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The Laplacian in Spherical Polar Coördinates

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I. SYNOPSIS

In treating the Hydrogen Atom's electron quantum mechanically, we normally convert the Hamiltonian from its Cartesian to its Spherical Polar form, since the problem is variable separable in the latter's coördinate system. This reading treats the brute-force method of effecting the transformation of the kinetic energy operator, normally called the Laplacian, from one to the other coördinate systems.

II. PRELIMINARY DEFINITIONS

We start with the primitive definitions

$$x = r \sin \theta \cos \phi$$

$$y = r \sin \theta \sin \phi$$

and

$$z = r \cos \theta$$

and their inverses

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \cos^{-1} \frac{z}{r} = \cos^{-1} \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

and

$$\phi = \tan^{-1} \frac{y}{x}$$

and attempt to write (using the chain rule)

$$\frac{\partial}{\partial x} = \left(\frac{\partial r}{\partial x} \right)_{y,z} \left(\frac{\partial}{\partial r} \right)_{\theta,\phi} + \left(\frac{\partial \theta}{\partial x} \right)_{y,z} \left(\frac{\partial}{\partial \theta} \right)_{r,\phi} + \left(\frac{\partial \phi}{\partial x} \right)_{y,z} \left(\frac{\partial}{\partial \phi} \right)_{r,\theta}$$

and

$$\frac{\partial}{\partial y} = \left(\frac{\partial r}{\partial y} \right)_{x,z} \left(\frac{\partial}{\partial r} \right)_{\theta,\phi} + \left(\frac{\partial \theta}{\partial y} \right)_{x,z} \left(\frac{\partial}{\partial \theta} \right)_{r,\phi} + \left(\frac{\partial \phi}{\partial y} \right)_{x,z} \left(\frac{\partial}{\partial \phi} \right)_{r,\theta}$$

and

$$\frac{\partial}{\partial z} = \left(\frac{\partial r}{\partial z} \right)_{x,y} \left(\frac{\partial}{\partial r} \right)_{\theta,\phi} + \left(\frac{\partial \theta}{\partial z} \right)_{x,y} \left(\frac{\partial}{\partial \theta} \right)_{r,\phi} + \left(\frac{\partial \phi}{\partial z} \right)_{x,y} \left(\frac{\partial}{\partial \phi} \right)_{r,\theta}$$

III. PRELIMINARY PARTIAL DERIVATIVES

$$\left(\frac{\partial r}{\partial z} \right)_{x,y} = \cos \theta \quad (3.3)$$

The needed (above) partial derivatives are:

$$\left(\frac{\partial r}{\partial x} \right)_{y,z} = \sin \theta \cos \phi \quad (3.1)$$

$$\left(\frac{\partial r}{\partial y} \right)_{x,z} = \sin \theta \sin \phi \quad (3.2)$$

and we have as a starting point for doing the θ terms,

$$\begin{aligned}
d \cos \theta &= -\sin \theta d\theta = \frac{dz}{r} + z \cdot d\left(\frac{1}{r}\right) = \frac{dz}{r} - z \cdot \left(\frac{1}{r^2}\right) dr \\
&= \frac{dz}{r} - \frac{z}{r^2} \frac{1}{r} (x dx + y dy + z dz)
\end{aligned} \tag{3.4}$$

so that, for example (when $dy = dz = 0$) we have

$$-\sin \theta d\theta = -\frac{z}{r^2} \frac{x}{r} dx$$

which is

$$-\sin \theta d\theta = -\frac{r \cos \theta}{r^2} \sin \theta \cos \phi dx = \frac{r^2 (1 - \cos^2 \vartheta)}{r^3} dz$$

so that

$$\left(\frac{\partial \theta}{\partial x}\right)_{y,z} = \frac{\cos \theta \cos \phi}{r} \tag{3.5}$$

$$\left(\frac{\partial \theta}{\partial y}\right)_{x,z} = \frac{\cos \theta \sin \phi}{r} \tag{3.6}$$

but, for the z-equation, we have

$$-\sin \theta d\theta = \frac{dz}{r} - \frac{z}{r^2} \frac{1}{r} z dz$$

which is

$$-\sin \theta d\theta = \left(\frac{1}{r} - \frac{z^2}{r^3}\right) dz = \frac{r^2 - z^2}{r^3} dz$$

$$-\sin \theta d\theta = \left(\frac{1}{r} - \frac{z^2}{r^3}\right) dz = \frac{r^2 \sin^2 \theta}{r^3} dz$$

so one has

$$\left(\frac{\partial \theta}{\partial z}\right)_{x,y} = -\frac{\sin \theta}{r} \tag{3.7}$$

Next, we have (as an example)

$$\tan \phi = \frac{\sin \phi}{\cos \phi} = \tan^{-1} \frac{y}{x}$$

and taking the partial derivatives on both sides, we obtain

$$\frac{1}{\cos \vartheta} d(\sin \phi) + \frac{\sin \phi}{-\cos^2 \phi} d(\cos \phi)$$

so

$$\left(1 + \frac{\sin^2 \phi}{\cos^2 \phi}\right) d\phi = \frac{dy}{x} - \frac{y}{x^2} dx$$

or

$$\left(\frac{1}{\cos^2 \phi}\right) d\phi = \frac{dy}{x} - \frac{y}{x^2} dx$$

so, after multiplying across by $\cos^2 \phi$ leads to (at constant x)

$$\left(\frac{\partial \phi}{\partial y}\right)_{x,z} = \frac{\cos \phi}{r \sin \theta} \tag{3.8}$$

and (at constant y)

$$\left(\frac{\partial \phi}{\partial x}\right)_{y,z} = -\frac{\sin \phi}{r \sin \theta} \tag{3.9}$$

$$\left(\frac{\partial \phi}{\partial z}\right)_{x,y} = 0 \tag{3.10}$$

IV. THE FIRST PARTIAL DERIVATIVE TERMS

Given these results (above) we write

$$\frac{\partial}{\partial z} = \cos \theta \frac{\partial}{\partial r} - \left(\frac{\sin \theta}{r}\right) \frac{\partial}{\partial \theta} \tag{4.1}$$

and

$$\frac{\partial}{\partial y} = (\sin \theta \sin \phi) \frac{\partial}{\partial r} + \left(\frac{\cos \theta \sin \phi}{r}\right) \frac{\partial}{\partial \theta} + \left(\frac{\cos \phi}{r \sin \theta}\right) \frac{\partial}{\partial \phi} \tag{4.2}$$

and

$$\frac{\partial}{\partial x} = (\sin \theta \cos \phi) \frac{\partial}{\partial r} + \left(\frac{\cos \theta \cos \phi}{r}\right) \frac{\partial}{\partial \theta} + \left(-\frac{\sin \phi}{r \sin \theta}\right) \frac{\partial}{\partial \phi} \tag{4.3}$$

V. GATHERING TERMS TO FORM THE LAPLACIAN

From Equation 4.1 we form

$$\frac{\partial^2}{\partial z^2} = \cos \theta \frac{\partial}{\partial r} \left[\cos \theta \frac{\partial}{\partial r} - \left(\frac{\sin \theta}{r} \right) \frac{\partial}{\partial \theta} \right] - \left(\frac{\sin \theta}{r} \right) \frac{\partial}{\partial \theta} \left(\cos \theta \frac{\partial}{\partial r} - \left(\frac{\sin \theta}{r} \right) \frac{\partial}{\partial \theta} \right) \quad (5.1)$$

while from Equation 4.2 we obtain

$$\begin{aligned} \frac{\partial^2}{\partial y^2} &= (\sin \theta \sin \phi) \frac{\partial}{\partial r} \left[\sin \theta \sin \phi \frac{\partial}{\partial r} + \left(\frac{\cos \theta \sin \phi}{r} \right) \frac{\partial}{\partial \theta} + \left(\frac{\cos \phi}{r \sin \theta} \right) \frac{\partial}{\partial \phi} \right] \\ &+ \left(\frac{\cos \theta \sin \phi}{r} \right) \frac{\partial}{\partial \theta} \left[\sin \theta \sin \phi \frac{\partial}{\partial r} + \left(\frac{\cos \theta \sin \phi}{r} \right) \frac{\partial}{\partial \theta} + \left(\frac{\cos \phi}{r \sin \theta} \right) \frac{\partial}{\partial \phi} \right] \\ &+ \left(\frac{\cos \phi}{r \sin \theta} \right) \frac{\partial}{\partial \phi} \left[\sin \theta \sin \phi \frac{\partial}{\partial r} + \left(\frac{\cos \theta \sin \phi}{r} \right) \frac{\partial}{\partial \theta} + \left(\frac{\cos \phi}{r \sin \theta} \right) \frac{\partial}{\partial \phi} \right] \end{aligned} \quad (5.2)$$

and from Equation 4.3 we obtain

$$\begin{aligned} \frac{\partial^2}{\partial x^2} &= (\sin \theta \cos \phi) \frac{\partial}{\partial r} \left[\sin \theta \cos \phi \frac{\partial}{\partial r} + \left(\frac{\cos \theta \cos \phi}{r} \right) \frac{\partial}{\partial \theta} - \left(\frac{\sin \phi}{r \sin \theta} \right) \frac{\partial}{\partial \phi} \right] \\ &+ \left(\frac{\cos \theta \cos \phi}{r} \right) \frac{\partial}{\partial \theta} \left[\sin \theta \cos \phi \frac{\partial}{\partial r} + \left(\frac{\cos \theta \cos \phi}{r} \right) \frac{\partial}{\partial \theta} - \left(\frac{\sin \phi}{r \sin \theta} \right) \frac{\partial}{\partial \phi} \right] \\ &- \left(\frac{\sin \phi}{r \sin \theta} \right) \frac{\partial}{\partial \phi} \left[\sin \theta \cos \phi \frac{\partial}{\partial r} + \left(\frac{\cos \theta \cos \phi}{r} \right) \frac{\partial}{\partial \theta} - \left(\frac{\sin \phi}{r \sin \theta} \right) \frac{\partial}{\partial \phi} \right] \end{aligned} \quad (5.3)$$

Expanding, we have

$$\begin{aligned} \frac{\partial^2}{\partial z^2} &= \cos^2 \theta \frac{\partial^2}{\partial r^2} + \frac{\cos \theta \sin \theta}{r^2} \frac{\partial}{\partial \theta} - \frac{\sin \theta \cos \theta}{r} \frac{\partial^2}{\partial r \partial \theta} \\ &- \left(\frac{\sin \theta}{r} \right) \left(-\sin \theta \frac{\partial}{\partial r} - \cos \theta \frac{\partial}{\partial \theta} \right) - \frac{\sin \theta \cos \theta}{r^2} \frac{\partial}{\partial \theta} + \left(\frac{\sin \theta}{r} \right)^2 \frac{\partial^2}{\partial \theta^2} \end{aligned} \quad (5.4)$$

while for the y-equation we have

$$\frac{\partial^2}{\partial y^2} = \sin^2 \theta \sin^2 \phi \frac{\partial^2}{\partial r^2} \quad (5.5)$$

$$+ \sin \theta \sin \phi \left[+ \left(\frac{\cos \theta \sin \phi}{r^2} \right) \frac{\partial}{\partial \theta} + \left(\frac{\cos \theta \sin \phi}{r} \right) \frac{\partial^2}{\partial r \partial \theta} \right] \quad (5.6)$$

$$+ \sin \theta \sin \phi \left[\left(-\frac{\cos \phi}{r^2 \sin \theta} \right) \frac{\partial}{\partial \phi} + \left(\frac{\cos \phi}{r \sin \theta} \right) \frac{\partial^2}{\partial r \partial \phi} \right] \quad (5.7)$$

$$+ \left(\frac{\cos \theta \sin \phi}{r} \right) \left[\cos \theta \sin \phi \frac{\partial}{\partial r} + \sin \theta \sin \phi \frac{\partial^2}{\partial r \partial \theta} \right] \quad (5.8)$$

$$+ \left(\frac{\cos \theta \sin \phi}{r} \right) \left[- \left(\frac{\sin \theta \sin \phi}{r} \right) \frac{\partial}{\partial \theta} + \left(\frac{\cos \theta \sin \phi}{r} \right) \frac{\partial^2}{\partial \theta^2} \right] \quad (5.9)$$

$$+ \left(\frac{\cos \theta \sin \phi}{r} \right) \left[- \left(\frac{\cos \phi \cos \theta}{r \sin^2 \theta} \right) \frac{\partial}{\partial \phi} + \left(\frac{\cos \phi}{r \sin \theta} \right) \frac{\partial^2}{\partial \phi \partial \theta} \right] \quad (5.10)$$

$$+ \left(\frac{\cos \phi}{r \sin \theta} \right) \left[\sin \theta \cos \phi \frac{\partial}{\partial r} + \sin \theta \sin \phi \frac{\partial^2}{\partial r \partial \phi} \right] \quad (5.11)$$

$$+ \left(\frac{\cos \phi}{r \sin \theta} \right) \left[+ \left(\frac{\cos \theta \cos \phi}{r} \right) \frac{\partial}{\partial \theta} + \left(\frac{\cos \theta \sin \phi}{r} \right) \frac{\partial^2}{\partial \theta \partial \phi} \right] \quad (5.12)$$

$$+ \left(\frac{\cos \phi}{r \sin \theta} \right) \left[- \left(\frac{\sin \phi \cos \phi}{r \sin \theta} \right) \frac{\partial}{\partial \phi} + \left(\frac{\cos \phi}{r \sin \theta} \right) \frac{\partial^2}{\partial \phi^2} \right] \quad (5.13)$$

and finally

$$\begin{aligned} \frac{\partial^2}{\partial x^2} &= (\sin \theta \cos \phi) \sin \theta \cos \phi \frac{\partial^2}{\partial r^2} \\ &+ (\sin \theta \cos \phi) \left[- \left(\frac{\cos \theta \cos \phi}{r^2} \right) \frac{\partial}{\partial \theta} + \left(\frac{\cos \theta \cos \phi}{r} \right) \frac{\partial^2}{\partial \theta \partial r} \right] \end{aligned} \quad (5.14)$$

$$- (\sin \theta \cos \phi) \left[- \left(\frac{\sin \phi}{r^2 \sin \theta} \right) \frac{\partial}{\partial \phi} + \left(\frac{\sin \phi}{r \sin \theta} \right) \frac{\partial^2}{\partial \phi \partial r} \right] \quad (5.15)$$

$$+ \left(\frac{\cos \theta \cos \phi}{r} \right) \left[\cos \theta \cos \phi \frac{\partial}{\partial r} + \sin \theta \cos \phi \frac{\partial^2}{\partial r \partial \theta} \right] \quad (5.16)$$

$$+ \left(\frac{\cos \theta \cos \phi}{r} \right) \left[- \left(\frac{\sin \theta \cos \phi}{r} \right) \frac{\partial}{\partial \theta} + \left(\frac{\cos \theta \cos \phi}{r} \right) \frac{\partial^2}{\partial \theta^2} \right] \quad (5.17)$$

$$+ \left(\frac{\cos \theta \cos \phi}{r} \right) \left[+ \left(\frac{\sin \phi}{r \sin^2 \theta} \right) \frac{\partial}{\partial \phi} - \left(\frac{\sin \phi}{r \sin \theta} \right) \frac{\partial^2}{\partial \phi \partial \theta} \right] \quad (5.18)$$

$$- \left(\frac{\sin \phi}{r \sin \theta} \right) \left[\sin \theta \sin \phi \frac{\partial}{\partial r} + \sin \theta \cos \phi \frac{\partial^2}{\partial r \partial \phi} \right] \quad (5.19)$$

$$- \left(\frac{\sin \phi}{r \sin \theta} \right) \left[- \left(\frac{\cos \theta \sin \phi}{r} \right) \frac{\partial}{\partial \theta} + \left(\frac{\cos \theta \cos \phi}{r} \right) \frac{\partial^2}{\partial \theta \partial \phi} \right] \quad (5.20)$$

$$- \left(\frac{\sin \phi}{r \sin \theta} \right) \left[- \left(\frac{\cos \phi}{r \sin \theta} \right) \frac{\partial}{\partial \phi} - \left(\frac{\sin \phi}{r \sin \theta} \right) \frac{\partial^2}{\partial \phi^2} \right] \quad (5.21)$$

Now, one by one, we expand completely each of these three terms. We have

$$\frac{\partial^2}{\partial z^2} = \cos^2 \theta \frac{\partial^2}{\partial r^2} \quad (5.22)$$

$$+ \frac{\cos \theta \sin \theta}{r^2} \frac{\partial}{\partial \theta} \quad (5.23)$$

$$- \frac{\sin \theta \cos \theta}{r} \frac{\partial^2}{\partial r \partial \theta} \quad (5.24)$$

$$+ \left(\frac{\sin^2 \theta}{r} \right) \frac{\partial}{\partial r} \quad (5.25)$$

$$- \left(\frac{\sin \theta \cos \theta}{r} \right) \frac{\partial^2}{\partial r \partial \theta} \quad (5.26)$$

$$+ \frac{\sin \theta \cos \theta}{r^2} \frac{\partial}{\partial \theta} \quad (5.27)$$

$$+ \left(\frac{\sin^2 \theta}{r^2} \right) \frac{\partial^2}{\partial \theta^2} \quad (5.28)$$

and, for the y-equation:

$$\frac{\partial^2}{\partial y^2} = \sin^2 \theta \sin^2 \phi \frac{\partial^2}{\partial r^2} \quad (5.29)$$

$$(5.6) \rightarrow + \left(\frac{\sin \theta \cos \theta \sin^2 \phi}{r^2} \right) \frac{\partial}{\partial \theta} \quad (5.30)$$

$$+ \left(\frac{\cos \theta \sin \theta \sin^2 \phi}{r} \right) \frac{\partial^2}{\partial r \partial \theta} \quad (5.31)$$

$$(5.7) \rightarrow - \left(\frac{\sin \phi \cos \phi}{r^2} \right) \frac{\partial}{\partial \phi} \quad (5.32)$$

$$+ \left(\frac{\cos \phi \sin \phi}{r} \right) \frac{\partial^2}{\partial r \partial \phi} \quad (5.33)$$

$$(5.8) \rightarrow + \left(\frac{\cos^2 \theta \sin^2 \phi}{r} \right) \frac{\partial}{\partial r} \quad (5.34)$$

$$+ \left(\frac{\cos \theta \sin \theta \sin^2 \phi}{r} \right) \frac{\partial^2}{\partial r \partial \theta} \quad (5.35)$$

$$- \left(\frac{\sin \theta \cos \theta \sin^2 \phi}{r^2} \right) \frac{\partial}{\partial \theta} \quad (5.36)$$

$$(5.9) \rightarrow + \left(\frac{\cos^2 \theta \sin^2 \phi}{r^2} \right) \frac{\partial^2}{\partial \theta^2} \quad (5.37)$$

$$- \left(\frac{\cos^2 \theta \cos \phi \sin \phi}{r \sin^2 \theta} \right) \frac{\partial}{\partial \phi} \quad (5.38)$$

$$+ \left(\frac{\cos \theta \cos \phi \sin \phi}{r^2 \sin \theta} \right) \frac{\partial^2}{\partial \phi \partial \theta} \quad (5.39)$$

$$(5.10) \rightarrow + \left(\frac{\cos^2 \phi}{r} \right) \frac{\partial}{\partial r} \quad (5.40)$$

$$+ \left(\frac{\cos \phi \sin \phi}{r} \right) \frac{\partial^2}{\partial r \partial \phi} \quad (5.41)$$

$$+ \left(\frac{\cos^2 \phi \cos \theta}{r^2 \sin \theta} \right) \frac{\partial}{\partial \theta} \quad (5.42)$$

$$(5.12) \rightarrow + \left(\frac{\cos \theta \cos \phi \sin \phi}{r^2 \sin \theta} \right) \frac{\partial^2}{\partial \theta \partial \phi} \quad (5.43)$$

$$(5.13) \rightarrow - \left(\frac{\cos^2 \phi \sin \phi}{r \sin^2 \theta} \right) \frac{\partial}{\partial \phi} \quad (5.44)$$

$$+ \left(\frac{\cos^2 \phi}{r^2 \sin^2 \theta} \right) \frac{\partial^2}{\partial \phi^2} \quad (5.45)$$

and finally, for the x-equation, we have

$$\frac{\partial^2}{\partial x^2} = \sin^2 \theta \cos^2 \phi \frac{\partial^2}{\partial r^2} \quad (5.46)$$

$$(5.14) \rightarrow - \left(\frac{\sin \theta \cos \theta \cos^2 \phi}{r^2} \right) \frac{\partial}{\partial \theta} \quad (5.47)$$

$$(5.14) \rightarrow + \left(\frac{\sin \theta \cos \theta \cos^2 \phi}{r} \right) \frac{\partial^2}{\partial \theta \partial r} \quad (5.48)$$

$$\left(\frac{\cos \phi \sin \phi}{r^2} \right) \frac{\partial}{\partial \phi} \quad (5.49)$$

$$- \left(\frac{\sin \phi \cos \phi}{r} \right) \frac{\partial^2}{\partial \phi \partial r} \quad (5.50)$$

$$(5.15) \rightarrow + \left(\frac{\cos^2 \theta \cos^2 \phi}{r} \right) \frac{\partial}{\partial r} \quad (5.51)$$

$$+ \left(\frac{\sin \theta \cos \theta \cos^2 \phi}{r} \right) \frac{\partial^2}{\partial r \partial \theta} \quad (5.52)$$

$$(5.15) \rightarrow - \left(\frac{\sin \theta \cos \theta \cos^2 \phi}{r^2} \right) \frac{\partial}{\partial \theta} \quad (5.53)$$

$$+ \left(\frac{\cos^2 \theta \cos^2 \phi}{r^2} \right) \frac{\partial^2}{\partial \theta^2} \quad (5.54)$$

$$(5.16) \rightarrow + \left(\frac{\cos \theta \cos \phi}{r} \right) \left(\frac{\cos \phi \sin \phi}{r \sin \theta} \right) \frac{\partial}{\partial \phi} \quad (5.55)$$

$$- \left(\frac{\sin \phi \cos \phi \cos \theta}{r^2 \sin \theta} \right) \frac{\partial^2}{\partial \phi \partial \theta} \quad (5.56)$$

$$(5.17) \rightarrow - \left(\frac{\sin^2 \phi}{r} \right) \frac{\partial}{\partial r} \quad (5.57)$$

$$- \left(\frac{\sin \phi \cos \phi}{r} \right) \frac{\partial^2}{\partial r \partial \phi} \quad (5.58)$$

$$(5.19) \rightarrow + \left(\frac{\cos \theta \sin^2 \phi}{r^2 \sin \theta} \right) \frac{\partial}{\partial \theta} \quad (5.59)$$

$$- \left(\frac{\cos \theta \sin \phi \cos \phi}{r^2 \sin \theta} \right) \frac{\partial^2}{\partial \theta \partial \phi} \quad (5.60)$$

$$(5.20) \rightarrow + \left(\frac{\sin \phi \cos \phi}{r \sin^2 \theta} \right) \frac{\partial}{\partial \phi} \quad (5.61)$$

$$+ \left(\frac{\sin^2 \phi}{r^2 \sin^2 \theta} \right) \frac{\partial^2}{\partial \phi^2} \quad (5.62)$$

Gathering terms as coefficients of partial derivatives, we obtain (from Equations 5.22, 5.29 and 5.46)

$$\frac{\partial^2}{\partial r^2} (\cos^2 \theta + \sin^2 \theta \sin^2 \phi + \sin^2 \theta \cos^2 \phi) \rightarrow \frac{\partial^2}{\partial r^2}$$

and (from Equations 5.23, 5.26, 5.30, 5.36, 5.42, 5.47, 5.53, and 5.59)

$$\begin{aligned} \frac{\partial}{\partial \theta} \left(+ \frac{\cos \theta \sin \theta}{r^2} + \frac{\sin \theta \cos \theta}{r^2} - \frac{\sin \theta \cos \theta \sin^2 \phi}{r^2} - \frac{\sin \theta \cos \theta \sin^2 \phi}{r^2} + \frac{\cos^2 \phi \cos \theta}{r^2 \sin \theta} \right. \\ \left. - \frac{\sin \theta \cos \theta \cos^2 \phi}{r^2} - \frac{\sin \theta \cos \theta \cos^2 \phi}{r^2} + \frac{\cos \theta \sin^2 \phi}{r^2 \sin \theta} \right) \\ \rightarrow \frac{\cos \theta}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \end{aligned} \quad (5.63)$$

while we obtain from Equations 5.28, 5.37, and 5.54:

$$\frac{\partial^2}{\partial \theta^2} \left(\frac{\sin^2 \theta}{r^2} + \frac{\cos^2 \theta \sin^2 \phi}{r^2} + \frac{\cos^2 \theta \cos^2 \phi}{r^2} \right) \rightarrow \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \quad (5.64)$$

From Equations 5.25, 5.34, 5.40, 5.51, 5.57,

$$\frac{\partial}{\partial r} \left(+ \frac{\sin^2 \theta}{r} + \frac{\cos^2 \theta \sin^2 \phi}{r} + \frac{\cos^2 \phi}{r} + \frac{\cos^2 \theta \cos^2 \phi}{r} - \frac{\sin^2 \phi}{r} \right) \rightarrow \frac{2}{r} \frac{\partial}{\partial r} \quad (5.65)$$

From Equations 5.32, 5.38, 5.44, 5.49, 5.55 and 5.61 we obtain

$$\begin{aligned} \frac{\partial}{\partial \phi} \left(- \frac{\sin \phi \cos \phi}{r^2} - \frac{\cos^2 \theta \cos \phi \sin \phi}{r \sin^2 \theta} - \frac{\cos^2 \theta \cos \phi \sin \phi}{r \sin^2 \theta} + \frac{\cos \phi \sin \phi}{r^2} + \left(\frac{\cos \theta \cos \phi}{r} \right) \right. \\ \left. + \left(\frac{\cos \theta \cos^2 \phi \sin \phi}{r^2 \sin \theta} \right) + \frac{\sin \phi \cos \phi}{r \sin^2 \theta} \right) \rightarrow 0 \end{aligned} \quad (5.66)$$

From Equations 5.45 and 5.62 we obtain

$$\frac{\partial^2}{\partial \phi^2} \left(\frac{\cos^2 \phi}{r^2 \sin^2 \theta} + \frac{\sin^2 \phi}{r^2 \sin^2 \theta} \right) \rightarrow \left(\frac{1}{r^2 \sin^2 \theta} \right) \frac{\partial^2}{\partial \phi^2} \quad (5.67)$$

The mixed derivatives yield, first, from Equations 5.33, 5.41, 5.50, and 5.58 leading to

$$\frac{\partial^2}{\partial r \partial \phi} \left(\frac{\cos \phi \sin \phi}{r} + \frac{\cos \phi \sin \phi}{r} - \frac{\sin \phi \cos \phi}{r} - \frac{\sin \phi \cos \phi}{r} \right) \rightarrow 0 \quad (5.68)$$

From Equations 5.24, 5.27, 5.35, 5.31 5.52, 5.48

$$\begin{aligned} & \frac{\partial^2}{\partial r \partial \theta} \left(-\frac{\sin \theta \cos \theta}{r} - \frac{\sin \theta \cos \theta}{r} + \frac{\cos \theta \sin \theta \sin^2 \phi}{r} \right. \\ & \left. + \frac{\sin \theta \cos \theta \cos^2 \phi}{r} + \frac{\cos \theta \sin \theta \sin^2 \phi}{r} + \frac{\sin \theta \cos \theta \cos^2 \phi}{r} \right) \rightarrow 0 \end{aligned} \quad (5.69)$$

From Equations 5.39 5.43 5.56 5.60

$$\begin{aligned} & \frac{\partial^2}{\partial \phi \partial \theta} \left(\frac{\cos \theta \cos \phi \sin \phi}{r^2 \sin \theta} + \frac{\cos \phi \sin \phi}{r^2 \sin \theta} \right. \\ & \left. - \left(\frac{\sin \phi \cos \phi \cos \theta}{r^2 \sin \theta} \right) - \left(\frac{\cos \theta \sin \phi \cos \phi}{r^2 \sin \theta} \right) \right) \rightarrow 0 \end{aligned} \quad (5.70)$$

Gathering together the non-vanishing terms, we obtain

$$\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\cos \theta}{r^2 \sin \theta} \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}$$

which is one of the two “classic” forms for ∇^2 . The other is

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin^2 \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \left(\frac{\sin \theta}{\partial \theta} \right) + \frac{\partial^2}{\partial \phi^2} \right)$$

VI. MAPLE EQUIVALENT

A. Example 1

Here is a set of Maple instructions adjusted from the 2-dimensional code [1] for our 3-dimensional case, which will get you the same result:

```
restart;
f:=g(r,theta,phi);
tx :=
sin(theta)*cos(phi)*diff(f,r)+((cos(theta)*cos(phi))/r)*diff(f,theta)
-(sin(phi)/(r*sin(theta)))*diff(f,phi);
tx2:=expand(
sin(theta)*cos(phi)*diff(tx,r)+((cos(theta)*cos(phi))/r)*diff(tx,theta)
-(sin(phi)/(r*sin(theta)))*diff(tx,phi));
ty :=
sin(theta)*sin(phi)*diff(f,r)+((cos(theta)*sin(phi))/r)*diff(f,theta)
+(cos(phi)/(r*sin(theta)))*diff(f,phi);

ty2:=expand(sin(theta)*sin(phi)*diff(ty,r)+((cos(theta)*sin(phi))/r)
*diff(ty,theta)+(cos(phi)/(r*sin(theta)))*diff(ty,phi));
tz := cos(theta)*diff(f,r)
-(sin(theta)/r)*diff(f,theta);
tz2 := expand(cos(theta)*diff(tz,r)-(sin(theta)/r)*diff(tz,theta));

del := tx2+ty2+tz2;
del := algsubs( cos(theta)^2=1-sin(theta)^2, del );
del := expand(algsubs( cos(phi)^2=1-sin(phi)^2, del ));
```

B. Example 2

Here is another version of the same thing:

```

> #CARTESIAN TO SPHERICAL POLAR
> restart;
> with(plots):

```

Warning, the name changecoords has been redefined

```

> uu:=u(sqrt(x^2+y^2+z^2),arccos(z/sqrt(x^2+y^2+z^2)),arctan(y,x));

```

$$uu := u(\sqrt{x^2 + y^2 + z^2}, \arccos(\frac{z}{\sqrt{x^2 + y^2 + z^2}}), \arctan(y, x))$$

```

> ux:=diff(uu,x):
> uy:=diff(uu,y):
> uz:=diff(uu,z):
> uxx:=diff(ux,x):
> uyy:=diff(uy,y):
> uzz:=diff(uz,z):
> Lapu:=simplify(uxx+uyy+uzz):
> assume(r,positive);
> Lapu:=simplify(subs(x=r*sin(theta)*cos(phi),
> y=r*sin(theta)*sin(phi),
> z = r*cos(theta),
> arctan(sin(theta)*sin(phi),sin(theta)*cos(phi))=phi,
> arccos(cos(theta))=theta,
> Lapu),trig):
> Lapu := subs(arctan(sin(theta)*sin(phi),sin(theta)*cos(phi))=phi,
> arccos(cos(theta))=theta,
> Lapu):
> Lapu := algsubs(-1+cos(theta)^2=-sin(theta)^2,Lapu):
> Lapu:=expand(Lapu);

```

$$Lapu := \frac{D_2(u)(r^{\sim}, \theta, \phi) \sin(\theta)^2 \cos(\theta)}{r^{\sim 2} (\sin(\theta)^2)^{(3/2)}} + \frac{D_{2,2}(u)(r^{\sim}, \theta, \phi)}{r^{\sim 2}} + \frac{D_{3,3}(u)(r^{\sim}, \theta, \phi)}{r^{\sim 2} \sin(\theta)^2} + \frac{2 D_1(u)(r^{\sim}, \theta, \phi)}{r^{\sim}} + D_{1,1}(u)(r^{\sim}, \theta, \phi)$$

It takes some getting used to Maple notation to see that this is the expected result.

are better ways to carry out the transformation from Cartesian to Spherical Polar (and indeed any orthogonal) coordinate system.

VII. COMMENTS

The reader should be aware that the brute force methods used here are primitive in the extreme, and that there

[1] Mathias Kawski, <http://math.la.asu.edu/~kawski/MAPLE/MAPLE.html>