{p}$ is

$$
\begin{aligned}
\vec{r}\otimes(\vec{L}\otimes\vec{p}) = \hat{\imath}\,&[y\,(L_xp_y - L_yp_x) - z\,(L_zp_x - L_xp_z)] \\
+\hat{\jmath}\,&[z\,(L_yp_z - L_zp_y) - x\,(L_xp_y - L_yp_x)] \\
+\hat{k}\,&[x\,(L_zp_x - L_xp_z) - y\,(L_yp_z - L_zp_y)]
\end{aligned} \tag{2.5}
$$

and with $\vec{A}\to\vec{r}$, $\vec{B}\to\vec{L}$ and $\vec{C}\to\vec{p}$ but with the order reversed, is

$$
\begin{aligned}
(\vec{L}\otimes\vec{p})\otimes\vec{r} = \hat{\imath}\,&[(L_zp_x - L_xp_z)\,z - (L_xp_y - L_yp_x)\,y] \\
+\hat{\jmath}\,&[(L_xp_y - L_yp_x)\,x - (L_yp_z - L_zp_y)\,z] \\
+\hat{k}\,&[(L_yp_z - L_zp_y)\,y - (L_zp_x - L_xp_z)\,x]
\end{aligned} \tag{2.6}
$$

For reference sake, in the next part of this work, we repeat this table from the earlier work:

$[L_x, x] = 0$	
$[L_y, y] = 0$	
$[L_z, z] = 0$	
$[L_x, y] = \imath\hbar z$	= -$[Ly, x]$
$[L_y, x] =$ -$\imath\hbar z$	$= L_x, y]$
$[L_x, z] = \imath\hbar x$	= - $[L_z, x]$
$[L_x, p_y] = \imath\hbar p_z$	
$[L_x, p_z] =$ -$\imath\hbar p_y$	
$[L_x, p_x] = 0$	

Adding the two, as requested, we have

$$
\begin{aligned}
&\vec{r}\otimes(\vec{L}\otimes\vec{p}) + (\vec{L}\otimes\vec{p})\otimes\vec{r} = \\
&\hat{\imath}\left[\underbrace{yL_xp_y} - \overbrace{yL_yp_x} - \underline{zL_zp_x} + \overline{zL_xp_z} + (\underline{L_zp_xz} - \overline{L_xp_zz} - \underbrace{L_xp_yy} + \overbrace{L_yp_xy})\right] \\
&+\hat{\jmath}\left[\underbrace{zL_yp_z} - \overbrace{zL_zp_y} - \underline{xL_xp_y} + \overline{xL_yp_x} + (\underline{L_xp_yx} - \overline{L_yp_xx} - \underbrace{L_yp_zz} + \overbrace{L_zp_yz})\right] \\
&+\hat{k}\left[\underbrace{xL_zp_x} - \overbrace{xL_xp_z} - \underline{yL_yp_z} + \overline{yL_zp_y} + (\underline{L_yp_zy} - \overline{L_zp_yy} - \underbrace{L_zp_xx} + \overbrace{L_xp_zx})\right]
\end{aligned} \tag{2.7}
$$

or, re-arranging

$$
\begin{aligned}
&\vec{r}\otimes(\vec{L}\otimes\vec{p}) + (\vec{L}\otimes\vec{p})\otimes\vec{r} = \\
&\hat{\imath}\left[\underbrace{yL_xp_y} - \underbrace{L_xp_yy} + \overbrace{\cancel{yL_yp_x}} - \overbrace{\cancel{L_yp_xy}} - \underline{\cancel{zL_zp_x}} + \underline{\cancel{L_zp_xz}} + \overline{zL_xp_z} - \overline{L_xp_zz}\right] \\
&+\hat{\jmath}\left[\underbrace{zL_yp_z} - \overbrace{\cancel{zL_zp_y}} + \overbrace{\cancel{L_zp_yz}} - \underbrace{L_yp_zz} - \underline{\cancel{xL_xp_y}} + \underline{\cancel{L_xp_yx}} + \overline{xL_yp_x} - \overline{L_yp_xx}\right] \\
&+\hat{k}\left[\underbrace{xL_zp_x} - \underbrace{L_zp_xx} - \overbrace{\cancel{xL_xp_z}} + \overbrace{\cancel{L_xp_zx}}) - \underline{\cancel{L_yp_zy}} + \underline{\cancel{yL_yp_z}} + \overline{yL_zp_y} - \overline{L_zp_yy}\right]
\end{aligned} \tag{2.8}
$$

which is, taking advantage of obvious cancellations:

$$
\vec{r}\otimes(\vec{L}\otimes\vec{p}) + (\vec{L}\otimes\vec{p})\otimes\vec{r} =
$$

$$
\hat{\imath}\left[\underbrace{yL_xp_y} + \overline{zL_xp_z} - \overline{L_xp_zz} - \underbrace{L_xp_yy}\right]
$$

$$+\hat{j}\left[\underbrace{zL_yp_z}+\overline{xL_yp_x}-\overline{L_yp_xx}-\underbrace{L_yp_zz}\right]$$
$$+\hat{k}\left[\underbrace{xL_zp_x}+\overline{yL_zp_y}-\overline{L_zp_yy}-\underbrace{L_zp_xx}\right] \tag{2.9}$$

which is

$$\begin{aligned}\vec{r}\otimes(\vec{L}\otimes\vec{p})+(\vec{L}\otimes\vec{p})\otimes\vec{r}=\\ \hat{i}\left[L_x(yp_y-p_yy+zp_z-p_zz)\right]\\ +\hat{j}\left[L_y(zp_z-p_zz+xp_x-p_xx)\right]\\ +\hat{k}\left[L_z(xp_x-p_xx+yp_y-p_yy)\right]\end{aligned} \tag{2.10}$$

which is

$$\begin{aligned}\vec{r}\otimes(\vec{L}\otimes\vec{p})+(\vec{L}\otimes\vec{p})\otimes\vec{r}=\\ \hat{i}\left[L_x(2\imath\hbar)\right]+\hat{j}\left[L_y(2\imath\hbar)\right]+\hat{k}\left[L_z(2\imath\hbar)\right]\end{aligned} \tag{2.11}$$

This means

$$\frac{1}{4Ze^2\mu}\left(\vec{r}\otimes(\vec{L}\otimes\vec{p})+(\vec{L}\otimes\vec{p})\otimes\vec{r}\right)=\frac{1}{4Ze^2\mu}\left(2\imath\hbar\vec{L}\right) \tag{2.12}$$

The other term we need (the obverse?) starts with $\vec{p}\otimes\vec{L}$ which would be

$$\vec{p}\otimes\vec{L}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ p_x & p_y & p_z\\ L_x & L_y & L_z\end{vmatrix}=\hat{i}(p_yL_z-p_zL_y)+\hat{j}(p_zL_x-p_xL_z)+\hat{k}(p_xL_y-p_yL_x)$$

Next, we need

$$\begin{aligned}\vec{r}\otimes(\vec{p}\otimes\vec{L})=&\ \hat{i}\left[y\left(p_xL_y-p_yL_x\right)-z\left(p_zL_x-p_xL_z\right)\right]\\ &+\hat{j}\left[z\left(p_yL_z-p_zL_y\right)-x\left(p_xL_y-p_yL_x\right)\right]\\ &+\hat{k}\left[x\left(p_zL_x-p_xL_z\right)-y\left(p_yL_z-p_zL_y\right)\right]\end{aligned} \tag{2.13}$$

and

$$\begin{aligned}(\vec{p}\otimes\vec{L})\otimes\vec{r}=&\ \hat{i}\left[\left(p_zL_x-p_xL_z\right)z-\left(p_xL_y-p_yL_x\right)y\right]\\ &+\hat{j}\left[\left(p_xL_y-p_yL_x\right)x-\left(p_yL_z-p_zL_y\right)z\right]\\ &+\hat{k}\left[\left(p_yL_z-p_zL_y\right)y-\left(p_zL_x-p_xL_z\right)x\right]\end{aligned} \tag{2.14}$$

$$\begin{aligned}\vec{r}\otimes(\vec{p}\otimes\vec{L})+(\vec{p}\otimes\vec{L})\otimes\vec{r}=\\ \hat{i}\left[y\left(p_xL_y-p_yL_x\right)-z\left(p_zL_x-p_xL_z\right)+\left(p_zL_x-p_xL_z\right)z-\left(p_xL_y-p_yL_x\right)y\right]\\ +\hat{j}\left[z\left(p_yL_z-p_zL_y\right)-x\left(p_xL_y-p_yL_x\right)+\left(p_xL_y-p_yL_x\right)x-\left(p_yL_z-p_zL_y\right)z\right]\\ +\hat{k}\left[x\left(p_zL_x-p_xL_z\right)-y\left(p_yL_z-p_zL_y\right)+\left(p_yL_z-p_zL_y\right)y-\left(p_zL_x-p_xL_z\right)x\right]\end{aligned} \tag{2.15}$$

which is

$$\begin{aligned}\vec{r}\otimes(\vec{p}\otimes\vec{L})+(\vec{p}\otimes\vec{L})\otimes\vec{r}=\\ \hat{i}\left[y\left(-p_yL_x\right)-z\left(p_zL_x\right)+\left(p_zL_x\right)z-\left(-p_yL_x\right)y\right]\\ +\hat{j}\left[z\left(-p_zL_y\right)-x\left(p_xL_y\right)+\left(p_xL_y\right)x-\left(-p_zL_y\right)z\right]\\ +\hat{k}\left[x\left(-p_xL_z\right)-y\left(p_yL_z\right)+\left(p_yL_z\right)y-\left(-p_xL_z\right)x\right]\end{aligned} \tag{2.16}$$

which is

$$\begin{aligned}\vec{r}\otimes(\vec{p}\otimes\vec{L})+(\vec{p}\otimes\vec{L})\otimes\vec{r}=\\ \hat{i}\left[y\left(-p_y(\underbrace{yp_z}-\overbrace{zp_y})\right)-z\left(p_z(\underline{yp_z}-\overline{zp_y})\right)+\left(p_z(\underline{yp_z}-\overline{zp_y})\right)z-\left(-p_y(\underbrace{yp_z}-\overbrace{zp_y})\right)y\right]\\ +\hat{j}\left[z\left(-p_z(\underbrace{zp_x}-\overbrace{xp_z})\right)-x\left(p_x(\underline{zp_x}-\overline{xp_z})\right)+\left(p_x(\underline{zp_x}-\overline{xp_z})\right)x-\left(-p_z(\underbrace{zp_x}-\overbrace{xp_z})\right)z\right]\\ +\hat{k}\left[x\left(-p_x(\underbrace{xp_y}-\overbrace{yp_x})\right)-y\left(p_y(\underline{xp_y}-\overline{yp_x})\right)+\left(p_y(\underline{xp_y}-\overline{yp_x})\right)y-\left(-p_x(\underbrace{xp_y}-\overbrace{yp_x})\right)x\right]\end{aligned} \tag{2.17}$$

or, expanding

$$
\begin{aligned}
&\vec{r}\otimes(\vec{p}\otimes\vec{L})+(\vec{p}\otimes\vec{L})\otimes\vec{r}= \\
&\hat{i}\left[-\underbrace{yp_yyp_z}+\overbrace{yp_yzp_y}-\underline{zp_zyp_z}+\overline{zp_zzp_y}+\underline{p_zyp_zz}-\overline{p_zzp_yz}+\underbrace{p_yyp_zy}-\overbrace{p_yzp_yy}\right] \\
&+\hat{j}\left[-\underbrace{zp_zzp_x}+\overbrace{zp_zxp_z}-\underline{xp_xzp_x}+\overline{xp_xxp_z}+\underline{p_xzp_xx}-\overline{p_xxp_zx}+\underbrace{p_zzp_xz}-\overbrace{p_zxp_zz}\right] \\
&+\hat{k}\left[-\underbrace{xp_xxp_y}+\overbrace{xp_xyp_x}-\underline{yp_yxp_y}+\overline{yp_yyp_x}+\underline{p_yxp_yy}-\overline{p_yyp_xy}+\underbrace{p_xxp_yx}-\overbrace{p_xyp_xx}\right]
\end{aligned}
\tag{2.18}
$$

or, rearranging

$$
\begin{aligned}
&\vec{r}\otimes(\vec{p}\otimes\vec{L})+(\vec{p}\otimes\vec{L})\otimes\vec{r}= \\
&\hat{i}\left[-\underbrace{yp_yyp_z+p_yyp_zy}+\overbrace{yp_yzp_y-p_yzp_yy}-\underline{zp_zyp_z+p_zyp_zz}+\overline{zp_zzp_y-p_zzp_yz}\right] \\
&+\hat{j}\left[-\underbrace{zp_zzp_x+p_zzp_xz}+\overbrace{zp_zxp_z-p_zxp_zz}-\underline{xp_xzp_x+p_xzp_xx}+\overline{xp_xxp_z-p_xxp_zx}\right] \\
&+\hat{k}\left[-\underbrace{xp_xxp_y+p_xxp_yx}+\overbrace{xp_xyp_x-p_xyp_xx}-\underline{yp_yxp_y+p_yxp_yy}+\overline{yp_yyp_x-p_yyp_xy}\right]
\end{aligned}
\tag{2.19}
$$

or

$$
\begin{aligned}
&\vec{r}\otimes(\vec{p}\otimes\vec{L})+(\vec{p}\otimes\vec{L})\otimes\vec{r}= \\
&\hat{i}\left[\underbrace{(-yp_yy+p_yy^2)p_z}+\overbrace{(yp_y^2-p_y^2y)z}+\underline{(-zp_z^2+p_z^2z)y}+\overline{(zp_zz-p_zz^2)p_y}\right] \\
&+\hat{j}\left[\underbrace{(-zp_z+p_zz)zp_x}+\overbrace{(zp_z^2-p_z^2z)x}+\underline{(-xp_x^2+p_x^2x)z}+\overline{(xp_xx-p_xx^2)p_z}\right] \\
&+\hat{k}\left[(-\underbrace{xp_xx+p_xx^2)p_y}+\overbrace{(xp_x^2-p_x^2x)y}-\underline{(yp_y^2+p_y^2y)x}+\overline{(yp_yy-p_yy^2)p_x}\right]
\end{aligned}
\tag{2.20}
$$

which, using the appropriate commutators, becomes

$$
\begin{aligned}
&\vec{r}\otimes(\vec{p}\otimes\vec{L})+(\vec{p}\otimes\vec{L})\otimes\vec{r}= \\
&\hat{i}\left[\underbrace{(-yp_y+p_yy)yp_z}+\overbrace{(yp_y^2-p_y^2y)z}+\underline{(-zp_z^2+p_z^2z)y}+\overline{(zp_z-p_zz)zp_y}\right] \\
&+\hat{j}\left[\underbrace{(-zp_z+p_zz)zp_x}+\overbrace{(zp_z^2-p_z^2z)x}+\underline{(-xp_x^2+p_x^2x)z}+\overline{(xp_x-p_xx)xp_z}\right] \\
&+\hat{k}\left[(-\underbrace{xp_x+p_xx)xp_y}+\overbrace{(xp_x^2-p_x^2x)y}-\underline{(yp_y^2+p_y^2y)x}+\overline{(yp_y-p_yy)yp_x}\right]
\end{aligned}
\tag{2.21}
$$

which is

$$
\begin{aligned}
&\vec{r}\otimes(\vec{p}\otimes\vec{L})+(\vec{p}\otimes\vec{L})\otimes\vec{r}= \\
&\hat{i}\left[\underbrace{(-\imath\hbar)p_zy}+\overbrace{(yp_y^2-p_y^2y)z}+\underline{(-zp_z^2+p_z^2z)y}+\overline{(\imath\hbar)p_yz}\right] \\
&+\hat{j}\left[\underbrace{(-\imath\hbar)zp_x}+\overbrace{(zp_z^2-p_z^2z)x}+\underline{(-xp_x^2+p_x^2x)z}+\overline{(\imath\hbar)xp_z}\right] \\
&+\hat{k}\left[(-\underbrace{\imath\hbar)xp_y}+\overbrace{(xp_x^2-p_x^2x)y}+\underline{(-yp_y^2+yp_y^2)x}+\overline{(\imath\hbar)yp_x}\right]
\end{aligned}
\tag{2.22}
$$

or

$$
\begin{aligned}
&\vec{r}\otimes(\vec{p}\otimes\vec{L})+(\vec{p}\otimes\vec{L})\otimes\vec{r}= \\
&\hat{i}\left[\underbrace{(-\imath\hbar)p_zy}+\overbrace{(\imath\hbar p_y)z}+\underline{(-\imath\hbar p_z)y}+\overline{(\imath\hbar)p_yz}\right]
\end{aligned}
$$

$$+\hat{\jmath}\left[\underbrace{(-\imath\hbar)zp_x}+\overbrace{(\imath\hbar p_z)x}+\underline{(-\imath\hbar p_x)z}+\overline{(\imath\hbar)xp_z}\right]$$
$$+\hat{k}\left[(-\underbrace{\imath\hbar)xp_y}+\overbrace{(\imath\hbar p_x)y}+\underline{(-\imath\hbar p_y)x}+\overline{(\imath\hbar)yp_x}\right] \quad (2.23)$$

or

$$\left(\frac{\vec{r}\otimes(\vec{p}\otimes\vec{L})+(\vec{p}\otimes\vec{L})\otimes\vec{r}}{-2\imath\hbar}\right)=$$
$$\hat{\imath}\left[yp_z-zp_y\right]+\hat{\jmath}\left[zp_x-xp_z\right]+\hat{k}\left[xp_y-yp_x\right]=$$
$$\hat{\imath}\left[L_x\right]+\hat{\jmath}\left[L_y\right]+\hat{k}\left[L_z\right]=\vec{L} \quad (2.24)$$

which becomes

$$\vec{r}\otimes(\vec{p}\otimes\vec{L})+(\vec{p}\otimes\vec{L})\otimes\vec{r}=-2\imath\hbar\vec{L} \quad (2.25)$$

We now substitute these two results (Equation 2.25 and Equation 2.12) into Equation 2.2:

$$\frac{1}{4Ze^2\mu}\left(\overline{\left(\vec{L}\otimes\vec{p}\right)\otimes\vec{r}+\vec{r}\otimes\left(\vec{L}\otimes\vec{p}\right)}\right.$$
$$\left.-\underline{\left(\vec{p}\otimes\vec{L}\right)\otimes\vec{r}+\vec{r}\otimes\left(\vec{p}\otimes\vec{L}\right)}\right) \quad (2.26)$$

Substituting, using Equation 2.12

$$\left(\vec{r}\otimes(\vec{L}\otimes\vec{p})+(\vec{L}\otimes\vec{p})\otimes\vec{r}\right)=2\imath\hbar\vec{L} \quad (2.27)$$

and Equation 2.25

$$\left(\vec{r}\otimes(\vec{p}\otimes\vec{L})+(\vec{p}\otimes\vec{L})\otimes\vec{r}\right)=-2\imath\hbar\vec{L} \quad (2.28)$$

one obtains

$$\frac{1}{2}\left(\vec{A}\otimes\vec{r}+\vec{r}\otimes\vec{A}\right)=\left(\frac{1}{2}\right)\frac{1}{4Ze^2\mu}\left(2\imath\hbar\vec{L}-(-2\imath\hbar\vec{L})\right)$$

which becomes

$$\frac{1}{2}\left(\vec{A}\otimes\vec{r}+\vec{r}\otimes\vec{A}\right)=\left(\frac{1}{2}\right)\frac{1}{Ze^2\mu}\left(\imath\hbar\vec{L}\right)$$

Unfortunately, Pauli's Equation 52 is

$$\frac{1}{2}\left(\vec{A}\otimes\vec{r}+\vec{r}\otimes\vec{A}\right)=-\frac{\hbar}{\imath}\frac{3}{2}\frac{1}{Ze^2m}\vec{L}$$

III.

We now assemble all our prior work into a set of commutators, sufficient to attack the Kepler problem. We start with the one already known, i.e., the angular momentum commutator rules.

We had

$$L_xL_y-L_yL_x=[L_x,L_z]=\imath\hbar L_z$$

and by cyclic permutation,

$$L_yL_z-L_zL_y=[L_y,L_z]=\imath\hbar L_x$$
$$L_zL_x-L_xL_z=[L_z,L_x]=\imath\hbar L_y$$

where $x\rightarrow y$, $y\rightarrow z$, and $z\rightarrow x$ corresponds to the cyclic permutation we are talking about.

There is a sensual formulation of these three rules expressed as a cross product, a highly stylized rendition:

$$\vec{L}\otimes\vec{L}=\imath\hbar\vec{L}$$

where of course the vectors are vector operators, so that the cross product refer to vectors and their components, and therefore, since these components are operators, do not commute and therefore *do not cancel*.

Let's expand the cross product to see what is happening in this condensed notation. We have

$$\vec{L}\otimes\vec{L}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ L_x & L_y & L_z\\ L_x & L_y & L_z\end{vmatrix}=\hat{\imath}\left(L_yL_z-L_zL_y\right)+\hat{\jmath}\left(L_zL_x-L_xL_z\right)+\hat{k}\left(L_xL_y-L_yL_x\right)$$

or

$$\vec{L}\otimes\vec{L}=\hat{\imath}\left(L_yL_z-L_zL_y\right)+\hat{\jmath}\left(L_zL_x-L_xL_z\right)+\hat{k}\left(L_xL_y-L_yL_x\right)=\hat{\imath}[L_y,L_z]+\hat{\jmath}[L_z,L_x]+\hat{k}[L_x,L_y]$$

Since

$$\hat{\imath}[L_y,L_z]+\hat{\jmath}[L_z,L_x]+\hat{k}[L_x,L_y]=\hat{\imath}\imath\hbar L_x+\hat{\jmath}\imath\hbar L_y+\hat{k}\imath\hbar L_z$$

we then have

$$\vec{L}\otimes\vec{L}=\imath\hbar\vec{L}$$

as asserted at the outset.

IV. THE RUNGE-LENZ VECTOR COMMUTATORS

A. Similar Components Commuting with Angular Momentum Components

We had (Equation 14.10 [2])

$$\vec{A} = \frac{1}{2Ze^2\mu}\left(\vec{L}\otimes\vec{p} - \vec{p}\otimes\vec{L}\right) + \hat{r} \tag{4.1}$$

(this is also Equation 13-31 in Borowitz, *loc cit*) as the defining equation for the operator form of the Runge-Lenz vector, and we wish now to see how this commutes with $\vec{L}$. First, we change over to a "better" (more normal) form.

We start with $[A_i, L_i]; \forall i$ which we think of as similar components. First, we need to evaluate

$$\vec{L}\otimes\vec{p} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ L_x & L_y & L_z \\ p_x & p_y & p_z \end{vmatrix} = \begin{pmatrix} \hat{i}(L_y p_z - L_z p_y) \\ \hat{j}(L_z p_x - L_x p_z) \\ \hat{k}(L_x p_y - L_y p_x) \end{pmatrix} \tag{4.2}$$

which is

$$\vec{L}\otimes\vec{p} = \begin{pmatrix} \hat{i}((zp_x - xp_z)p_z - (xp_y - yp_x)p_y) \\ \hat{j}((xp_y - yp_x)p_x - (yp_z - zp_y)p_z) \\ \hat{k}((yp_z - zp_y)p_y - (zp_x - xp_z)p_x) \end{pmatrix} \tag{4.3}$$

In the reverse order, we had

$$\vec{p}\otimes\vec{L} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ p_x & p_y & p_z \\ yp_z - zp_y & zp_x - xp_z & xp_y - yp_x \end{vmatrix} = \begin{pmatrix} \hat{i}(p_y(xp_y - yp_x) - p_z(zp_x - xp_z)) \\ \hat{j}(p_z(yp_z - zp_y) - p_x(xp_y - yp_x)) \\ \hat{k}(p_x(zp_x - xp_z) - p_y(yp_z - zp_y)) \end{pmatrix} \tag{4.4}$$

so, combining these two we have (don't worry, we're not making an error here relative to earlier work)

$$\vec{L}\otimes\vec{p} + \vec{p}\otimes\vec{L} = \begin{pmatrix} \hat{i}((zp_x - xp_z)p_z - (xp_y - yp_x)p_y) \\ \hat{j}((xp_y - yp_x)p_x - (yp_z - zp_y)p_z) \\ \hat{k}((yp_z - zp_y)p_y - (zp_x - xp_z)p_x) \end{pmatrix} + \begin{pmatrix} \hat{i}(p_y(xp_y - yp_x) - p_z(zp_x - xp_z)) \\ \hat{j}(p_z(yp_z - zp_y) - p_x(xp_y - yp_x)) \\ \hat{k}(p_x(zp_x - xp_z) - p_y(yp_z - zp_y)) \end{pmatrix} \tag{4.5}$$

which is, combining term

$$\vec{L}\otimes\vec{p} + \vec{p}\otimes\vec{L} = \begin{pmatrix} \hat{i}[(zp_x - xp_z)p_z - (xp_y - yp_x)p_y + \{p_y(xp_y - yp_x) - p_z(zp_x - xp_z)\}] \\ \hat{j}[(xp_y - yp_x)p_x - (yp_z - zp_y)p_z + \{p_z(yp_z - zp_y) - p_x(xp_y - yp_x)\}] \\ \hat{k}[(yp_z - zp_y)p_y - (zp_x - xp_z)p_x + \{p_x(zp_x - xp_z) - p_y(yp_z - zp_y)\}] \end{pmatrix} \tag{4.6}$$

which is, upon expansion,

$$\vec{L}\otimes\vec{p} + \vec{p}\otimes\vec{L} = \begin{pmatrix} \hat{i}[zp_x p_z - \cancel{xp_z^2} - \bcancel{xp_y^2} + yp_x p_y + \bcancel{xp_y^2} - p_y y p_x - p_z z p_x + \cancel{xp_z^2}] \\ \hat{j}[xp_y p_x i - \cancel{yp_x^2} - \bcancel{yp_z^2} + zp_y p_z + \bcancel{yp_z^2} - p_z z p_y - p_x x p_y + \cancel{yp_x^2}] \\ \hat{k}[yp_z p_y - \cancel{zp_y^2} - \bcancel{zp_x^2} + xp_z p_x + \bcancel{zp_x^2} - p_x x p_z - p_y y p_z + \cancel{zp_y^2}] \end{pmatrix} \tag{4.7}$$

or

$$\vec{L}\otimes\vec{p} + \vec{p}\otimes\vec{L} = \begin{pmatrix} \hat{i}[zp_x p_z + yp_x p_y - p_y y p_x - p_z z p_x] \\ \hat{j}[xp_y p_x + zp_y p_z - p_z z p_y - p_x x p_y] \\ \hat{k}[yp_z p_y + xp_z p_x - p_x x p_z - p_y y p_z] \end{pmatrix} \tag{4.8}$$

or

$$\vec{L}\otimes\vec{p} + \vec{p}\otimes\vec{L} = \begin{pmatrix} \hat{i}(p_x(zp_z - p_z z) + p_x(yp_y + p_y y)) \\ \hat{j}(p_y(xp_x - p_x x) + p_y(zp_z + p_z z)) \\ \hat{k}(p_z(yp_y - p_y y) + p_z(xp_x - p_x x)) \end{pmatrix} \tag{4.9}$$

$$\vec{L}\otimes\vec{p} + \vec{p}\otimes\vec{L} = \begin{pmatrix} \hat{i}2\imath\hbar p_x) \\ \hat{j}2\imath\hbar p_y \\ \hat{k}2\imath\hbar p_z) \end{pmatrix} \tag{4.10}$$

i.e.,

$$\vec{L}\otimes\vec{p} + \vec{p}\otimes\vec{L} = 2\imath\hbar\vec{p} \tag{4.11}$$

so we can rewrite this as

$$\vec{p}\otimes\vec{L} = -\vec{L}\otimes\vec{p} + 2\imath\hbar\vec{p} \tag{4.12}$$

which allows us to use a slightly simpler form in future work for the Runge Lenz vector, i.e.,

$$\vec{A} = \frac{1}{2Ze^2\mu}\left(\vec{L}\otimes\vec{p} - \left(-\vec{L}\otimes\vec{p} + 2\imath\hbar\vec{p}\right)\right) + \hat{r} \tag{4.13}$$

which results in our "final" version:

$$\vec{A} = \frac{1}{Ze^2\mu}\left(\vec{L}\otimes\vec{p} - \imath\hbar\vec{p}\right) + \hat{r} \tag{4.14}$$

This is the standard form used in most discussions of the Ringe-Lenz vector operator, i.e., not the original form cited and used before.

V. COMMUTATOR OF A_{op} WITH L_i, ETC.

It is a labor of love to obtain the commutators of the Runge-Lenz vector with respect to other quantum-mechanical operators needed for dealing with the hydrogen atom. The Runge-Lenz vector form used here is from Equation 4.14. What follows is a Maple session showing the commutator relationships needed to proceed.

VI. ADDENDUM

Given these commutator relations, one can now proceed on to the ladder operator solution to the H-atom's electronic energy levels (C. W. David, Am. J. Phys., 34, 984 (1966). This was the only time that I ever encountered the "thrill of discovery" first hand. It was exhilarating. It was even **cited:**Blinder, S. M., J. Chem. Educ. 2001, 78, 391.

```
> #Commutator example 2 Zhang
```

> From Hong-Tao Zhang, arxiv.org/PS_cache/quant-ph/pdf/0204/0204081v1.pdf, A Simple Method of Calculating Comtators in Hamiltonian System with Mathematica Software.

```
> restart;
> assume(r>0);
> Comm3D := proc(f,g)local t1 , t2, t3, t4;
> t1 := diff(f,q1)*diff(g,p1)-diff(f,p1)*diff(g,q1);
> t2 := diff(f,q2)*diff(g,p2)-diff(f,p2)*diff(g,q2);
> t3 := diff(f,q3)*diff(g,p3)-diff(f,p3)*diff(g,q3);
> t4 := subs(
> q1*p2-q2*p1=L_3,q2*p3-q3*p2=L_1,q3*p1-q1*p3=L_2,t1+t2+t3);
> t4 := subs(sqrt(q1^2+q2^2+q3^2)=r,t4);
> t4 := algsubs(q1^2+q2^2+q3^2=r^2,t4);
> t4 := expand(t4);
> return(I*hbar*t4);
> end proc;
```

Comm3D := **proc**(f, g)
local $t1$, $t2$, $t3$, $t4$;
$t1 := \text{diff}(f,\ q1) * \text{diff}(g,\ p1) - \text{diff}(f,\ p1) * \text{diff}(g,\ q1)$;
$t2 := \text{diff}(f,\ q2) * \text{diff}(g,\ p2) - \text{diff}(f,\ p2) * \text{diff}(g,\ q2)$;
$t3 := \text{diff}(f,\ q3) * \text{diff}(g,\ p3) - \text{diff}(f,\ p3) * \text{diff}(g,\ q3)$;
$t4 := \text{subs}(q1 * p2 - q2 * p1 = L\_3,\ q2 * p3 - q3 * p2 = L\_1,\ q3 * p1 - q1 * p3 = L\_2,$
$t1 + t2 + t3)$;
$t4 := \text{subs}(\text{sqrt}(q1^2 + q2^2 + q3^2) = r,\ t4)$;
$t4 := \text{algsubs}(q1^2 + q2^2 + q3^2 = r^2,\ t4)$;
$t4 := \text{expand}(t4)$;
return $hbar * t4 * I$
end proc

```
> ham := proc(p1,p2,p3,q1,q2,q3)
> return((p1^2+p2^2+p3^2)/2 - 1/sqrt(q1^2+q2^2+q3^2));
> end proc;
```

ham := **proc**($p1$, $p2$, $p3$, $q1$, $q2$, $q3$)
return $1/2 * p1^2 + 1/2 * p2^2 + 1/2 * p3^2 - 1/\text{sqrt}(q1^2 + q2^2 + q3^2)$
end proc

> The angular momentum vector is defined here:

```
> L1 := q2*p3-q3*p2:
> L2 := q3*p1-q1*p3:
> L3 := q1*p2-q2*p1:
```

```
> The Runge-Lenz vector is defined here:
> RL1 := L3*p2-L2*p3-I*hbar*p1-q1/sqrt(q1^2+q2^2+q3^2):
> RL2 := L1*p3-L3*p1-I*hbar*p2-q2/sqrt(q1^2+q2^2+q3^2):
> RL3 := L2*p1-L1*p2-I*hbar*p3-q3/sqrt(q1^2+q2^2+q3^2):
> To show that L1 x L2 = i hbar L3
> print (' Comm3D(L1,L2) = ');
> comm1 := Comm3D(L1,L2);
```

$$Comm3D(L1, L2) \ =$$

$$comm1 := hbar\ L\_3\ I$$

To show that RL1 x L1 equals zero

```
> print (' Comm3D(RL1,L1) = ');
> comm2 := Comm3D(RL1,L1):
> comm2 := simplify(comm2);
```

$$Comm3D(RL1, L1) \ =$$

$$comm2 := 0$$

```
> To show that RL1 x RL2 = - K L3
> comm3 := Comm3D(RL1,RL2):
> print (' Comm3D(RL1,RL2) = ');
> comm3a := expand(simplify(comm3/(I*hbar)));
> comm3b := subs(q1^3=q1*(r^2-q2^2-q3^2),comm3a);
> comm3b := subs(q2^3=q2*(r^2-q1^2-q3^2),comm3b);
> comm3b := subs(q3^3=q3*(r^2-q1^2-q2^2),comm3b);
> comm3c := expand(subs(p3^3 =p3*(2*(En+1/r)-(p1^2+p2^2)),comm3b));
> comm3d := algsubs(q1*p2-q2*p1=L_3,comm3c);#we know that L_1 and L_2
> are not involved
> comm3e := expand(subs(p3^2 = 2*(En- ((p1^2+p2^2)/2 - 1/r)),comm3d));
> January 25, 2007 SUCCESS
```

$$Comm3D(RL1, RL2) \ =$$

$$comm3a := -\frac{q3^2\ q2\ p1}{r\tilde{}^3} - \frac{q1^2\ q2\ p1}{r\tilde{}^3} - p2^3\ q1 + q2\ p1^3 - \frac{q2^3\ p1}{r\tilde{}^3} + p2^2\ q2\ p1 + p3^2\ q2\ p1 - p3^2\ q1\ p2 - p1^2\ p2\ q1 + \frac{2\ q1\ p2}{r\tilde{}} - \frac{q2\ p1}{r\tilde{}}$$

$$comm3b := -\frac{q3^2\ q2\ p1}{r\tilde{}^3} - \frac{q1^2\ q2\ p1}{r\tilde{}^3} - p2^3\ q1 + q2\ p1^3 - \frac{q2^3\ p1}{r\tilde{}^3} + p2^2\ q2\ p1 + p3^2\ q2\ p1 - p3^2\ q1\ p2 - p1^2\ p2\ q1 + \frac{2\ q1\ p2}{r\tilde{}} - \frac{q2\ p1}{r\tilde{}}$$

$$comm3b := -\frac{q3^2\ q2\ p1}{r\tilde{}^3} - \frac{q1^2\ q2\ p1}{r\tilde{}^3} - p2^3\ q1 + q2\ p1^3 - \frac{q2\ (r\tilde{}^2 - q1^2 - q3^2)\ p1}{r\tilde{}^3} + p2^2\ q2\ p1 + p3^2\ q2\ p1 - p3^2\ q1\ p2 - p1^2\ p2\ q1 + \frac{2\ q1\ p2}{r\tilde{}} - \frac{q2\ p1}{r\tilde{}}$$

$$comm3b := -\frac{q3^2\ q2\ p1}{r\tilde{}^3} - \frac{q1^2\ q2\ p1}{r\tilde{}^3} - p2^3\ q1 + q2\ p1^3 - \frac{q2\ (r\tilde{}^2 - q1^2 - q3^2)\ p1}{r\tilde{}^3} + p2^2\ q2\ p1 + p3^2\ q2\ p1 - p3^2\ q1\ p2 - p1^2\ p2\ q1 + \frac{2\ q1\ p2}{r\tilde{}} - \frac{q2\ p1}{r\tilde{}}$$

$$comm3c := -p2^3\ q1 + q2\ p1^3 - \frac{2\ q2\ p1}{r\tilde{}} + p2^2\ q2\ p1 + p3^2\ q2\ p1 - p3^2\ q1\ p2 - p1^2\ p2\ q1 + \frac{2\ q1\ p2}{r\tilde{}}$$

$$comm3d := -L\_3\ p1^2 - \frac{L\_3\ (-2 + p3^2\ r\tilde{})}{r\tilde{}} - p2^2\ L\_3$$

$$comm3e := -2\ L\_3\ En$$

```
> To show that RL1 x r = - K L3
> print('L cross r calculation');
> vec1 := L1*(Comm3D(q2,p2)+Comm3D(q3,p3));
> vec2 := L2*(Comm3D(q3,p3)+Comm3D(q1,p1));
> vec3 := L3*(Comm3D(q1,p1)+Comm3D(q2,p2));#this is Eqn 1.11 in h_lad1a
> manuscript.
> January 30, 2007 SUCCESS
```

$$\textit{L cross r calculation}$$
$$\mathit{vec1} := 2\,I\,(\mathit{q2}\;\mathit{p3} - \mathit{q3}\;\mathit{p2})\,\mathit{hbar}$$
$$\mathit{vec2} := 2\,I\,(\mathit{q3}\;\mathit{p1} - \mathit{q1}\;\mathit{p3})\,\mathit{hbar}$$
$$\mathit{vec3} := 2\,I\,(\mathit{q1}\;\mathit{p2} - \mathit{q2}\;\mathit{p1})\,\mathit{hbar}$$

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[1] http://digitalcommons.uconn.edu/chem_educ/14
[2] http://digitalcommons.uconn.edu/chem_educ op cit